Numerical Methods Solving ODE

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1 Introduction

At present, Ordinary Differential Equation (ODE) has been widely used in many aspects such as physics, biology and economics. In practical problems, it is particularly important to find the solution of ordinary differential equations. Despite the analytical solution can find the accurate answer, in some practical problems either it is hard to find the analytic solutions or the solving process is too complicated. In this case, it is extremely important to find the numerical solution of ODE by some methods (such as Euler method, Runge-Kutta method and Adams method). In this report, several methods – Euler method, Runge-Kutta method and Adams method—have been compared in the process of solving the two practical problems.

1.1 Euler's Method

Euler's method approximates the value of $\Phi(t)$ in the region $[t_n, t_{n+1}]$ of the curve by a straight line. In this report we mainly talk about four methods—the Forward Euler's Method, the Backward Euler's Method, the Trapezium Euler's Method and the Improved Euler's Method.

The Forward Euler's Method approximate the original curve with a tangent line of slope $f(t_n, y_n)$. The formula is in the form:

$$y_{n+1} = y_n + hf(t_n, y_n),\ \ n = 0, 1, 2, ... \tag{1}$$

The Backward Euler's Method approximate the original curve with a tangent line of slope $f(t_{n+1}, y_{n+1})$. The formula is in the form:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}),\ \ n = 0, 1, 2, ... \tag{2}$$

The Trapezium Euler's Method approximate the original curve with a tangent line of slope $1/over2(f(t_n) + f(t_(n+1)))$. The formula is in the form:

$$y_{n+1} = y_n + \frac{h}{2}(f(t_n) + f(t_(n+1))),\ \ n = 0, 1, 2, ... \tag{3}$$

The Improved Euler's Method combines the Forward Euler's Method and the Trapezium Euler Method. The formula is in the form:

$$y_{n+1} = y_n + \frac{f_n + f(t_n + h, y_n + hf(t_n, y_n))}{2}h,\ \ , n = 0, 1, 2, ... \tag{4}$$

1.2 The Runge-Kutta Method

The Runge-Kutta Method is similar to the Euler's Method, however, in this method, we try to find the slope with a few more points on the region $[t_n, t_{n+1}]$, then taking a weighted average of the slope values at these points, and using the resulting value as the approximate slope k. In this report we discuss the third-order three-stage Runge-Kutta method and fourth-order four-stage Runge-Kutta method. The formula of third-order three-stage Runge-Kutta method is in the form:

$$y_{n+1} = y_n + h(\frac{k_{n1} + 4k_{n2} + k_{n3}}{6}),\ \ n = 0, 1, 2, ... \tag{5}$$

Where

$$
\begin{aligned}
k_{n1} &= f(t_n, y_n),\\
k_{n2} &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}),\\
k_{n3} &= f(t_n + h, y_n - hk_{n1} + \frac{1}{2}hk_{n2}),
\end{aligned}
$$

The formula of fourth-order four-stage Runge-Kutta method is in the form:

$$
y_{n+1} = y_n + h(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}),\ \ n = 0, 1, 2, ... \tag{6}
$$

Where

$$
\begin{aligned}
k_{n1} &= f(t_n, y_n),\\
k_{n2} &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}),\\
k_{n3} &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}),\\
k_{n4} &= f(t_n + h, y_n + hk_{n3}),
\end{aligned}
$$

1.3 Adam's Method

Adam's Method makes use of a polynomial $P_k(t)$ of degree k to approximate $phi'(t)$ and evaluates the integral of $phi'(t)$. Second-order Adams-Bashforth formula is

$$
y_{n+1} = y_n + \frac{3}{2}hf_n - \frac{1}{2}f_{n-1},\ n = 0, 1, 2, ... \tag{7}
$$

Fourth-order Adams-Bashforth formula is

$$
y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}),\ n = 0, 1, 2, ... \tag{8}
$$

Adams-Moulton formulas is

$$
y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}),\ n = 0, 1, 2, ... \tag{9}
$$

2 Numerical Solution

2.1 Problem 1

For problem 1, we are supposed to compute the solution of such a first-order nonlinear ODE:

$$
y' = y^2 + ty + t^2 \tag{ODE1}
$$

2.1.1 Domain

To acquire a maximal solution of this problem, an relatively accurate domain should be computed. The estimation of the boundaries can be done both in analytical and numerical way. A pure analytical way can be Taylor Series, which yet is not the point of this report.

At the beginning, some graphic methods could be made use of to provide a big picture of how the system works, e.g. a slope field as Figure 1.

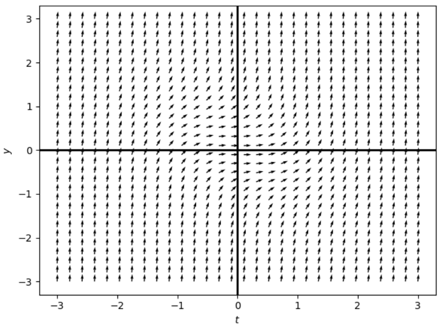

Figure 1: The slope field of $f(t, y)$ of Problem 1

In Figure 1 it seems that the solution curve of problem 1 is monotonic increasing. In fact, the derivative $y' = y^2 + ty + t^2 = (y + \frac{1}{2}t)^2 + \frac{3}{4}t^2 \geq 0$, thus it's globally increasing.

Also, Figure 1 turns out that the maximal solution might be restricted between two vertical asymptotes. Therefore determining the domain is to determine these two essential asymptotes, and at this moment a using numerical method to make some attempts can be helpful.

We make use of three numerical methods, i.e. the Euler Improved Method, the Runge-Kutta Method (4^{th} order) and the Adams-Monlton Method to determine the value of t when y is infinity.

	Euler improved		**RK 4th**		**Adams-Monlton**	
h	a_1	b_1	a_1	b_1	a_1	b_1
0.01	-2.17	0.91	-2.14	0.88	-2.12	0.85
0.005	-2.15	0.885	-2.13	0.87	-2.1201	0.855
0.001	-2.126	0.864	-2.123	0.861	-2.1202	0.858
0.0005	-2.1235	0.8615	-2.122	0.86	-2.1205	0.8585
0.0002	-2.1218	0.86	-2.121	0.8594	-2.1206	0.8587
0.0001	-2.1212	0.8594	-2.1209	0.8591	-2.1207	0.8588

Table 1: Determine t When y is infinity

It seems that the left bound of the solution is around $a_1 = -2.12$ and the right around $b_1 = 0.85$. To be more accurate, some analytical approaches should be adopted.

We will use two functions f_1 and f_2 to squeeze f and reduce the range. The sign relationship between t and y should be determined first. We use the numerical results of the Adams-Monlton method at $t = -2.120$ and $t = 0.858$ in Table 2 to illustrate some details.

	$t = -2.120$	$t = 0.858$
h	y	y
0.01	$-\infty$	∞
0.005	$-\infty$	∞
0.001	$-\infty$	∞
0.0005	-1657.12353516	1177.6400201
0.0002	-1523.43140641	1142.81318658
0.0001	-1518.10313846	1141.29183391

Table 2: y of the results of the Adams-Monlton method when t = -2.120 and t = 0.858

It's obvious that t and y have the same sign, plus the monotonicity and $|y| > |t|$ near the border, we can have such relationship: $y^2 < y^2 + ty + t^2 < 3y^2$. Therefore we obtain two solvable ODEs $y' = f_1(t, y) = y^2$ and $y' = f_2(t, y) = 3y^2$ to locate the border accurately. By solving these two ODEs we obtain:

$$\Phi_1(t) = -\frac{1}{t + C_1} \tag{10}$$

$$\Phi_2(t) = -\frac{1}{3t + C_2} \tag{11}$$

For the left boundary, we substitude the initial problem with $y(-2.120) = -1518.10313846$ from the results of **??** and obtain $C_1 = 2.12065871$ and $C_2 = 6.36065871$, which turns out that the left bound should be restricted between -2.12065871 and -2.12021957, i.e., $-2.12065871 < a_1 < -2.12021957$.

For the right boundary, similarly, we take the initial problem as $y(0.858) = 1141.29183391$ and obtain $C_1 = -0.85887620$ and $C_2 = -2.57487620$, which turns out that the right boundary should locate between 0.85829206 and 0.85887620, i.e., $0.85829206 < b_1 < 0.85887620$.

In conclusion, the domain of the solution of problem 1 is:

$$\begin{cases} t \in (a_1, b_1) \\ a_1 \in (-2.12065871, -2.12021957) \\ b_1 \in (0.85829206, 0.85887620) \end{cases}$$

!!!!!!NOTE: Now the domain part is over, should start the error(?) part.

We can estimate that there might be two vertical asymptotes, a_1, b_1. Since the slope of the vector tends to be infinity near $t = -2$ and $t = 1$, so we can have a try: let a_1 very close to -2 and b_1 very close to 1 to find accurate vertical asymptotes.

	Euler Implicit		**Euler Explicit**		**Euler Improved**		**Euler Trapezium**	
h	a_1	b_1	a_1	b_1	a_1	b_1	a_1	b_1
0.01	-2.07	0.82	-2.26	1	-2.17	0.91	-2.11	0.85
0.005	-2.09	0.83	-2.195	0.935	-2.15	0.885	-2.115	0.855
0.001	-2.115	0.853	-2.137	0.875	-2.126	0.864	-2.1201	0.859
0.0005	-2.118	0.856	-2.129	0.8675	-2.1235	0.8615	-2.1206	0.8591
0.0002	-2.1196	0.8578	-2.1242	0.8624	-2.1218	0.86	-2.1208	0.8592
0.0001	-2.1202	0.8584	-2.1225	0.8607	-2.1212	0.85964	-2.1209	0.8592

Table 3: Using Euler Method to Determine t When y is infinity

2.1.2 The Euler Method

When $h = 0.01$, vertical asymptotes are not stable. As h is decreasing, a_1, b_1 gradually reach to a stable value. When $h = 0.0001$, the value of a_1, b_1 calculated by these four methods is absolutely close. Digitally, we can solve that $a_1 = -2.1212$, $b_1 = 0.8594$.

To be more vivid, we can basically draw $y = f(t)$ as Figure 2.

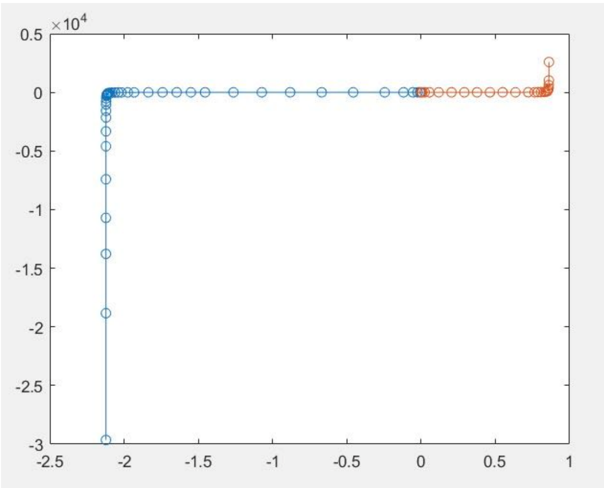

Figure 2: The Solution of Problem 1

2.1.3 The Runge-Kutta Method

The stable value of a_1, b_1 can be got when $h = 0.0001$. RK 4th has more accuracy, but in this particular problem, compared with RK 3rd, it doesn't fully manifest. When $h = 0.01$ or 0.005, the value of a_1, b_1 is not so accurate, but not far from stable value. Therefore, we can solve $a_1 = -2.1209$, $b_1 = 0.8591$.

	RK 3rd		RK 4th	
h	**a1**	**b1**	**a2**	**b2**
0.01	-2.15	0.89	-2.14	0.88
0.005	-2.135	0.875	-2.13	0.87
0.001	-2.124	0.862	-2.123	0.861
0.0005	-2.122	0.8605	-2.122	0.86
0.0002	-2.1212	0.8596	-2.121	0.8594
0.0001	-2.121	0.8592	-2.1209	0.8591

Table 4: Using Runge-Kutta Method to Determine t When y is infinity

2.1.4 The Adams Method

For Adams Bashforth method, when h is not small enough, the value of a_1, b_1 can't be stable. Just when h is smaller than the level of 10^{-3}, a_1, b_1 will tend to be stable. However, for Adams Molton method, no matter what value h is, in other words, for $h = 0.01, 0.005, 0.001, 0.0005, 0.0002, 0.0001$, in each condition, the value of a_1, b_1 have little difference. So it can be concluded that Adams Monlton has more accuracy than Adams Bashforth. As a result, we can solve $a_1 = -2.1207$, $b_1 = 0.8588$.

	Adams-Bashforth		Adams-Monlton	
h	**a1**	**b1**	**a2**	**b2**
0.01	-2.21	0.95	-2.12	0.85
0.005	-2.17	0.905	-2.1201	0.855
0.001	-2.13	0.868	-2.1202	0.858
0.0005	-2.1255	0.8635	-2.1205	0.8585
0.0002	-2.1226	0.8608	-2.1206	0.8587
0.0001	-2.1216	0.8598	-2.1207	0.8588

Table 5: Using Adams Method to Determine t When y is infinity

2.2 Problem 2

For problem 2, we are supposed to compute the solution of such a first-order nonlinear ODE:

$$y' = y^3 + ty^2 + t^2y + t^3 \tag{ODE2}$$

2.2.1 Domain

We start with the determination of the domain of solution. Same as problem 1, a slope field can be constructed to help understanding the tendency. According to the slope field Figure 3 shown below, the slopes seem not approach to infinite on the left side. Also, it seems that all the points below $y = -t$ have negative slope and all the points upon have positive slope. The numerical results of improved Euler method in Table 6 also shows the tendency that it approaches to $y = -t$ when t is approaching to negative infinity.

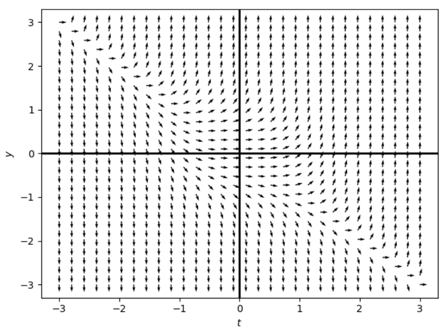

Figure 3: The slope field of $f(t, y)$ of Problem 2

t	-50	-45	-40	-35	-30	-25
y	49.9998	44.9997531	39.9996875	34.9995918	29.99944441	24.99919992

Table 6: Using Four Euler Method to Compute y Value

This qualitative tendency is proved as follows. Although Equation ODE2 is not an autonomous system, a "phase space" with t as a parameter can still be constructed, as shown in the left-hand side in Figure 4. The graph illustrates that the graph of $y' = f(y)$ has only one zero-crossing at $(-t, 0)$, and points near the line $y \equiv -t$ tend to go away from it (the blue arrows show the tendency), meaning this line is asymptotically unstable. In other words, if we go from $t = 0$ to $t = -\infty$, the sign of the derivative $y' = \frac{dy}{dt}$ should change, i.e. the graph becomes the right-hand side of Figure 4, and the line $y \equiv -t$ now becomes an "asymptote".

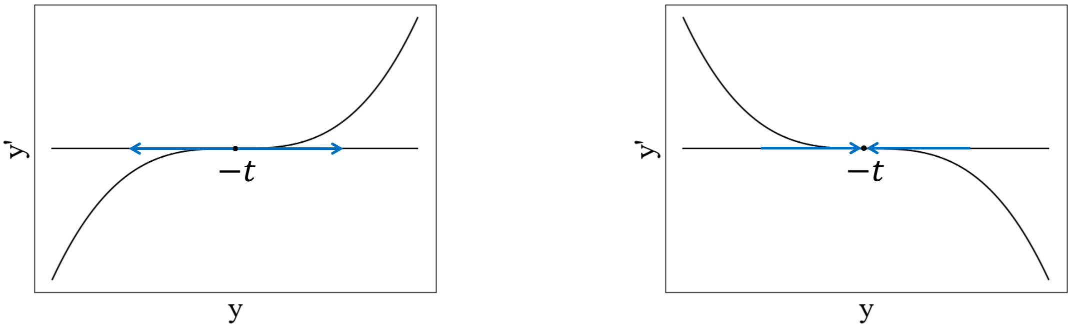

Figure 4: The phase space of Equation ODE2

Thus, the ODE does not have vertical asymptote on the left side.

For the right vertical asymptote, we use four Euler methods to estimate. The t value where y values first return inf from computation is considered as the approximate location of the vertical asymptote under that condition. Same as problem 1, a table can be constructed as Table 7.

	Euler_explicit	**Euler_implicit**	**Euler_improved**	**Euler_trapezium**
h = 0.01	0.5	0.4	0.48	0.43
h = 0.005	0.495	0.42	0.46	0.435
h = 0.001	0.452	0.436	0.444	0.44
h = 0.0005	0.446	0.438	0.442	0.4405
h = 0.0002	0.4426	0.4392	0.4408	0.4402
h = 0.0001	0.4413	0.4396	0.4403	0.4401

Table 7: Using Four Euler Methods to Compute the Nearest Value t Where y is INF

According to this numerical result, the difference of t is less than 0.001 when h = 0.0002 in both improved Euler method and trapezium Euler method's calculation. Thus, t = 0.440 can be considered as the right vertical asymptote location of the ODE if only use the numerical methods.

h	**0.01**	**0.005**	0.001	0.0005	0.0002	0.0001
y	∞	∞	25.62721138	23.60040387	23.43528374	23.4279037

Table 8: y of the results of the Adams-Monlton method when t = 0.439

However, analytical approaches similar to problem 1 can still be adopted to acquire more accuracy. To illustrate the details and relationship between t and y we use the Adams-Monlton method at $t = 0.439$ again. Obviously, near the right border t and y are both positive and $y > t$, therefore, we can obtain such an relationship: $y^3 < y^3 + ty^2 + t^2y + t^3 < 4y^3$, which consists of two solvable ODEs $y' = f_1(t, y) = y^2$ and $y' = f_2(t, y) = 4y^3$. By solving them we obtain:

$$\Phi_1(t) = \frac{1}{\sqrt{2}\sqrt{C_1 - t}} \tag{12}$$

$$\Phi_2(t) = \frac{1}{\sqrt{2}\sqrt{C_2 - 4t}} \tag{13}$$

Now we can choose one pair of (t, y) from Table 8 as a new initial value. we take $y(0.439) = 23.4279037$ and obtain $C_1 = 0.4399109$ and $C_2 = 1.7569109$, which leads to the results that the right boundary of problem 2 should locate between 0.4392277 and 0.4399109, i.e., $0.4392277 < b_2 < 0.4399109$.

In conclusion, the domain of the solution of problem 2 is:

$$\begin{cases} t \in (-\infty, b_2) \\ b_2 \in (0.4392277, 0.4399109) \end{cases}$$

2.2.2 Tthe Euler Method

We solve the ODE with four different Euler methods, and compute its values with different step h and at $t = 0.25$, $t = 0.30$, $t = 0.35$, $t = 0.40$, which close to its right vertical asymptote.

	Euler_explicit				**Euler_implicit**			
h	**t = 0.25**	**t = 0.3**	**t = 0.35**	**t = 0.4**	**t = 0.25**	**t = 0.3**	**t = 0.35**	**t = 0.4**
h = 0.01	1.47770259	1.70913766	2.09188224	2.87954605	1.52806345	1.8192135	2.40057921	∞
h = 0.005	1.48930857	1.73296161	2.14955647	3.08663402	1.51450889	1.78702797	2.29743739	3.92122702
h = 0.001	1.49920013	1.75384897	2.20325749	3.31926539	1.49920013	1.75384897	2.20903007	3.38833119
h = 0.0005	1.50047775	1.75658869	2.21054691	3.35489343	1.50047775	1.75658869	2.21054691	3.37260573
h = 0.0002	1.50124891	1.75824716	2.21498915	3.37718304	1.50124891	1.75824716	2.21498915	3.37718304
h = 0.0001	1.50150673	1.75880245	2.21648159	3.38477411	1.50150673	1.75880245	2.21648159	3.38477411
	Euler_improved				**Euler_trapezium**			
h	**t = 0.25**	**t = 0.3**	**t = 0.35**	**t = 0.4**	**t = 0.25**	**t = 0.3**	**t = 0.35**	**t = 0.4**
h = 0.01	1.50151241	1.75868118	2.21550691	3.37109489	1.50151241	1.75911757	2.21895347	3.41527979
h = 0.005	1.50170122	1.75918727	2.21734767	3.38677614	1.49778798	1.75310181	2.20573518	3.35601484
h = 0.001	1.50176237	1.75935206	2.21795425	3.39221267	1.49920013	1.75384897	2.20346482	3.34431531
h = 0.0005	1.5017643	1.75935726	2.2179735	3.3923893	1.50047775	1.75658869	2.21054691	3.35929849
h = 0.0002	1.50176484	1.75935871	2.2179789	3.392439	1.50124891	1.75824716	2.21498915	3.37718304
h = 0.0001	1.50176491	1.75935892	2.21797968	3.39244612	1.50150673	1.75880245	2.21648159	3.38477411

Table 9: Using Four Euler Methods to Compute y Value with different h

Data from the table shows that the solution computed by Improved Euler method is stable when step $h = 0.0005$. Comparing the y values computed by the same method shows that smaller step h leads to higher degree of accuracy. Therefore, for the other three Euler methods, the solutions could reach the stable condition when it reach a step h which is less than 0.0001.

2.2.3 The Runge-Kutta Method

We use Runger-Kutta method by 3rd order and 4th order to solve ODE and compute its y values at $t = 0.15$ to $t = 0.35$.

	Runge-Kutta_3rd					**Runge-Kutta_4th**				
h	**t = 0.15**	**t = 0.2**	**t =0.25**	**t = 0.3**	**t = 0.35**	**t = 0.15**	**t = 0.2**	**t =0.25**	**t = 0.3**	**t = 0.35**
h = 0.01	1.21432129	1.33362603	1.50176378	1.75935475	2.21795648	1.21432143	1.33362641	1.50176494	1.75935898	2.21797975
h = 0.005	1.21432141	1.33362636	1.50176479	1.75935842	2.2179767	1.21432143	1.33362641	1.50176494	1.75935899	2.21797993
h = 0.001	1.21432143	1.33362641	1.50176494	1.75935899	2.21797991	1.21432143	1.33362641	1.50176494	1.75935899	2.21797993
h = 0.0005	1.21432143	1.33362641	1.50176494	1.75935899	2.21797993	1.21432143	1.33362641	1.50176494	1.75935899	2.21797993
h = 0.0002	1.21432143	1.33362641	1.50176494	1.75935899	2.21797993	1.21432143	1.33362641	1.50176494	1.75935899	2.21797993
h = 0.0001	1.21432143	1.33362641	1.50176494	1.75935899	2.21797993	1.21432143	1.33362641	1.50176494	1.75935899	2.21797993

Table 10: Using the Runge-Kutta Method by 3rd and 4th order to Compute y Value

The results at $t = 0.15$ to 0.35 have been stable at 5th decimal places for 3rd order Runge-Kutta method and 7th decimal places for 4th order one when h is 0.005. It shows that with the same method and the same step h, the higher order computation has higher degree accuracy and stability. From the table we could find that the decrease of h also leads to a more stable solution in this method.

2.2.4 Adams-Bashforth and Adams-Monlton Method

The y values determined by the Adams-Bashforth method and Adams-Monlton method are shown in the table below. Adams-Bashforth method reaches stability at 8th decimal places when $h = 0.0002$, and $h = 0.0005$ by Adams-Monlton method. Which shows that Adams-Bashforth method has higher stability and accuracy than Adams-Monlton in this problem. Compare the result with those from previous methods, it shows that both Adams-Bashforth and Adams-Monlton methods solve the problem better than other methods in stability and accuracy.

	Adams_bashforth					**Adams_monlton**				
h	**t = -0.2**	**t = -0.1**	**t = 0.1**	**t = 0.2**	**t = 0.3**	**t = -0.2**	**t = -0.1**	**t = 0.1**	**t = 0.2**	**t = 0.3**
h = 0.01	0.85718229	0.91666778	1.12500093	1.33361711	1.75925478	0.85718192	0.91666752	1.12500238	1.33362721	1.75936873
h = 0.005	0.85718197	0.91666756	1.12500218	1.33362574	1.75935095	0.85718192	0.91666752	1.12500238	1.33362646	1.75935967
h = 0.001	0.85718197	0.91666756	1.12500218	1.33362641	1.75935899	0.85718192	0.91666752	1.12500238	1.33362646	1.75935899
h = 0.0005	0.85718197	0.91666756	1.12500218	1.33362641	1.75935899	0.85718192	0.91666752	1.12500238	1.33362646	1.75935899
h = 0.0002	0.85718197	0.91666756	1.12500218	1.33362641	1.75935899	0.85718192	0.91666752	1.12500238	1.33362646	1.75935899
h = 0.0001	0.85718197	0.91666756	1.12500218	1.33362641	1.75935899	0.85718192	0.91666752	1.12500238	1.33362646	1.75935899

Table 11: Using the Adams-Bashforth and Adams-Monlton Method to Compute y Value

2.2.5 Error Analysis

According to the previous result analysis with different methods, Adams-Bashforth method has the highest degree stability. In order to compare the accuracy of each methods, we compare the stable solution with the accurate one solved by ode113() function in MATLAB and calculate their percentage error difference. The result is shown below.

	t = -25			**t = 0.35**		
	y_stable	**y_ode113()**	**percentage error**	**y_stable**	**y_ode113()**	**percentage error**
Euler Explicit	24.99919992	25.01208677	-0.0515225%	2.21648159	2.21136738	0.2312690%
Euler Implicit	24.99919992	25.01208677	-0.0515225%	2.21648159	2.21136738	0.2312690%
Euler Improved	24.99919992	25.01208677	-0.0515225%	2.21797968	2.21136738	0.2990140%
Euler Trapezium	24.99919992	25.01208677	-0.0515225%	2.21648159	2.21136738	0.2312690%
RK 3rd	24.99919992	25.01208677	-0.0515225%	2.21797993	2.21136738	0.2990253%
RK 4th	24.99919992	25.01208677	-0.0515225%	2.21797993	2.21136738	0.2990253%
Adams Monlton	24.99919992	25.01208677	-0.0515225%	2.21797984	2.21136738	0.2990212%
Adams Bashforth	24.99919992	25.01208677	-0.0515225%	2.21797993	2.21136738	0.2990253%

Table 12: percentage error between stable solution and accurate solution at t = -25 and t = 0.35 with h = 0.0001

3 Results

Comparing the data and solutions of two questions IVP1 and IVP2, we can find out some typical properties for these methods.

3.1 Accuracy comparison

The first conclusion is, for all these methods, the numerical solutions will be more accurate with the step size h decreasing in a certain range. However, the global truncation error of each method is different because of the formula inside.

The second conclusion is the accuracy comparison of different methods. From the error comparison graphs of IVP1, IVP2 below , we can get that the Adams-Moulton method is the most accurate and the Euler Method has the largest error. The errors of improved Euler method, Euler trapezium method and Runge-Kutta method are quite similar.

In theoretical, for Euler Method, the global truncation error is proportional to the 1st power of h, so the error might be the largest among these methods. For improved Euler method, the global truncation error is proportional to the 2nd power of h. The fourth-order four-stage Runge-Kutta method has a global truncation error of h^4. The Adams-Moulton method has a global truncation error of h^5.

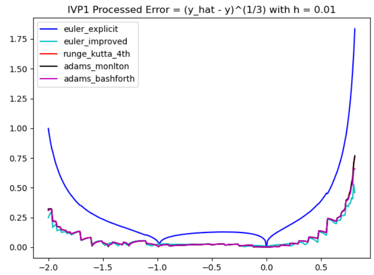

Figure 5:

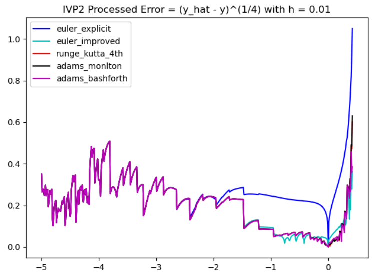

Figure 6:

Figure 7: Error Comparison Graphs of IVP1, IVP2

From this comparison graph another conclusion come into being: it is not the case that the more accurate a method is, the better it is. Since the RK4 method has a global truncation error of h^4 and improved Euler method has a global truncation error of h^3, while they have quite close solutions. So accuracy is not the only factor we need to consider, to judge whether a method is suitable or not for one ODE, we need to consider if it is worthwhile spending much more time on calculation to increase only very small amount of accuracy.

In this case, computation complexity should also be considered.

3.2 Computation complexity comparison

In general, "Computation Complexity" can be composed of "Time Complexity" and "Space Complexity ". Since the two can convert to each other and in numerical methods the intermediate processes of each method mainly have effect on the total time. So time complexity will be the main part discussed here.

	Euler_exp	**Eul_imp**	**Eul_trapezium**	**Eul_imp**	**Rungekutta_3rd**	**RK_4th**	**Adams_Monlt**	**Adm_bashfth**
0.01	0.19939	3.893579	3.191479	0.398877	0.498486	0.797803	22.0448	1.196753
0.005	0.399097	6.683203	6.183374	0.797827	1.097021	1.495996	43.68333	2.393555
0.001	1.996265	29.11707	30.41863	4.089111	5.485376	7.830151	197.6515	14.06235
0.0005	3.889575	54.87087	54.06863	8.577002	10.97065	15.75786	406.0816	29.12207
0.0002	10.27258	143.0175	132.258	19.64763	27.33054	40.19226	955.9132	63.23076
0.0001	24.43809	290.3356	232.6368	38.99563	56.57815	85.29304	1956.909	143.8662

Table 13: Time Consuming with different h in IVP1

Eul_exp	**Eul_imp**	**Eul_trape**	**Eul_imp**	**RK_3rd**	**RK_4th**	**Adams_monlt**	**Adams_bashfth**
0.01	1.19260254	48.57026367	43.97243652	2.78720703	3.49047852	5.28737793	315.6517334
0.005	2.59301758	59.17590332	48.36687012	5.88430176	7.67946777	10.57209473	443.2843994
0.001	12.9652832	30.43623047	33.61003418	30.71765137	41.64165039	54.85251465	493.8230713
0.0005	28.42741699	54.05429687	67.07429199	61.84191895	77.99611816	109.7824219	806.3147949
0.0002	75.06044922	140.6254883	178.127832	160.380542	198.1198486	261.8647461	1660.448975
0.0001	162.1139404	289.7324219	379.6040283	339.4297119	399.6322754	538.4823975	3068.49563

Table 14: Time Consuming with different h in IVP2

The Euler method has the advantage that the formula is simple and easy to run in computer. It costs least time to finish the calculation. The computation complexity of the improved Euler method's algorithm increases compared to the Euler method, since $f(t, y)$ should be evaluated twice in the formula to go from t_n to t_n+1 in the formula. The fourth-order four-stage Runge-Kutta method, weighing four steps with $\frac{1}{6}(k_{n1}+2k_{n2}+2k_{n3}+k_{n4})$. An increasing in accuracy will lead to a more complex algorithm. The Adams-Moulton method and the Adams-Bashforth method are multi-step methods, it would take much more time to evaluate. Using a "Big-O" method to evaluate these methods, the conclusion is that all methods are in the order of $O(mn) \in O(n)$,$m \in O(1)$

In a certain range, there exists a negative correlation between computation complexity and accuracy. However, this is not always true.

In addition, the time consuming might also influenced by the detailed ODE functions. For IVP1, the Adams-Moulton method is most time-consuming, instead for IVP2, the Adams-Bashforth method is the most time-consuming one.

4 Conclusion

Usually a high accuracy method would cost more time and space to get the solution, however, within the tolerable margin of error, a simpler method would be more suggested.

In a conclusion, every method has its own merit and demerit. In reality Which one to choose should depend more on the detailed ODE function and the balance between accuracy and computation complexity requirements.

5 Reference List and Appendix

CN DOTA, BEST DOTA![? ? ?]

References