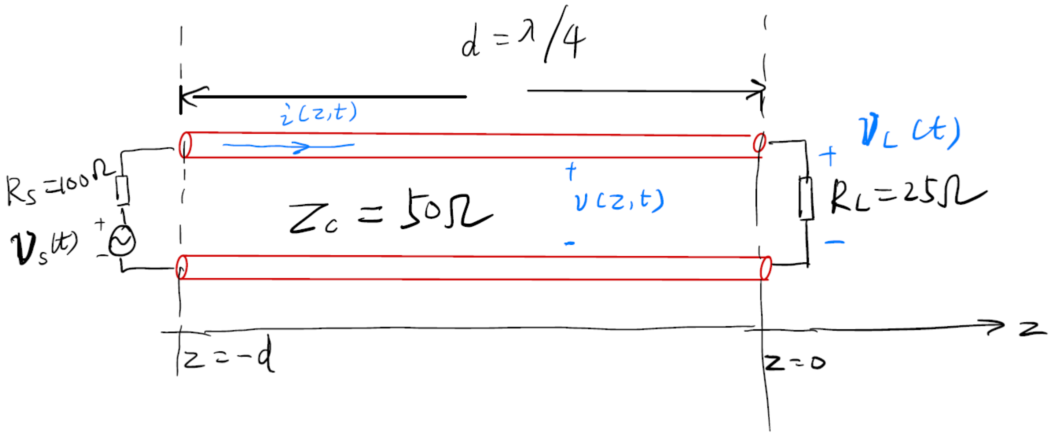
Project 2 – Solve the Transmission Line Equations with The Finite Difference in Time Domain (FDTD) Method

(release: Dec. 17, 2020; Due: 11:59pm, Jan. 3, 2021)

This is an extra homework. If you work on this, it will help you obtain as much as 5 points extra credits in your final grade. That is more than the average of two regular homework.
For this project, please submit both your solution and your code. Please submit your code in a zip file named with your name.

Problem: (100 pts)

Consider the circuit as illustrated in the diagram, a lossless transmission line of characteristic impedance $Z_c = 50\Omega$ of length $d = \lambda/4$ is connecting a source $v_s(t) = V_0 \sin(\omega t)\, u(t)$ with source impedance $R_s = 100\Omega$ to a resistive load $R_L = 25\Omega$. Assume $V_0 = 1V$. The coordinate is defined such that the load is at $z = 0$, and the source is at $z = -d$. The propagation wavelength is $\lambda = 2\pi/\beta$ where β is the propagation constant of the transmission line. The angular frequency is $\omega = 2\pi/T$ where T is the period of the sinusoidal signal. $u(t)$ is the step function that is 0 for $t < 0$ and 1 for $t \geq 0$. The solution of the problem does not depend on the values of λ and T, thus you can normalize the distance and time with respect to λ and T, respectively.



We want to compute the volage $v(z,t)$ and the current $i(z,t)$ along the transmission line for $-d \leq z \leq 0$ and $t \geq 0$ by solving the following coupled transmission line equations using the FDTD method.

$$\frac{dv}{dz} + L\frac{di}{dt} = 0$$
$$\frac{di}{dz} + C\frac{dv}{dt} = 0$$

where $L = Z_c/v_p$ and $C = 1/(Z_c v_p)$, and the phase velocity $v_p = \omega/\beta = \lambda/T$.

1. (20 pts) Please derive the analytical solution of $v(z,t)$ and $i(z,t)$ assuming $v_s(t) = V_0 \sin(\omega t)$ so that you do not need to worry about the transient effects. Please show your derivation.
2. Please find the load voltage $v_L(t) = v(z = 0, t)$, for $0 \le t \le 10T$ using the FDTD method given $v_s(t) = V_0 \sin(\omega t)\, u(t)$.
   (i) (30 pts) Please plot out the load voltage $v_L(t)$ from the FDTD simulation for $0 \le t \le 10T$. Please discuss the effects of choosing different Δz and Δt to the accuracy and stability of the numerical results in the discretization. You can plot and compare the results with different discretization parameters. You can then choose proper Δz and Δt that yield accurate and stable results in the following exercises.
   (ii) (5 pts) Please compare your results with analytical results using the transmission line theory assuming $v_s(t) = V_0 \sin(\omega t)$ so that you do not need to worry about the transient effects.
   (iii) (5 pts) Please also compare your results with the lumped circuit theory results assuming $d = 0$.

3. Please calculate and plot out the volage $v(z,t)$ and the current $i(z,t)$ along the transmission line for $-d \le z \le 0$. Do this at time $t = \frac{T}{8}, \frac{T}{4}, \frac{3T}{8}, \frac{T}{2}, T,$ and $10T$, respectively.
   (i) (30 pts) Please compute the results from the FDTD simulation given $v_s(t) = V_0 \sin(\omega t)\, u(t)$.
   (ii) (10 pts) Please compare the results with analytical results using the transmission line theory assuming $v_s(t) = V_0 \sin(\omega t)$.

Hints:
1. Please self-study Lecture 30 for the transmission line theory, and Lecture 31 for the FDTD method.
2. In deriving the analytical results of $v(z,t)$ and $i(z,t)$, you may want to convert between the phasor representation and the time harmonic representation.
3. In deriving the updating scheme for $v(z,t)$ at the two ends of the transmission line, try to discretize the equation $\frac{di}{dz} + C\frac{dv}{dt} = 0$ at the two ends at time $t = \left(n + \frac{1}{2}\right)\Delta t$. Then approximate di/dz, e.g., at the load side, by

$$\left.\frac{di}{dz}\right|_{z=N\Delta z}^{t=\left(n+\frac{1}{2}\right)\Delta z} \approx \frac{i^{\left(n+\frac{1}{2}\right)}(N) - i^{\left(n+\frac{1}{2}\right)}\left(N - \frac{1}{2}\right)}{\frac{1}{2}\Delta z}$$

then link $i^{\left(n+\frac{1}{2}\right)}(N)$ to $v^{\left(n+\frac{1}{2}\right)}(N)$ by the Ohm's law, and approximate $v^{\left(n+\frac{1}{2}\right)}(N)$ by

$$v^{\left(n+\frac{1}{2}\right)}(N) \approx \frac{v^{(n+1)}(N) + v^{(n)}(N)}{2}.$$

4. Please choose the spatial discretization Δz properly to ensure accuracy. A rule of thumb is to choose $\Delta z \le \lambda/20$. To ensure smoothness in the results, please choose $\Delta z \le \lambda/40$ in this problem.
5. Please choose the time step Δt properly to ensure stability of the algorithm. A rule of thumb is to choose $\Delta t < \Delta z/v_p$.