Price a European Up-and-out Call Option

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Abstract

This report presents the results of the simulation of a European up-and-out call option over twelve months. The Black-Scholes-Merton model will be used to implement Monte Carlo estimations using Python code. The estimations will simulate paths for the underlying share and the counterparty's firm value using varying sample sizes, and determine estimates of both the default-free value of the option and the Credit Valuation Adjustment (CVA). Following from these it is simple to price the option incorporating counterparty risk, given by the default-free price less the CVA.

1 Introduction

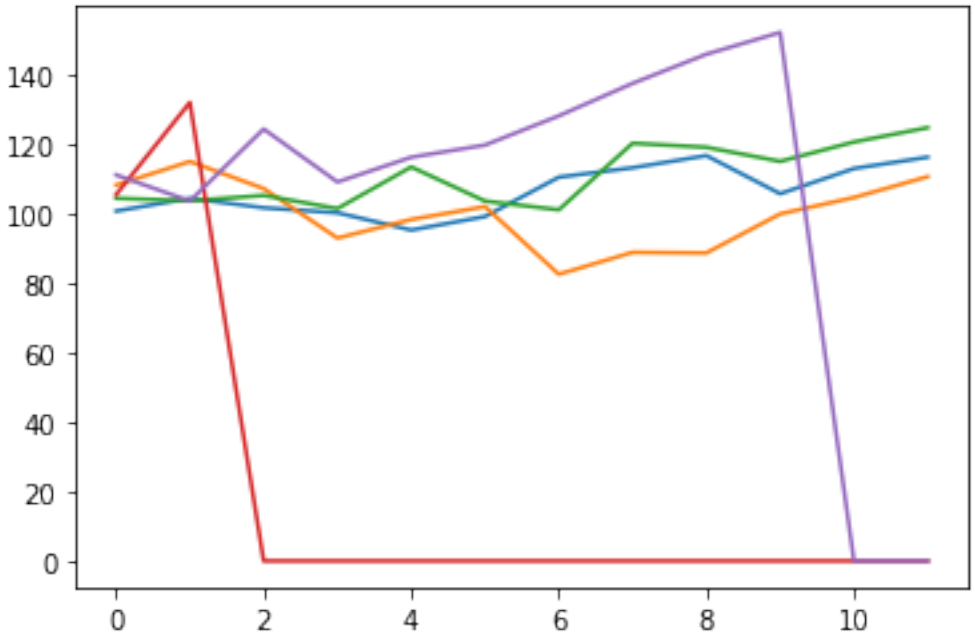
A European up-and-out call option is type of a Barrier Option, these options are consider path-dependant because barrier option's payoff is based on the underlying stock's price path. The payoff of an up-and-out option at maturity T is given by,

$$Payoff_T = (S_T - K)^+ given, max_{t\in[0,T]} S_t < L$$

Where, where K is the strike of the option, L is the barrier level, and S_t is the share price at time t. This is a type of call option whose payoff is reduced to 0 if the share price crosses the barrier level.

The seminal work of Merton [1] pioneered the formula for pricing barrier options. This led pricing formulas under the geometric Brownian motion (GBM) framework for one-asset barrier options by Rich [2] and multi-asset barrier options by Wong and Kwok [3]. Although the return dynamics of underlying shares are not sufficiently well described by the GBM process proposed by Black and Scholes [4], we are going to assume both the stock and counterparty firm values follow GBM with constant drift and volatilities and default only occurs at maturity.

Possible paths for the up-and-out barrier option payoff would look similar to the chart below, where two of the paths have crossed the barrier and resulted in zero payoff, but the other three have not crossed the barrier and behave like a vanilla option.



We initialize most variables as given by the question. We assume the counterparty firm's value is at 200, as this is above the counterparty's debt due in one year. Also, we calculate the default-free value of our option to be 6.60, so we select a firm value not too distant such that the CVA value we calculate is not negligible. For example, if we chose counterparty firm value as 1,000, the CVA would be 0.000.

[2]:
```
### Initialize problem parameters
T = 1 # option maturity
L = 150 # up-and-out barrier
S0 = 100 # current share price
K = 100 # strike price, at-the-money
risk_free = .08 # risk-free rate
sigma = .3 # volatility
v_0 = 200 # counterparty firm current value (Our assumption)
sigma_firm = .25 # volatility for the counterparty's firm
debt = 175 # counterparty's debt, due in one year
corr = .2 # correlation
recovery_rate = 0.25 # recovery rate
########
corr_matrix = np.array([[1, corr], [corr, 1]])
sample_sizes = range(1000, 50001, 1000)
```

2 Simulated Paths for the Underlying Share and Counterparty Firm

In order to simulate paths for the underlying share and the firm, we use a Cholesky decomposition to generate the correlated price paths.

[3]:
```
def share_path(S_0, risk_free_rate, sigma, Z, dT):
    return S_0*np.exp(np.cumsum((risk_free_rate-sigma**2/2)*dT + sigma*np.
 ↪sqrt(dT)*Z,1))


def generate_share_and_firm_price(S0, v_0, risk_free, sigma, sigma_firm, corr,␣
 ↪T, sample_size = 10, timesteps = 12):
    corr_matrix = np.array([[1, corr], [corr, 1]])
    norm_matrix = stats.norm.rvs(size = np.array([sample_size, 2, timesteps]))
    corr_norm_matrix = np.matmul(np.linalg.cholesky(corr_matrix), norm_matrix)

    share_price_path = pd.DataFrame(share_path(S0, risk_free, sigma,␣
 ↪Z=corr_norm_matrix[:,0,], dT=1/timesteps))
    share_price_path = share_price_path.transpose()

    firm_price_path = pd.DataFrame(share_path(v_0, risk_free, sigma_firm,␣
 ↪Z=corr_norm_matrix[:,1,], dT=1/timesteps))
    firm_price_path = firm_price_path.transpose()

    return [share_price_path,firm_price_path]
```

To double check that the stock prices and firm values monthly returns are correlated, we check them as follows, generating 20 different price paths with 10,000 timesteps.

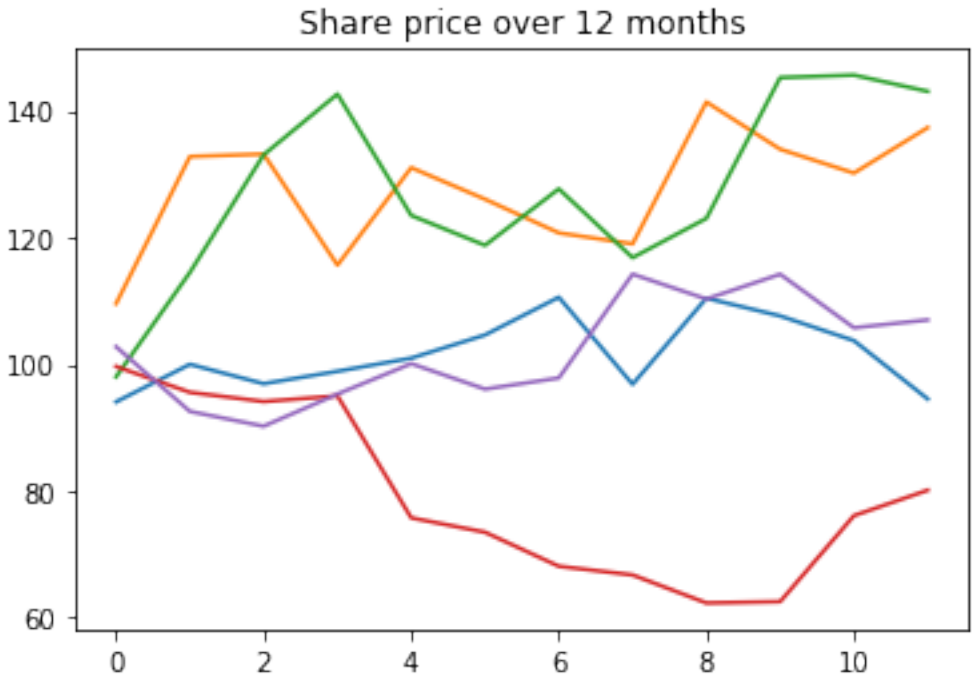
[4]:
```
#Testing share and firm price correlation
sample_size = 20
test = generate_share_and_firm_price(S0, v_0, risk_free, sigma, sigma_firm,␣
 ↪corr, T, sample_size, timesteps = 10000)

share_ret = np.log(test[0])

for i in range(sample_size):
    test[0]['sharelog'] = np.log(test[0][i])
    test[1]['firmlog'] = np.log(test[1][i])
    pearson, p_value = stats.pearsonr(test[0]['sharelog'].diff().dropna(),␣
 ↪test[1]['firmlog'].diff().dropna())
    print("Pearson correlation coefficient : {:.3f}, Two-tailed p-value {:.3f}".
 ↪format(pearson, p_value))
```

```
Pearson correlation coefficient : 0.208, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.198, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.184, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.208, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.196, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.203, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.204, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.209, Two-tailed p-value 0.000
```

```
Pearson correlation coefficient : 0.198, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.215, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.187, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.196, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.208, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.196, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.196, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.192, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.186, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.204, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.205, Two-tailed p-value 0.000
Pearson correlation coefficient : 0.188, Two-tailed p-value 0.000
```

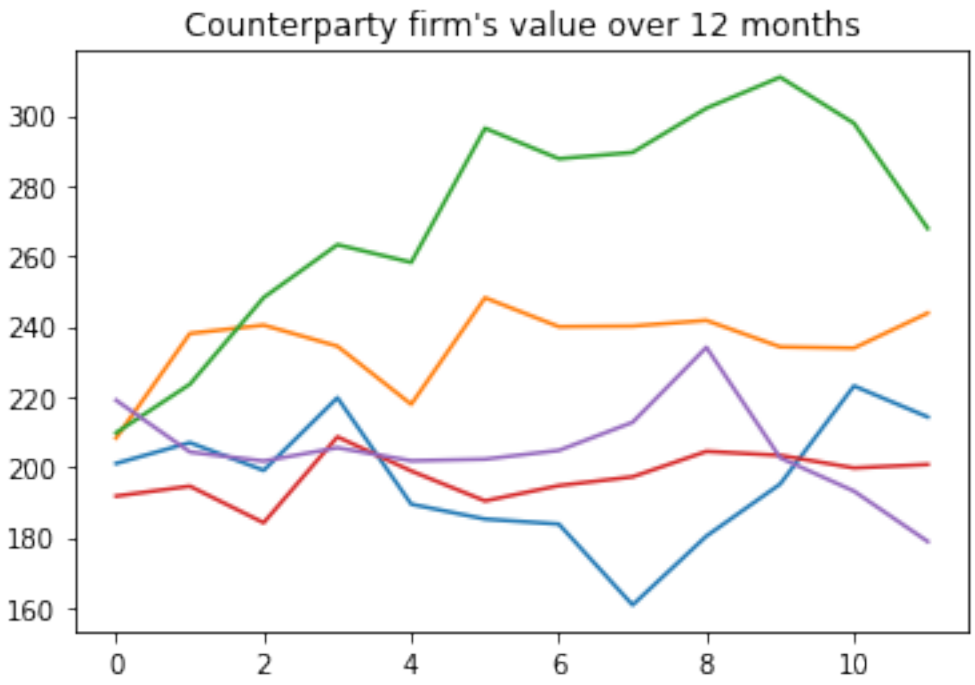
Let's try to simulate the share price with a small number of sample paths and visualize them over the course of 12 months

[5]:
```
share_and_firm_price_12_months = generate_share_and_firm_price(S0, v_0,␣
↪risk_free, sigma, sigma_firm, corr, T, sample_size = 5, timesteps = 12)
share_price_12_months = share_and_firm_price_12_months[0]
share_price_12_months.plot(title='Share price over 12 months', legend=False);
```



We can do the same thing to simulate counterparty firm's value

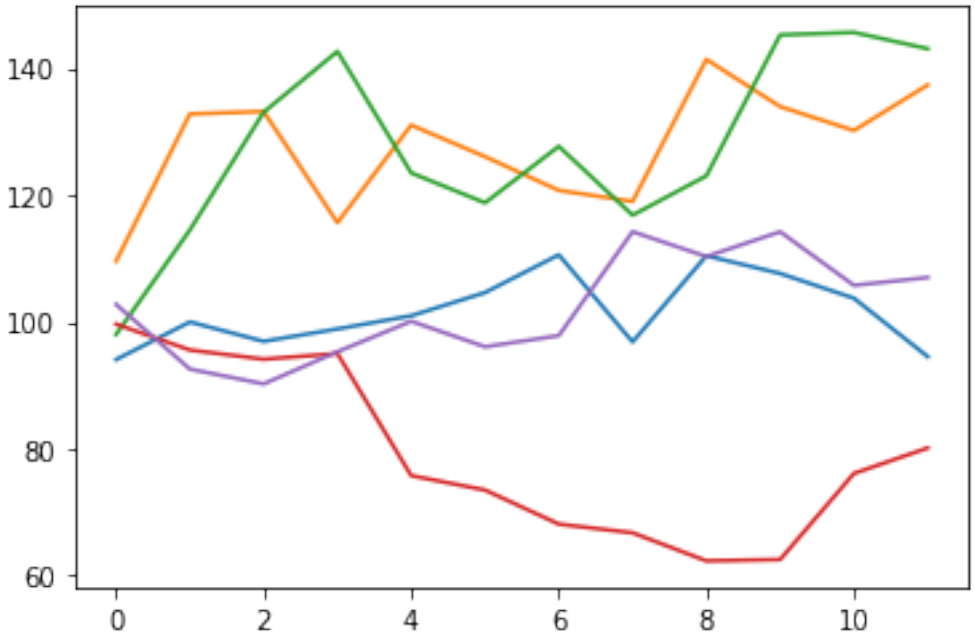
[6]:
```
firm_value_12_months = share_and_firm_price_12_months[1]
firm_value_12_months.plot(title="Counterparty firm's value over 12 months",␣
↪legend=False);
```



Let's visualize when the stopped process is appied

[8]:
```
def stop(s, cond):
    ret = s.copy()
    r = ret[cond]
    if len(r) > 0:
        print(r)
        ret[r.idxmin():] = 0
    return ret
```

[9]:
```
share_price_12_months.apply(lambda s: stop(s, s>L), axis=0).plot(legend=False);
```



We then define a function to calculate the payoff for the up-and-out call option.

[10]:
```
# define payoff for up-and-out call option
def payoff(S_t, K, L):
    stopped_S = S_t.iloc[-1].where((S_t < L).all(), 0)
    return np.maximum(stopped_S - K, 0).to_numpy()

payoff(share_price_12_months, K, L)
```

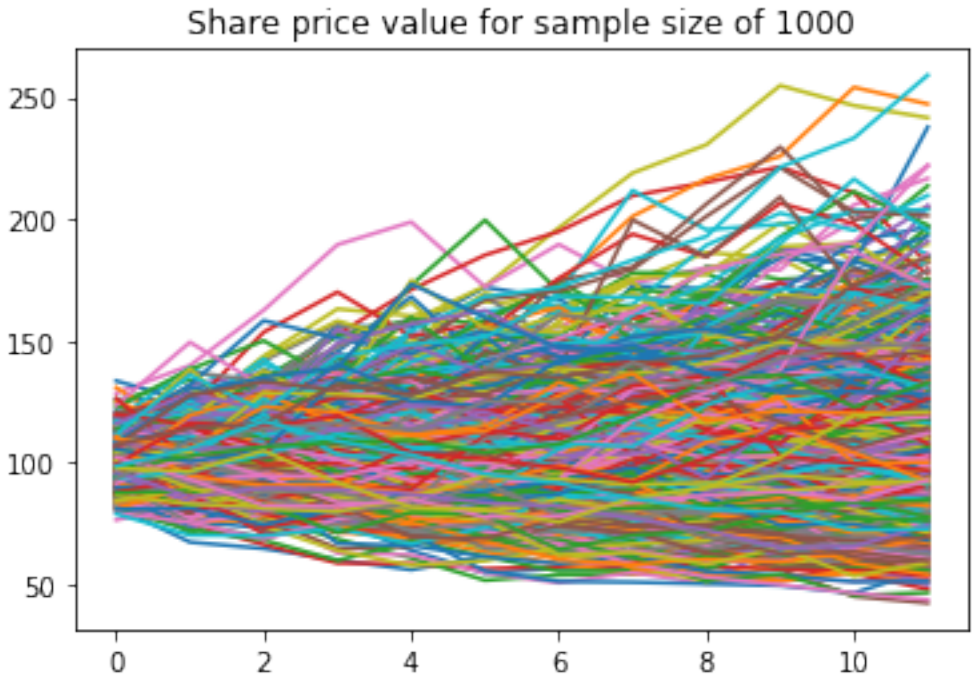
[10]:
```
array([ 0.        , 37.34484527, 43.05073004,  0.        ,  7.00008561])
```

We now create the share and firm price paths for sample sizes 1000, 2000, ..., 50000, as required for part 1 of the assignment.

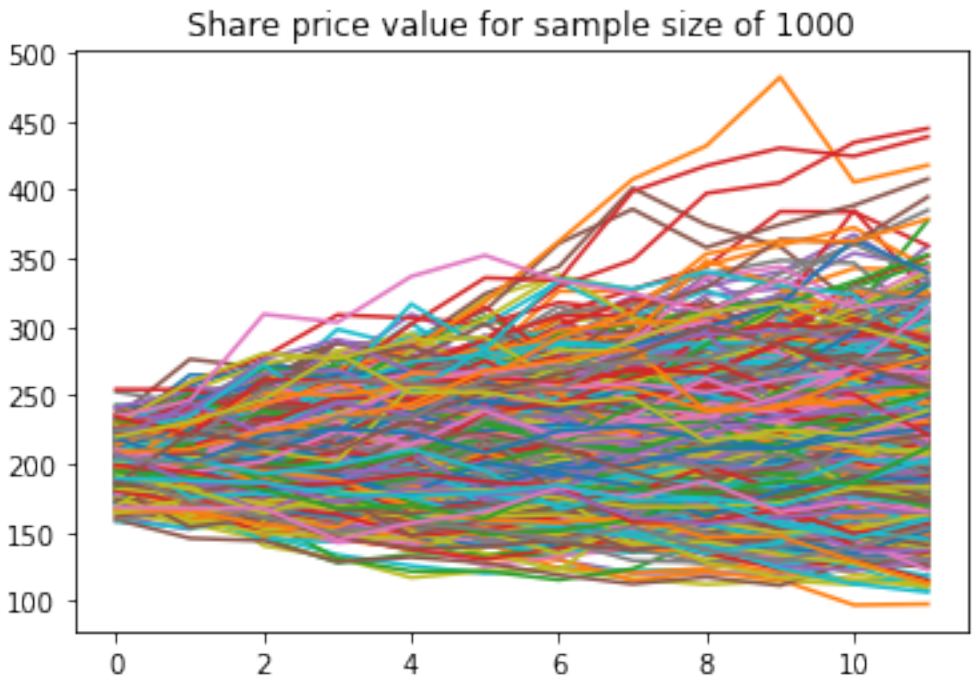
[11]:
```
share_price_paths = {}
firm_val_paths = {}

sample_sizes = range(1000, 50001, 1000)

for sample_size in sample_sizes:
    share_val, firm_val = generate_share_and_firm_price(S0, v_0, risk_free,␣
 ↪sigma, sigma_firm, corr, T, sample_size = sample_size, timesteps = 12)

    share_price_paths[sample_size] = share_val
    firm_val_paths[sample_size] = firm_val
```

We now plot the share and firm price paths for sample size of 1000.

[12]:
```
share_price_paths[1000].plot(title="Share price value for sample size of 1000",␣
↪legend=False);
```



[13]:
```
firm_val_paths[1000].plot(title="Share price value for sample size of 1000",␣
↪legend=False);
```



3 Monte Carlo Estimations

3.1 Default-Free Value Estimation

To calculate Monte Carlo estimate of the default-free option value, we calculate the average payoff of the 1000s of sample price paths, to estimate the price of the option

[14]:
```
# Estimate the default-free value of the option:
option_estimate = []
option_std = []

for sample_size, paths in share_price_paths.items():
    payoffs = payoff(paths, K, L)
    option_price = np.exp(-risk_free*T)*payoffs
    option_estimate.append(option_price.mean())
    option_std.append(option_price.std()/np.sqrt(sample_size))
```

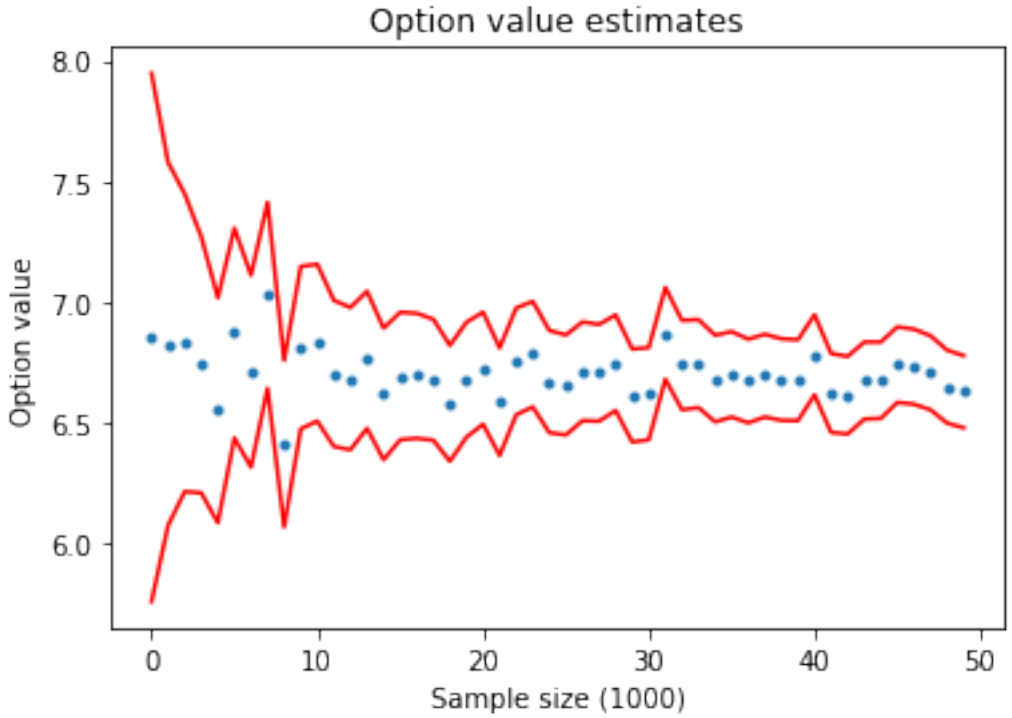
Prices of default-free option at various sample sizes:

[15]:
```
for i in range(len(option_estimate)):
    print("sample size: {}, Option value: {:.3f}".
 ↪format((i+1)*1000,option_estimate[i]))
```

```
sample size: 1000, Option value: 6.853
sample size: 2000, Option value: 6.826
sample size: 3000, Option value: 6.831
sample size: 4000, Option value: 6.739
sample size: 5000, Option value: 6.551
sample size: 6000, Option value: 6.871
sample size: 7000, Option value: 6.713
sample size: 8000, Option value: 7.028
sample size: 9000, Option value: 6.412
sample size: 10000, Option value: 6.810
sample size: 11000, Option value: 6.831
sample size: 12000, Option value: 6.704
sample size: 13000, Option value: 6.682
sample size: 14000, Option value: 6.760
sample size: 15000, Option value: 6.619
sample size: 16000, Option value: 6.693
sample size: 17000, Option value: 6.694
sample size: 18000, Option value: 6.678
sample size: 19000, Option value: 6.580
sample size: 20000, Option value: 6.678
sample size: 21000, Option value: 6.726
sample size: 22000, Option value: 6.587
sample size: 23000, Option value: 6.754
sample size: 24000, Option value: 6.784
sample size: 25000, Option value: 6.670
sample size: 26000, Option value: 6.656
sample size: 27000, Option value: 6.713
sample size: 28000, Option value: 6.706
sample size: 29000, Option value: 6.748
sample size: 30000, Option value: 6.612
sample size: 31000, Option value: 6.620
sample size: 32000, Option value: 6.870
sample size: 33000, Option value: 6.738
sample size: 34000, Option value: 6.744
sample size: 35000, Option value: 6.683
sample size: 36000, Option value: 6.700
sample size: 37000, Option value: 6.672
sample size: 38000, Option value: 6.694
sample size: 39000, Option value: 6.678
sample size: 40000, Option value: 6.676
sample size: 41000, Option value: 6.781
sample size: 42000, Option value: 6.623
sample size: 43000, Option value: 6.614
sample size: 44000, Option value: 6.674
sample size: 45000, Option value: 6.676
sample size: 46000, Option value: 6.740
sample size: 47000, Option value: 6.732
sample size: 48000, Option value: 6.706
```

```
sample size: 49000, Option value: 6.649
sample size: 50000, Option value: 6.628
```

[16]:
```
plt.plot(option_estimate, '.')
plt.plot(option_estimate + 3 * np.array(option_std), 'r')
plt.plot(option_estimate - 3 * np.array(option_std), 'r')
plt.title("Option value estimates")
plt.xlabel("Sample size (1000)")
plt.ylabel("Option value")
plt.show()
```



3.2 Credit Valuation Adjustment Estimation

To calculate Monte Carlo estimate of the credit value adjustment, we calculate the average loss of the 1000s of sample price paths, which happens when we both see a gain in the option with hold and when the counterparty firm value is less than its debt.

As per notes, we assume that default can only occur at time T, and firm defaults if the firm value is below firm debt amount

[17]:
```
def terminal_value(S_0, risk_free_rate, sigma, Z, T): #applies to both firm and␣
↪stock
```

```
    return S_0 * np.exp((risk_free_rate - sigma**2/2) * T + sigma * np.sqrt(T) *␣
 ↪Z)
```

[18]:
```
cva_estimate = []
cva_std = []

for sample_size, paths in share_price_paths.items():
    payoffs = payoff(paths, K, L)
    term_firm_vals = firm_val_paths[sample_size].iloc[-1].to_numpy()
    amount_lost = np.exp(-risk_free*T)*(1-recovery_rate)*(term_firm_vals <␣
 ↪debt)*payoffs
    cva_estimate.append(amount_lost.mean())
    cva_std.append(amount_lost.std()/np.sqrt(sample_size))
```

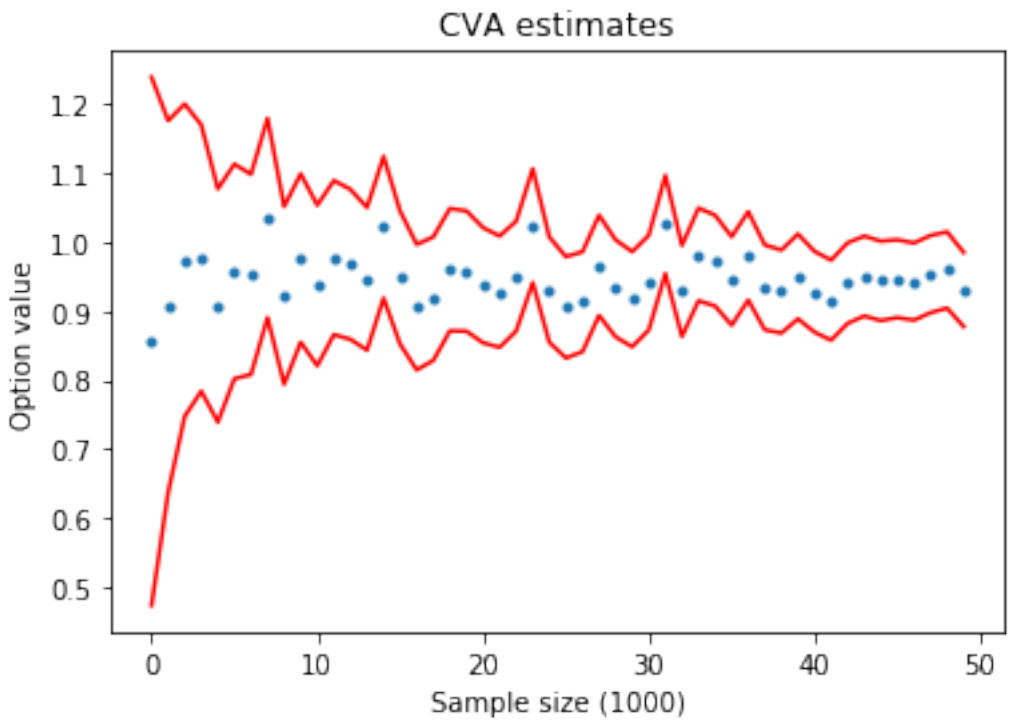
Credit value adjustment at various sample sizes:

[19]:
```
for i in range(len(cva_estimate)):
    print("Sample size: {}, CVA: {:.3f}".format((i+1)*1000,cva_estimate[i]))
```

```
Sample size: 1000, CVA: 0.856
Sample size: 2000, CVA: 0.906
Sample size: 3000, CVA: 0.974
Sample size: 4000, CVA: 0.977
Sample size: 5000, CVA: 0.908
Sample size: 6000, CVA: 0.957
Sample size: 7000, CVA: 0.953
Sample size: 8000, CVA: 1.035
Sample size: 9000, CVA: 0.923
Sample size: 10000, CVA: 0.977
Sample size: 11000, CVA: 0.937
Sample size: 12000, CVA: 0.977
Sample size: 13000, CVA: 0.968
Sample size: 14000, CVA: 0.947
Sample size: 15000, CVA: 1.021
Sample size: 16000, CVA: 0.948
Sample size: 17000, CVA: 0.906
Sample size: 18000, CVA: 0.918
Sample size: 19000, CVA: 0.960
Sample size: 20000, CVA: 0.958
Sample size: 21000, CVA: 0.937
Sample size: 22000, CVA: 0.928
Sample size: 23000, CVA: 0.951
Sample size: 24000, CVA: 1.024
Sample size: 25000, CVA: 0.931
Sample size: 26000, CVA: 0.905
Sample size: 27000, CVA: 0.913
```

```
Sample size: 28000, CVA: 0.966
Sample size: 29000, CVA: 0.933
Sample size: 30000, CVA: 0.917
Sample size: 31000, CVA: 0.941
Sample size: 32000, CVA: 1.025
Sample size: 33000, CVA: 0.929
Sample size: 34000, CVA: 0.982
Sample size: 35000, CVA: 0.973
Sample size: 36000, CVA: 0.944
Sample size: 37000, CVA: 0.980
Sample size: 38000, CVA: 0.934
Sample size: 39000, CVA: 0.928
Sample size: 40000, CVA: 0.950
Sample size: 41000, CVA: 0.928
Sample size: 42000, CVA: 0.916
Sample size: 43000, CVA: 0.941
Sample size: 44000, CVA: 0.951
Sample size: 45000, CVA: 0.944
Sample size: 46000, CVA: 0.947
Sample size: 47000, CVA: 0.943
Sample size: 48000, CVA: 0.954
Sample size: 49000, CVA: 0.959
Sample size: 50000, CVA: 0.931
```

[20]:
```
plt.plot(cva_estimate, '.')
plt.plot(cva_estimate + 3 * np.array(cva_std), 'r')
plt.plot(cva_estimate - 3 * np.array(cva_std), 'r')
plt.title("CVA estimates")
plt.xlabel("Sample size (1000)")
plt.ylabel("Option value")
plt.show()
```



4 Incorporating Counterparty Risk

We calculate the Monte Carlo estimates for the price of the option incorporating counterparty risk, given by the default-free price less the CVA.

[21]:
```
option_cva_adjusted_prices = []
option_cva_adjusted_std = []

for sample_size, paths in share_price_paths.items():
    payoffs = payoff(paths, K, L)
    option_price = np.exp(-risk_free*T)*payoffs

    term_firm_vals = firm_val_paths[sample_size].iloc[-1].to_numpy()
    amount_lost = np.exp(-risk_free*T)*(1-recovery_rate)*(term_firm_vals <␣
 ↪debt)*payoffs

    option_cva_price = option_price - amount_lost

    option_cva_adjusted_prices.append(option_cva_price.mean())
    option_cva_adjusted_std.append(option_cva_price.std()/np.sqrt(sample_size))
```

Credit value adjustment at various sample sizes:

[22]:
```
for i in range(len(option_cva_adjusted_prices)):
    print("Sample size: {}, CVA-adjusted option value: {:.3f}".
 ↪format((i+1)*1000,option_cva_adjusted_prices[i]))
```

```
Sample size: 1000, CVA-adjusted option value: 5.997
Sample size: 2000, CVA-adjusted option value: 5.920
Sample size: 3000, CVA-adjusted option value: 5.857
Sample size: 4000, CVA-adjusted option value: 5.763
Sample size: 5000, CVA-adjusted option value: 5.643
Sample size: 6000, CVA-adjusted option value: 5.913
Sample size: 7000, CVA-adjusted option value: 5.760
Sample size: 8000, CVA-adjusted option value: 5.993
Sample size: 9000, CVA-adjusted option value: 5.490
Sample size: 10000, CVA-adjusted option value: 5.834
Sample size: 11000, CVA-adjusted option value: 5.894
Sample size: 12000, CVA-adjusted option value: 5.726
Sample size: 13000, CVA-adjusted option value: 5.715
Sample size: 14000, CVA-adjusted option value: 5.814
Sample size: 15000, CVA-adjusted option value: 5.597
Sample size: 16000, CVA-adjusted option value: 5.745
Sample size: 17000, CVA-adjusted option value: 5.788
Sample size: 18000, CVA-adjusted option value: 5.760
Sample size: 19000, CVA-adjusted option value: 5.620
Sample size: 20000, CVA-adjusted option value: 5.720
Sample size: 21000, CVA-adjusted option value: 5.788
Sample size: 22000, CVA-adjusted option value: 5.659
Sample size: 23000, CVA-adjusted option value: 5.803
Sample size: 24000, CVA-adjusted option value: 5.760
Sample size: 25000, CVA-adjusted option value: 5.739
Sample size: 26000, CVA-adjusted option value: 5.751
Sample size: 27000, CVA-adjusted option value: 5.799
Sample size: 28000, CVA-adjusted option value: 5.740
Sample size: 29000, CVA-adjusted option value: 5.815
Sample size: 30000, CVA-adjusted option value: 5.695
Sample size: 31000, CVA-adjusted option value: 5.678
Sample size: 32000, CVA-adjusted option value: 5.845
Sample size: 33000, CVA-adjusted option value: 5.809
Sample size: 34000, CVA-adjusted option value: 5.762
Sample size: 35000, CVA-adjusted option value: 5.709
Sample size: 36000, CVA-adjusted option value: 5.756
Sample size: 37000, CVA-adjusted option value: 5.693
Sample size: 38000, CVA-adjusted option value: 5.760
Sample size: 39000, CVA-adjusted option value: 5.750
Sample size: 40000, CVA-adjusted option value: 5.726
Sample size: 41000, CVA-adjusted option value: 5.853
Sample size: 42000, CVA-adjusted option value: 5.707
Sample size: 43000, CVA-adjusted option value: 5.673
```

```
Sample size: 44000, CVA-adjusted option value: 5.724
Sample size: 45000, CVA-adjusted option value: 5.732
Sample size: 46000, CVA-adjusted option value: 5.793
Sample size: 47000, CVA-adjusted option value: 5.789
Sample size: 48000, CVA-adjusted option value: 5.753
Sample size: 49000, CVA-adjusted option value: 5.689
Sample size: 50000, CVA-adjusted option value: 5.697
```

[23]:
```
plt.plot(option_cva_adjusted_prices, '.')
plt.plot(option_cva_adjusted_prices + 3 * np.array(option_cva_adjusted_std), 'r')
plt.plot(option_cva_adjusted_prices - 3 * np.array(option_cva_adjusted_std), 'r')
plt.title("CVA-adjusted option prices estimates")
plt.xlabel("Sample size (1000)")
plt.ylabel("Option value")
plt.show()
```

5 Conclusion

References

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