Homework 1

ECE 358 Foundations of Computing
Fall Semester, 2020

Due: Monday, September 28, 5PM

This homework is designed so you can practice your background in introductory discrete mathematics and combinatorics. You need to have this background material practiced well because future homeworks will use it extensively!

- All page numbers are from 2009 edition of Cormen, Leiserson, Rivest and Stein.
- For each algorithm you asked to design you should give a detailed *description* of the idea, proof of algorithm correctness, termination, analysis of time and space complexity. If not, your answer will be incomplete and you will miss credit. You are allowed to refer to pages in the textbook.
- Do not write C code! When asked to describe an algorithm give analytical pseudocode.
- **Read the handout with the instructions on how to submit your Homework online. Failure to adhere to those instructions may disqualify you and you may receive a mark of zero on your HW.**
- Write *clearly*, if we cannot understand what you write you may not get credit for the question. Be as formal as possible in your answers. Don't forget to include your name(s) and student number(s) on the front page!
- No Junk Clause: For any question, if you don't know the answer, you may write "I DON'T KNOW" in order to receive 20% of the marks.

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1. **[Permutations and Combinations, 10+10 points]**

   (a) Give a combinatorial argument to prove that

   $$\sum_{i=0}^{n} \binom{n}{i} 4^i = 5^n.$$

   (b) For integers n and k where $n > k \geq 1$, give a combinatorial argument to prove that:

   $$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

   Hint: consider counting subsets of $\{1, \dots, n+1\}$. How many such subsets have $(i+1)$ as their largest element?

2. **[Recurrences, 15 points]**

   Solve the following recurrences by giving tight Θ-notation bounds. (*Hint*: apply the Master Theorem on the first three.)

   (a) $T(n) = 3T(n/4) + n!$
   (b) $T(n) = 2T(n/2) + n \log n$
   (c) $T(n) = 6T(n/3) + n^2 \log n$
   (d) $T(n) = T(n/3) + T(2n/3) + \Theta(n)$

3. **[Asymptotics, 25 points]**

   Sort the following 24 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any. You do not need to turn in proofs, but you should do them anyway just for practice.

$n^{4.5} - (n-2)^{4.5}$	n	$n^{1+(1/\log n)}$	$\log^*(n/2)$	$\sum_{i=1}^{n} \log i$
$\sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1}\right) + 2$	$\log^*(\log^* n)$	2^n	$(\log n)^{\log^* n}$	n^5
$\log^* 2^n$	$2^{\log^* n}$	e^n	$\lfloor \log\log(n!) \rfloor$	
$\left(1 - \log \frac{1}{1-1/n}\right)^n$	$n^{(\log\log n)/(\log n)}$	$(\log n)^{(n/2)}$	$(\log n)^{\log n}$	$\left(1 + \frac{1}{200n}\right)^{200n}$
$n^{1/\log\log n}$	$n^{\log\log n}$	$\log^{(200)} n$	$\log^2 n$	$n(\log n)^2$

   To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions n^2, n, $\binom{n}{2}$, n^3 could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.

4. **[Induction, 5+5 points]**

   Use induction to prove the following statements. Make sure to show clearly all three steps of induction (inductive basis, inductive hypothesis and inductive step) or you will miss credit.

   (a)
$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

   where F_n is the n^{th} Fibonacci number. The Fibonacci numbers are defined recursively by

$$F_n = \begin{cases} 1 & , n = 1 \\ 1 & , n = 2 \\ F_{n-2} + F_{n-1} & , otherwise \end{cases}$$

   and

$$\phi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$$

   for all positive integers n.

(b) If $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_i \subseteq B_i$ for $i = 1, 2, \ldots, n$, then,

$$A_1 \cup A_2 \cup \ldots \cup A_n \subseteq B_1 \cup B_2 \cup \ldots \cup B_n$$

for all positive integers n.

5. **[Probability, 10 points]**

   During a committee election, a person places their vote on 3 of 20 candidiates. Assume that all candidate have equal chance to be selected into the committee. At the end of the election, 5 of the candidate will form the committee. What is the probability that at least one of the candidates elected by this person will be on the committee?

6. **[Proof by contradiction, 10 points]** Assume you are in the final round of a quiz contest, and the final True or False question is so extremely difficult that you have no clue which one of the two choices is correct. There are two people who know the answer that you can potentially get help from. However, one of them *always* lies and the other *always* tells the truth. You cannot distinguish which one is the liar. You are allowed to ask *one* question to *one* of the potential helpers, without knowing which one you are talking to. Come up with the single question you will ask the helper, and decide which answer you will choose based on their response. Use a proof by contradiction to show your strategy is indeed correct. Make sure to show clearly all the steps of the proof to receive full credit.

7. **[Trees, 10 points]**

   George has a 26-node binary tree, with each node labeled by a unique letter of the alphabet. The preorder and postorder sequences of nodes are as follows:

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   preorder:   o v e f g t l s d n x a y u m k p j i c w b z h q r
   postorder:  t s l g f e n u m y a k x d v j z b h w c r q i p o
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   Draw George's binary tree.