1. Prove the following identity:

$$Pr(\alpha_1, \ldots, \alpha_n \mid \beta) = Pr(\alpha_1 \mid \alpha_2, \ldots, \alpha_n, \beta) Pr(\alpha_2 \mid \alpha_3, \ldots, \alpha_n, \beta) \ldots Pr(\alpha_n \mid \beta).$$

Base Case: Suppose $n = 1$,

$Pr(\alpha_1 \mid \beta) = Pr(\alpha_1 \mid \beta)$

The base case is holded.

Induction step: Let $n \in \mathbb{Z}^+$ be arbitrary, assume the case also could be holded,

Prove LHS, $Pr(\alpha_1, \ldots, \alpha_n \mid \beta)$

$$= \frac{Pr(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \beta)}{Pr(\beta)}$$

Prove RHS,

Applying chain Rule,

$Pr(\alpha_1 \wedge \cdots \wedge \alpha_n \wedge \beta)$

$= Pr(\alpha_1 \mid \alpha_2 \wedge \cdots a_n \wedge \beta)\, Pr(\alpha_2 \mid \alpha_3 \wedge \cdots \wedge a_n \wedge \beta) \cdots Pr(\alpha_n \mid \beta)\, Pr(\beta)$

$$\Rightarrow \frac{Pr(\alpha_1 \wedge \cdots \wedge \alpha_n \wedge \beta)}{Pr(\beta)} = Pr(\alpha_1 \mid \alpha_2 \wedge \cdots \wedge \alpha_n \wedge \beta) \cdots Pr(\alpha_n \mid \beta)$$

(Note that $Pr(\beta) \neq 0$)

Therefore, we have

$$Pr(\alpha_1, \ldots, \alpha_n \mid \beta) = \frac{Pr(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \beta)}{Pr(\beta)} = Pr(\alpha_1 \mid \alpha_2, \cdots, \alpha_n, \beta) Pr(\alpha_2 \mid \alpha_3, \cdots, \alpha_n, \beta) \cdots Pr(\alpha_n \mid \beta)$$

Proved.

2. A well is being drilled on a farm. Based on what has happened to similar farms, we judge the probability of oil being present to be 0.5, the probability of natural gas being present to be 0.2, and the probability of neither being present to be 0.3. If oil is present, a geological test will give a positive result with probability 0.9; if natural gas is present, it will give a positive result with probability 0.3; and if neither are present, the test will be positive with probability 0.1. Suppose the test comes back positive. What's the probability that oil is present?

Given the data:

Probability:
Pr(Oil) = 0.5
Pr(Gas) = 0.2
Pr(¬ Oil, ¬Gas) = 0.3
Pr(Positive |Oil) = 0.9
Pr(Positive |Gas) = 0.3
Pr(Positive | (¬ Oil, ¬Gas)) = 0.1

We need to find Pr (Oil | Positive)

Denote T as event that the test come out positive. O → Oil is presented, G → Natural Gas is presented.

As $Pr(\neg O, \neg G) =$
$= Pr(\neg O \wedge \neg G)$
$= Pr(\neg(O \vee G))$
$= 0.3$

$Pr(O \vee G) = 1 - Pr(\neg(O \vee G)) = 0.7$

$Pr(O \wedge G) = Pr(O) + Pr(G) - Pr(O \vee G) = 0 = Pr(O, G)$

$Pr(T) = Pr(T|O,G)\, P(O,G) + Pr(T|\neg O, \neg G)\, Pr(\neg O, \neg G) + Pr(T|O)\, P(O) + Pr(T|G)\, P(G)$
$= 0 + 0.1 * 0.3 + 0.9 * 0.5 + 0.3 * 0.2$
$= 0 + 0.03 + 0.45 + 0.06$
$= 0.54$

$$Pr(O|T) = \frac{Pr(T|O) \cdot Pr(O)}{Pr(T)} = \frac{0.9 * 0.5}{0.54} = 0.833$$

3. We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 40%, and 80%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X_1, X_2, and X_3. A bell will ring "on" if all coins flips come out the same. Draw the Bayesian network corresponding to this setup and define the necessary CPTs (Conditional Probability Tables).

3.

Let C denote Coin, $B \to$ Bell $\{1 \text{ (on)}, 0 \text{ (off)}\}$.

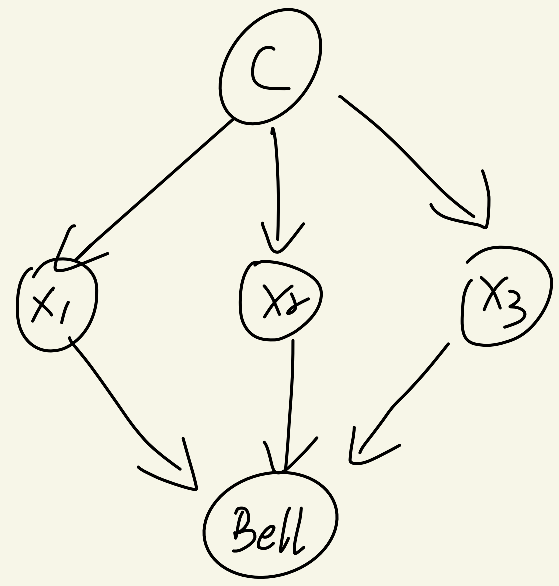
Given data:

Random variable C : $\{a, b, c\}$

Coin	P(coin)
a	20%
b	40%
c	80%

The network has C at the root and X_1, X_2, X_3 as children.

Bayesian Network:



CPT for C:

C	θ_C
a	1/3
b	1/3
c	1/3

CPT for X_i given C are the same, and equal to:

C	X_i	$\theta_{X_i \mid C}$
a	Head	0.2
a	Tail	0.8
b	Head	0.4
b	Tail	0.6
C	Head	0.8
C	Tail	0.2

$(i = 1, 2, 3)$

CPT for B - Bell :

Denote Head as H
Tail as T
B-Bell {1 (on), 0 (off)}

X_1	X_2	X_3	B	$\theta_{B \mid X_1, X_2, X_3}$
H	H	H	1	1
H	H	H	0	0
H	H	T	1	0
H	H	T	0	1
H	T	H	1	0
H	T	H	0	1
H	T	T	1	0
H	T	T	0	1
T	H	H	1	0
T	H	H	0	1
T	H	T	1	0
T	H	T	0	1
T	T	H	1	0
T	T	H	0	1
T	T	T	1	1
T	T	T	0	0

4. Consider the DAG in Figure 1:

(a) List the Markovian assumptions asserted by the DAG.

(b) True or false? Why?

- $d\_separated(A, F, E)$
- $d\_separated(G, B, E)$
- $d\_separated(AB, CDE, GH)$

(c) Express $Pr(a, b, c, d, e, f, g, h)$ in factored form using the chain rule for Bayesian networks.

(d) Compute $Pr(A = 1, B = 1)$ and $Pr(E = 0 \mid A = 0)$. Justify your answers.

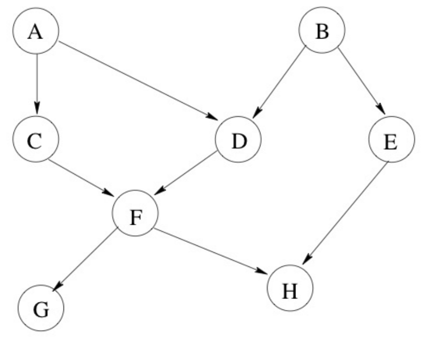


Figure 1: The DAG of a Bayesian network.

$Pr(A = 0)$	$Pr(A = 1)$
.8	.2

$Pr(B = 0)$	$Pr(B = 1)$
.3	.7

	$Pr(D = 0 \mid A, B)$	$Pr(D = 1 \mid A, B)$
$A = 0, B = 0$	.2	.8
$A = 0, B = 1$	.9	.1
$A = 1, B = 0$	.4	.6
$A = 1, B = 1$	.5	.5

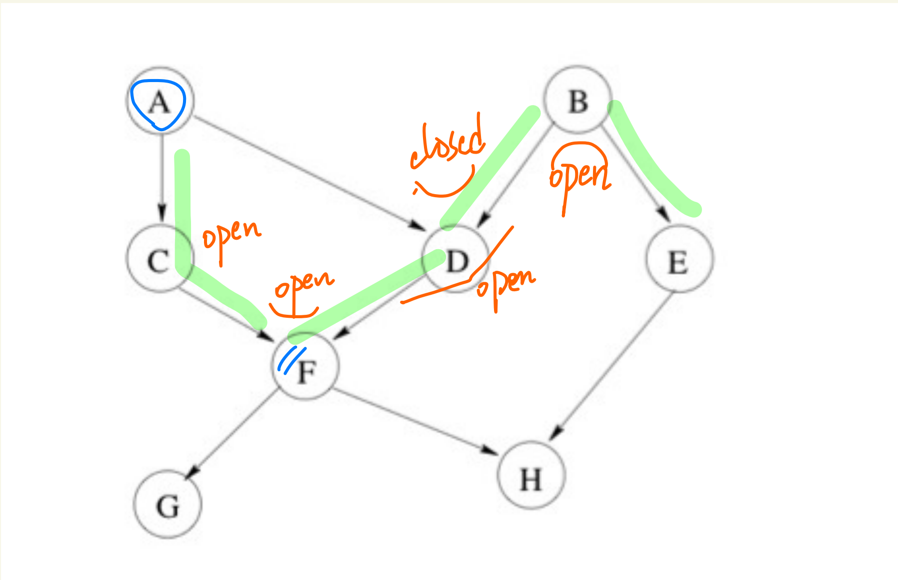
	$Pr(E = 0 \mid B)$	$Pr(E = 1 \mid B)$
$B = 0$	.1	.9
$B = 1$	.9	.1

a) :

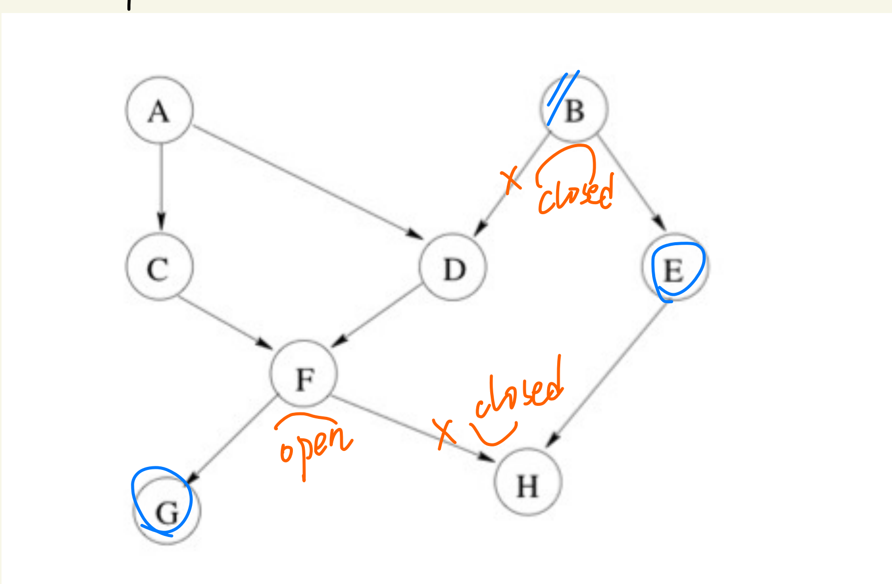
I (V, parents(V), Non-descendants(v))

- I (A, ∅ , BE)
- I (B, ∅, AC)
- I (C, A, BDE)
- I (D, AB, CE)
- I (E, B, ACDFG)
- I (F, CD, ABE)
- I (G, F, ABCDEH)
- I (H, EF, ACBDG)

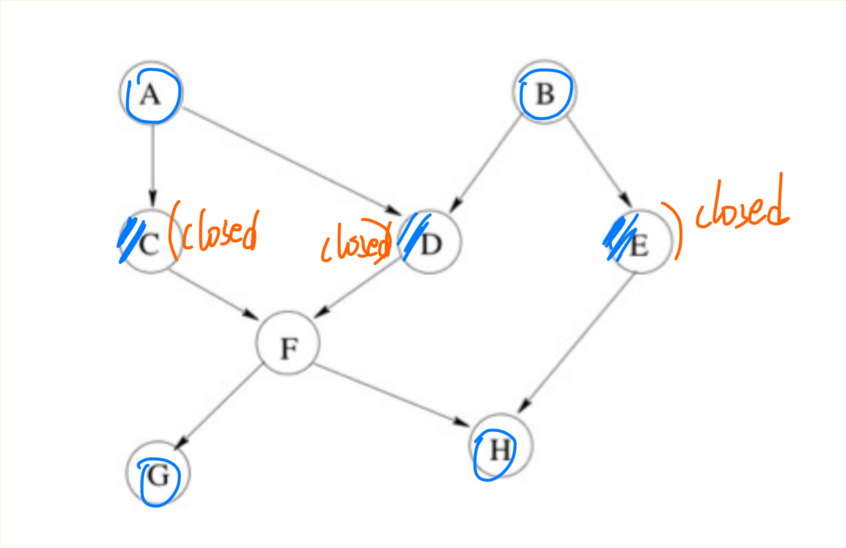
b) · d_separated (A, F, E) is false.
As the path A C F D B E is not blocked by Z = {F}.



· d_separated (G, B, E) = True. Since there are 2 paths between G and E
{ G F H E
{ G F D B E
and they both are block by Z = {B}



• d_separated (AB, CDE, GH) = True. All paths from AB to GH are blocked by $Z = \{CDE\}$.



As the image shown, there are 4 paths that must go through

$$\left\{ \begin{array}{l} ADF \text{ is closed} \\ BDF \text{ is closed} \\ ACF \text{ is closed} \\ BEH \text{ is closed} \end{array} \right.$$

c)

$$Pr(a,b,c,d,e,f,g,h)$$
$$= Pr(a)Pr(b)Pr(c|a)Pr(d|a,b)Pr(e|b)Pr(f|c,d)Pr(g|f)Pr(h|e,f)$$

d)

A and B are marginally independent, so $Pr(A=1, B=1) = Pr(A=1)Pr(B=1)$
$= 0.2 * 0.7$
$= 0.14$

Similarly, from DAG, A and E are marginally independent.

So $Pr(E=0 | A=0) = Pr(E=0) = Pr(E=0 | B=0)Pr(B=0) + Pr(E=0 | B=1)Pr(B=1)$
$= 0.1 * 0.3 + 0.9 * 0.7$
$= 0.03 + 0.63$
$= 0.66$

5. Consider the joint probability distribution in Table 1 and the propositional sentence $\alpha : A \Rightarrow B$.

both true

(a) List the models of α.

(b) Compute the probability $Pr(\alpha)$.

(c) Compute the conditional probability distribution $Pr(A, B \mid \alpha)$ as in Table 1.

(d) Compute the probability $Pr(A \Rightarrow \neg B \mid \alpha)$.

	A	B	$Pr(A, B)$
w_0	T	T	0.3
w_1	T	F	0.2
w_2	F	T	0.1
w_3	F	F	0.4

Table 1: A joint probability distribution.

a)

	A	B	$\alpha : A \Rightarrow B$	
w_0	T	T	T	✓
w_1	T	F	F	×
w_2	F	T	T	✓
w_3	F	F	T	✓

Models of α: w_0, w_2, w_3

b)

$$Pr(\alpha) = Pr(w_0) + Pr(w_2) + Pr(w_3) = 0.3 + 0.1 + 0.4 = 0.8$$

c)

A	B	$Pr(A, B \mid \alpha)$	
T	T	$0.3/0.8 = 0.375$	✓
T	F	0	
F	T	$0.1/0.8 = 0.125$	
F	F	$0.4/0.8 = 0.5$	

$Pr(A, B \mid \alpha) = 0.375$

d)

	A	B	α	$A \Rightarrow \neg B$	
w_0	T	T	T	F	
w_1	T	F	F	T	
w_2	F	T	T	T	✓
w_3	T	T	T	T	✓

Modules of $\alpha \wedge (A \Rightarrow \neg B)$: w_2 and w_3

$$\therefore Pr(A \Rightarrow \neg B \mid \alpha) = \frac{Pr(A \Rightarrow \neg B, \alpha)}{Pr(\alpha)} = \frac{0.1 + 0.4}{0.8} = 0.625$$