1. For each of the following languages

   $\mathbf{L}_1 = \{ a^p; p \text{ is a prime number } \}$,

   $\mathbf{L}_2 = \{ a^p; p \text{ is a prime number, } m \text{ is a fixed number and } m \geq p \geq 0 \}$,

   $\mathbf{L}_3 = \{ a^m b^{2m} c^{3m}; m \geq 0 \}$,

   $\mathbf{L}_4 = \{ a^m b^2 c^{3m}; m \geq 0 \}$,

   find if it is:

a) a regular language;

b) a context-free language;

c) a recursively enumerable language.

In case (a) for the language $\mathbf{L}_i$, build a finite automaton A such that $L(A) = \mathbf{L}_i$.

In case (b), prove that $\mathbf{L}_i$ is not a regular language and build a formal grammar G such that $L(G) = \mathbf{L}_i$.

In case (c), explain why $\mathbf{L}_i$ is not a regular or context-free language and how to build a Turing machine T such that $L(T) = \mathbf{L}_i$.

> Pumping Lemma for CFL:
>
> If L is CFL, then L has a pumping lemma p such that any string w, where $|w| \geq p$. w may be divided into 5 pieces $w = uvxyz$ such that
>
> - uv^ixy^iz is in L for all $i \geq 0$
> - $|vy| > 0$
> - $|vxy| \leq p$

1. L_1:

L_1 is not a regular language, not a context-free language, it is a recursively enumerable language.

Proof:

Suppose L_1 is context-free language, then the pumping lemma should be hold. Suppose a pumping length p, then consider some prime number $n \geq p+2$. (Such an *n* must exist since there are an infinite number of primes)

Let a string $w = a^n$, split into $uvxyz$.

Let $|vy| = m$, then $|uxz| = n - |vy| = n - m$.

By the pumping lemma, $uv^{n-m}xy^{n-m}z \in L_2$.

$$
\begin{aligned}
|uv^{n-m}xy^{n-m}z| &= |uxz| + (n-m) \times (|v| + |y|) \\
&= n - m + (n-m)m \\
&= (n-m)(m+1)
\end{aligned}
$$

Which is not prime unless one of the above factores is 1.

However, $(n-m) > 1$ since $n \geq p+2$ was chosen and $m \leq p$ since $m = |vy| \leq |vxy| \leq p$.

And $(m+1) > 1$ since $m = |vy|, and |vy| > 0$.

Hence, $|uv^{n-m}xy^{n-m}z|$ is composite number, it is not in L_1. A contradiction.

Therefore, L_1 is not context-free language. At the same time, since it is not CFL, then it must not a regular language.

To prove it is RFL, we need to show a Turning machine T. T has two tapes: one for the input, another to store all the numbers between 2 and $n-1$. T Acts as follow.

1) First T counts the number p of a's on tape 1.Iif $p = 0$ or $p = 1$, reject. If p=2, accept. T Writes 2 a's on tape 2.

2) T scans both tapes from left to right. It checks if the numebr of a's on tape 1 is divisible by the number of a's on tape 2. If yes, reject. Otherwise:

3) Add one a to tape 2. If the number of a's on both tapes are the same, accept. Otherwise, go to (2).

So $L(T) = L_1$. Hence L_1 is REL.

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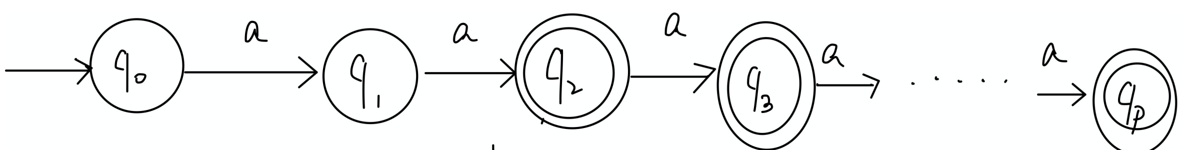
2. L_2 **is regular language** since every finite language is regular. A NFA could be constructed as follow:



p is a prime number such that $m \geq p \geq 0$. (m is fixed)

Note that for states q_i ($\forall i \leq p$). when i is prime number, the state q_i is final state.

Since it is regular language, **it must be a context-free language** as well.

Similar, since it is regular language, **it must be a recursively enumerable language.** Since RL ⊆ CFL ⊆ REL.

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3. L_3 **is not regualr language, and it is not context-free language.** L_3

L_3 is not context free language proof:

Suppose L_3 is CFL, then the pumping lemma could be hold.

Choose pumping length n, let string $w = a^m b^{2m} c^{3m}$

Split into $uvwxy$. where $|vwx| \le n$.

$(uvwx) \le m$, vwx only contains 'a'

Since $uv^iwx^iy \in L_3$.

when $i=0$, $uwy \in L_3$

$uwy = a^{m-uvx} b^{2m} c^{3m}$

$\because$ $vx \ne \varepsilon$

$\therefore$ $|vx| \geq 1$

$\therefore$ $uwy \notin L_3$,

contradiction.

Since L_3 is not CFL, then it must be not RL as well. Since RL ⊆ CFL.

L_3 is RFL.

The turning machine T acts as follow:

∧	a	b	b	c	c	c	∧

One tape for the input. Place the right and left endmarker "∧".
T scan the tape from left to right.
1) If head is at ∧, then accepts. If head is at "a", then erase it. Then change the rightmost 2 b's to c's, and then remove the rightmost 1 c's.
If the number of b's or c's is insufficient, reject.
2) Scan left until ∧ is found, then take 1 step right, go to (1).

So $L(T) = L_3$. It is REL.

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4. L_4:

It is not regular language.

Proof:

Suppose L_4 is regular language. Then pumping lemma should be hold.

Suppose a pumping length n, let string $w = a^m b^2 c^{3m}$, $m \geq$.n

Split into xyz.

$$xyz = \underbrace{a^p}_{xy} \; \underbrace{a^{m-p} b^2 c^{3m}}_{z}$$

$$= \underbrace{a^{p-q}}_{x} \; \underbrace{a^q}_{y} \; \underbrace{a^{m-p} b^2 c^{3m}}_{z} \qquad (q>0)$$

$$= a^{m+q} b^2 c^{3m} \notin L_4$$

A contradiction, hence L_4 is not RL.

It is context-free language.

A CFG as below:

$E \to I$

$I \to aIccc$

$I \to bb$

Since it is CFL, then **it must be recursively enumerable language** as well.