Find if the following problems are algorithmically decidable and prove that your answers are correct.

1. (50 pts) Given two context-free languages L and M, find if the language L∪M is empty.

Decidable. Proof:

Claim that if L and M both are context-free languages, then the language $L \cup M$ is context-free as well. (By Closure property under union). Proof for this:

Let $L = L(g), M = L(h). \quad g, h$ are CFG.

Let $S(g) = S, S(h) = E$

Construct a new CFG k:

$S(k) = Q,$

add $Q \to S$, $Q \to E$ and all rules from g and h to k

$\Rightarrow L(k) = L \cup M$

Therefore, $L \cup M$ is CFL.

And clearly, $L \cup M \neq \emptyset$ Iff at least one of them is not empty. In other words, $L \cup M$ is empty if both L and M are empty.

Hence, to solve this problem, we need to test if any of L and M is empty . According to the lecture, we know that the problem of "L is CFL, is L empty? " is decidable. Clearly the problem in the lecture could be extended to the given problem in homework. i.e., the given problem could be solved by speartating into two problems of determine if a CFL L is empty. If both are empty, then $L \cup M$ is empty; otherwise, it is not empty.

So there is a valid algorithm to solve the given problem, the answer could be yes or no. Therefore, the problem is decidable. Proved.

2. (50 pts) Given three context-free languages N, L and M, find if the language $(L \cap N) \cup (N \cap M)$ is empty.

Undecidable:

Proof:

Similary like question 1, $(L \cap N) \cup (N \cap M)$ is empty iff both of $L \cap N$ and $N \cap M$ are empty.

Support the problem is decidable.

Support $M = \emptyset$, M still is CFL.

Then

$(L \cap N) \cup (N \cap M)$

= $(L \cap N) \cup (N \cap \emptyset)$

= $(L \cap N) \cup \emptyset$

= $L \cap N$

From above, “L, N are CFL, does $L \cap N = \emptyset$” is decidable.

However, as from the lecture, we know that the problem of “ L, M are CFL, does $L \cap M \neq \emptyset ?$ ” is undecidable.

Contradiction!

Therefore, the given problem is undecidable.