Bipartite Graph Matching Algorithms

A Comparative Study of Hungarian and Hopcroft-Karp Algorithms

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2025-08-07

Abstract

This report presents a comprehensive study of two fundamental algorithms for bipartite graph matching: the Hungarian algorithm and the Hopcroft-Karp algorithm. We provide detailed theoretical analysis, implementation insights, and empirical performance comparisons. The Hungarian algorithm solves the maximum weight matching problem in $O(n^3)$ time, while the Hopcroft-Karp algorithm finds maximum cardinality matchings in $O\left(\sqrt{V} \cdot E\right)$ time. Through extensive benchmarking on various graph types and sizes, we demonstrate the complementary strengths of both algorithms and provide practical guidance for algorithm selection in real-world applications.

Keywords: bipartite graphs, matching algorithms, Hungarian algorithm, Hopcroft-Karp, complexity analysis, performance evaluation

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1. Introduction

The problem of finding optimal matchings in bipartite graphs is one of the most fundamental and well-studied problems in combinatorial optimization and graph theory. A **bipartite graph** $G = (L \cup R, E)$ consists of two disjoint sets of vertices L and R, where edges only connect vertices between the two sets. A **matching** $M \subseteq E$ is a set of edges with no common vertices.

Two primary variants of the bipartite matching problem have driven significant algorithmic development:

1. **Maximum Cardinality Matching**: Find a matching with the maximum number of edges
2. **Maximum Weight Matching**: In a weighted bipartite graph, find a matching with maximum total weight

These problems arise naturally in numerous applications including job assignment, resource allocation, network flows, computer vision, and operations research. The theoretical importance of these problems has led to the development of several fundamental algorithms, among which the Hungarian algorithm and the Hopcroft-Karp algorithm stand as cornerstones of the field.

1.1. Research Objectives

This study aims to:

- Provide rigorous theoretical analysis of both algorithms including complexity bounds and correctness proofs
- Present clean, well-documented implementations suitable for educational and practical use
- Conduct comprehensive empirical evaluation across diverse graph types and problem scales
- Offer practical guidance for algorithm selection based on problem characteristics

1.2. Contributions

Our main contributions include:

- From-scratch implementations of both algorithms without reliance on external graph libraries
- Comprehensive benchmarking framework evaluating performance across multiple dimensions
- Detailed analysis of algorithmic trade-offs and practical considerations
- Visualization tools for understanding algorithm behavior and matching quality

2. Theoretical Foundations

2.1. Bipartite Graphs and Matchings

Let $G = (L \cup R, E)$ be a bipartite graph where L and R are disjoint vertex sets with $|L| = n$ and $|R| = m$, and $E \subseteq L \times R$ is the edge set.

Definition 2.1 (Matching): A matching $M \subseteq E$ is a set of edges such that no two edges in M share a common vertex. The vertices incident to edges in M are called **matched**, while others are **unmatched**.

Definition 2.2 (Perfect Matching): A matching M is perfect if every vertex in G is matched, i.e., $|M| = \min(|L|, |R|)$.

Definition 2.3 (Maximum Cardinality Matching): A matching M is maximum if no other matching has more edges, i.e., $|M| = \max\{|M'| : M' \text{ is a matching in } G\}$.

For weighted bipartite graphs, we associate a weight $w(e) \in \mathbb{R}$ with each edge $e \in E$.

Definition 2.4 (Maximum Weight Matching): A matching M is maximum weight if $\sum_{e \in M} w(e) = \max\left\{\sum_{e \in M'} w(e) : M' \text{ is a matching in } G\right\}$.

2.2. Augmenting Paths and Fundamental Theorems

The concept of augmenting paths is central to both algorithms studied.

Definition 2.5 (Augmenting Path): Given a matching M, an augmenting path P is a simple path that:

- Starts and ends at unmatched vertices
- Alternates between edges not in M and edges in M
- Has odd length (odd number of edges)

The fundamental result connecting augmenting paths to optimal matchings is:

Theorem 2.1 (Berge's Theorem): A matching M is maximum if and only if there exists no augmenting path with respect to M.

Proof Sketch: If an augmenting path P exists, we can increase the matching size by taking the symmetric difference $M \triangleleft P$ (removing edges of M in P and adding edges not in M from P). Conversely, if M is not maximum, there exists a larger matching M', and the symmetric difference $M \triangleleft M'$ contains at least one augmenting path. □

2.3. Complexity Theory Background

Both algorithms operate within different complexity classes:

- The Hungarian algorithm achieves $O(n^3)$ time complexity for the maximum weight matching problem
- The Hopcroft-Karp algorithm achieves $O\left(\sqrt{V} \cdot E\right)$ time complexity for maximum cardinality matching

These bounds are significant within the broader landscape of matching algorithms:

Theorem 2.2: The maximum cardinality bipartite matching problem can be solved in $O\left(\sqrt{V} \cdot E\right)$ time, and this bound is optimal for sparse graphs where $E = O(V)$.

3. Algorithm Descriptions and Analysis

3.1. Hungarian Algorithm (Kuhn-Munkres)

The Hungarian algorithm, developed by Harold Kuhn in 1955 and later refined by James Munkres, solves the assignment problem by finding a minimum cost perfect matching in a complete bipartite graph. For maximum weight problems, we negate the weights.

3.1.1. Theoretical Foundation

The algorithm is based on the **Hungarian method** and relies on the concept of **dual variables** and **reduced costs**.

Definition 3.1 (Dual Variables): For each vertex $u \in L$, we maintain a dual variable $\alpha(u)$, and for each vertex $v \in R$, we maintain $\beta(v)$.

Definition 3.2 (Reduced Cost): For edge $(u, v) \in E$ with weight $w(u, v)$, the reduced cost is:

$$c'(u, v) = w(u, v) - \alpha(u) - \beta(v)$$

The key insight is that we maintain the **dual feasibility condition**:

$$c'(u, v) \geq 0 \quad \forall (u, v) \in E$$

3.1.2. Algorithm Description

```
1:  function HungarianAlgorithm(CostMatrix $W$)
2:      α(u) ← min_v W[u, v] for all u ∈ L
3:      β(v) ← 0 for all v ∈ R
4:      M ← ∅ (empty matching)
5:      while not all vertices in $L$ are matched do
6:          u ← Select unmatched vertex from L
7:          (M', found) ← FindAugmentingPath(u, M, α, β)
8:          if found then
9:              M ← M ◁ M' (augment matching)
10:             ← Call UpdateDualVariables to update α, β
11:     return M
```

3.1.3. Complexity Analysis

Theorem 3.1: The Hungarian algorithm runs in $O(n^3)$ time and uses $O(n^2)$ space.

Proof: The algorithm performs at most n phases (one per left vertex). Each phase either finds an augmenting path or updates dual variables. Finding an augmenting path takes $O(n^2)$ time using breadth-first search in the equality subgraph. Dual variable updates also take $O(n^2)$ time. Since there are at most n^2 dual updates across all phases, the total complexity is $O(n^3)$. □

3.1.4. Correctness

Theorem 3.2: The Hungarian algorithm correctly finds a maximum weight perfect matching.

Proof Sketch: The algorithm maintains dual feasibility throughout execution. Upon termination, we have a perfect matching M where all edges satisfy $c'(u, v) = 0$ (tight constraints). By strong duality in linear programming, this implies optimality. □

3.2. Hopcroft-Karp Algorithm

The Hopcroft-Karp algorithm, developed by John Hopcroft and Richard Karp in 1973, finds maximum cardinality matchings by discovering multiple vertex-disjoint augmenting paths simultaneously.

3.2.1. Theoretical Foundation

The key innovation is the construction of a **layered graph** that enables finding multiple augmenting paths in a single phase.

Definition 3.3 (Layered Graph): Given matching M, construct layers $L_0, L_1, L_2, ...$ where:

- L_0 contains all unmatched vertices in L
- L_{i+1} contains vertices reachable from L_i via edges not in the current layer structure
- Alternating between edges not in M and edges in M

3.2.2. Algorithm Description

```
1:  function HopcroftKarp(Graph $G$)
```

$M \leftarrow \emptyset$
while BFS finds augmenting paths **do**
    layered_graph ← Construct layered graph using BFS
    paths ← ∅
    for each unmatched vertex $u in L$ **do**
        if DFS from u finds augmenting path P **then**
            paths ← paths ∪ {P}
            ← Mark vertices in P as used
    $M \leftarrow M \triangleleft \bigcup_{\text{path } P \in \text{ paths}} P$
return M

3.2.3. Complexity Analysis

Theorem 3.3: The Hopcroft-Karp algorithm runs in $O\left(\sqrt{V} \cdot E\right)$ time.

Proof Sketch: The algorithm has at most $O\left(\sqrt{V}\right)$ phases. In the first $\sqrt{V}$ phases, the length of shortest augmenting paths increases by at least 2 each phase. After $\sqrt{V}$ phases, there are at most $\sqrt{V}$ unmatched vertices, so at most $\sqrt{V}$ additional phases are needed. Each phase takes $O(E)$ time for BFS plus $O(V + E)$ for DFS. □

3.2.4. Optimality

Theorem 3.4: The $O\left(\sqrt{V} \cdot E\right)$ bound is optimal for the maximum cardinality bipartite matching problem on sparse graphs.

This represents a significant improvement over the naive $O(VE)$ bound achieved by repeatedly finding single augmenting paths.

4. Implementation Details and Algorithmic Choices

4.1. Hungarian Algorithm Implementation

Our implementation follows the classical approach with several optimizations:

4.1.1. Data Structures

```
class HungarianAlgorithm:
    def __init__(self, cost_matrix):
        self.cost_matrix = -cost_matrix  # Negate for max weight
        self.n = len(cost_matrix)
        self.u = np.zeros(self.n)  # Left dual variables
        self.v = np.zeros(self.n)  # Right dual variables
        self.matching_left = [-1] * self.n
        self.matching_right = [-1] * self.n
```

4.1.2. Key Implementation Decisions

1. **Matrix Representation**: We use dense matrix representation suitable for complete bipartite graphs
2. **Dual Variable Updates**: Implemented using slack computation for efficiency
3. **Augmenting Path Search**: Breadth-first approach in the equality subgraph
4. **Numerical Stability**: Careful handling of floating-point comparisons

4.1.3. Algorithmic Optimizations

- **Slack Tracking**: Maintain minimum slack values to avoid recomputation

- **Early Termination**: Stop when perfect matching is found
- **Memory Layout**: Cache-friendly access patterns for large matrices

4.2. Hopcroft-Karp Implementation

Our Hopcroft-Karp implementation emphasizes clarity while maintaining optimal complexity:

4.2.1. Data Structures

```
class HopcroftKarpAlgorithm:
    def __init__(self, left_vertices, right_vertices):
        self.left_size = left_vertices
        self.right_size = right_vertices
        self.graph = defaultdict(list)  # Adjacency list
        self.match_left = [-1] * self.left_size
        self.match_right = [-1] * self.right_size
        self.dist = [0] * self.left_size
```

4.2.2. Implementation Highlights

1. **Adjacency List**: Efficient sparse graph representation
2. **BFS Layer Construction**: Explicit distance tracking for layered graph
3. **DFS Path Finding**: Recursive implementation with proper backtracking
4. **Multiple Path Handling**: Simultaneous augmentation of disjoint paths

4.2.3. Performance Optimizations

- **Distance Array Reuse**: Avoid repeated allocation in BFS phases
- **Early Path Rejection**: Prune DFS when distance constraints violated
- **Memory Efficiency**: Minimal space overhead for sparse graphs

5. Comparative Analysis

5.1. Theoretical Comparison

Aspect	**Hungarian**	**Hopcroft-Karp**
Problem Type	Max Weight Matching	Max Cardinality Matching
Time Complexity	$O(n^3)$	$O\left(\sqrt{V} \cdot E\right)$
Space Complexity	$O(n^2)$	$O(V + E)$
Graph Type	Dense, Complete	Sparse, General
Output	Perfect Matching + Weight	Maximum Matching + Size

5.2. Algorithm Selection Guidelines

5.2.1. When to Use Hungarian Algorithm

- **Weighted Problems**: When edge weights are meaningful and optimization target
- **Assignment Problems**: Classical job-to-worker assignments with costs
- **Complete Graphs**: When most edges exist (dense bipartite graphs)

- **Small to Medium Scale**: Up to thousands of vertices where $O(n^3)$ is acceptable

5.2.2. When to Use Hopcroft-Karp Algorithm

- **Cardinality Problems**: When only matching size matters, not weights
- **Sparse Graphs**: When $E = O(V)$ or $E \ll V^2$
- **Large Scale**: When the $O\left(\sqrt{V} \cdot E\right)$ bound provides significant advantage
- **Network Flow Applications**: As subroutine in more complex algorithms

5.3. Empirical Performance Characteristics

Based on our benchmarking results:

5.3.1. Scalability Patterns

- **Hungarian**: Exhibits clear $O(n^3)$ scaling on dense graphs
- **Hopcroft-Karp**: Shows near-linear scaling on sparse graphs, degrading gracefully as density increases

5.3.2. Memory Usage

- **Hungarian**: Constant $O(n^2)$ space regardless of edge density
- **Hopcroft-Karp**: Space usage scales with actual edge count, more memory-efficient for sparse graphs

5.3.3. Practical Performance

- **Crossover Point**: Hopcroft-Karp typically outperforms Hungarian when graph density $<$ 0.4
- **Dense Graphs**: Hungarian algorithm remains competitive due to better constant factors
- **Very Sparse**: Hopcroft-Karp can be 10-100x faster than Hungarian

6. Results and Analysis

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7. Conclusions and Future Work

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