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Optimal Investment of DC Pension Plan with S-shaped Utility and Trading and VaR Constraints

Yinghui Dong* and Harry Zheng†

Abstract

In this paper we investigate an optimal investment problem under loss aversion (S-shaped utility) and with trading and Value-at-Risk (VaR) constraints faced by a defined contribution (DC) pension fund manager. We apply the concavification and dual control method to solve the problem and derive the closed-form representation of the optimal terminal wealth in terms of a controlled dual state variable. We propose a simple and effective algorithm for computing the initial dual state value, the Lagrange multiplier and the optimal terminal wealth. Theoretical and numerical results show that the VaR constraint can significantly impact the distribution of the optimal terminal wealth and may greatly reduce the risk of losses in bad economic states due to loss aversion.

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1 Introduction

Defined contribution (DC) pension schemes are becoming increasingly important in the pension systems of many countries. In a DC plan, the member contributes part of the salary to the plan and bears the financial risk. The retirement benefit of a DC plan is mainly determined by the performance of its fund portfolios before retirement. It is essential for the contributors to find optimal investment strategies during the accumulation phase to build sufficient funds on retirement. The optimal investment problem for the DC pension plan has attracted extensive research, see Boulier et al. (2001), Cairns et al. (2006), Zhang et al. (2007), Zhang and Ewald (2010), Yao et al. (2013) and Blake et al. (2013, 2014).

Most literature on investment problems focus on maximizing the expectation of a smooth utility of terminal wealth. Loss aversion, first proposed by Kahneman and Tversky (1979) within the framework of prospect theory (PT), is defined over gains and losses in wealth relative to a pre-defined reference point, rather than in terms of changes in the absolute level of total wealth itself. Every investor has a reference point that defines relative 'losses' and 'gains'. Kahneman and

*Department of Mathematics and Physics, Suzhou University of Science and Technology, Suzhou 215009, P. R. China. Email: dongyinghui1030@163.com

†Department of Mathematics, Imperial College, London SW7 2AZ, UK. Email: h.zheng@imperial.ac.uk

Tversky (1992) also demonstrate the loss aversion and risk seeking behaviour by an asymmetric S-shaped utility function, convex in the domain of losses and concave in the domain of gains. The martingale method is mainly used to solve the S-shaped utility maximization problem when the market is complete. For example, Berkelaar et al. (2004) derive the optimal investment strategies with two utility functions under loss aversion in a continuous-time case. Guan and Liang (2016) find the optimal allocation of a DC plan under the S-shaped utility function. Chen et al. (2017) obtain the optimal investment strategy in a DC plan with inflation risk by maximizing the expected S-shaped utility of the terminal wealth exceeding the minimum performance. Extending Chen et al. (2017) to incomplete market setting, Dong and Zheng (2018) include both short-selling and portfolio insurance (PI) constraints in the model and apply the dual control method to derive the optimal portfolio and wealth processes for a DC pension plan.

The PI constraint can well protect the members' benefits by keeping the optimal terminal wealth always above the minimum guarantee, but it may lead to over-cautious investment strategy and a relatively low expected terminal wealth. Value-at-Risk (VaR), defined as the worst expected loss given a pre-set confidence level, is a quantile measure that controls the tail risk of the terminal wealth. Due to its prominence in current regulatory frameworks for banks (Basel II) as well as for insurance companies (Solvency II), VaR plays an important role in pensions, insurance companies and other financial institutions. In particular, VaR-based risk management (VaR-RM) has a convenient property that it nests the PI-based risk management (PI-RM). The problem of best allocations under a VaR constraint has been extensively studied in the literature. For example, Basak and Shapiro (2001) derive the optimal investment strategies to manage VaR risk by a martingale method. Boyle and Tian (2007) generalize the VaR constraint to the case where the wealth must exceed a stochastic, but hedgeable, benchmark with a given probability. Kraft and Steffensen (2013) provide a dynamic programming approach to solve constrained portfolio problems. Guan and Liang (2016) find the optimal allocation under a smooth concave utility with a VaR constraint. Recently, the problem of non-concave utility maximization has been considered by many researchers, see Bernard et al. (2015), El Karoui et al. (2005), Bichuch and Sturm (2014), Chen et al. (2018) and He and Kou (2018).

The aforementioned references all assume that the market is complete, which is equivalent to the existence of a unique pricing kernel. The martingale approach is commonly used to solve optimal investment problems as one may first find the optimal terminal wealth by solving a simplified static optimization problem and then find the replicating feasible trading strategy (the optimal control) with the martingale representation theorem, see, for example, Pliska (1986), Cox and Huang (1989), and Karatzas et al. (1986). He and Zhou (2011) investigate a concave utility maximization problem in an incomplete market model which has infinitely many pricing kernels and show that there exists a unique pricing kernel, called the minimal pricing kernel, in the presence of closed convex cone control constraints and one may solve the problem in the same way as in the complete market case by using the minimal pricing kernel. The optional decomposition theorem (see Follmer and Kramkov (1997)) ensures the existence of a feasible replicating trading strategy. To actually find the replicating trading strategy, one may first find the optimal wealth process by computing the conditional expectation of the optimal terminal wealth with the minimal pricing kernel and then derive the stochastic differential equation for the optimal wealth process with Ito's formula. The key step in this procedure is to find explicitly the optimal wealth process, which is in general difficult when the utility function is not a simple strictly concave function (see Chen et al. (2018)), let alone any additional constraints on the terminal wealth.

In this paper, we investigate an S-shaped utility maximization problem for a DC plan under trading and VaR constraints. It is in general highly challenging to solve such a problem. We circumvent the difficulty by introducing a Lagrange multiplier to reflect the bindingness of VaR constraint, which leads to a utility maximization problem with a non-concave discontinuous utility function. We apply the concavification and the dual control method, together with the pathwise differentiation and likelihood ratio method, to solve the resulting VaR constraint-free maximization problem with a fixed Lagrange multiplier and give the explicit characterizations of the optimal portfolio and wealth processes in terms of the optimal dual control and state processes. To find the correct Lagrange multiplier that makes the optimal solution of the unconstrained problem the same one for the constrained problem, we need to solve two fully coupled nonlinear equations (binding budget and VaR constraints) with Lagrange multiplier and initial dual state value as variables, again a highly nontrivial problem. We overcome the difficulty by finding the explicit relation of the Lagrange multiplier and the initial dual value from the binding VaR constraint and then solve the binding budget constraint equation to find the unique initial dual state value. The key to our success is the dual control method which is effective in solving portfolio optimization problems with control constraints as it relates the original stochastic optimal control problem to a dual problem which may be relatively easier to solve, see Xu and Shreve (1992) and Bian et al. (2011). Unlike He and Zhou (2011), we derive the explicit representation of the optimal wealth process and investment strategy in terms of the dual value function, its derivatives, and the optimal dual state process, which makes possible either to find their closed form expressions or to compute their numerical values with the Monte Carlo simulation method. It is virtually impossible to solve the problem in this paper directly in its primal form. We propose a numerical algorithm for constructively computing the initial dual state value, the Lagrange multiplier and the optimal terminal wealth.

Our theoretical and numerical results show that the loss aversion as well as trading and VaR constraints have significant effects on the optimization problem. It is well known that, a VaR constraint leads to heavier losses in bad market scenarios than in the case of no VaR constraint under a smooth concave utility. Loss aversion leads the manager to be risk averse in the gain domain and risk-seeking in the loss domain. We find that introducing a VaR constraint under an S-shaped utility may not lead to more losses than in the case of no VaR constraint. On the contrary, a VaR constraint may strictly improve the risk management for bad economic states. It is crucially important to choose a reasonable confidence level and protection level in a VaR constraint. If the confidence level is too low or the protection level is too high, then more states would need to be insured against and the optimal terminal wealth would become less volatile which results in a relatively low expected terminal wealth. If the confidence level is too high or the protection level is too low, then a VaR constraint would not well protect the members' benefits.

The main contribution of this paper is that we solve an optimal investment problem of a DC plan with S-shaped utility and trading and VaR constraints, prove the existence and uniqueness of the optimal solution, characterize explicitly the optimal terminal wealth and the Lagrange multiplier, and identify the joint impact of the S-shaped utility and VaR and control constraints on the distribution of the optimal terminal wealth. To the best of the authors' knowledge, we are the first in the literature to have completely solved the aforementioned constrained optimization problem based on the concavification and dual control method.

The rest of the paper is organized as follows. In Section 2 we formulate a DC investment problem with S-shaped utility and trading and VaR constraints, and convert the constrained optimization problem (2.7) into an equivalent unconstrained one (2.8) coupled with the feasibility and complementary slackness condition (2.9). In Section 3 we apply the concavification and dual control method to solve the unconstrained nonconcave discontinuous optimization problem (2.8) and characterize explicitly the optimal solution and optimal control in Propositions 3.1 and 3.2 and Theorem 3.6. In Section 4 we state the main result of the paper, Theorem 4.1, on the existence and uniqueness of the optimal solution for the constrained optimization problem (2.7) and give a constructive proof which leads to a simple and effective algorithm to compute the initial dual state variable and Lagrange multiplier. In Section 5 we present a numerical example with short-selling constraints and discuss the impact of VaR constraint on the distribution of the optimal terminal wealth. Section 6 concludes. The appendix gives a technical lemma that is used in constructing the concave envelope of a nonconcave discontinuous function.

2 The investment problem for a DC pension fund

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space with the filtration $\mathbb{F} := \{\mathcal{F}_t | 0 \le t \le T\}$ being the natural filtration generated by an n-dimensional standard Brownian motion $W(t) = (W_1(t), \cdots, W_n(t))^\top$, where $W_1(t), \cdots, W_n(t)$ are independent and $a^\top$ is the transpose of a, and satisfying the usual conditions. The pension fund starts at time 0 and the retirement time is T.

Let the financial market consist of $n+1$ tradable securities: one riskless savings account $S_0(t)$ and n risky assets $S_i(t), i = 1, \cdots, n$. The riskless savings account evolves as

$$dS_0(t) = rS_0(t)dt, \tag{2.1}$$

where r is a riskless interest rate. The price processes of the n risky assets are modelled by

$$dS(t) = \mathrm{diag}(S(t))(\mu dt + \sigma dW(t)), \tag{2.2}$$

where $S(t) = (S_1(t), \cdots, S_n(t))^\top$, $\mathrm{diag}(S(t))$ is an $n \times n$ matrix with diagonal elements $S_i(t)$ and all other elements 0, $\mu = (\mu_1, \cdots, \mu_n)^\top$ is a constant vector representing the stock growth rate with $\mu_i > r, i = 1, \cdots, n$, and $\sigma = (\sigma_{ij})$ is an $n \times n$ nonsingular constant matrix representing the volatility and correlation information of $S(t)$.

In a DC plan, the pension members contribute a proportion of salary to the pension plan before retirement time T and the amount of money contributed at time t is assumed to be $c(t) > 0$. Based on the fact that, in the labor market, the average salary of employees and the contribution rate often steadily increase in the long run, we describe $c(t)$ by a deterministic, nondecreasing function. Assume that there are no transaction costs or taxes in the financial market. The pension account is endowed with an initial endowment $x_0 \ge 0$. The wealth process $X^\pi(t)$ satisfies the following controlled stochastic differential equation (SDE):

$$dX^\pi(t) = (rX^\pi(t) + \pi^\top(t)\sigma\xi)dt + \pi^\top(t)\sigma dW(t) + c(t)dt,\ t \ge 0, \tag{2.3}$$

with initial condition $X^\pi(0) = x_0$, where $\pi(t) = (\pi_1(t), \cdots, \pi_n(t))^\top$ and $\pi_i(t)$ is the amout of wealth invested in the ith risky asset for $i = 1, \cdots, n$, $\xi = \sigma^{-1}(\mu - r\mathbf{1})$ is the market price of risk vector and $\mathbf{1}$ is a vector with all components 1. We next define the set of admissible trading strategies.

Definition 2.1. *Let K be a closed convex cone. A portfolio strategy $\pi = (\pi_1, \cdots, \pi_n)^\top$ is said to be admissible if it is a progressively measurable, $\mathcal{F}$-adapted process which satisfies $E[\int_0^T \|\pi(t)\|^2 dt] < \infty$, $\pi(t) \in K$, a.s., and there exists a unique strong solution $X^\pi(t)$ to (2.3). The set of all admissible portfolio strategies is denoted by $\mathcal{A}$.*

The goal of the pension manager is to find the optimal investment strategies within $[0, T]$ under loss aversion with trading and VaR constraints. Kahneman and Tversky (1979) claim that people tend to be risk averse to gains and risk seeking to losses. Furthermore, people make decisions relative to some reference levels rather than absolute values directly. Based on experiments, Kahneman and Tversky (1979) propose an S-shaped utility function to characterize different behaviors of people over gains and losses relative to a reference point. Here we extend their S-shaped utility to a more general utility.

Let θ be a reference point, which is chosen in advance. Consider a utility function defined by:

$$U(x) = \begin{cases} -U_2(\theta - x), & 0 \le x < \theta, \\ U_1(x - \theta), & x \ge \theta, \end{cases} \tag{2.4}$$

where U_1 and U_2 are two strictly increasing, strictly concave, continuously differentiable, real-valued functions defined on $[0, \infty)$ satisfying

$$U_i(0) = 0, \lim_{x \to +\infty} U_i(x) = +\infty, \lim_{x \to 0^+} U_i'(x) = +\infty, \lim_{x \to +\infty} U_i'(x) = 0, \tag{2.5}$$

and

$$0 \le U_i(x) \le M_i(1 + x^{p_i}), \ x \ge 0, \tag{2.6}$$

for some constants $M_i > 0, 0 < p_i < 1, i = 1, 2$. Let $U(x) = -\infty$ if $x < 0$. Note that U is convex when x is less than θ and concave when x is greater than θ, which gives an S-shaped graph.

Condition (2.5) ensures that the strictly decreasing function U_1' has a strictly decreasing inverse $I_1 : (0, \infty) \to (0, \infty)$, that is,

$$U_1'(I_1(y)) = y, \forall y > 0, \ I_1(U_1'(x)) = x, \forall x > 0.$$

Let $L \ge \theta$ be a given level. In order to provide a downside protection, the pension manager is to find the optimal investment strategy to maximize the expected utility of the wealth at time T under a VaR constraint:

$$\begin{cases} \max_{\pi \in \mathcal{A}} E[U(X^\pi(T))], \\ s.t. \ X^\pi(t) \text{ satisfies (2.3)}, \\ \ P(X^\pi(T) \ge L) \ge 1 - \varepsilon, \end{cases} \tag{2.7}$$

where $0 \le \varepsilon \le 1$ is a given constant. The VaR constraint requires that the probability of the terminal wealth above the level L is at least $1 - \varepsilon$. When $\varepsilon = 1$, the constraint is not binding. When $\varepsilon = 0$, (2.7) recovers the case of portfolio insurance, see Basak (1995).

We may use the Lagrange multiplier method to solve problem (2.7). Define

$$\tilde{U}_\lambda(x) = U(x) + \lambda 1_{\{x \ge L\}},$$

where $\lambda \geq 0$ is a Lagrange multiplier to be determined. Note that $\tilde{U}_\lambda$ is a nonconcave discontinuous function.

Consider the following VaR constraint-free version of problem (2.7):

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}} E[\tilde{U}_\lambda(X^\pi(T))], \\ s.t.\ X^\pi(t) \text{ satisfies } (2.3). \end{cases} \tag{2.8}$$

If we can find the optimal solution $X^{\pi^*,\lambda^*}(T)$ of problem (2.8) for some admissible control π^* and nonnegative constant λ^* such that

$$\begin{cases} P(X^{\pi^*,\lambda^*}(T) \geq L) \geq 1 - \varepsilon, \\ \lambda^*(P(X^{\pi^*,\lambda^*}(T) \geq L) - 1 + \varepsilon) = 0, \end{cases} \tag{2.9}$$

then $X^{\pi^*,\lambda^*}(T)$ is the optimal solution of problem (2.7), π^* is the optimal control and λ^* is the Lagrange multiplier.

3 Solving unconstrained optimization problem (2.8)

For a fixed $\lambda \geq 0$, the dual function of $\tilde{U}_\lambda$ is defined by:

$$V_\lambda(y) = \sup_{x \geq 0}\{\tilde{U}_\lambda(x) - xy\}, y > 0. \tag{3.1}$$

Denote by f^c the concave envelope of a function f with domain D, that is, f^c is the smallest concave function that is greater than or equal to f, defined by

$$f^c(x) := \inf\{g(x) : D \to R |\ g \text{ is a concave function}, g(t) \geq f(t), \forall t \in D\}, \forall x \in D.$$

Since U is not concave, we first derive the concave envelope of U. To simplify the formulation, we introduce the notation

$$c_x := U_1'(x - \theta), x > \theta.$$

Note that c_x is a decreasing function of $x > \theta$ and is the slope of the tangent line to the curve $U_1(x - \theta)$ at point $x > \theta$.

Let z be the tangent point of the straight line starting at $(0, -U_2(\theta))$ to the curve $U_1(x), x \geq \theta$. Simple calculus shows that there exists a unique solution $z > \theta$ to the equation

$$U_1(x - \theta) + U_2(\theta) - xU_1'(x - \theta) = 0, \tag{3.2}$$

and the concave envelope of U is given by (see Figure 1)

$$U^c(x) = \begin{cases} c_z x - U_2(\theta), & 0 \leq x < z, \\ U_1(x - \theta), & x \geq z, \end{cases} \tag{3.3}$$

where z is the solution to (3.2), see Carpenter (2000).

Consider the concavified version of problem (3.1):

$$V_\lambda^c(y) = \sup_{x \geq 0}\{\tilde{U}_\lambda^c(x) - xy\}, y > 0. \tag{3.4}$$

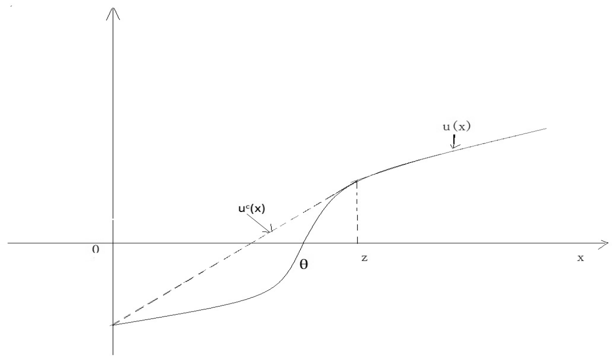

Figure 1: Concave envelope of $U(x)$

From Lemma 2.9 of Reichlin (2013), we have for $y > 0$,

$$V_\lambda(y) = V_\lambda^c(y) = \tilde{U}_\lambda^c(x^{*,\lambda}(y)) - x^{*,\lambda}(y)y = \tilde{U}_\lambda(x^{*,\lambda}(y)) - x^{*,\lambda}(y)y, \tag{3.5}$$

where $x^{*,\lambda}(y)$ solves both (3.1) and (3.4). For $\lambda \geq 0$ and $L \geq \theta$, denote by

$$k_\lambda = \frac{U_1(L-\theta) + \lambda + U_2(\theta)}{L}. \tag{3.6}$$

Note that k_λ is the slope of the straight line linking points $(0, -U_2(\theta))$ and $(L, U_1(L-\theta) + \lambda)$ and k_λ depends on λ as well as parameters L, θ and utility functions U_1, U_2. The next two results characterize $\tilde{U}_\lambda^c$ and $x^{*,\lambda}(y)$ for two cases: $L \geq z$ and $\theta \leq L < z$.

Proposition 3.1. *Let $L \geq z$ with z determined by (3.2) and k_λ be defined by (3.6). Then we have $k_\lambda \geq k_0 \geq c_L$ and $c_z \geq k_0 \geq c_L$. For $y > 0$,*

Case I: If $k_\lambda > c_z$, then

$$\tilde{U}_\lambda^c(x) = \begin{cases} k_\lambda x - U_2(\theta), & 0 \leq x < L, \\ U_1(x-\theta) + \lambda, & x \geq L, \end{cases} \tag{3.7}$$

and

$$x^{*,\lambda}(y) = \begin{cases} \theta + I_1(y), & y < c_L, \\ L, & c_L \leq y < k_\lambda, \\ 0, & y \geq k_\lambda. \end{cases} \tag{3.8}$$

Case II: If $c_L \leq k_\lambda \leq c_z$, then

$$\tilde{U}_\lambda^c(x) = \begin{cases} c_z x - U_2(\theta), & 0 \leq x < z, \\ U_1(x-\theta), & z \leq x < L_0, \\ c_{L_0}(x-L) + U_1(L-\theta) + \lambda, & L_0 \leq x < L, \\ U_1(x-\theta) + \lambda, & x \geq L, \end{cases} \tag{3.9}$$

and

$$x^{*,\lambda}(y) = \begin{cases} \theta + I_1(y), & y < c_L, \\ L, & c_L \leq y < c_{L_0}, \\ \theta + I_1(y), & c_{L_0} \leq y < c_z, \\ 0, & y \geq c_z, \end{cases} \tag{3.10}$$

where L_0 is the unique solution in the interval $[z, L]$ of the equation

$$U_1(x-\theta) - U_1(L-\theta) - \lambda - (x-L)U_1'(x-\theta) = 0. \tag{3.11}$$

In particular, if $\lambda = 0$, then $L_0 = L$, $\tilde{U}^c_\lambda(x)$ is the same as $\tilde{U}^c(x)$ given by (3.3) and

$$x^{*,0}(y) = \begin{cases} \theta + I_1(y), & y < c_z, \\ 0, & y \geq c_z. \end{cases} \tag{3.12}$$

Proof. Since $L \geq z$, we have $c_z \geq k_0 = \frac{U_1(L-\theta)+U_2(\theta)}{L} \geq c_L$, which yields $k_\lambda \geq k_0 \geq c_L$ for $\lambda \geq 0$.

Case I: When $k_\lambda > c_z$, we have that $\tilde{U}_\lambda(x) \leq k_\lambda x - U_2(\theta)$ for $0 \leq x \leq L$ and $\tilde{U}_\lambda(x)$ is concave on (L, ∞). Then Lemma A.1 gives the concave envelope of $\tilde{U}_\lambda$ given by (3.7)(see Figure 2). Note that $\tilde{U}^c_\lambda(x) = \tilde{U}_\lambda(x)$ when $x \in \{0\} \cup [L, \infty)$ and the solution $x^{*,\lambda}(y)$ which solves both (3.1) and (3.4) should satisfy $x^{*,\lambda}(y) \in \{x | \tilde{U}_\lambda(x) = \tilde{U}^c_\lambda(x)\}$. Then it is easy to find the point $x^{*,\lambda}(y)$ for which 0 is in the superdifferential of $\tilde{U}^c_\lambda(x) - xy$ given by (3.8).

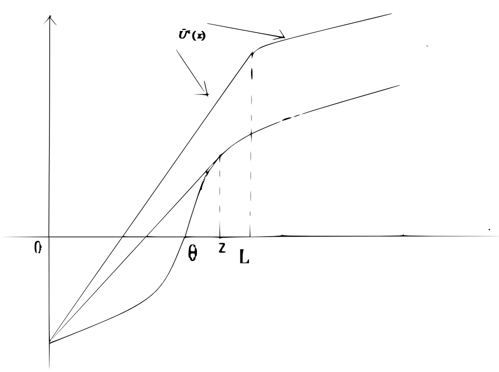

Figure 2: Concave envelope of $\tilde{U}_\lambda$, $k_\lambda > c_z$

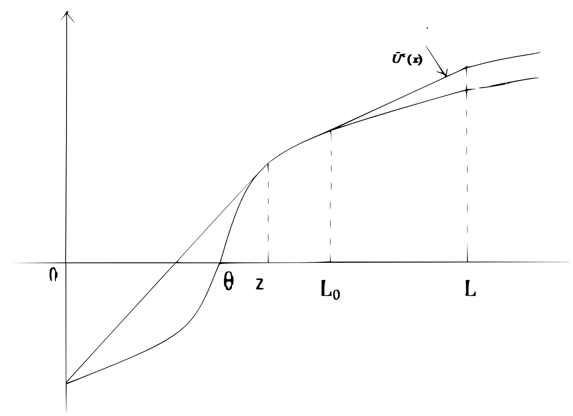

Figure 3: Concave envelope of $\tilde{U}_\lambda$, $k_\lambda \leq c_z$

Case II: When $c_L \leq k_\lambda \leq c_z$, we let L_0 be the tangent point of the straight line starting at $(L, U_1(L-\theta) + \lambda)$ to the curve $U_1(x-\theta), \theta \leq x \leq L$ (see Figure 3). Straightforward calculation shows that there exists a unique solution L_0 in the interval $[z, L]$ to the equation (3.11). Lemma A.1 gives the concave envelope of $\tilde{U}_\lambda$ represented by (3.9). Similar to deriving (3.8), one can easily find $x^{*,\lambda}(y)$ given by (3.10). $\square$

It is clear that $x^{*,0}(y)$ in (3.12) is bounded above by $x^{*,\lambda}$ in (3.10) that is bounded above by $x^{*,\lambda}$ in (3.8).

Proposition 3.2. *Let $\theta \leq L < z$ with z determined by (3.2) and k_λ be defined by (3.6). Then we have $k_0 < c_L$. For $y > 0$,*

Case I: If $k_\lambda \geq c_L$, then $\tilde{U}^c_\lambda(x)$ and $x^{*,\lambda}(y)$ are given by (3.7) and (3.8), respectively.

Case II: If $k_\lambda < c_L$, then

$$\tilde{U}^c_\lambda(x) = \begin{cases} c_{z_0}x - U_2(\theta), & 0 \leq x < z_0, \\ U_1(x-\theta) + \lambda, & x \geq z_0, \end{cases} \tag{3.13}$$

and

$$x^{*,\lambda}(y) = \begin{cases} \theta + I_1(y), & y < c_{z_0}, \\ 0, & y \geq c_{z_0}, \end{cases} \tag{3.14}$$

where z_0 is the unique solution in the interval $(L, z]$ of the equation

$$U_1(x-\theta) + U_2(\theta) + \lambda - xU_1'(x-\theta) = 0. \tag{3.15}$$

In particular, if $\lambda = 0$, then $z_0 = z$, $\tilde{U}^c_\lambda(x)$ and $x^{*,0}(y)$ are given by (3.3) and (3.12), respectively.

Proof. Since $\theta \leq L < z$, we have $k_0 = \frac{U_1(L-\theta)+U_2(\theta)}{L} < c_L$.

Case I: If $k_\lambda \geq c_L$, then the expressions for $\tilde{U}^c_\lambda(x)$ and $x^{*,\lambda}(y)$ are the same as those in Case I of Proposition 3.1(see Figure 4).

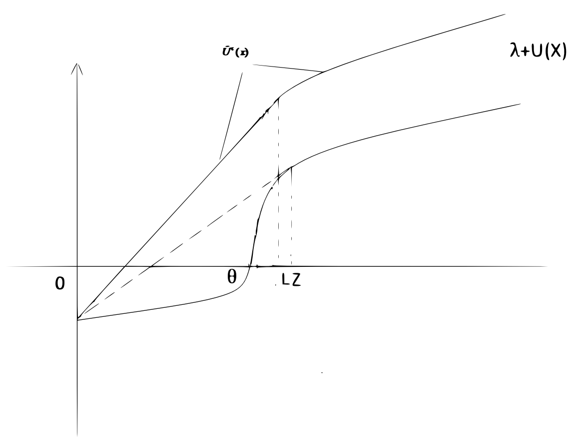

Figure 4: Concave envelope of $\tilde{U}_\lambda$, $k_\lambda \geq c_L$

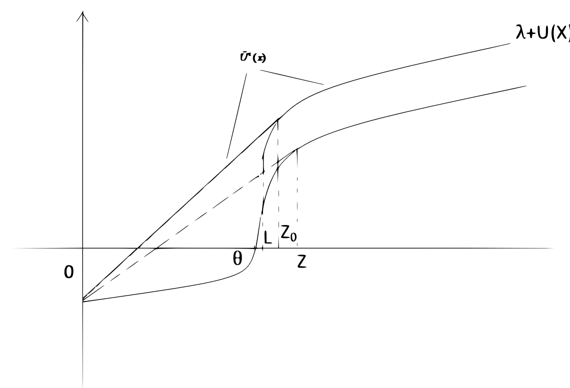

Figure 5: Concave envelope of $\tilde{U}_\lambda$, $k_\lambda < c_L$

Case II: If $k_\lambda < c_L$, then we let z_0 be the tangent point of the straight line starting at $(0, -U_2(\theta))$ to the curve $U_1(x-\theta)+\lambda, x \geq \theta$ (see Figure 5). It is easy to verify that there exists a unique solution $L < z_0 \leq z$ to the equation (3.15) and $\tilde{U}_\lambda(x) \leq c_{z_0}x - U_2(\theta)$ for $0 \leq x < z_0$. From Lemma A.1, the concave envelope of $\tilde{U}_\lambda$ is given by (3.13). Then one can easily derive $x^{*,\lambda}(y)$ given by (3.14). □

Note that the wealth process X^π given by (2.3) is not self-financing due to the contribution term in a DC pension plan. Therefore, the problem (2.8) is not a classical utility maximization problem with control constraints. To apply the existing results on the optimization problem with control constraints, we introduce an auxiliary process as follows:

$$\tilde{X}^\pi(t) = X^\pi(t) + C(t), \tag{3.16}$$

where

$$C(t) = \int_t^T c(s)e^{-r(s-t)}ds, \tag{3.17}$$

is the discounted value at time t of total pension contribution from t to T. Using (2.3), we have

$$d\tilde{X}^\pi(t) = (r\tilde{X}^\pi(t) + \pi^\top(t)\sigma\xi)dt + \pi^\top(t)\sigma dW(t), \tilde{X}^\pi(0) = \tilde{x}_0 \geq 0, \tag{3.18}$$

with $\tilde{x}_0 = x_0 + C(0) = x_0 + \int_0^T c(s)e^{-rs}ds$. Note that $\tilde{X}^\pi(T) = X^\pi(T)$. Therefore, the optimization problem (2.8) is equivalent to the following problem:

$$\begin{cases} \max_{\pi \in \mathcal{A}} E[\tilde{U}_\lambda(\tilde{X}^\pi(T))], \\ s.t.\ \tilde{X}^\pi(t) \text{ satisfies (3.18)}. \end{cases} \tag{3.19}$$

Define the value functions of the primal problem and its concavified version by

$$u_\lambda(t, \tilde{x}) = \max_{\pi \in \mathcal{A}} E[\tilde{U}_\lambda(\tilde{X}^\pi(T)) | \tilde{X}^\pi(t) = \tilde{x}], \tag{3.20}$$

and

$$u_\lambda^c(t, \tilde{x}) = \max_{\pi \in \mathcal{A}} E[\tilde{U}_\lambda^c(\tilde{X}^\pi(T)) | \tilde{X}^\pi(t) = \tilde{x}]. \tag{3.21}$$

We next use the dual control method to solve the optimization problems $u_\lambda(t, \tilde{x})$ and $u_\lambda^c(t, \tilde{x})$. First, we define the dual control set.

Definition 3.3. *Let $\tilde{K}$ be the positive polar cone of K, i.e., $\tilde{K} = \{p : p^\top v \geq 0$ for all $v \in K\}$. A dual control process is a progressively measurable, $\mathcal{F}$-adapted process $\nu = (\nu_1, \cdots, \nu_n)^\top$ which satisfies $E[\int_0^T \|\nu(t)\|^2 dt] < \infty$ and $\nu(t) \in \tilde{K}$ a.s. for all t. We denote the set of all dual control processes by $\mathcal{A}_0$.*

Following a similar argument in Bian et al. (2011), we choose a nonnegative supermartingale process Y^ν such that $\tilde{X}^\pi(t)Y^\nu(t)$ is a supermartingale for the wealth process $\tilde{X}^\pi$ given by (3.18). For $\nu \in \mathcal{A}_0$, define the dual process

$$dY^\nu(t) = Y^\nu(t)(-rdt - (\sigma^{-1}\nu(t) + \xi)^\top dW(t)), Y^\nu(0) = y_0.$$

Consider the dual minimization problem

$$\min_{\nu \in \mathcal{A}_0} E[V_\lambda^c(Y^\nu(T))].$$

The dual value function is defined by

$$v_\lambda(t, y) = \min_{v \in \mathcal{A}_0} E[V_\lambda^c(Y^\nu(T)) | Y^\nu(t) = y].$$

The dual HJB equation is given by

$$\begin{cases} \frac{\partial v_\lambda}{\partial t}(t, y) - ry\frac{\partial v_\lambda}{\partial y}(t, y) + \frac{1}{2}y^2 \min_{\nu \in \tilde{K}} \|\xi + \sigma^{-1}\nu\|^2 \frac{\partial^2 v_\lambda}{\partial y^2}(t, y) = 0, y > 0, t < T, \\ v_\lambda(T, y) = V_\lambda(y). \end{cases} \tag{3.22}$$

Here we have used $V_\lambda(y) = V_\lambda^c(y)$ for $y > 0$. There exists a unique minimizer $\hat{\nu} \in \tilde{K}$ for convex quadratic function

$$f(\nu) = \|\xi + \sigma^{-1}\nu\|^2 \tag{3.23}$$

over $\nu \in \tilde{K}$, see Xu and Shreve (1992). Denote by

$$\hat{\xi} = \xi + \sigma^{-1}\hat{\nu}. \tag{3.24}$$

We assume the following condition to ensure the linear parabolic PDE (3.22) is non-degenerate.

Assumption 3.4. $\hat{\xi} \neq 0$.

Under the Assumption 3.4, the solution to (3.22) is given by

$$\begin{cases} v_\lambda(t,y) = E[V_\lambda(Y^{\hat{\nu}}(T))|Y^{\hat{\nu}}(t) = y], \\ Y^{\hat{\nu}}(s) = y\frac{H^{\hat{\nu}}(s)}{H^{\hat{\nu}}(t)}, t \leq s \leq T, \end{cases} \tag{3.25}$$

where

$$H^{\hat{\nu}}(t) = e^{-(r+\frac{\|\hat{\xi}\|^2}{2})t - \hat{\xi}^\top W(t)} \tag{3.26}$$

is a state-price density process in the fictitious market (see Cox and Huang (1989)). If there is no limitation on the trading strategy, then $\hat{\nu} = 0$ and $H^{\hat{\nu}}(t)$ is exactly the pricing kernel in a complete market. Reichlin (2013) investigates the relationship between $u^c_\lambda(t,\tilde{x})$ and $u_\lambda(t,\tilde{x})$ (Theorem 5.1 of Reichlin (2013)) and gives the following result.

Theorem 3.5. *(Reichlin (2013)) Assume that $u_\lambda(t,\tilde{x})$ and $u^c_\lambda(t,\tilde{x})$ are given by (3.20) and (3.21). Then it holds that $u_\lambda(t,\tilde{x}) = u^c_\lambda(t,\tilde{x})$.*

Bian et al. (2011) and Xu and Shreve (1992) give the relationship between the optimization problem $u^c_\lambda(t,\tilde{x})$ and the dual optimization problem. Combining with Theorem 3.5, we have the following result.

Theorem 3.6. *(Bian et al. (2011)) Assume that $v_\lambda(t,y)$ is given by (3.25) and the conditions (2.5) and (2.6) hold, then we have, for $0 \leq t < T$,*

$$u_\lambda(t,\tilde{x}) = v_\lambda(t, y(t,\tilde{x})) + \tilde{x}y(t,\tilde{x}), \tilde{x} \geq 0,$$

where $y = y(t,\tilde{x})$ satisfies the equation

$$\frac{\partial v_\lambda}{\partial y}(t,y) + \tilde{x} = 0. \tag{3.27}$$

The optimal feedback control is given by

$$\pi^{*,\lambda}(t,\tilde{x}) = (\sigma^\top)^{-1}\hat{\xi}y(t,\tilde{x})\frac{\partial^2 v_\lambda}{\partial y^2}(t, y(t,\tilde{x})), \tag{3.28}$$

and $\pi^{*,\lambda}(t,\tilde{x}) \in K$. Furthermore, starting with the initial wealth $\tilde{x}_0$ at time 0, the optimal wealth process is given by

$$\tilde{X}^{\pi^{*,\lambda}}(t) = -\frac{\partial v_\lambda}{\partial y}(t, y_0 H^{\hat{\nu}}(t)), \tag{3.29}$$

where y_0 is the solution to the equation

$$\frac{\partial v_\lambda}{\partial y}(0, y_0) + \tilde{x}_0 = 0. \tag{3.30}$$

Note that $V_\lambda(y)$ is continuous for $y > 0$ and continuously differentiable for $y > 0$ except at finitely many points. Using (3.5) and pathwise differentiation, we have

$$\frac{\partial v_\lambda}{\partial y}(t,y) = -E[x^{*,\lambda}(Y^{\hat{\nu}}(T))\frac{Y^{\hat{\nu}}(T)}{y}|Y^{\hat{\nu}}(t) = y], \tag{3.31}$$

where $x^{*,\lambda}(y)$ is defined in (3.8),(3.10) for $L \geq z$ and (3.8),(3.14) for $\theta \leq L < z$, respectively. Since $x^{*,\lambda}(y)$ is not continuous everywhere for $y > 0$, we cannot use the pathwise differentiation, but we may apply the likelihood ratio method (see Broadie and Glasserman (1996)) to obtain

$$\frac{\partial^2 v_\lambda}{\partial y^2}(t, y) = -E[x^{*,\lambda}(Y^{\hat{\nu}}(T))\frac{Y^{\hat{\nu}}(T)(\ln(\frac{Y^{\hat{\nu}}(T)}{y}) + \beta(t))}{\alpha(t)y^2}|Y^{\hat{\nu}}(t) = y], \tag{3.32}$$

where

$$\alpha(t) = \|\hat{\xi}\|^2(T - t), \beta(t) = (r - \frac{\|\hat{\xi}\|^2}{2})(T - t). \tag{3.33}$$

Since $Y^{\hat{\nu}}(T)$ is a lognormal variable, some lengthy but straightforward calculations will lead to closed-form expressions for $\pi^{*,\lambda}(t, \tilde{x})$, $\tilde{X}^{\pi^{*,\lambda}}(t)$ in (3.28) and (3.29), which are omitted in this paper. Once $\tilde{X}^{\pi^{*,\lambda}}(t)$ is derived, the optimal wealth process $X^{\pi^*,\lambda}(t)$ can be easily obtained from (3.16).

4 Solving constrained optimization problem (2.7)

In the previous section, we have applied the dual control method to solve the unconstrained optimization problem (2.8) for every fixed $\lambda \geq 0$. We now show there exists a $\lambda^* \geq 0$ such that condition (2.9) holds. Applying Theorem 3.6, we can easily find the optimal wealth process $X^{\pi^*,\lambda^*}(t)$ and the optimal investment strategy $\pi^{*,\lambda^*}(t)$ for $0 \leq t \leq T$. We next state the main result of the paper.

Theorem 4.1. *Assume $\hat{\nu} \in \tilde{K}$ is the unique minimizer of (3.23). Assume $L \geq \theta$ and $x_0 + C(0) > E[LH^{\hat{\nu}}(T)1_{\{H^{\hat{\nu}}(T)<H^*\}}]$, where H^* solves*

$$P(H^{\hat{\nu}}(T) > H^*) = \varepsilon, \tag{4.1}$$

for $0 \leq \varepsilon \leq 1$. Then there exists a unique $\lambda^* \geq 0$ such that $X^{\pi^*,\lambda^*}(T)$ is the optimal solution of unconstrained problem (2.8), λ^* and $X^{\pi^*,\lambda^*}(T)$ satisfy condition (2.9). Therefore, $X^{\pi^*,\lambda^*}(T)$ is the optimal solution of VaR constrained problem (2.7) and λ^* is the Lagrange multiplier.

Proof. We first consider the case $L \geq z$, where z is the solution to (3.2). For a fixed $\lambda \geq 0$, we can obtain from equations (3.29), (3.31) that the optimal terminal wealth is given by

$$X^{\pi^*,\lambda}(T) = \tilde{X}^{\pi^{*,\lambda}}(T) = x^{*,\lambda}(y_0 H^{\hat{\nu}}(T)), \tag{4.2}$$

where $x^{*,\lambda}(y)$ is defined in (3.8), (3.10) and (3.12), and y_0 is determined by the binding budget constraint

$$E[X^{\pi^*,\lambda}(T)H^{\hat{\nu}}(T)] = x_0 + C(0), \tag{4.3}$$

If we can find a unique solution (y_0, λ^*) to equations (2.9) and (4.3), then $X^{\pi^*,\lambda^*}(T)$ is the solution to the problem (2.7). To solve equations (2.9) and (4.3), let H^* be defined by (4.1). We now choose λ and check the bindingness of the VaR constraint by comparing the solution to the optimal terminal wealth without VaR constraint with the threshold of the VaR constraint H^*. If $\lambda = 0$, then we have $x^{*,\lambda}(y)$ in (4.2) is defined by (3.12) and the optimal terminal wealth without VaR constraint, denoted by $X^{\pi^*}(T)$, is given by

$$X^{\pi^*}(T) = (\theta + I_1(y_0 H^{\hat{\nu}}(T)))1_{\{H^{\hat{\nu}}(T)<\frac{c_z}{y_0}\}}, \tag{4.4}$$

where y_0 is determined by (4.3) with $X^{\pi^{*,\lambda}}(T)$ replaced by $X^{\pi^*}(T)$. Note that for any $\omega \in \Omega$, $y_0 \to X^{\pi^*}(T)$ is a decreasing function of y_0 since I_1 is strictly decreasing. Then $V(y_0) = E[H^{\hat{\nu}}(T)X^{\pi^*}(T)]$ is continuous and strictly decreasing in y_0. Furthermore, for any $\omega \in \Omega$, we have $\lim\limits_{y_0 \to 0^+} X^{\pi^*}(T) = \infty$ and $\lim\limits_{y_0 \to \infty} X^{\pi^*}(T) = 0$, which yields

$$\lim_{y_0 \to 0^+} V(y_0) = \infty, \lim_{y_0 \to \infty} V(y_0) = 0 < x_0 + C(0).$$

Thus, there exists a unique solution y_0 to equation (4.3).

If $H^* \leq \frac{c_L}{y_0}$, then

$$P(X^{\pi^*}(T) \geq L) = P(H^{\hat{\nu}}(T) \leq \frac{c_L}{y_0}) \geq P(H^{\hat{\nu}}(T) \leq H^*) = 1 - \varepsilon.$$

We can choose $\lambda^* = 0$ as $X^{\pi^*}(T)$ naturally satisfies the VaR constraint and maximizes problem (2.7).

If $H^* > \frac{c_L}{y_0}$, then the VaR constraint is binding and it should hold that $\lambda > 0$, which implies that $X^{\pi^{*,\lambda}}(T)$ satisfies

$$P(X^{\pi^{*,\lambda}}(T) \geq L) = 1 - \varepsilon. \tag{4.5}$$

Then we can choose λ as a function of y_0 by using (4.1) and (2.9).

(i) If $\frac{c_L}{y_0} < H^* \leq \frac{c_z}{y_0}$, then $X^{\pi^*,\lambda}(T) = x^{*,\lambda}(y_0 H^{\hat{\nu}}(T))$ takes a four-region form, where $x^{*,\lambda}(y)$ is defined in (3.10), that is,

$$X^{\pi^{*,\lambda}}(T) = (\theta + I_1(y_0 H^{\hat{\nu}}(T)))(1_{\{H^{\hat{\nu}}(T) < \frac{c_L}{y_0}\}} + 1_{\{H^* \leq H^{\hat{\nu}}(T) < \frac{c_z}{y_0}\}}) + L 1_{\{\frac{c_L}{y_0} \leq H^{\hat{\nu}}(T) < H^*\}}. \tag{4.6}$$

Define L_0 by the relation $H^* = \frac{c_{L_0}}{y_0}$, which is to ensure (4.5) holds. Since $c_L < c_{L_0} \leq c_z$, we have $z \leq L_0 < L$. Define

$$\lambda = U(L_0) - U(L) - U'(L_0)(L_0 - L) \hat{=} g_1(y_0).$$

Since $\frac{d}{dx}(U(x) - U(L) - U'(x)(x - L)) = -U''(x)(x - L) < 0$ for $\theta < x < L$, we conclude that $\lambda > U(L) - U(L) - U'(L)(L - L) = 0$.

(ii) If $H^* > \frac{c_z}{y_0}$, then $X^{\pi^*,\lambda}(T) = x^{*,\lambda}(y_0 H^{\hat{\nu}}(T))$ takes a three-region form, where $x^{*,\lambda}(y)$ is defined in (3.8), that is,

$$X^{\pi^{*,\lambda}}(T) = (\theta + I_1(y_0 H^{\hat{\nu}}(T)))1_{\{H^{\hat{\nu}}(T) < \frac{c_L}{y_0}\}} + L 1_{\{\frac{c_L}{y_0} \leq H^{\hat{\nu}}(T) < H^*\}}. \tag{4.7}$$

Define k_λ by the relation $H^* = \frac{k_\lambda}{y_0}$, which is again to ensure (4.5) holds. We have $k_\lambda > c_z$. Define

$$\lambda = k_\lambda L - U_1(L - \theta) - U_2(\theta) \hat{=} g_2(y_0).$$

It is easy to check $\lambda = L(k_\lambda - \frac{U_1(L-\theta)+U_2(\theta)}{L}) \geq L(k_\lambda - c_z) > 0$.

Therefore, when the VaR constraint is binding, the multiplier λ can be chosen as a function of y_0:

$$\lambda = g_1(y_0) 1_{\{\frac{c_L}{y_0} < H^* \leq \frac{c_z}{y_0}\}} + g_2(y_0) 1_{\{H^* > \frac{c_z}{y_0}\}} \hat{=} g(y_0).$$

It remains to show that there is a unique root y_0 to (4.3). Note that $V_1(y_0) = E[H^{\hat{\nu}}(T)X^{\pi^*,\lambda}(T)]$ is continuous and strictly decreasing in y_0. Furthermore, for any $\omega \in \Omega$, we have $\lim\limits_{y_0 \to 0^+} X^{\pi^*,\lambda}(T) = \infty$ and $\lim\limits_{y_0 \to \infty} X^{\pi^*,\lambda}(T) = L1_{\{H^{\hat{\nu}}(T)<H^*\}}$, which yields

$$\lim_{y_0 \to 0^+} V_1(y_0) = \infty, \lim_{y_0 \to \infty} V_1(y_0) = E[H^{\hat{\nu}}(T)L1_{\{H^{\hat{\nu}}(T)<H^*\}}] < x_0 + C(0).$$

Thus, there exists a unique solution y_0 to equation (4.3). We conclude that $X^{\pi^*,\lambda^*}(T)$ solves problem (2.7), $\lambda^* = g(y_0)$ is the Lagrange multiplier and y_0 is the unique solution of equation (4.3).

We next consider the case $L < z$. The proof is similar. For a fixed $\lambda \geq 0$, the optimal terminal wealth is $X^{\pi^*,\lambda}(T) = x^{*,\lambda}(y_0 H^{\hat{\nu}}(T))$, where $x^{*,\lambda}(y)$ is defined in (3.8), (3.14), (3.12) and y_0 is determined by (4.3). It remains to find the unique solution (y_0, λ^*) to equations (2.9) and (4.3). If $H^* \leq \frac{c_z}{y_0}$, then $X^{\pi^*}(T)$ naturally achieves the VaR constraint and the multiplier λ^* should be 0. If $H^* > \frac{c_z}{y_0}$, the VaR constraint is binding and we choose λ as a function of y_0.

(i) If $\frac{c_z}{y_0} < H^* < \frac{c_L}{y_0}$, then $X^{\pi^*,\lambda}(T) = x^{*,\lambda}(y_0 H^{\hat{\nu}}(T))$ takes a two-region form, where $x^{*,\lambda}(y)$ is defined in (3.14), that is,

$$X^{\pi^*,\lambda}(T) = (\theta + I_1(y_0 H^{\hat{\nu}}(T)))1_{\{H^{\hat{\nu}}(T)<H^*\}}. \tag{4.8}$$

Define z_0 by the relation $H^* = \frac{c_{z_0}}{y_0}$, which is to ensure ensure (4.5) holds. Since $c_z < c_{z_0} < c_L$, we have $L < z_0 < z$. Define

$$\lambda = z_0 U'(z_0) - U_2(\theta) - U(z_0) \hat{=} \tilde{g}_1(y_0).$$

Since $\frac{d}{dx}(xU'(x) - U(x)) = U''(x)x < 0$ for $x > \theta$, we have that $\lambda > zU'(z) - U_2(\theta) - U(z) = 0$.

(ii) If $H^* \geq \frac{c_L}{y_0}$, then $X^{\pi^*,\lambda}(T)$ is the same as (4.7) and $\lambda = g_2(y_0)$.

Similarly, we can prove that equation (4.3) has a unique root with $X^{\pi^*,\lambda}(T)$ given by (4.7), (4.8) and

$$\lambda \hat{=} \tilde{g}(y_0) := \tilde{g}_1(y_0)1_{\{\frac{c_z}{y_0}<H^*<\frac{c_L}{y_0}\}} + g_2(y_0)1_{\{H^* \geq \frac{c_L}{y_0}\}}.$$

Therefore, when the VaR constraint is binding, $X^{\pi^*,\lambda^*}(T)$ solves problem (2.7), $\lambda^* = \tilde{g}(y_0)$ is the Lagrange multiplier and y_0 is the unique solution of equation (4.3). □

Remark 4.2. *Similar to Basak and Shapiro (2001), from (4.4), (4.6), (4.7) and (4.8), it is easy to see that if* $E[LH^{\hat{\nu}}(T)1_{\{H^{\hat{\nu}}(T)<H^*\}}] > x_0 + C(0)$, *then the optimization problem (2.7) is infeasible. For* $E[LH^{\hat{\nu}}(T)1_{\{H^{\hat{\nu}}(T)<H^*\}}] = x_0 + C(0)$, *there is only one admissible solution* $X^{\pi^*,\lambda^*}(T) = L1_{\{H^{\hat{\nu}}(T)<H^*\}}$. *In particular, if* $\varepsilon = 0$, *then* $H^* = \infty$ *and* $X^{\pi^*,\lambda^*}(T) = L = e^{rT}(x_0 + C(0))$, *which implies that one should only invest in the riskless savings account. The assumption* $E[LH^{\hat{\nu}}(T)1_{\{H^{\hat{\nu}}(T)<H^*\}}] < x_0 + C(0)$ *ensures there is a set of nontrivial admissible solutions.*

The optimal terminal wealth $X^{\pi^*,\lambda^*}(T)$ may take a two-, three- or four-region form for $L \geq z$ and may take a two- or three-region form for $\theta \leq L < z$ according to the value of y_0. The values of y_0 in different forms of $X^{\pi^*,\lambda^*}(T)$ are different. Denote by $y_0^2, y_0^3, y_0^4, \tilde{y}_0^2$ the values of y_0 determined by (4.4), (4.6), (4.7) and (4.8), respectively. It is easy to conclude that $y_0^2 < y_0^4 < y_0^3$ and $y_0^2 < \tilde{y}_0^2 < y_0^3$ from the budget constraint. With the help of these observations, we can design a simple algorithm to find the optimal terminal wealth and the Lagrange multiplier.

Algorithm for finding the optimal terminal wealth and Lagrange multiplier.

Step 1: Let $X^{\pi^*,\lambda^*}(T)$ be given by (4.4) and compute $y_0 = y_0^2$ from the budget constraint (4.3). If $y_0^2 \le \frac{c_L}{H^*}$ for $L \ge z$ ($y_0^2 \le \frac{c_z}{H^*}$ for $\theta \le L < z$), then $X^{\pi^*}(T)$ naturally satisfies the VaR constraint and is the optimal terminal wealth and set $\lambda^* = 0$.

Step 2 (for $L \ge z$): If $y_0^2 > \frac{c_L}{H^*}$, then let $X^{\pi^*,\lambda^*}(T)$ be given by (4.6) and compute $y_0 = y_0^4$ from (4.3). If $\frac{c_L}{H^*} < y_0^4 \le \frac{c_z}{H^*}$, then the optimal terminal wealth is given by (4.6) with y_0 replaced by y_0^4 and the Lagrange multiplier $\lambda^* = g_1(y_0^4)$.

(Step 2' (for $\theta \le L < z$): If $y_0^2 > \frac{c_z}{H^*}$, then let $X^{\pi^*,\lambda^*}(T)$ be given by (4.8) and compute $\tilde{y}_0^2$ from (4.3). If $\frac{c_z}{H^*} < \tilde{y}_0^2 < \frac{c_L}{H^*}$, then the optimal terminal wealth is given by (4.8) with y_0 replaced by $\tilde{y}_0^2$ and the Lagrange multiplier $\lambda^* = \tilde{g}_1(\tilde{y}_0^2)$.)

Step 3: If $y_0^4 > \frac{c_z}{H^*}$ for $L \ge z$ ($\tilde{y}_0^2 \ge \frac{c_L}{H^*}$ for $\theta \le L < z$), then let $X^{\pi^*,\lambda^*}(T)$ be given by (4.7) and compute $y_0 = y_0^3$ from (4.3). The optimal terminal wealth is given by (4.7) with y_0 replaced by y_0^3 and the Lagrange multiplier $\lambda^* = g_2(y_0^3)$. □

Since λ^* can be expressed as a function of y_0 and λ^* is related to H^*, which is determined by ε, we use the superscript ε in place of λ^* in $X^{\pi^*,\lambda^*}(T)$ in the following analysis, that is, the optimal terminal wealth is written as $X^{\pi^*,\varepsilon}(T)$.

Remark 4.3. *When the utility is a smooth concave increasing function, we can obtain the optimal wealth process with the VaR constraint from Theorem 4.1 by setting $\theta = 0$, which results in $z = 0$ and $c_z = \infty$. Guan and Liang (2016) apply the martingale method to derive it in a complete market. However, the martingale method can not be used here due to trading constraints. If there are only two risky assets and the constraint set K is the whole space, then the optimal terminal wealth is given by (4.6) with $\theta = 0$ and $\hat{\nu} = (0,0)^\top$, which is the same as (3.20) in Guan and Liang (2016).*

Remark 4.4. *From Theorem 4.1 we can conclude that if $\varepsilon = 0$, then $H^* = \infty$, which implies that all the bad states are insured against and the VaR constraint becomes the PI constraint. The optimal terminal wealth takes a two-region form given by (4.7) with $H^* = \infty$, that is*

$$X^{\pi^*,0}(T) = (\theta + I_1(y_0 H^{\hat{\nu}}(T)))1_{\{H^{\hat{\nu}}(T) < \frac{c_L}{y_0}\}} + L1_{\{H^{\hat{\nu}}(T) \ge \frac{c_L}{y_0}\}}. \tag{4.9}$$

If $\varepsilon = 1$, then $H^* = 0$ and $\lambda = 0$, which implies that the VaR constraint vanishes. The optimal terminal wealth $X^{\pi^*,1}(T)$ is the same as (4.4), that is,

$$X^{\pi^*,1}(T) = X^{\pi^*}(T). \tag{4.10}$$

Similar to the optimal terminal wealth under a smooth concave utility in Basak and Shapiro (2001), in contrast to $X^{\pi^*}(T)$, $X^{\pi^*,\varepsilon}(T)$ is not modified in the good- and bad-states regions and $X^{\pi^*,\varepsilon}(T)$ equals to the bound L in the intermediate-states region $\{\max\{\frac{c_L}{y_0}, \frac{c_z}{y_0}\} \le H^{\hat{\nu}}(T) < H^*\}$ in order to achieve the VaR constraint.

We next analyze how ε and L impact the optimal terminal wealth. Following Proposition 1 of Basak and Shapiro (2001), we can deduce from the budget constraint (4.3) that for a fixed ε, y_0 increases in L and the intermediate-states region grows at the expense of the good-states region. Accordingly, to attain a higher protection level, the optimal terminal wealth in the good-states region decreases. Similarly, for a fixed L, y_0 decreases in ε. With ε decreasing, the intermediate-states region $\{\frac{c_L}{y_0} \le H^{\hat{\nu}}(T) < H^*\}$ enlarges as more states need to be insured against, while the unmodified regions $\{H^{\hat{\nu}}(T) < \frac{c_L}{y_0}\}$ and $\{H^{\hat{\nu}}(T) \ge H^*\}$ both shrink. Furthermore, to meet the VaR constraint, the optimal terminal wealth in the good-states region must decrease.

As shown in Theorem 4.1, the constrained optimal terminal wealth $X^{\pi^{*,\varepsilon}}(T)$ takes different forms depending on the relative position of H^* and $\frac{c_L}{y_0}$ and $\frac{c_z}{y_0}$. We now analyse how ε determines the form of $X^{\pi^{*,\varepsilon}}(T)$ for $L \geq z$. Denote by

$$\varepsilon^* = P(X^{\pi^*}(T) < L) = P(H^{\hat{\nu}}(T) > \frac{c_L}{y_0}),$$

where $X^{\pi^*}(T)$ is the unconstrained optimal terminal wealth given by (4.4) and y_0 is determined by the binding budget constraint (4.3). When $\varepsilon \geq \varepsilon^*$, $X^{\pi^*}(T)$ naturally satisfies the VaR constraint and therefore, $X^{\pi^{*,\varepsilon}}(T)$ is the same as $X^{\pi^*}(T)$. When $\varepsilon < \varepsilon^*$, the VaR constraint is binding and the uninsured loss-states region $\{H^{\hat{\nu}}(T) \geq H^*\}$ may take a one- or two-region form. More precisely, the states in the region $\{H^* \leq H^{\hat{\nu}}(T) < \max\{H^*, \frac{c_z}{y_0}\}\}$ have wealth in $[z, L)$ and the states in the region $\{H^{\hat{\nu}}(T) \geq \max\{H^*, \frac{c_z}{y_0}\}\}$ have wealth 0. Let

$$\bar{\varepsilon}^* = P(X^{\pi^{*,\varepsilon}}(T) < L) = P(H^{\hat{\nu}}(T) > \frac{c_z}{y_0}),$$

where $X^{\pi^{*,\varepsilon}}(T)$ is given by (4.7) with $H^* = \frac{c_z}{y_0}$ and y_0 is determined by the binding budget constraint (4.3). When $\varepsilon = \bar{\varepsilon}^*$, all the states with wealth in $[z, L)$ are insured against and the manager leaves all the states with wealth 0 uninsured. For $\varepsilon \in (\bar{\varepsilon}^*, \varepsilon^*)$, we have $H^* \in (\frac{c_L}{y_0}, \frac{c_z}{y_0})$, which implies that some states with wealth in $[z, L)$ and all the states with wealth 0 are left uninsured, so $X^{\pi^{*,\varepsilon}}(T)$ takes a two-region form in the loss-states region and takes a four-region form given by (4.6). For $\varepsilon \in (0, \bar{\varepsilon}^*]$, we have $H^* \in [\frac{c_z}{y_0}, \infty)$, which implies that the manager chooses to leave some states with wealth 0 uninsured, so $X^{\pi^{*,\varepsilon}}(T)$ only takes 0 in the loss-states region and takes a three-region form given by (4.7).

One disadvantage of the VaR constraint under a smooth concave utility is that it leads to higher losses in bad market scenarios than in the case of no VaR constraint, see Basak and Shapiro (2001). Loss aversion leads $X^{\pi^*}(T)$ to be 0 in the region $\{H^{\hat{\nu}}(T) \geq \frac{c_z}{y_0}\}$. For a relatively large ε, a VaR constraint brings higher losses for those states in the region $(H^*, \frac{c_z}{y_0})$, where the optimal terminal wealth takes $\theta + I_1(y_0 H^{\hat{\nu}}(T))$. However, if ε is small enough such that $H^* \geq \frac{c_z}{y_0}$, then we have $X^{\pi^{*,\varepsilon}}(T) \geq X^{\pi^*}(T)$ in the region where $X^{\pi^*}(T) \leq L$. Therefore, a VaR constraint with a relatively small ε strictly improves risk management in bad economic states.

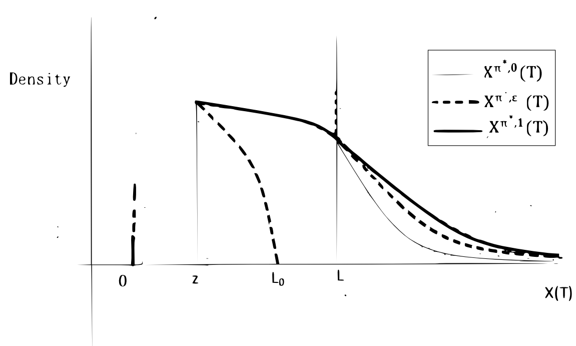

Figure 6: Probability density of optimal terminal wealth with a relatively large ε

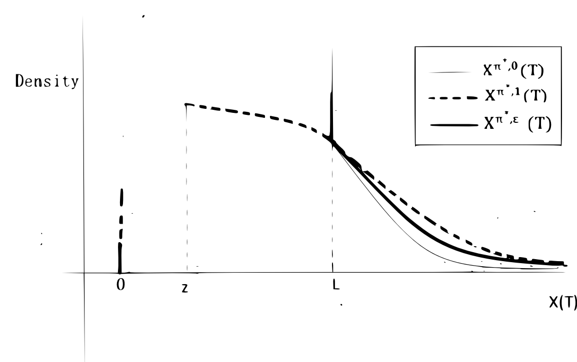

Figure 7: Probability density of optimal terminal wealth with a relatively small ε

Figures 6 and 7 depict the distributions of $X^{\pi^*,\varepsilon}(T)$ for $L \geq z$ with relatively large and small ε. The distribution of the optimal terminal wealth is not continuous. There is a probability mass at L when the VaR constraint is binding and there is a probability mass at 0 for $\varepsilon > 0$. For a relatively large ε (see Figure 6), states with wealth 0 have a higher probability than that in case of no VaR constraint and there are some states with wealth less than that in case of no VaR constraint, which implies that a VaR constraint leads to more losses. However, for a relatively small ε (see Figure 7), there are no states with wealth between (z, L) and states with wealth 0 have a lower probability than that in case of no VaR constraint, which implies that the VaR constrained optimal terminal wealth dominates the unconstrained one for bad economic states.

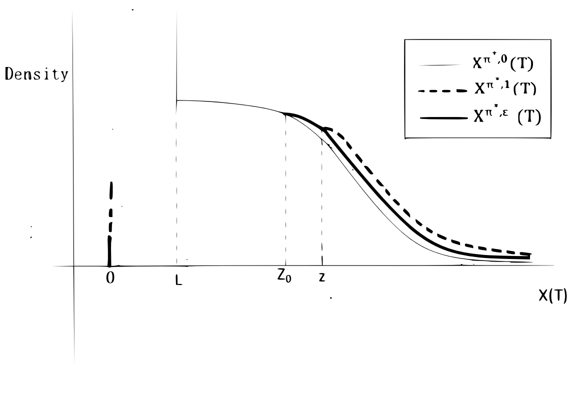

Figure 8: Probability density of optimal terminal wealth with a relatively large ε

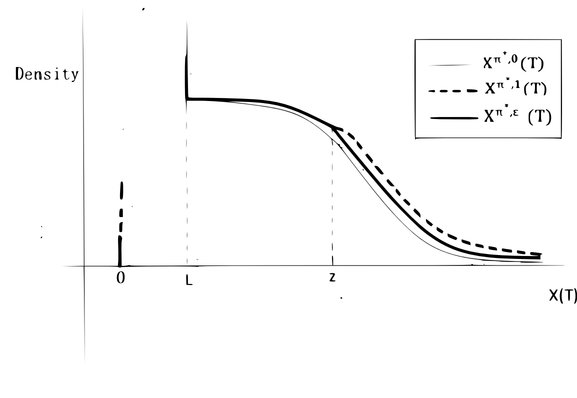

Figure 9: Probability density of optimal terminal wealth with a relatively small ε

Similarly, for $\theta \leq L < z$, denote by

$$\varepsilon_1^* = P(X^{\pi^*}(T) < L) = P(H^{\hat{\nu}}(T) > \frac{c_z}{y_0}),$$

where y_0 is determined by the binding budget constraint (4.3). When $\varepsilon \geq \varepsilon_1^*$, $X^{\pi^*}(T)$ given by (4.4) naturally satisfies the VaR constraint and $X^{\pi^*,\varepsilon}(T) = X^{\pi^*}(T)$. Let

$$\bar{\varepsilon}_1^* = P(X^{\pi^*,\varepsilon}(T) < L) = P(H^{\hat{\nu}}(T) > \frac{c_L}{y_0}),$$

where $X^{\pi^*,\varepsilon}(T)$ is given by (4.8) with $H^* = \frac{c_L}{y_0}$ and y_0 is determined by the binding budget constraint (4.3). For $\varepsilon \in [\bar{\varepsilon}_1^*, \varepsilon_1^*)$, we have $H^* \in (\frac{c_z}{y_0}, \frac{c_L}{y_0}]$ and the pension manager insures against the region $\{\frac{c_z}{y_0} \leq H^{\hat{\nu}}(T) < H^*\}$ by letting $X^{\pi^*,\varepsilon}(T) = \theta + I_1(y_0 H^{\hat{\nu}}(T))$, so $X^{\pi^*,\varepsilon}(T)$ takes a two-region form given by (4.8). For $\varepsilon \in (0, \bar{\varepsilon}_1^*)$, we have $H^* \in (\frac{c_L}{y_0}, \infty)$ and the pension manager insures against the regions $\{\frac{c_z}{y_0} \leq H^{\hat{\nu}}(T) < \frac{c_L}{y_0}\}$ and $\{\frac{c_L}{y_0} \leq H^{\hat{\nu}}(T) < H^*\}$ by letting $X^{\pi^*,\varepsilon}(T) = \theta + I_1(y_0 H^{\hat{\nu}}(T))$ and $X^{\pi^*,\varepsilon}(T) = L$, respectively, so $X^{\pi^*,\varepsilon}(T)$ takes a three-region form given by (4.7).

Figures 8 and 9 depict the distributions of $X^{\pi^*,\varepsilon}(T)$ for $\theta \leq L < z$ with relatively large and small ε. The distribution of the optimal terminal wealth is not continuous. There is a probability mass at 0 for $\varepsilon > 0$ and there is a probability mass at L for a relatively small ε. Comparing with the case $L \geq z$, a VaR constraint with any $\varepsilon > 0$ under the case $\theta \leq L < z$ provides a genuine improvement of the risk management for the loss states. Furthermore, relative to the unconstrained optimal terminal wealth, a VaR constraint shifts the distribution of good states to the left.

5 Numerical analysis

In this section, we do some numerical calculations to investigate the influence of a VaR constraint on the optimal terminal wealth.

Assume that

$$U(x) = \begin{cases} -A(\theta - x)^{\gamma_1}, & x < \theta, \\ (x-\theta)^{\gamma}, & x \geq \theta, \end{cases}$$

where $A > 1, 0 < \gamma, \gamma_1 < 1$. Assume that the financial market consists of three tradable assets, whose price processes are modelled by (2.1)-(2.2) with $n = 2$, and $\sigma_{11} = \sigma_1, \sigma_{12} = 0$, $\sigma_{21} = \rho\sigma_2, \sigma_{22} = \sqrt{1-\rho^2}\sigma_2$, ρ is a correlation coefficient, and $\vartheta_{S_1} = \frac{\mu_1 - r}{\sigma_1} > 0, \vartheta_{S_2} = \frac{\mu_2 - r}{\sigma_2} > 0$ are the market prices of risk of the two risky assets, respectively. Let $K = [0,\infty)^2$, which means short selling is not allowed. The positive polar cone of K is given by $\tilde{K} = [0,\infty)^2$. To compute $\hat{\xi}$ in (3.24), we need to find the minimizer $\hat{\nu} \in \tilde{K}$ of (3.23). The Kuhn-Tucker optimality condition implies there exists a Lagrange multiplier $u = (u_1, u_2)^\top$ such that $\hat{\nu}$ and u satisfy the following set of equations:

$$\begin{cases} \frac{2}{\sigma_1}(\frac{\hat{\nu}_1}{\sigma_1} + \vartheta_{S_1}) - \frac{2\rho}{(1-\rho^2)\sigma_1}(\frac{\hat{\nu}_2}{\sigma_2} + \vartheta_{S_2} - \rho(\frac{\hat{\nu}_1}{\sigma_1} + \vartheta_{S_1})) - u_1 = 0, \\ \frac{2}{(1-\rho^2)\sigma_2}(\frac{\hat{\nu}_2}{\sigma_2} + \vartheta_{S_2} - \rho(\frac{\hat{\nu}_1}{\sigma_1} + \vartheta_{S_1})) - u_2 = 0, \\ u_i\hat{\nu}_i = 0, u_i \geq 0, \hat{\nu}_i \geq 0, i = 1, 2. \end{cases}$$

Solving the above equation gives

$$\hat{\xi} = \begin{cases} (\vartheta_{S_1}, \frac{\vartheta_{S_2} - \rho\vartheta_{S_1}}{\sqrt{1-\rho^2}})^\top, & \vartheta_{S_2} > \rho\vartheta_{S_1}, \vartheta_{S_1} > \rho\vartheta_{S_2}, \\ (\vartheta_{S_1}, 0)^\top, & \rho\vartheta_{S_1} \geq \vartheta_{S_2}, \\ (\rho\vartheta_{S_2}, \sqrt{1-\rho^2}\vartheta_{S_2})^\top, & \rho\vartheta_{S_2} \geq \vartheta_{S_1}. \end{cases}$$

Note that when $K = (-\infty,\infty)^2$, we have $\tilde{K} = \{(0,0)^\top\}$ and $\hat{\xi} = (\vartheta_{S_1}, \frac{\vartheta_{S_2} - \rho\vartheta_{S_1}}{\sqrt{1-\rho^2}})^\top$, which is different from the case $K = [0,\infty)^2$. Therefore, the trading constraint impacts the optimal terminal wealth through the pricing kernel $H^{\hat{\nu}}(T)$.

For all numerical computations, the benchmark data used are the following: $c(t) = 0.1, x_0 = 13, L = 80, \varepsilon = 0.025, \mu_1 = 0.06, \mu_2 = 0.065, r = 0.05, \sigma_1 = 0.1, \sigma_2 = 0.4, \rho = 0.5, T = 40, \theta = 40, A = 2.25, \gamma_1 = 0.2, \gamma = 0.5$. From (3.2), we have $z = 50.1 < L$ and $c_z = 0.1574$.

Figure 10 shows the optimal terminal value $X^{\pi^{*,\varepsilon}}(T)$ versus $H^{\hat{\nu}}(T)$ for different ε. We see that $X^{\pi^{*,\varepsilon}}(T)$ takes a two-, three- or four-region form according to the value of ε. When ε decreases, the intermediate-states region enlarges while the good- and bad-states regions both shrink. In order to meet the protection level in the intermediate-states region, the VaR constraint leads to a decrease in the optimal terminal wealth of good states. We also note that $X^{\pi^*}(T) = X^{\pi^{*,1}}(T)$ is dominated by $X^{\pi^{*,0.025}}(T)$ in the region where $X^{\pi^*}(T) < L$, while $X^{\pi^{*,0.15}}(T)$ is dominated by $X^{\pi^*}(T)$ in the region where $X^{\pi^{*,0.15}}(T) < L$, which numerically confirms the result presented in Section 4: when $L \geq z$, a VaR constraint with a relatively small ε can reduce risk exposure in bad market conditions whereas with a relatively large ε can incur heavier losses.

Figure 11 presents the optimal terminal wealth $X^{\pi^{*,0.025}}(T)$ versus $H^{\hat{\nu}}(T)$ for different L. We observe that the bad-states region $\{H^{\hat{\nu}}(T) \geq H^*\}$ for different L remains unchanged, due to H^*

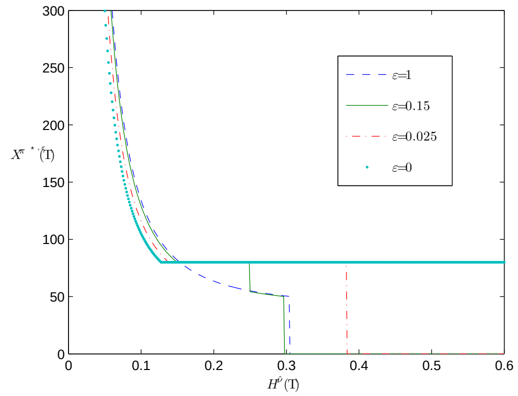

Figure 10: $X^{\pi^{*,\varepsilon}}(T)$ versus $H^{\hat{\nu}}(T)$ for different ε

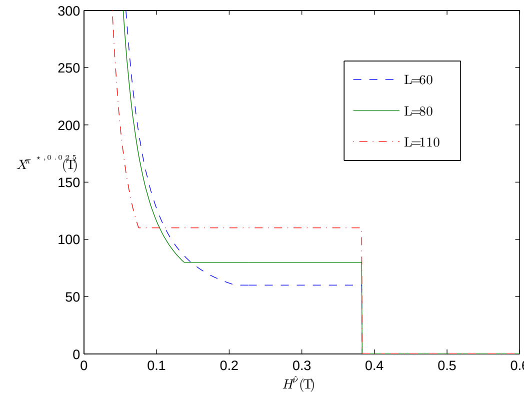

Figure 11: $X^{\pi^{*,0.025}}(T)$ versus $H^{\hat{\nu}}(T)$ for different L

Table 1: (conditional) expectations, standard deviations, quantile values and probabilities

(ε, L)	ε $(L = 80)$				L $(\varepsilon = 0.025)$		
	1	0.15	0.025	0	60	80	110
mean	207.65	204.55	182.83	166.04	197.54	182.83	141.84
std dev	339.77	329.88	271.66	230.39	313.78	271.66	134.52
0.1 quantile	55.12	54.70	80	80	60	80	110
0.9 quantile	426.82	416.21	352.79	307.31	398.48	352.79	203.66
$P(X^{\pi^{*,\varepsilon}}(T) = 0)$	0.055	0.057	0.025	0	0.025	0.025	0.025
$P(X^{\pi^{*,\varepsilon}}(T) \in (z, L))$	0.249	0.093	0	0	0	0	0
$P(X^{\pi^{*,\varepsilon}}(T) = L)$	0	0.162	0.340	0.413	0.134	0.340	0.704
$P(X^{\pi^{*,\varepsilon}}(T) > L)$	0.696	0.688	0.635	0.587	0.841	0.635	0.271
$E(X^{\pi^{*,\varepsilon}}(T) \vert X^{\pi^{*,\varepsilon}}(T) > L)$	275.36	267.14	245.16	226.54	225.25	245.16	237.68

only depending on ε and the distribution of $H^{\hat{\nu}}(T)$, that the intermediate region increases with L at the expense of the good-states region as more states need to be insured against, and that $X^{\pi^{*,0.025}}(T)$ in the good-states region decreases to attain a higher L in the intermediate region. It is noted that $X^{\pi^{*,0.025}}(T)$ with a lower L is dominated by that with a higher L in the region where both the optimal terminal wealth value are below the higher protection level, which implies that a higher protection level leads to a lower left tail risk for a small ε.

Table 1 lists some probabilities, expectations, standard deviations, conditional expectations and quantile values at low end and high end of $X^{\pi^{*,\varepsilon}}(T)$ for different ε with $L = 80$ and different L with $\varepsilon = 0.025$. Numerical results further illustrate different forms of the terminal wealth according to the value of ε presented in Section 4: for $\varepsilon \geq \varepsilon^* = 0.304$, the VaR constraint is not binding and $X^{\pi^{*,\varepsilon}}(T) = X^{\pi^*}(T)$ takes either 0 or in (z, ∞); for $0.062 = \varepsilon_1^* < \varepsilon < 0.304$, $X^{\pi^{*,\varepsilon}}(T)$ takes either 0 or L or in (z, L) or in (L, ∞); for $0 < \varepsilon \leq 0.062$, $X^{\pi^{*,\varepsilon}}(T)$ takes either 0 or L or in (L, ∞); and for $\varepsilon = 0$, $X^{\pi^{*,0}}(T)$ takes either L or in (L, ∞). We observe that the probability $P(X^{\pi^{*,\varepsilon}}(T) = L)$ decreases in ε and increases in L, which is in line with what has been discussed in Figures 10-11: when ε decreases or L increases, more states need to be insured against and therefore the intermediate-states region grows at the expense of unmodified regions.

We observe from Table 1 that the expectation and the standard deviation decrease in L and increase in ε, respectively, which implies that in order to achieve the VaR constraint with a smaller ε or a higher protection level L, the manager takes more prudent strategies such that $X^{\pi^{*,\varepsilon}}(T)$ becomes less volatile. The quantile value at high end decreases in L and increases in ε, consistent with the observations from Figures 10-11: the optimal terminal wealth in good states decreases in L and increases in ε. For $\varepsilon = 0.025$, the quantile value at low end increases with L, since the optimal terminal wealth with a lower L in the loss states is dominated by that with a higher L for a relatively low ε, which has been revealed in Figure 11. However, there is no monotonicity in ε for the quantile value at low end since a VaR constraint leads to more losses for a relatively high ε. We also note that for $\varepsilon = 0.025$, the conditional expectation is not a monotonic function of L. The reason is that as observed from Figure 11, changing L does not lead to monotone changes in the optimal terminal wealth in good states. However, for $L = 80$, a smaller ε leads to a lower right tail risk and a smaller conditional expectation, consistent with the observation from Figure 10: the optimal terminal wealth with a lower ε in good states is dominated by that with a higher ε.

6 Conclusions

In this paper, we investigate the optimal portfolio selection problem for a DC plan manager under loss aversion and with trading and VaR constraints. We solve the problem in two steps: First, we solve the unconstrained optimization problem (2.8) with a fixed Lagrange multiplier. By using a concavification technique and a dual control method, we derive the closed-form optimal wealth process and the optimal trading strategy. Second, we solve the constrained optimization problem (2.7) by finding the solutions of two coupled nonlinear equations (binding budget and VaR constraints). We propose a simple algorithm to constructively compute the initial dual state value and the Lagrange multiplier which are used to find the optimal terminal wealth. Theoretical and numerical results show that the VaR constraint has significant impact on the distribution of optimal terminal wealth and may provide an effective improvement in bad states due to loss aversion. There remain many open questions for optimal allocation of a DC pension plan, for example, the DC contribution rate $c(t)$ may be modelled by a stochastic process, the financial market may include some credit-related products. We leave these and other questions for future research.

A Appendix

We give a useful result which is used in constructing the concave envelope of $\tilde{U}_\lambda$.

Lemma A.1. Let $0 \le z_1 \le z_2 < z_3$ be given constants and f be right continuous on $[z_1, \infty)$ satisfying

1. f is concave on intervals $[z_1, z_2)$ and (z_3, ∞),
2. $f(x) \le k(x - z_2) + f(z_2)$ on $[z_1, z_3]$ with $k = \frac{f(z_3)-f(z_2)}{z_3-z_2} \ge f'_+(z_3) > 0$,

where f'_+ is the right derivative of f at z_3. Then the concave envelope of f is given by

$$f^c(x) = \begin{cases} f(x), & z_1 \le x < z_2, \\ k(x - z_2) + f(z_2), & z_2 \le x < z_3, \\ f(x), & x \ge z_3. \end{cases} \tag{A.1}$$

In particular, for $z_1 = z_2$, the concave envelope of f is given by

$$f^c(x) = \begin{cases} k(x - z_1) + f(z_1), & z_1 \le x < z_3, \\ f(x), & x \ge z_3. \end{cases} \tag{A.2}$$

Proof. By definition $f^c \ge f$. Let g be concave with $g \ge f$. Then $g \ge f^c$ on $[z_1, z_2) \cup [z_3, \infty)$. Assume $x = uz_2 + (1-u)z_3 \in (z_2, z_3)$ for $u \in (0,1)$. By concavity of g, we have

$$g(x) \ge ug(z_2) + (1-u)g(z_3) \ge uf(z_2) + (1-u)(k(z_3 - z_2) + f(z_2)) = f(z_2) + k(x - z_2) = f^c(x).$$

It remains to prove that f^c is concave. Let $z_1 \le x_0 < x_1 < \infty$ and $x_u = ux_0 + (1-u)x_1$ with $u \in (0,1)$. It is easy to conclude that if $x_1 \le z_2$, or $x_0 \ge z_3$, and or $z_2 \le x_0 < x_1 \le z_3$, then

$$f^c(x_u) \ge uf^c(x_0) + (1-u)f^c(x_1).$$

If $z_2 \le x_0 \le z_3 \le x_1$, then we have

$$f^c(x_0) = k(x_0 - z_2) + f(z_2), f^c(z_3) = k(z_3 - z_2) + f(z_2). \tag{A.3}$$

Note that by concavity

$$f^c(x_1) = f(x_1) \le f'_+(z_3)(x_1 - z_3) + f(z_3) \le k(x_1 - z_3) + f(z_3) = k(x_1 - z_2) + f(z_2). \tag{A.4}$$

Equations (A.3) and (A.4) imply that the slope of the line through $(z_3, f^c(z_3))$ and $(x_1, f^c(x_1))$ is less than the slope of the line through $(x_0, f^c(x_0))$ and $(x_1, f^c(x_1))$, that is,

$$\frac{f^c(x_1) - f^c(x_0)}{x_1 - x_0} \ge \frac{f^c(x_1) - f^c(z_3)}{x_1 - z_3}. \tag{A.5}$$

If $x_u \in (z_2, z_3)$, then

$$\begin{aligned} f^c(x_u) &= k(x_u - z_2) + f(z_2) = k(u(x_0 - z_2) + (1-u)(x_1 - z_2)) + f(z_2) \\ &= uf^c(x_0) + (1-u)(k(x_1 - z_2) + f(z_2)) \ge uf^c(x_0) + (1-u)f^c(x_1), \end{aligned}$$

where the last inequality follows from (A.4).

If $x_u \in (z_3, \infty)$, then

$$\begin{aligned} f^c(x_u) &\ge \frac{f^c(x_1) - f^c(z_3)}{x_1 - z_3}(x_u - x_1) + f^c(x_1) \\ &\ge \frac{f^c(x_1) - f^c(x_0)}{x_1 - x_0}(x_u - x_1) + f^c(x_1) \\ &= uf^c(x_0) + (1-u)f^c(x_1). \end{aligned}$$

where the first inequality holds since $f^c(x)$ is concave on $[z_3, \infty)$, and the second inequality follows from (A.5). Therefore, we can conclude that f^c is concave on $[z_2, \infty)$.

If $z_1 \le x_0 \le z_2 \le x_1$, then we have

$$f^c(x_0) \le k(x_0 - z_2) + f^c(z_2), f^c(z_2) < f^c(x_1) \le k(x_1 - z_2) + f^c(z_2),$$

which implies that the slope of the line through $(x_0, f^c(x_0))$ and $(x_1, f^c(x_1))$ is less than the slope of the line through $(x_0, f^c(x_0))$ and $(z_2, f^c(z_2))$, and is greater than the slope of the line through $(z_2, f^c(z_2))$ and $(x_1, f^c(x_1))$, that is,

$$\frac{f^c(x_1) - f^c(z_2)}{x_1 - z_2} \le \frac{f^c(x_1) - f^c(x_0)}{x_1 - x_0} \le \frac{f^c(z_2) - f^c(x_0)}{z_2 - x_0}. \tag{A.6}$$

If $x_u \in (z_2, x_1)$, then

$$\begin{aligned} f^c(x_u) &\ge \frac{f^c(x_1) - f^c(z_2)}{z_2 - x_1}(x_u - x_1) + f^c(x_1) \\ &\ge \frac{f^c(x_1) - f^c(x_0)}{z_2 - x_1}(x_u - x_1) + f^c(x_1) \\ &= uf^c(x_0) + (1-u)f^c(x_1), \end{aligned}$$

where the first inequality holds from the fact that f^c is concave on $[z_2, \infty)$, and the second inequality follows from (A.6).

If $x_u \in (x_0, z_2)$, then by concavity

$$\begin{aligned} f^c(x_u) &\ge \frac{f^c(z_2) - f^c(x_0)}{z_2 - x_0}(x_u - x_0) + f^c(x_0) \\ &\ge \frac{f^c(x_1) - f^c(x_0)}{x_1 - x_0}(x_u - x_0) + f^c(x_0) \\ &= uf^c(x_0) + (1-u)f^c(x_1), \end{aligned}$$

which concludes the proof. □

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