Foundations of Artificial Intelligence

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Exercise Sheet 6 — Solutions

Exercise 6.1 (Semantics of Predicate Logic)

Consider the Interpretation $\mathcal{I} =< \mathcal{D}, \cdot^{\mathcal{I}} >$ with

- $\mathcal{D} = \{0, 1, 2, 3\}$
- $even^{\mathcal{I}} = \{0, 2\}$
- $odd^{\mathcal{I}} = \{1, 3\}$
- $lessThan^{\mathcal{I}} = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$
- $two^{\mathcal{I}} = 2$
- $plus^{\mathcal{I}} : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}, plus^{\mathcal{I}}(a, b) = (a + b) \mod 4$

and the variable assignment $\alpha = \{(x, 0), (y, 1)\}$.
Decide for the following formulae θ_i if $\mathcal{I}$ is a model for θ_i under α, i.e. if $\mathcal{I}, \alpha \models \theta_i$.
Explain your answer by formally applying the semantics.

(a) $\theta_1 = odd(y) \wedge even(two)$

(b) $\theta_2 = \forall x\ (even(x) \vee odd(x))$

(c) $\theta_3 = \forall x \exists y\ lessThan(x, y)$

(d) $\theta_4 = \forall x\ (even(x) \Rightarrow \exists y\ lessThan(x, y))$

(e) $\theta_5 = \forall x\ (odd(x) \Rightarrow even(plus(x, y)))$

Solution:

(a) $\theta_1^{\mathcal{I}} =$
$(odd(y) \wedge even(two))^{\mathcal{I}} =$
$odd^{\mathcal{I}}(y^{\mathcal{I}}) \wedge even^{\mathcal{I}}(two^{\mathcal{I}}) =$
$odd^{\mathcal{I}}(1) \wedge even^{\mathcal{I}}(2) =$
$\top \wedge \top = \top$, since $1 \in odd^{\mathcal{I}}$ and $2 \in even^{\mathcal{I}}$.
Thus, $\mathcal{I}, \alpha \models \theta_1$.

(b) Let $\theta_2 = \forall x \phi_2$ with $\phi_2 = even(x) \vee odd(x)$.
Then $\mathcal{I}, \alpha \models \theta_2$ iff $\mathcal{I}, \alpha[x/d](\phi_2) = \top$ for all $d \in \mathcal{D}$.
Notice that here x is bound by a quantifier and therefore the variable assignment in the definition of α is overridden by $\alpha[x/d]$.

Since our universe $\mathcal{D}$ is finite (and small), we can evaluate the above for all $d \in \mathcal{D}$:

$\mathcal{I}, \alpha[x/0](\phi_2) = \mathit{even}^{\mathcal{I}}(0) \vee \mathit{odd}^{\mathcal{I}}(0) = \top \vee \bot = \top$
$\mathcal{I}, \alpha[x/1](\phi_2) = \mathit{even}^{\mathcal{I}}(1) \vee \mathit{odd}^{\mathcal{I}}(1) = \bot \vee \top = \top$
$\mathcal{I}, \alpha[x/2](\phi_2) = \mathit{even}^{\mathcal{I}}(2) \vee \mathit{odd}^{\mathcal{I}}(2) = \top \vee \bot = \top$
$\mathcal{I}, \alpha[x/3](\phi_2) = \mathit{even}^{\mathcal{I}}(3) \vee \mathit{odd}^{\mathcal{I}}(3) = \bot \vee \top = \top$

Thus, $\mathcal{I}, \alpha \models \theta_2$.

(c) Let $\theta_3 = \forall x \exists y \phi_3$ with $\phi_3 = \mathit{lessThan}(x, y)$.
Then $\mathcal{I}, \alpha \models \theta_3$ iff for all $d_1 \in \mathcal{D}$ there exists a $d_2 \in \mathcal{D}$ so that $\mathcal{I}, \alpha[x/d_1, y/d_2](\phi_3) = \top$.
In this case it is clear that for $d_1 = 3$ no such d_2 exists, more formally, $\mathit{lessThan}^{\mathcal{I}}(3, d_2) = \bot$ for all $d_2 \in \mathcal{D}$ and therefore $I, \alpha \not\models \theta_3$.

(d) Let $\theta_4 = \forall x \phi_4$ with $\phi_4 = \mathit{even}(x) \Rightarrow \exists y\ \mathit{lessThan}(x, y)$.
Then $\mathcal{I}, \alpha \models \theta_4$ iff $\mathcal{I}, \alpha[x/d_1](\phi_4) = \top$ for all $d_1 \in \mathcal{D}$
We again consider all cases, starting odd numbers, i.e., $d_1 \in \{1, 3\}$: $(\mathit{even}(x) \Rightarrow \exists y\ \mathit{lessThan}(x, y))^{\mathcal{I}} = \neg \mathit{even}^{\mathcal{I}}(d_1) \vee ... =$ $\neg\bot \vee ... = \top \vee ... = \top$.
For $d_1 \in \{0, 2\}$ we have $\neg \mathit{even}^{\mathcal{I}}(d_1) = \bot$, thus we have to find a corresponding $d_2 \in \mathcal{D}$ with $\mathcal{I}, \alpha[x/d_1, y/d_2](\mathit{lessThan}(x, y)) = \top$. Here, we have $(0, 1) \in \mathit{lessThan}^{\mathcal{I}}$ and $(2, 3) \in \mathit{lessThan}^{\mathcal{I}}$.
Summing up all of the above, we showed that $\mathcal{I}, \alpha \models \theta_4$.

(e) Let $\theta_5 = \forall x \phi_5$ with $\phi_5 = \mathit{odd}(x) \Rightarrow \mathit{even}(\mathit{plus}(x, y))$.
Then $\mathcal{I}, \alpha \models \theta_5$ iff $\mathcal{I}, \alpha[x/d](\phi_5) = \top$ for all $d \in \mathcal{D}$
$\mathcal{I}, \alpha[x/d](\phi_5) = \neg \mathit{odd}^{\mathcal{I}}(d) \vee \mathit{even}^{\mathcal{I}}(\mathit{plus}^{\mathcal{I}}(d, 1))$
Notice that here y is a free variable and therefore the assignment $[y/1]$ is applied.
Now we consider all cases for d:
$\mathcal{I}, \alpha[x/0](\phi_5) = \neg \mathit{odd}^{\mathcal{I}}(0) \vee \mathit{even}^{\mathcal{I}}(\mathit{plus}^{\mathcal{I}}(0, 1)) = \neg \mathit{odd}^{\mathcal{I}}(0) \vee \mathit{even}^{\mathcal{I}}(1) = \top \vee \bot = \top$
$\mathcal{I}, \alpha[x/1](\phi_5) = \neg \mathit{odd}^{\mathcal{I}}(1) \vee \mathit{even}^{\mathcal{I}}(\mathit{plus}^{\mathcal{I}}(1, 1)) = \neg \mathit{odd}^{\mathcal{I}}(1) \vee \mathit{even}^{\mathcal{I}}(2) = \bot \vee \top = \top$
$\mathcal{I}, \alpha[x/2](\phi_5) = \neg \mathit{odd}^{\mathcal{I}}(2) \vee \mathit{even}^{\mathcal{I}}(\mathit{plus}^{\mathcal{I}}(2, 1)) = \neg \mathit{odd}^{\mathcal{I}}(2) \vee \mathit{even}^{\mathcal{I}}(3) = \top \vee \bot = \top$
$\mathcal{I}, \alpha[x/3](\phi_5) = \neg \mathit{odd}^{\mathcal{I}}(3) \vee \mathit{even}^{\mathcal{I}}(\mathit{plus}^{\mathcal{I}}(3, 1)) = \neg \mathit{odd}^{\mathcal{I}}(3) \vee \mathit{even}^{\mathcal{I}}(0) = \bot \vee \top = \top$
Thus, $\mathcal{I}, \alpha \models \theta_5$.

Exercise 6.2 (Semantics of Predicate Logic)
Consider the Interpretation $\mathcal{I} =< \mathcal{D}, \cdot^{\mathcal{I}} >$ and the variable assignment α:

- $\mathcal{D} = \{a, b, c\}$
- $P^{\mathcal{I}} = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, a)\}$
- $Q^{\mathcal{I}} = \{a, b\}$
- $R^{\mathcal{I}} = \{(a, a), (a, b), (a, c), (b, c), (c, b)\}$

- $\alpha = \{(v, a), (w, b)\}$

Decide for the following formulae θ_i if $\mathcal{I}$ is a model for θ_i under α, i.e. if $\mathcal{I}, \alpha \models \theta_i$. Explain your answer by formally applying the semantics.

(a) $\theta_1 = \forall x\big(P(x, w) \Rightarrow Q(x)\big)$

(b) $\theta_2 = \exists x\big(R(v, x) \Rightarrow P(x, x)\big)$

(c) $\theta_3 = \forall x \forall y\big(R(x, y) \iff Q(y)\big)$

(d) $\theta_4 = \Big[\neg\forall x\forall y\big(Q(y) \vee P(x, y)\big)\Big] \wedge \Big[\exists z\big(Q(z) \vee P(w, z)\big)\Big]$

Solution:

(a) Let $\theta_1 = \forall x\phi_1$ with $\phi_1 = \big(P(x, w) \Rightarrow Q(x)\big)$.
Then $\mathcal{I}, \alpha \models \theta_1$ iff $\mathcal{I}, \alpha[x/d](\phi_1) = \top$ for all $d \in \mathcal{D}$.
Notice that here x is bound by a quantifier and therefore the variable assignment in the definition of α is overridden by $\alpha[x/d]$.
Also w is a free variable and therefore the assignment $[w/b]$ is applied.
Since our universe $\mathcal{D}$ is finite (and small), we can evaluate the above for all $d \in \mathcal{D}$:
$\mathcal{I}, \alpha[x/a](\phi_1) = \neg P^{\mathcal{I}}(a, b) \vee Q^{\mathcal{I}}(a) = \bot \vee \top = \top$
$\mathcal{I}, \alpha[x/b](\phi_1) = \neg P^{\mathcal{I}}(b, b) \vee Q^{\mathcal{I}}(b) = \bot \vee \top = \top$
$\mathcal{I}, \alpha[x/c](\phi_1) = \neg P^{\mathcal{I}}(c, b) \vee Q^{\mathcal{I}}(c) = \top \vee \bot = \top$
Thus, $\mathcal{I}, \alpha \models \theta_1$.

(b) Let $\theta_2 = \exists x\phi_2$ with $\phi_2 = \big(R(v, x) \Rightarrow P(x, x)\big)$.
Then $\mathcal{I}, \alpha \models \theta_3$ if there exists a $d \in \mathcal{D}$
so that $\mathcal{I}, \alpha[x/d](\phi_2) = \top$.
Also v is a free variable and therefore the assignment $[w/a]$ is applied.
One case which can be applied is $d = b$.
$\mathcal{I}, \alpha[x/b](\phi_2) = \neg R^{\mathcal{I}}(a, b) \vee P^{\mathcal{I}}(b, b) = \bot \vee \top = \top$
Thus $\mathcal{I}, \alpha \models \theta_2$.

(c) Let $\theta_3 = \forall x\forall y\phi_3$ with $\phi_3 = \big(R(x, y) \iff Q(y)\big)$.
Then $\mathcal{I}, \alpha \models \theta_3$ iff $\mathcal{I}, \alpha[x/d1][y/d2](\phi_1) = \top$ for all $d1$ and $d2 \in \mathcal{D}$.
However, it can does not satisfy by assigning x=a and y = c,
$\mathcal{I}, \alpha[x/a][y/c](\phi_3) = \big(R^{\mathcal{I}}(a, c) \iff Q^{\mathcal{I}}(c)\big) = \bot \iff \top = \bot$
Hence $\mathcal{I}, \alpha \not\models \theta_3$.

(d) $\mathcal{I}, \alpha \models \theta_4$
Let $\theta_4 = \neg\theta_4^a \wedge \theta_4^b$ with
$\theta_4^a = \forall x\forall y\big(Q(y) \vee P(x, y)\big)$ and
$\theta_4^b = \exists z\big(Q(z) \vee P(w, z)\big)$
Thus, $\mathcal{I}, \alpha$ must model θ_4^b but not θ_4^a.
$\mathcal{I}, \alpha \not\models \theta_4^a$, since for $y = c$, $c \notin Q^{\mathcal{I}}$ and $(c, c) \notin P^{\mathcal{I}}$
$\mathcal{I}, \alpha \models \theta_4^b$, since, for instance, $a \in Q^{\mathcal{I}}$.