Sapienza University of Rome

Master in Artificial Intelligence and Robotics
Master in Engineering in Computer Science

Machine Learning

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7. Linear models for classification

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Overview

- Linearly separable data
- Linear models
- Least squares
- Fisher's linear discriminant
- Perceptron
- Support Vector Machines

References
C. Bishop. Pattern Recognition and Machine Learning. Sect. 4.1, 7.1
T. Mitchell. Machine Learning. Section 4.4

Linear Models for Classification

Learning a function $f : X \to Y$, with ...

- $X \subseteq \Re^d$
- $Y = \{C_1, \dots, C_k\}$

assuming *linearly separable* data.

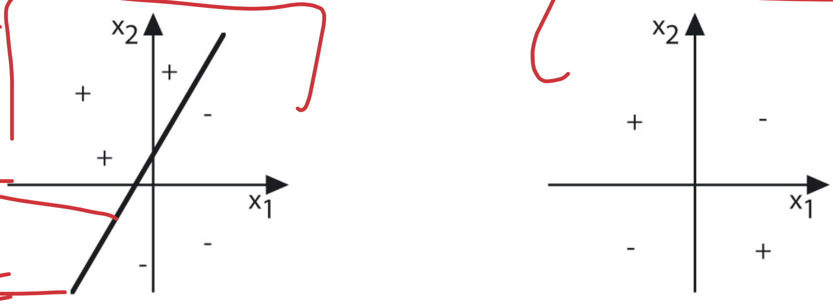
Cont. space input
Supervised
Multiclass

Linearly separable data

Instances in a data set are *linearly separable* iff there exists a hyperplane that separates the instance space into two regions, such that differently classified instances are separated

Stream at 11:51

Separate classe.



(a) (b)

Linear discriminant functions

Take an instance and represent a boundary that separate in classes.

Linear discriminant function

$$y : X \rightarrow \{C_1, \ldots, C_K\}$$

Stream at 11:1 $\boxed{29/10}$

Two classes:

$$y(\mathbf{x}) = \mathbf{w}^T \mathbf{x} + w_0$$

$w^T x + w_0 = 0$ è la def. della funzione

K-class:

$$y_1(\mathbf{x}) = \mathbf{w}_1^T \mathbf{x} + w_{10}$$
$$\ldots$$
$$y_K(\mathbf{x}) = \mathbf{w}_K^T \mathbf{x} + w_{K0}$$

Con 2 classi è una linea, con >2 è un piano.

Compact notation

Two classes:

$$y(\mathbf{x}) = \mathbf{w}^T \mathbf{x} + w_0 = \tilde{\mathbf{w}}^T \tilde{\mathbf{x}}, \text{ with:}$$

$$\tilde{\mathbf{w}} = \begin{pmatrix} w_0 \\ \mathbf{w} \end{pmatrix}, \tilde{\mathbf{x}} = \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$$

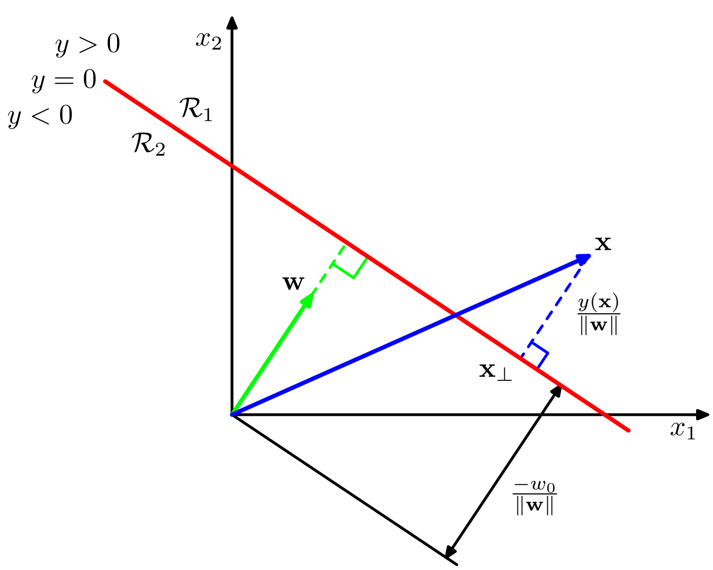
K-class:

$$\mathbf{y}(\mathbf{x}) = \begin{pmatrix} y_1(\mathbf{x}) \\ \cdots \\ y_K(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{w}_1^T \mathbf{x} + w_{10} \\ \cdots \\ \mathbf{w}_K^T \mathbf{x} + w_{K0} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{w}}_1^T \\ \cdots \\ \tilde{\mathbf{w}}_K^T \end{pmatrix} \tilde{\mathbf{x}} = \tilde{\mathbf{W}}^T \tilde{\mathbf{x}}, \text{ with:}$$

K linear function

$$\tilde{\mathbf{W}}^T = \begin{pmatrix} \tilde{\mathbf{w}}_1^T \\ \cdots \\ \tilde{\mathbf{w}}_K^T \end{pmatrix}, \text{ i.e.: } \tilde{\mathbf{W}} = (\tilde{\mathbf{w}}_1, \cdots, \tilde{\mathbf{w}}_K)$$

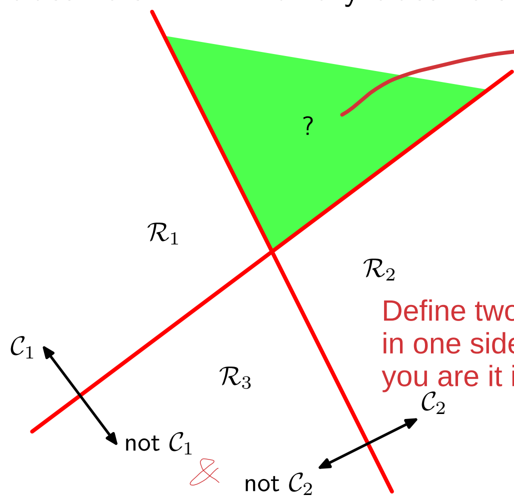
Linear discriminant functions



Multiple classes

Cannot use combinations of binary linear models.

One-versus-the-rest classifiers: $K-1$ binary classifiers: C_k vs. not-C_k



Both class 1 & 2

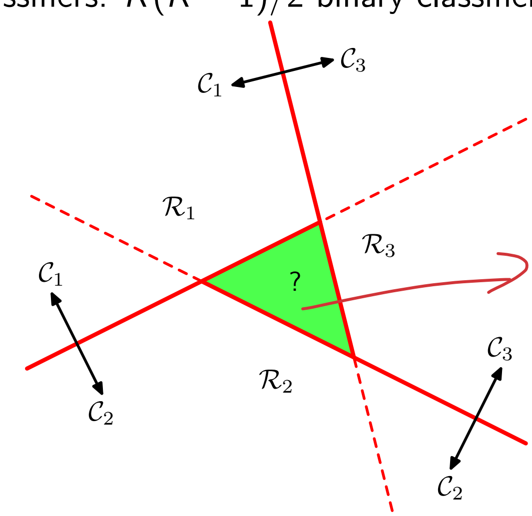
NOT THIS WAY TO GENERALIZE TO MULTI REGION.

Define two linear functions and then if you are in one side you are in one classe, otherwise you are it in that class, not matter which other

Multiple classes

Cannot use combinations of binary linear models.

One-versus-one classifiers: $K(K-1)/2$ binary classifiers: C_k vs. C_j



doesn't know how to classify it

Multiple classes

K-class discriminant comprising K linear functions ($\mathbf{x}$ not in dataset)

$$\mathbf{y}(\mathbf{x}) = \begin{pmatrix} y_1(\mathbf{x}) \\ \cdots \\ y_K(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{w}}_1^T \tilde{\mathbf{x}} \\ \cdots \\ \tilde{\mathbf{w}}_K^T \tilde{\mathbf{x}} \end{pmatrix} = \tilde{\mathbf{W}}^T \tilde{\mathbf{x}}$$

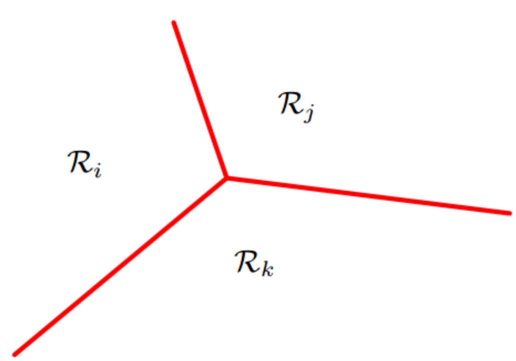
Classify $\mathbf{x}$ as C_k if $y_k(\mathbf{x}) > y_j(\mathbf{x})$ for all $j \neq k$ $(j, k = 1, \ldots, K)$

Decision boundary between C_k and C_j (hyperplane in $\Re^{D-1}$):

$$(\tilde{\mathbf{w}}_k - \tilde{\mathbf{w}}_j)^T \, \tilde{\mathbf{x}} = 0$$

Multiple classes

Example of K-class discriminant



Learning linear discriminants

Given a multi-class classification problem and data set D with linearly separable data,

determine $\tilde{\mathbf{W}}$ such that $\mathbf{y}(\mathbf{x}) = \tilde{\mathbf{W}}^T \tilde{\mathbf{x}}$ is the K-class discriminant.

Approaches to learn linear discriminants

- Least squares
- Fisher's linear discriminant
- Perceptron
- Support Vector Machines

Least squares

C_k - ...

Given $D = \{(\mathbf{x}_n, \mathbf{t}_n)_{n=1}^N\}$, find the linear discriminant

$$\mathbf{y}(\mathbf{x}) = \tilde{\mathbf{W}}^T \tilde{\mathbf{x}}$$

1-of-K coding scheme for $\mathbf{t}$: $\mathbf{x} \in C_k \rightarrow t_k = 1, t_j = 0$ for all $j \neq k$.
E.g., $\mathbf{t}_n = (0, \ldots, 1, \ldots, 0)^T$

$$\tilde{\mathbf{X}} = \begin{pmatrix} \tilde{\mathbf{x}}_1^T \\ \cdots \\ \tilde{\mathbf{x}}_N^T \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} \mathbf{t}_1^T \\ \cdots \\ \mathbf{t}_N^T \end{pmatrix}$$

Screen at: 12:24

Least squares

Trace = sum of diagonal.

Minimize sum-of-squares error function

$$E(\tilde{\mathbf{W}}) = \frac{1}{2} Tr\left\{(\tilde{\mathbf{X}}\tilde{\mathbf{W}} - \mathbf{T})^T (\tilde{\mathbf{X}}\tilde{\mathbf{W}} - \mathbf{T})\right\}$$

Closed-form solution:

$$\tilde{\mathbf{W}} = \underbrace{(\tilde{\mathbf{X}}^T\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}^T}_{\tilde{\mathbf{X}}^\dagger} \mathbf{T}$$

Pseudoinverse

$$\mathbf{y}(\mathbf{X}) = \tilde{\mathbf{W}}^T \tilde{\mathbf{X}} = \mathbf{T}^T (\tilde{\mathbf{X}}^\dagger)^T \tilde{\mathbf{X}}$$

Least squares

Classification of new instance $\mathbf{x}$ not in dataset:

Use learnt $\tilde{\mathbf{W}}$ to compute:

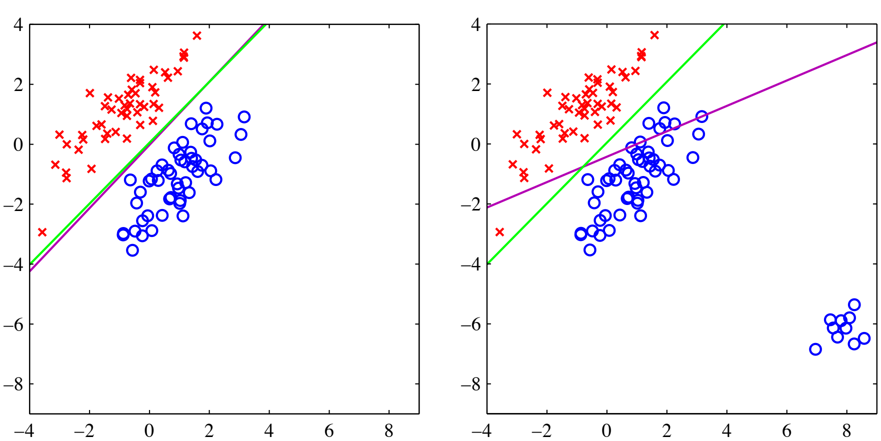
$$\mathbf{y}(\mathbf{x}) = \tilde{\mathbf{W}}^T \tilde{\mathbf{x}} = \begin{pmatrix} y_1(\mathbf{x}) \\ \cdots \\ y_K(\mathbf{x}) \end{pmatrix}$$

Assign class C_k to $\mathbf{x}$, where:

$$k = \underset{i \in \{1,\ldots,k\}}{\operatorname{argmax}} \{y_i(\mathbf{x})\}$$

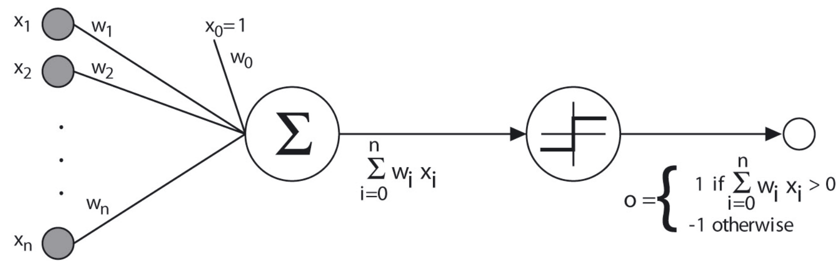
Issues with least squares

Assume Gaussian conditional distributions. Not robust to outliers!



Perceptron

end of 60'
Another method.



$$o(x_1, \ldots, x_d) = \begin{cases} 1 & \text{if } w_0 + w_1 x_1 + \cdots + w_d x_d > 0 \\ -1 & \text{otherwise.} \end{cases}$$

$$o(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{w}^T \mathbf{x} > 0 \\ -1 & \text{otherwise.} \end{cases} = sign(\mathbf{w}^T \mathbf{x})$$

Perceptron training rule

Consider the *unthresholded linear unit*, where

Minimizes error with respect to output

$$o = w_0 + w_1 x_1 + \cdots + w_d x_d = \mathbf{w}^T \mathbf{x}$$

Let's learn w_i from training examples $D = \{(\mathbf{x}_n, t_n)_{n=1}^N\}$ that minimize the squared error (*loss function*)

$$E(\mathbf{w}) \equiv \frac{1}{2} \sum_{n=1}^{N} (t_n - o_n)^2 = \frac{1}{2} \sum_{n=1}^{N} (t_n - \mathbf{w}^T \mathbf{x}_n)^2$$

Perceptron training rule

$$
\begin{aligned}
\frac{\partial E}{\partial w_i} &= \frac{\partial}{\partial w_i} \frac{1}{2} \sum_{n=1}^{N} (t_n - \mathbf{w}_n^T \mathbf{x}_n)^2 = \frac{1}{2} \sum_{n=1}^{N} \frac{\partial}{\partial w_i} (t_n - \mathbf{w}^T \mathbf{x}_n)^2 \\
&= \frac{1}{2} \sum_{n=1}^{N} 2(t_n - \mathbf{w}^T \mathbf{x}_n) \frac{\partial}{\partial w_i} (t_n - \mathbf{w}^T \mathbf{x}_n) \\
&= \sum_{n=1}^{N} (t_n - \mathbf{w}^T \mathbf{x}_n) \frac{\partial}{\partial w_i} (t_n - \mathbf{w}^T \mathbf{x}_n) \\
&= \sum_{n=1}^{N} (t_n - \mathbf{w}^T \mathbf{x}_n)(-x_{i,n})
\end{aligned}
$$

Perceptron training rule

Unthresholded unit:

Update of weights $\mathbf{w}$

$$
\begin{aligned}
w_i &\leftarrow w_i + \Delta w_i \\
\Delta w_i &= -\eta \frac{\partial E}{\partial w_i} = \eta \sum_{n=1}^{N} (t_n - \mathbf{w}^T \mathbf{x}_n)\, x_{i,n}
\end{aligned}
$$

η is a small constant (e.g., 0.05) called *learning rate*

Perceptron training rule

Thresholded unit:

Update of weights **w**

$$\begin{aligned} w_i &\leftarrow w_i + \Delta w_i \\ \Delta w_i &= -\eta \frac{\partial E}{\partial w_i} = \eta \sum_{n=1}^{N} (t_n - sign(\mathbf{w}^T \mathbf{x}_n))\, x_{i,n} \end{aligned}$$

Perceptron algorithm

Given perceptron model $o(\mathbf{x}) = sign(\mathbf{w}^T \mathbf{x})$ and data set D, determine weights **w**.

1. Initialize $\hat{\mathbf{w}}$ (e.g. small random values)
2. Repeat until termination condition
    - $\hat{w}_i \leftarrow \hat{w}_i + \Delta w_i$
3. Output $\hat{\mathbf{w}}$

Perceptron algorithm

Batch mode: Consider all dataset D

$$\Delta w_i = \eta \sum_{(\mathbf{x},t) \in D} (t - o(\mathbf{x}))\, x_i$$

Mini-Batch mode: Choose a small subset $S \subset D$

$$\Delta w_i = \eta \sum_{(\mathbf{x},t) \in S} (t - o(\mathbf{x}))\, x_i$$

Incremental mode: Choose one sample $(\mathbf{x}, t) \in D$

$$\Delta w_i = \eta\, (t - o(\mathbf{x}))\, x_i$$

$o(\mathbf{x}) = \mathbf{w}^T\mathbf{x}$ for unthresholded, $o(\mathbf{x}) = sign(\mathbf{w}^T\mathbf{x})$ for thresholded
Incremental and mini-batch modes speed up convergence and are less sensitive to local minima.

Perceptron algorithm

Termination conditions

- Predefined number of iterations
- Threshold on changes in the loss function $E(\mathbf{w})$

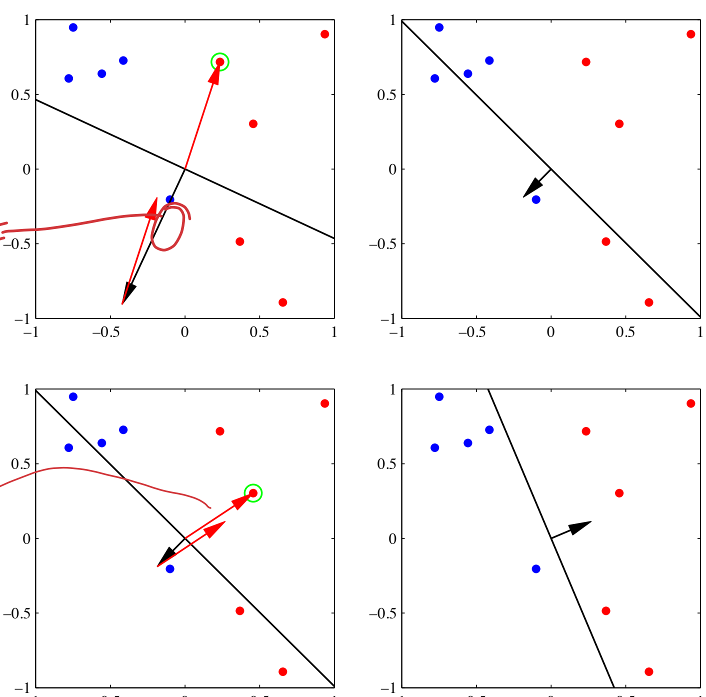
Perceptron training rule

Example:
$\eta = 0.1$, $x_i = 0.8$

- if $t = 1$ and $o = -1$ then $\Delta w_i = 0.16$
- if $t = -1$ and $o = 1$ then $\Delta w_i = -0.16$

Perceptron training rule

misclassified



augmented Δw
$w = w + \Delta w$

Perceptron training rule

Can prove it will converge:

- if training data is linearly separable
- and η sufficiently small

Small $\eta \rightarrow$ slow convergence.

Perceptron: Prediction

Classification of new instance $\mathbf{x}$ not in dataset:

Classify $\mathbf{x}$ as C_k, for $k = sign(\mathbf{w}^T\mathbf{x})$, using learnt $\mathbf{w}$

Fisher's linear discriminant

Consider two classes case.

Determine $y = \mathbf{w}^T \mathbf{x}$
and classify $\mathbf{x} \in C_1$ if $y \geq -w_0$, $\mathbf{x} \in C_2$ otherwise.

Corresponding to the projection on a line determined by $\mathbf{w}$.

Fisher's linear discriminant

Adjusting $\mathbf{w}$ to find a direction that maximizes class separation.

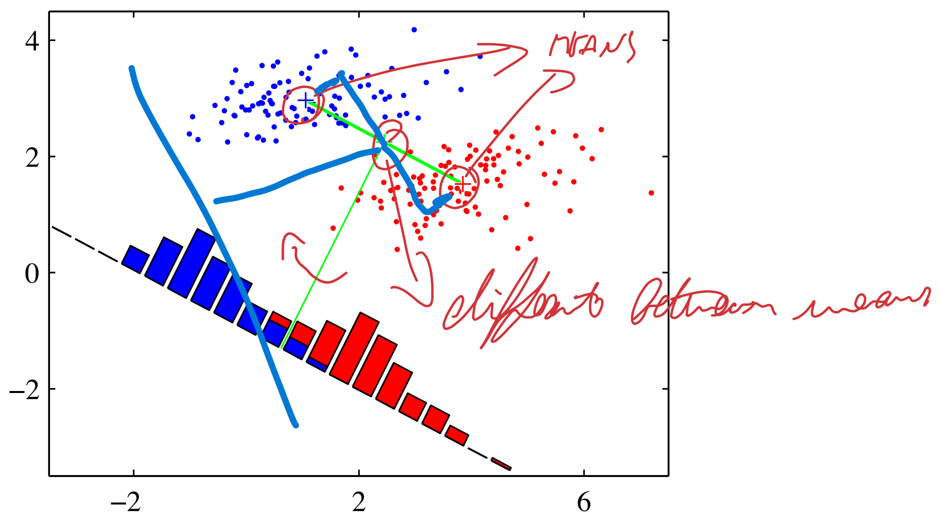
Consider a data set with N_1 points in C_1 and N_2 points in C_2

$$\mathbf{m}_1 = \frac{1}{N_1} \sum_{n \in C_1} \mathbf{x}_n \qquad \mathbf{m}_2 = \frac{1}{N_2} \sum_{n \in C_2} \mathbf{x}_n$$

Choose $\mathbf{w}$ that maximizes $J(\mathbf{w}) = \mathbf{w}^T(\mathbf{m}_2 - \mathbf{m}_1)$, subject to $||\mathbf{w}|| = 1$.

Fisher's linear discriminant

$$\mathbf{w} \propto (\mathbf{m}_2 - \mathbf{m}_1)$$



Fisher's linear discriminant

Fisher criterion

$$J(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{S}_B \mathbf{w}}{\mathbf{w}^T \mathbf{S}_W \mathbf{w}}$$

with
$\mathbf{S}_B = (\mathbf{m}_2 - \mathbf{m}_1)(\mathbf{m}_2 - \mathbf{m}_1)^T$
Between class scatter

$\mathbf{S}_W = \sum_{n \in C_1} (\mathbf{x}_n - \mathbf{m}_1)(\mathbf{x}_n - \mathbf{m}_1)^T + \sum_{n \in C_2} (\mathbf{x}_n - \mathbf{m}_2)(\mathbf{x}_n - \mathbf{m}_2)^T$
Within class scatter

Choose $\mathbf{w}$ that maximizes $J(\mathbf{w})$.

Fisher's linear discriminant

Find $\mathbf{w}$ that maximizes

$$J(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{S}_B \mathbf{w}}{\mathbf{w}^T \mathbf{S}_W \mathbf{w}}$$

by solving

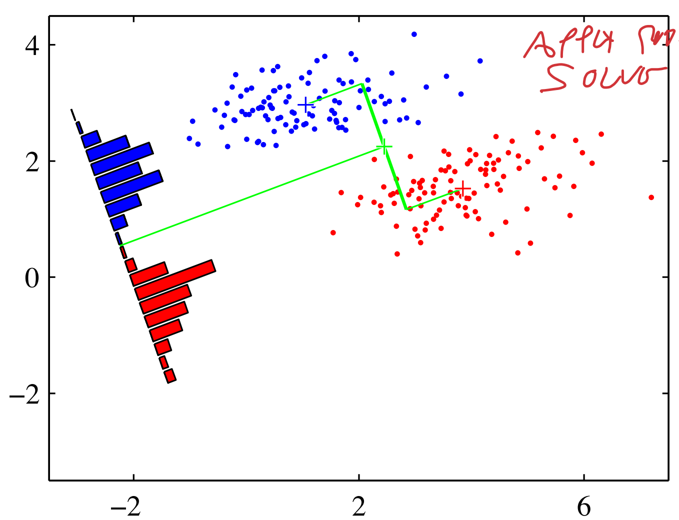
$$\frac{d}{d\mathbf{w}} J(\mathbf{w}) = 0$$

$$\Rightarrow \mathbf{w}^* \propto \mathbf{S}_W^{-1}(\mathbf{m}_2 - \mathbf{m}_1)$$

Fisher's linear discriminant

PROPORTIONAL $\mathbf{w} \propto \mathbf{S}_W^{-1}(\mathbf{m}_2 - \mathbf{m}_1)$ RESULTING VECTOR.

APPLY ROTATION. TO SOLVE OVERLAPPING



Fisher's linear discriminant

Summarizing, given a two classes classification problem, Fisher's linear discriminant is given by the function $y = \mathbf{w}^T \mathbf{x}$ and the classification of new instances is given by $y \geq -w_0$
where

$$\mathbf{w} = \mathbf{S}_W^{-1}(\mathbf{m}_2 - \mathbf{m}_1)$$

$$w_0 = \mathbf{w}^T \mathbf{m}$$

$\mathbf{m}$ is the global mean of all the data set.

Fisher's linear discriminant

Multiple classes.

$$\mathbf{y} = \mathbf{W}^T \mathbf{x}$$

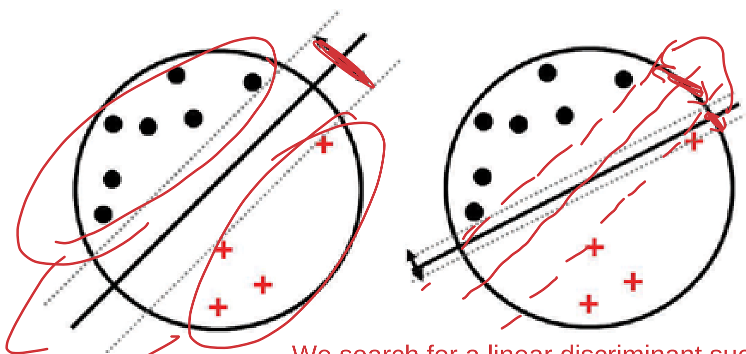
Maximizing

$$J(\mathbf{W}) = Tr\left\{(\mathbf{W}\mathbf{S}_W\mathbf{W}^T)^{-1}(\mathbf{W}\mathbf{S}_B\mathbf{W}^T)\right\}$$

...

Support Vector Machines

Support Vector Machines (SVM) for Classification aims at maximum margin providing for better accuracy.



DATASET

We search for a linear discriminant such that the distance of the point closest to the discriminant is maximized.
Distance is Called margin.

We have to maximize the margin.

Support Vector Machines

ALSO MULTI-CLASS

Let's consider binary classification $y : X \to \{+1, -1\}$ with data set $D = \{(\mathbf{x}_n, t_n)_{n=1}^N\}$, $t_n \in \{+1, -1\}$ and a linear model

$$y(\mathbf{x}) = \mathbf{w}^T \mathbf{x} + w_0$$

LIN. DISCRIMINANT.

Assume D is linearly separable (Exist the line that divide)

FIND $\exists \mathbf{w}, w_0 \ s.t.$ $\begin{array}{l} y(\mathbf{x}_n) > 0, \ if \ t_n = +1 \\ y(\mathbf{x}_n) < 0, \ if \ t_n = -1 \end{array}$

We have to fine the weights such that, this occur.

$$t_n \, y(\mathbf{x}_n) > 0 \ \forall n = 1, \dots N$$

Screen at 15:5

and 15:02

Support Vector Machines

Let $\mathbf{x}_k$ be the closest point of the data set D to the hyperplane $\bar{h} : \bar{\mathbf{w}}^T \mathbf{x} + \bar{w}_0 = 0$

the *margin* (smallest distance between $\mathbf{x}_k$ and $\bar{h}$) is $\frac{|y(\mathbf{x}_k)|}{||\mathbf{w}||}$

Given data set D and hyperplane $\bar{h}$, the margin is computed as

$$\min_{n=1,\dots,N} \frac{|y(\mathbf{x}_n)|}{||\mathbf{w}||} = \cdots = \frac{1}{||\mathbf{w}||} \min_{n=1,\dots,N} [t_n(\bar{\mathbf{w}}^T \mathbf{x}_n + \bar{w}_0)]$$

using the property $|y(\mathbf{x}_n)| = t_n \, y(\mathbf{x}_n)$

Support Vector Machines

Given data set D, the hyperplane $h^* : \mathbf{w}^{*T} \mathbf{x} + w_0^* = 0$ with maximum margin is computed as

$$\mathbf{w}^*, w_0^* = \underset{\mathbf{w}, w_0}{\operatorname{argmax}} \frac{1}{||\mathbf{w}||} \min_{n=1,\dots,N} [t_n(\mathbf{w}^T \mathbf{x}_n + w_0)]$$

THAT MAXIMIZES

Support Vector Machines

NORMALIZATION

Rescaling all the points does not affect the solution.

15:06
15:11

Rescale in such a way that for the closet point $\mathbf{x}_k$ we have

$$t_k(\mathbf{w}^T\mathbf{x}_k + w_0) = 1$$

→ if < 1 multiply to obtain = 1

Canonical representation:

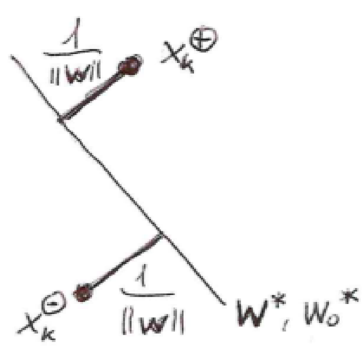
$$t_n(\mathbf{w}^T\mathbf{x}_n + w_0) \geq 1 \;\; \forall n = 1, \ldots, N$$

THE CLOSEST POINT HAS DISTANCE 1

Support Vector Machines

When the maxim margin hyperplane $\mathbf{w}^*, {w_0}^*$ is found, there will be at least 2 closest points $\mathbf{x}_k^{\oplus}$ and $\mathbf{x}_k^{\ominus}$ (one for each class).

$$\mathbf{w}^{*T}\mathbf{x}_k^{\oplus} + w_0^* = +1$$
$$\mathbf{w}^{*T}\mathbf{x}_k^{\ominus} + w_0^* = -1$$



Support Vector Machines

In the canonical representation of the problem the maxim margin hyperplane can be found by solving the optimization problem

MARGIN

$$\max \frac{1}{||\mathbf{w}||} = \min \frac{1}{2}||\mathbf{w}||^2$$

subject to

$$t_n(\mathbf{w}^T\mathbf{x}_n + w_0) \geq 1 \;\; \forall n = 1, \ldots, N$$

ALL INSTANCES ARE CORRECTLY CLASSIFIED

Quadratic programming problem solved with Lagrangian method.

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Support Vector Machines

Solution

REQUIRED TO SOLVE MAX. PROBLEM

$$\mathbf{w}^* = \sum_{n=1}^{N} a_n \, t_n \, \mathbf{x}_n$$

WE KNOW IT

a_i (Lagrange multipliers): results of the Lagrangian optimization problem

$$\tilde{L}(\mathbf{a}) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} a_n \, a_m \, t_n \, t_m \, \mathbf{x}_n^T \mathbf{x}_m$$

subject to

$$a_n \geq 0 \;\; \forall n = 1, \ldots, N$$

Screen on IPAD.

$$\sum_{n=1}^{N} a_n t_n = 0$$

Support Vector Machines

SCRIBE ON IPAD

Karush-Kuhn-Tucker (KKT) condition:
for each $\mathbf{x}_n \in X_D$*, either* $a_n = 0$ *or* $t_n\, y(\mathbf{x}_n) = 1$

$\mathbf{x}_n$ for which $a_m = 0$ do not contribute to the solution

Support vectors: x_k such that $a_k \neq 0$ and $t_k\, y(\mathbf{x}_k) = 1$

$$SV \equiv \{\mathbf{x}_k \in X_D \mid t_k\, y(\mathbf{x}_k) = 1\}$$

Support Vector Machines

Hyperplanes expressed with support vectors

$$y(\mathbf{x}) = \overset{w^T x}{\boxed{\sum_{\mathbf{x}_j \in SV} a_j\, t_j\, \mathbf{x}^T \mathbf{x}_j}} + w_0 = 0$$

Note: other vectors $\mathbf{x}_n \notin SV$ do not contribute ($a_n = 0$)

Support Vector Machines

To compute w_0:

Support vector $\mathbf{x}_k \in SV$ satisfies $t_k \, y(\mathbf{x}_k) = 1$

$$t_k \left(\sum_{\mathbf{x}_j \in SV} a_j t_j \mathbf{x}_k^T \mathbf{x}_j + w_0 \right) = 1$$

Multiplying by t_k and using $t_k^2 = 1$

$$w_0 = t_k - \sum_{\mathbf{x}_j \in SV} a_j t_j \mathbf{x}_k^T \mathbf{x}_j$$

Support Vector Machines

Instead of using one particular support vector $\mathbf{x}_k$ to determine w_0

$$w_0 = t_k - \sum_{\mathbf{x}_j \in SV} a_j t_j \mathbf{x}_k^T \mathbf{x}_j$$

a more stable solution is obtained by averaging over all the support vectors

$$w_0 = \frac{1}{|SV|} \sum_{\mathbf{x}_k \in SV} \left(t_k - \sum_{\mathbf{x}_j \in S} a_j t_j \mathbf{x}_k^T \mathbf{x}_j \right)$$

Support Vector Machines

Given the maximum margin hyperplane determined by a_k^*, w_0^*

Classification of a new instance $\mathbf{x}'$

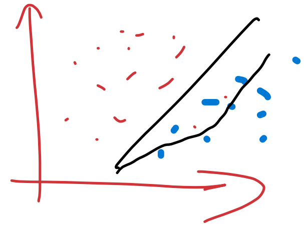
$$sign(y(\mathbf{x}')) = sign\left(\sum_{\mathbf{x}_k \in SV} a_k^* \, t_k \, \mathbf{x}'^T \mathbf{x}_k + w_0^*\right)$$

Support Vector Machines

Optimization problem for determining $\mathbf{w}$ (dimension $|X|$) transformed in an optimization problem for determining $\mathbf{a}$ (dimension $|D|$)

Efficient when $|X| < |D|$ (most of a_i will be zero).
Very useful when $|X|$ is large or infinite.

Support Vector Machines with soft margin constraints



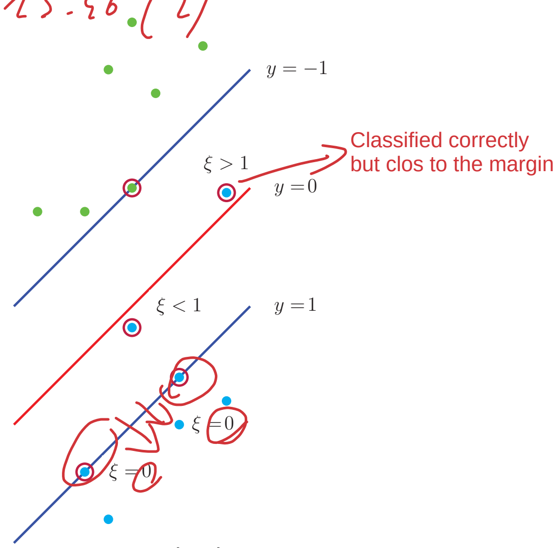
What if data are "almost" linearly separable (e.g., a few points are on the "wrong side")

Let us introduce *slack variables* $\xi_n \geq 0 \; n = 1, \ldots, N$

We allow for the misclassification but associate a price to misclassification and want to minimize the price, to solve the problem anyway.

Support Vector Machines with soft margin constraints

See chat at 15:46 (2)



- $\xi_n = 0$ if point on or inside the correct margin boundary
- $0 < \xi_n \leq 1$ if point inside the margin but correct side
- $\xi_n > 1$ if point on wrong side of boundary

when $\xi_n = 1$, the sample lies on the decision boundary $y(\mathbf{x}_n) = 0$
when $\xi_n > 1$, the sample will be mis-classified

Support Vector Machines with soft margin constraints

Soft margin constraint

$$t_n\, y(\mathbf{x}_n) \geq 1 - \xi_n\,, \quad n = 1, \ldots, N$$

Optimization problem with soft margin constraints

MAX. THE MARGIN (INVERSE)

$$\min \frac{1}{2}||\mathbf{w}||^2 + C \sum_{n=1}^{N} \xi_n$$

MINIMAL PRICE TO PAY.

subject to

Right classified normalized by the price
We are allowed to right classify but pay a price.

$$t_n\, y(\mathbf{x}_n) \geq 1 - \xi_n\,, \quad n = 1, \ldots, N$$

$$\xi_n \geq 0\,, \quad n = 1, \ldots, N$$

C is a constant (inverse of a regularization coefficient)

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Support Vector Machines with soft margin constraints

Solution similar to the case of linearly separable data.

$$\mathbf{w}^* = \sum_{n=1}^{N} a_n\, t_n\, \mathbf{x}_n$$

$$w_0^* = \ldots.$$

with a_n computed as solution of a Lagrangian optimization problem.

Basis functions

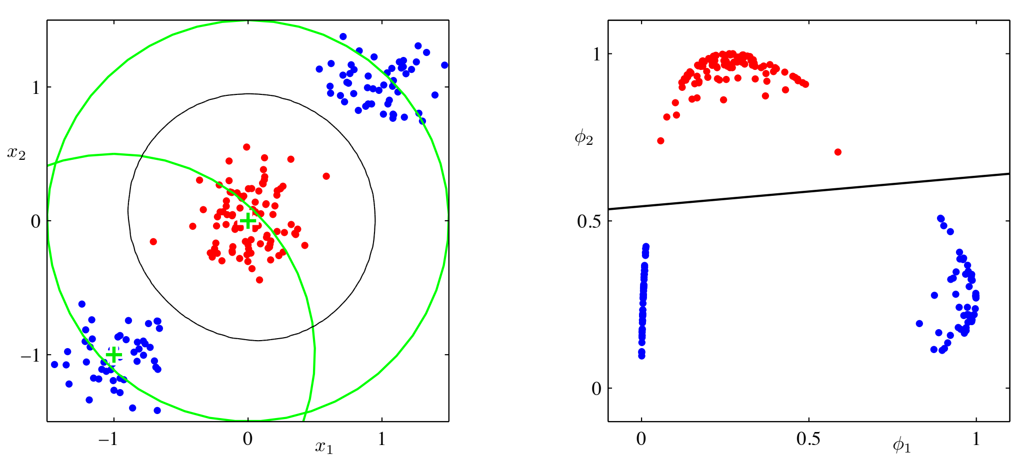
So far we considered models working directly on **x**.

vaw All the results hold if we consider a non-linear transformation of the inputs $\phi(\mathbf{x})$ (*basis functions*).

Decision boundaries will be linear in the feature space ϕ and non-linear in the original space **x**

Classes that are linearly separable in the feature space ϕ may not be separable in the input space **x**.

Basis functions example



Basis functions examples

- Linear
- Polynomial
- Radial Basis Function (RBF)
- Sigmoid
- ...

Linear models for non-linear functions

Learning non-linear function

$$y : X \to \{C_1, \ldots, C_K\}$$

from data set D non-linearly separable.

Find a non-linear transformation ϕ and learn a linear model

$$y(\mathbf{x}) = \mathbf{w}^T \phi(\mathbf{x}) + w_0 \;\; \textit{(two classes)}$$

$$y_k(\mathbf{x}) = \mathbf{w}_k^T \phi(\mathbf{x}) + w_{k0} \;\; \textit{(multiple classes)}$$

Summary

- Basic methods for learning linear classification functions
- Based on solution of an optimization problem
- Closed form vs. iterative solutions
- Sensitivity to outliers
- Learning non-linear functions with linear models using basis functions
- Further developed as kernel methods