Foundations of Artificial Intelligence

6. Constraint Satisfaction Problems

CSPs as Search Problems, Solving CSPs, Problem Structure

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A motivating example of CSP (here: graph coloring)

- FCC Spectrum Auction in 2017
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  - US Federal Communications Commission (FCC) held 13-month auction

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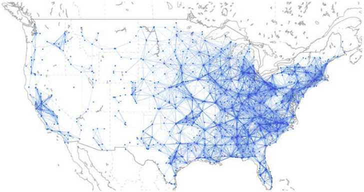
  - 2991 stations (nodes) &
    2.7 million interference constraints:
    stations in neighboring regions cannot use too similar frequencies
  - Need to check feasibility whenever an offer is made
  - More instances checkable: higher revenue



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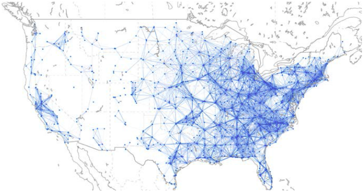


- Formulated as a CSP and solved with SAT solvers (improved by meta-algorithmics, see future lecture)
  - Improved ratio of instances solved from 73% to 99.6%
  - Net income for US government: $7 billion (used to pay down national debt)

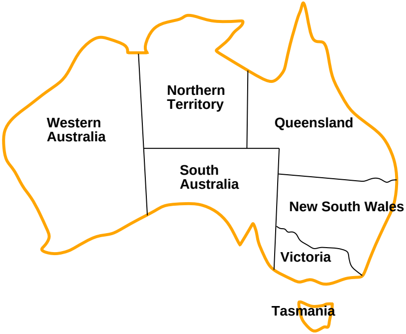
Contents

Lecture Overview

Constraint Satisfaction Problems

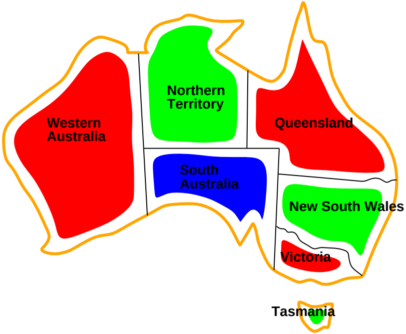
- A Constraint Satisfaction Problems (CSP) is given by
  - a set of variables $\{x_1, x_2, \ldots, x_n\}$,
  - an associated set of value domains $\{dom_1, dom_2, \ldots, dom_n\}$, and
  - a set of constraints. i.e., relations, over the variables.
  - An assignment of values to variables that satisfies all constraints is a solution of such a CSP.
- If CSPs are viewed as search problems, states are explicitly represented as variable assignments. CSP search algorithms take advantage of this structure.
- The main idea is to exploit the constraints to eliminate large portions of search space.
- *Formal representation language* with associated general inference algorithms

Example: Map-Coloring



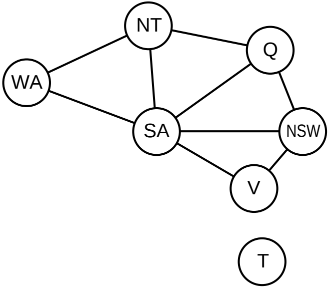
- Variables: WA, NT, SA, Q, NSW, V, T
- Values: $\{red, green, blue\}$
- Constraints: adjacent regions must have different colors, e.g., $NSW \neq V$

One Solution



- Solution assignment:
  - $\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

Constraint Graph



- a constraint graph can be used to visualize binary constraints
- for higher order constraints, hyper-graph representations might be used
- Nodes = variables, arcs = constraints

Variations

- Binary, ternary, or even higher arity (e.g., ALL_DIFFERENT)
- Finite domains (d values) $\rightarrow d^n$ possible variable assignments
- Infinite domains (reals, integers)
  - *linear constraints (each variable occurs only in linear form)*: solvable (in P if real)
  - *nonlinear constraints*: unsolvable

Applications

- Timetabling (classes, rooms, times)
- Configuration (hardware, cars, . . .)
- Nurse rostering
- Scheduling (sports, etc)
- Sudoku
- . . .

Lecture Overview

Backtracking Search over Assignments

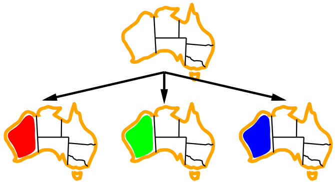
- Assign values to variables step by step (order does not matter)
- Consider only one variable per search node!
- DFS with single-variable assignments is called backtracking search

Algorithm

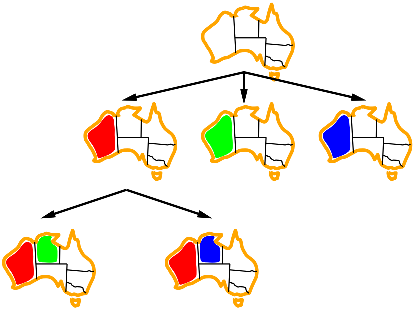
```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
  return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
     if value is consistent with assignment then
        add {var = value} to assignment
        inferences ← INFERENCE(csp, var, value)
        if inferences ≠ failure then
           add inferences to assignment
           result ← BACKTRACK(assignment, csp)
           if result ≠ failure then
              return result
     remove {var = value} and inferences from assignment
  return failure
```

Example (1)

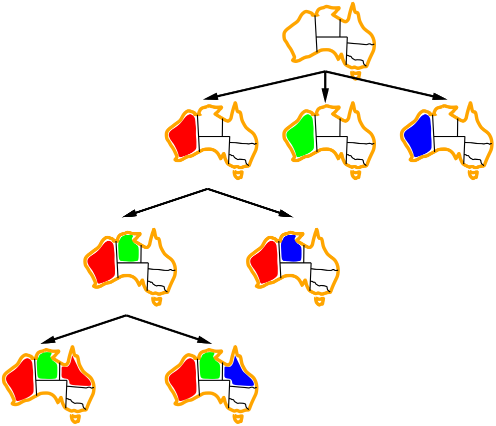
Example (2)



Example (3)



Example (4)



Lecture Overview

Improving Efficiency: CSP Heuristics & Pruning Techniques

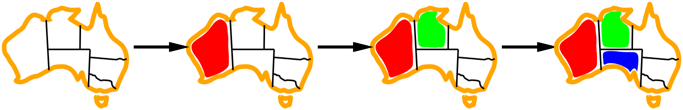
- Variable ordering: Which one to assign first?
- Value ordering: Which value to try first?
- Try to detect failures early on
- Try to exploit problem structure

→ Note: all this is not problem-specific!

Variable Ordering: Most constrained first

- Most constrained variable:
  - choose the variable with the fewest remaining legal values

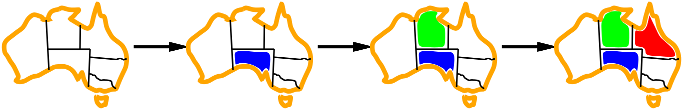
$\rightarrow$ reduces branching factor!



Variable Ordering:
Most Constraining Variable First

- Break ties among variables with the same number of remaining legal values:
  - choose variable with the most constraints on remaining unassigned variables

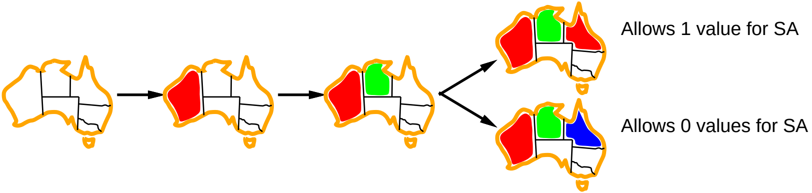
$\rightarrow$ reduces branching factor in the next steps



Value Ordering: Least Constraining Value First

- Given a variable,
  - choose first a value that rules out the fewest values in the remaining unassigned variables

→ We want to find an assignment that satisfies the constraints (of course, this does not help if the given problem is unsatisfiable.)



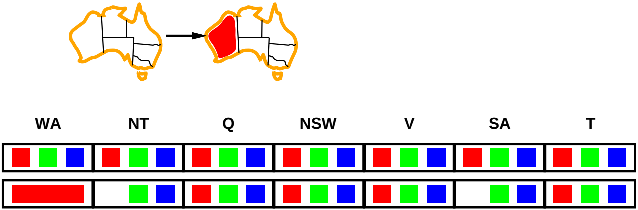
Rule out Failures early on: Forward Checking

- Whenever a value is assigned to a variable, values that are now illegal for other variables are removed
- Implements what the ordering heuristics implicitly compute
- $WA =$ *red*, then NT cannot become *red*
- If all values are removed for one variable, we can stop!

- Keep track of remaining values
- Stop if all have been removed

Forward Checking (2)

- Keep track of remaining values
- Stop if all have been removed



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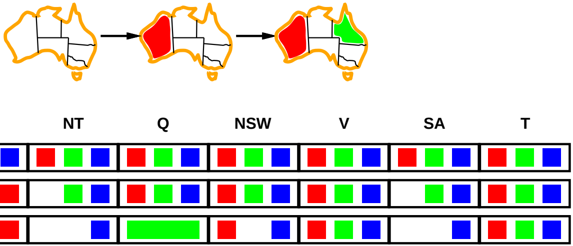

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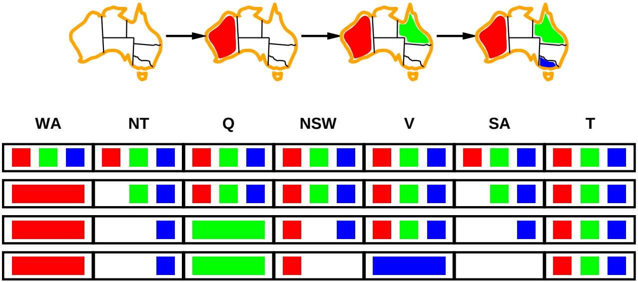
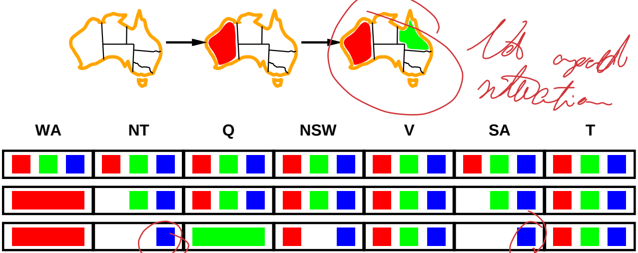


Lecture Overview

Forward Checking: Sometimes it Misses Something

Not perfect

- Forward Checking propagates information from assigned to unassigned variables
- However, there is no propagation between unassigned variables

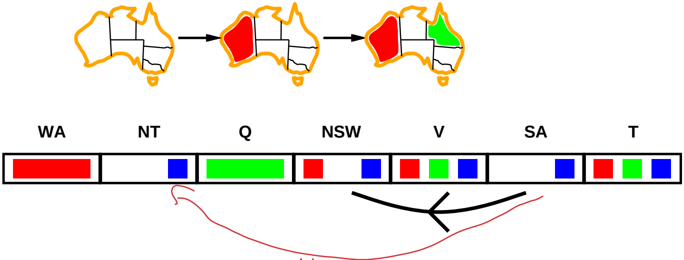


Not a good situation

Arc Consistency

- A directed arc $X \rightarrow Y$ is "consistent" iff
  - for every value x of X, there exists a value y of Y, such that (x, y) satisfies the constraint between X and Y
- Remove values from the domain of X to enforce arc-consistency
- Arc consistency detects failures earlier
- Can be used as preprocessing technique or as a propagation step during backtracking

Arc Consistency Example



Non - Consist.

AC-3 Algorithm

function AC-3(csp) **returns** false if an inconsistency is found and true otherwise
  inputs: csp, a binary CSP with components $(X,\ D,\ C)$
  local variables: $queue$, a queue of arcs, initially all the arcs in csp

  while $queue$ is not empty **do**
    $(X_i,\ X_j) \leftarrow$ REMOVE-FIRST($queue$)
    if REVISE(csp, $X_i,\ X_j$) **then**
      if size of $D_i\ =\ 0$ **then return** $false$
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
        add $(X_k,\ X_i)$ to $queue$
  return $true$

Check wether i have ti Remo e a valle from this domani

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function REVISE(csp, $X_i,\ X_j$) **returns** true iff we revise the domain of X_i
  $revised \leftarrow false$
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      $revised \leftarrow true$
  return $revised$

Properties of AC-3

- What is the computational complexity of AC-3?
  - Let n denote the number of nodes, and let d denote the maximal number of elements in a domain
  - Hint: what is the complexity of function REVISE, how often can it return true in the worst case, and how often is it thus called in the worst case?

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- AC-3 runs in $O(d^3n^2)$ time
  - REVISE takes $O(d^2)$ (for each element $x \in D_i$, you need to check each element $y \in D_j$)
  - Each time REVISE returns true one element of X_i is eliminated; there are only max. d elements for each of the n variables
  - Each time REVISE returns true up to n constraints are added to the queue
  - Alltogether, in the worst case, REVISE can only be called a maximum of $O(n^2d)$ times, each taking time $O(d^2)$

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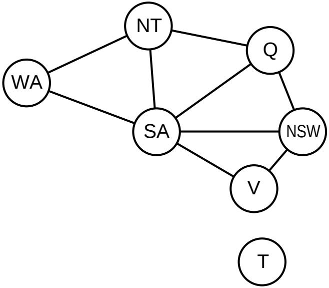
  - Alltogether, in the worst case, REVISE can only be called a maximum of $O(n^2 d)$ times, each taking time $O(d^2)$
- Of course, AC-3 does not detect all inconsistencies (which is an NP-hard problem)

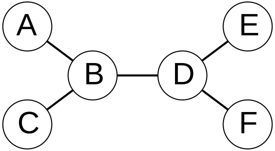
Lecture Overview

Problem Structure (1)



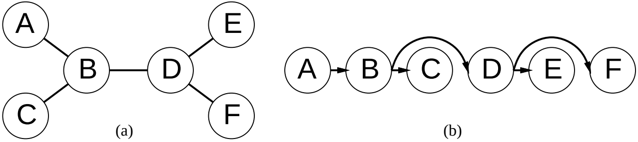
- This example CSP has two independent components
- Identifiable as connected components of constraint graph
- Can reduce the search space dramatically

Problem Structure (2): Tree-structured CSPs



- If the CSP graph is a tree, then it can be solved in $O(nd^2)$ (general CSPs need in the worst case $O(d^n)$).
- *Idea:* Pick root, order nodes, apply arc consistency from leaves to root, and assign values starting at root.

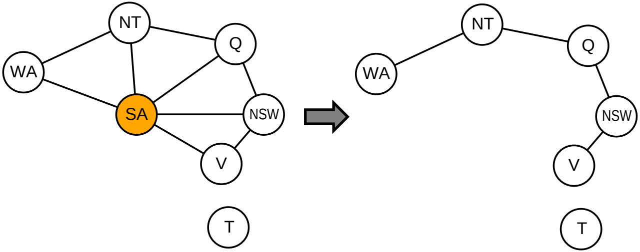
Problem Structure (2): Tree-structured CSPs



- Pick any variable as root; choose an ordering such that each variable appears after its parent in the tree.
- Apply arc-consistency to (x_i, x_k) when x_i is the parent of x_k for all $k = n$ down to 2 (any tree with n nodes has $n-1$ arcs and per arc d^2 comparisons are needed, which results in a complexity of $O(n\,d^2)$).
- Now we can start at x_1 assigning values from the remaining domains without creating any conflict in one sweep through the tree!
- This algorithm is linear in n.

Problem Structure (3): Almost Tree-structured

Idea: Reduce the graph structure to a tree by fixing values in a reasonably chosen subset

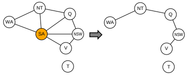


Instantiate a variable and prune values in neighboring variables is called Conditioning

Problem Structure (4): Almost Tree-structured

Algorithm Cutset Conditioning:

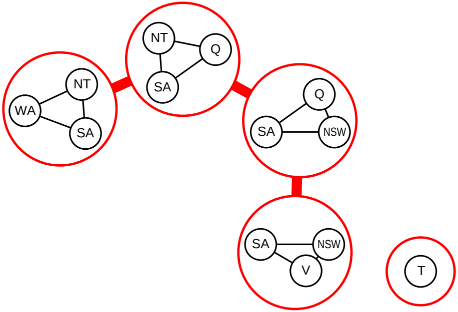
1. Choose a subset S of the CSPs variables such that the constraint graph becomes a tree after removal of S. The set S is called a cycle cutset.
2. For each possible assignment of variables in S that satisfies all constraints on S
   1. remove from the domains of the remaining variables any values that are inconsistent with the assignments for S, and
   2. if the remaining CSP has a solution, return it together with the assignment for S



Note: Finding the smallest cycle cutset is NP hard, but several efficient approximation algorithms are known.

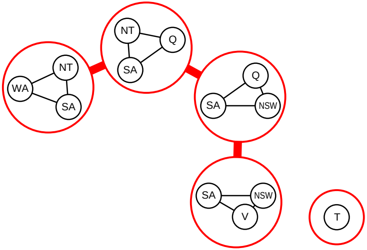
Another Method: Tree Decomposition (1)

- Decompose the problem into a set of connected sub-problems, where two sub-problems are connected when they share a constraint
- Solve the sub-problems independently and then combine the solutions



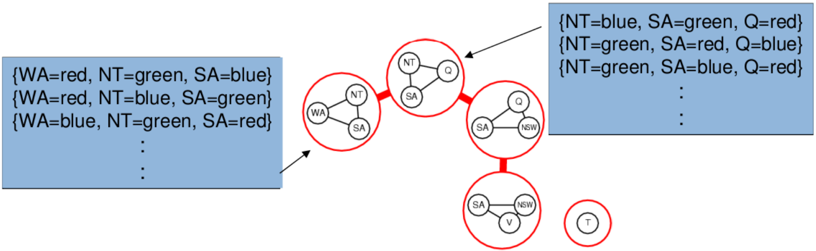
Another Method: Tree Decomposition (2)

- A tree decomposition must satisfy the following conditions:
  - Every variable of the original problem appears in at least one sub-problem
  - Every constraint appears in at least one sub-problem
  - If a variable appears in two sub-problems, it must appear in all sub-problems on the path between the two sub-problems
  - The connections form a tree



Another Method: Tree Decomposition (3)

- Consider sub-problems as new mega-variables, which have values defined by the solutions to the sub-problems
- Use technique for tree-structured CSP to find an overall solution (constraint is to have identical values for the same variable)



Tree Width

- The aim is to make the subproblems as small as possible. The tree width w of a tree decomposition is the size of largest sub-problem minus 1
- Tree width of a graph is minimal tree width over all possible tree decompositions
- If a graph has tree width w and we know a tree decomposition with that width, we can solve the problem in $O(nd^{w+1})$
- Unfortunately, finding a tree decomposition with minimal tree width is NP-hard. However, there are heuristic methods that work well in practice.

Summary

- CSPs are a special kind of search problem:
  - states are value assignments
  - goal test is defined by constraints
- Backtracking = DFS with one variable assigned per node. Other intelligent backtracking techniques possible
- Variable/value ordering heuristics can help dramatically
- Constraint propagation prunes the search space
- Tree structure of CSP graph simplifies problem significantly
- Cutset conditioning and tree decomposition are two ways to transform part of the problem into a tree
- CSPs can also be solved using local search