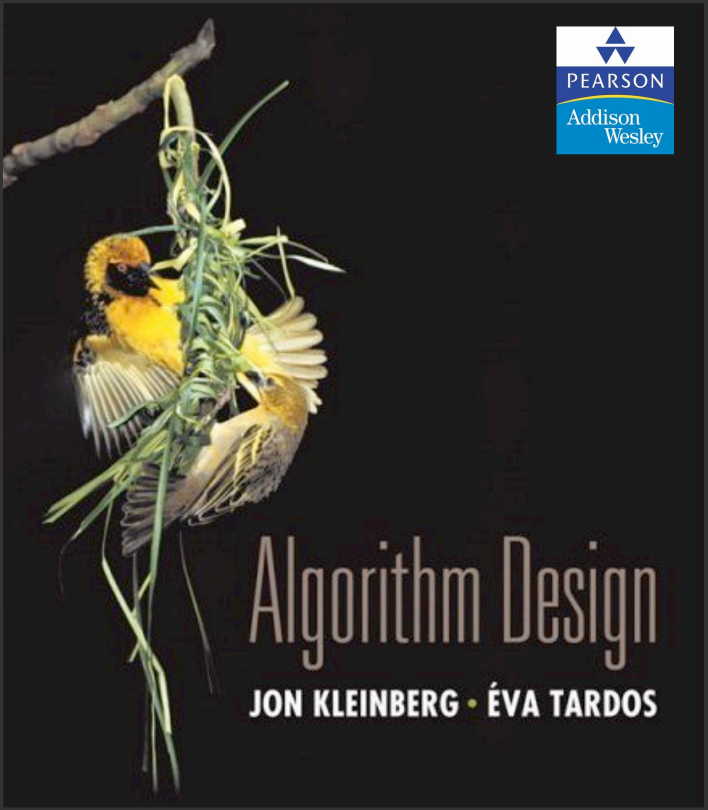


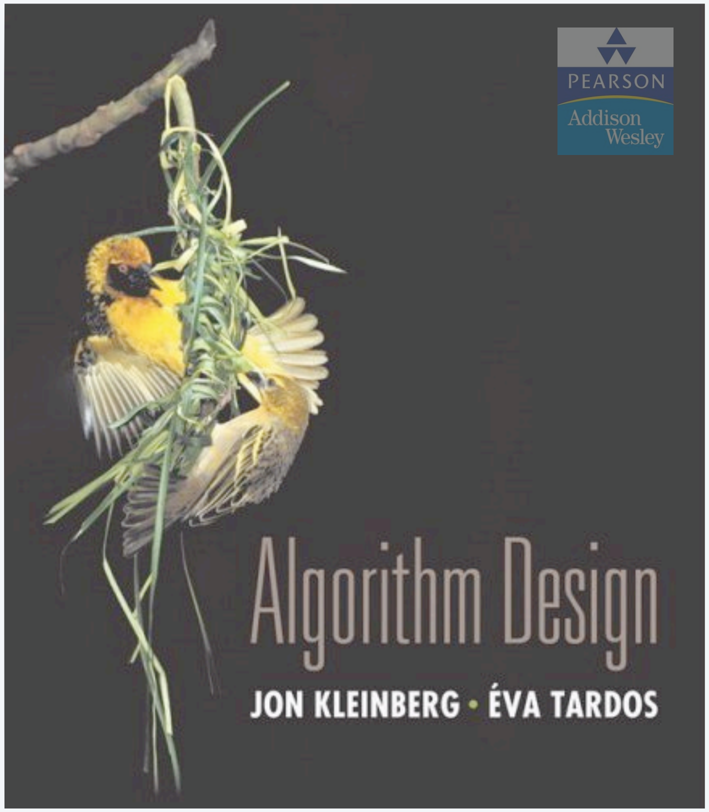
Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- *min-cost arborescences*

Last updated on Sep 8, 2013 6:30 AM



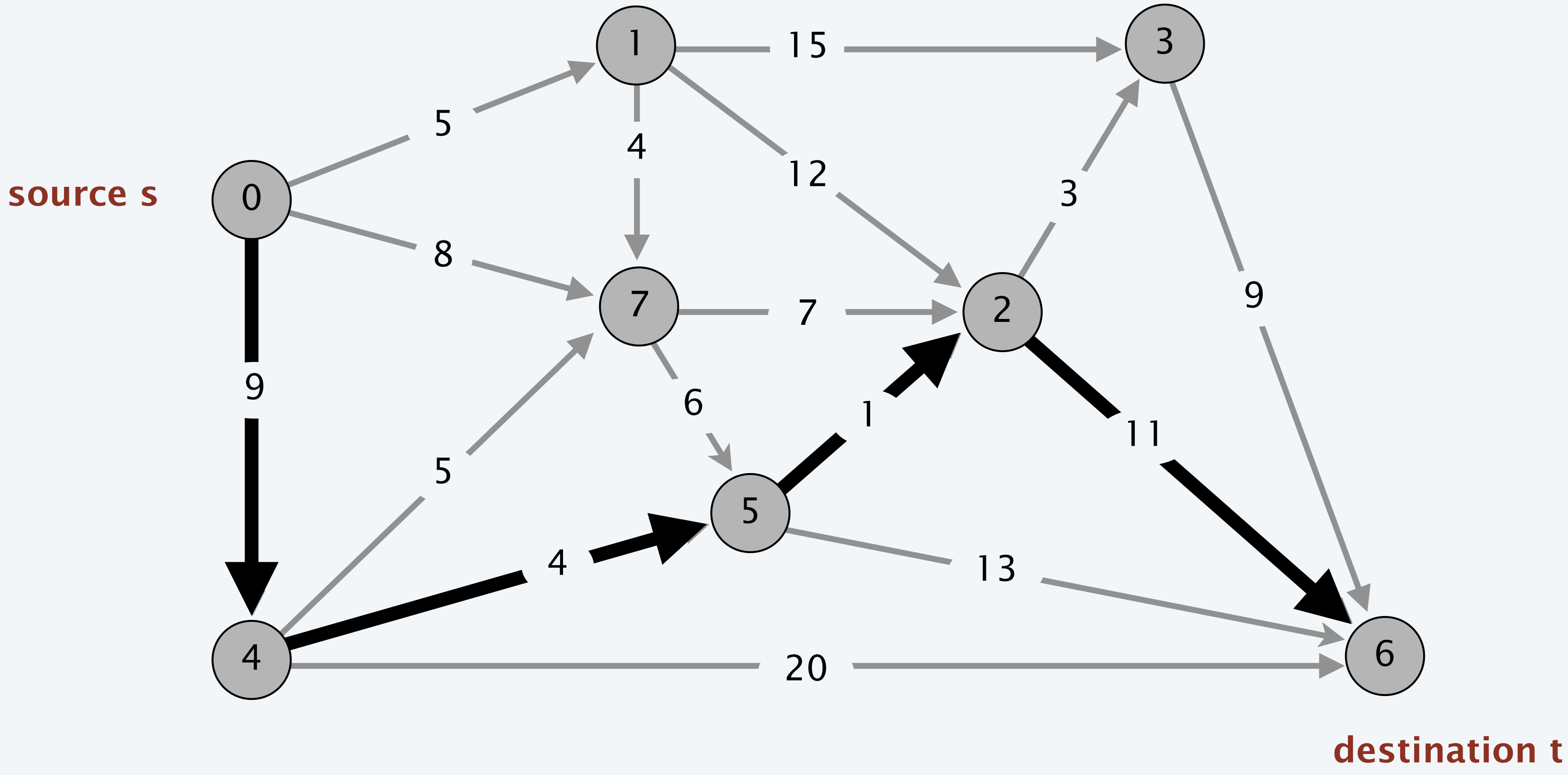
SECTION 4.4

4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- *min-cost arborescences*

Shortest-paths problem

Problem. Given a digraph $G = (V, E)$, edge lengths $\ell_e \geq 0$, source $s \in V$, and destination $t \in V$, find the shortest directed path from s to t.



length of path = 9 + 4 + 1 + 11 = 25

Car navigation

Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in LaTeX.
- Urban traffic planning.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Optimal truck routing through given traffic congestion pattern.

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

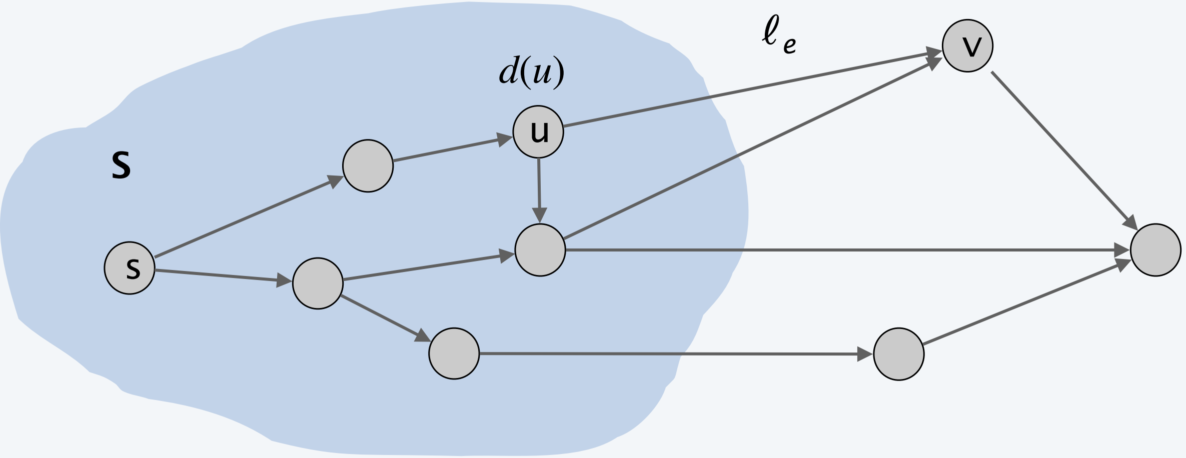
Dijkstra's algorithm

Greedy approach. Maintain a set of explored nodes S for which algorithm has determined the shortest path distance $d(u)$ from s to u.

- Initialize $S = \{ s \},\ d(s) = 0$.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v)\,:\,u \in S} d(u) + \ell_e ,$$

shortest path to some node u in explored part, followed by a single edge (u, v)



Dijkstra's algorithm

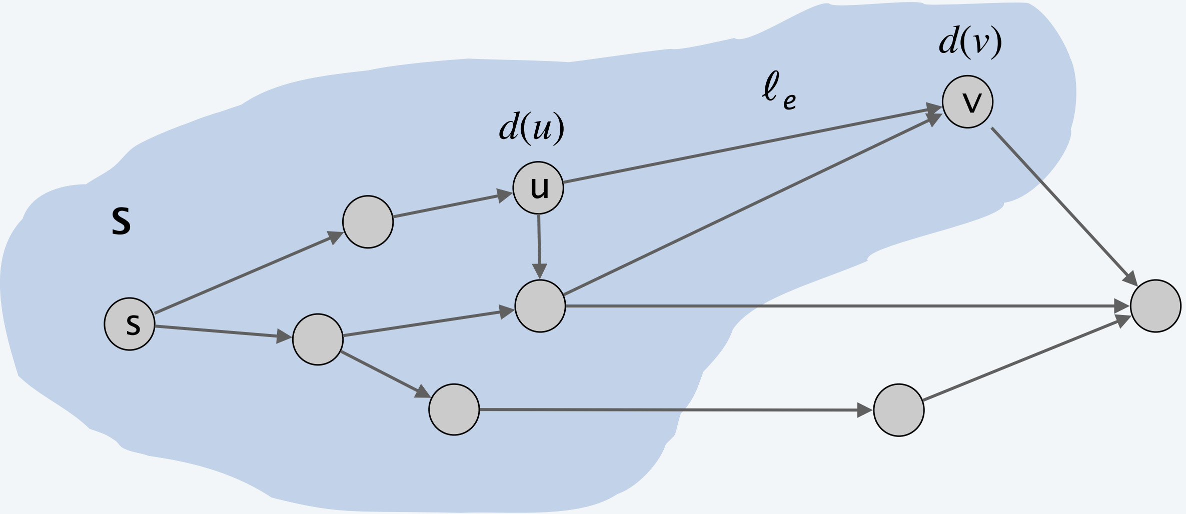
Greedy approach. Maintain a set of explored nodes S for which algorithm has determined the shortest path distance $d(u)$ from s to u.

- Initialize $S = \{ s \},\ d(s) = 0$.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v)\,:\,u \in S} d(u) + \ell_e\,,$$

add v to S, and set $d(v) = \pi(v)$.

shortest path to some node u in explored part, followed by a single edge (u, v)



Dijkstra's algorithm: proof of correctness

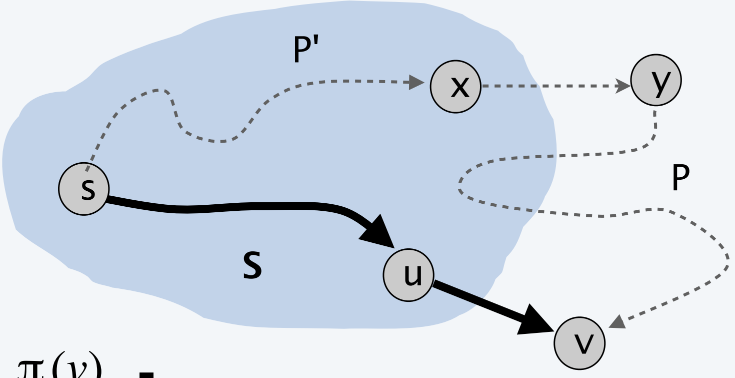
Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest $s \rightarrow u$ path.

Pf. [by induction on $|S|$]

Base case: $|S| = 1$ is easy since $S = \{ s \}$ and $d(s) = 0$.

Inductive hypothesis: Assume true for $|S| = k \geq 1$.

- Let v be next node added to S, and let (u, v) be the final edge.
- The shortest $s \rightarrow u$ path plus (u, v) is an $s \rightarrow v$ path of length $\pi(v)$.
- Consider any $s \rightarrow v$ path P. We show that it is no shorter than $\pi(v)$.
- Let (x, y) be the first edge in P that leaves S, and let P' be the subpath to x.
- P is already too long as soon as it reaches y.



$$\ell(P) \;\geq\; \ell(P') + \ell(x, y) \;\geq\; d(x) + \ell(x, y) \;\geq\; \pi(y) \;\geq\; \pi(v) \quad \blacksquare$$

↑ nonnegative lengths

↑ inductive hypothesis

↑ definition of π(y)

↑ Dijkstra chose v instead of y

Dijkstra's algorithm: efficient implementation

Critical optimization 1. For each unexplored node v, explicitly maintain $\pi(v)$ instead of computing directly from formula:

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e .$$

- For each $v \notin S$, $\pi(v)$ can only decrease (because S only increases).
- More specifically, suppose u is added to S and there is an edge (u, v) leaving u. Then, it suffices to update:

$$\pi(v) = \min \{ \pi(v), \ d(u) + \ell(u, v) \}$$

Critical optimization 2. Use a priority queue to choose the unexplored node that minimizes $\pi(v)$.

Dijkstra's algorithm: efficient implementation

Implementation.

- Algorithm stores $d(v)$ for each explored node v.
- Priority queue stores $\pi(v)$ for each unexplored node v.
- Recall: $d(u) = \pi(u)$ when u is deleted from priority queue.

```
DIJKSTRA ($V, E, s$)

Create an empty priority queue.
FOR EACH $v \neq s$ : $d(v) \leftarrow \infty$; $d(s) \leftarrow 0$.
FOR EACH $v \in V$ : insert $v$ with key $d(v)$ into priority queue.
WHILE (the priority queue is not empty)
    $u \leftarrow$ delete-min from priority queue.
    FOR EACH edge $(u, v) \in E$ leaving $u$:
        IF $d(v) > d(u) + \ell(u, v)$
            decrease-key of $v$ to $d(u) + \ell(u, v)$ in priority queue.
            $d(v) \leftarrow d(u) + \ell(u, v)$.
```

Dijkstra's algorithm: which priority queue?

Performance. Depends on PQ: n insert, n delete-min, m decrease-key.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci/Brodal best in theory, but not worth implementing.

PQ implementation	insert	delete-min	decrease-key	total
unordered array	$O(1)$	$O(n)$	$O(1)$	$O(n^2)$
binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(m \log n)$
d-way heap (Johnson 1975)	$O(d \log_d n)$	$O(d \log_d n)$	$O(\log_d n)$	$O(m \log_{m/n} n)$
Fibonacci heap (Fredman-Tarjan 1984)	$O(1)$	$O(\log n)$ †	$O(1)$ †	$O(m + n \log n)$
Brodal queue (Brodal 1996)	$O(1)$	$O(\log n)$	$O(1)$	$O(m + n \log n)$

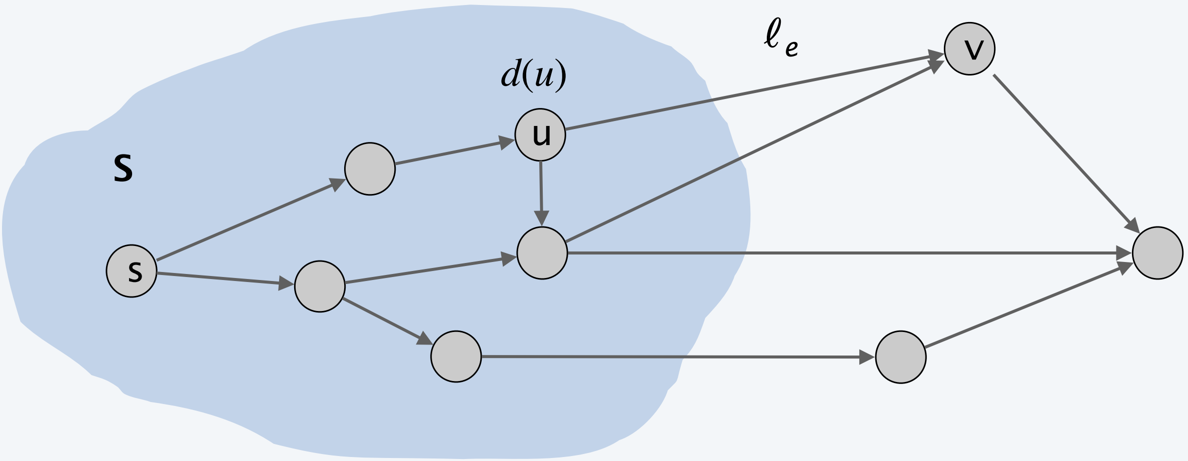
† amortized

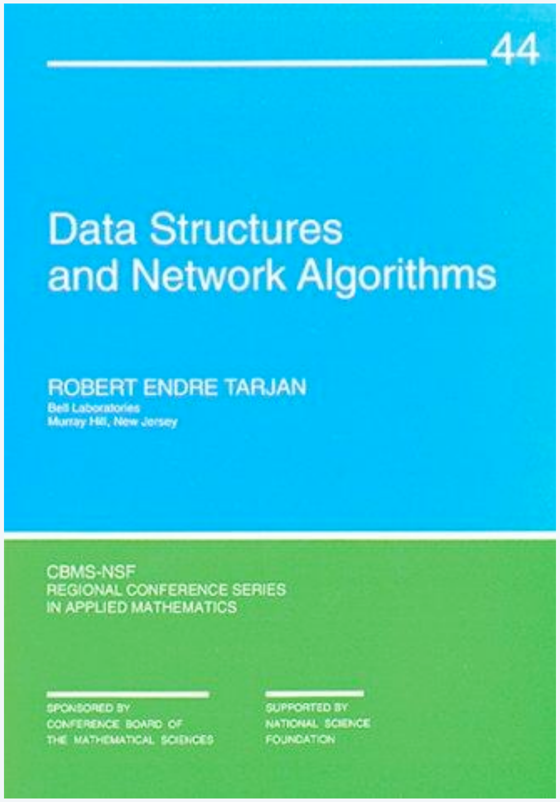
Extensions of Dijkstra's algorithm

Dijkstra's algorithm and proof extend to several related problems:

- Shortest paths in undirected graphs: $d(v) \le d(u) + \ell(u, v)$.
- Maximum capacity paths: $d(v) \ge \min\{\, \pi(u),\ c(u, v) \,\}$.
- Maximum reliability paths: $d(v) \ge d(u) \times \gamma(u, v)$.
- …

Key algebraic structure. Closed semiring (tropical, bottleneck, Viterbi).





SECTION 6.1

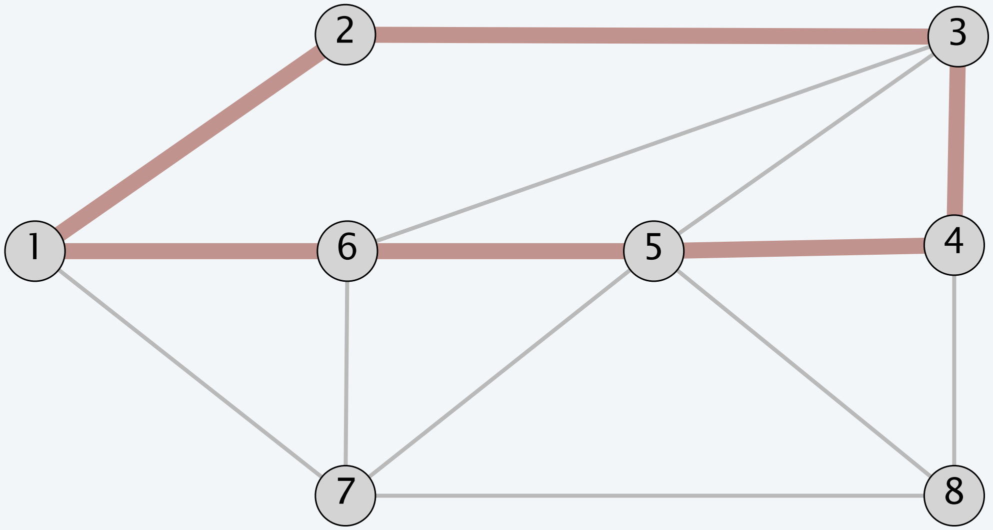
4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- *min-cost arborescences*

Cycles and cuts

Def. A path is a sequence of edges which connects a sequence of nodes.

Def. A cycle is a path with no repeated nodes or edges other than the starting and ending nodes.

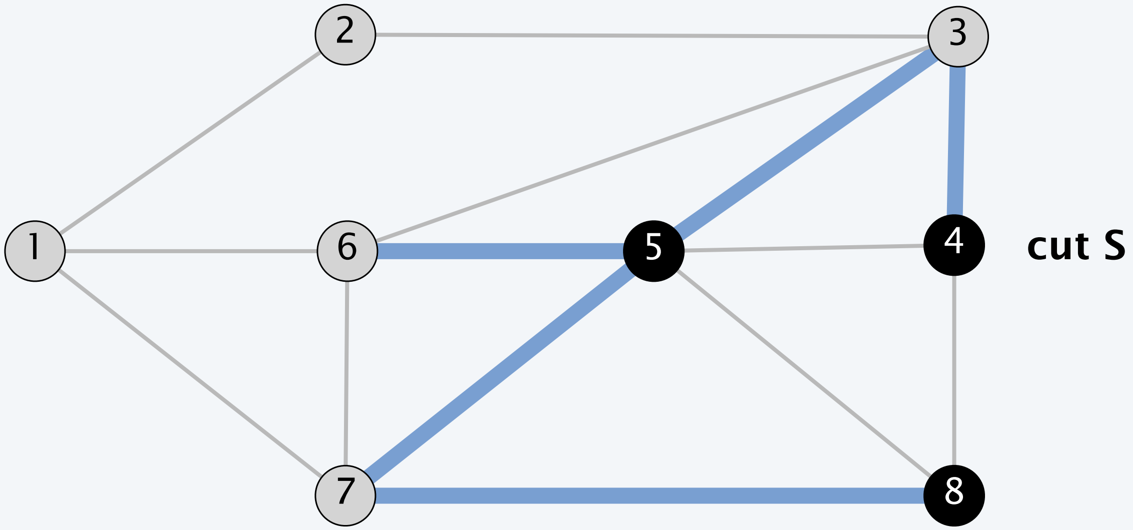


cycle C = { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1) }

Cycles and cuts

Def. A cut is a partition of the nodes into two nonempty subsets S and $V - S$.

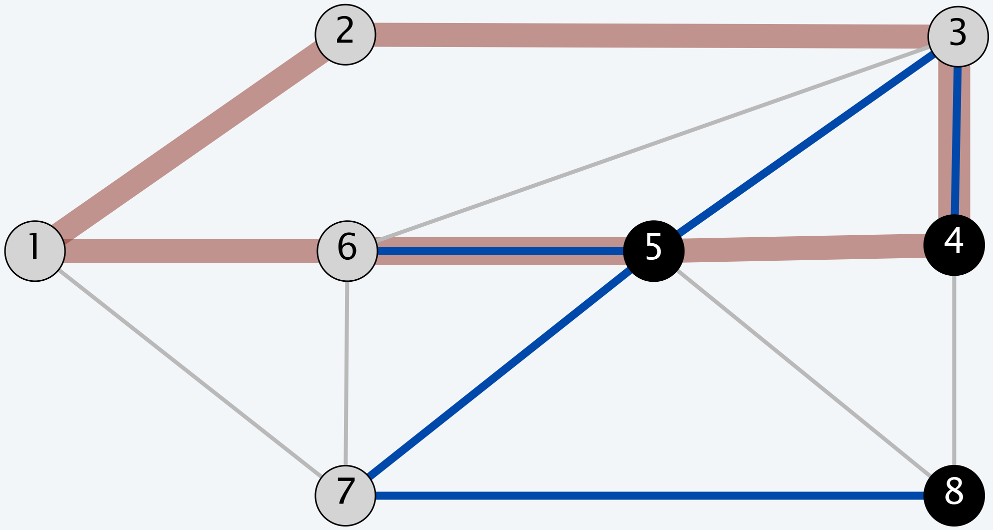
Def. The cutset of a cut S is the set of edges with exactly one endpoint in S.



cutset D = { (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) }

Cycle-cut intersection

Proposition. A cycle and a cutset intersect in an even number of edges.



cutset D = { (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) }

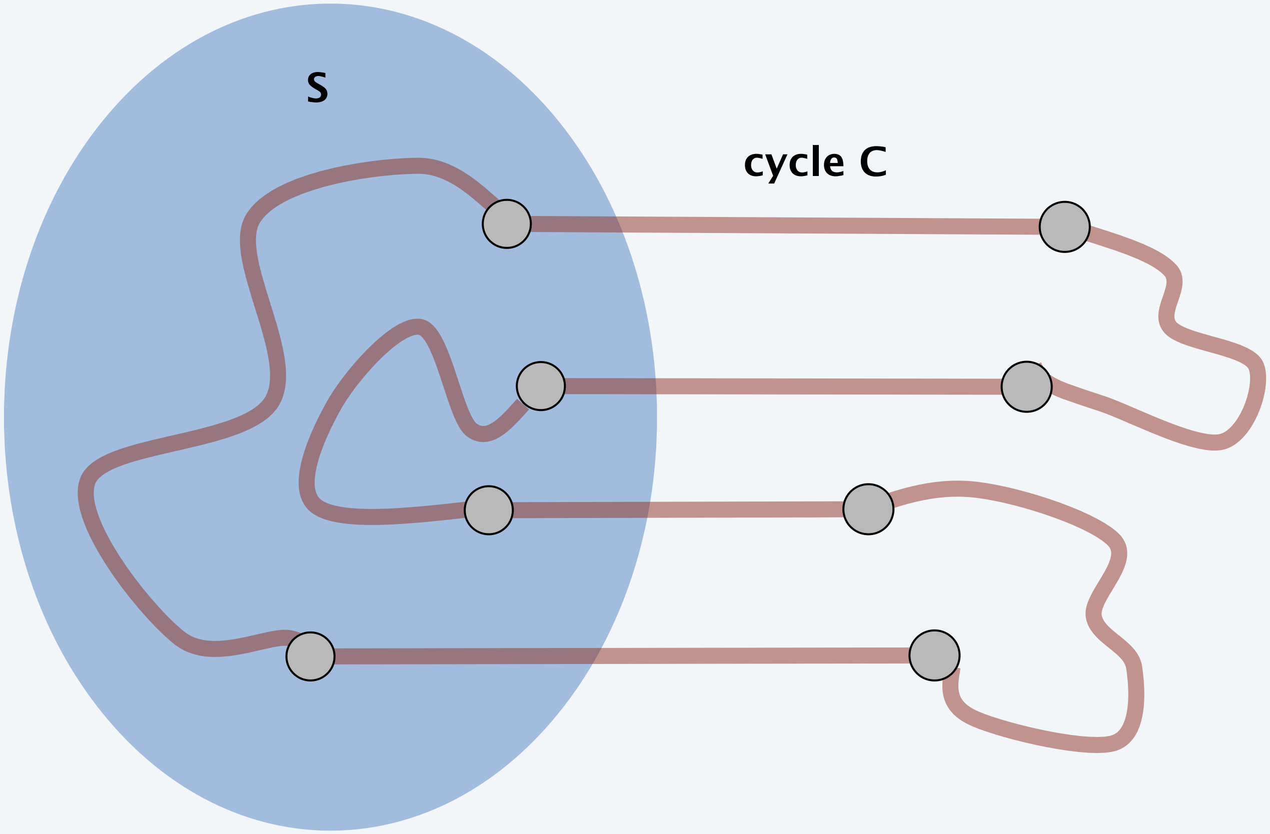
cycle C = { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1) }

intersection C ∩ D = { (3, 4), (5, 6) }

Cycle-cut intersection

Proposition. A cycle and a cutset intersect in an even number of edges.

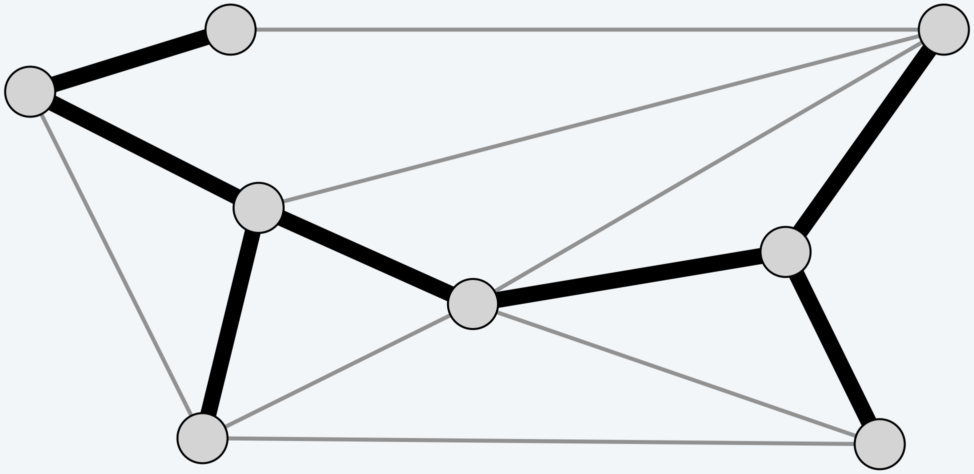
Pf. [by picture]



Spanning tree properties

Proposition. Let $T = (V, F)$ be a subgraph of $G = (V, E)$. TFAE:

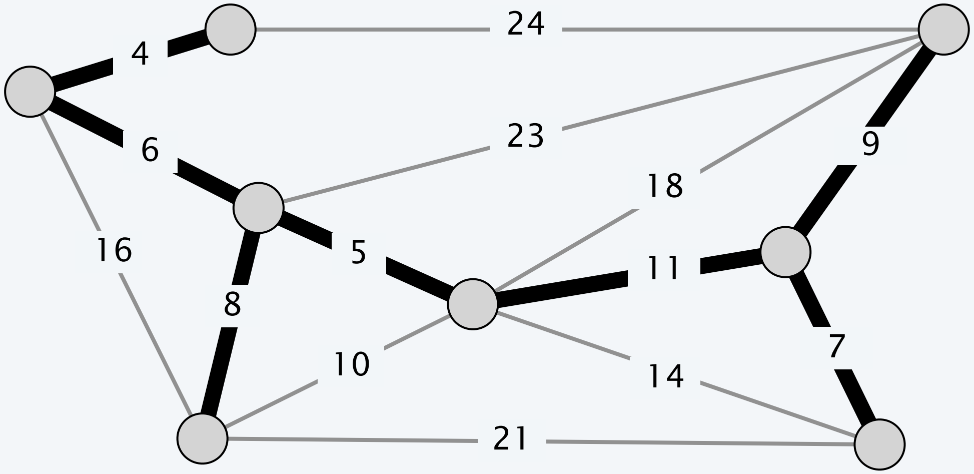
- T is a spanning tree of G.
- T is acyclic and connected.
- T is connected and has $n - 1$ edges.
- T is acyclic and has $n - 1$ edges.
- T is minimally connected: removal of any edge disconnects it.
- T is maximally acyclic: addition of any edge creates a cycle.
- T has a unique simple path between every pair of nodes.



T = (V, F)

Minimum spanning tree

Given a connected graph $G = (V, E)$ with edge costs c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge costs is minimized.



MST cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

Cayley's theorem. There are n^{n-2} spanning trees of K_n. ← can't solve by brute force

Applications

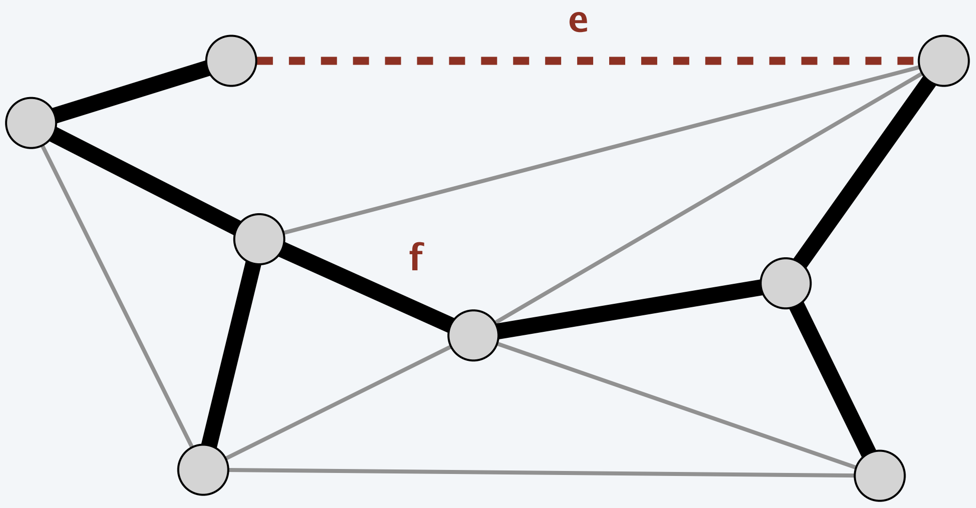
MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, computer, road).

Fundamental cycle

Fundamental cycle.

- Adding any non-tree edge e to a spanning tree T forms unique cycle C.
- Deleting any edge $f \in C$ from $T \cup \{ e \}$ results in new spanning tree.



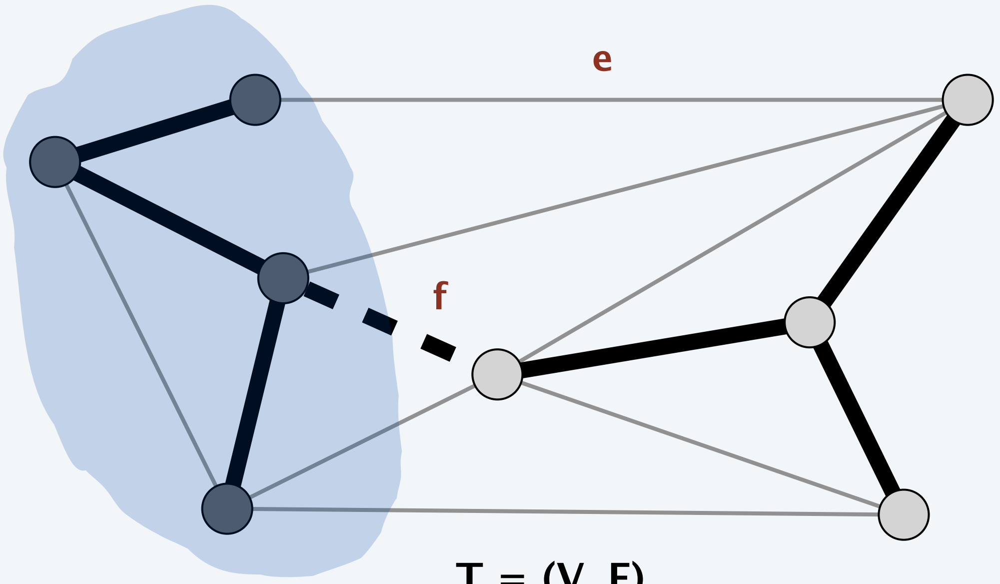
T = (V, F)

Observation. If $c_e < c_f$, then T is not an MST.

Fundamental cutset

Fundamental cutset.

- Deleting any tree edge f from a spanning tree T divide nodes into two connected components. Let D be cutset.
- Adding any edge $e \in D$ to $T - \{ f \}$ results in new spanning tree.



T = (V, F)

Observation. If $c_e < c_f$, then T is not an MST.

The greedy algorithm

Red rule.

- Let C be a cycle with no red edges.
- Select an uncolored edge of C of max weight and color it red.

Blue rule.

- Let D be a cutset with no blue edges.
- Select an uncolored edge in D of min weight and color it blue.

Greedy algorithm.

- Apply the red and blue rules (non-deterministically!) until all edges are colored. The blue edges form an MST.
- Note: can stop once $n - 1$ edges colored blue.

Greedy algorithm: proof of correctness

Color invariant. There exists an MST T^* containing all of the blue edges and none of the red edges.

Pf. [by induction on number of iterations]

Base case. No edges colored ⇒ every MST satisfies invariant.

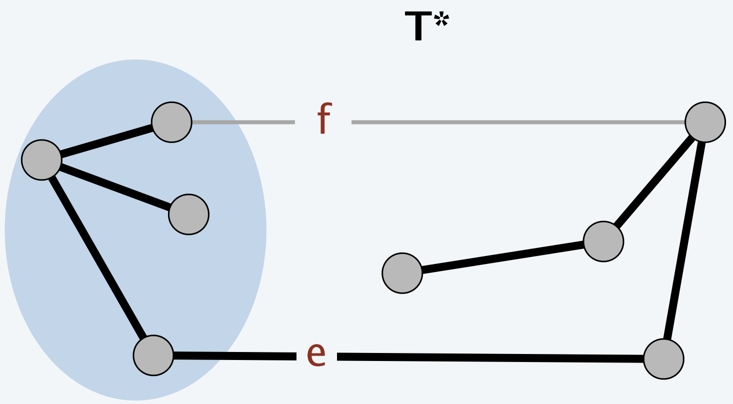
Greedy algorithm: proof of correctness

Color invariant. There exists an MST T^* containing all of the blue edges and none of the red edges.

Pf. [by induction on number of iterations]

Induction step (blue rule). Suppose color invariant true before blue rule.

- let D be chosen cutset, and let f be edge colored blue.
- if $f \in T^*$, T^* still satisfies invariant.
- Otherwise, consider fundamental cycle C by adding f to T^*.
- let $e \in C$ be another edge in D.
- e is uncolored and $c_e \geq c_f$ since
  - $e \in T^* \Rightarrow e$ not red
  - blue rule $\Rightarrow e$ not blue and $c_e \geq c_f$
- Thus, $T^* \cup \{ f \} - \{ e \}$ satisfies invariant.



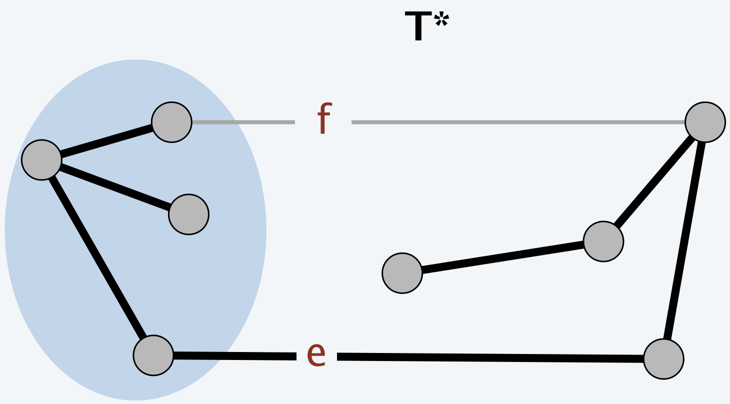
Greedy algorithm: proof of correctness

Color invariant. There exists an MST T^* containing all of the blue edges and none of the red edges.

Pf. [by induction on number of iterations]

Induction step (red rule). Suppose color invariant true before red rule.

- let C be chosen cycle, and let e be edge colored red.
- if $e \notin T^*$, T^* still satisfies invariant.
- Otherwise, consider fundamental cutset D by deleting e from T^*.
- let $f \in D$ be another edge in C.
- f is uncolored and $c_e \geq c_f$ since
  - $f \notin T^* \Rightarrow f$ not blue
  - red rule $\Rightarrow f$ not red and $c_e \geq c_f$
- Thus, $T^* \cup \{ f \} - \{ e \}$ satisfies invariant. ▪



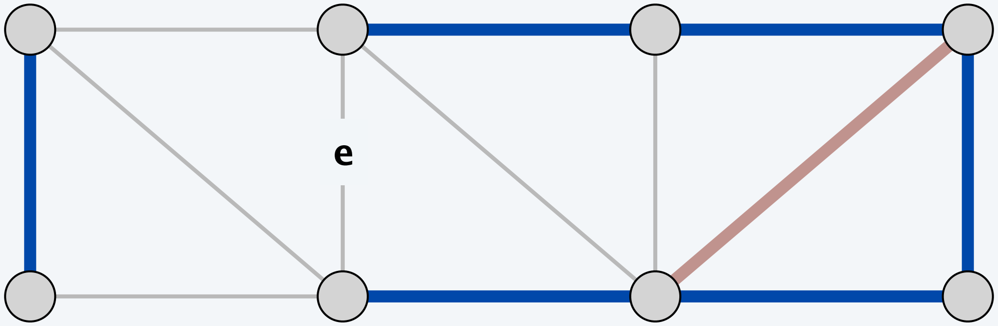
Greedy algorithm: proof of correctness

Theorem. The greedy algorithm terminates. Blue edges form an MST.

Pf. We need to show that either the red or blue rule (or both) applies.

- Suppose edge e is left uncolored.
- Blue edges form a forest.
- Case 1: both endpoints of e are in same blue tree.

  $\Rightarrow$ apply red rule to cycle formed by adding e to blue forest.



Case 1

Greedy algorithm: proof of correctness

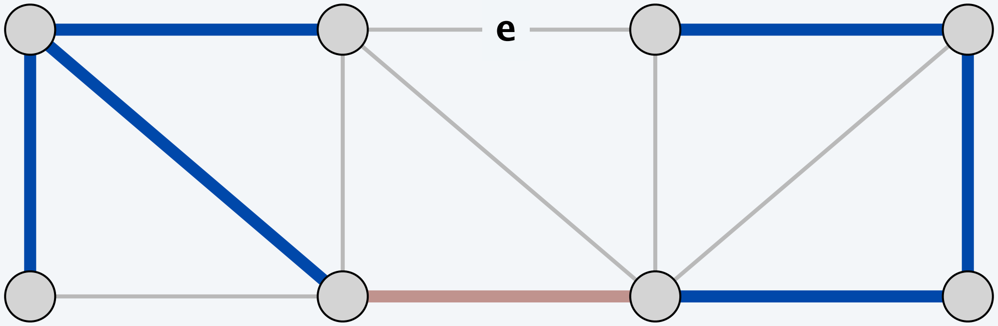
Theorem. The greedy algorithm terminates. Blue edges form an MST.

Pf. We need to show that either the red or blue rule (or both) applies.

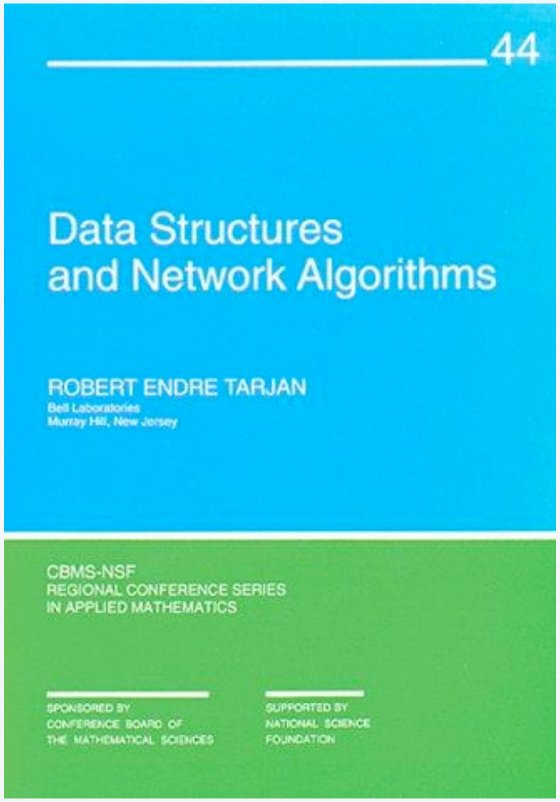
- Suppose edge e is left uncolored.
- Blue edges form a forest.
- Case 1: both endpoints of e are in same blue tree.

  ⇒ apply red rule to cycle formed by adding e to blue forest.
- Case 2: both endpoints of e are in different blue trees.

  ⇒ apply blue rule to cutset induced by either of two blue trees. ▪



Case 2



SECTION 6.2

4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- *min-cost arborescences*

Prim's algorithm

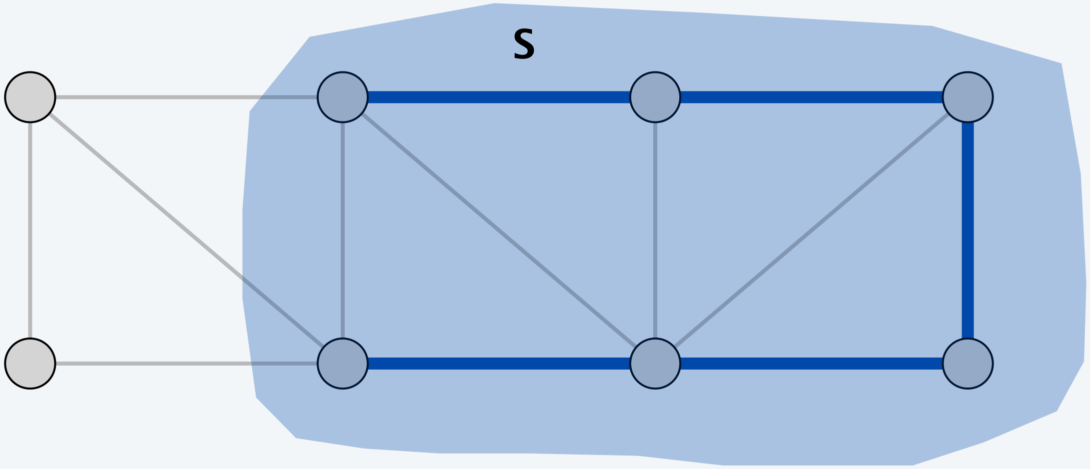
Initialize S = any node.

Repeat $n - 1$ times:

- Add to tree the min weight edge with one endpoint in S.
- Add new node to S.

Theorem. Prim's algorithm computes the MST.

Pf. Special case of greedy algorithm (blue rule repeatedly applied to S). ▪



Prim's algorithm: implementation

Theorem. Prim's algorithm can be implemented in $O(m \log n)$ time.

Pf. Implementation almost identical to Dijkstra's algorithm.

[$d(v)$ = weight of cheapest known edge between v and S]

```
PRIM $(V, E, c)$

Create an empty priority queue.
$s \leftarrow$ any node in $V$.
FOR EACH $v \neq s$ : $d(v) \leftarrow \infty$; $d(s) \leftarrow 0$.
FOR EACH $v$ : insert $v$ with key $d(v)$ into priority queue.
WHILE (the priority queue is not empty)
    $u \leftarrow$ delete-min from priority queue.
    FOR EACH edge $(u, v) \in E$ incident to $u$:
        IF $d(v) > c(u, v)$
            decrease-key of $v$ to $c(u, v)$ in priority queue.
            $d(v) \leftarrow c(u, v)$.
```

Kruskal's algorithm

Consider edges in ascending order of weight:

- Add to tree unless it would create a cycle.

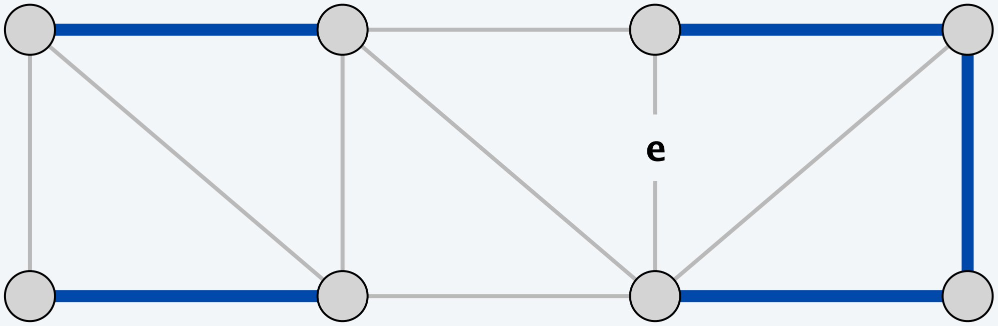
Theorem. Kruskal's algorithm computes the MST.

Pf. Special case of greedy algorithm.

- Case 1: both endpoints of e in same blue tree.
  - $\Rightarrow$ color red by applying red rule to unique cycle.
- Case 2. If both endpoints of e are in different blue trees.
  - $\Rightarrow$ color blue by applying blue rule to cutset defined by either tree. ▪

all other edges in cycle are blue

no edge in cutset has smaller weight
(since Kruskal chose it first)



Kruskal's algorithm: implementation

Theorem. Kruskal's algorithm can be implemented in $O(m \log m)$ time.

- Sort edges by weight.
- Use union-find data structure to dynamically maintain connected components.

```
KRUSKAL (V, E, c)
SORT m edges by weight so that $c(e_1) \leq c(e_2) \leq \ldots \leq c(e_m)$
$S \leftarrow \phi$
FOREACH $v \in V$: MAKESET(v).
FOR i = 1 TO m
    $(u, v) \leftarrow e_i$
    IF FINDSET(u) ≠ FINDSET(v)        ← are u and v in same component?
        $S \leftarrow S \cup \{ e_i \}$
        UNION(u, v).                 ← make u and v in same component
RETURN S
```

Reverse-delete algorithm

Consider edges in descending order of weight:

- Remove edge unless it would disconnect the graph.

Theorem. The reverse-delete algorithm computes the MST.

Pf. Special case of greedy algorithm.

- Case 1: removing edge e does not disconnect graph.

  ⇒ apply red rule to cycle C formed by adding e to existing path between its two endpoints

  any edge in C with larger weight would have been deleted when considered

- Case 2: removing edge e disconnects graph.

  ⇒ apply blue rule to cutset D induced by either component. ▪

  e is the only edge in the cutset (any other edges must have been colored red / deleted)

Fact. [Thorup 2000] Can be implemented in $O(m \log n (\log \log n)^3)$ time.

Review: the greedy MST algorithm

Red rule.

- Let C be a cycle with no red edges.
- Select an uncolored edge of C of max weight and color it red.

Blue rule.

- Let D be a cutset with no blue edges.
- Select an uncolored edge in D of min weight and color it blue.

Greedy algorithm.

- Apply the red and blue rules (non-deterministically!) until all edges are colored. The blue edges form an MST.
- Note: can stop once $n - 1$ edges colored blue.

Theorem. The greedy algorithm is correct.

Special cases. Prim, Kruskal, reverse-delete, ...

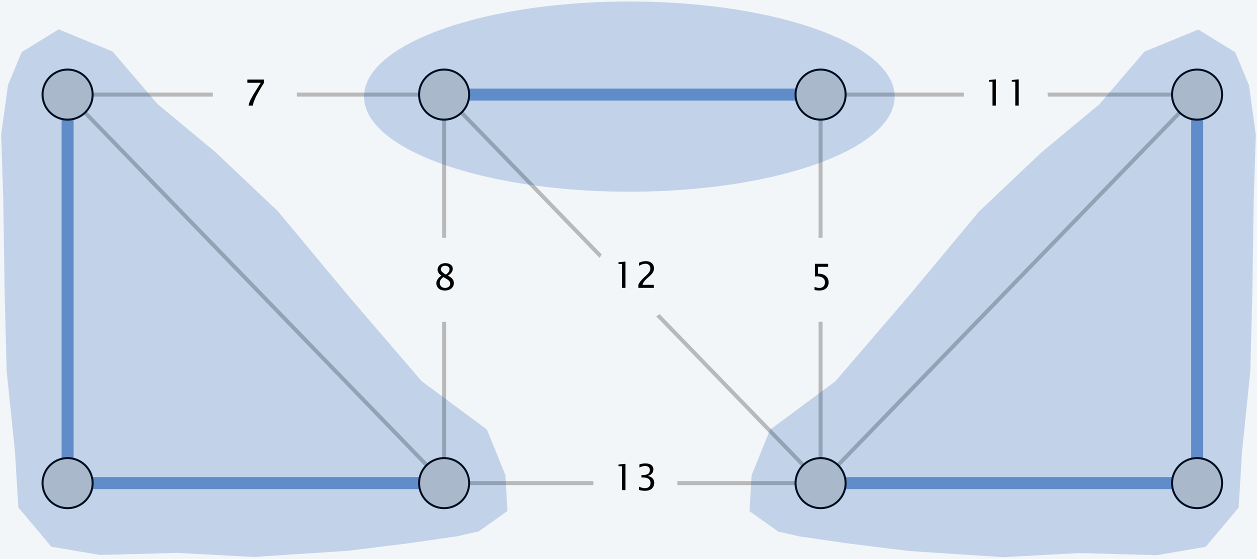
Borůvka's algorithm

Repeat until only one tree.

- Apply blue rule to cutset corresponding to each blue tree.
- Color all selected edges blue.

Theorem. Borůvka's algorithm computes the MST. ← assume edge costs are distinct

Pf. Special case of greedy algorithm (repeatedly apply blue rule). ▪

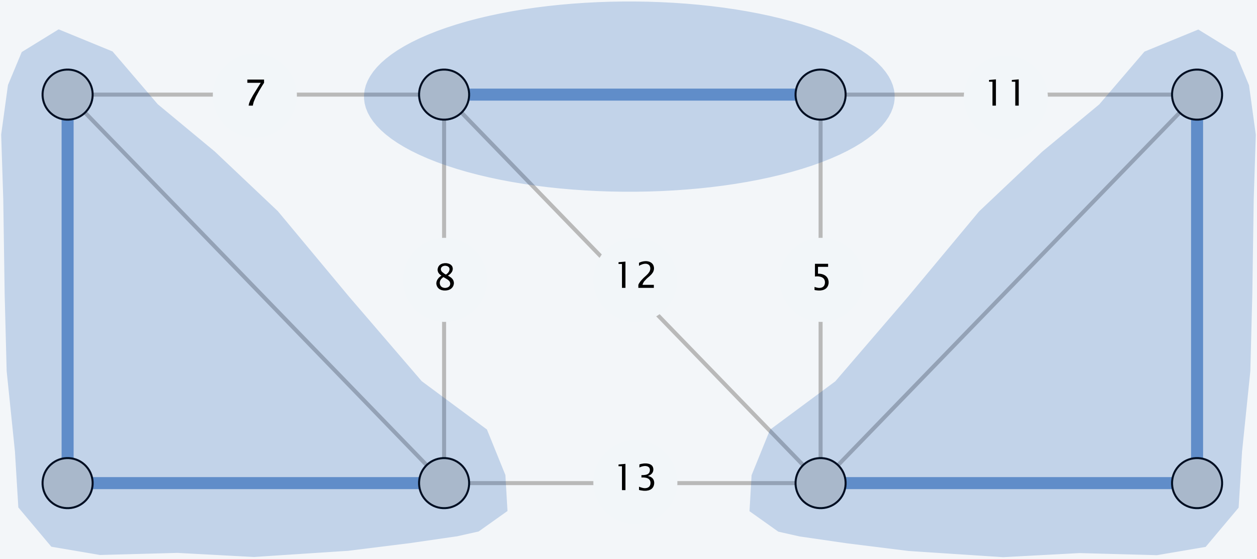


Borůvka's algorithm: implementation

Theorem. Borůvka's algorithm can be implemented in $O(m \log n)$ time.

Pf.

- To implement a phase in $O(m)$ time:
  - compute connected components of blue edges
  - for each edge $(u, v) \in E$, check if u and v are in different components; if so, update each component's best edge in cutset
- At most $\log_2 n$ phases since each phase (at least) halves total # trees. ▪

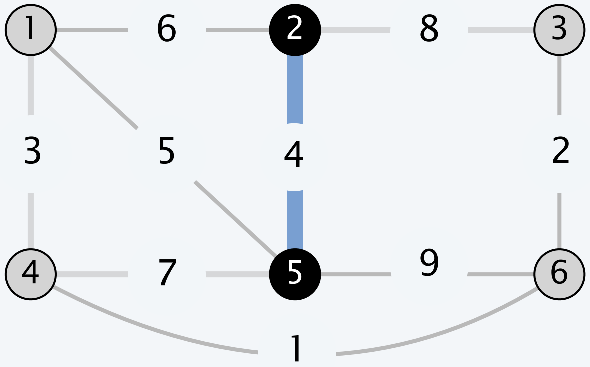


Borůvka's algorithm: implementation

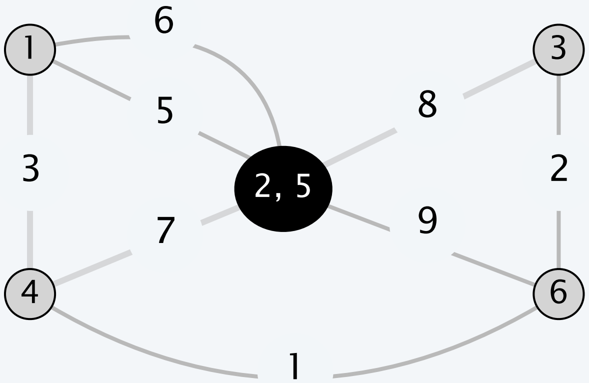
Node contraction version.

- After each phase, contract each blue tree to a single supernode.
- Delete parallel edges (keeping only cheapest one) and self loops.
- Borůvka phase becomes: take cheapest edge incident to each node.

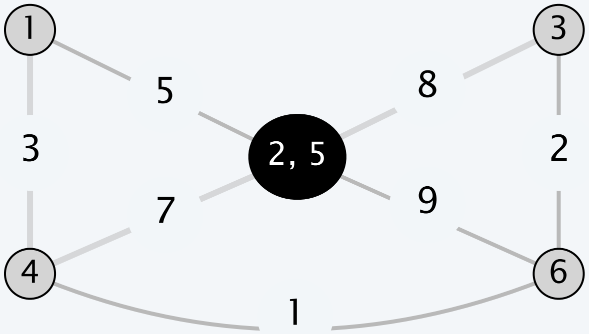
graph G



contract nodes 2 and 5



delete parallel edges and self loops

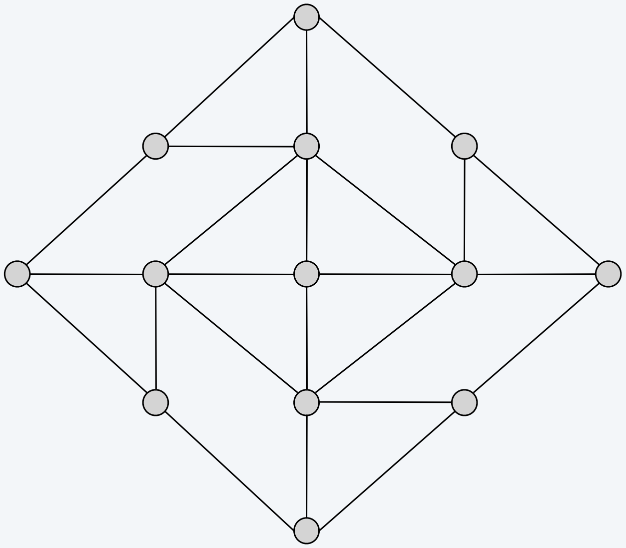


Borůvka's algorithm on planar graphs

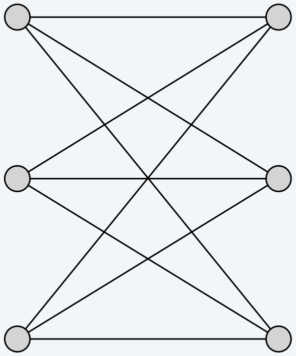
Theorem. Borůvka's algorithm runs in $O(n)$ time on planar graphs.

Pf.

- To implement a Borůvka phase in $O(n)$ time:
  - use contraction version of algorithm
  - in planar graphs, $m \leq 3n - 6$.
  - graph stays planar when we contract a blue tree
- Number of nodes (at least) halves.
- At most $\log_2 n$ phases: $cn + cn/2 + cn/4 + cn/8 + \ldots = O(n)$. ▪



planar



not planar

Borůvka-Prim algorithm

Borůvka-Prim algorithm.

- Run Borůvka (contraction version) for $\log_2 \log_2 n$ phases.
- Run Prim on resulting, contracted graph.

Theorem. The Borůvka-Prim algorithm computes an MST and can be implemented in $O(m \log \log n)$ time.

Pf.

- Correctness: special case of the greedy algorithm.
- The $\log_2 \log_2 n$ phases of Borůvka's algorithm take $O(m \log \log n)$ time; resulting graph has at most $n / \log_2 n$ nodes and m edges.
- Prim's algorithm (using Fibonacci heaps) takes $O(m + n)$ time on a graph with $n / \log_2 n$ nodes and m edges. ▪

$$O\left(m + \frac{n}{\log n} \log\left(\frac{n}{\log n}\right)\right)$$

Does a linear-time MST algorithm exist?

deterministic compare-based MST algorithms

year	worst case	discovered by
1975	$O(m \log \log n)$	Yao
1976	$O(m \log \log n)$	Cheriton-Tarjan
1984	$O(m \log^* n)$ $O(m + n \log n)$	Fredman-Tarjan
1986	$O(m \log (\log^* n))$	Gabow-Galil-Spencer-Tarjan
1997	$O(m\, \alpha(n) \log \alpha(n))$	Chazelle
2000	$O(m\, \alpha(n))$	Chazelle
2002	*optimal*	Pettie-Ramachandran
20xx	$O(m)$	???

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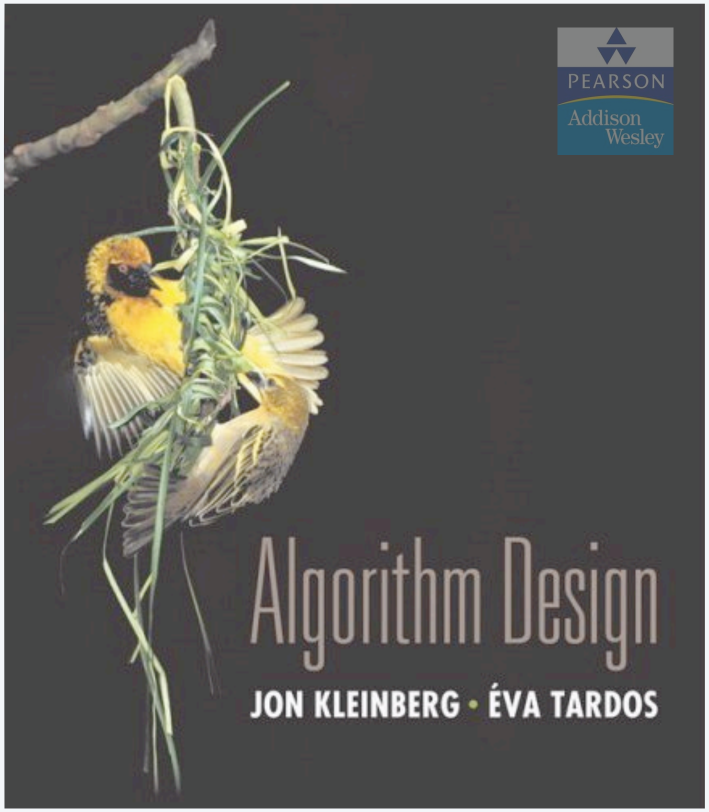
Remark 1. $O(m)$ randomized MST algorithm. [Karger-Klein-Tarjan 1995]

Remark 2. $O(m)$ MST verification algorithm. [Dixon-Rauch-Tarjan 1992]



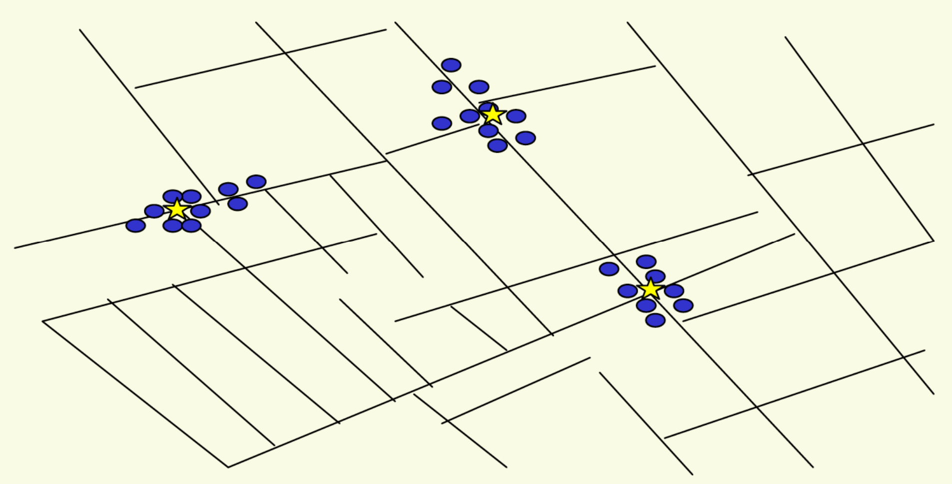
SECTION 4.7

4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- *min-cost arborescences*

Clustering

Goal. Given a set U of n objects labeled $p_1, \ldots, p_n$, partition into clusters so that objects in different clusters are far apart.



outbreak of cholera deaths in London in 1850s (Nina Mishra)

Applications.

- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.
- ...

Clustering of maximum spacing

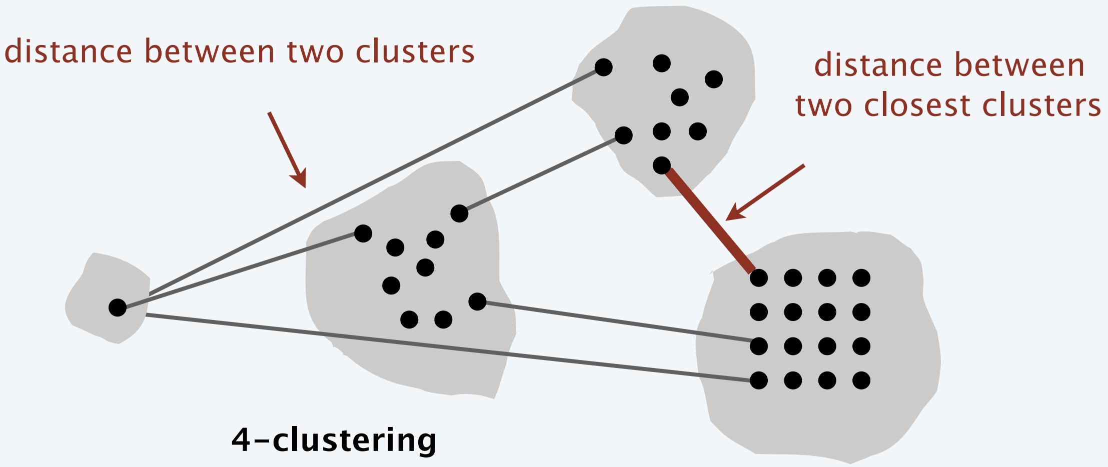
k-clustering. Divide objects into k non-empty groups.

Distance function. Numeric value specifying "closeness" of two objects.

- $d(p_i, p_j) = 0$ iff $p_i = p_j$ [identity of indiscernibles]
- $d(p_i, p_j) \geq 0$ [nonnegativity]
- $d(p_i, p_j) = d(p_j, p_i)$ [symmetry]

Spacing. Min distance between any pair of points in different clusters.

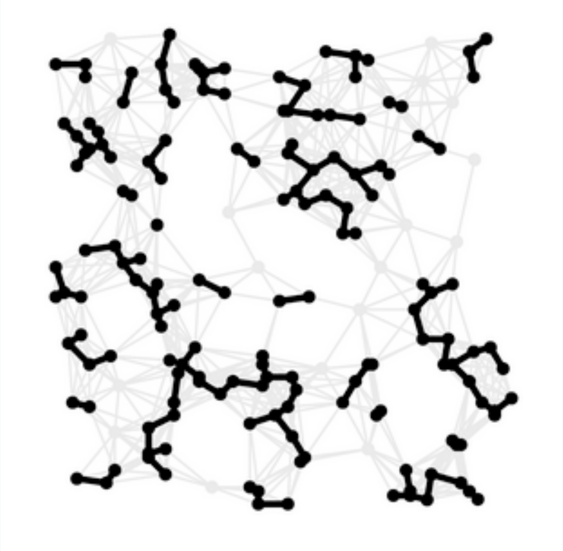
Goal. Given an integer k, find a k-clustering of maximum spacing.



Greedy clustering algorithm

"Well-known" algorithm in science literature for single-linkage k-clustering:

- Form a graph on the node set U, corresponding to n clusters.
- Find the closest pair of objects such that each object is in a different cluster, and add an edge between them.
- Repeat $n - k$ times until there are exactly k clusters.



Key observation. This procedure is precisely Kruskal's algorithm (except we stop when there are k connected components).

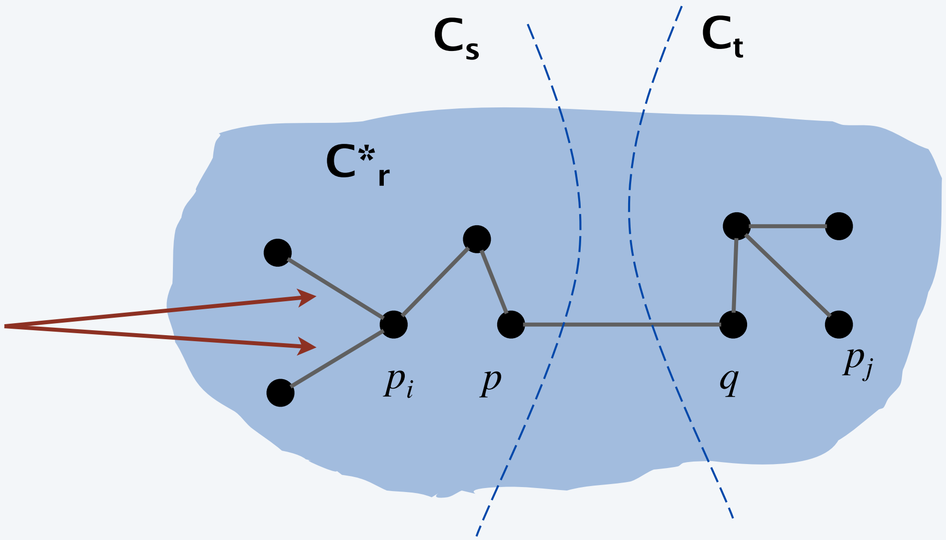
Alternative. Find an MST and delete the $k - 1$ longest edges.

Greedy clustering algorithm: analysis

Theorem. Let C^* denote the clustering $C^*_1, \ldots, C^*_k$ formed by deleting the $k-1$ longest edges of an MST. Then, C^* is a k-clustering of max spacing.

Pf. Let C denote some other clustering $C_1, \ldots, C_k$.

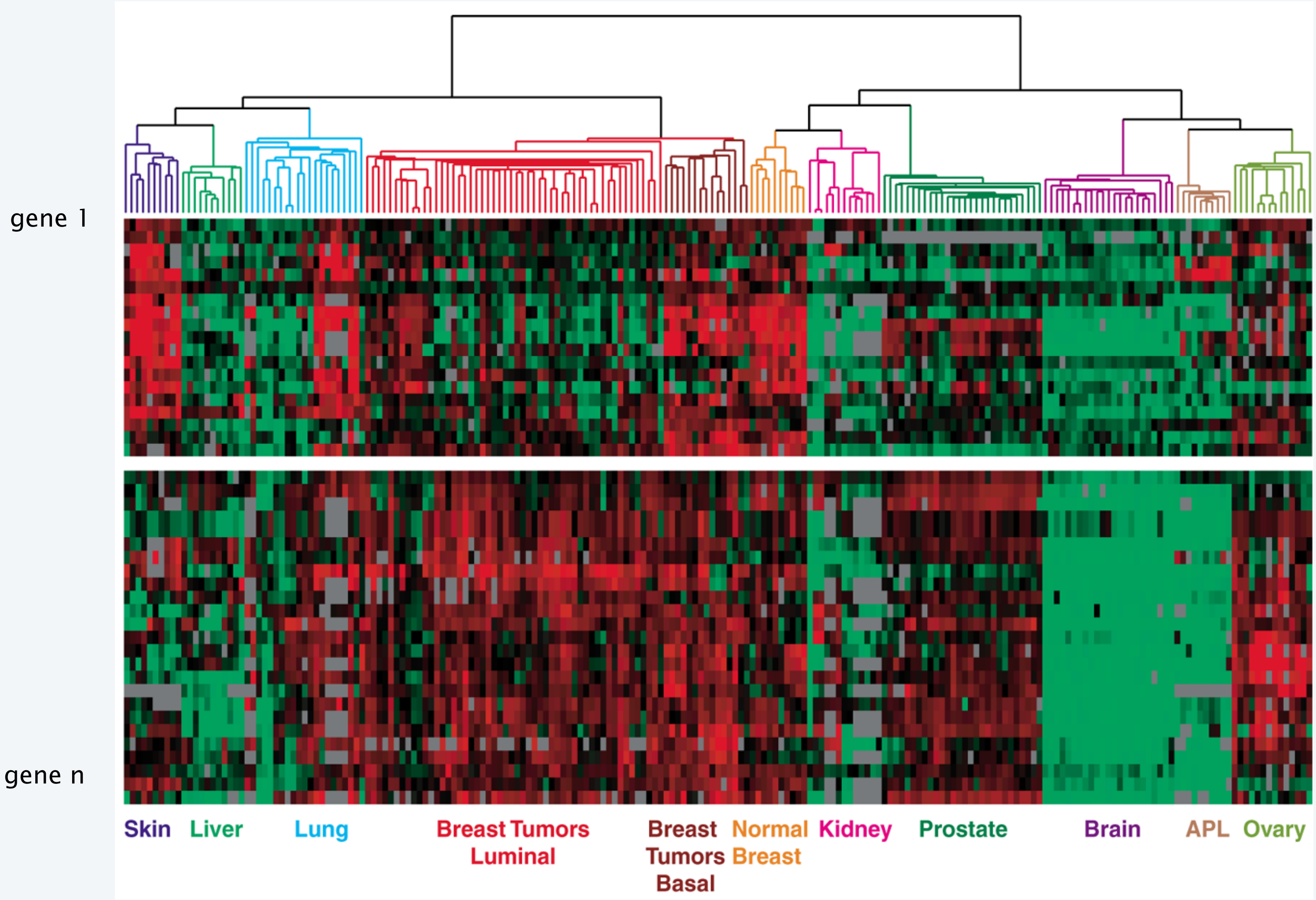
- The spacing of C^* is the length d^* of the $(k-1)^{\text{st}}$ longest edge in MST.
- Let p_i and p_j be in the same cluster in C^*, say C^*_r, but different clusters in C, say C_s and C_t.
- Some edge (p, q) on $p_i - p_j$ path in C^*_r spans two different clusters in C.
- Edge (p, q) has length $\leq d^*$ since it wasn't deleted.
- Spacing of C is $\leq d^*$ since p and q are in different clusters. ▪



edges left after deleting
k – 1 longest edges
from a MST

Dendrogram of cancers in human

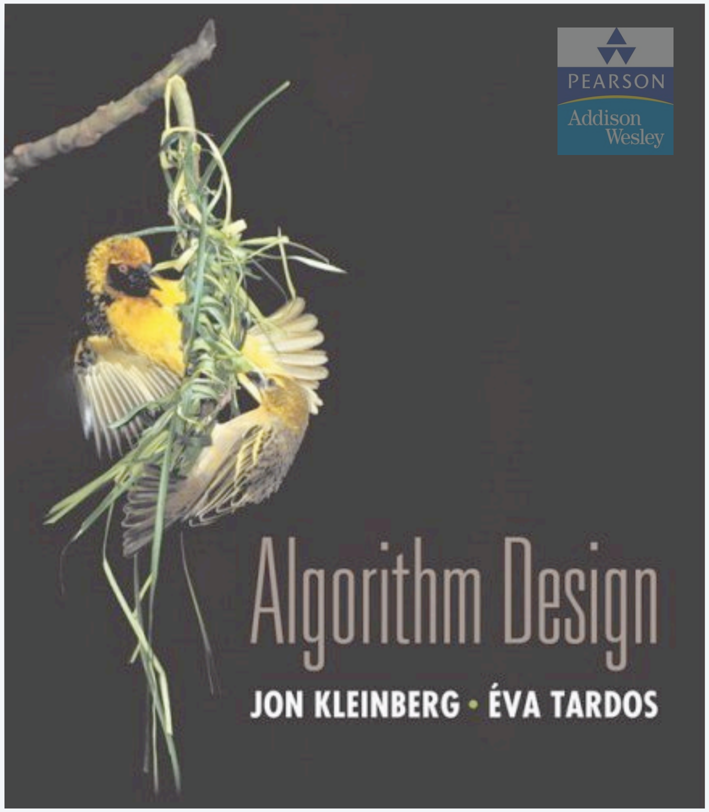
Tumors in similar tissues cluster together.



Reference: Botstein & Brown group

🟥 gene expressed

🟩 gene not expressed



SECTION 4.9

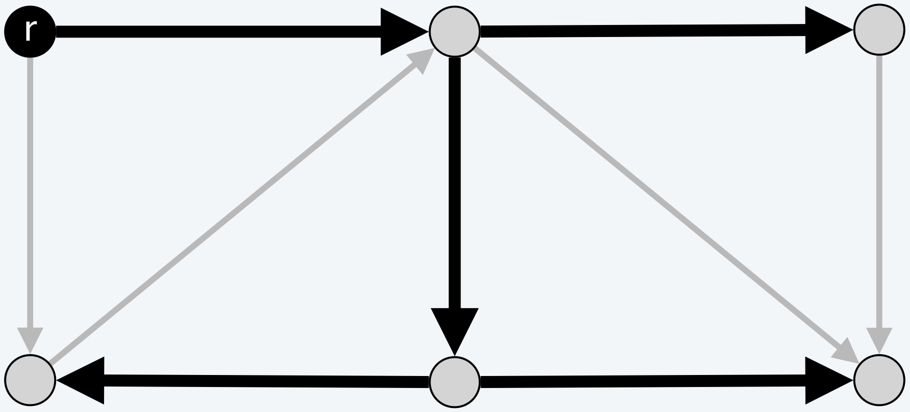
4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- *min-cost arborescences*

Arborescences

Def. Given a digraph $G = (V, E)$ and a root $r \in V$, an arborescence (rooted at r) is a subgraph $T = (V, F)$ such that

- T is a spanning tree of G if we ignore the direction of edges.
- There is a directed path in T from r to each other node $v \in V$.



Warmup. Given a digraph G, find an arborescence rooted at r (if one exists).

Algorithm. BFS or DFS from r is an arborescence (iff all nodes reachable).

Arborescences

Def. Given a digraph $G = (V, E)$ and a root $r \in V$, an arborescence (rooted at r) is a subgraph $T = (V, F)$ such that

- T is a spanning tree of G if we ignore the direction of edges.
- There is a directed path in T from r to each other node $v \in V$.

Proposition. A subgraph $T = (V, F)$ of G is an arborescence rooted at r iff T has no directed cycles and each node $v \neq r$ has exactly one entering edge.

Pf.

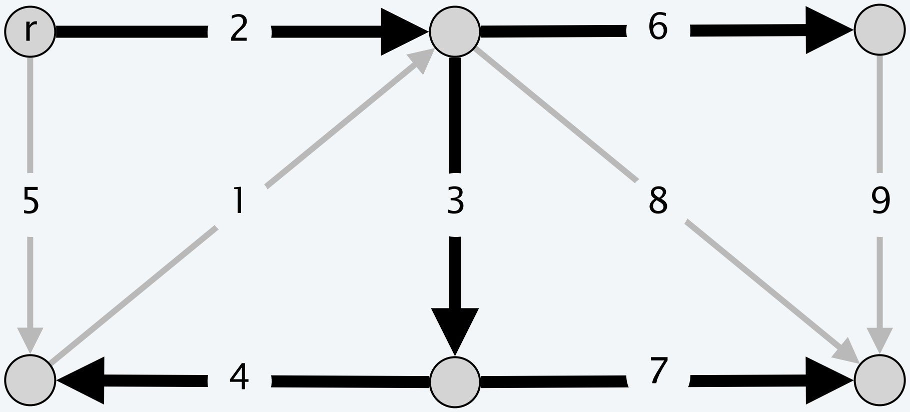
⇒ If T is an arborescence, then no (directed) cycles and every node $v \neq r$ has exactly one entering edge—the last edge on the unique $r \rightarrow v$ path.

⇐ Suppose T has no cycles and each node $v \neq r$ has one entering edge.

- To construct an $r \rightarrow v$ path, start at v and repeatedly follow edges in the backward direction.
- Since T has no directed cycles, the process must terminate.
- It must terminate at r since r is the only node with no entering edge. ▪

Min-cost arborescence problem

Problem. Given a digraph G with a root node r and with a nonnegative cost $c_e \geq 0$ on each edge e, compute an arborescence rooted at r of minimum cost.



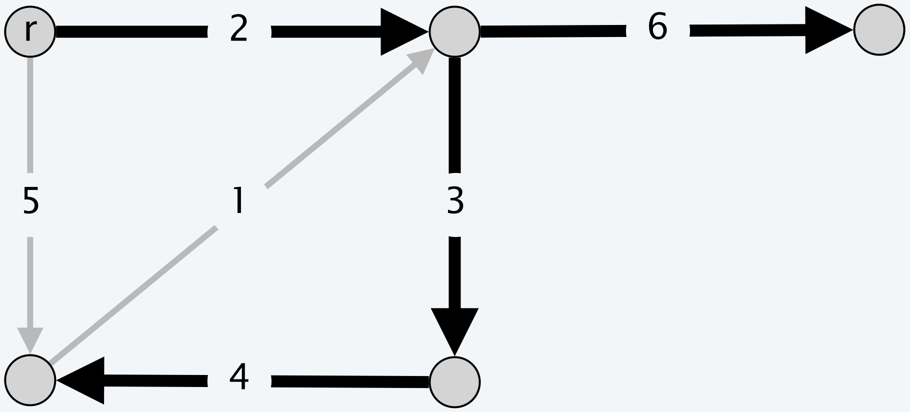
Assumption 1. G has an arborescence rooted at r.

Assumption 2. No edge enters r (safe to delete since they won't help).

Simple greedy approaches do not work

Observations. A min-cost arborescence need not:

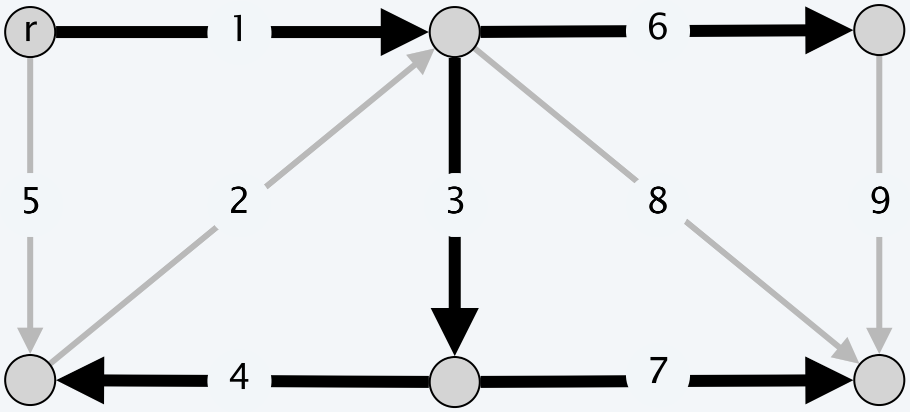
- Be a shortest-paths tree.
- Include the cheapest edge (in some cut).
- Exclude the most expensive edge (in some cycle).



A sufficient optimality condition

Property. For each node $v \neq r$, choose one cheapest edge entering v and let F^* denote this set of $n - 1$ edges. If (V, F^*) is an arborescence, then it is a min-cost arborescence.

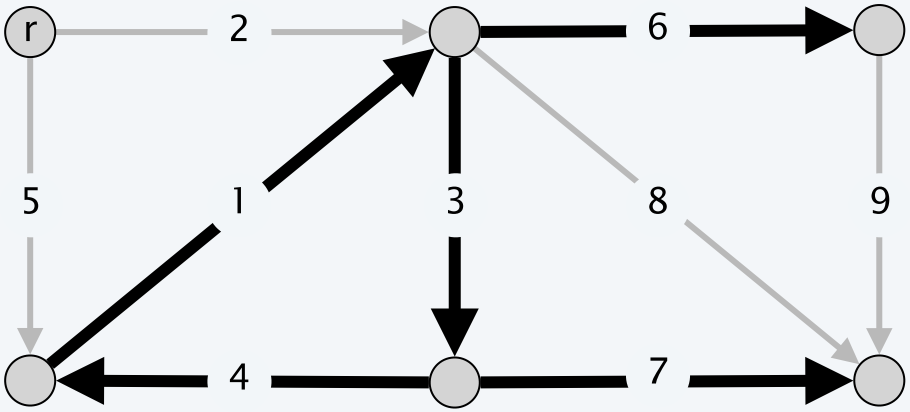
Pf. An arborescence needs exactly one edge entering each node $v \neq r$ and (V, F^*) is the cheapest way to make these choices. ▪



A sufficient optimality condition

Property. For each node $v \neq r$, choose one cheapest edge entering v and let F^* denote this set of $n - 1$ edges. If (V, F^*) is an arborescence, then it is a min-cost arborescence.

Note. F^* may not be an arborescence (since it may have directed cycles).

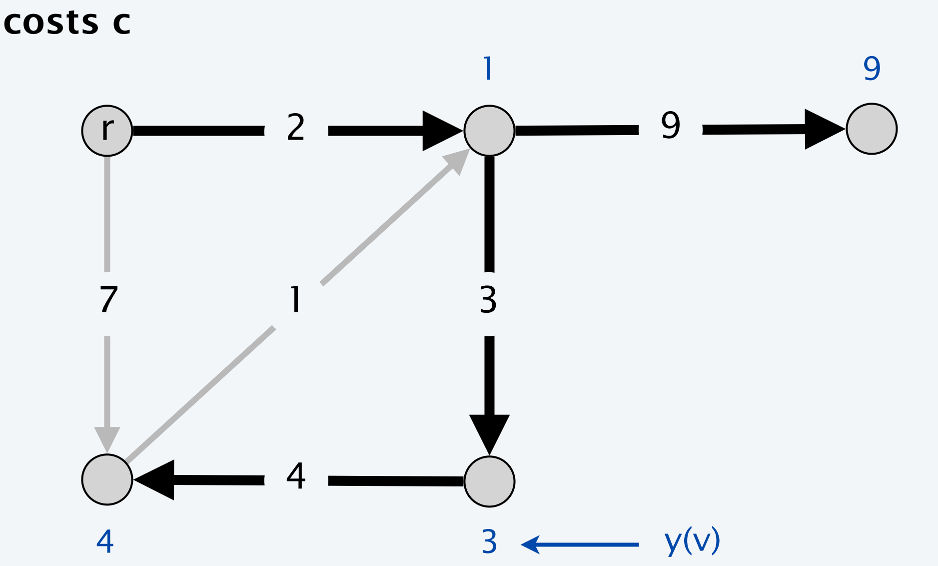


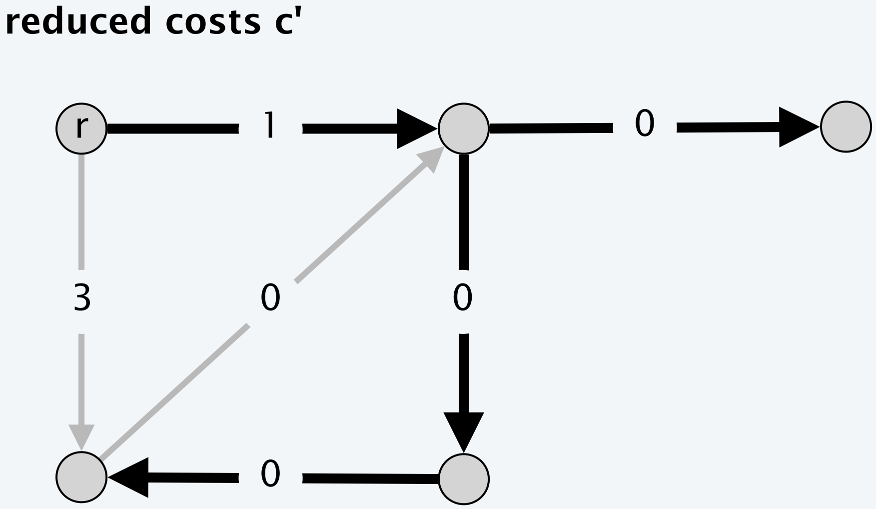
Reduced costs

Def. For each $v \neq r$, let $y(v)$ denote the min cost of any edge entering v.
The reduced cost of an edge (u, v) is $c'(u, v) = c(u, v) - y(v) \ \geq 0$.

Observation. T is a min-cost arborescence in G using costs c iff
T is a min-cost arborescence in G using reduced costs c'.

Pf. Each arborescence has exactly one edge entering v.

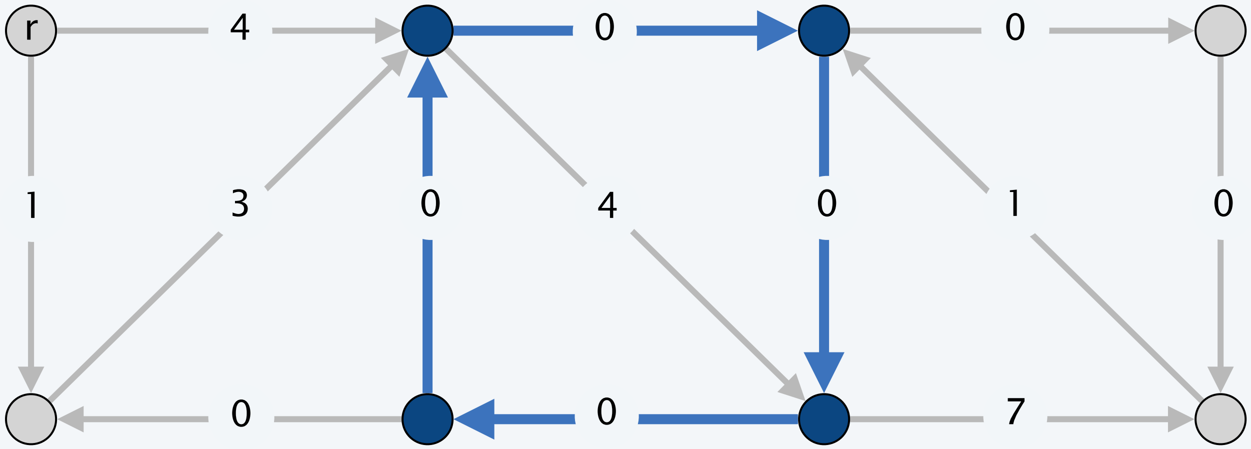




Edmonds branching algorithm: intuition

Intuition. Recall F^* = set of cheapest edges entering v for each $v \neq r$.

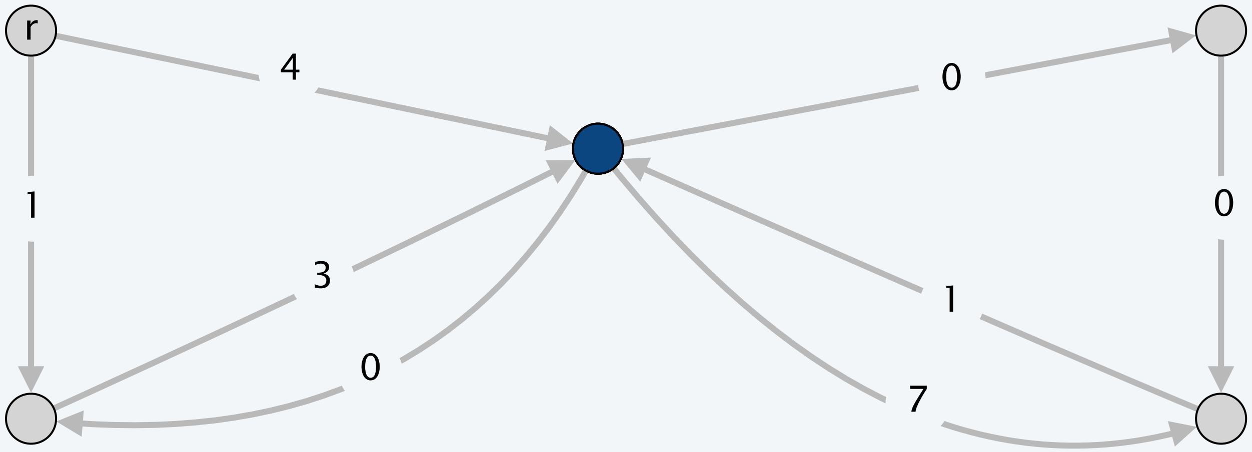
- Now, all edges in F^* have 0 cost with respect to costs $c'(u, v)$.
- If F^* does not contain a cycle, then it is a min-cost arborescence.
- If F^* contains a cycle C, can afford to use as many edges in C as desired.
- Contract nodes in C to a supernode.
- Recursively solve problem in contracted network G' with costs $c'(u, v)$.



Edmonds branching algorithm: intuition

Intuition. Recall F^* = set of cheapest edges entering v for each $v \neq r$.

- Now, all edges in F^* have 0 cost with respect to costs $c'(u, v)$.
- If F^* does not contain a cycle, then it is a min-cost arborescence.
- If F^* contains a cycle C, can afford to use as many edges in C as desired.
- Contract nodes in C to a supernode (removing any self-loops).
- Recursively solve problem in contracted network G' with costs $c'(u, v)$.



Edmonds branching algorithm

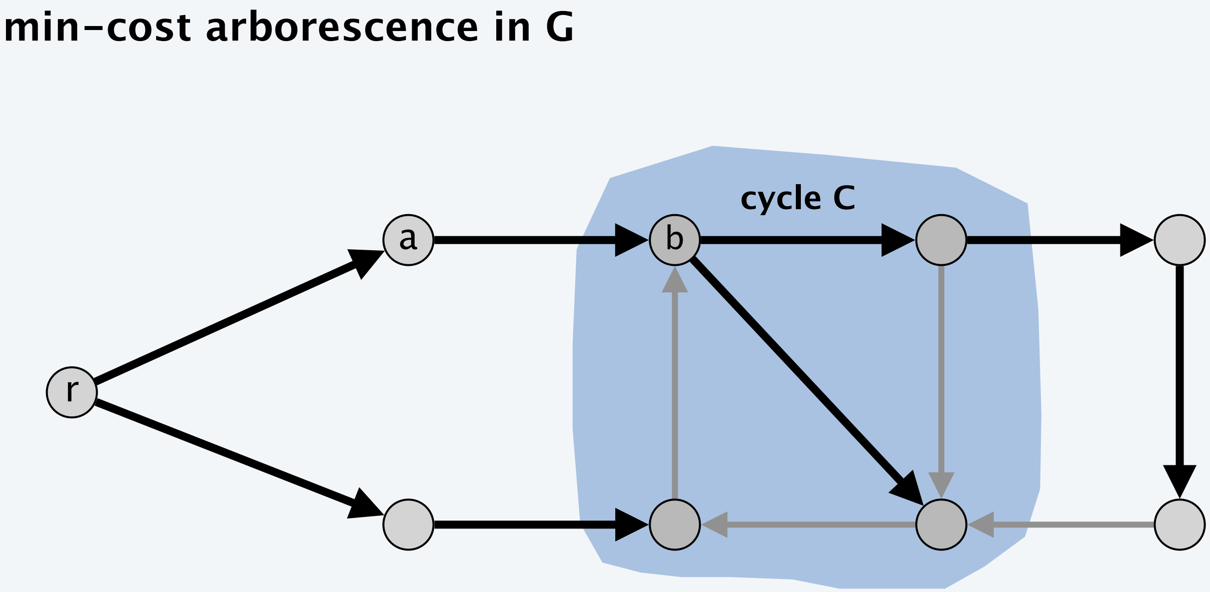
```
EDMONDSBRANCHING($G, r, c$)

FOREACH $v \neq r$
    $y(v) \leftarrow$ min cost of an edge entering $v$.
    $c'(u, v) \leftarrow c'(u, v) - y(v)$ for each edge $(u, v)$ entering $v$.
FOREACH $v \neq r$: choose one 0-cost edge entering $v$ and let $F^*$
be the resulting set of edges.
IF $F^*$ forms an arborescence, RETURN $T = (V, F^*)$.
ELSE
    $C \leftarrow$ directed cycle in $F^*$.
    Contract $C$ to a single supernode, yielding $G' = (V', E')$.
    $T' \leftarrow$ EDMONDSBRANCHING($G', r, c'$)
    Extend $T'$ to an arborescence $T$ in $G$ by adding all but one edge of $C$.
    RETURN $T$.
```

Edmonds branching algorithm

Q. What could go wrong?

A.

- Min-cost arborescence in G' has exactly one edge entering a node in C (since C is contracted to a single node)
- But min-cost arborescence in G might have more edges entering C.



Edmonds branching algorithm: key lemma

Lemma. Let C be a cycle in G consisting of 0-cost edges. There exists a min-cost arborescence rooted at r that has exactly one edge entering C.

Pf. Let T be a min-cost arborescence rooted at r.

Case 0. T has no edges entering C.
Since T is an arborescence, there is an $r \to v$ path fore each node v $\Rightarrow$
at least one edge enters C.

Case 1. T has exactly one edge entering C.
T satisfies the lemma.

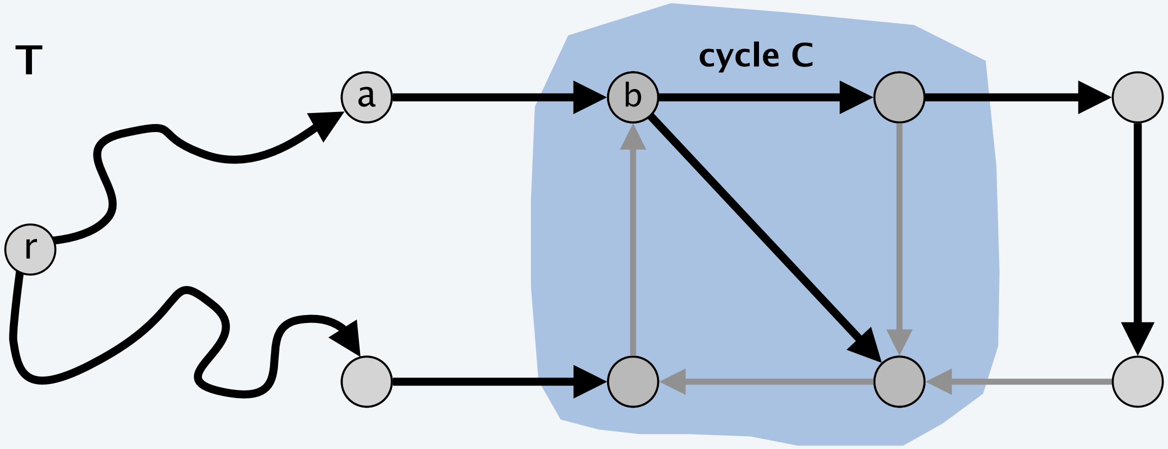
Case 2. T has more than one edge that enters C.
We construct another min-cost arborescence T' that has exactly one edge entering C.

Edmonds branching algorithm: key lemma

Case 2 construction of T'.

- Let (a, b) be an edge in T entering C that lies on a shortest path from r.
- We delete all edges of T that enter a node in C except (a, b).
- We add in all edges of C except the one that enters b.

path from r to C uses only one node in C



Edmonds branching algorithm: key lemma

Case 2 construction of T'.

- Let (a, b) be an edge in T entering C that lies on a shortest path from r.
- We delete all edges of T that enter a node in C except (a, b).
- We add in all edges of C except the one that enters b.

path from r to C uses only one node in C

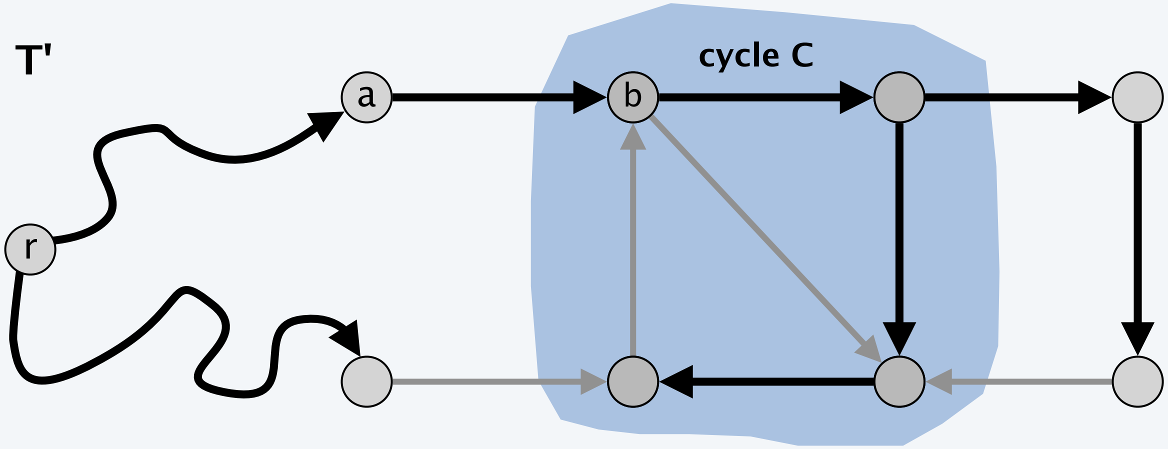
Claim. T' is a min-cost arborescence.

- The cost of T' is at most that of T since we add only 0-cost edges.
- T' has exactly one edge entering each node $v \neq r$.
- T' has no directed cycles.

T is an arborescence rooted at r

(T had no cycles before; no cycles within C; now only (a, b) enters C)

and the only path in T' to a is the path from r to a (since any path must follow unique entering edge back to r)



Edmonds branching algorithm: analysis

Theorem. [Chu-Liu 1965, Edmonds 1967] The greedy algorithm finds a min-cost arborescence.

Pf. [by induction on number of nodes in G]

- If the edges of F^* form an arborescence, then min-cost arborescence.
- Otherwise, we use reduced costs, which is equivalent.
- After contracting a 0-cost cycle C to obtain a smaller graph G', the algorithm finds a min-cost arborescence T' in G' (by induction).
- Key lemma: there exists a min-cost arborescence T in G that corresponds to T'. ▪

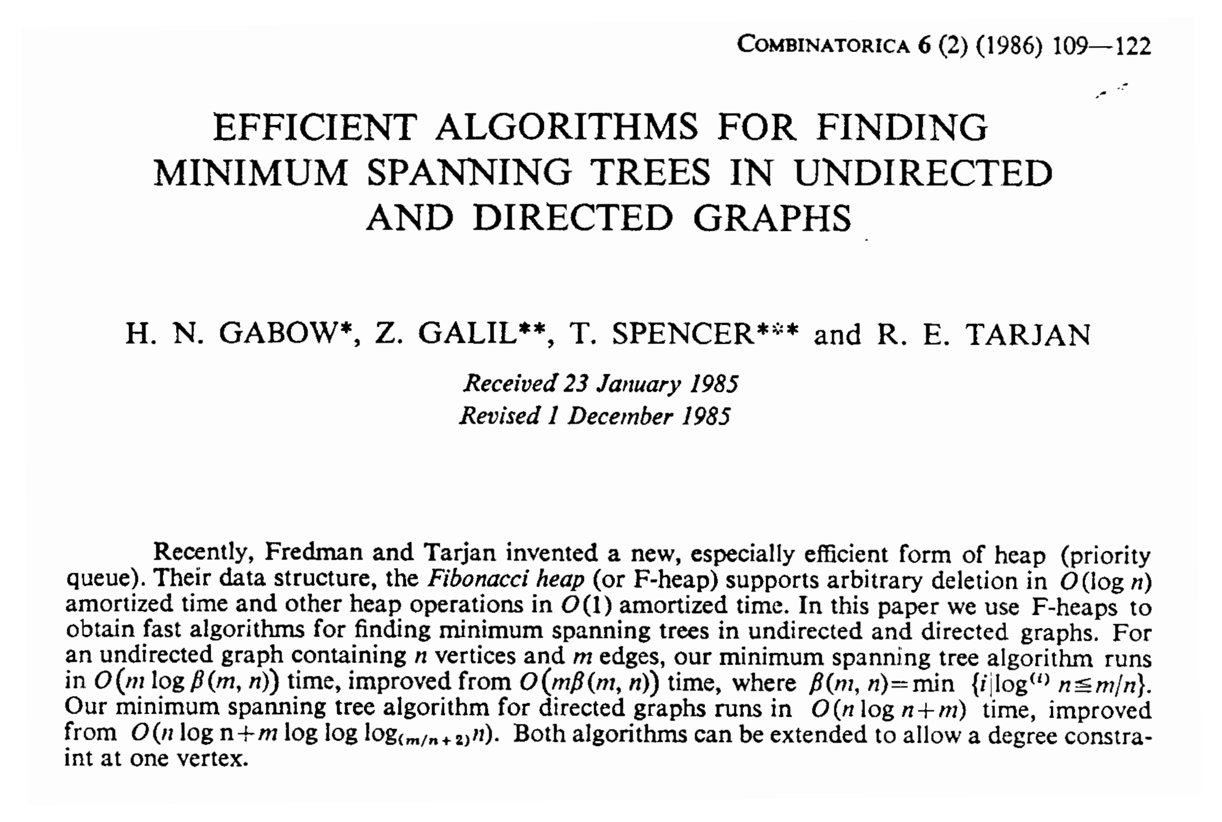
Theorem. The greedy algorithm can be implemented in $O(m\,n)$ time.

Pf.

- At most n contractions (since each reduces the number of nodes).
- Finding and contracting the cycle C takes $O(m)$ time.
- Transforming T' into T takes $O(m)$ time. ▪

Min-cost arborescence

Theorem. [Gabow-Galil-Spencer-Tarjan 1985] There exists an $O(m + n \log n)$ time algorithm to compute a min-cost arborescence.



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EFFICIENT ALGORITHMS FOR FINDING MINIMUM SPANNING TREES IN UNDIRECTED AND DIRECTED GRAPHS

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Received 23 January 1985
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Recently, Fredman and Tarjan invented a new, especially efficient form of heap (priority queue). Their data structure, the *Fibonacci heap* (or F-heap) supports arbitrary deletion in $O(\log n)$ amortized time and other heap operations in $O(1)$ amortized time. In this paper we use F-heaps to obtain fast algorithms for finding minimum spanning trees in undirected and directed graphs. For an undirected graph containing n vertices and m edges, our minimum spanning tree algorithm runs in $O(m \log \beta(m, n))$ time, improved from $O(m\beta(m, n))$ time, where $\beta(m, n) = \min \{i \mid \log^{(i)} n \leq m/n\}$. Our minimum spanning tree algorithm for directed graphs runs in $O(n \log n + m)$ time, improved from $O(n \log n + m \log \log \log_{(m/n+2)} n)$. Both algorithms can be extended to allow a degree constraint at one vertex.