Exercise 1 - DL

Exercise 1 Given the following $\mathcal{ALC}$ TBox:

$$
\begin{array}{rcl}
A & \sqsubseteq & C \\
D & \sqsubseteq & \exists R.C \\
E & \sqsubseteq & \forall R.F \\
E & \sqsubseteq & B \\
F & \sqsubseteq & \neg B \\
G \sqcap B & \sqsubseteq & \exists R.A \\
H & \sqsubseteq & G \\
H & \sqsubseteq & \exists R.B
\end{array}
$$

(a) tell whether the TBox $\mathcal{T}$ is satisfiable, and if so, show a model for $\mathcal{T}$;

(b) tell whether the concept $E \sqcap G$ is satisfiable with respect to $\mathcal{T}$, and if so, show a model for $\mathcal{T}$ where $E \sqcap G$ is satisfiable;

(c) tell whether the concept $E \sqcap H$ is satisfiable with respect to $\mathcal{T}$, and if so, show a model for $\mathcal{T}$ where $E \sqcap H$ is satisfiable;

(d) given the ABox $\mathcal{A} = \{E(a), R(a, b)\}$, use the tableau method to establish whether the knowledge base $\langle \mathcal{T}, \mathcal{A} \rangle$ entails the assertion $F(b)$.

(a)

Let $\mathcal{I}$ be the interpretation over the domain $\Delta^{\mathcal{I}} = \{d\}$ such that $A^{\mathcal{I}} = B^{\mathcal{I}} = C^{\mathcal{I}} = D^{\mathcal{I}} = E^{\mathcal{I}} = F^{\mathcal{I}} = G^{\mathcal{I}} = H^{\mathcal{I}} = R^{\mathcal{I}} = \emptyset$. We can see that all the axiom of T are satisfied in $\mathcal{I}$. That means that $\mathcal{I}$ is a model for T, which implies that T is satisfiable.

(b)

To prove that the concept $E \sqcap G$ is satisfiable with respect to T we have to show a model for T where $E \sqcap G$ is not empty. Let $\mathcal{I}'$ be the interpretation over the domain $\Delta^{\mathcal{I}'} = \{d_0, d_1\}$ such that $A = C = F = \{d_1\}$, $B = D = E = G = \{d_0\}$, $H = \emptyset$ and $R = \{(d_0, d_1)\}$. We can see that all the axiom of T are satisfied in $\mathcal{I}$. That means that $\mathcal{I}$ is a model for T, $(E \sqcap G)^{\mathcal{I}}$ is not empty $(= \{d_0\})$ so $E \sqcap G$ is satisfiable

(c)

To prove that the concept $E \sqcap H$ is satisfiable with respect to T we have to show a model for T where $E \sqcap H$ is not empty. Since the TBox contain the axioms '$E \sqsubseteq \forall R.F$' and '$H \sqsubseteq \exists R.B$' we can say that '$E \sqcap H \sqsubseteq \forall R.F \sqcap \exists R.B$' that means that it has to exist at least one couple in R i.e. (x,y) such that $x \in E \sqcap H$ and $y \in F \sqcap B$, but since for the axiom '$F \sqsubseteq \neg B$' the intersection $F \sqcap B = \emptyset$. Consequently no model I for T exist such that $(E \sqcap H)^I$ is not empty

d

Since the TBox is not unfoldable we need to compute C_{GCI}:

$$C_{GCI} = (\neg A \sqcup C) \sqcap (\neg D \sqcup \exists R.C) \sqcap (\neg E \sqcup \forall R.F) \sqcap (\neg E \sqcup B) \sqcap (\neg F \sqcup \neg B) \sqcap (\neg G \sqcup \neg B \sqcup \exists R.A) \sqcap (\neg H \sqcup G) \sqcap (\neg H \sqcup \exists R.B)$$

We start the tableaux method from the Abox A, and the negation of the assertion we want to prove the entailment $(F(b))$:

$$A_0 = A = \{ E(a),\ R(a,b),\ \neg F(b) \}$$

CGI - RULE

$$A_1 = A_0 \cup \{ C_{GCI}(a), \; C_{GCI}(b) \} =$$

$$= \left\{ \begin{array}{l} \big((\neg A \sqcup C) \sqcap (\neg D \sqcup \exists R.C) \sqcap (\neg E \sqcup \forall R.F) \sqcap (\neg E \sqcup B) \sqcap (\neg F \sqcup \neg B) \sqcap \\ \qquad (\neg G \sqcup \neg B \sqcup \exists R.A) \sqcap (\neg H \sqcup G) \sqcap (\neg H \sqcup \exists R.B)\big)(a), \\ \big((\neg A \sqcup C) \sqcap (\neg D \sqcup \exists R.C) \sqcap (\neg E \sqcup \forall R.F) \sqcap (\neg E \sqcup B) \sqcap (\neg F \sqcup \neg B) \sqcap \\ \qquad (\neg G \sqcup \neg B \sqcup \exists R.A) \sqcap (\neg H \sqcup G) \sqcap (\neg H \sqcup \exists R.B)\big)(b) \end{array} \right\}$$

OR - RULE' :

$A_2 = A_1 \cup \{ \neg E(a) \}$ × CLASH with $E(a)$ in A_0

$A_3 = A_1 \cup \{ \forall R.F \}$ × CLASH with $R(a,b), \neg F(b)$ in A_0

The Tableaux is closed, it doesn't exist a possible interpretation where $\neg F(b)$ is satisfiable, that means that $F(b)$ is entailed in the $KB = \langle T, A \rangle$.