Approximation Algorithms: Basic Concepts

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Thanks to Lap-Chi Lau

Comp efficiently and solve problem close to optimal solution

NP-completeness

Majority of problems cannot be solved in polynomial time

There many polynomial time solvable optimization problems.
e.g. maximum matching, min-cost flow, minimum cut, etc.

However, there are much more optimization problems that are NP-complete: we do not know how to solve in polynomial time.
e.g. traveling salesman, graph colorings, maximum independent set, set cover, maximum clique, maximum cut, minimum Steiner tree, satisfiability, etc.

Vertex Cover

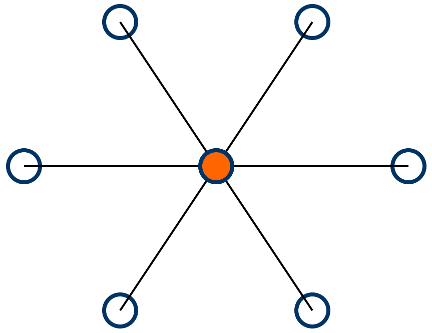
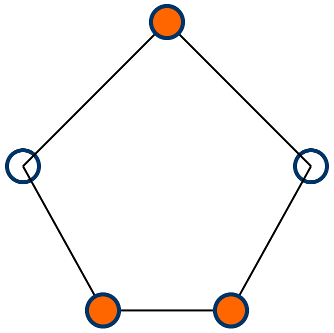
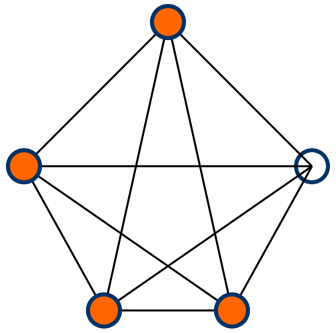
min. problem.

Vertex cover: a subset of vertices which "covers" every edge.
An edge is covered if one of its endpoint is chosen.

Approx vertex cover to independent set is difficult because inep is a max, this is min

The Minimum Vertex Cover Problem: Given a graph find a vertex cover with minimum number of vertices. (optimization problem)

n nodes
n/2 cycles to solve.

Some Alternatives

- Special graph classes

  e.g. vertex cover in bipartite graphs, perfect graphs.

- Fixed parameter algorithms

  find a vertex cover of size k efficiently for small k.

- Average case analysis

  find an algorithm which works well on average.

- **Approximation algorithms**

  find an algorithm which return solutions that are guaranteed to be close to an optimal solution.

Approximation Algorithms

Key: provably close to optimal.

Let OPT be the value of an optimal solution,
and let SOL be the value of the solution that our algorithm returned.

optimal solution time.

Additive approximation algorithms: SOL <= OPT + c for some constant c.
Very few examples known: Only to min. problem.
edge coloring, minimum maximum-degree spanning tree, bin packing

Constant factor approximation algorithms:
SOL <= c OPT for some constant c.
Many more examples known.

Different approximation factors

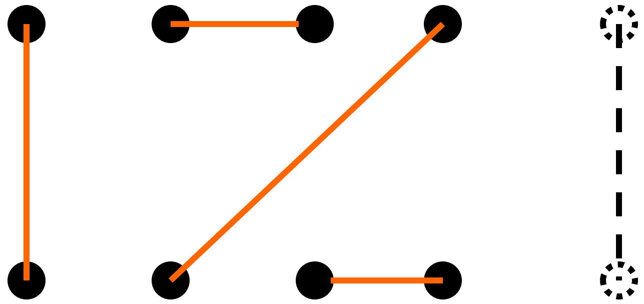
Constant approximation algorithms: SOL <= c OPT
e.g.: Vertex Cover, TSP, Max SAT, Steiner Tree, Facility Location, Max Cut

Logarithmic approximation algorithms: SOL = O(log n)^c OPT
e.g.: Set Cover, Multi-Cut, Dominating Set,..

Polinomial approximation: SOL = O(n^c) OPT, c<=1
e.g: Max Clique, Independent Set, Coloring

Polynomial Approximation Schemes: SOL <= (1+ϵ) OPT, for each ϵ >0
Fully Polynomial Approximation schemes if running time O(poly(1/ϵ))
e.g.: Knapsack, Scheduling, Budgeted MST, Euclidean TSP, Bin packing

Vertex Cover: 2 apx algorithm



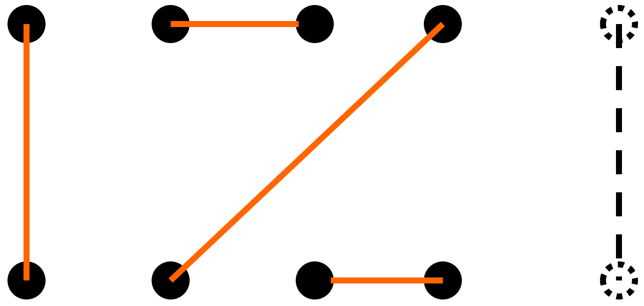
Fix a maximum matching. Call the vertices involved **black**.

Since the matching is maximum, every edge must have a black endpoint.

So, by choosing all the black vertices, we have a vertex cover.

SOL <= $\bigcirc\!\!\!\!\!2$ * size of a maximum matching

Vertex Cover: 2-apx algorithm



What about an optimal solution?

Each edge in the matching has to be covered by a **different** vertex!

Lower bound to the optimal solution:
OPT >= size of a maximum matching

So, SOL <= 2 OPT, and we have a 2-approximation algor.!

Vertex Cover

Approximate min-max theorem:

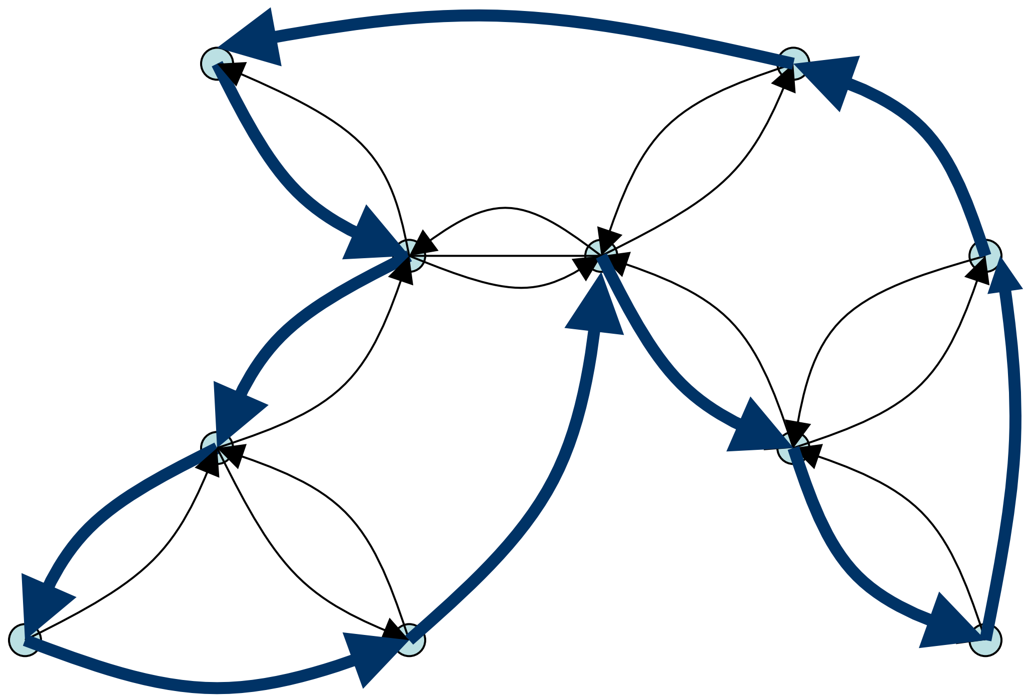
Maximum matching <= minimum vertex cover <= 2*maximum matching

Is the analysis tight? Yes. Consider for instance a bipartite graph.

Major open question:
Can we obtain a c-approximation algorithm, c <2?

Hardness result:
It is NP-complete even to ***approximate*** within a factor of 1.36!!

Approximation Algorithms for (Min) Traveling Salesman Problem (TSP)



The Hamiltonian Path Problem

Hamiltonian Path Problem:
Given an undirected graph, find a cycle visiting every vertex exactly once.

Eulerian Path Problem:
Given an undirected graph, find a walk visiting every edge exactly once.
Notice that in a walk some vertices may have been visited more than once.

The Eulerian Path problem is polynomial time solvable.
A graph has an Eulerian path if and only if every vertex has an even degree.

The Hamiltonian Path problem is NP-complete.

The Traveling Salesman Problem

Traveling Salesman Problem (TSP):
Given a complete graph with nonnegative edge costs,
Find a minimum cost cycle visiting every vertex exactly once.

Given a number of cities and the costs of traveling from any city to any other city, what is the cheapest round-trip route that visits each city exactly once and then returns to the starting city?

One of the most well-studied problem in combinatorial optimization.

Inapproximability of Traveling Salesman Problem

Theorem: There is no constant factor approximation algorithm for TSP, unless P=NP.

Idea: Use the Hamiltonian path problem.

- For each edge, we add an edge of cost 1.
- For each non-edge, we add an edge of cost **nk**.

- If there is a Hamiltonian path, then there is a cycle of cost n.
- If there is no Hamiltonian path, then every cycle has cost greater than **nk**.

So, if you have a k-approximation algorithm for TSP, one just needs to check if the returned solution is **at most nk**.

- If yes, then the original graph has a Hamiltonian path.
- Otherwise, the original graph has no Hamiltonian path.

Inapproximability of Traveling Salesman Problem

Theorem: There is no constant factor approximation algorithm for TSP, unless P=NP.

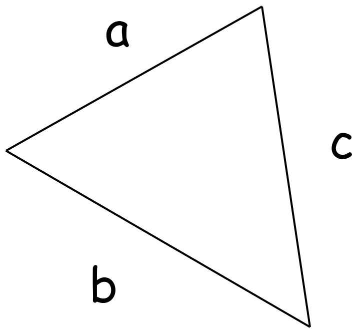
This type of theorem is called "hardness result" in the literature.
Just like their names, usually they are very hard to obtain.

- If there is a Hamiltonian path, then there is a cycle of cost n.
- If there is no Hamiltonian path, then every cycle has cost greater than **nk**.

The strategy is usually like this.
This creates a gap between yes and no instances.
The bigger the gap, the problem is harder to approximate.

Approximation Algorithm for Metric TSP

Metric Traveling Salesman Problem (metric TSP):
Given a complete graph with edge costs satisfying triangle inequalities,
Find a minimum cost cycle visiting every vertex exactly once.



$\Rightarrow$

$a + b \geq c$

For example, think of cost of an edge as the distance between two points.

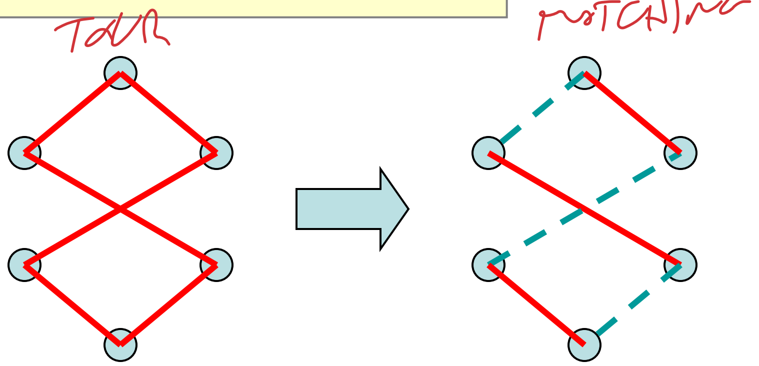
How could triangle inequalities help in finding approximation algorithm?

Lower Bounds for TSP

What can be a good lower bound to the cost of TSP?

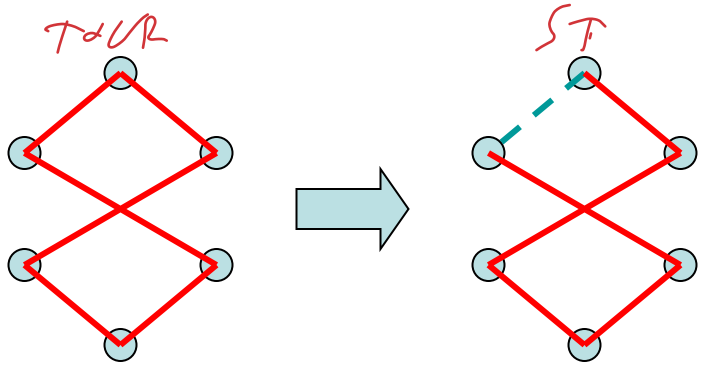
In a tour of 6 edges we have 2 matching

A tour contains a matching.



Let OPT be the cost of an optimal tour, since a tour contains two matchings, the cost of a **minimum weight perfect matching** is at most OPT/2.

A tour contains a spanning tree.



So, the cost of a **minimum spanning tree** is at most OPT.

Spanning Tree and TSP

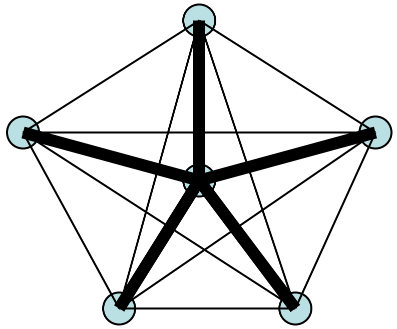
Let the thick edges have cost 1,
And all other edges have cost greater than 1.

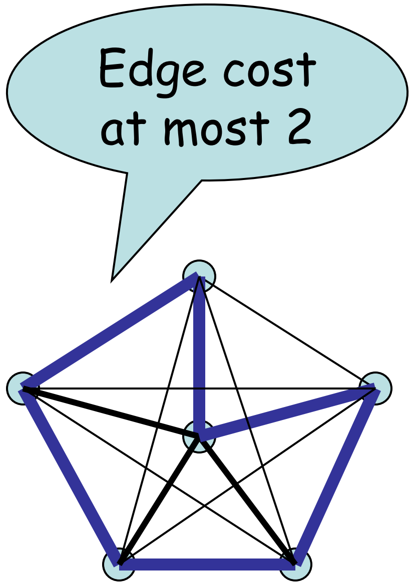
So the thick edges form a minimum spanning tree.

But it doesn't look like a Hamiltonian cycle at all!

Consider a Hamiltonian cycle.
The costs of the edges which are not in the minimum spanning tree might have very high costs.

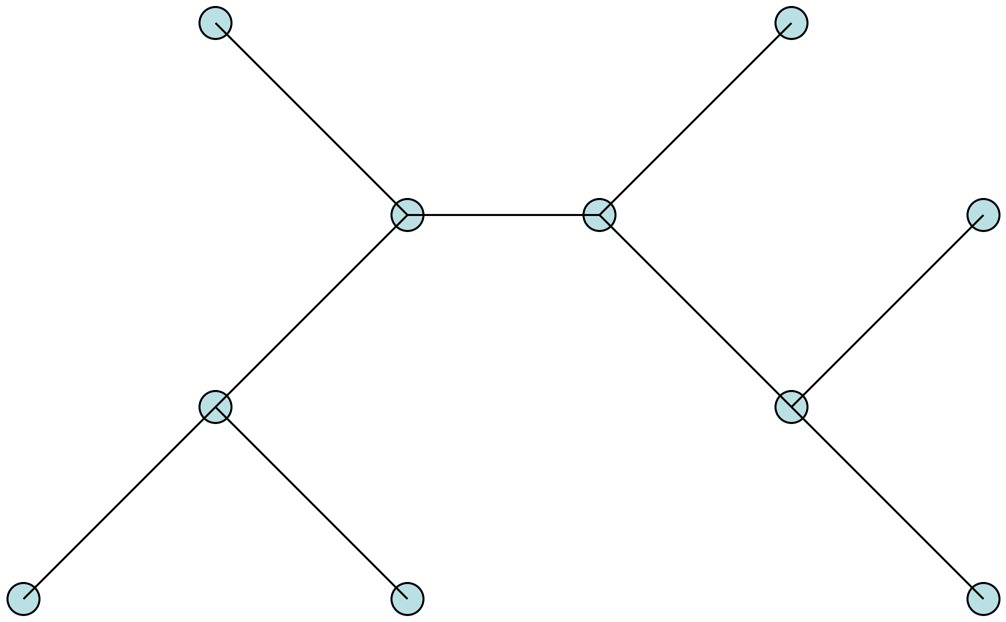
Not really! Each such edge has cost at most 2 because of the **triangle inequality**.





Spanning Tree and TSP

How to formalize the idea of "following" a minimum spanning tree?



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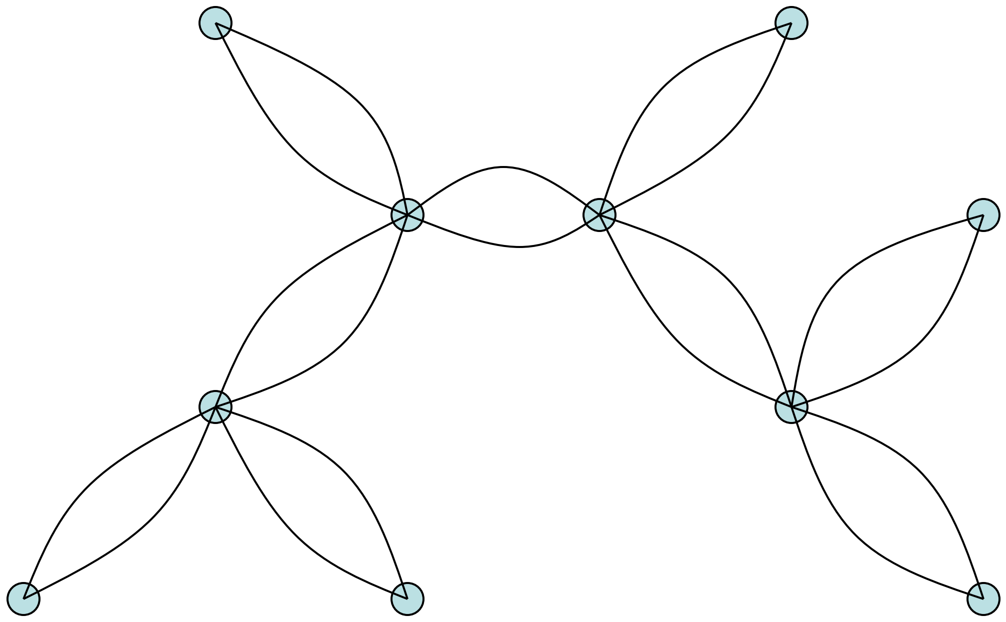

Key idea: double all the edges and find an Eulerian tour.

This graph has cost 2MST.

Spanning Tree and TSP

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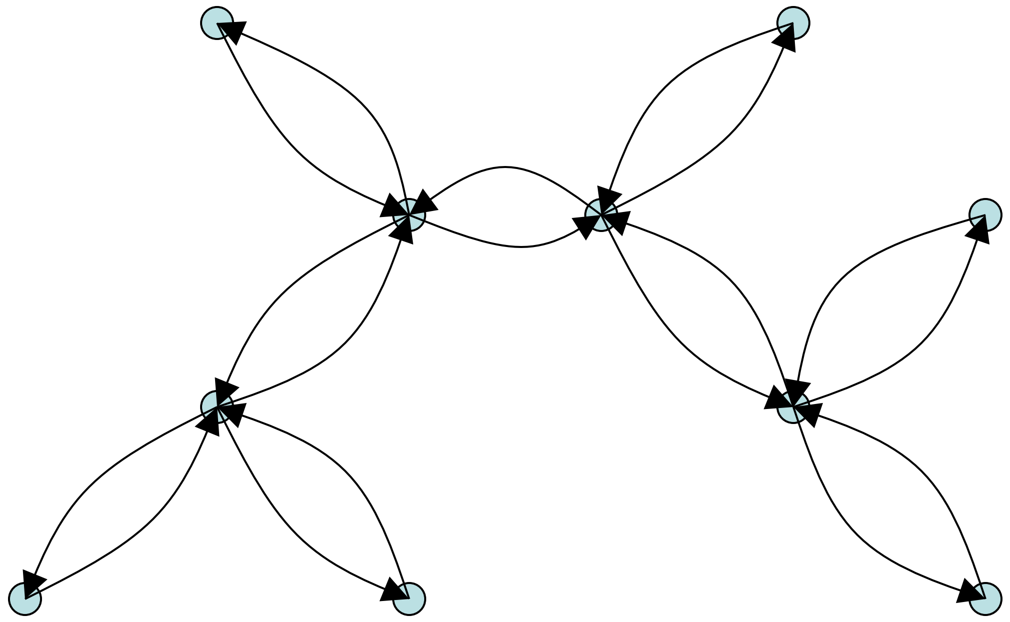

Key idea: double all the edges and find an Eulerian tour.

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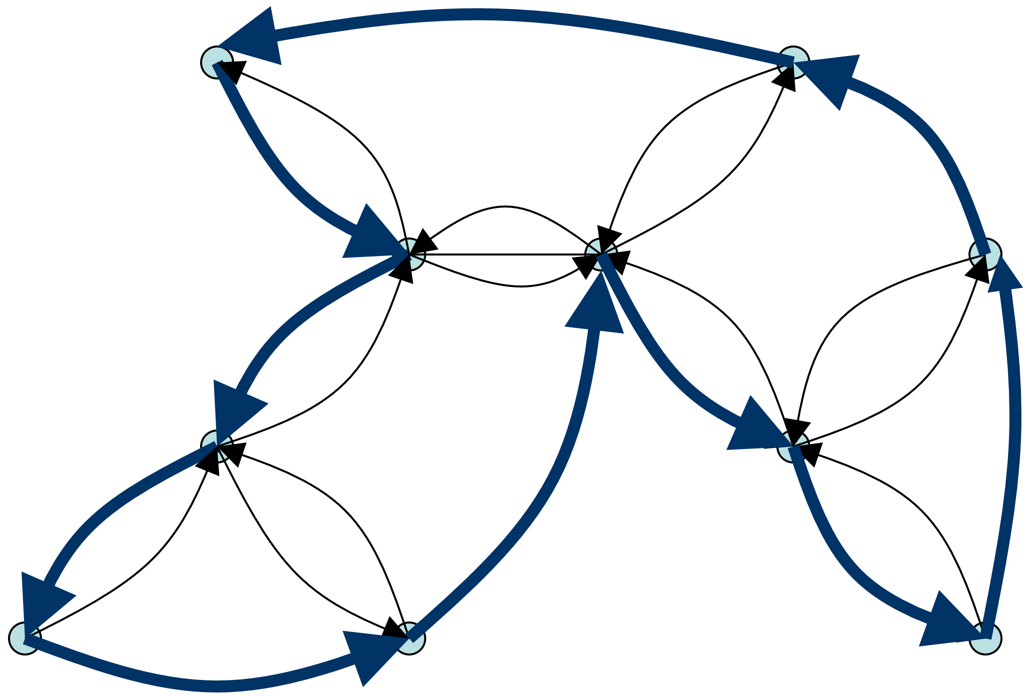
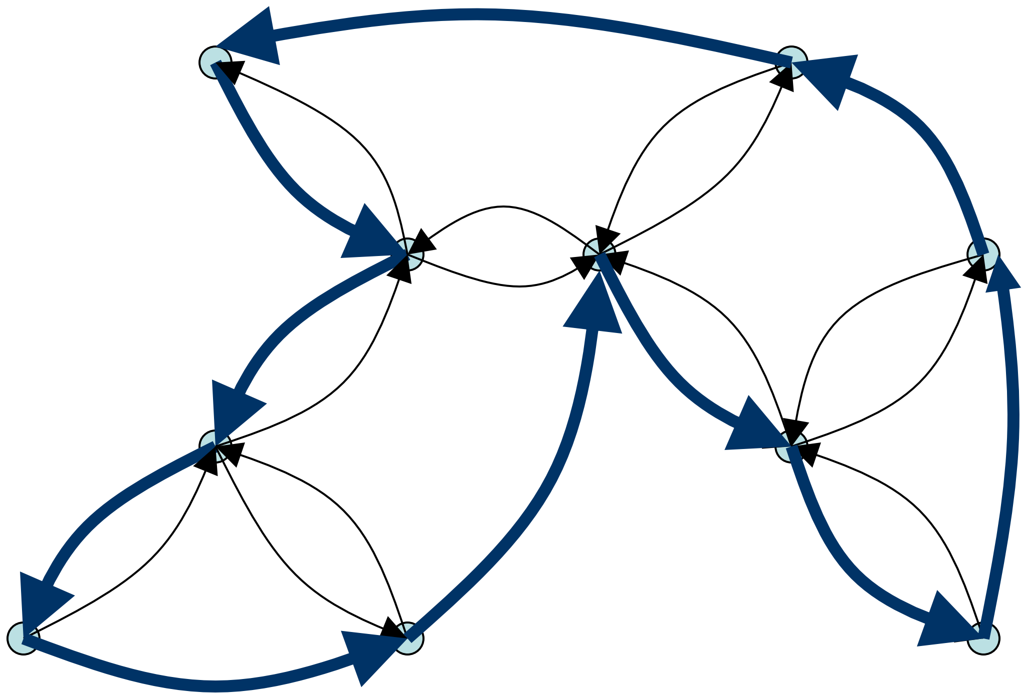
Spanning Tree and TSP

Strategy: shortcut this Eulerian tour.



Spanning Tree and TSP

By **triangle inequalites**, the shortcut tour is not longer than the Eulerian tour.



Each directed edge is used exactly once in the shortcut tour.

A 2-Approximation Algorithm for Metric TSP

(Metric TSP – Factor 2)

1. Find an MST, T, of G.
2. Double every edge of the MST to obtain an Eulerian graph.
3. Find an Eulerian tour, T*, on this graph.
4. Output the tour that visits vertices of G in the order of their first appearance in T*. Let C be this tour.
   (That is, shortcut T*)

Analysis:

1. cost(T) ≤ OPT (because MST is a lower bound of TSP)
2. cost(T*) = 2cost(T) (because every edge appears twice)
3. cost(C) ≤ cost(T*) (because of triangle inequalities, **shortcutting**)
4. So, cost(C) ≤ 2OPT

Better approximation?

There is a 1.5 approximation algorithm for metric TSP.

Hint: use a minimum spanning tree and a maximum matching
(instead of double a minimum spanning tree). See textbook

Major open problem: Improve this to 4/3?

An aside: hardness result is not an excuse to stop working,
but to guide us to identify interesting cases.

Matching

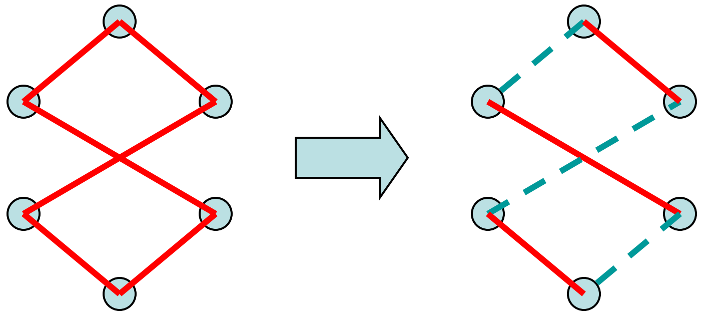
Matching problems

Given a graph with n vertices and edge costs,

Find set of edges of edges of a maximum total cost

s.t. each vertex is covered by at most one edge

Note: if number of vertices odd then there is at least one unmatched vertex



Matching is solvable in polynomial time

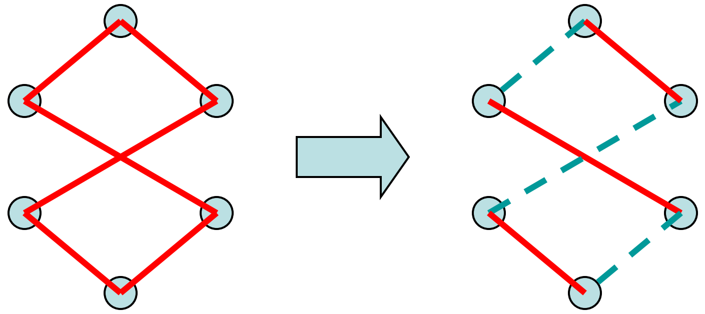
Compare with vertex cover

How could matching helps in finding approximation algorithm?

Lower Bounds for TSP

What can be a good lower bound to the cost of TSP?

A tour contains a matching.



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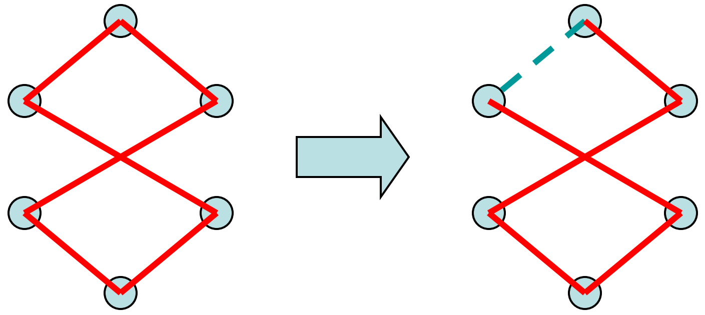
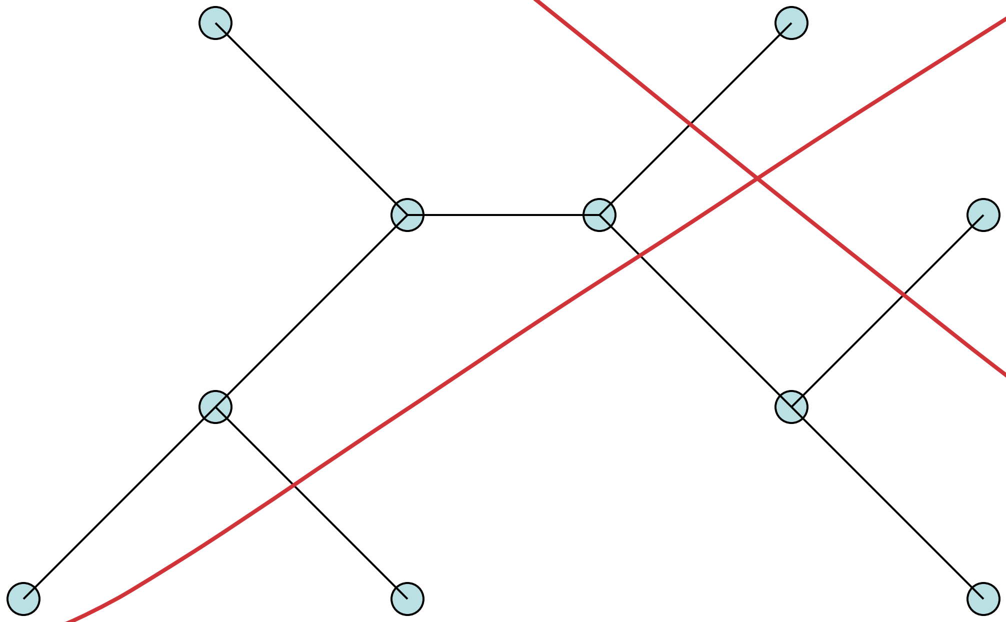


So, the cost of a **minimum spanning tree** is at most OPT.

Spanning Tree and TSP

How to formalize the idea of "following" a minimum spanning tree?



Spanning Tree and TSP

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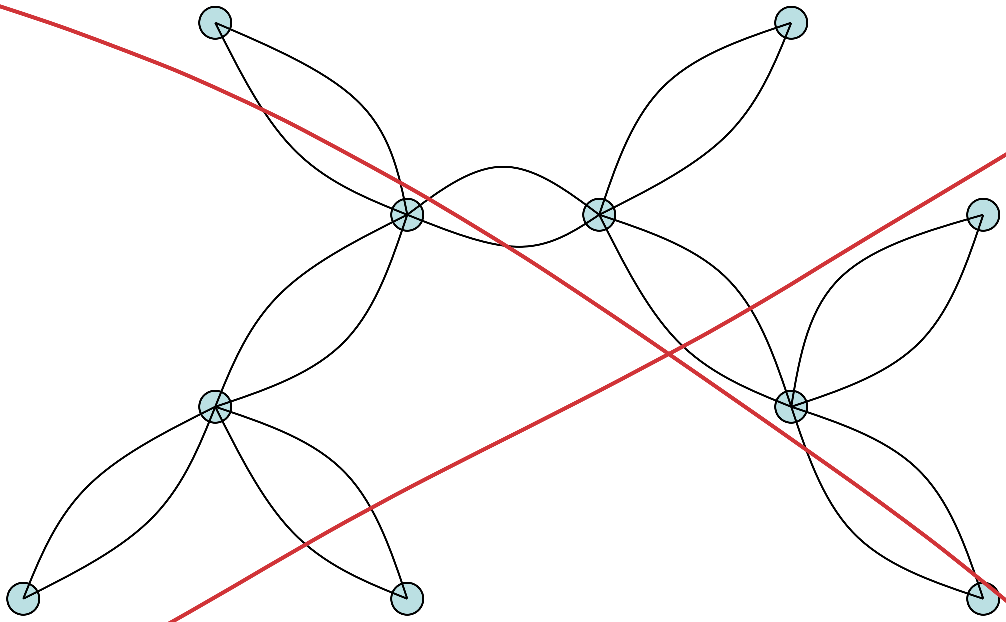

Key idea: double all the edges and find an Eulerian tour.

This graph has cost 2MST.

Spanning Tree and TSP

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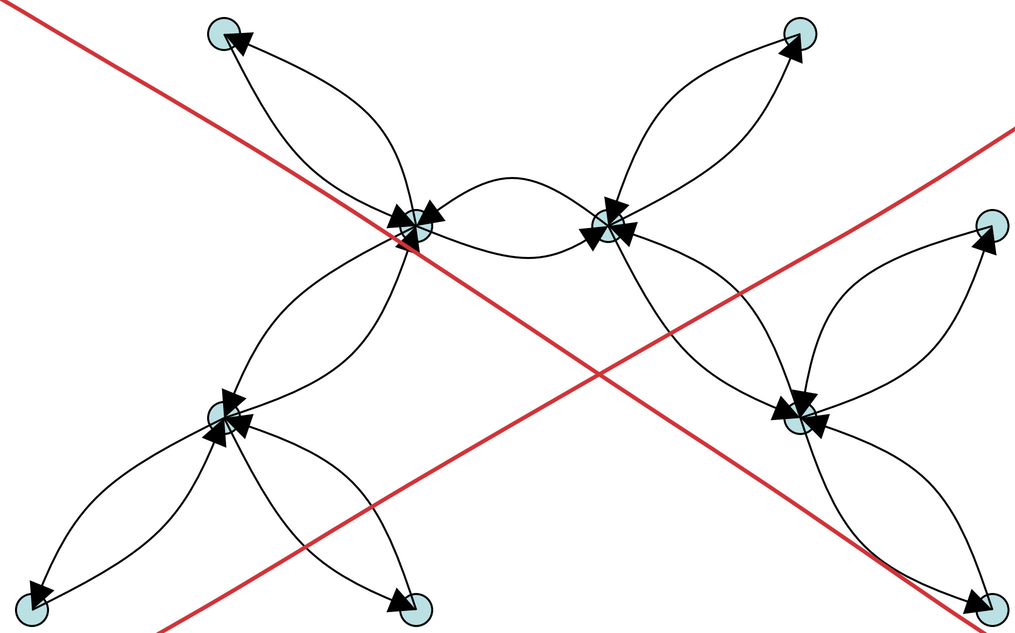

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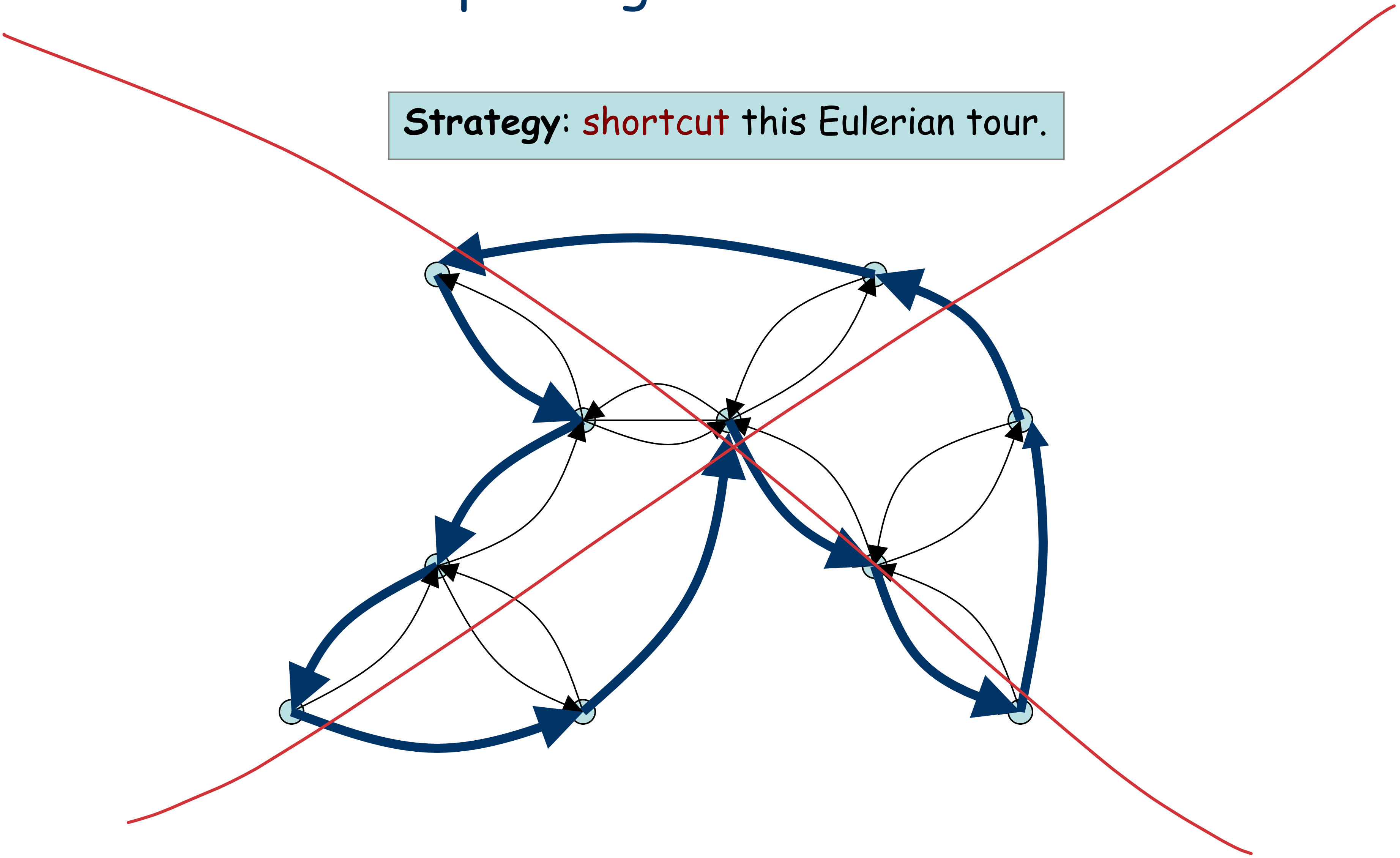
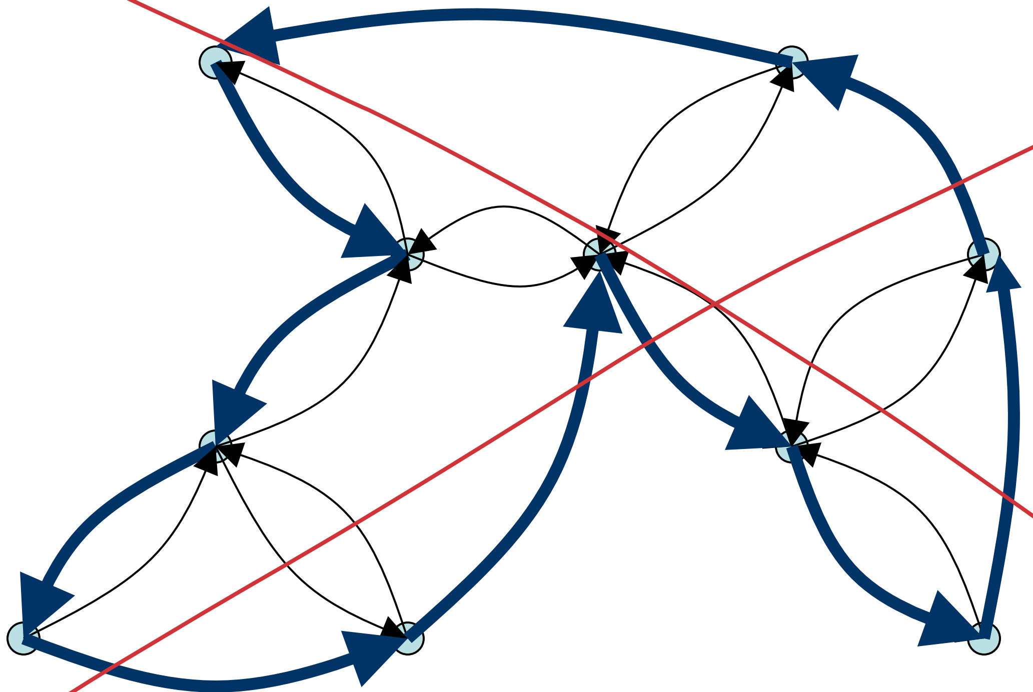
Spanning Tree and TSP

Strategy: shortcut this Eulerian tour.



Spanning Tree and TSP

By **triangle inequalites**, the shortcut tour is not longer than the Eulerian tour.



Each directed edge is used exactly once in the shortcut tour.

A 2-Approximation Algorithm for Metric TSP

(Metric TSP – Factor 2)

1. Find an MST, T, of G.
2. Double every edge of the MST to obtain an Eulerian graph.
3. Find an Eulerian tour, T*, on this graph.
4. Output the tour that visits vertices of G in the order of their first appearance in T*. Let C be this tour. (That is, shortcut T*)

Analysis:

1. cost(T) ≤ OPT (because MST is a lower bound of TSP)
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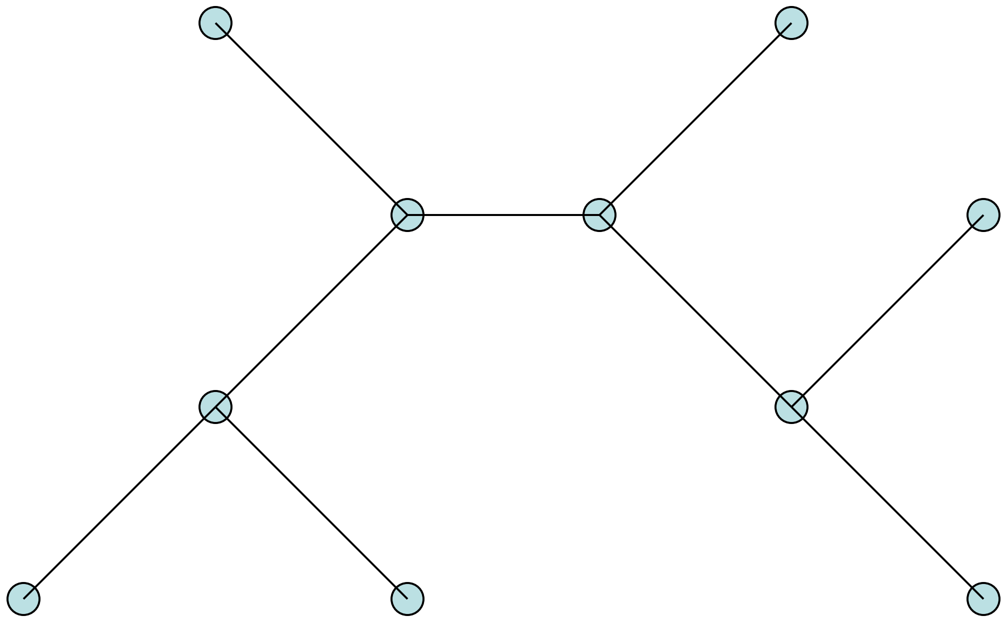
4. So, cost(C) ≤ 2OPT

Spanning Tree matching and TSP

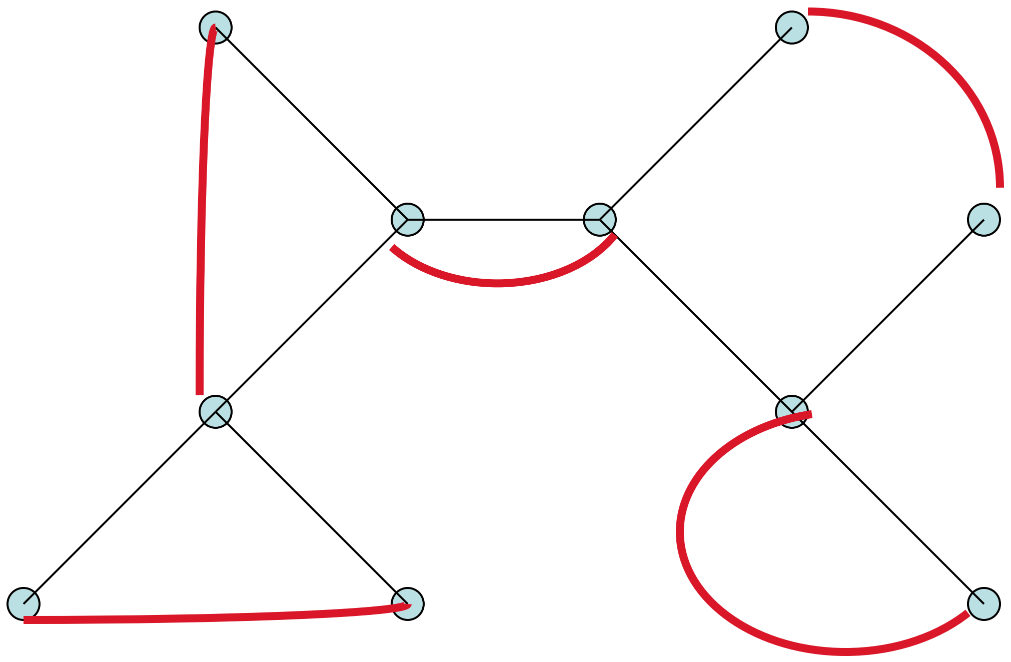
How to generalize the idea of "following" a minimum spanning tree?



Spanning Tree matching and TSP

How to generalize the idea of "following" a matching and a minimum spanning tree?

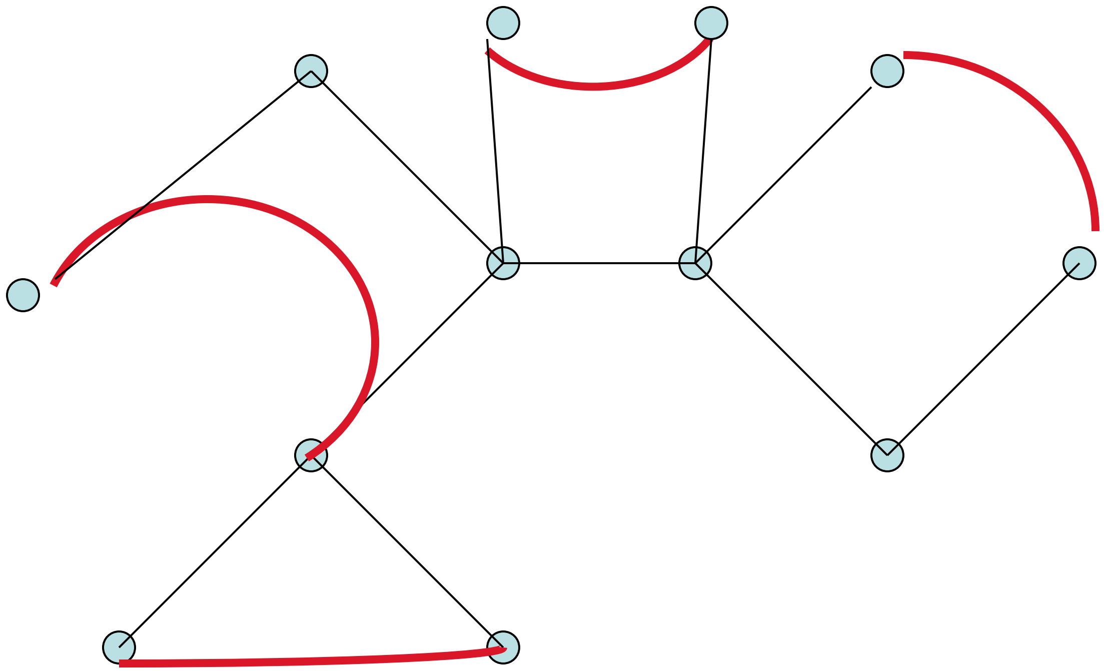
nothing



IDEA: obtain an Eulerian graph using a spanning tree and an matching

Spanning Tree matching and TSP

Eulerian graph: each vertex has even degree



Christofides: apply matching to vertices with odd degree in the minimum spanning tree!

A Christofides' Algorithm for Metric TSP

(Metric TSP – Factor 2)

1. Find an MST, T, of G.
2. Find a matching M among **odd** degree vertices.
3. Find an Eulerian tour E, on the graph with edges from T and M.
4. Output the tour C that visits vertices of G in the order of their first appearance. Let C be this tour.

   (That is, shortcut T*)

Analysis:

1. cost(T) ≤ OPT (because MST is a lower bound of TSP)
2. cost(M) = 0.5 OPT (because every edge appears twice)
3. cost(C) ≤ cost(T)+ cost(M) (because of triangle inequalities, **shortcutting**)
4. So, cost(C) ≤ 1.5 OPT

Better approximation?

Christofides:there is a 1.5 approximation algorithm for metric TSP.

There exists a PAS for metric TSP

Approximation criteria

An algorithm is c approximation algorithm if

- SOL ≤ c OPT for a minimization problem
- SOL ≥ c OPT for a maximization problem

An algorithm ***A*** is an **approximation scheme** if for every ϵ > 0,

A runs in polynomial time (which may depend on ϵ) and return a solution:

- SOL ≤ (1+ϵ)OPT for a minimization problem
- SOL ≥ (1-ϵ)OPT for a maximization problem

Knapsack

Given n objects with

size(a_i) and profit(a_i), $i=1,2,\ldots n$

Knapsack can only hold a

- total weight of B

Goal: to pick a subset which can fit into the knapsack and **maximize the value** of this subset.

Knapsack

Given a set $S = \{a_1, \ldots, a_n\}$ of objects,
with specified sizes and profits, $\mathrm{size}(a_i)$ and $\mathrm{profit}(a_i)$,
and a knapsack capacity B, find a subset of objects whose
total size is bounded by B and total profit is maximized.

Assume $\mathrm{size}(a_i)$, $\mathrm{profit}(a_i)$, and B are all integers.
Integer programming formulation

Max $\Sigma_i \ \mathrm{profit}(a_i)x_i$

s.t. $\Sigma_i \ \mathrm{size}(a_i)\, x_i \le B$, $x_i \varepsilon \{0,1\}$

Greedy methods

General greedy method:

Sort the objects by some rule,

and then put the objects into the knapsack according to this order

Sort

- G1: Sort by object size in non-decreasing order
- G2:Sort by profit in non-increasing order
- G3: Sort by profit/object size in non-increasing order

Fact: All previous ordering do not work to find a good approximate solution

Theorem: For all instances G3 returns a solution SOL-G3 such that

max(profit(a_i), SOL-G3) ≥= 1/2 OPT

Exercise: prove the Fact and the Theorem

Dynamic Programming

Dynamic programming is just exhaustive search with polynomial number of subproblems.

We only need to compute each subproblem once,
and each subproblem is looked up at most a polynomial number of times,
and so the total running time is at most a polynomial.

Dynamic Programming for Knapsack

Suppose we have considered object 1 to object i.
We want to remember what profits are achievable.
For each achievable profit, we want to minimize the size.

Let S(i,p) denote a subset of {a1,…,ai} whose total profit is **exactly** p and total size is **minimized**.
Let A(i,p) denote the size of the set S(i,p)
(A(i,p) = ∞ if no such set exists).

For example, A(1,p) = size(a1) if p=profit(a1),
Otherwise A(1,p) = ∞ (if p $\neq$ profit(a1)).

Recurrence

Remember: $A(i,p)$ denote the minimum size to achieve profit p
using objects from 1 to i.

Goal: we know $A(i,q)$ for all q and we want to compute $A(i+1,p)$

Two possibilities:

If we do not choose object $i+1$:
then $A(i+1,p) = A(i,p)$.

If we choose object $i+1$:
then $\underbrace{A(i+1,p) = \text{size}(a_{i+1}) + A(i,p-\text{profit}(a_{i+1})) \quad \text{if } p >}$ $\text{profit}(a_{i+1})$

$A(i+1,p) =$ minimum of these two values.

An Example

Remember: A(i,p) denote the minimize size to achieve profit p using objects from 1 to i.

Optimal Solution: max{ p | A(n,p) ≤ B} where B is the size of the knapsack.

size(a1)=2, profit(a1)=4; size(a2)=3, profit(a2)=5;
size(a3)=2, profit(a3)=3; size(a4)=1, profit(a4)=2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	∞	∞	∞	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
3	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
4	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

An Example

$A(i+1,p) = \min\{A(i,p), \text{size}(a_{i+1}) + A(i,p-\text{profit}(a_{i+1}))\}.$

$A(2,p) = \min\{A(1,p), \quad A(1,p-5)+3\}.$

size(a1)=2, profit(a1)=4; size(a2)=3, profit(a2)=5;
size(a3)=2, profit(a3)=3; size(a4)=1, profit(a4)=2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	∞	∞	∞	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
3	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
4	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

An Example

$A(i+1,p) = \min\{A(i,p),\ size(a_{i+1}) + A(i,p-profit(a_{i+1}))\}.$

$A(3,p) = \min\{A(2,p),\ A(2,p-3)+2\}.$

size(a1)=2, profit(a1)=4; size(a2)=3, profit(a2)=5;
size(a3)=2, profit(a3)=3; size(a4)=1, profit(a4)=2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	∞	∞	∞	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	0	∞	∞	∞	**2**	**3**	∞	∞	∞	**5**	∞	∞	∞	∞	∞
3	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
4	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

An Example

$A(i+1,p) = \min\{A(i,p), \text{size}(a_{i+1}) + A(i,p-\text{profit}(a_{i+1}))\}.$

$A(4,p) = \min\{A(3,p),\ A(3,p-2)+1\}.$

size(a1)=2, profit(a1)=4; size(a2)=3, profit(a2)=5;
size(a3)=2, profit(a3)=3; size(a4)=1, profit(a4)=2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	∞	∞	∞	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	0	∞	∞	∞	**2**	**3**	∞	∞	∞	**5**	∞	∞	∞	∞	∞
3	0	∞	∞	**2**	**2**	**3**	∞	**4**	**5**	**5**	∞	∞	**7**	∞	∞
4	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

An Example

$A(i+1,p) = \min\{A(i,p),\ size(a_{i+1}) + A(i,p-profit(a_{i+1}))\}.$

$A(4,p) = \min\{A(3,p),\ \ A(3,p-2)+1\}.$

size(a1)=2, profit(a1)=4; size(a2)=3, profit(a2)=5;
size(a3)=2, profit(a3)=3; size(a4)=1, profit(a4)=2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	∞	∞	∞	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	0	∞	∞	∞	2	3	∞	∞	∞	5	∞	∞	∞	∞	∞
3	0	∞	∞	2	2	3	∞	4	5	5	∞	∞	7	∞	∞
4	0	∞	1	2	2	3	3	4	5	5	6	6	7	∞	8

An Example

Remember: A(i,p) denote the minimize size to achieve profit p using objects from 1 to i.

Optimal Solution: max{ p | A(n,p) ≤ B} where B is the size of the knapsack.

For example, if B=8, OPT=14, if B=7, OPT=12, if B=6, OPT=11.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	∞	∞	∞	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	0	∞	∞	∞	2	3	∞	∞	∞	5	∞	∞	∞	∞	∞
3	0	∞	∞	2	2	3	∞	4	5	5	∞	∞	7	∞	∞
4	0	∞	1	2	2	3	3	4	5	5	6	6	7	∞	8

Running Time

The input has 2n numbers, so the input total lenght is 2n
Assume P= max profit(ai) i=1,2,…,n and S= P= max size(ai) i=1,2,…,n
So the input has total length 2nlog(max(P,S)).

For the dynamic programming algorithm,

there are n rows and at most nP columns.

Each entry can be computed in constant time (look up two entries).

So the total time complexity is $O(n^2P)$.

The running time is not polynomial if P is very large compared to n

In fact the Knapsack problem is NP-complete and we do not expect

That there exists a polynomial time algorithm

Scaling Down

Idea: use dynamic programming to scale down the numbers and compute the optimal solution in this modified instance

- Suppose P ≥ 1000n, (P max profit).
- Then OPT ≥ 1000n (each single object must enter the knapsack).
- Now scale down each element by 100 times (profit*:=profit/100).
- Compute the optimal solution using this new profit.
- Can't distinguish between element of size, say 2199 and 2100.
- Each element contributes at most an error of 100.
- So total error is at most 100n.
- This is at most 1/10 of the optimal solution.
- However, the running time is 100 times faster.

Approximation Scheme

Goal: to find a solution which is at least (1- ϵ)OPT for any ϵ > 0.

Approximation Scheme for Knapsack

1. Given ϵ > 0, let K = ϵP/n, where P is the largest profit of an object.
2. For each object ai, define profit*(ai) = $\lfloor$profit(ai)/K$\rfloor$.
3. With these as profits of objects, using the dynamic programming algorithm, find the most profitable set, say S'.
4. Output S' as the approximate solution.

Quality of Solution

Theorem. Let S denote the set returned by the algorithm. Then,

$$\boxed{\text{profit}(S) \geq (1-\epsilon)\text{OPT}.}$$

Proof. Let O denote the optimal set.

For each object a, because of rounding down,

$K \cdot \text{profit}^*(a)$ can be smaller than $\text{profit}(a)$, but by not more than K.

Since there are at most n objects in O,

$$\text{profit}(O) - K \cdot \text{profit}^*(O) \leq nK.$$

Since the algorithm return an optimal solution under the new profits,

$$\text{profit}(S) \geq K \cdot \text{profit}^*(S) \geq K \cdot \text{profit}^*(O) \geq \text{profit}(O) - nK$$
$$= \text{OPT} - \epsilon P \geq (1 - \epsilon)\text{OPT}$$

because $\text{OPT} \geq P$.

Running Time

For the dynamic programming algorithm,

there are n rows and at most $n\lfloor P/K \rfloor$ columns.

Each entry can be computed in constant time (look up two entries).

So the total time complexity is $O(n^2\lfloor P/K \rfloor) = O(n^3/\epsilon)$.

Therefore, we have an approximation scheme for Knapsack.

Claim: Strong NP-hard problems do not admit FPAS assuming P≠NP

Knapsack is not Strong NP-hard

Approximation Scheme

Quick Summary

1. Modify the instance by rounding the numbers.
2. Use dynamic programming to compute an optimal solution S in the modified instance.
3. Output S as the approximate solution.

Other examples: bin packing, Euclidean TSP.

Set covering

Given n subsets S1,S2,…,Sm of U
with U=1,2,…n (so |U|=n)

Find a minimum set C of {1,2,…m}
Such that $\bigcup_{i \text{ in } C} Si = U$

NP-hard;

Goal: to pick a subset which cover all items
and **minimize the cardinality** of this subset.

Greedy for set covering

General greedy method:

```
Sol = emptyset
While not finished
    choose the set that covers most elements not yet covered
```

Example

U={1,2,3,4,5,6}

Sets: –S^1={1,2} –S^2={3,4} –S^3={5,6} –S^4={1,3,5}

Algorithm picks {4,1,2,3}

Not optimal!

Greedy for set covering

Notation: C^{OPT} = optimal cover let $k=|C^{OPT}|$

Fact: At any iteration of the algorithm, there exists Sj which contains at ≥ 1/k fraction of yet-not-covered elements

Proof: by contradiction.

If all sets cover <1/k fraction of yet-not-covered elements, there is no way to cover them using k sets

But C^{OPT} does that !

Therefore, at each iteration greedy covers ≥1/k fraction of yet-not-covered elements

Greedy for set covering

Fact: At any iteration of the algorithm, there exists Sj which contains at $\geq 1/k$ fraction of yet-not-covered elements

Let C^i be the number of yet-not-covered elements at the end of step i=0,1,2,…

We have $C^{i+1} \leq C^i(1-1/k)$ $C^0=n$

Therefore, after t=k ln n steps, we have

$C^t \leq C^0 (1-1/k)^t \leq n (1-1/k)^{k \ln n} < n \, 1/e^{\ln n} =1$

I.e., all elements are covered by the k ln n sets chosen by greedy algorithm

Opt size is k $\Rightarrow$ greedy is ln(n)-approximate

The weighted set cover problem

- Assume set S has cost c(S).
- Find a min cost collection of sets that cover all elements $U=\{e_1,e_2,\ldots, e_n\}$
- Ci be the set of elements not yet covered at the beginning of iteration i
- The greedy algorithm repeatedly selects the most cost-effective set, i.e. the set that maximizes c(Si)/(Si∧Ci)
- Repeat till all elements are covered

- For an element ej covered by set Si define price(ej)= c(Si)/(Si∧Ci)

Claim: $C^{ALG} = \sum_j price(ej)$

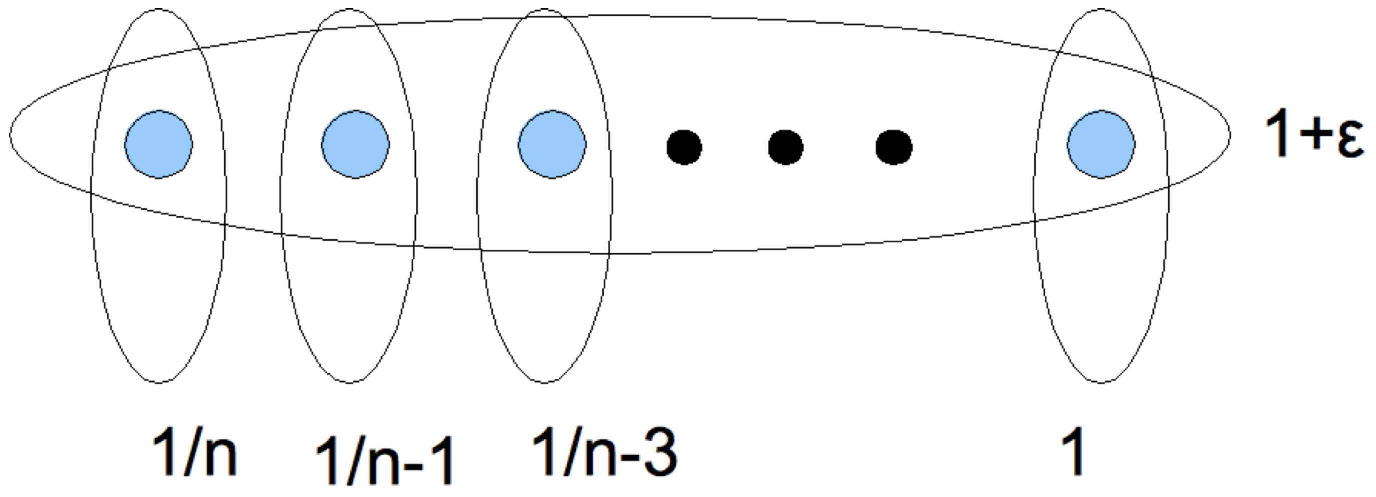
Analysis of Greedy Set Cover

- Claim: When set Si is selected there exists a set of OPT with cost effectiveness <= C^{OPT}/Ci
- For ej covered by set Si it holds $price(ej) \le \frac{C^{OPT}}{Ci} \le \frac{C^{OPT}}{n-j+1}$
- We conclude $C^{ALG} = \sum_j price(ej) \le \sum \frac{C^{OPT}}{n-j+1} \le H_n C^{OPT}$

Lower bound for Greedy

Optimal cost is $1 + \varepsilon$

Greedy cost is Hn



Better algorithm for Set covering?

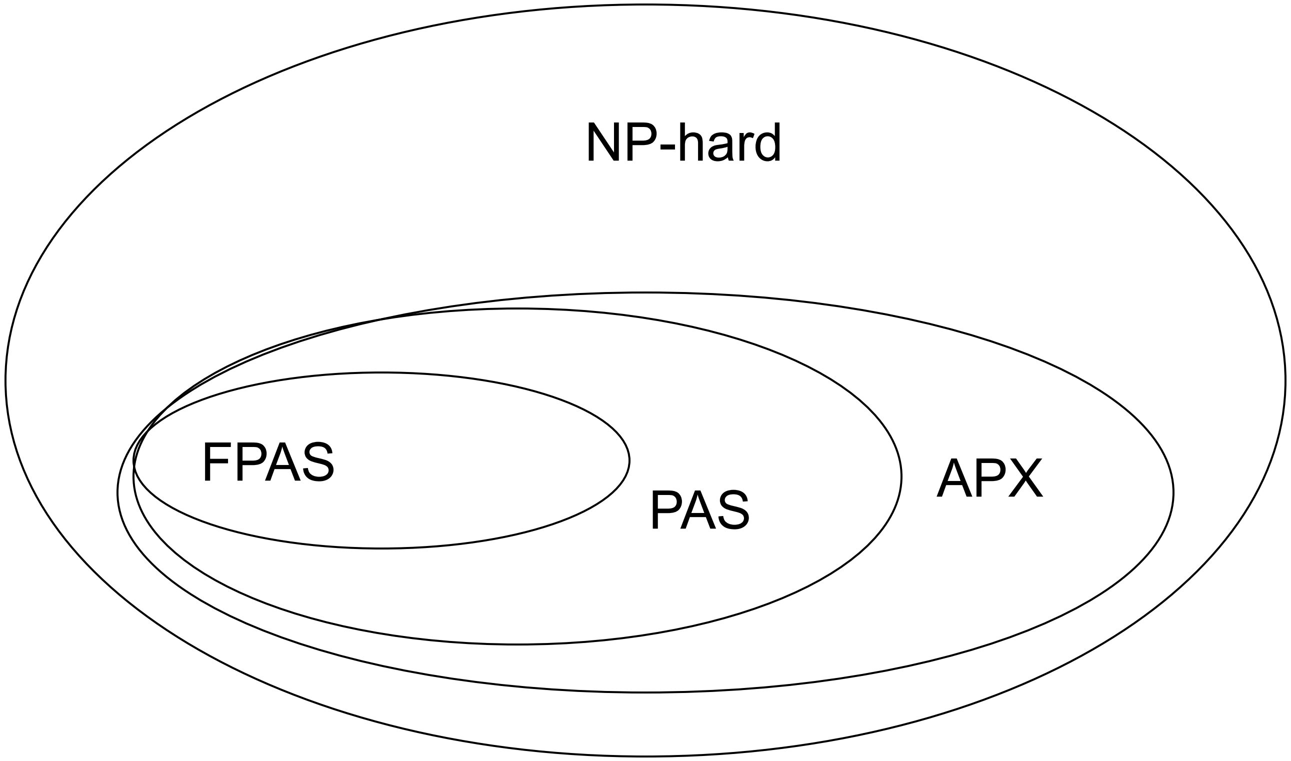
It is possible to show that approximating Set covering better than 0.999.. ln(m) is NP-hard!!

APX-hard problem: a problem for which there is a constant c such that it is NP-hard to find an approximation algorithm with approximation ratio better than c

Equivalently: there exists constant c such that finding an approximation better than c is as hard as finding the optimal solution

Claim: APX hard problems do not admit PAS (Polynomial Approximation Schemes)

APX, PAS and FPAS



PAS and FPAS

Class of APX problems: problems that have a Polynomial Approximation Algorithm: for some constant c running time polynomial in input lenght

Class of PAS problems: problem that have a Polynomial Approximation Schemes: for any given ϵ running time polynomial in input lenght

Class of FPAS problems: problems that have a Fully Polynomial Approximation Schemes: for any given ϵ running time polynomial in input lenght and $1/\epsilon$

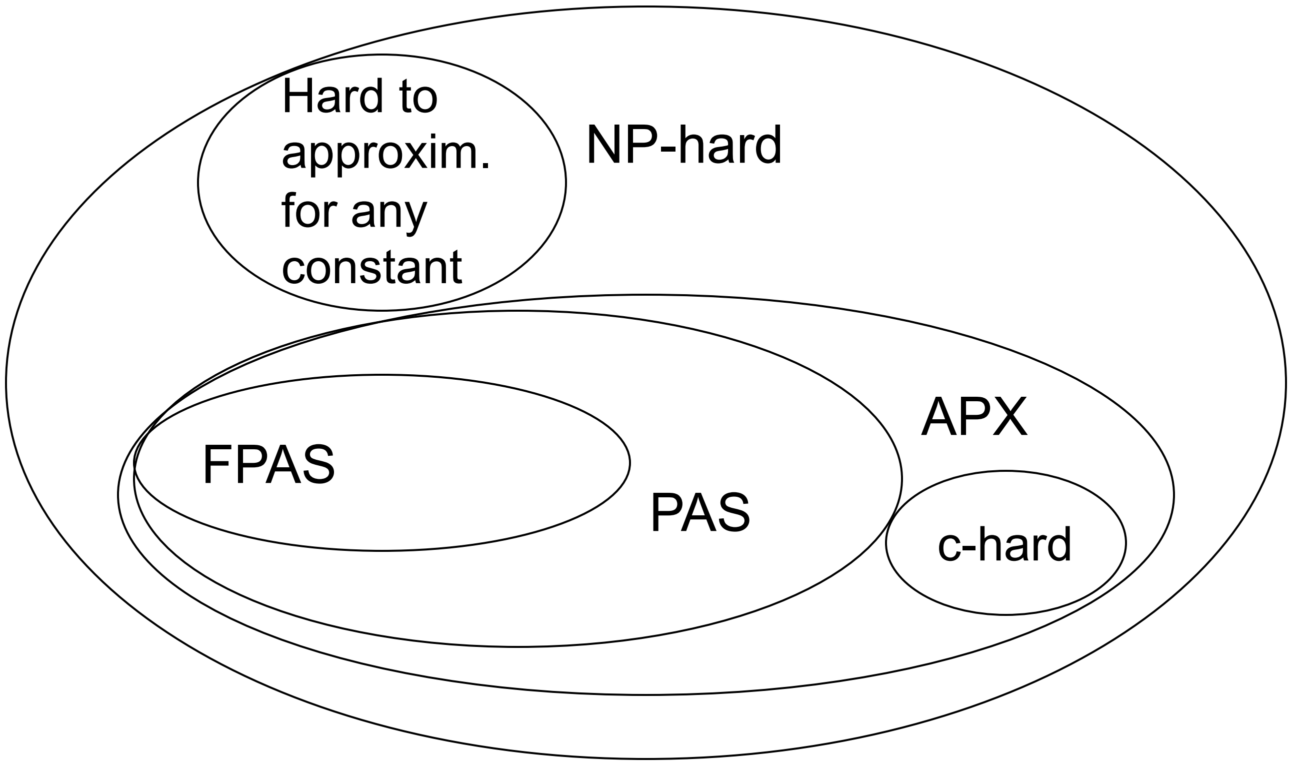
APX-hard problems

There are problems that are hard to approximate:

For any contant: TSP in the general case, Set covering

For some constant c: Vertex cover, Max Sat (maximize no. of satisfied clauses)

APX hard problems do not admit PAS or FPAS



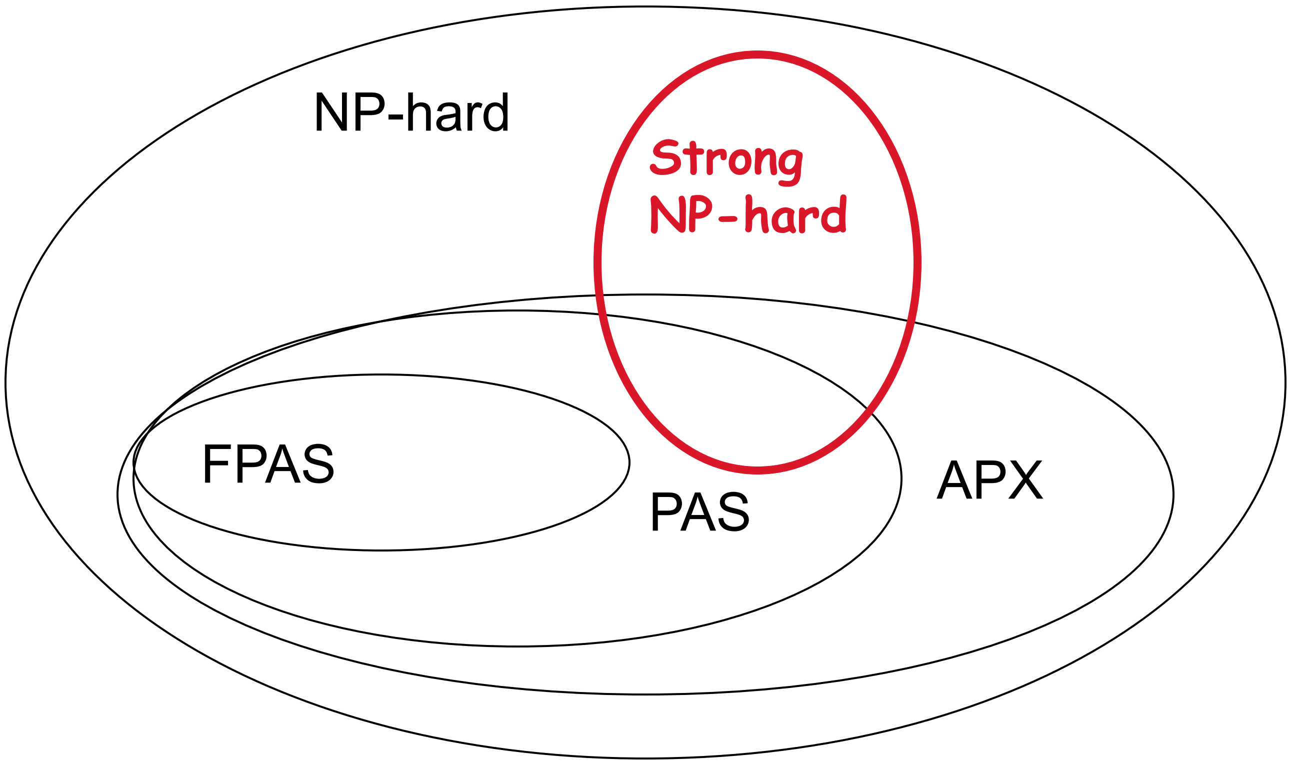
Strong NP-hard problems

There are Strong NP-hard problems that admit a PAS

But finding a FPAS is as hard as finding the optimum:

Generalization of Knapsack with two constraints

Note: not all Strong NP-hard admit a good approximation


NP-hard
Strong NP-hard
FPAS
PAS
APX