

# Foundations of Artificial Intelligence

## 11. Making Simple Decisions under Uncertainty

Probability Theory, Bayesian Networks, Other Approaches

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# Lecture Overview

- 1 Motivation
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- In many cases, our knowledge of the world is incomplete (not enough information) or uncertain (sensors are unreliable).
- Often, rules about the domain are incomplete or even incorrect
  - e.g., *qualification problem*: what are the preconditions for an action?
- We have to act in spite of this!
- Drawing conclusions under uncertainty

# Example

- **Goal:** Be in Freiburg at 9:15 to give a lecture.
- There are several **plans** that achieve the goal:
  - $P_1$ : Get up at 7:00, take the bus at 8:15, the train at 8:30, arrive at 9:00 ...
  - •  $P_2$ : Get up at 6:00, take the bus at 7:15, the train at 7:30, arrive at 8:00 ...
  - ...
- All these plans are correct, but
- They imply different **costs** and different **probabilities** of actually achieving the goal.
- $P_2$  eventually is the plan of choice, since giving a lecture is very important, and the success rate of  $P_1$  is only 90-95%.

# Uncertainty in Logical Rules (1)

Example: Expert dental diagnosis system.

$$\forall p[\text{Symptom}(p, \text{toothache}) \Rightarrow \text{Disease}(p, \text{cavity})]$$

→ This rule is *incorrect*! Better:

$$\forall p[\text{Symptom}(p, \text{toothache}) \Rightarrow \text{Disease}(p, \text{cavity}) \vee \text{Disease}(p, \text{gum\_disease}) \vee \dots]$$

... however, we do not know all the causes.

Perhaps a *causal* rule is better?

$$\forall p[\text{Disease}(p, \text{cavity}) \Rightarrow \text{Symptom}(p, \text{toothache})]$$

→ Does not allow to reason from symptoms to causes & is still wrong!

# Uncertainty in Logical Rules (2)

- We cannot enumerate all possible causes, and even if we could ...
- We do not know how correct the rules are (in medicine)
- ... and even if we did, there will always be uncertainty about the patient (the coincidence of having a toothache and a cavity that are unrelated, or the fact that not all tests have been run)
- Without perfect knowledge, logical rules do not help much!

Let us suppose we wanted to support the localization of a robot with (constant) landmarks. With the availability of landmarks, we can narrow down on the area.

Problem: Sensors can be imprecise.

- From the fact that a landmark was perceived, we cannot conclude with certainty that the robot is at that location.
- The same is true when no landmark is perceived.
- Only the probability increases or decreases.



# Degree of Belief and Probability Theory

- We (and other agents) are convinced by facts and rules only up to a certain degree.
- One possibility for expressing the degree of belief is to use probabilities.
- Probabilities as frequencies / subjective beliefs
  - e.g., the agent is 90% (or 0.9) convinced by its sensor information means that it believes that in 9 out of 10 cases, the information is correct
- Probabilities quantify the uncertainty that stems from lack of knowledge.
- Probabilities are not to be confused with vagueness. The predicate *tall* is vague; the statement, "A man is 1.75–1.80m tall" is uncertain.

# Uncertainty and Rational Decisions

And we want to make an action

- We have a choice of actions (or plans).
- These can lead to different results (worlds) with different probabilities.
- The actions have different (subjective) costs.
- The results have different (subjective) utilities.
- It would be rational to choose the action with the maximum expected total utility!

Decision Theory = Utility Theory + Probability Theory

$$\arg \max_a \sum_w p(w|a) \cdot [U(w) - c(a)]$$

# Decision-Theoretic Agent

**function** DT-AGENT(*percept*) **returns** an *action*

**persistent:** *belief\_state*, probabilistic beliefs about the current state of the world  
*action*, the agent's action

update *belief\_state* based on *action* and *percept*

calculate outcome probabilities for actions,

given action descriptions and current *belief\_state*

select *action* with highest expected utility

given probabilities of outcomes and utility information

**return** *action*

*Decision theory:* An agent is rational exactly when it chooses the action with the maximum expected utility taken over all results of actions.

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# Axiomatic Probability Theory

## Axioms of Probability Theory

A function  $P$  mapping from formulae in propositional logic to the set  $[0, 1]$  is a probability measure if for all propositions  $\phi$ ,  $\psi$  (whereby propositions are the equivalence classes formed by logically equivalent formulae):

1  $0 \leq P(\phi) \leq 1$

2  $P(\text{true}) = 1$

3  $P(\text{false}) = 0$

4  $P(\phi \vee \psi) = P(\phi) + P(\psi) - P(\phi \wedge \psi)$



All other properties can be derived from these axioms, for example:

$$P(\neg\phi) = 1 - P(\phi)$$

since  $1 \stackrel{(2)}{=} P(\phi \vee \neg\phi) \stackrel{(4)}{=} P(\phi) + P(\neg\phi) - \cancel{P(\phi \wedge \neg\phi)} \stackrel{(3)}{=} P(\phi) + P(\neg\phi).$

# Why are the Axioms Reasonable?

- If  $P$  represents an objectively observable probability, the axioms clearly make sense.
- But why should an agent respect these axioms when it models its own degree of belief?

→ *Objective* vs. *subjective* probabilities

The axioms limit the set of beliefs that an agent can maintain.

One of the most convincing arguments for why subjective beliefs should respect the axioms was put forward by de Finetti in 1931. It is based on the connection between actions and degree of belief:

- If the beliefs do not follow the axioms, then there exists a betting strategy (the so-called “dutch book”) against the agent, where he will definitely lose!

We use **random variables** such as *Weather* (capitalized word), which has a **domain** of ordered **values**. In our case that could be sunny, rain, cloudy, snow (lower case words).

A proposition might then be: *Weather = cloudy*.

If the random variable is Boolean, e.g., *Headache*, we may write either *Headache = true* or equivalently headache (lowercase!). Similarly, we may write *Headache = false* or equivalently  $\neg$ headache.

Further, we can of course use Boolean connectors, e.g.,  $\neg$ headache  $\wedge$  *Weather = cloudy*.

# Unconditional Probabilities (1)

$P(a)$  denotes the **unconditional** probability that it will turn out that  $A = \text{true}$  in the absence of any other information, for example:

$$P(\text{cavity}) = 0.1$$

$$P(\neg \text{cavity}) = 0.9$$

In case of non-Boolean random variables:

$$P(\text{Weather} = \text{sunny}) = 0.7$$

$$P(\text{Weather} = \text{rain}) = 0.2$$

$$P(\text{Weather} = \text{cloudy}) = 0.08$$

$$P(\text{Weather} = \text{snow}) = 0.02$$

1



# Unconditional Probabilities (2)

$\mathbf{P}(X)$  is the vector of probabilities for the (ordered) domain of the random variable  $X$ :

$$\mathbf{P}(\text{Headache}) = \langle 0.1, 0.9 \rangle$$

$$\mathbf{P}(\text{Weather}) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$$

define the probability distributions for the random variables *Headache* and *Weather*.

$\mathbf{P}(\text{Headache}, \text{Weather})$  is a  $4 \times 2$  table of probabilities of all combinations of the values of a set of random variables.

H \ W	t	f
s	0.3	0.4
c		?
s		0.6
s		

	Headache = <u>true</u>	Headache = <u>false</u>
Weather = <u>sunny</u>	$P(W = \text{sunny} \wedge \text{headache})$	$P(W = \text{sunny} \wedge \neg \text{headache})$
Weather = <u>rain</u>		
Weather = <u>cloudy</u>		
Weather = <u>snow</u>		

# Conditional Probabilities (1)

New information can change the probability.

Example: The probability of a cavity increases if we know the patient has a toothache.

If additional information is available, we can no longer use the prior probabilities!

$P(a|b)$  is the conditional or posterior probability of  $a$  given that *all* we know is  $b$ :

$$P(\text{cavity} | \text{toothache}) \neq 0.8$$

$P(X | Y)$  is the table of all conditional probabilities over all values of  $X$  and  $Y$ .

## Conditional Probabilities (2)

$P(\text{Weather} \mid \text{Headache})$  is a  $4 \times 2$  table of conditional probabilities of all combinations of the values of a set of random variables.

	<i>Headache = true</i>	<i>Headache = false</i>
<i>Weather = sunny</i>	$P(W = \text{sunny} \mid \text{headache})$	$P(W = \text{sunny} \mid \neg \text{headache})$
<i>Weather = rain</i>		
<i>Weather = cloudy</i>		
<i>Weather = snow</i>		

Conditional probabilities result from unconditional probabilities (if  $P(b) > 0$ ) (by definition):

$$\underline{P(a \mid b)} = \frac{P(a \wedge b)}{\underline{P(b)}}$$

$$P(a \wedge b) = P(a \mid b) \cdot P(b)$$

# Conditional Probabilities (3)

$\mathbf{P}(X, Y) = \mathbf{P}(X \mid Y)\mathbf{P}(Y)$  corresponds to a system of equations:

- $P(W = \textit{sunny} \wedge \textit{headache}) = P(W = \textit{sunny} \mid \textit{headache})P(\textit{headache})$
- $P(W = \textit{rain} \wedge \textit{headache}) = P(W = \textit{rain} \mid \textit{headache})P(\textit{headache})$
- $\dots = \dots$
- $P(W = \textit{snow} \wedge \neg \textit{headache}) = P(W = \textit{snow} \mid \neg \textit{headache})P(\neg \textit{headache})$
-

## Conditional Probabilities (4)

$$P(a \mid b) = \frac{P(a \wedge b)}{P(b)}$$

- Product rule:  $P(\underline{a \wedge b}) = P(a \mid b)P(b)$
- Similarly:  $P(a \wedge b) = P(b \mid a)P(a)$
- $a$  and  $b$  are **independent** iff  $P(a \mid b) = P(a)$   
(equiv.  $P(b \mid a) = P(b)$ ).  
Then (and only then) it holds that  $P(a \wedge b) = P(a)P(b)$ .

Making this assumption: what is the effect wrt. computational efficiency?

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# Joint Probability

The agent assigns probabilities to every proposition in the domain.

An **atomic event** assigns a value to every random variable  $X_1, \dots, X_n$  (= complete specification of a state).

Example: Let  $X$  and  $Y$  be Boolean variables. Then we have the following 4 atomic events:  $x \wedge y$ ,  $x \wedge \neg y$ ,  $\neg x \wedge y$ ,  $\neg x \wedge \neg y$ .

The **joint probability distribution**  $\mathbf{P}(X_1, \dots, X_n)$  assigns a probability to every *atomic event*. Example of such a complete instantiation:

	<i>toothache</i>	$\neg$ <i>toothache</i>
<i>cavity</i>	0.04	0.06
$\neg$ <i>cavity</i>	0.01	0.89

Observe: The sum of all fields is 1 (disjunction of events). Since all atomic events are disjoint, the conjunction of any two atomic events is necessarily *false*.

# Working with the Joint Probability

All relevant probabilities can be computed using the joint probability by expressing them as a disjunction of atomic events.

Examples:

$$\begin{aligned}P(\text{cavity} \vee \text{toothache}) &= P(\text{cavity} \wedge \text{toothache}) \\&\quad + P(\neg \text{cavity} \wedge \text{toothache}) \\&\quad + P(\text{cavity} \wedge \neg \text{toothache})\end{aligned}$$

We obtain marginal probabilities by adding across a row or column:

*only this*

$$P(\text{cavity}) = P(\text{cavity} \wedge \text{toothache}) + P(\text{cavity} \wedge \neg \text{toothache})$$

We obtain conditional probabilities by using a marginal probability:

$$P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \wedge \text{toothache})}{P(\text{toothache})} = \frac{0.04}{0.04 + 0.01} = 0.80$$



For any sets of variables  $Y$  and  $Z$  we have

$$\underline{P(Y)} = \sum_{\underline{z}} P(Y, \underline{z}) = \sum_{\underline{z}} P(Y | \underline{z}) P(\underline{z})$$

law of total probability

# Problems with Joint Probabilities

We can easily obtain all probabilities from the joint probability.

The joint probability, however, involves  $k^n$  values, if there are  $n$  random variables with  $k$  values.

→ Difficult to represent

→ Difficult to assess

Questions:

→ Is there a more compact way of representing joint probabilities?

→ Is there an efficient method to work with this representation?

Answer: Not in general, but it can work in many cases. Modern systems work directly with conditional probabilities and make assumptions on the independence of variables (→ conditional independence) to simplify calculations.

# Representing Joint Probabilities

Using the product rule  $P(a \wedge b) = P(a | b) P(b)$ , joint probabilities can be expressed as products of conditional probabilities.

$$P(x_1, \dots, x_n) = P(x_n | \dots, x_1)$$

# Representing Joint Probabilities

Using the product rule  $P(a \wedge b) = P(a|b)P(b)$ , joint probabilities can be expressed as products of conditional probabilities.

$$P(x_1, \dots, x_n) = P(x_n, \dots, x_1) = P(x_n | x_{n-1} \dots, x_1) P(x_{n-1}, \dots, x_1)$$

# Representing Joint Probabilities

Using the product rule  $P(a \wedge b) = P(a|b)P(b)$ , joint probabilities can be expressed as products of conditional probabilities.

$$\begin{aligned} P(x_1, \dots, x_n) &= P(x_n, \dots, x_1) = P(x_n | x_{n-1} \dots, x_1) P(x_{n-1}, \dots, x_1) \\ &= P(x_n | x_{n-1} \dots, x_1) P(x_{n-1} | x_{n-2} \dots, x_1) P(x_{n-2}, \dots, x_1) \end{aligned}$$

# Representing Joint Probabilities

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Can these transformations change the required storage size?

# Bayes' Rule

We know (product rule):

$$P(\underline{a \wedge b}) = P(a | b)P(b) \text{ and } P(\underline{a \wedge b}) = P(b | a)P(a)$$

By equating the right-hand sides, we get

$$P(a | b)P(b) = P(b | a)P(a)$$

$$\Rightarrow P(\underline{a | b}) = \frac{P(\underline{b | a})P(a)}{P(b)}$$

$$P(a | b) \uparrow$$

For multi-valued variables we get a set of equalities:

$$P(Y | X) = \frac{P(X | Y)P(Y)}{P(X)}$$

Generalization (conditioning on background evidence  $e$ ):

$$P(\underline{Y | X, e}) = \frac{P(X | Y, \underline{e})P(Y | \underline{e})}{P(X | \underline{e})}$$

# Applying Bayes' Rule

X

$$P(\text{toothache} \mid \text{cavity}) = 0.4$$

$$P(\text{cavity}) = 0.1$$

$$P(\text{toothache}) = 0.05$$

$$\longrightarrow P(\text{cavity} \mid \text{toothache}) = \frac{0.4 \times 0.1}{0.05} = 0.8$$

Why do we not try to assess  $P(\text{cavity} \mid \text{toothache})$  directly?

$P(\text{toothache} \mid \text{cavity})$  (causal) is more robust than  $P(\text{cavity} \mid \text{toothache})$  (diagnostic):

- $P(\text{toothache} \mid \text{cavity})$  is independent from the prior probabilities  $P(\text{toothache})$  and  $P(\text{cavity})$ .
- If there is a cavity epidemic and  $P(\text{cavity})$  increases,  $P(\text{toothache} \mid \text{cavity})$  does not change, but  $P(\text{toothache})$  and  $P(\text{cavity} \mid \text{toothache})$  will change proportionally.



# Relative Probability

Let's say we would also like to consider the probability that our patient has gum disease.

$$P(\text{toothache} \mid \text{gumdisease})$$

$$= 0.7$$

$$P(\text{gumdisease})$$

$$= 0.02$$

Which diagnosis is more probable? Cavity or gum disease?

$$P(c \mid t) = \frac{P(t \mid c)P(c)}{P(t)} \quad \text{or} \quad P(g \mid t) = \frac{P(t \mid g)P(g)}{P(t)}$$

If we are only interested in the relative probability, we need not assess  $P(t)$ :

$$\begin{aligned} \frac{P(c \mid t)}{P(g \mid t)} &= \frac{P(t \mid c)P(c)}{P(t)} \times \frac{P(t)}{P(t \mid g)P(g)} = \frac{P(t \mid c)P(c)}{P(t \mid g)P(g)} \\ &= \frac{0.4 \times 0.1}{0.7 \times 0.02} = 2.857 \end{aligned}$$

→ We elegantly excluded other possible diagnoses for toothache.

# Normalization (1)

If we wish to determine the absolute probability of  $P(c | t)$  but do not know  $P(t)$ , we can alternatively carry out a **complete** case analysis (e.g., for  $c$  and  $\neg c$ ) and use the fact that  $P(c | t) + P(\neg c | t) = 1$  (here Boolean variables):

$$P(c | t) = \frac{P(t | c)P(c)}{P(t)}$$

$$P(\neg c | t) = \frac{P(t | \neg c)P(\neg c)}{P(t)}$$

$$1 = P(c | t) + P(\neg c | t) = \frac{P(t | c)P(c)}{P(t)} + \frac{P(t | \neg c)P(\neg c)}{P(t)}$$

$$P(t) = P(t | c)P(c) + P(t | \neg c)P(\neg c) \quad *$$

## Normalization (2)

By substituting into the first equation:

$$P(c | t) = \frac{P(t | c)P(c)}{P(t | c)P(c) + P(t | \neg c)P(\neg c)}$$

*Handwritten notes: A green circle around  $P(c | t)$ , a green asterisk below it, a green circle around the denominator, and  $P(c|t)$  written to the right.*

For random variables with multiple values:

$$P(Y | X) = \alpha P(X | Y)P(Y)$$

*Handwritten notes:  $\propto P(X|Y) \cdot P(Y)$  in red above the equation, a green circle around  $\alpha$ , and a red underline under  $P(X|Y)P(Y)$ .*

$$\frac{P(c|t)}{2}$$

*Handwritten notes:  $P(c|t)$  written to the right of the fraction, and a green underline under the denominator 2.*

where  $\alpha$  is the **normalization constant** needed to make the entries in  $P(Y | X)$  sum to 1 for each value of  $X$ .

Example:  $\alpha(.1, .1, .3)$  =  $(.2, .2, .6)$ .

Remark: In ML, relative probabilities often are sufficient.

## Example

Your doctor tells you that you have tested positive for a serious but rare (1/10000) disease. This test ( $t$ ) is correct to 99% (1% false positive & 1% false negative results).

What does this mean for you?

$$P(d | t) = \frac{P(t | d) \cdot P(d)}{P(t)}$$

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What does this mean for you?

$$P(d | t) = \frac{P(t | d)P(d)}{P(t)} = \frac{P(t | d)P(d)}{P(t | d)P(d) + P(t | \neg d)P(\neg d)}$$

*Handwritten annotations:* 0.99 (above  $P(t | d)$ ), 1/10000 (above  $P(d)$ ). Red circles highlight  $P(t)$ ,  $P(t | d)P(d)$ ,  $P(t | \neg d)P(\neg d)$ , and  $P(\neg d)$ .

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Your doctor tells you that you have tested positive for a serious but rare (1/10000) disease. This test ( $t$ ) is correct to 99% (1% false positive & 1% false negative results).

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$$P(d) = 0.0001 \quad P(t | d) = 0.99 \quad P(t | \neg d) = 0.01$$

$$\begin{aligned} P(d | t) &= \frac{0.99 \times 0.0001}{0.99 \times 0.0001 + 0.01 \times 0.9999} = \frac{0.000099}{0.000099 + 0.009999} \\ &= \frac{0.000099}{0.010088} \approx 0.01 \end{aligned}$$

**Moral:** If the test imprecision is much greater than the rate of occurrence of the disease, then a positive result is not as threatening as you might think.

# Multiple Evidence (1)

A probe by the dentist catches ( $Catch = true$ ) in the aching tooth ( $Toothache = true$ ) of a patient. We already know that  $P(cavity | toothache) = 0.8$ . Furthermore, using Bayes' rule, we can calculate:

$$P(cavity | catch) = 0.95$$

But how does the combined evidence ( $tooth \wedge catch$ ) help?

Using Bayes' rule, the dentist could establish:

$$\begin{aligned} P(cav | tooth \wedge catch) &= \frac{P(tooth \wedge catch | cav) \times P(cav)}{P(tooth \wedge catch)} \\ &= \alpha P(tooth \wedge catch | cav) \times P(cav) \end{aligned}$$

## Multiple Evidence (2)

Problem: The dentist needs  $P(\text{tooth} \wedge \text{catch} \mid \text{cav})$ , i.e., diagnostic knowledge of all combinations of symptoms in the general case.

It would be nice if tooth and catch were independent but they are not:  $P(\text{tooth} \mid \text{catch}) \neq P(\text{tooth})$  - if a probe catches in the tooth, it probably has a cavity which probably causes toothache.



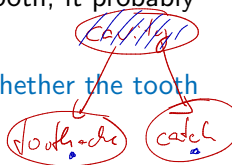
# Multiple Evidence (2)

Problem: The dentist needs  $P(\text{tooth} \wedge \text{catch} \mid \text{cav})$ , i.e., diagnostic knowledge of all combinations of symptoms in the general case.

It would be nice if *tooth* and *catch* were independent but they are not:  $P(\text{tooth} \mid \text{catch}) \neq P(\text{tooth})$  - if a probe catches in the tooth, it probably has a cavity which probably causes toothache.

They are conditionally independent given that we know whether the tooth has a cavity:

$$P(\text{tooth} \mid \text{catch}, \text{cav}) = P(\text{tooth} \mid \text{cav})$$



If one already knows that there is a cavity, then the additional knowledge of the probe catching does not change the probability.

$$P(\text{tooth} \wedge \text{catch} \mid \text{cav}) = P(\text{tooth} \mid \text{catch}, \text{cav})P(\text{catch} \mid \text{cav}) = P(\text{tooth} \mid \text{cav})P(\text{catch} \mid \text{cav})$$

Thus our diagnostic problem turns into:

$$P(cav \mid tooth \wedge catch) = \alpha P(tooth \wedge catch \mid cav)P(cav)$$

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$$\begin{aligned}P(cav \mid tooth \wedge catch) &= \alpha P(tooth \wedge catch \mid cav)P(cav) \\&= \alpha P(tooth \mid catch, cav)P(catch \mid cav)P(cav)\end{aligned}$$

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$$= \alpha P(tooth \mid catch, cav)P(catch \mid cav)P(cav)$$

$$= \alpha P(tooth \mid cav)P(catch \mid cav)P(cav)$$

The general definition of conditional independence of two variables  $X$  and  $Y$  given a third variable  $Z$  (a common cause) is:

$$\mathbf{P}(X, Y \mid Z) = \underline{\mathbf{P}(X \mid Z)} \underline{\mathbf{P}(Y \mid Z)}$$

# Conditional Independence - Further Example

Eating icecream and observing sunshine is not independent

$$P(\textit{ice} \mid \textit{sun}) \neq P(\textit{ice})$$

The variables *Ice* and *Sun* are not independent.

But if the reason for eating icecream is simply that it is hot outside, then the additional observation of sunshine does not make a difference:

$$P(\textit{ice} \mid \textit{sun}, \textit{hot}) = P(\textit{ice} \mid \textit{hot})$$

The variables *Ice* and *Sun* are conditionally independent given that *Hot = true* is observed.

The knowledge about independence often comes from insight of the domain and is part of the modelling of the problem. Conditional independence can often be exploited to make things simpler (see later).

Problem: we would like to avoid calculating the full joint probability table!

Assuming conditional independence, multiple evidence can be reduced to prior probabilities and conditional probabilities.

The general combination rule, if  $Z_1$  and  $Z_2$  are independent given  $X$  is

$$\mathbf{P}(X \mid Z_1, Z_2) = \alpha \mathbf{P}(X) \mathbf{P}(Z_1 \mid X) \mathbf{P}(Z_2 \mid X)$$

where  $\alpha$  is the normalization constant.

# Recursive Bayesian Updating

Problem: we would like to avoid calculating the full joint probability table!

Assuming conditional independence, multiple evidence can be reduced to prior probabilities and conditional probabilities.

The general combination rule, if  $Z_1$  and  $Z_2$  are independent given  $X$  is

$$\mathbf{P}(X \mid Z_1, Z_2) = \alpha \mathbf{P}(X) \mathbf{P}(Z_1 \mid X) \mathbf{P}(Z_2 \mid X)$$

where  $\alpha$  is the normalization constant.

Generalization: Recursive Bayesian Updating

$$\mathbf{P}(X \mid Z_1, \dots, Z_n) = \alpha \mathbf{P}(X) \prod_{i=1}^n \mathbf{P}(Z_i \mid X)$$



# Types of Variables

- Variables can be discrete or continuous:
- Discrete variables
  - *Weather: sunny, rain, cloudy, snow*
  - *Cavity: true, false* (Boolean)
- Continuous variables
  - Tomorrow's maximum temperature in Freiburg
  - Domain can be the entire real line or any subset.
  - Distributions for continuous variables are typically given by probability density functions.

# Summary

- **Uncertainty** is unavoidable in complex, dynamic worlds in which agents are ignorant.
- **Probabilities** express the agent's inability to reach a definite decision. They summarize the agent's beliefs.
- **Conditional** and **unconditional** probabilities can be formulated over propositions.
- If an agent disrespects the theoretical probability **axioms**, it is likely to demonstrate irrational behaviour.
- **Bayes' rule** allows us to calculate known probabilities from unknown probabilities.
- **Multiple evidence** (assuming independence) can be effectively incorporated using **recursive Bayesian updating**.

# Lecture Overview

- 1 Motivation
- 2 Foundations of Probability Theory
- 3 Probabilistic Inference
- 4 Bayesian Networks**
- 5 Alternative Approaches

Example domain: I am at work. My neighbour John calls me to tell me, that my alarm is ringing. My neighbour Mary doesn't call. Sometimes, the alarm is started by a slight earthquake.

Question: Is there a burglary?

Variables: *Burglary*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*.

Domain knowledge/ assumptions:

- Events *Burglary* and *Earthquake* are independent. (of course, to be discussed: a burglary does not cause an earthquake, but a burglar might use an earthquake to do the burglary. Then the independence assumption is not true. This is a design decision!)
- *Alarm* might be activated by burglary or earthquake
- John calls if and only if he heard the alarm. His call probability is not influenced by the fact, that there is an earthquake at the same time. Same for Mary.

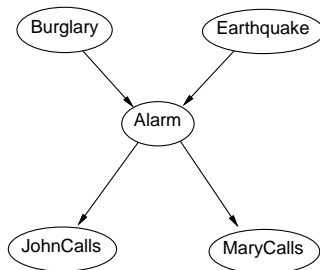
How to model this domain efficiently? Goal: Answer questions.

# Bayesian Networks

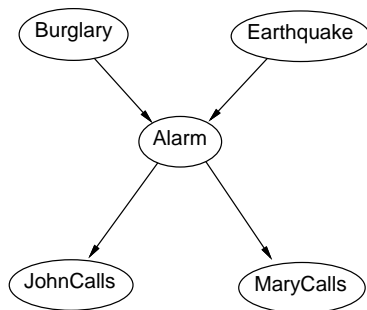
(also belief networks, probabilistic networks, causal networks)

- The *random variables* are the **nodes**.
- **Directed edges** between nodes represent *direct influence*.
- A **table of conditional probabilities** (CPT) is associated with every node, in which the effect of the **parent** nodes is quantified.
- The graph is **acyclic** (a DAG).

Remark: *Burglary* and *Earthquake* are denoted as the **parents** of *Alarm*



# The Meaning of Bayesian Networks



- *Alarm* depends on *Burglary* and *Earthquake*.
- *MaryCalls* only depends on *Alarm*.

$$P(\text{maryCalls} \mid \text{alarm}, \text{burglary}) = P(\text{maryCalls} \mid \text{alarm}) \text{ and}$$
$$P(\text{maryCalls} \mid \text{alarm}, \text{burglary}, \text{johnCalls}, \text{earthquake}) = P(\text{maryCalls} \mid \text{alarm})$$

→ Bayesian Networks can be considered as sets of (conditional) independence assumptions.

# Bayesian Networks and the Joint Probability

Bayesian networks can be seen as a more compact representation of joint probabilities.

Let all nodes  $X_1, \dots, X_n$  be ordered topologically according to the arrows in the network. Let  $x_1, \dots, x_n$  be the values of the variables. Then

$$\begin{aligned} P(x_1, \dots, x_n) &= P(x_n \mid x_{n-1}, \dots, x_1) \cdot \dots \cdot P(x_2 \mid x_1) P(x_1) \\ &= \prod_{i=1}^n P(x_i \mid x_{i-1}, \dots, x_1) \end{aligned}$$

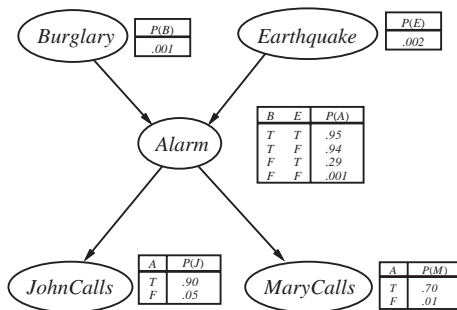
According to the independence assumption, this is equivalent to

$$P(x_1, \dots, x_n) = \prod_{i=1}^n P(x_i \mid \text{parents}(x_i))$$

We can calculate the joint probability from the network topology and the conditional probability tables (CPTs)!



# Example



Only prob. for pos. events are given, negative:  $P(\neg x) = 1 - P(x)$ . Note: the size of the table depends on the number of parents!

$$P(j, m, a, \neg b, \neg e) =$$

$$P(j \mid m, a, \neg b, \neg e)P(m \mid a, \neg b, \neg e)P(a \mid \neg b, \neg e)P(\neg b \mid \neg e)P(\neg e)$$

$$= P(j \mid a)P(m \mid a)P(a \mid \neg b, \neg e)P(\neg b)P(\neg e)$$

$$= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 = 0.00062$$

# Compactness of Bayesian Networks

- For the explicit representation of Bayesian networks, we need a table of size  $2^n$  where  $n$  is the number of variables.
- In the case that every node in a network has at most  $k$  parents, we only need  $n$  tables of size  $2^k$  (assuming Boolean variables).
- *Example:*  $n = 20$  and  $k = 5$ 
  - $2^{20} = 1,048,576$  and  $20 \times 2^5 = 640$  different explicitly-represented probabilities!
  - In the worst case, a Bayesian network can become exponentially large, for example if every variable is directly influenced by all the others.
  - The size depends on the application domain (local vs. global interaction) and the skill of the designer.

# Naive Design of a Network

- Order all variables
- Take the first from those that remain
- Assign all direct influences from nodes already in the network to the new node (Edges + CPT).
- If there are still variables in the list, repeat from step 2.

# Example 1

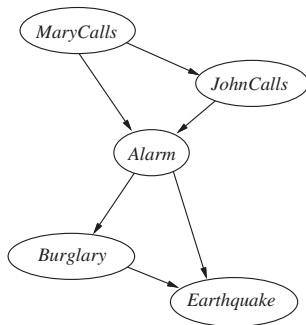
$M, J, A, B, E$

## Example 2

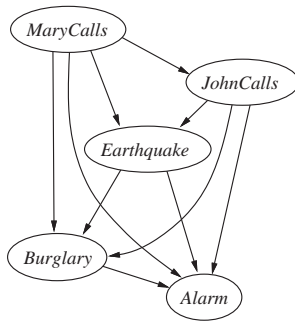
$M, J, E, B, A$

# Example

left =  $M, J, A, B, E$ , right =  $M, J, E, B, A$



(a)

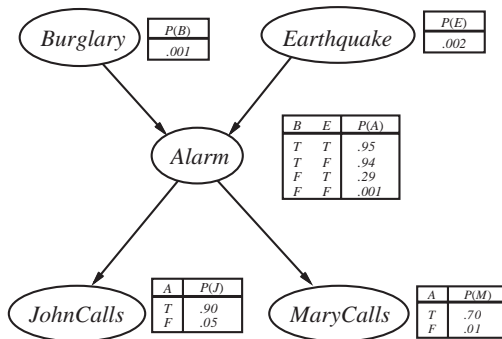


(b)

→ Appears to be an attempt to build a diagnostic model of symptoms and causes, which always leads to dependencies between causes that are actually independent and symptoms that appear separately.

# Inference in Bayesian Networks

Instantiating evidence variables and sending queries to nodes.



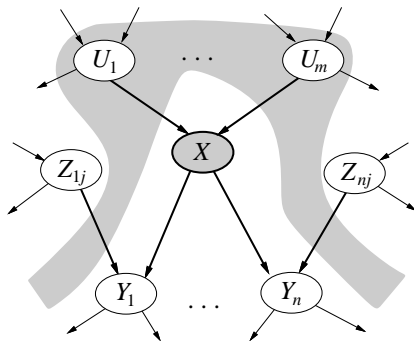
What is  
or

$$P(\text{burglary} \mid \text{johncalls})$$

$$P(\text{burglary} \mid \text{johnCalls}, \text{maryCalls})?$$

# Conditional Independence Relations in Bayesian Networks (1)

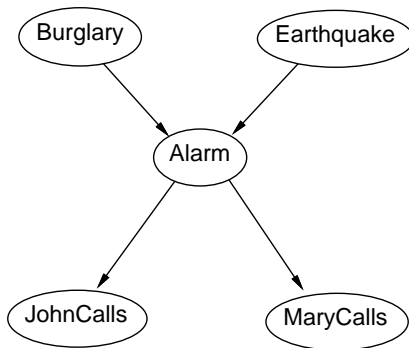
A node is conditionally independent of its non-descendants given its parents.





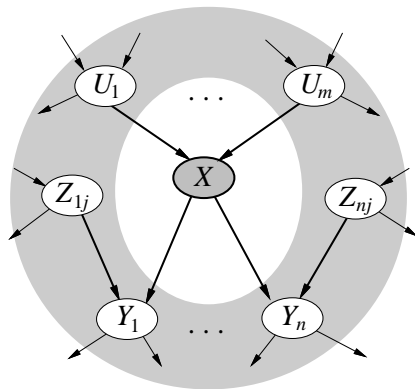
# Example

*JohnCalls* is independent of *Burglary* and *Earthquake* given the value of *Alarm*.



# Conditional Independence Relations in Bayesian Networks (2)

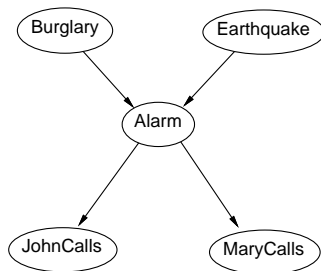
A node is conditionally independent of all other nodes in the network given the **Markov blanket**, i.e., its parents, children and children's parents.



# Example

*Burglary* is independent of *JohnCalls* and *MaryCalls*, given the values of *Alarm* and *Earthquake*, i.e.,

$$\begin{aligned} P(\textit{Burglary} \mid \textit{JohnCalls}, \textit{MaryCalls}, \textit{Alarm}, \textit{Earthquake}) \\ = P(\textit{Burglary} \mid \textit{Alarm}, \textit{Earthquake}) \end{aligned}$$



# Exact Inference in Bayesian Networks

- Compute the **posterior probability** distribution for a **set of query variables**  $X$  given an observation, i.e., the values of a **set of evidence variables**  $E$ .
- Complete set of variables is  $X \cup E \cup Y$
- $Y$  are called the **hidden variables**
- Typical query  $P(X \mid e)$  where  $e$  are the observed values of  $E$ .
- In the remainder:  $X$  is a singleton

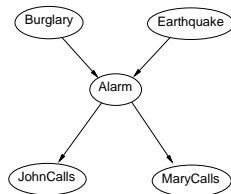
Example:

$$\mathbf{P}(\textit{Burglary} \mid \textit{JohnCalls} = \textit{true}, \textit{MaryCalls} = \textit{true}) = (0.284, 0.716)$$

- $P(X \mid e) = \alpha P(X, e) = \sum_y \alpha P(X, e, y)$
- The network gives a complete representation of the full joint distribution.
- A query can be answered using a Bayesian network by computing sums of products of conditional probabilities from the network.
- We sum over the hidden variables.

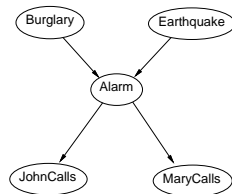
# Example

- Consider  $\mathbf{P}(\text{Burglary} \mid \text{JohnCalls} = \text{true}, \text{MaryCalls} = \text{true})$
- The evidence variables are



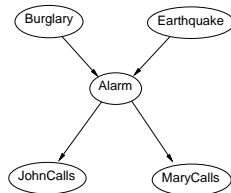
# Example

- Consider  $\mathbf{P}(\text{Burglary} \mid \text{JohnCalls} = \text{true}, \text{MaryCalls} = \text{true})$
- The evidence variables are *JohnCalls* and *MaryCalls*.
- The hidden variables are



# Example

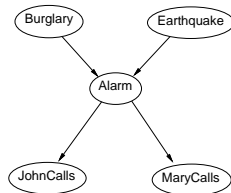
- Consider  $\mathbf{P}(Burglary \mid JohnCalls = true, MaryCalls = true)$
- The evidence variables are *JohnCalls* and *MaryCalls*.
- The hidden variables are *Earthquake* and *Alarm*.
- We have:  $\mathbf{P}(B \mid j, m) = \alpha \mathbf{P}(B, j, m)$





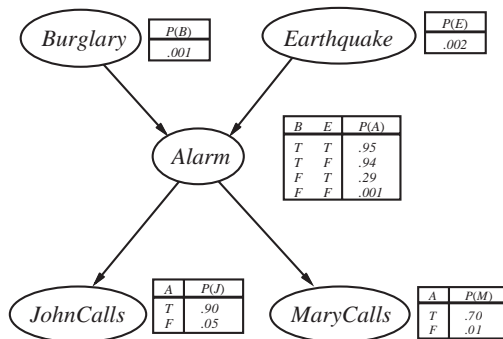
# Example

- Consider  $\mathbf{P}(\text{Burglary} \mid \text{JohnCalls} = \text{true}, \text{MaryCalls} = \text{true})$
- The evidence variables are *JohnCalls* and *MaryCalls*.
- The hidden variables are *Earthquake* and *Alarm*.



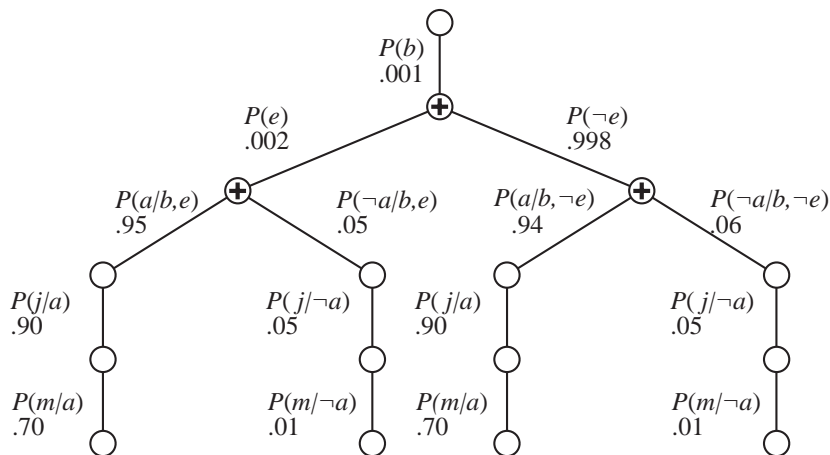
- We have: 
$$\mathbf{P}(B \mid j, m) = \alpha \mathbf{P}(B, j, m)$$
$$= \alpha \sum_e \sum_a \mathbf{P}(B, j, m, e, a)$$
- If we consider the independence of variables, we obtain for  $B = \text{true}$ 
$$P(b \mid j, m) = \alpha \sum_e \sum_a P(j \mid a) P(m \mid a) P(a \mid e, b) P(e) P(b)$$
- Reorganization of the terms yields:
$$P(b \mid j, m) = \alpha P(b) \sum_e P(e) \sum_a P(a \mid e, b) P(j \mid a) P(m \mid a)$$

# Recall Bayesian Network for Domain



# Evaluation of $P(b \mid j, m)$

$$P(b \mid j, m) = \alpha P(b) \sum_e P(e) \sum_a P(a \mid e, b) P(j \mid a) P(m \mid a)$$



$$\mathbf{P}(B \mid j, m) = \alpha(0.0006, 0.0015) = (0.284, 0.716)$$

# Enumeration Algorithm for Answering Queries on Bayesian Networks

**function** ENUMERATION-ASK( $X, \mathbf{e}, bn$ ) **returns** a distribution over  $X$

**inputs:**  $X$ , the query variable

$\mathbf{e}$ , observed values for variables  $\mathbf{E}$

$bn$ , a Bayes net with variables  $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$  /\*  $\mathbf{Y} = \text{hidden variables}$  \*/

$\mathbf{Q}(X) \leftarrow$  a distribution over  $X$ , initially empty

**for each** value  $x_i$  of  $X$  **do**

$\mathbf{Q}(x_i) \leftarrow \text{ENUMERATE-ALL}(bn.\text{VARS}, \mathbf{e}_{x_i})$

where  $\mathbf{e}_{x_i}$  is  $\mathbf{e}$  extended with  $X = x_i$

**return** NORMALIZE( $\mathbf{Q}(X)$ )

---

**function** ENUMERATE-ALL( $vars, \mathbf{e}$ ) **returns** a real number

**if** EMPTY?( $vars$ ) **then return** 1.0

$Y \leftarrow \text{FIRST}(vars)$

**if**  $Y$  has value  $y$  in  $\mathbf{e}$

**then return**  $P(y \mid \text{parents}(Y)) \times \text{ENUMERATE-ALL}(\text{REST}(vars), \mathbf{e})$

**else return**  $\sum_y P(y \mid \text{parents}(Y)) \times \text{ENUMERATE-ALL}(\text{REST}(vars), \mathbf{e}_y)$

where  $\mathbf{e}_y$  is  $\mathbf{e}$  extended with  $Y = y$

# Properties of the ENUMERATION-ASK Algorithm

- The ENUMERATION-ASK algorithm evaluates the trees in a depth-first manner.
- Space complexity is linear in the number of variables.
- Time complexity for a network with  $n$  Boolean variables is  $O(2^n)$ , since in the worst case, all terms must be evaluated for the two cases (“true” and “false”)

- The enumeration algorithm can be improved significantly by **eliminating repeating or unnecessary calculations**.
- The key idea is to **evaluate expressions from right to left (bottom-up) and to save results for later use**.
- Additionally, **unnecessary expressions can be removed**.

# Example

- Let us consider the query  $P(\text{JohnCalls} \mid \text{Burglary} = \text{true})$ .
- The nested sum is

$$P(j, b) = \alpha P(b) \sum_e P(e) \sum_a P(a \mid b, e) P(j, a) \sum_m P(m \mid a)$$

# Example

- Let us consider the query  $P(\text{JohnCalls} \mid \text{Burglary} = \text{true})$ .
- The nested sum is

$$P(j, b) = \alpha P(b) \sum_e P(e) \sum_a P(a \mid b, e) P(j, a) \sum_m P(m \mid a)$$

- Obviously, the rightmost sum equals 1 so that it can safely be dropped.
- general observation: variables, that are not query or evidence variables and not **ancestor nodes** of **query or evidence variables** can be removed. **Variable elimination repeatedly removes these variables** and this way speeds up computation.



# Example

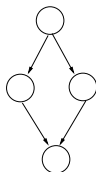
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- general observation: variables, that are not query or evidence variables and not **ancestor nodes** of **query or evidence variables** can be removed. **Variable elimination repeatedly removes these variables** and this way speeds up computation.
- within example: *Alarm* and *Earthquake* are ancestor nodes of query variable *JohnCalls* and cannot be removed. *MaryCalls* is neither a query nor an evidence variable and no ancestor node. Therefore it can be removed.

# Complexity of Exact Inference

- If the network is singly connected or a **polytree** (at most one undirected path between two nodes in the graph), the time and space complexity of exact inference is linear in the size of the network.
- The burglary example is a typical singly connected network.
- For multiply connected networks **inference in Bayesian Networks is NP-hard**.



- There are **approximate inference methods** for **multiply connected networks** such as sampling techniques or Markov chain Monte Carlo.

# Lecture Overview

- 1 Motivation
- 2 Foundations of Probability Theory
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# Other Approaches (1)

- Rule-based methods with “certainty factors”.
  - Logic-based systems with weights attached to rules, which are combined using inference.
  - Had to be designed carefully to avoid undesirable interactions between different rules.
  - Might deliver incorrect results through overcounting of evidence.
  - Their use is no longer recommended.

- Dempster-Shafer Theory

- Allows the representation of *ignorance* as well as uncertainty.
- Example: If a coin is fair, we assume  $P(Heads) = 0.5$ . But what if we do not know if the coin is fair?  $\rightarrow Bel(Heads) = 0, Bel(Tails) = 0$ . If the coin is 90% fair,  $0.5 \times 0.9$ , i.e.  $Bel(Heads) = 0.45$ .

- $\rightarrow$  Interval of probabilities is  $[0.45, 0.55]$  with the evidence,  $[0, 1]$  without.
- $\rightarrow$  The notion of utility is not yet well understood in Dempster-Shafer Theory.

- Fuzzy logic and fuzzy sets
  - A means of representing and working with **vagueness**, not uncertainty.
  - Example: The car is *fast*.
  - Used especially in control and regulation systems.
  - In such systems, it can be interpreted as an *interpolation technique*.

# Summary

- Bayesian Networks allow a compact representation of joint probability distribution.
- Bayesian Networks provide a concise way to represent conditional independence in a domain.
- Inference in Bayesian networks means computing the probability distribution of a set of query variables, given a set of evidence variables.
- Exact inference algorithms such as variable elimination are efficient for poly-trees.
- In complexity of belief network inference depends on the network structure.
- In general, Bayesian network inference is NP-hard.