Foundations of Artificial Intelligence

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Exercise Sheet 5 — Solutions

Exercise 5.1 (Resolution Method)

Given clause set K:

$$K = \{\{A, B, \neg C\}, \{\neg A, C\}, \{\neg A, \neg B\}, \{A, C\}\},$$

using the resolution method, show whether $K \models (\neg B \rightarrow (A \wedge C))$ holds.

Solution:

In order to show that $K \models \varphi$, it is sufficient to show that $K \cup \{\neg\varphi\} \models \bot$. Therefore, we first have to extend K by the clauses corresponding to $\neg(\neg B \rightarrow (A \wedge C))$, and then derive a contradiction (empty set) using resolution. Transformation of $\neg(\neg B \rightarrow (A \wedge C))$ to CNF:

$$\begin{aligned} \neg(\neg B \rightarrow (A \wedge C)) &\equiv \neg B \wedge \neg(A \wedge C) \\ &\equiv \neg B \wedge (\neg A \vee \neg C) \end{aligned}$$

i.e., $\{\{\neg B\}, \{\neg A, \neg C\}\}$.
Resolution (one of many possibilities, recommended notation):

$$\Big\{A, B, \neg C\Big\} \tag{1}$$
$$\Big\{\neg A, C\Big\} \tag{2}$$
$$\Big\{\neg A, \neg B\Big\} \tag{3}$$
$$\Big\{A, C\Big\} \tag{4}$$
$$\Big\{\neg B\Big\} \tag{5}$$
$$\Big\{\neg A, \neg C\Big\} \tag{6}$$
$$(1) + (5): \quad \Big\{A, \neg C\Big\} \tag{7}$$
$$(2) + (6): \quad \Big\{\neg A\Big\} \tag{8}$$
$$(4) + (7): \quad \Big\{A\Big\} \tag{9}$$
$$(8) + (9): \quad \emptyset \tag{10}$$

Exercise 5.2 (Modeling, Proofs)

Consider the following knowledge base:

If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is immortal or a mammal, then it is horned. The unicorn is magical if it is horned.

Using this knowledge base, can you prove that the unicorn is (a) mythical, (b) magical or (c) horned? First, formalize the knowledge base with propositional logic. If a statement is valid or unsatisfiable, use resolution for prove. Else, write down one satisfying and one unsatisfying interpretation.

Solution:

The above statements can be formalized using the atomic propositions *mythical*, *mortal*, *mammal*, *magical* and *horned*:

$$\begin{aligned} \mathit{mythical} &\rightarrow \neg \mathit{mortal} && \text{(i)} \\ \neg \mathit{mythical} &\rightarrow \mathit{mortal} \wedge \mathit{mammal} && \text{(ii)} \\ \neg \mathit{mortal} \vee \mathit{mammal} &\rightarrow \mathit{horned} && \text{(iii)} \\ \mathit{horned} &\rightarrow \mathit{magical} && \text{(iv)} \end{aligned}$$

Let $KB = (i) \wedge (ii) \wedge (iii) \wedge (iv)$ be the set of those four propositions, in CNF:

$$\left\{\neg \mathit{mythical}, \neg \mathit{mortal}\right\} \tag{1}$$

$$\left\{\mathit{mythical}, \mathit{mortal}\right\} \tag{2}$$

$$\left\{\mathit{mythical}, \mathit{mammal}\right\} \tag{3}$$

$$\left\{\mathit{mortal}, \mathit{horned}\right\} \tag{4}$$

$$\left\{\neg \mathit{mammal}, \mathit{horned}\right\} \tag{5}$$

$$\left\{\neg \mathit{horned}, \mathit{magical}\right\} \tag{6}$$

(a) We cannot tell from KB whether the unicorn is mythical or not, since there exist two models $I_{mythical}$ and $I_{\neg mythical}$ of KB with $I_{mythical} \models \mathit{mythical}$ and $I_{\neg mythical} \models \neg \mathit{mythical}$.

More specific:

$I_{mythical} = \{\mathit{mythical} \mapsto 1, \mathit{mortal} \mapsto 0, \mathit{mammal} \mapsto 0, \mathit{magical} \mapsto 1, \mathit{horned} \mapsto 1\}$

and

$I_{\neg mythical} = \{\mathit{mythical} \mapsto 0, \mathit{mortal} \mapsto 1, \mathit{mammal} \mapsto 1, \mathit{magical} \mapsto 1, \mathit{horned} \mapsto 1\}$.

Thus, $KB \not\models \neg \mathit{mythical}$ and $KB \not\models \mathit{mythical}$, since neither *mythical* nor $\neg$*mythical* are true in *all* models of KB. In other words, the unicorn could be mythical or not, depending on the choice of model for KB.

(b) $KB \models \mathit{horned}$ holds, since $KB \cup \{\neg \mathit{horned}\}$ is unsatisfiable. Proof:

Let

$$\left\{\neg \mathit{horned}\right\} \tag{7a}$$

then

$$(5) + (7a): \quad \Big\{\neg mammal\Big\} \tag{8a}$$

$$(4) + (7a): \quad \Big\{mortal\Big\} \tag{9a}$$

$$(3) + (8a): \quad \Big\{mythical\Big\} \tag{10a}$$

$$(1) + (10a): \quad \Big\{\neg mortal\Big\} \tag{11a}$$

$$(11a) + (9a): \quad \emptyset$$

(c) $KB \models magical$ holds, since $KB \cup \{\neg magical\}$ is unsatisfiable. Proof:

Let

$$\Big\{\neg magical\Big\} \tag{7b}$$

then

$$(6) + (7b): \quad \Big\{\neg horned\Big\} \tag{8b}$$

$$(5) + (8b): \quad \Big\{\neg mammal\Big\} \tag{9b}$$

$$(4) + (8b): \quad \Big\{mortal\Big\} \tag{10b}$$

$$(3) + (9b): \quad \Big\{mythical\Big\} \tag{11b}$$

$$(1) + (11b): \quad \Big\{\neg mortal\Big\} \tag{12b}$$

$$(12b) + (10b): \quad \emptyset$$

Exercise 5.3 (DPLL)

Use the Davis-Putnam-Logemann-Loveland (DPLL) procedure to check whether the given formulae ϕ_1 and ϕ_2 are satisfiable or not. Write down all steps carried out by the algorithm during the process. If you have to apply a splitting rule, split on variables in alphabetical order, trying *true* first, then *false*. Should the formula be satisfiable, please indicate the satisfying assignment.

(a)

$$\phi_1 = (D \vee C) \wedge (\neg A \vee \neg D \vee B \vee \neg C) \wedge (A \vee C) \wedge (\neg C \vee \neg B) \wedge (\neg C \vee B \vee \neg A \vee D) \wedge (C \vee \neg D \vee B) \wedge (\neg D \vee \neg B \vee \neg A)$$

Solution:

Splitting $A \to 1$	$(D \vee C) \wedge (\neg D \vee B \vee \neg C) \wedge (\neg C \vee \neg B) \wedge (\neg C \vee B \vee D) \wedge$ $(C \vee \neg D \vee B) \wedge (\neg D \vee \neg B)$
Splitting $B \to 1$	$(D \vee C) \wedge \neg C \wedge \neg D$
Unit-propagation $C \to 0$	$D \wedge \neg D$
Unit-propagation $D \to 1$	$\perp$
Backtracking $B \to 0$	$(D \vee C) \wedge (\neg D \vee \neg C) \wedge (\neg C \vee D) \wedge (C \vee \neg D)$
Splitting $C \to 1$	$\neg D \wedge D$
Unit-propagation $D \to 1$	$\perp$
Backtracking $C \to 0$	$D \wedge \neg D$
Unit-propagation $D \to 1$	$\perp$
Backtracking $A \to 0$	$(D \vee C) \wedge C \wedge (\neg C \vee \neg B) \wedge (C \vee \neg D \vee B)$
Unit-propagation $C \to 1$	$\neg B$
Unit-propagation $B \to 0$	$\top$

Satisfying assignment: $A \to 0; B \to 0; C \to 1; D \to 1$ or 0;

(b)

$$\phi_2 = (D \vee \neg A \vee B) \wedge (\neg B \vee \neg C \vee A \vee D) \wedge (\neg B \vee \neg A) \wedge (B \vee \neg D) \wedge (A \vee C \vee \neg B) \wedge (\neg D \vee \neg C) \wedge (B \vee D)$$

Solution:

Splitting $A \to 1$	$(D \vee B) \wedge \neg B \wedge (B \vee \neg D) \wedge (\neg D \vee \neg C) \wedge (B \vee D)$
Unit-propagation $B \to 0$	$D \wedge \neg D \wedge (\neg D \vee \neg C) \wedge D$
Unit-propagation $D \to 1$	$\perp$
Backtracking $A \to 0$	$(\neg B \vee \neg C \vee D) \wedge (B \vee \neg D) \wedge (C \vee \neg B) \wedge (\neg D \vee \neg C) \wedge (B \vee D)$
Splitting $B \to 1$	$(\neg C \vee D) \wedge C \wedge (\neg D \vee \neg C)$
Unit-propagation $C \to 1$	$D \wedge \neg D$
Unit-propagation $D \to 1$	$\perp$
Backtracking $B \to 0$	$\neg D \wedge (\neg D \vee \neg C) \wedge D$
Unit-propagation $D \to 1$	$\perp$

The formula is unsatisfiable.