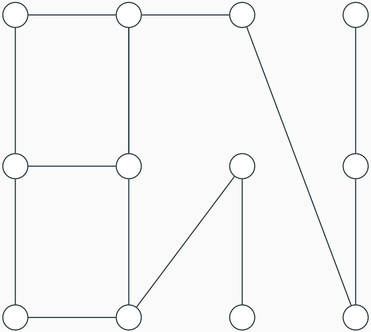
Hopcroft-Karp Algorithm for Bipartite Matching

(Matching Results Chris Likes)

Maximum Matching

Given a graph $G(V, E)$, a matching is a set of edges $M \subseteq E$, such that no two edges of M share a common node.



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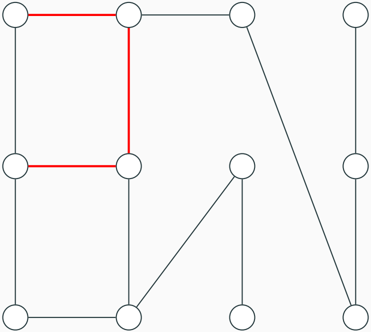

not a matching

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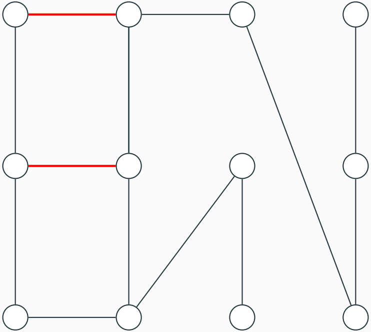

a valid matching

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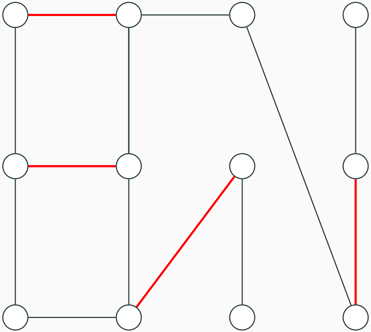

a maximal matching

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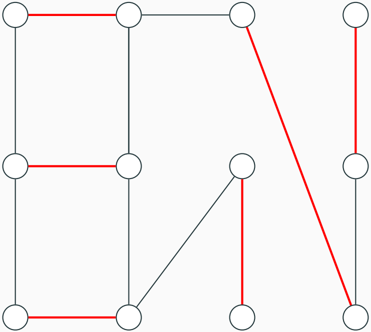

a maximum matching

Relationship Between Maximum and Maximal

For any two maximal matchings A and B, we have

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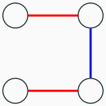

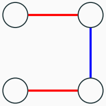

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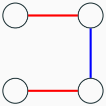

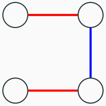

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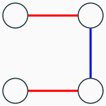

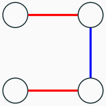

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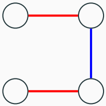
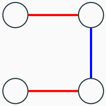


$$|A| \;=\; |A \cap B| + |A \setminus B|$$



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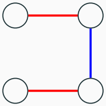

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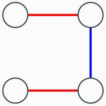

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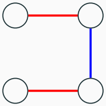

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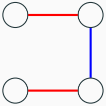

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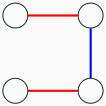

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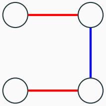

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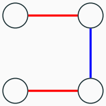

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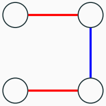


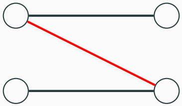
Algorithms for Maximum Cardinality Matching

General: $O(m\sqrt{n})$, Micali, Vazirani, 1980
$O(n^{\omega})$[1] Mucha, Sankowski, 2004

Bipartite: $O(m\sqrt{n})$, Hopcroft, Karp, 1973
$O(n^{\omega})$, Mucha, Sankowski, 2004
$\tilde{O}(m^{10/7})$, Madry, 2013

[1] $\omega \approx 2.37286...$

Augmenting Paths in Flows



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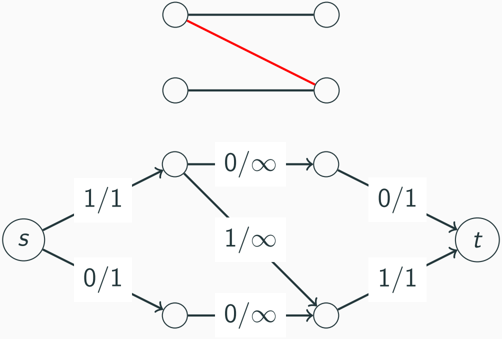

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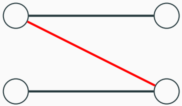

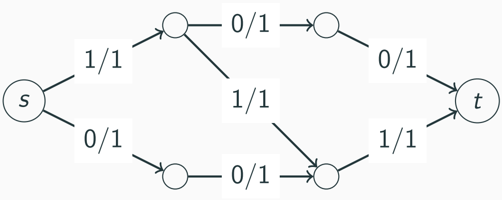

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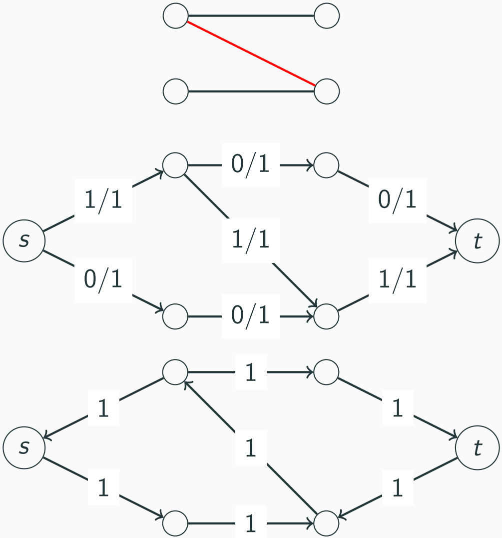

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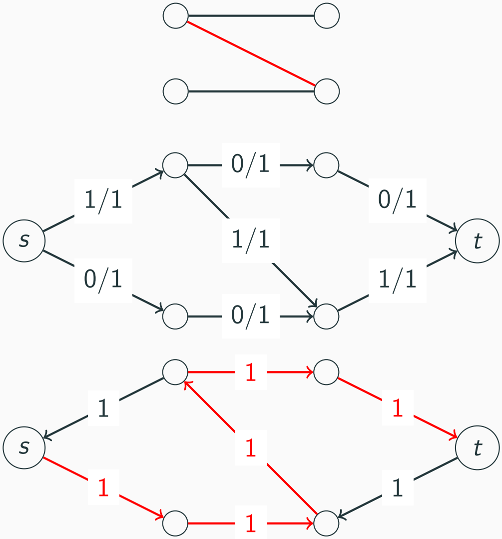

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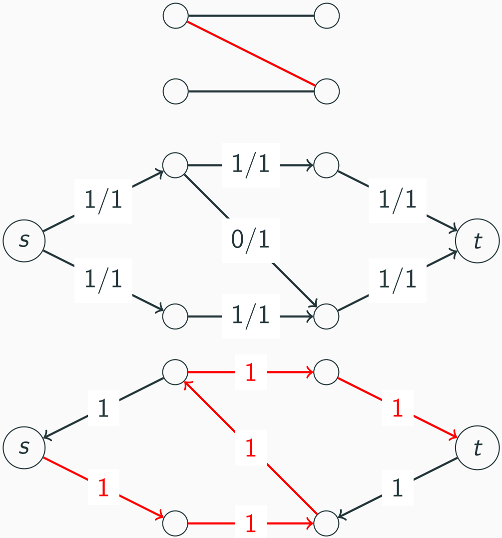

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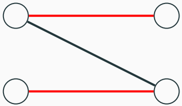

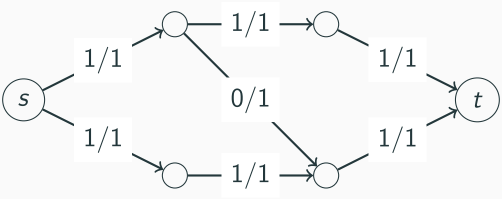

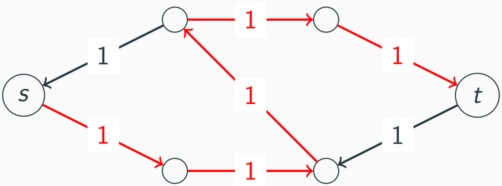

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Augmenting Paths: A path starting and ending with two free nodes such that matched and unmatched edges alternate.

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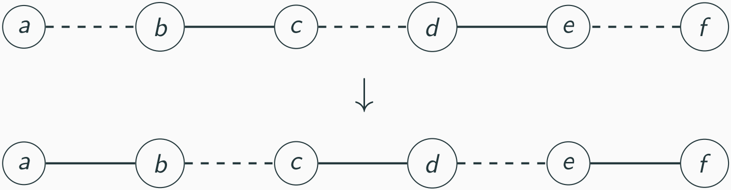

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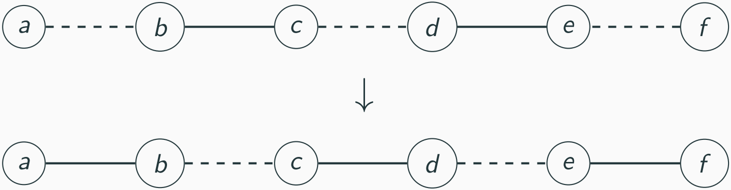


Given a matching M and an augmenting path p, we obtain a larger matching $M' = M \bigoplus p$ with $|M'| = |M| + 1$.

Optimum and Augmenting Paths

Lemma 0.5 (Berge's Theorem)

M is an optimal matching if and only if there exist no augmenting paths with respect to M.

$\Rightarrow$: Augmenting paths increase matching by 1.

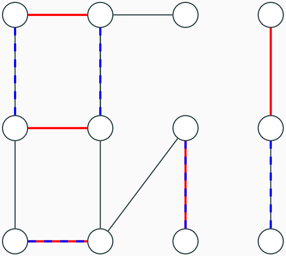
$\Leftarrow$: Let M' be an optimum.

$M' \bigoplus M$ can consist of:

1. Isolated vertices
2. Even length cycles
3. Even length paths
3. Odd length paths

M' and M agree in size on everything except for the odd length paths.

Augmenting paths are odd length paths.



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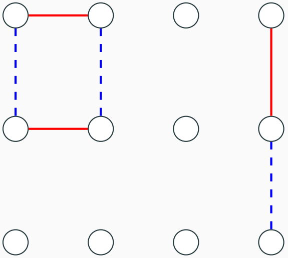

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First Attempt

Successive Augmenting Path

Compute a maximal matching M

For $i = 1$ **to** $n/2$

  For every unmatched node v

    Find an augmenting path p starting at v.

    $M = M \bigoplus p$

Running time: $O(mn^2)$

Correctness: Berge's Theorem

Giving the Algorithm a Bit of Structure

Lemma 1

Let M be a matching where the shortest augmenting path with respect to M has length k. Let P be a maximal set of edge disjoint augmenting paths of length k. Then the shortest augmenting path with respect to $M' = M \bigoplus P$ has length at least $k + 2$.

Let π be an augmenting path with respect to M'.

If π does not intersect any path from P, its length is greater than k.

Otherwise its existence contradicts the assumption that P is maximal or that the paths in P are shortest.

Proof of Lemma 1 Continued

Let π intersect with path $p \in P$ in $M \bigoplus P$.

Then $\pi \bigoplus p$ were two augmenting paths a and b with respect to M.

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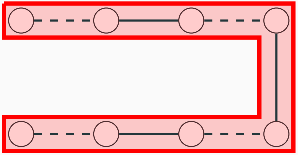

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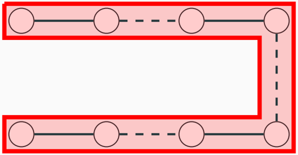

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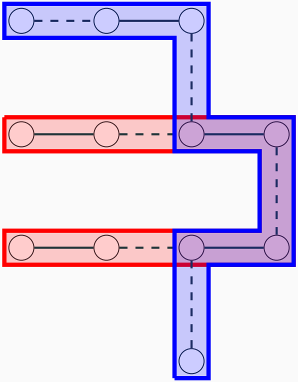

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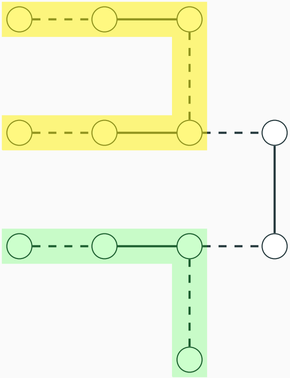

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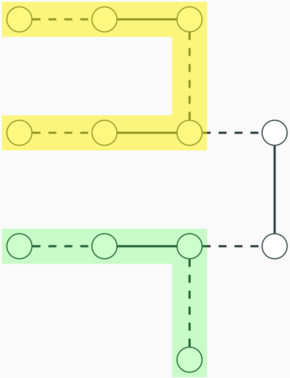
$|a| + |b| < |p| + |\pi| \leq 2|p|$



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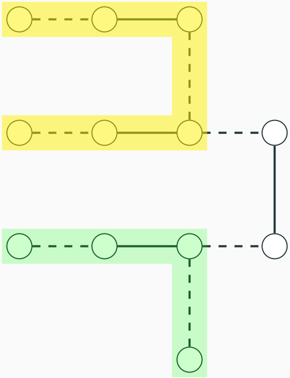
Contradiction to the assumption that p was shortest



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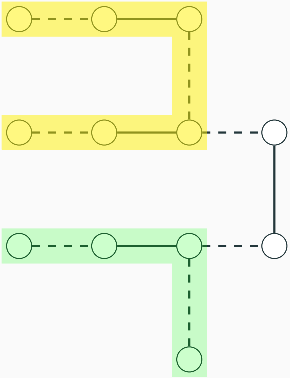
Contradiction to the assumption that p was shortest

By induction, a similar claim holds if π intersects with $p_1, \ldots p_j \in P$



Second Attempt

Successive Shortest Augmenting Path

Compute a maximal matching M

For $i = 1$ **to** n

$P \leftarrow \emptyset$

    While augmenting paths of length $2i + 1$ exist

        Find an augmenting path p of length $2i + 1$.

        Add p to P

        Remove p and incident edges from the graph G

    $M = M \bigoplus P$

    Restore G

Naive way to determine P: Run a BFS for every free node.

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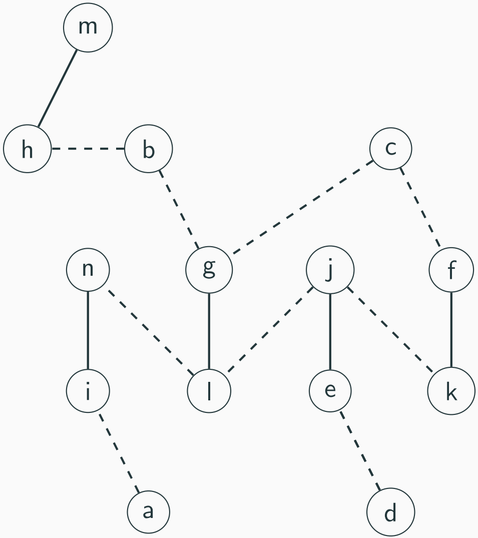
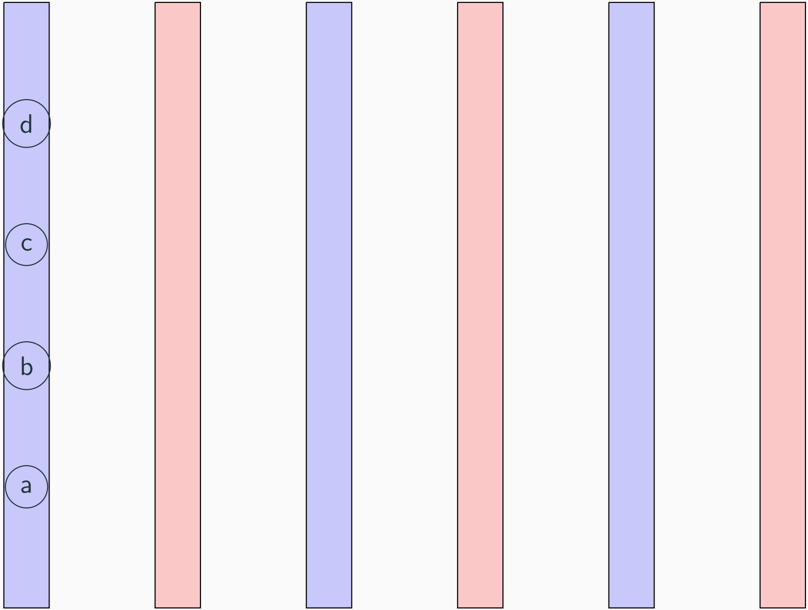
Compute P with just one *BFS* and one *DFS*

Running Example



Hopcroft-Karp Trees



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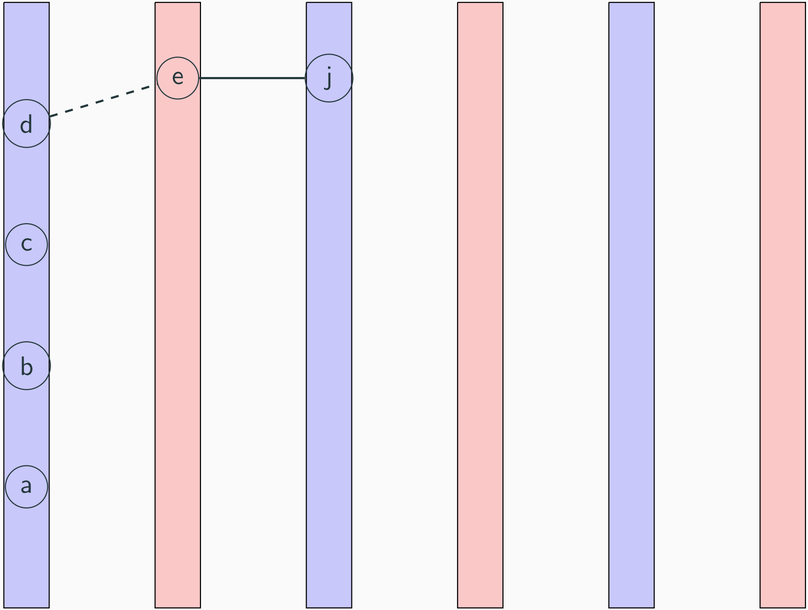

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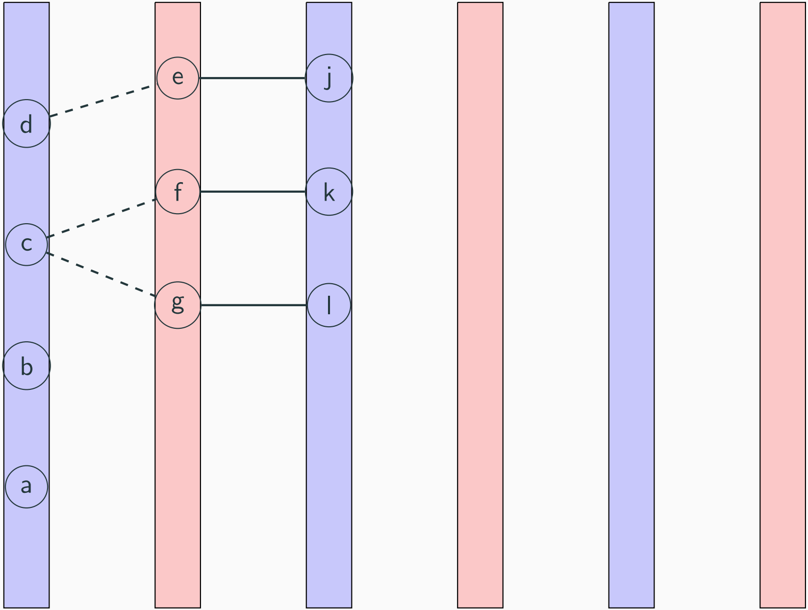

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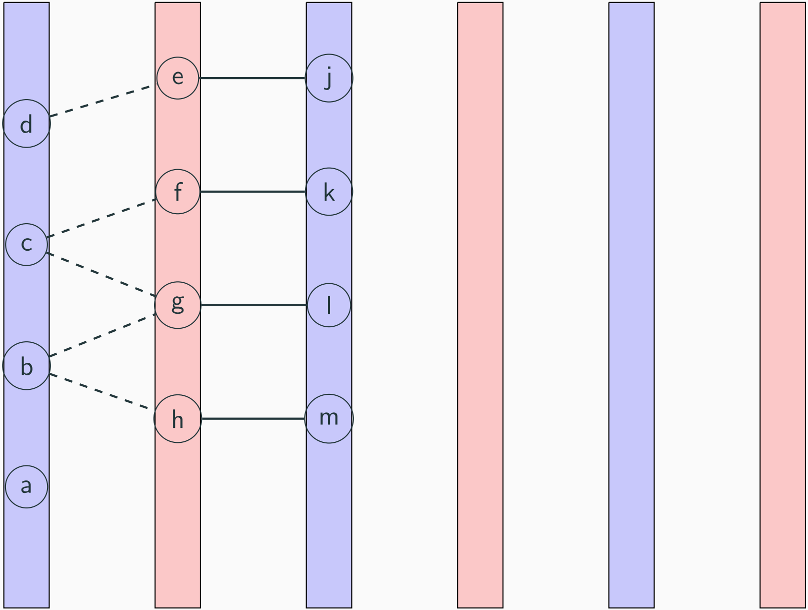

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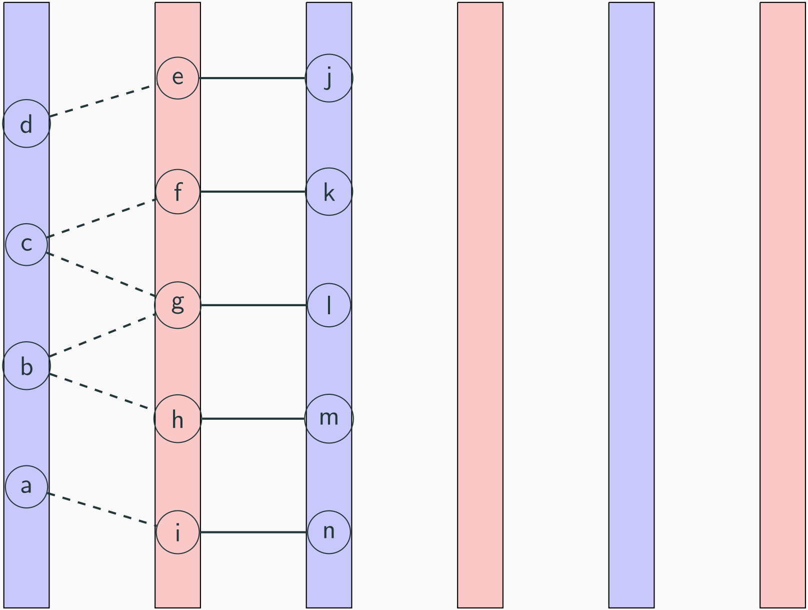

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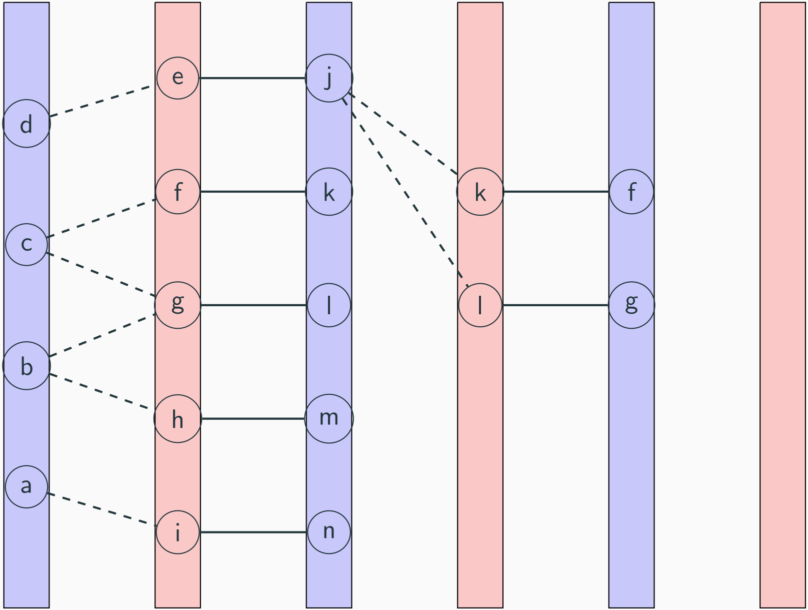

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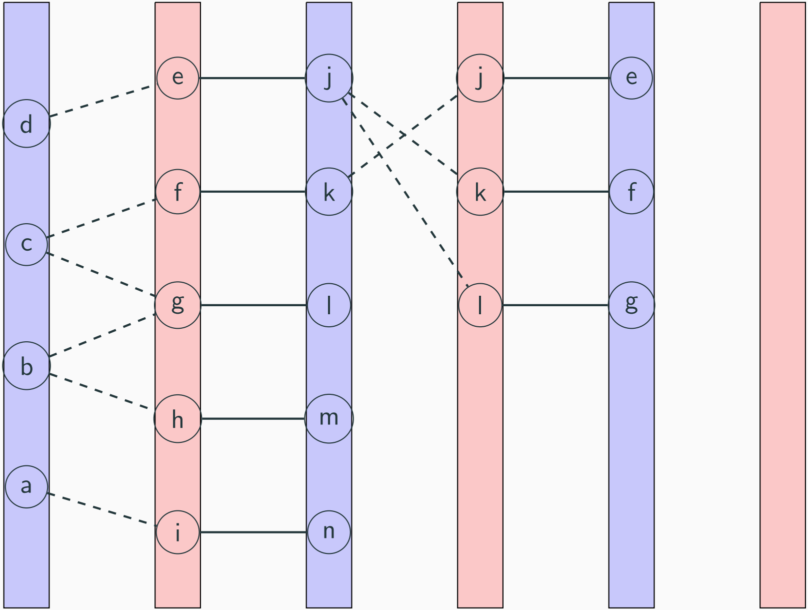

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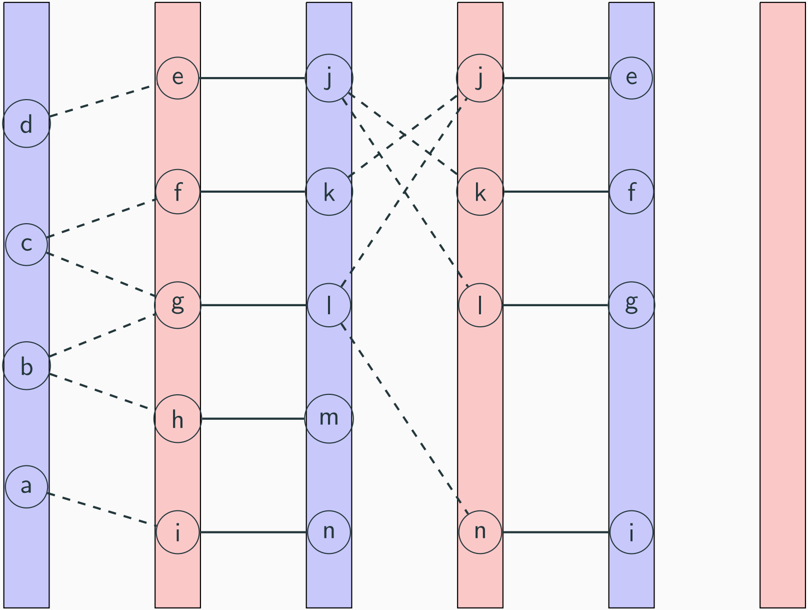

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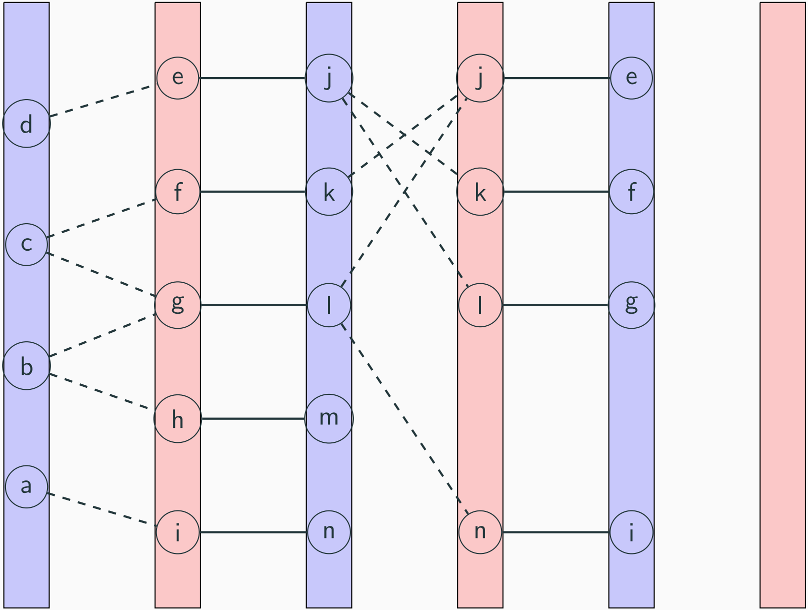

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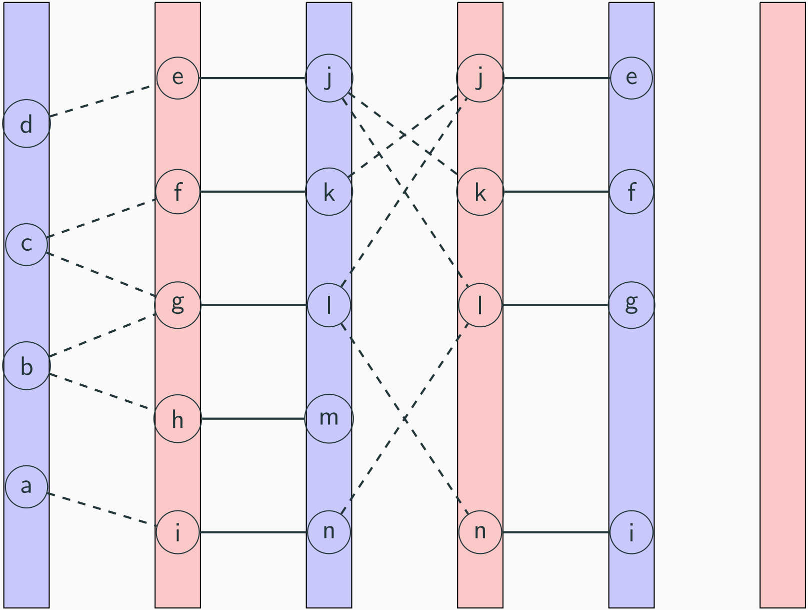

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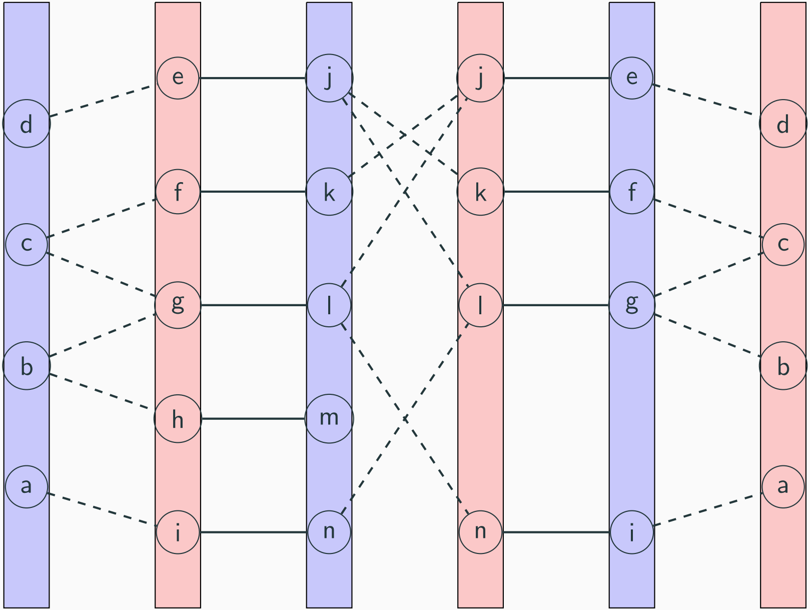


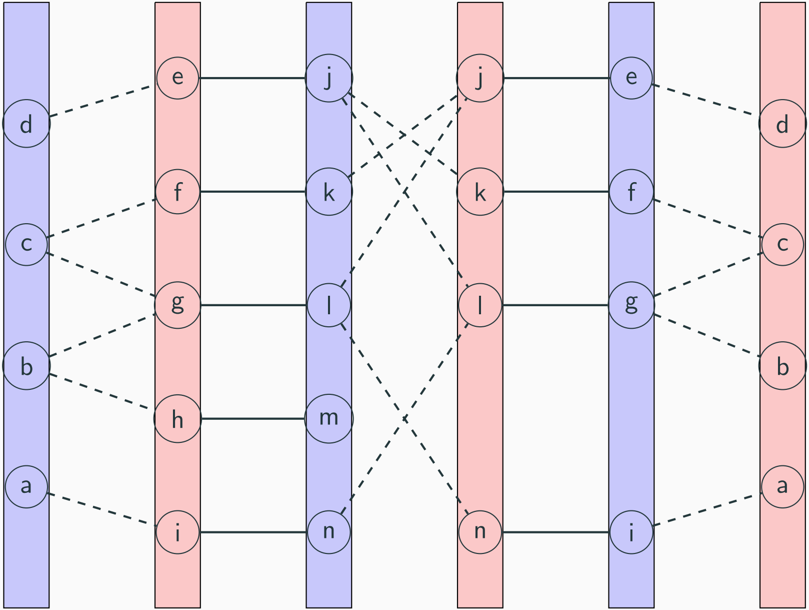
Formal Intermission

Hopcroft-Karp Tree Initialization

Put all free nodes into a queue
Run alternating path BFS
Place no node into more than one even (red) or odd (blue) level

Running Time: $O(m + n)$

Hopcroft-Karp Trees



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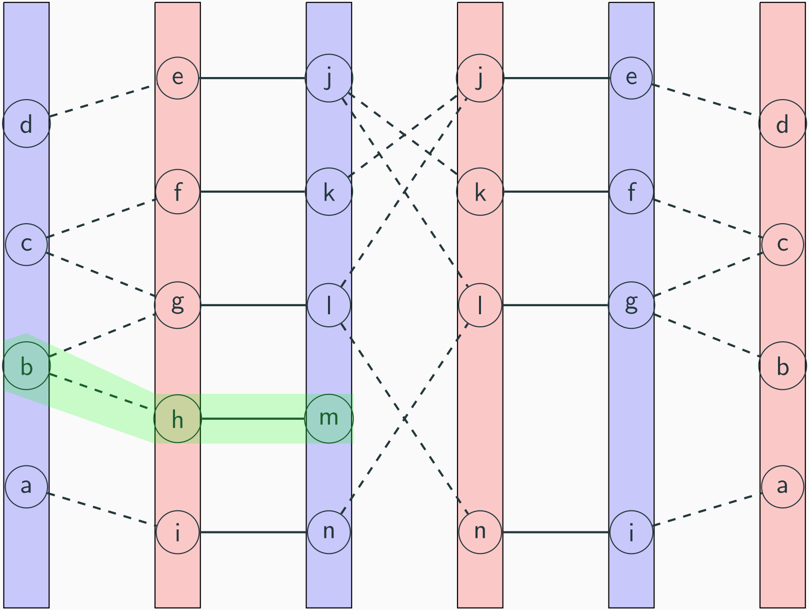

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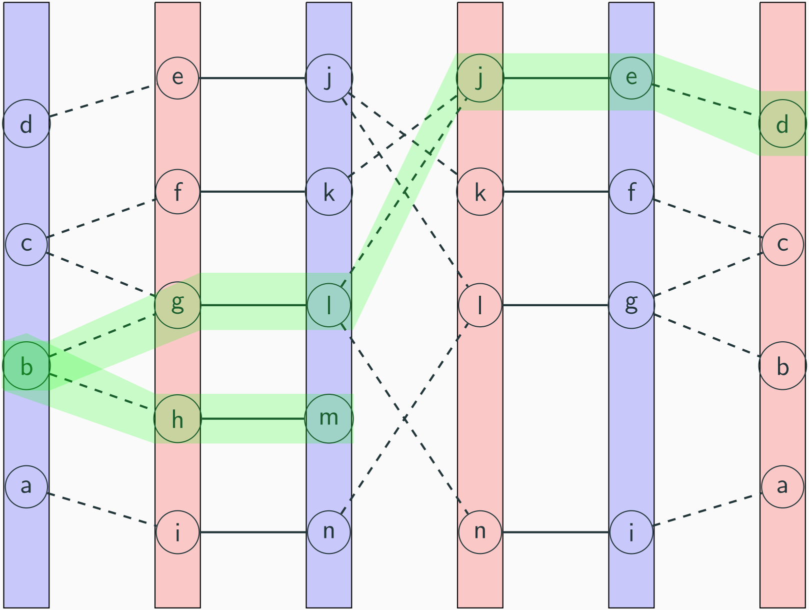

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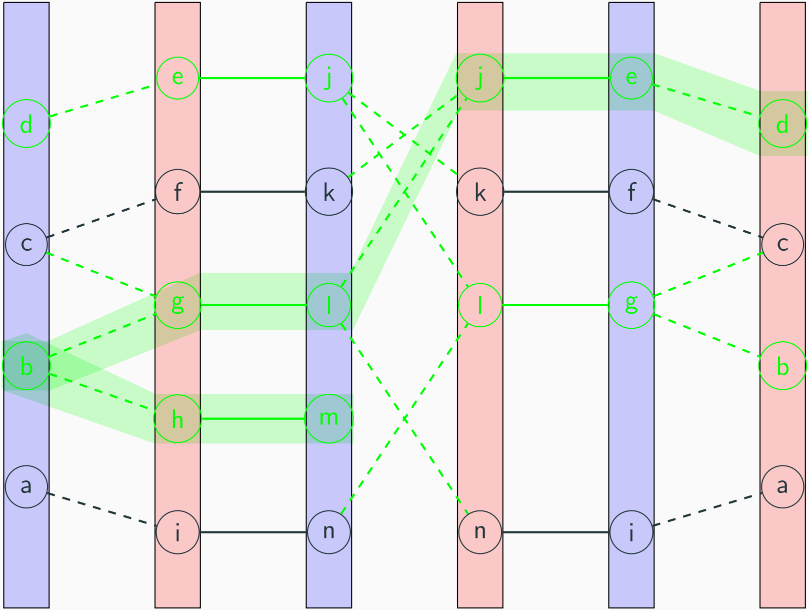

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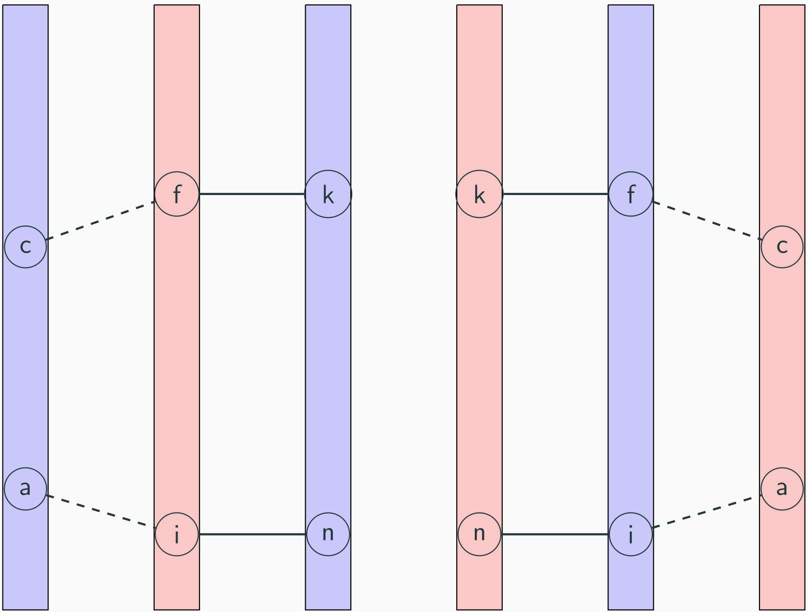

Disjoint Augmenting Path Detection

Hopcroft-Karp Tree Initialization

Put all free nodes into a queue

Run alternating path BFS

Place no node into more than one even (red) or odd (blue) level

Path Extraction

$P \leftarrow \emptyset$

For all free nodes in the first level

  Run a DFS in the Hopcroft-Karp Tree

  If path p is found, add p to P and stop.

  Remove every node and any incident edges of the DFS traversal.

Running Time: $O(m + n)$

Analysis of Hopcroft-Karp Trees

Claim

We can find a maximal set of disjoint augmenting paths in $O(m+n)$ time.

Hopcroft-Karp Trees contain *all* augmenting paths up to a given depth.

A DFS from a free node u finds an augmenting path starting at u, or none exist.

If a node a was visited by a DFS traversal, but not included in an augmenting path, then no augmenting path containing a exists (that does not overlap with P).

If a DFS rooted at u yields an augmenting path p, then no augmenting path disjoint from p at v can use one of the nodes in p.

Third Attempt

Hopcroft-Karp

Compute a maximal matching M

For $i = 1$ **to** n

$\quad P \leftarrow \emptyset$

$\quad$ Find a maximal set P of disjoint augmenting paths of length $2i + 1$.

$\quad M = M \bigoplus P$

Running Time: $O(m \cdot n)$

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I promised you $O(m\sqrt{n})$.

Detailed View on Berge's Theorem

Lemma 2

Let M' be an optimum matching and let M be any matching. If the length of the shortest augmenting path with respect to M is k, then $|M'| - |M| \leq \frac{n}{k}$.

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There are are no more then $\frac{n}{k}$ such paths.

Applications

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We run Hopcroft-Karp until $k \geq \sqrt{n}$.

We have at most $\frac{n}{\sqrt{n}} = \sqrt{n}$ paths left.

The algorithm now only computes $\sqrt{n}$ remaining iterations of BFS/DFS searches.

Final Algorithm

Hopcroft-Karp

Compute a maximal matching M

For $i = 1$ **to** $\sqrt{n}$

$P \leftarrow \emptyset$

Find a maximal set P of disjoint augmenting paths in Hopcroft-Karp trees of depth $2i + 1$.

$M = M \bigoplus P$

For $i = 1$ **to** $\sqrt{n}$

$P \leftarrow \emptyset$

Find a maximal set P of disjoint augmenting paths in Hopcroft-Karp trees of depth n.

$M = M \bigoplus P$