Computer Graphics
Parametric Curves - 1

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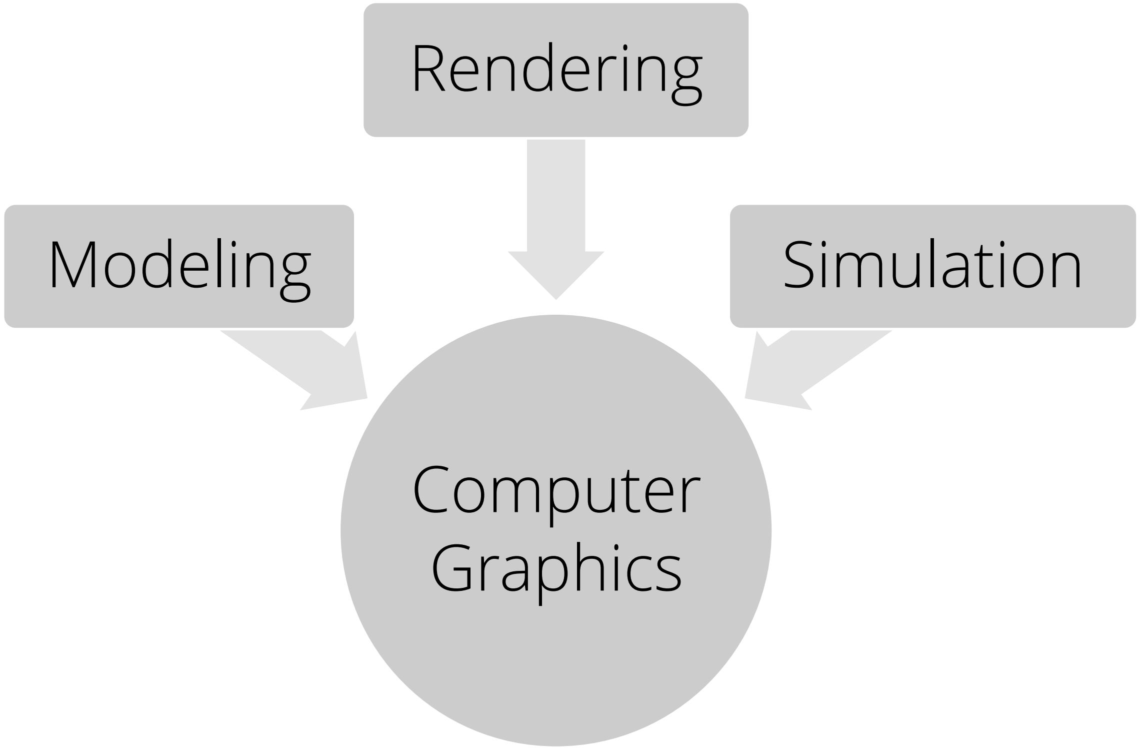
Outline

- Introduction
- Polynomial curves
- Bézier curves
- Matrix notation
- Curve subdivision
- Differential curve properties
- Piecewise polynomial curves
- B-spline curves

Course Topics

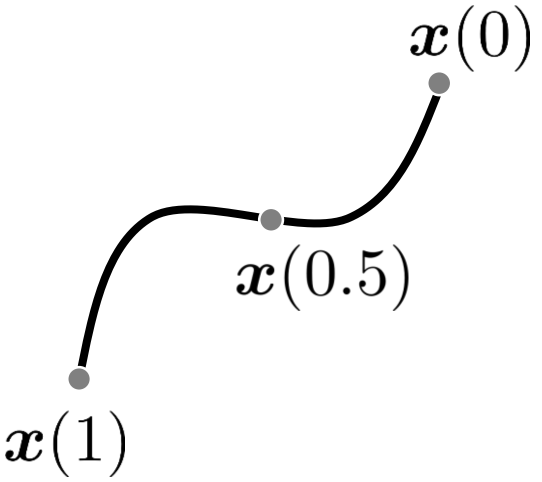
- Rendering

  Study of light transportation
  - What is visible at a sensor?
    - Ray casting
    - Rasterization / Depth test
  - Which color does it have?
    - Phong
- Modeling
  - Parametric curves



Idea

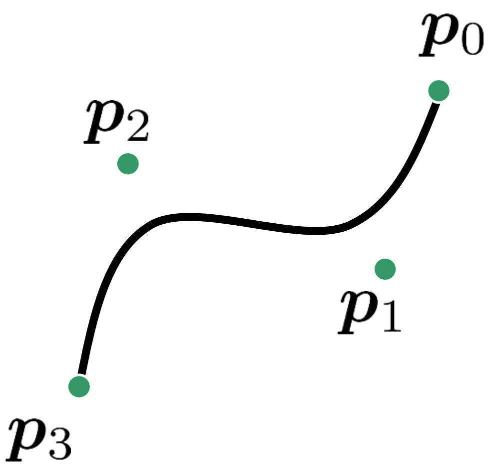
Using parametric curves for modeling purposes.



$$\boldsymbol{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \sum_{i=0}^{3} \boldsymbol{c}_i t^i$$

Curve is defined by functions.
Unintuitive coefficients c_i.

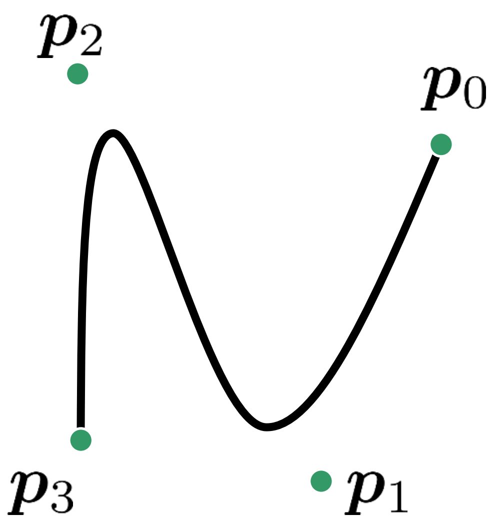
Specifying the curve with a small number of control points.



$$\boldsymbol{x}(t) = \sum_{i=0}^{3} \boldsymbol{p}_i w_i(t)$$

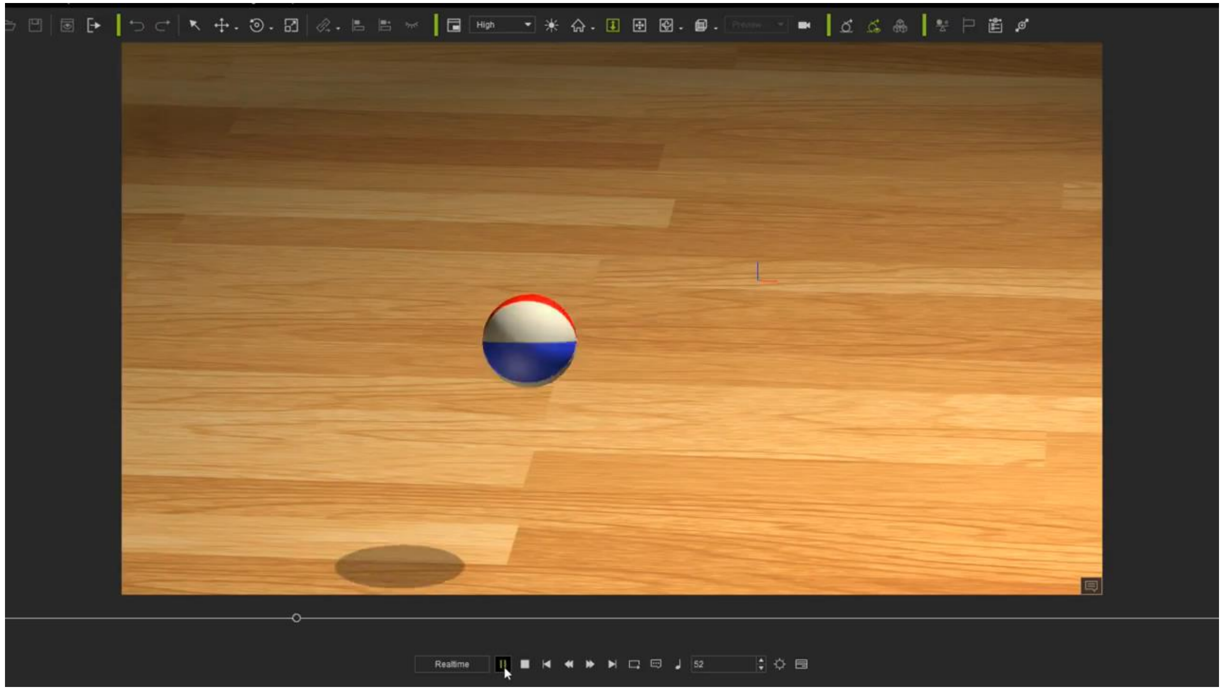
Curve is computed as weighted sum of control points.
Intuitive coefficients p_i.

Modifying the curve by moving the control points should be intuitive.



Applications

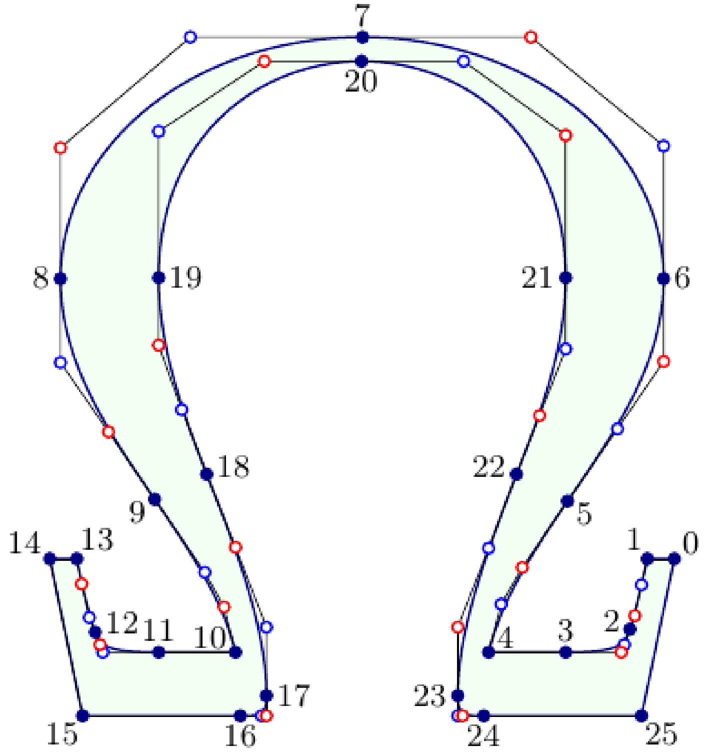
- Animation
  - Simple, flexible and intuitive user interaction

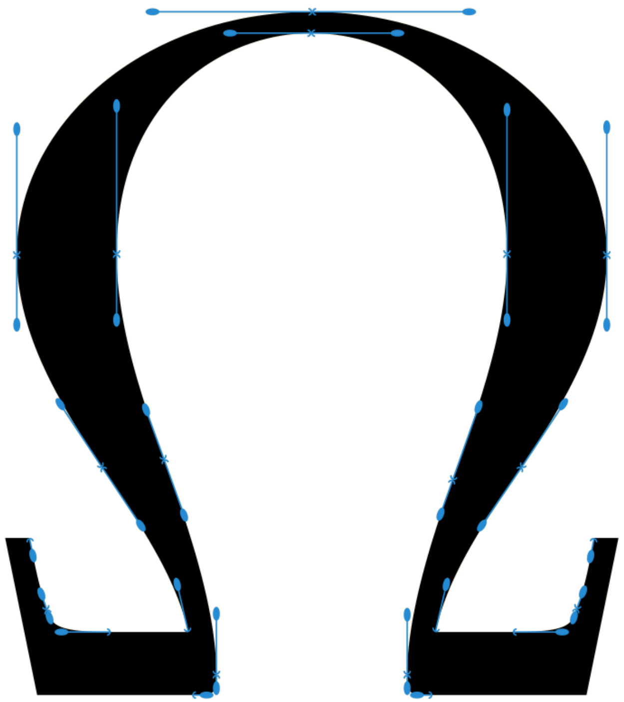


iClone Animation Curve Editor

Applications

- Font modeling
  - High-quality rendering in case of scaling or shearing





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Polynomial Curves

- Parametric curve in the plane $\boldsymbol{x}(t) = (x(t), y(t))^\mathsf{T}$
- Parametric curve in 3D space $\boldsymbol{x}(t) = (x(t), y(t), z(t))^\mathsf{T}$
- If $x(t)$ and $y(t)$ are polynomials, $\boldsymbol{x}(t)$ is a polynomial curve
- Highest power of t is the degree of the curve
- If the functions have the form $\frac{p(t)}{q(t)}$ with $p(t)$ and $q(t)$ being polynomials, $\boldsymbol{x}(t)$ is a rational curve

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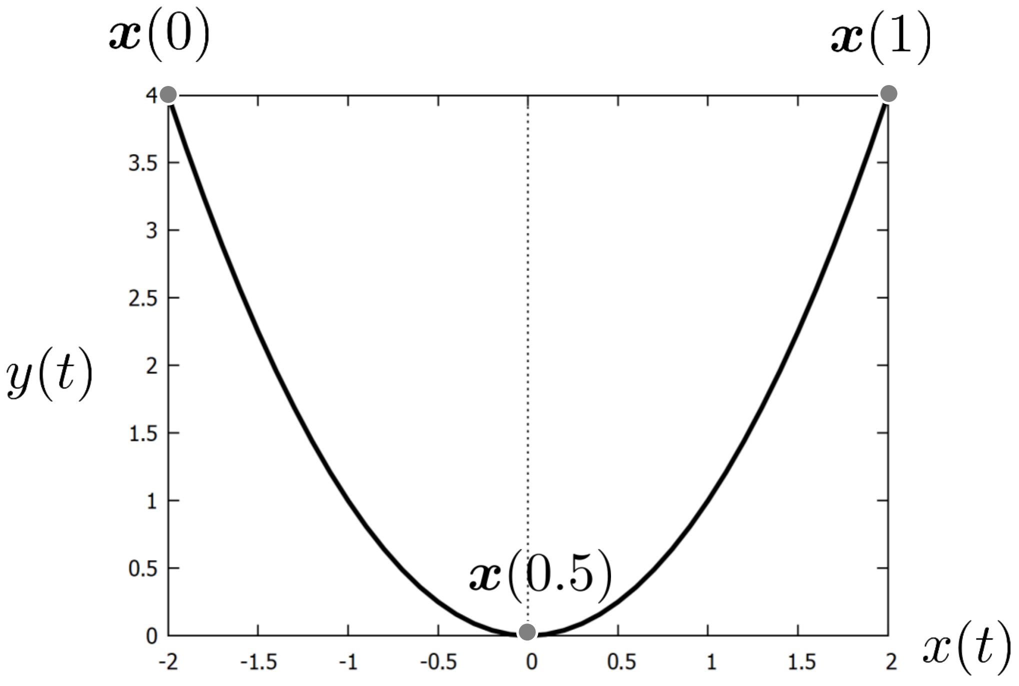
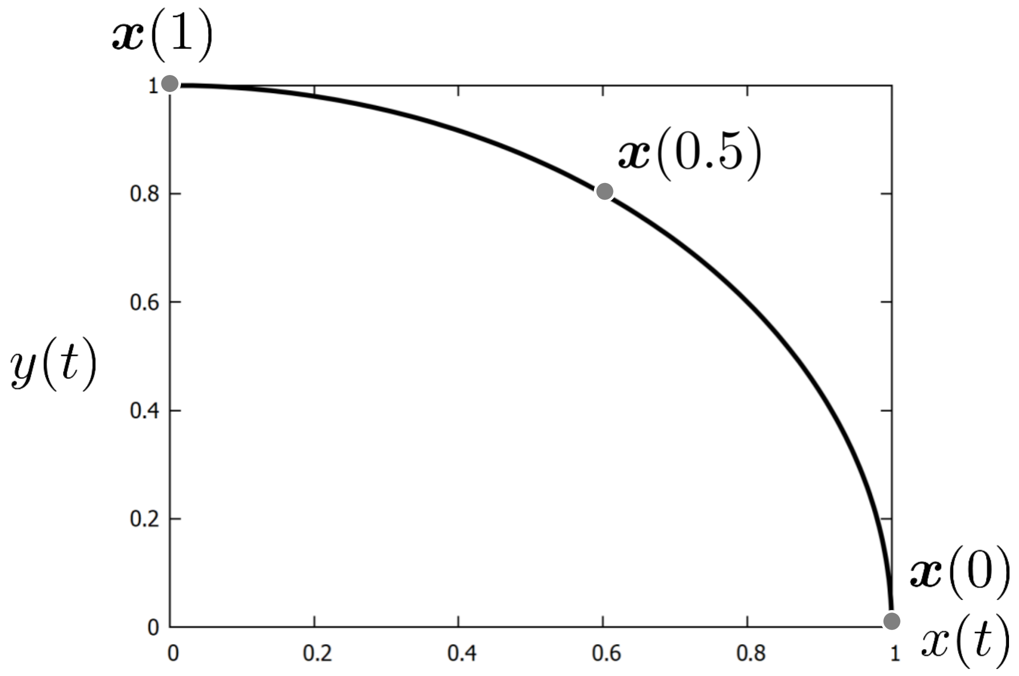
Examples



$$\boldsymbol{x}(t) = (4t-2, (4t-2)^2)^\mathsf{T}$$

Polynomial curve
of degree 2



$$\boldsymbol{x}(t) = \left(\tfrac{1-t^2}{1+t^2}, \tfrac{2t}{1+t^2}\right)^\mathsf{T}$$

Rational curve
of degree 2

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Bézier Curves

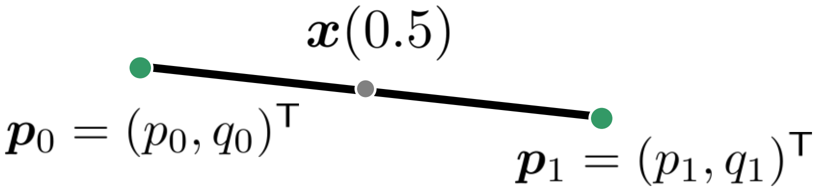
- Are polynomial curves
- Represented by control points
  - n+1 control points for a curve of degree n
- Have various mathematical properties which support their processing and analysis
- Simple and intuitive usage

Low-Degree Bézier Curves

- Constant Bézier curve (degree 0) $\boldsymbol{x}(t) = \boldsymbol{p}_0 \quad t \in [0,1] \quad \boldsymbol{p}_0 = (p_0, q_0)^\mathsf{T}$
- Linear Bézier curve (degree 1)

  $$\boldsymbol{x}(t) = (1-t)\boldsymbol{p}_0 + t\boldsymbol{p}_1 \qquad t \in [0,1]$$

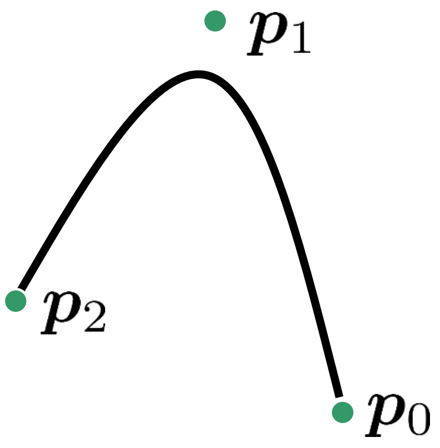
  $$\boldsymbol{x}(t) = ((1-t)p_0 + tp_1, (1-t)q_0 + tq_1)^\mathsf{T}$$



- Quadratic Bézier curve (degree 2)

  $$\boldsymbol{x}(t) = (1-t)^2\boldsymbol{p}_0 + 2(1-t)t\boldsymbol{p}_1 + t^2\boldsymbol{p}_2 \quad t \in [0,1]$$

- Control points $\boldsymbol{p}_i$
  - First and last control point are interpolated
  - Other control points are approximated



Examples

- Linear Bézier curve
  - Control points: $\boldsymbol{p}_0 = (1,2)^\mathsf{T} \quad \boldsymbol{p}_1 = (3,4)^\mathsf{T}$
  - Curve: $\boldsymbol{x}(t) = (1-t)\boldsymbol{p}_0 + t\boldsymbol{p}_1$
    $= (1 - t + 3t, 2(1-t) + 4t)^\mathsf{T} = (1+2t, 2+2t)^\mathsf{T}$
- Quadratic Bézier curve
  - Control points: $\boldsymbol{p}_0 = (1,2)^\mathsf{T} \quad \boldsymbol{p}_1 = (4,-1)^\mathsf{T} \quad \boldsymbol{p}_2 = (8,6)^\mathsf{T}$
  - Curve: $\boldsymbol{x}(t) = (1-t)^2\boldsymbol{p}_0 + 2(1-t)t\boldsymbol{p}_1 + t^2\boldsymbol{p}_2$
    $= (1 + 6t + t^2, 2 - 6t + 10t^2)^\mathsf{T}$
- Control points define a parametric curve in *t*
- Bézier curves are polynomials in *t*

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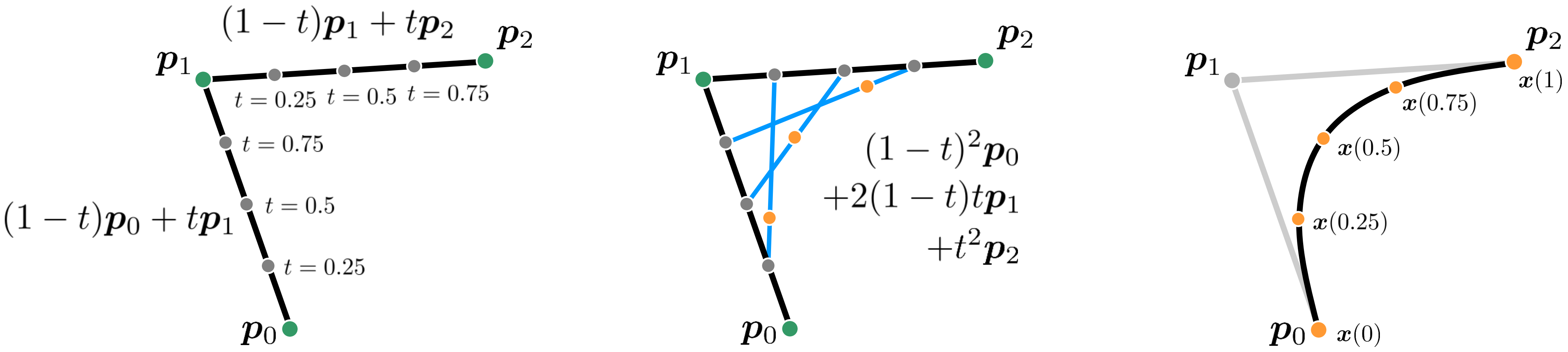
Illustration

- Linear: $\boldsymbol{x}(t) = (1-t)\boldsymbol{p}_0 + t\boldsymbol{p}_1$ $= (1-t)t + (1-t)t$
  - Interpolation between two points
- Quadratic: $\boldsymbol{x}(t) = (1-t)^2\boldsymbol{p}_0 + 2(1-t)t\boldsymbol{p}_1 + t^2\boldsymbol{p}_2$

$$= (1-t)\big[(1-t)\boldsymbol{p}_0 + t\boldsymbol{p}_1\big] + t\big[(1-t)\boldsymbol{p}_1 + t\boldsymbol{p}_2\big]$$

  - Interpolation between the interpolation results of two points



Cubic Bézier Curves

- Interpolation of the interpolation results of the interpolation results of two control points

3 times

- $\boldsymbol{x}(t) = (1-t)\Big\{(1-t)\big[(1-t)\boldsymbol{p}_0 + t\boldsymbol{p}_1\big] + t\big[(1-t)\boldsymbol{p}_1 + t\boldsymbol{p}_2\big]\Big\}$
  $\quad + t\Big\{(1-t)\big[(1-t)\boldsymbol{p}_1 + t\boldsymbol{p}_2\big] + t\big[(1-t)\boldsymbol{p}_2 + t\boldsymbol{p}_3\big]\Big\}$
- $\boldsymbol{x}(t) = (1-t)^3\boldsymbol{p}_0 + 3(1-t)^2 t\boldsymbol{p}_1 + 3(1-t)t^2\boldsymbol{p}_2 + t^3\boldsymbol{p}_3 \qquad t \in [0,1]$

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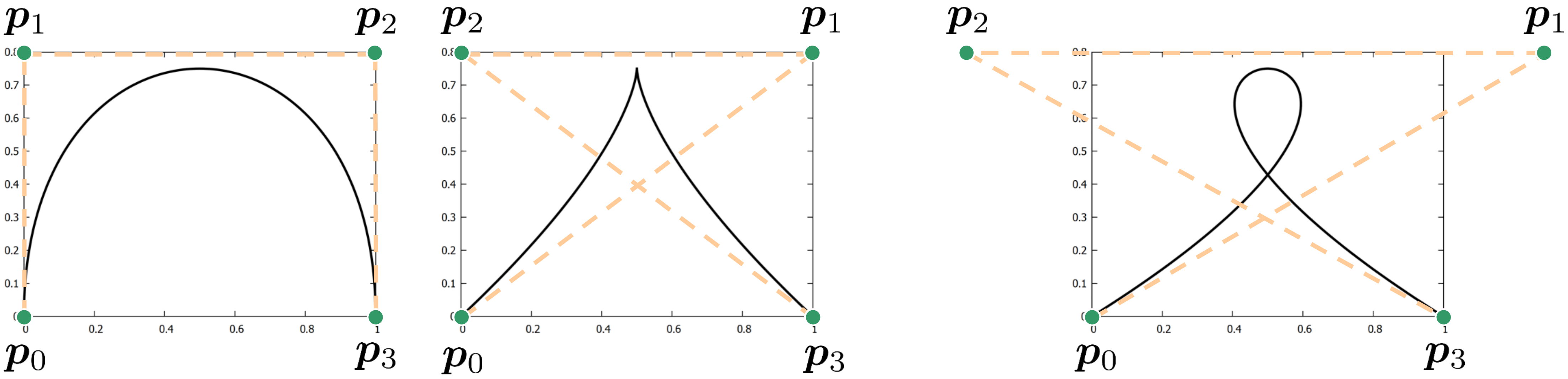
Cubic Bézier Curves

– Four control points $\boldsymbol{p}_i$

– Larger variety of shapes compared to linear and quadratic Bézier curves



General Bézier Curves

– Bézier curve of degree *n* with *n*+1 control points $\boldsymbol{p}_i$

$$\boldsymbol{x}(t) = \sum_{i=0}^{n} B_{i,n}(t)\boldsymbol{p}_i \qquad t \in [0,1]$$

$$B_{i,n}(t) = \frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i \qquad 0 \leq i \leq n$$

– Binomial coefficients: $\frac{n!}{(n-i)!i!} = \binom{n}{i}$

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\ & & & & \cdots & & & & \end{array} = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \cdots & & & & \end{array}$$

Curves of degree larger three are not often used. Designing a curve with more than four control points gets more difficult. Instead, piecewise cubic or quadratic Bézier curves are used.

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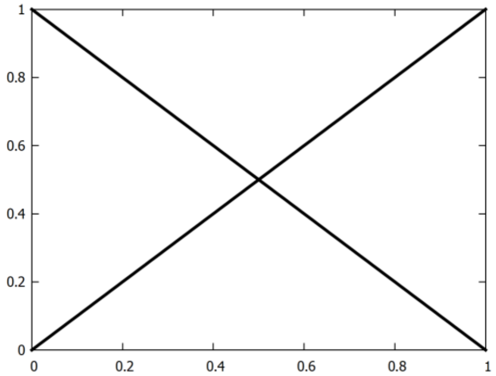
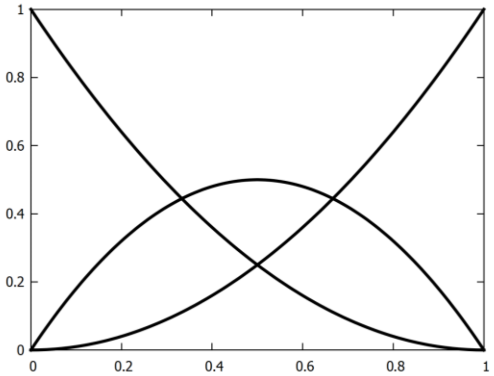
Bernstein Polynomials

Classical logic of quadratic/cubic bynomials



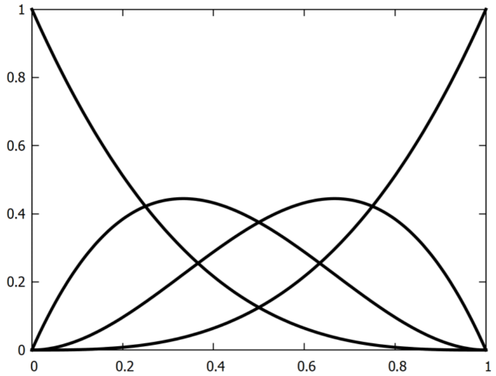
$$B_{0,1}(t) = (1-t)$$
$$B_{1,1}(t) = t$$

$= 1$



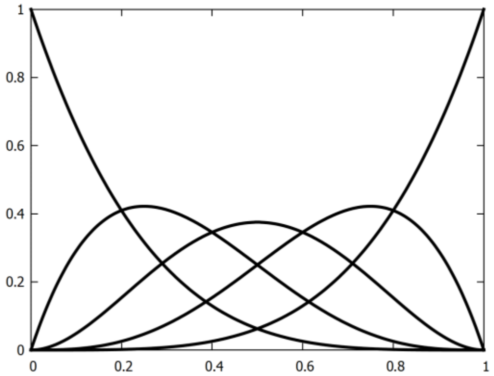
$$B_{0,2}(t) = (1-t)^2$$
$$B_{1,2}(t) = 2(1-t)t$$
$$B_{2,2}(t) = t^2$$

$= 1$



$$B_{0,3}(t) = (1-t)^3$$
$$B_{1,3}(t) = 3(1-t)^2 t$$
$$B_{2,3}(t) = 3(1-t)t^2$$
$$B_{3,3}(t) = t^3$$

$= 1$



$$B_{0,4}(t) = (1-t)^4$$
$$B_{1,4}(t) = 4(1-t)^3 t$$
$$B_{2,4}(t) = 6(1-t)^2 t^2$$
$$B_{3,4}(t) = 4(1-t)t^3$$
$$B_{4,4}(t) = t^4$$

$= 1$

Bernstein Polynomials - Properties

- Partition of unity: $\sum_{i=0}^{n} B_{i,n}(t) = 1 \quad t \in [0,1]$
- Positivity: $B_{i,n}(t) \geq 0 \quad t \in [0,1]$
- Symmetry: $B_{n-i,n}(t) = B_{i,n}(1-t) \quad i = 0, \ldots, n$
- Recursion: $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$

  $i = 0, \ldots, n \quad B_{-1,n-1}(t) = B_{n,n-1} = 0$

Bézier Curves - Properties

- Endpoint interpolation:

$$\boldsymbol{x}(0) = \sum_{i=0}^{n} B_{i,n}(0)\boldsymbol{p}_i = \boldsymbol{p}_0 \qquad \boldsymbol{x}(1) = \sum_{i=0}^{n} B_{i,n}(1)\boldsymbol{p}_i = \boldsymbol{p}_n$$

- Endpoint tangent:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(0) = n(\boldsymbol{p}_1 - \boldsymbol{p}_0) \qquad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(1) = n(\boldsymbol{p}_n - \boldsymbol{p}_{n-1})$$

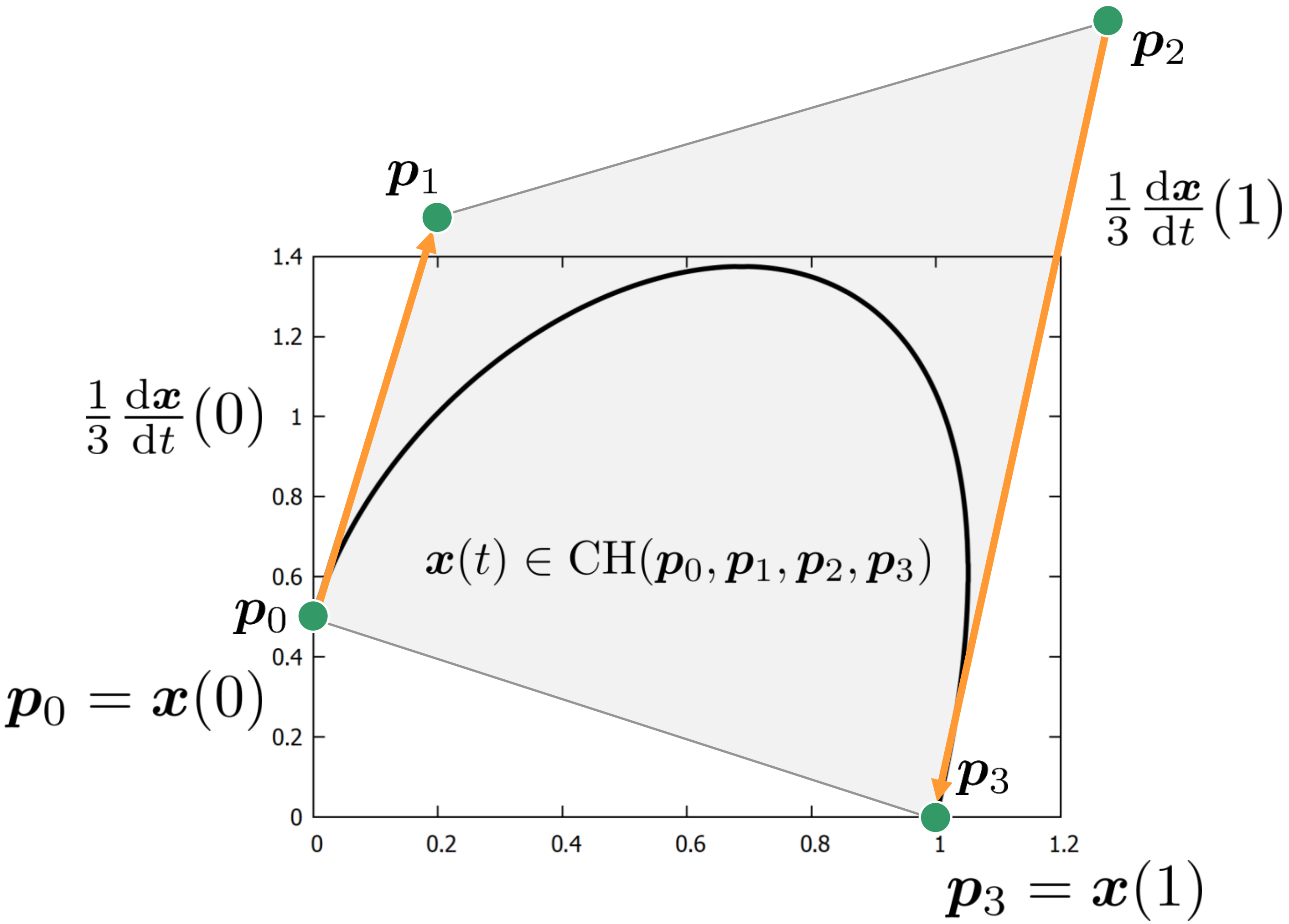
- Convex hull:

$$\boldsymbol{x}(t) \in \mathrm{CH}(\boldsymbol{p}_0, \dots, \boldsymbol{p}_n) \qquad t \in [0, 1]$$

$$\mathrm{CH}(\boldsymbol{p}_0, \dots, \boldsymbol{p}_n) = \left\{ \sum_{i=0}^{n} a_i \boldsymbol{p}_i \;\middle|\; \sum_{i=0}^{n} a_i = 1, a_i \geq 0 \right\}$$

Smallest convex set that
contains the shape

Bézier Curves - Properties



Bézier Curves - Properties

– Invariance under affine transformations
  – $\boldsymbol{M}\left(\sum_{i=0}^{n} B_{i,n}(t)\boldsymbol{p}_i\right) = \sum_{i=0}^{n} B_{i,n}(t)\boldsymbol{M}\boldsymbol{p}_i$
  – $\boldsymbol{M}$ is a transformation matrix
  – $\boldsymbol{p}_i$ are the control points
  – Transforming a point on the curve corresponds to computing the point on the curve from the transformed control points
  – Bézier curves can be transformed by transforming their control points

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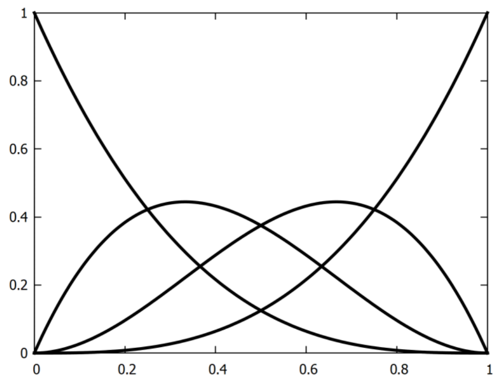
Bézier Curves - Properties

- Points $\boldsymbol{x}(t)$ on a Bézier curve are a linear combination of the control points $\boldsymbol{p}_i$ weighted with Bernstein polynomials at t
- Cubic Bézier curve

$$\boldsymbol{x}(t) = \boldsymbol{p}_0 B_{0,3}(t) + \boldsymbol{p}_1 B_{1,3}(t) + \boldsymbol{p}_2 B_{2,3}(t) + \boldsymbol{p}_3 B_{3,3}(t)$$



$$B_{0,3}(t) = (1-t)^3$$
$$B_{1,3}(t) = 3(1-t)^2 t$$
$$B_{2,3}(t) = 3(1-t)t^2$$
$$B_{3,3}(t) = t^3$$

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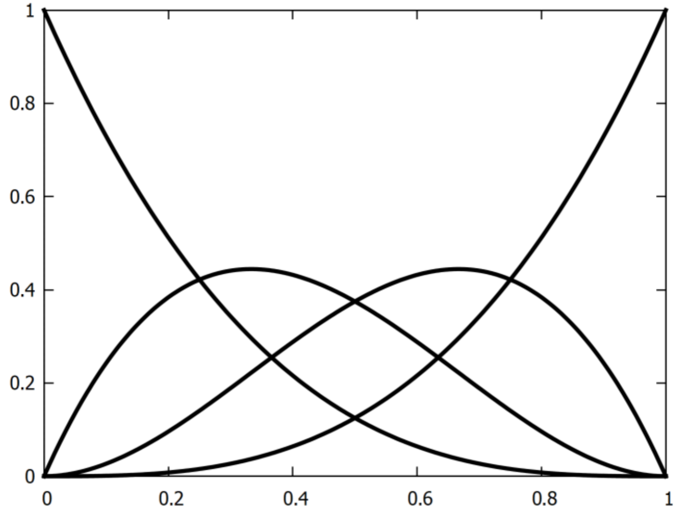
Bézier Curves - Properties

- Cubic Bézier curve
  - $B_{i,3}(t)$ describes the influence of control point $\boldsymbol{p}_i$
  - All points $\boldsymbol{x}(t)$ on the curve with $t \in (0, 1)$ are influenced by all control points $B_{i,3}(t)$
  - $\boldsymbol{x}(0) = B_{0,3}(0)\boldsymbol{p}_0 = 1 \cdot \boldsymbol{p}_0$
  - $\boldsymbol{x}(1) = B_{3,3}(1)\boldsymbol{p}_3 = 1 \cdot \boldsymbol{p}_3$



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Bernstein Polynomials – Matrix Notation

– Quadratic

$$B_{0,2}(t) = (1-t)^2$$
$$B_{1,2}(t) = 2(1-t)t$$
$$B_{2,2}(t) = t^2$$

$$\begin{pmatrix} B_{0,2}(t) \\ B_{1,2}(t) \\ B_{2,2}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}_2^{\text{Bez}}} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$$

– Cubic

$$B_{0,3}(t) = (1-t)^3$$
$$B_{1,3}(t) = 3(1-t)^2 t$$
$$B_{2,3}(t) = 3(1-t)t^2$$
$$B_{3,3}(t) = t^3$$

$$\begin{pmatrix} B_{0,3}(t) \\ B_{1,3}(t) \\ B_{2,3}(t) \\ B_{3,3}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}_3^{\text{Bez}}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Polynomial Bases

- $\{1, t, t^2, t^3\}$ is the canonical basis for cubic polynomials
  - Elements (monomials) are linearly independent
  - All cubic polynomials are linear combinations of the elements
- $\{B_{0,3}(t), B_{1,3}(t), B_{2,3}(t), B_{3,3}(t)\}$ is an (alternative) Bernstein basis for cubic polynomials
- $\boldsymbol{S}_3^{\text{Bez}}$ with $\begin{pmatrix} B_{0,3}(t) \\ B_{1,3}(t) \\ B_{2,3}(t) \\ B_{3,3}(t) \end{pmatrix} = \boldsymbol{S}_3^{\text{Bez}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$ represents a basis transform

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Polynomial Bases

– Basis transforms

$$\begin{pmatrix} B_{0,3}(t) \\ B_{1,3}(t) \\ B_{2,3}(t) \\ B_{3,3}(t) \end{pmatrix} = \boldsymbol{S}_3^{\text{Bez}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = (\boldsymbol{S}_3^{\text{Bez}})^{-1} \begin{pmatrix} B_{0,3}(t) \\ B_{1,3}(t) \\ B_{2,3}(t) \\ B_{3,3}(t) \end{pmatrix}$$

$$(\boldsymbol{S}_3^{\text{Bez}})^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$1 = B_{0,3}(t) + B_{1,3}(t) + B_{2,3}(t) + B_{3,3}(t)$$
$$t = \tfrac{1}{3}B_{1,3}(t) + \tfrac{2}{3}B_{2,3}(t)$$
$$t^2 = \tfrac{1}{3}B_{2,3}(t) + B_{3,3}(t)$$
$$t^3 = B_{3,3}(t)$$

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Bézier Curves

– Cubic in 2D

$$\boldsymbol{x}(t) = B_{0,3}(t)\boldsymbol{p}_0 + B_{1,3}(t)\boldsymbol{p}_1 + B_{2,3}(t)\boldsymbol{p}_2 + B_{3,3}(t)\boldsymbol{p}_3$$

$$\boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} B_{0,3}(t) \\ B_{1,3}(t) \\ B_{2,3}(t) \\ B_{3,3}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{pmatrix} \boldsymbol{S}_3^{\mathsf{Bez}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} B_{0,3}(t) \\ B_{1,3}(t) \\ B_{2,3}(t) \\ B_{3,3}(t) \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Bézier Curves

- Cubic in 2D

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Curve | Geometry matrix | Spline matrix (Bernstein) | Basis (canonical)

- General spline formulation
  - Piecewise polynomial function
  - $\boldsymbol{x}(t) = \boldsymbol{GST}(t)$ Curve = Geometry · Spline basis · Power basis

General Spline Formulation

- $\boldsymbol{x}(t) = \boldsymbol{GST}(t)$
- Examples
  - 2D cubic Bézier curve

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

  - 3D quadratic Bézier curve

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \\ r_0 & r_1 & r_2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$$

General Spline Formulation

- Examples
  - 3D cubic Bézier spline

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \\ r_0 & r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

  - Transformed 3D cubic Bézier spline

$$\boldsymbol{M} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \left[\boldsymbol{M} \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \\ r_0 & r_1 & r_2 & r_3 \end{pmatrix} \right] \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

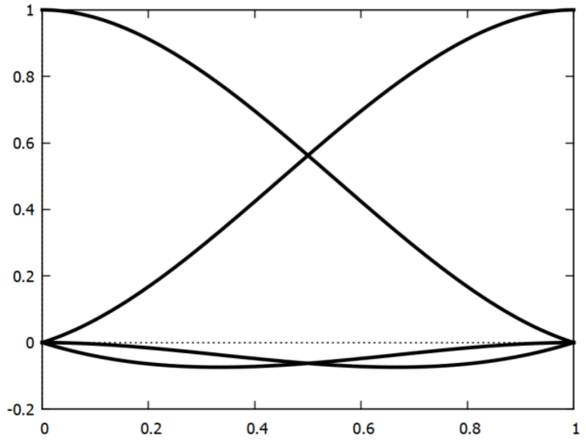
The curve can be transformed by transforming the control points.

General Spline Formulation

- Examples
  - 2D cubic Catmull-Rom spline
  - Interpolates control points $\boldsymbol{p}_1, \boldsymbol{p}_2 : \boldsymbol{x}(0) = \boldsymbol{p}_1$ and $\boldsymbol{x}(1) = \boldsymbol{p}_2$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} \underbrace{\frac{1}{2} \begin{pmatrix} 0 & -1 & 2 & -1 \\ 2 & 0 & -5 & 3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{\boldsymbol{S}_3^{\mathsf{CR}}} \underbrace{\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}}_{\boldsymbol{T}_3}$$

Catmull-Rom Spline



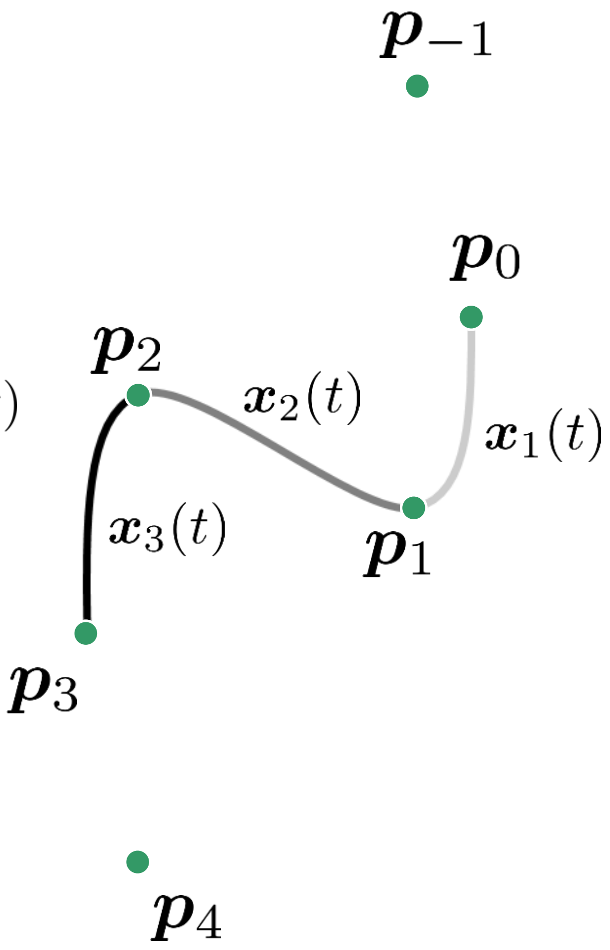
Basis functions

$\boldsymbol{S}_3^{\mathsf{CR}}\boldsymbol{T}(t)$

$$\boldsymbol{x}_1(t) = \begin{pmatrix}\boldsymbol{p}_{-1} & \boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2\end{pmatrix} \boldsymbol{S}_3^{\mathsf{CR}}\boldsymbol{T}_3(t)$$

$$\boldsymbol{x}_2(t) = \begin{pmatrix}\boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3\end{pmatrix} \boldsymbol{S}_3^{\mathsf{CR}}\boldsymbol{T}_3(t)$$

$$\boldsymbol{x}_3(t) = \begin{pmatrix}\boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4\end{pmatrix} \boldsymbol{S}_3^{\mathsf{CR}}\boldsymbol{T}_3(t)$$



Each curve interpolates between two control points using four control points

Conversion From Canonical to Bézier

– Given a curve in canonical form $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$

– How to compute the control points $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}, \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \begin{pmatrix} p_3 \\ q_3 \end{pmatrix}$

– We have

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Conversion From Canonical to Bézier

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix}$$

– Example

$$\boldsymbol{x}(t) = \begin{pmatrix} 1 + t + t^2 + t^3 \\ 1 + t + t^2 + t^3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{p}_0 = (1,1)^\mathsf{T}, \boldsymbol{p}_1 = \left(\tfrac{4}{3}, \tfrac{4}{3}\right)^\mathsf{T}, \boldsymbol{p}_2 = (2,2)^\mathsf{T}, \boldsymbol{p}_3 = (4,4)^\mathsf{T}$$

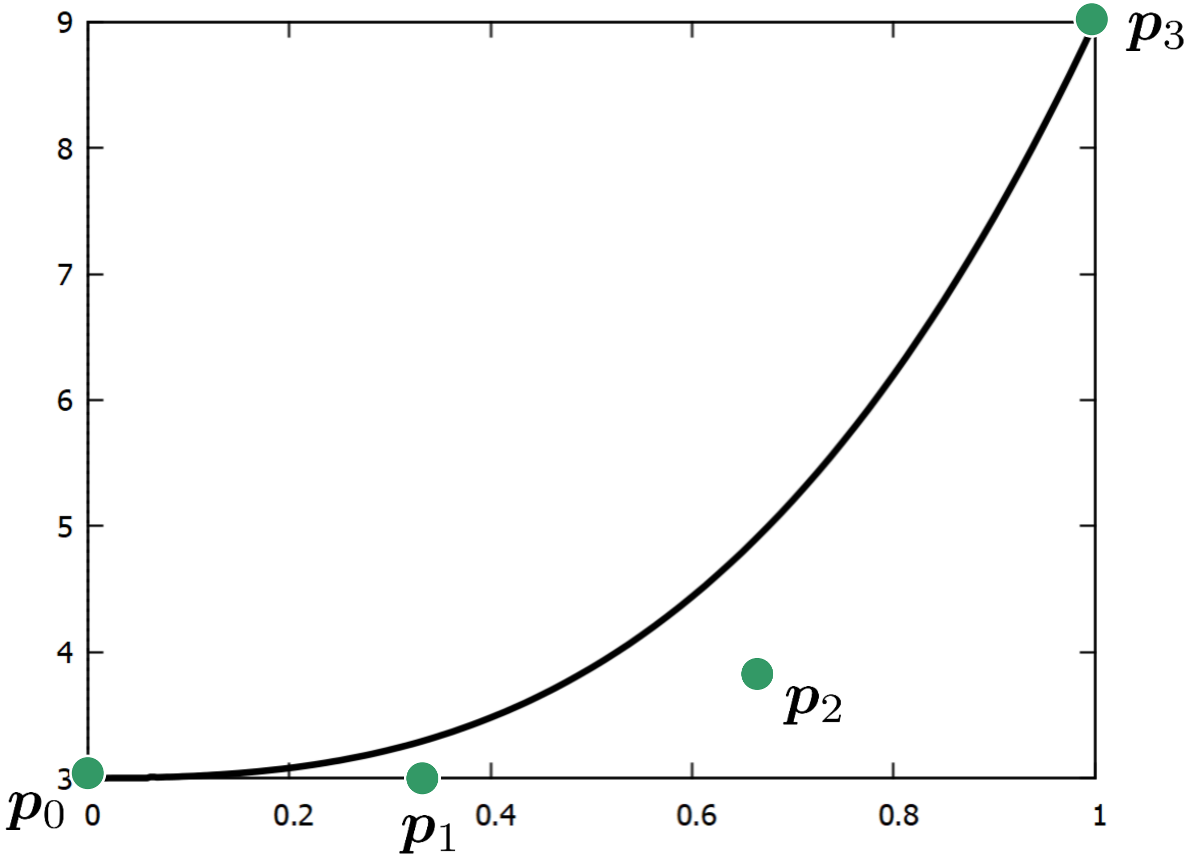
Conversion From Canonical to Bézier

– Example

$$\boldsymbol{x}(t) = \begin{pmatrix} t \\ 3 + t^2 + 5t^3 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{p}_0 = (0,3)^\mathsf{T}, \boldsymbol{p}_1 = \left(\tfrac{1}{3}, 3\right)^\mathsf{T},$$

$$\boldsymbol{p}_2 = \left(\tfrac{2}{3}, \tfrac{10}{3}\right)^\mathsf{T}, \boldsymbol{p}_3 = (1,9)^\mathsf{T}$$



Outline

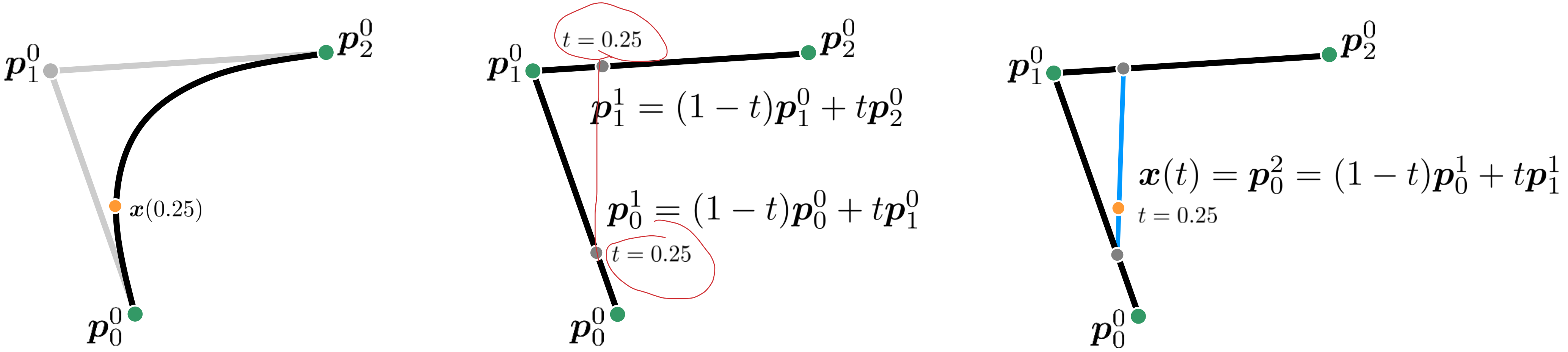
- Introduction
- Polynomial curves
- Bézier curves
- Matrix notation
- Curve subdivision
- Differential curve properties
- Piecewise polynomial curves
- B-spline curves

De Casteljau Algorithm

- Evaluation of a curve point $\boldsymbol{x}(t)$ for a given $t \in [0, 1]$
- Illustration for $\boldsymbol{x}(t) = \boldsymbol{G}\boldsymbol{S}_2^{\text{Bez}}\boldsymbol{T}_2(t)$

$$\boldsymbol{x}(t) = (1-t)\big[\underbrace{(1-t)\boldsymbol{p}_0^0 + t\boldsymbol{p}_1^0}_{\boldsymbol{p}_0^1}\big] + t\big[\underbrace{(1-t)\boldsymbol{p}_1^0 + t\boldsymbol{p}_2^0}_{\boldsymbol{p}_1^1}\big]$$

$$\boldsymbol{x}(t) = \boldsymbol{p}_0^2 = (1-t)\boldsymbol{p}_0^1 + t\boldsymbol{p}_1^1$$



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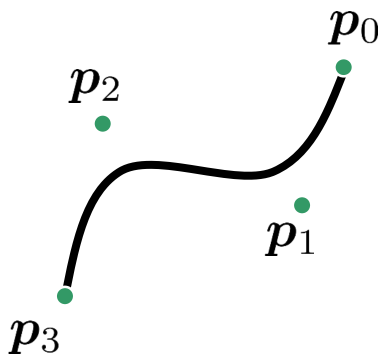
De Casteljau Algorithm

- Cubic Bézier curve with control points $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$
  - $\boldsymbol{p}_i^0 = \boldsymbol{p}_i \quad i = 0, 1, 2, 3$
  - $\boldsymbol{p}_i^j = (1-t)\boldsymbol{p}_i^{j-1} + t\boldsymbol{p}_{i+1}^{j-1} \qquad \boldsymbol{x}(t) = \boldsymbol{p}_0^3$

    $j = 1, 2, 3 \quad i = 0, \ldots, 3-j$



$$\boldsymbol{p}_0^0 = \boldsymbol{p}_0 \qquad \boldsymbol{p}_1^0 = \boldsymbol{p}_1 \qquad \boldsymbol{p}_2^0 = \boldsymbol{p}_2 \qquad \boldsymbol{p}_3^0 = \boldsymbol{p}_3$$

$$\boldsymbol{p}_0^1 = (1-t)\boldsymbol{p}_0^0 + t\boldsymbol{p}_1^0 \quad \boldsymbol{p}_1^1 = (1-t)\boldsymbol{p}_1^0 + t\boldsymbol{p}_2^0 \quad \boldsymbol{p}_2^1 = (1-t)\boldsymbol{p}_2^0 + t\boldsymbol{p}_3^0$$

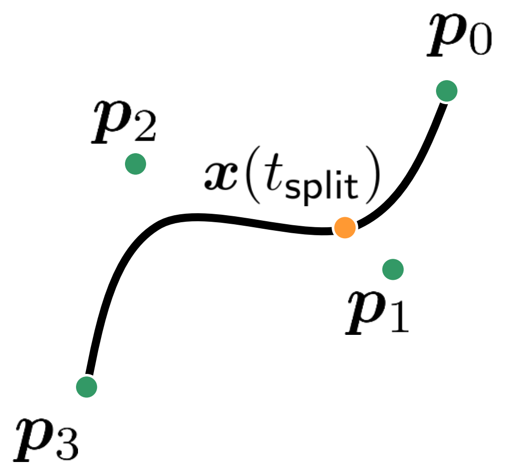
$$\boldsymbol{p}_0^2 = (1-t)\boldsymbol{p}_0^1 + t\boldsymbol{p}_1^1 \quad \boldsymbol{p}_1^2 = (1-t)\boldsymbol{p}_1^1 + t\boldsymbol{p}_2^1$$

$$\boldsymbol{p}_0^3 = (1-t)\boldsymbol{p}_0^2 + t\boldsymbol{p}_1^2$$

three levels of interpolation

Subdivision of a Cubic Bézier

- Given a curve from $\boldsymbol{p}_0$ to $\boldsymbol{p}_3$, generate two curves from $\boldsymbol{p}_0$ to $\boldsymbol{x}(t_{\text{split}})$ and from $\boldsymbol{x}(t_{\text{split}})$ to $\boldsymbol{p}_3$ given a value $0 \leq t_{\text{split}} \leq 1$



- Applications
  - Rendering: Subdivide a curve towards quasi linear segments.
  - Modeling: Modify a part of a curve without changing the other one. Adding degrees of freedom without increasing the curve degree.

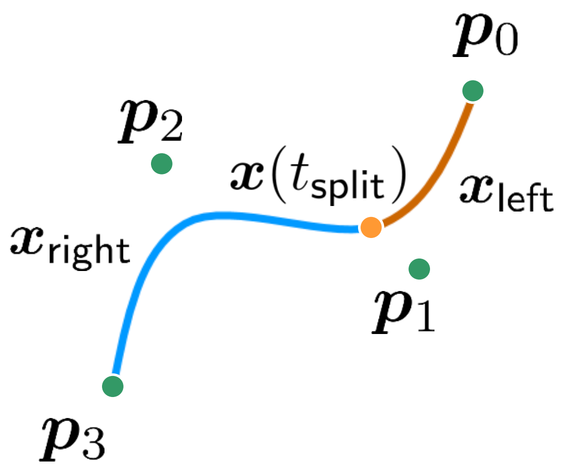
Subdivision of a Cubic Bézier

– Use de Casteljau algorithm

$$\boldsymbol{p}_i^0 = \boldsymbol{p}_i \quad i = 0, 1, 2, 3$$

$$\boldsymbol{p}_i^j = (1 - t_{\text{split}})\boldsymbol{p}_i^{j-1} + t_{\text{split}}\boldsymbol{p}_{i+1}^{j-1} \qquad \boldsymbol{x}(t_{\text{split}}) = \boldsymbol{p}_0^3$$

$$j = 1, 2, 3 \quad i = 0, \ldots, 3 - j$$



– Two resulting curves after split

$$\boldsymbol{x}_{\text{left}}(t) = (\boldsymbol{p}_0^0 \quad \boldsymbol{p}_0^1 \quad \boldsymbol{p}_0^2 \quad \boldsymbol{p}_0^3)\, \boldsymbol{S}_3^{\text{Bez}}\boldsymbol{T}_3(t)$$

$$\boldsymbol{x}_{\text{right}}(t) = (\boldsymbol{p}_0^3 \quad \boldsymbol{p}_1^2 \quad \boldsymbol{p}_2^1 \quad \boldsymbol{p}_3^0)\, \boldsymbol{S}_3^{\text{Bez}}\boldsymbol{T}_3(t)$$

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Subdivision of a Quadratic Bézier

$$\boldsymbol{x}_{\text{left}}(t) = B_{0,2}(t)\boldsymbol{p}_0 + B_{1,2}(t)\boldsymbol{p}_1 + B_{2,2}(t)\boldsymbol{p}_2 \qquad t \in [0, t_{\text{split}}]$$

$$\boldsymbol{x}_{\text{left}}(t_l) = B_{0,2}(t_l \cdot t_{\text{split}})\boldsymbol{p}_0 + B_{1,2}(t_l \cdot t_{\text{split}})\boldsymbol{p}_1 + B_{2,2}(t_l \cdot t_{\text{split}})\boldsymbol{p}_2 \qquad t_l \in [0, 1]$$

In matrix notation

$$\boldsymbol{x}_{\text{left}}(t_l) = \begin{pmatrix}\boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2\end{pmatrix} \boldsymbol{S}_2^{\text{Bez}} \begin{pmatrix} 1 \\ t_l \cdot t_{\text{split}} \\ (t_l \cdot t_{\text{split}})^2 \end{pmatrix}$$

Goal: Compute control points $\boldsymbol{p}_{l,0}, \boldsymbol{p}_{l,1}, \boldsymbol{p}_{l,2}$ with

$$\boldsymbol{x}_{\text{left}}(t_l) = \begin{pmatrix}\boldsymbol{p}_{l,0} & \boldsymbol{p}_{l,1} & \boldsymbol{p}_{l,2}\end{pmatrix} \boldsymbol{S}_2^{\text{Bez}} \begin{pmatrix} 1 \\ t_l \\ t_l^2 \end{pmatrix} \qquad t_l \in [0, 1]$$

Subdivision of a Quadratic Bézier

$$\boldsymbol{x}_{\text{left}}(t_l) = \begin{pmatrix}\boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2\end{pmatrix} \boldsymbol{S}_2^{\text{Bez}} \begin{pmatrix} 1 \\ t_l \cdot t_{\text{split}} \\ (t_l \cdot t_{\text{split}})^2 \end{pmatrix}$$

$$\boldsymbol{x}_{\text{left}}(t_l) = \begin{pmatrix}\boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2\end{pmatrix} \boldsymbol{S}_2^{\text{Bez}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_{\text{split}} & 0 \\ 0 & 0 & t_{\text{split}}^2 \end{pmatrix} \begin{pmatrix} 1 \\ t_l \\ t_l^2 \end{pmatrix}$$

Rewriting the curve with the canonical basis

$$\boldsymbol{x}_{\text{left}}(t_l) = \underbrace{\begin{pmatrix}\boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2\end{pmatrix} \boldsymbol{S}_2^{\text{Bez}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_{\text{split}} & 0 \\ 0 & 0 & t_{\text{split}}^2 \end{pmatrix} (\boldsymbol{S}_2^{\text{Bez}})^{-1}}_{\begin{pmatrix}\boldsymbol{p}_{l,0} & \boldsymbol{p}_{l,1} & \boldsymbol{p}_{l,2}\end{pmatrix}} \boldsymbol{S}_2^{\text{Bez}} \begin{pmatrix} 1 \\ t_l \\ t_l^2 \end{pmatrix}$$

Rewriting the curve with the Bernstein basis functions

Geometry matrix

Subdivision of a Quadratic Bézier

$$\boldsymbol{S}_2^{\text{Bez}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_{\text{split}} & 0 \\ 0 & 0 & t_{\text{split}}^2 \end{pmatrix} (\boldsymbol{S}_2^{\text{Bez}})^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_{\text{split}} & 0 \\ 0 & 0 & t_{\text{split}}^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 - t_{\text{split}} & (1 - t_{\text{split}})^2 \\ 0 & t_{\text{split}} & 2t_{\text{split}}(1 - t_{\text{split}}) \\ 0 & 0 & t_{\text{split}}^2 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{p}_{l,0} & \boldsymbol{p}_{l,1} & \boldsymbol{p}_{l,2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2 \end{pmatrix} \begin{pmatrix} 1 & 1 - t_{\text{split}} & (1 - t_{\text{split}})^2 \\ 0 & t_{\text{split}} & 2t_{\text{split}}(1 - t_{\text{split}}) \\ 0 & 0 & t_{\text{split}}^2 \end{pmatrix}$$

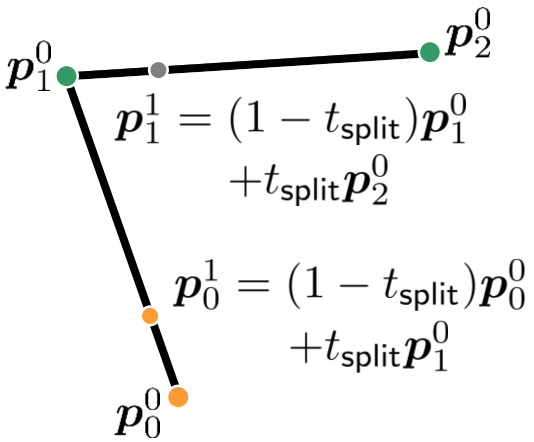
Transformation from old control points to new control points

Subdivision of a Quadratic Bézier

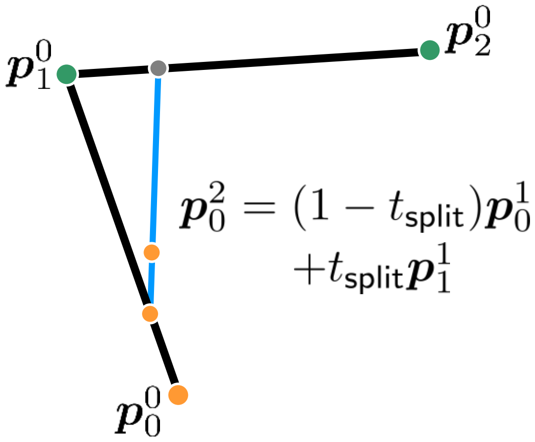
$$\boldsymbol{p}_{l,0} = \boldsymbol{p}_0$$

$$\boldsymbol{p}_{l,1} = (1 - t_{\text{split}})\boldsymbol{p}_0 + t_{\text{split}}\boldsymbol{p}_1$$

$$\begin{aligned}\boldsymbol{p}_{l,2} &= (1 - t_{\text{split}})^2\boldsymbol{p}_0 + 2t_{\text{split}}(1 - t_{\text{split}})\boldsymbol{p}_1 + t_{\text{split}}^2\boldsymbol{p}_2 \\ &= (1 - t_{\text{split}})\big[(1 - t_{\text{split}})\boldsymbol{p}_0 + t_{\text{split}}\boldsymbol{p}_1\big] \\ &\quad + t\big[(1 - t_{\text{split}})\boldsymbol{p}_1 + t_{\text{split}}\boldsymbol{p}_2\big]\end{aligned}$$



$$\boldsymbol{p}_{l,0} = \boldsymbol{p}_0^0 \quad \boldsymbol{p}_{l,1} = \boldsymbol{p}_0^1 \quad \boldsymbol{p}_{l,2} = \boldsymbol{p}_0^2$$



$$\boldsymbol{x}_{\text{left}}(t) = \begin{pmatrix}\boldsymbol{p}_0^0 & \boldsymbol{p}_0^1 & \boldsymbol{p}_0^2\end{pmatrix} \boldsymbol{S}_2^{\text{Bez}}\boldsymbol{T}_2(t)$$

$$\boldsymbol{x}_{\text{right}}(t) = \begin{pmatrix}\boldsymbol{p}_0^2 & \boldsymbol{p}_1^1 & \boldsymbol{p}_2^0\end{pmatrix} \boldsymbol{S}_2^{\text{Bez}}\boldsymbol{T}_2(t)$$

Right sub-curve derived in the same way.

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Outline

- Introduction
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- Differential curve properties
- Piecewise polynomial curves
- B-spline curves

Computer Graphics
Parametric Curves - 2

Matthias Teschner

Outline

- Introduction
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Differential Curve Properties

- Derivatives: velocity / tangent, acceleration
- Can be considered when connecting polynomials to splines, e.g. continuous velocity, acceleration in-between adjacent polynomials

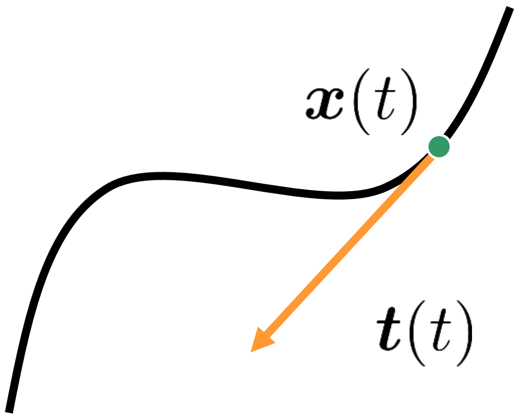
Tangent

– Tangent vector $\boldsymbol{t}(t)$ at a curve point $\boldsymbol{x}(t) = (x(t), y(t))^\mathsf{T}$ is the direction of the curve at that point

$$\boldsymbol{t}_{\Delta t}(t) = \frac{(x(t+\Delta t), y(t+\Delta t))^\mathsf{T} - (x(t), y(t))^\mathsf{T}}{\Delta t}$$

$$\boldsymbol{t}(t) = \lim_{\Delta t \to 0} \boldsymbol{t}_{\Delta t}(t)$$

$$= \left(\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}\right)^\mathsf{T}$$



$$\boldsymbol{t}(t) = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(t) = \left(\frac{\mathrm{d}x}{\mathrm{d}t}(t), \frac{\mathrm{d}y}{\mathrm{d}t}(t)\right)^\mathsf{T}$$

If *x(t)* and *y(t)* are differentiable.

Tangent - Bézier Curves

– Linear Bézier curve

Given by interpolation between point p0 and p1

$$\boldsymbol{x}(t) = (1-t)\boldsymbol{p}_0 + t\boldsymbol{p}_1 \qquad \boldsymbol{p}_i = (p_i, q_i)^\mathsf{T}$$

$$\boldsymbol{t}(t) = (\tfrac{\mathrm{d}x}{\mathrm{d}t}(t), \tfrac{\mathrm{d}y}{\mathrm{d}t}(t))^\mathsf{T}$$

$$= (p_1 - p_0, q_1 - q_0)^\mathsf{T} = \boldsymbol{p}_1 - \boldsymbol{p}_0$$

– Quadratic Bézier curve

$$\boldsymbol{x}(t) = (1-t)^2\boldsymbol{p}_0 + 2(1-t)t\boldsymbol{p}_1 + t^2\boldsymbol{p}_2$$

$$\boldsymbol{t}(t) = -2(1-t)\boldsymbol{p}_0 + 2(1-t)\boldsymbol{p}_1 - 2t\boldsymbol{p}_1 + 2t\boldsymbol{p}_2$$

Tangent - Bézier Curves

- Cubic Bézier curve
  - $\boldsymbol{x}(t) = (1-t)^3\boldsymbol{p}_0 + 3(1-t)^2 t\boldsymbol{p}_1 + 3(1-t)t^2\boldsymbol{p}_2 + t^3\boldsymbol{p}_3$
  - Tangent: $\boldsymbol{t}(t) = -3(1-t)^2\boldsymbol{p}_0 + 3(1-t)^2\boldsymbol{p}_1 - 6(1-t)t\boldsymbol{p}_1 + 6(1-t)t\boldsymbol{p}_2 - 3t^2\boldsymbol{p}_2 + 3t^2\boldsymbol{p}_3$
- Tangents $\boldsymbol{t}(0)$ and $\boldsymbol{t}(1)$
  - Linear: $\boldsymbol{t}(0) = \boldsymbol{p}_1 - \boldsymbol{p}_0$ $\quad \boldsymbol{t}(1) = \boldsymbol{p}_1 - \boldsymbol{p}_0$
  - Quadratic: $\boldsymbol{t}(0) = 2(\boldsymbol{p}_1 - \boldsymbol{p}_0)$ $\quad \boldsymbol{t}(1) = 2(\boldsymbol{p}_2 - \boldsymbol{p}_1)$
  - Cubic: $\boldsymbol{t}(0) = 3(\boldsymbol{p}_1 - \boldsymbol{p}_0)$ $\quad \boldsymbol{t}(1) = 3(\boldsymbol{p}_3 - \boldsymbol{p}_2)$
  - Degree *n*: $\boldsymbol{t}(0) = n(\boldsymbol{p}_1 - \boldsymbol{p}_0)$ $\quad \boldsymbol{t}(1) = n(\boldsymbol{p}_n - \boldsymbol{p}_{n-1})$

Tangent - Bézier Curves

– Matrix notation

Canonical basis

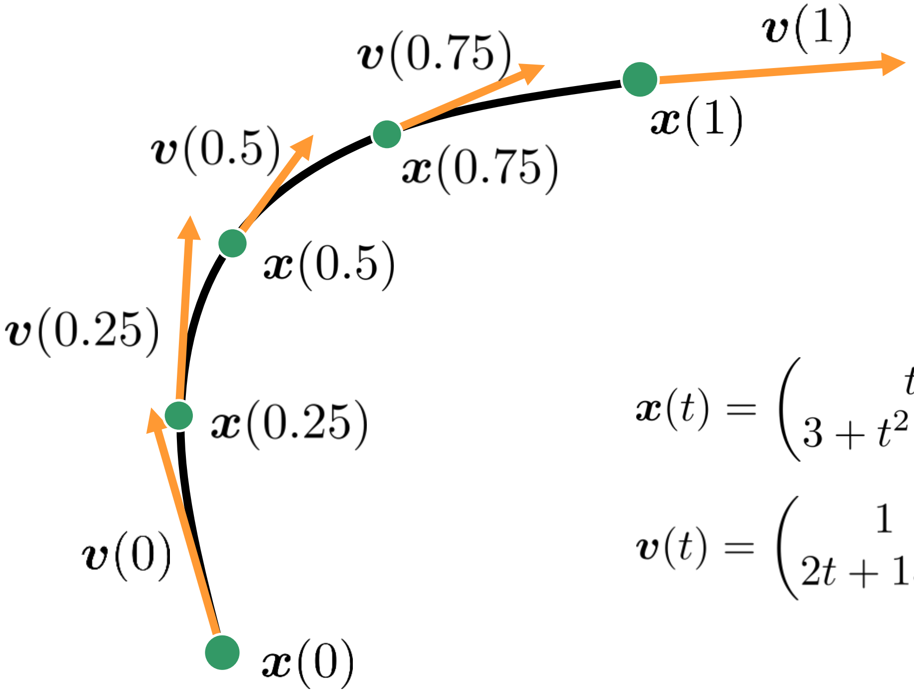
$$\boldsymbol{t}(t) = \begin{pmatrix} \boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} -3 & 6 & -3 \\ 3 & -12 & 9 \\ 0 & 6 & -9 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$$

Velocity

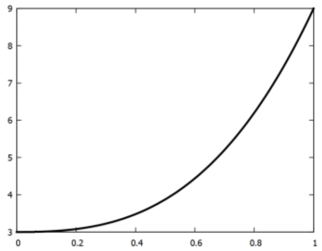
- If t is interpreted as time, $\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(t)$ is a velocity, i.e. position change per time
- Magnitude of the velocity is $v(t) = \|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(t)\|$



Illustration

$$\boldsymbol{x}(t) = \begin{pmatrix} t \\ 3 + t^2 + 5t^3 \end{pmatrix}$$

$$\boldsymbol{v}(t) = \begin{pmatrix} 1 \\ 2t + 15t^2 \end{pmatrix}$$



Example

Acceleration

- If t is interpreted as time, $\boldsymbol{a}(t) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}(t) = (\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t), \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}(t))^\mathsf{T}$ is an acceleration, i.e. velocity change per time
- Linear Bézier curve
  - $\boldsymbol{x}(t) = (1-t)\boldsymbol{p}_0 + t\boldsymbol{p}_1$
  - $\boldsymbol{a}(t) = \mathbf{0}$
- Cubic Bézier curve
  - $\boldsymbol{x}(t) = (1-t)^3\boldsymbol{p}_0 + 3(1-t)^2 t\boldsymbol{p}_1 + 3(1-t)t^2\boldsymbol{p}_2 + t^3\boldsymbol{p}_3$
  - $\boldsymbol{a}(t) = 6(1-t)\boldsymbol{p}_0 - 12(1-t)\boldsymbol{p}_1 + 6t\boldsymbol{p}_1 + 6(1-t)\boldsymbol{p}_2 - 12t\boldsymbol{p}_2 + 6t\boldsymbol{p}_3$

Derivatives - Bézier Curves

– General forms

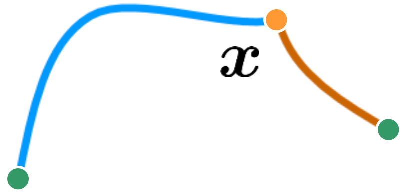
$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(t) = \textstyle\sum_{i=0}^{n-1} n(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) B_{i,n-1}(t)$$

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}(t) = \textstyle\sum_{i=0}^{n-2} n(n-1)(\boldsymbol{p}_{i+2} - 2\boldsymbol{p}_{i+1} + \boldsymbol{p}_i) B_{i,n-2}(t)$$

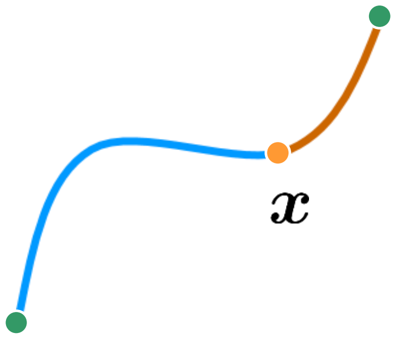
C^k Continuity

– A parametric curve $\boldsymbol{x}(t) = (x(t), y(t))^\mathsf{T}$ is C^k continuous, if the first k derivatives of $x(t)$ and $y(t)$ exist and are continuous

– Used to characterize seams for piecewise polynomial curves



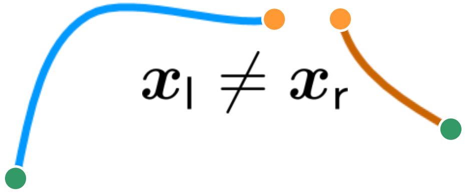
C^0 continuity at x



C^1 continuity at x

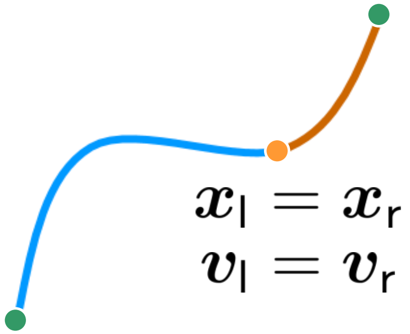
Continuity at Seams

- C^{-1} – continuity
  - Curve endpoint positions are not equal
- C^{0} – continuity
  - Curve endpoint positions are *equal*
- C^{1} – continuity
  - Tangent continuity
  - C^{0} and first derivatives at endpoints are *equal*
- C^{2} – continuity
  - Curvature continuity
  - C^{1} and second derivatives at endpoints are *equal*



C^{-1} continuity at *x*

C^{0} continuity at *x*



C^{1} continuity at *x*

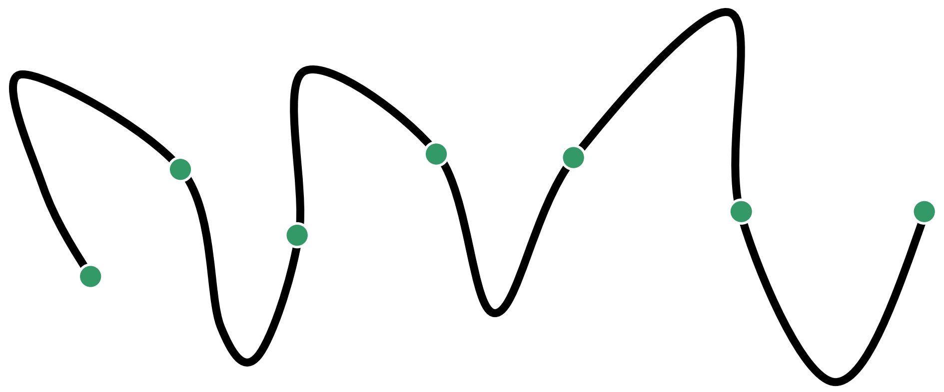
Outline

- Introduction
- Polynomial curves
- Bézier curves
- Matrix notation
- Curve subdivision
- Differential curve properties
- Piecewise polynomial curves
- B-spline curves

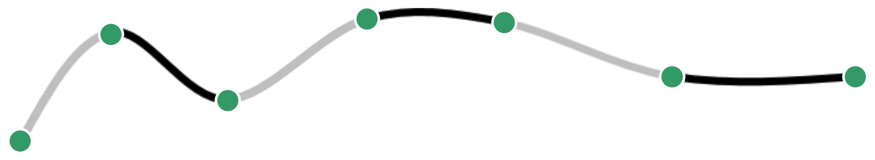
Motivation

Why connect polynomial curves of lower degree to larger piecewise polynomial ones

- Interpolation of *n* control points
  - Higher-order polynomials suffer from oscillations

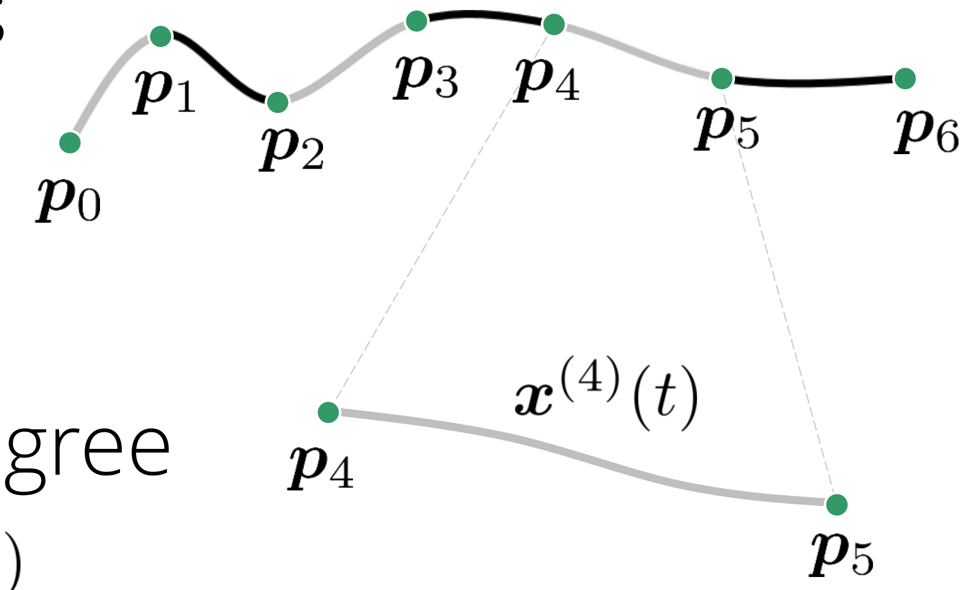


  - Connect *n*-1 polynomials of lower degree instead



Setting

- Cubic piecewise polynomials $\boldsymbol{x}^{(i)}(t)$ connect two control points $\boldsymbol{p}_i$ and $\boldsymbol{p}_{i+1}$
- Smooth connections can be obtained up to a relevant degree
  - C^0 continuity: $\boldsymbol{x}^{(i)}(1) = \boldsymbol{x}^{(i+1)}(0)$
  - C^1 continuity: $\boldsymbol{v}^{(i)}(1) = \boldsymbol{v}^{(i+1)}(0)$
  - G^1 continuity: $\boldsymbol{v}^{(i)}(1) = \alpha \boldsymbol{v}^{(i+1)}(0)$



$$\boldsymbol{p}_4 = \boldsymbol{x}^{(3)}(1) = \boldsymbol{x}^{(4)}(0)$$
$$\boldsymbol{p}_5 = \boldsymbol{x}^{(4)}(1) = \boldsymbol{x}^{(5)}(0)$$

Geometric continuity G^1: Same velocity direction, but not necessarily the same velocity magnitude. proportional.

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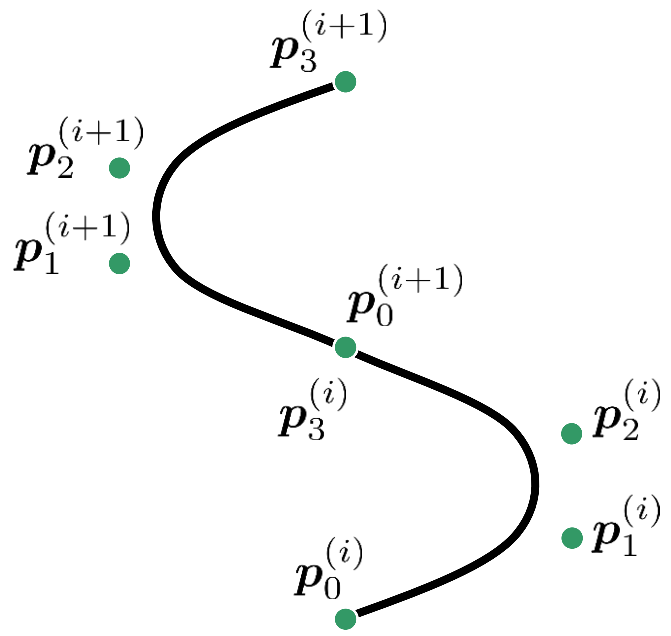
Cubic Bézier Spline

- Connect cubic Bézier curves to Bézier splines
- Curve $\boldsymbol{x}^{(i)}(t)$ interpolates $\boldsymbol{p}_0^{(i)}, \boldsymbol{p}_3^{(i)}$
- Curve $\boldsymbol{x}^{(i+1)}(t)$ interpolates $\boldsymbol{p}_0^{(i+1)}, \boldsymbol{p}_3^{(i+1)}$
- *C⁰* continuity: $\boldsymbol{p}_3^{(i)} = \boldsymbol{p}_0^{(i+1)}$
- Intermediate control points $\boldsymbol{p}_1^{(i)}, \boldsymbol{p}_2^{(i)}$ and $\boldsymbol{p}_1^{(i+1)}, \boldsymbol{p}_2^{(i+1)}$ can be used to obtain C^1 continuity



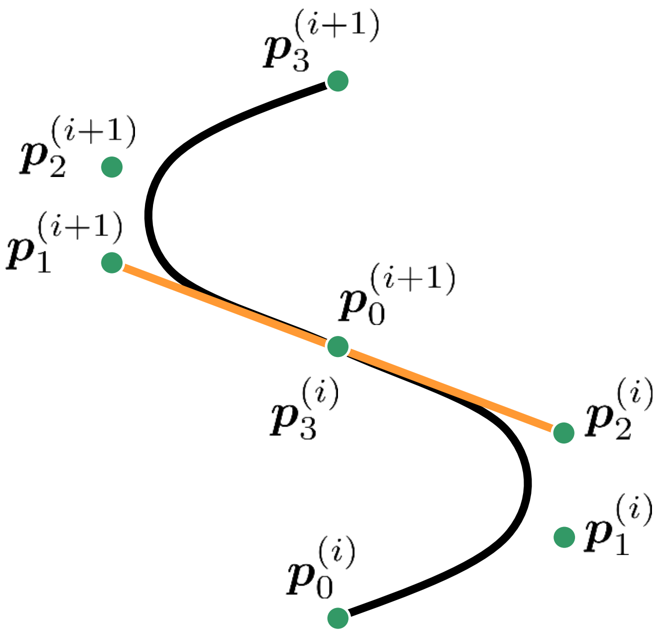
A Bézier spline formed by two Bézier curves

Cubic Bézier Spline – C^1 Continuity

- C^1 continuity: $\boldsymbol{v}^{(i)}(1) = \boldsymbol{v}^{(i+1)}(0)$
- Velocity:

$$\boldsymbol{v}(t) = -3(1-t)^2\boldsymbol{p}_0 + 3(1-t)^2\boldsymbol{p}_1$$
$$-6(1-t)t\boldsymbol{p}_1 + 6(1-t)t\boldsymbol{p}_2 - 3t^2\boldsymbol{p}_2 + 3t^2\boldsymbol{p}_3$$
$$\boldsymbol{v}^{(i)}(1) = 3(\boldsymbol{p}_3^{(i)} - \boldsymbol{p}_2^{(i)})$$
$$\boldsymbol{v}^{(i+1)}(0) = 3(\boldsymbol{p}_1^{(i+1)} - \boldsymbol{p}_0^{(i+1)})$$

- C^1 continuity: $\boldsymbol{p}_3^{(i)} - \boldsymbol{p}_2^{(i)} = \boldsymbol{p}_1^{(i+1)} - \boldsymbol{p}_0^{(i+1)}$
- Can be enforced locally for each connection



Cubic Polynomial in Canonical Form

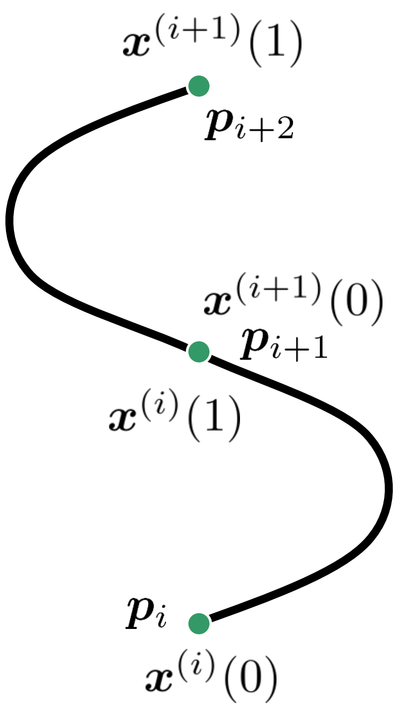
- Curve $\boldsymbol{x}^{(i)}(t) = \boldsymbol{a}_i + \boldsymbol{b}_i t + \boldsymbol{c}_i t^2 + \boldsymbol{d}_i t^3$ interpolates $\boldsymbol{p}_i, \boldsymbol{p}_{i+1}$
- Curve $\boldsymbol{x}^{(i+1)}(t) = \boldsymbol{a}_{i+1} + \boldsymbol{b}_{i+1} t + \boldsymbol{c}_{i+1} t^2 + \boldsymbol{d}_{i+1} t^3$ interpolates $\boldsymbol{p}_{i+1}, \boldsymbol{p}_{i+2}$
- Constraints:

$$\boldsymbol{x}^{(i)}(0) = \boldsymbol{p}_i \qquad \frac{\mathrm{d}\boldsymbol{x}^{(i)}}{\mathrm{d}t}(1) = \frac{\mathrm{d}\boldsymbol{x}^{(i+1)}}{\mathrm{d}t}(0)$$

$$\boldsymbol{x}^{(i)}(1) = \boldsymbol{p}_{i+1} \qquad \frac{\mathrm{d}^2\boldsymbol{x}^{(i)}}{\mathrm{d}t^2}(1) = \frac{\mathrm{d}^2\boldsymbol{x}^{(i+1)}}{\mathrm{d}t^2}(0)$$

$$\boldsymbol{x}^{(i+1)}(0) = \boldsymbol{p}_{i+1} \qquad \frac{\mathrm{d}^2\boldsymbol{x}^{(i)}}{\mathrm{d}t^2}(0) = \boldsymbol{0}$$

$$\boldsymbol{x}^{(i+1)}(1) = \boldsymbol{p}_{i+2} \qquad \frac{\mathrm{d}^2\boldsymbol{x}^{(i+1)}}{\mathrm{d}t^2}(1) = \boldsymbol{0}$$



Typically, minimal velocity change, i.e. minimal curvature changes are desired.

Cubic Polynomial in Canonical Form

– Linear system for unknown coefficients

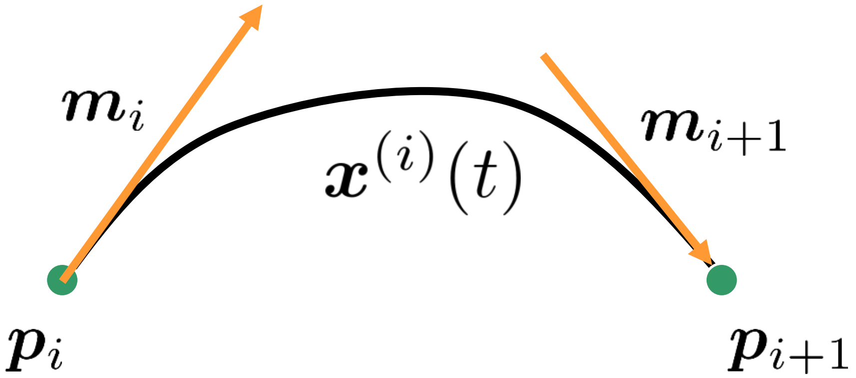
$$\begin{pmatrix} \boldsymbol{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{I}_2 & \boldsymbol{I}_2 & \boldsymbol{I}_2 & \boldsymbol{I}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{I}_2 & \boldsymbol{I}_2 & \boldsymbol{I}_2 & \boldsymbol{I}_2 \\ 0 & \boldsymbol{I}_2 & 2\boldsymbol{I}_2 & 3\boldsymbol{I}_2 & 0 & -\boldsymbol{I}_2 & 0 & 0 \\ 0 & 0 & 2\boldsymbol{I}_2 & 6\boldsymbol{I}_2 & 0 & 0 & -2\boldsymbol{I}_2 & 0 \\ 0 & 0 & 2\boldsymbol{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\boldsymbol{I}_2 & 6\boldsymbol{I}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_i \\ \boldsymbol{b}_i \\ \boldsymbol{c}_i \\ \boldsymbol{d}_i \\ \boldsymbol{a}_{i+1} \\ \boldsymbol{b}_{i+1} \\ \boldsymbol{c}_{i+1} \\ \boldsymbol{d}_{i+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_i \\ \boldsymbol{p}_{i+1} \\ \boldsymbol{p}_{i+1} \\ \boldsymbol{p}_{i+2} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}$$

Cubic Hermite

- Works with positions of and derivatives at control points
- Given:

$$\boldsymbol{x}^{(i)}(0) = \boldsymbol{p}_i \qquad \boldsymbol{x}^{(i)}(1) = \boldsymbol{p}_{i+1}$$

$$\frac{\mathrm{d}\boldsymbol{x}^{(i)}}{\mathrm{d}t}(0) = \boldsymbol{m}_i \qquad \frac{\mathrm{d}\boldsymbol{x}^{(i)}}{\mathrm{d}t}(1) = \boldsymbol{m}_{i+1}$$



How do basis functions H look like that use p_i, p_{i+1}, m_i, m_{i+1} as coefficients?

$$\boldsymbol{x}^{(i)}(t) = \boldsymbol{p}_i H_{0,3}(t) + \boldsymbol{p}_{i+1} H_{1,3}(t) + \boldsymbol{m}_i H_{2,3}(t) + \boldsymbol{m}_{i+1} H_{3,3}(t)$$

Cubic Hermite Basis - Derivation

- One coefficient: $$x^{(i)}(t) = a^{(i)} + b^{(i)}t + c^{(i)}t^2 + d^{(i)}t^3$$
$$\frac{\mathrm{d}x^{(i)}}{\mathrm{d}t}(t) = b^{(i)} + 2c^{(i)}t + 3d^{(i)}t^2$$
- Constraints:

$$x^{(i)}(0) = p_i \quad \Rightarrow \quad a^{(i)} = p_i$$
$$x^{(i)}(1) = p_{i+1} \quad \Rightarrow \quad a^{(i)} + b^{(i)} + c^{(i)} + d^{(i)} = p_{i+1}$$
$$\frac{\mathrm{d}x^{(i)}}{\mathrm{d}t}(0) = m_i \quad \Rightarrow \quad b^{(i)} = m_i$$
$$\frac{\mathrm{d}x^{(i)}}{\mathrm{d}t}(1) = m_{i+1} \quad \Rightarrow \quad b^{(i)} + 2c^{(i)} + 3d^{(i)} = m_{i+1}$$

Cubic Hermite Basis - Derivation

– Constraints in matrix notation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} a^{(i)} \\ b^{(i)} \\ c^{(i)} \\ d^{(i)} \end{pmatrix} = \begin{pmatrix} p_i \\ p_{i+1} \\ m_i \\ m_{i+1} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_i \\ p_{i+1} \\ m_i \\ m_{i+1} \end{pmatrix} = \begin{pmatrix} a^{(i)} \\ b^{(i)} \\ c^{(i)} \\ d^{(i)} \end{pmatrix}$$

– General spline formulation (arbitrary dimension)

$$\underbrace{\boldsymbol{x}^{(i)}(t)}_{\text{Curve}} = \underbrace{\begin{pmatrix} \boldsymbol{p}_i & \boldsymbol{p}_{i+1} & \boldsymbol{m}_i & \boldsymbol{m}_{i+1} \end{pmatrix}}_{\text{Geometry matrix}} \underbrace{\begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{\text{Spline matrix (Hermite)}} \underbrace{\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}}_{\text{Basis (canonical)}}$$

Cubic Hermite

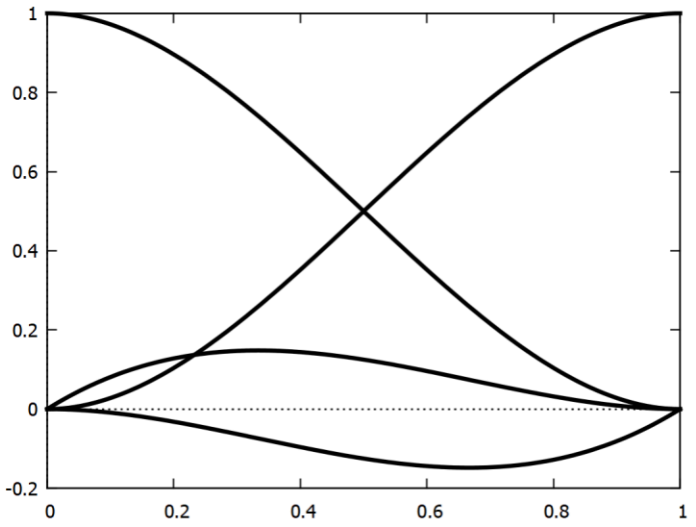
– Basis functions

$$H_{0,3}(t) = 1 - 3t^2 + 2t^3$$

$$H_{1,3}(t) = 3t^2 - 2t^3$$

$$H_{2,3}(t) = t - 2t^2 + t^3$$

$$H_{3,3}(t) = -t^2 + t^3$$



– Curve

$$\boldsymbol{x}^{(i)}(t) = \boldsymbol{p}_i H_{0,3}(t) + \boldsymbol{p}_{i+1} H_{1,3}(t) + \boldsymbol{m}_i H_{2,3}(t) + \boldsymbol{m}_{i+1} H_{3,3}(t)$$

Cubic Hermite - Example

$$H_{0,3}(t) = 1 - 3t^2 + 2t^3$$
$$H_{1,3}(t) = 3t^2 - 2t^3$$
$$H_{2,3}(t) = t - 2t^2 + t^3$$
$$H_{3,3}(t) = -t^2 + t^3$$

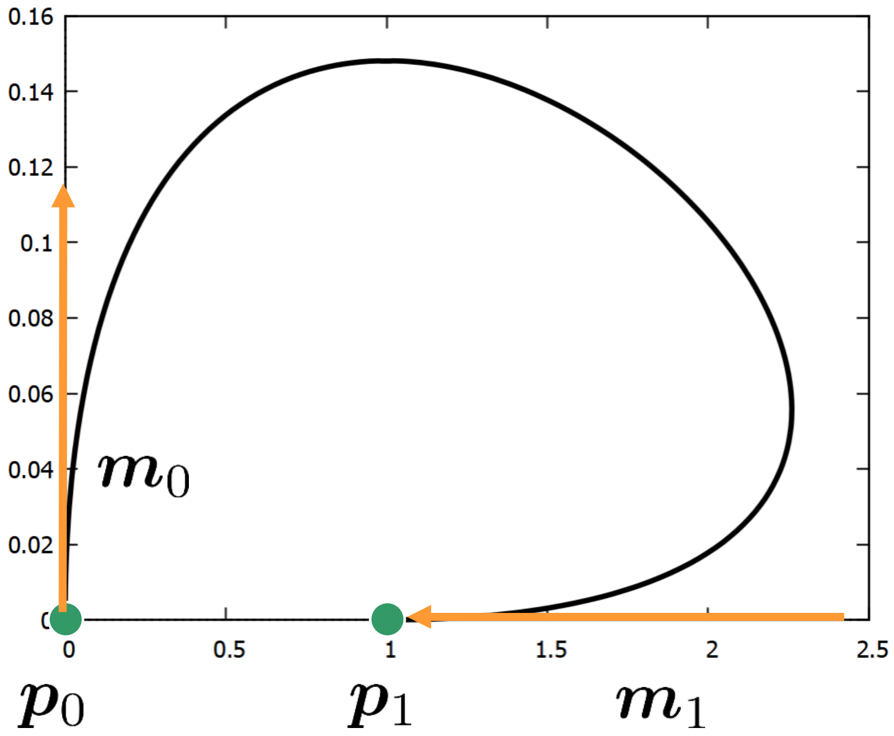
Basis functions

$$\boldsymbol{p}_0 = (0,0)^\mathsf{T}$$
$$\boldsymbol{p}_1 = (1,0)^\mathsf{T}$$
$$\boldsymbol{m}_0 = (0,1)^\mathsf{T}$$
$$\boldsymbol{m}_1 = (-10,0)^\mathsf{T}$$

Geometry

$$\boldsymbol{x}^{(i)}(t) = (0,0)^\mathsf{T} + (3t^2 - 2t^3, 0)^\mathsf{T}$$
$$+(0, t - 2t^2 + t^3)^\mathsf{T} + (10t^2 - 10t^3, 0)^\mathsf{T}$$

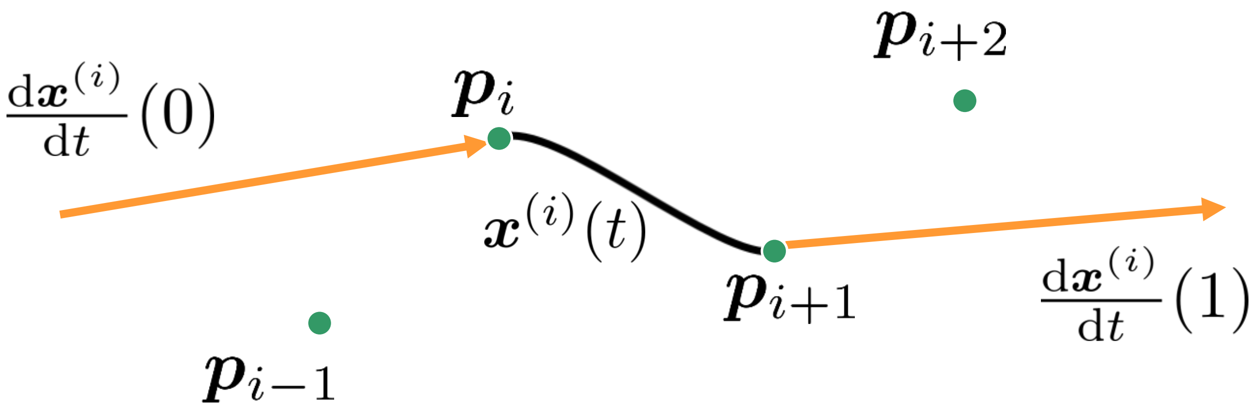
Curve



Catmull-Rom Spline

- Variant of the Hermite spline
- Formulate derivatives with control points
- Given $\boldsymbol{x}^{(i)}(0) = \boldsymbol{p}_i \qquad \boldsymbol{x}^{(i)}(1) = \boldsymbol{p}_{i+1}$

$$\frac{\mathrm{d}\boldsymbol{x}^{(i)}}{\mathrm{d}t}(0) = \frac{1}{2}(\boldsymbol{p}_{i+1} - \boldsymbol{p}_{i-1}) \qquad \frac{\mathrm{d}\boldsymbol{x}^{(i)}}{\mathrm{d}t}(1) = \frac{1}{2}(\boldsymbol{p}_{i+2} - \boldsymbol{p}_i)$$



Catmull-Rom Spline

– Spline formulation

$$\boldsymbol{x}^{(i)}(t) = \begin{pmatrix} \boldsymbol{p}_i & \boldsymbol{p}_{i+1} & \frac{1}{2}(\boldsymbol{p}_{i+1} - \boldsymbol{p}_{i-1}) & \frac{1}{2}(\boldsymbol{p}_{i+2} - \boldsymbol{p}_i) \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Hermite
geometry matrix

Hermite
spline matrix

$$\boldsymbol{x}^{(i)}(t) = \begin{pmatrix} \boldsymbol{p}_{i-1} & \boldsymbol{p}_i & \boldsymbol{p}_{i+1} & \boldsymbol{p}_{i+2} \end{pmatrix} \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Catmull-Rom
geometry matrix

Catmull-Rom
spline matrix

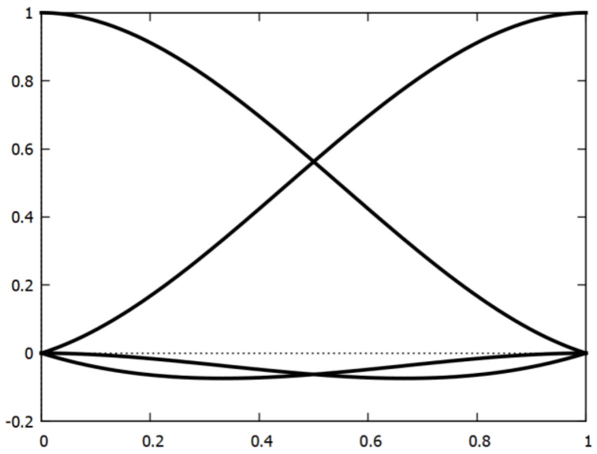
Catmull-Rom Spline

– Spline formulation

$$\boldsymbol{x}^{(i)}(t) = \begin{pmatrix} \boldsymbol{p}_{i-1} & \boldsymbol{p}_i & \boldsymbol{p}_{i+1} & \boldsymbol{p}_{i+2} \end{pmatrix} \underbrace{\frac{1}{2}\begin{pmatrix} 0 & -1 & 2 & -1 \\ 2 & 0 & -5 & 3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{\boldsymbol{S}_3^{\mathsf{CR}}} \underbrace{\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}}_{\boldsymbol{T}_3(t)}$$

– Basis functions

$$CR_{0,3}(t) = \tfrac{1}{2}(-t + 2t^2 - t^3)$$
$$CR_{1,3}(t) = \tfrac{1}{2}(2 - 5t^2 + 3t^3)$$
$$CR_{2,3}(t) = \tfrac{1}{2}(t + 4t^2 - 3t^3)$$
$$CR_{3,3}(t) = \tfrac{1}{2}(-t^2 + t^3)$$



Catmull-Rom Spline - Illustration

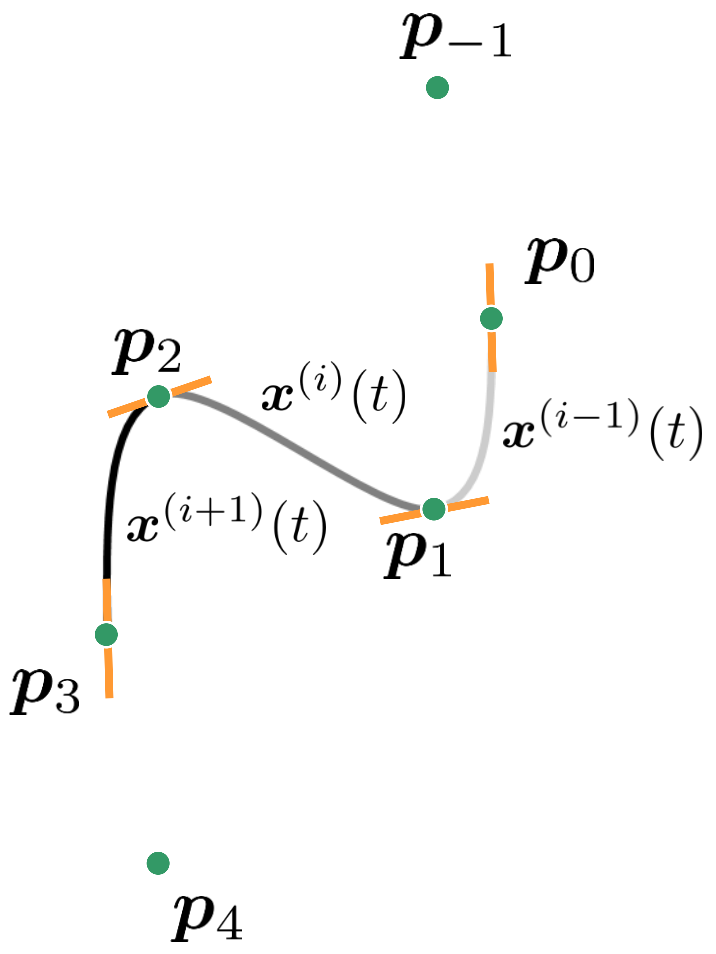
- Catmull-Rom splines are C^1 continuous
  - First derivatives are equal at connections

$$\boldsymbol{x}^{(i-1)}(t) = \begin{pmatrix}\boldsymbol{p}_{-1} & \boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2\end{pmatrix} \boldsymbol{S}_3^{\mathsf{CR}} \boldsymbol{T}_3(t)$$

$$\boldsymbol{x}^{(i)}(t) = \begin{pmatrix}\boldsymbol{p}_0 & \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3\end{pmatrix} \boldsymbol{S}_3^{\mathsf{CR}} \boldsymbol{T}_3(t)$$

$$\boldsymbol{x}^{(i+1)}(t) = \begin{pmatrix}\boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4\end{pmatrix} \boldsymbol{S}_3^{\mathsf{CR}} \boldsymbol{T}_3(t)$$

Each curve interpolates between two control points using four control points



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