Knowledge Representation and Semantic Technologies
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Exercises on Description Logics

Exercise 1 Given the following TBox:

$$\begin{array}{rcl} A & \sqsubseteq & B \\ B & \sqsubseteq & C \\ C & \sqsubseteq & \exists r.D \\ D & \sqsubseteq & \neg A \end{array}$$

1. tell whether the TBox $\mathcal{T}$ is satisfiable, and if so, show a model for $\mathcal{T}$;
2. tell whether the concept D is satisfiable with respect to $\mathcal{T}$, and if so, show a model for $\mathcal{T}$ where the interpretation of D is non-empty;
3. tell whether the concept expression $A \sqcap D$ is satisfiable with respect to $\mathcal{T}$, and if so, show a model for $\mathcal{T}$ where the interpretation of $A \sqcap D$ is non-empty.

Solution

1. Let $\mathcal{I}$ be the interpretation over the domain $\Delta^{\mathcal{I}} = \{d\}$ such that $A^{\mathcal{I}} = B^{\mathcal{I}} = C^{\mathcal{I}} = D^{\mathcal{I}} = r^{\mathcal{I}} = \emptyset$. It is immediate to see that all the axioms of $\mathcal{T}$ are satisfied in $\mathcal{I}$: e.g., since $A^{\mathcal{I}}$ is empty, it is obviously true that $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$, hence the first axiom of $\mathcal{T}$ is satisfied by $\mathcal{I}$. Consequently, $\mathcal{I}$ is a model for $\mathcal{T}$, which implies that $\mathcal{T}$ is satisfiable.
2. To prove that the concept D is satisfiable with respect to $\mathcal{T}$ we have to show a model for $\mathcal{T}$ where the interpretation of D is non-empty. Now, the above model $\mathcal{I}$ does not show that D is satisfiable with respect to $\mathcal{T}$, because $D^{\mathcal{I}}$ is empty. So, we define a new interpretation $\mathcal{J}$, over the domain $\Delta^{\mathcal{J}} = \{d\}$, such that $A^{\mathcal{J}} = B^{\mathcal{J}} = C^{\mathcal{J}} = r^{\mathcal{J}} = \emptyset$ and $D^{\mathcal{J}} = \{d\}$. Again, it is immediate to verify that all the axioms of $\mathcal{T}$ are satisfied in $\mathcal{J}$. In particular, $D \sqsubseteq \neg A$ is satisfied since $(\neg A)^{\mathcal{J}} = \Delta^{\mathcal{J}} = \{d\}$. Consequently, $\mathcal{J}$ is a model for $\mathcal{T}$.
3. Since the TBox $\mathcal{T}$ contains the axiom $D \sqsubseteq \neg A$, it follows that every model $\mathcal{I}$ for $\mathcal{T}$ is such that $D^{\mathcal{I}} \subseteq (\neg A)^{\mathcal{I}}$, i.e., $D^{\mathcal{I}} \cap A^{\mathcal{I}} = \emptyset$. Consequently, no model $\mathcal{I}$ for $\mathcal{T}$ exists such that $(A \sqcap D)^{\mathcal{I}}$ is non-empty.

Exercise 2 Given the knowledge base (KB) $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{T}$ is the following TBox:

$$\begin{array}{ll} \text{(Ax1)} & A \sqsubseteq B \\ \text{(Ax2)} & B \sqsubseteq C \\ \text{(Ax3)} & C \sqsubseteq \exists r.D \\ \text{(Ax4)} & D \sqsubseteq \neg A \\ \text{(Ax5)} & A \sqsubseteq \forall r.A \end{array}$$

and $\mathcal{A}$ is the following ABox:

$$\{ A(a), D(c), r(a,b), r(b,c) \}$$

1. using the tableau method, tell whether the KB $\mathcal{K}$ is satisfiable (i.e., consistent), and if so, show a model for $\mathcal{K}$;
2. now consider the KB $\mathcal{K}'$ obtained from $\mathcal{K}$ by deleting axiom (Ax1) in the TBox. Tell whether the concept assertion $\neg A \sqcup \neg D(c)$ is entailed by $\mathcal{K}'$, using the tableau method.

Solution, point 1

We start by considering point 1 of the exercise. First, C_{GCI} for the given TBox is the following concept expression:

$$C_{GCI} = (\neg A \sqcup B) \sqcap (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists r.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall r.A)$$

Now, we start the tableau from the initial ABox $\mathcal{A}_0 = \mathcal{A}$:

$$\mathcal{A}_0 = \{ A(a), D(c), r(a,b), r(b,c) \}$$

We then apply the tableau C_{GCI}-rule to the individual a, obtaining

$$\mathcal{A}_1 = \mathcal{A}_o \cup \{ ((\neg A \sqcup B) \sqcap (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists r.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall r.A))(a) \}$$

We then apply the tableau and-rule to the last assertion, obtaining

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{ (\neg A \sqcup B)(a), (\neg B \sqcup C)(a), (\neg C \sqcup \exists r.D)(a), (\neg D \sqcup \neg A)(a), (\neg A \sqcup \forall r.A)(a) \}$$

We then apply the tableau or-rule to the assertion $(\neg A \sqcup B)(a)$, obtaining

$$\begin{aligned} \mathcal{A}_3 &= \mathcal{A}_2 \cup \{ \neg A(a) \} \\ \mathcal{A}_4 &= \mathcal{A}_2 \cup \{ B(a) \} \end{aligned}$$

Now, $\mathcal{A}_3$ contains the clash $\{A(a), \neg A(a)\}$ (since $A(a) \in \mathcal{A}_0$), so it is closed. We then consider $\mathcal{A}_4$ and apply the tableau or-rule to the assertion $(\neg B \sqcup C)(a)$, obtaining

$$\begin{aligned} \mathcal{A}_5 &= \mathcal{A}_4 \cup \{ \neg B(a) \} \\ \mathcal{A}_6 &= \mathcal{A}_4 \cup \{ C(a) \} \end{aligned}$$

Now, $\mathcal{A}_5$ contains the clash $\{B(a), \neg B(a)\}$ (since $B(a) \in \mathcal{A}_4$), so it is closed. We then consider $\mathcal{A}_6$ and apply the tableau or-rule to the assertion $(\neg C \sqcup \exists r.D)(a)$, obtaining

$$\begin{aligned} \mathcal{A}_7 &= \mathcal{A}_6 \cup \{ \neg C(a) \} \\ \mathcal{A}_8 &= \mathcal{A}_6 \cup \{ \exists r.D(a) \} \end{aligned}$$

Now, $\mathcal{A}_7$ contains the clash $\{C(a), \neg C(a)\}$ (since $C(a) \in \mathcal{A}_6$), so it is closed. We then consider $\mathcal{A}_8$ and apply the tableau $\exists$-rule to the assertion $\exists r.D(a)$, obtaining

$$\mathcal{A}_9 = \mathcal{A}_8 \cup \{ r(a,x), D(x) \}$$

We now apply the tableau or-rule to the assertion $(\neg D \sqcup \neg A)(a)$, obtaining

$$\mathcal{A}_{10} = \mathcal{A}_9 \cup \{ \neg D(a) \}$$
$$\mathcal{A}_{11} = \mathcal{A}_9 \cup \{ \neg A(a) \}$$

Now, $\mathcal{A}_{11}$ contains the clash $\{A(a), \neg A(a)\}$ (since $A(a) \in \mathcal{A}_0$), so it is closed. We then consider $\mathcal{A}_{10}$ and apply the tableau or-rule to the assertion $(\neg A \sqcup \forall r.A)(a)$, obtaining

$$\mathcal{A}_{12} = \mathcal{A}_{10} \cup \{ \neg A(a) \}$$
$$\mathcal{A}_{13} = \mathcal{A}_{10} \cup \{ \forall r.A(a) \}$$

Again, $\mathcal{A}_{12}$ contains the clash $\{A(a), \neg A(a)\}$ (since $A(a) \in \mathcal{A}_0$), so it is closed. We now consider $\mathcal{A}_{13}$ and apply the tableau $\forall$-rule to the assertion $\forall r.A(a)$ (notice that $A(a)$ and $r(a,x)$ belong to $\mathcal{A}_{13}$), obtaining

$$\mathcal{A}_{14} = \mathcal{A}_{13} \cup \{ A(x) \}$$

We then apply the tableau C_{GCI}-rule to the individual x, obtaining

$$\mathcal{A}_{15} = \mathcal{A}_{14} \cup \{ ((\neg A \sqcup B) \sqcap (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists r.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall r.A))(x) \}$$

We then apply the tableau and-rule to the last assertion, obtaining

$$\mathcal{A}_{16} = \mathcal{A}_{15} \cup \{ (\neg A \sqcup B)(x), (\neg B \sqcup C)(x), (\neg C \sqcup \exists r.D)(x), (\neg D \sqcup \neg A)(x), (\neg A \sqcup \forall r.A)(x) \}$$

We then apply the tableau or-rule to the assertion $(\neg D \sqcup A)(x)$, obtaining

$$\mathcal{A}_{17} = \mathcal{A}_{16} \cup \{ \neg D(x) \}$$
$$\mathcal{A}_{18} = \mathcal{A}_{16} \cup \{ \neg A(x) \}$$

Now, notice that $D(x) \in \mathcal{A}_9$, therefore $\mathcal{A}_{17}$ contains the clash $D(x), \neg D(x)$. Moreover, notice that $A(x) \in \mathcal{A}_{14}$, therefore $\mathcal{A}_{18}$ contains the clash $A(x), \neg A(x)$.

Consequently, all the ABoxes (branches) generated by the tableau are closed. We can thus conclude that the knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ of point 1 is inconsistent (unsatisfiable).

Solution, point 2

We now consider point 2 of the exercise. First, C_{GCI} for the given TBox is the following concept expression:

$$C_{GCI} = (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists r.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall r.A)$$

Now, we start the tableau from the initial ABox $\mathcal{A}_0$ obtained by the ABox $\mathcal{A}$ of point 1 adding the negation of the assertion $\neg A \sqcup \neg D(c)$:

$$\mathcal{A}_0 = \mathcal{A} \cup \{ A \sqcap D(c) \}$$

We apply the tableau and-rule to the above assertion, obtaining

$$\mathcal{A}_1 = \mathcal{A}_0 \cup \{ A(c), D(c) \}$$

We then apply the tableau C_{GCI}-rule to the individual c, obtaining

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{ (\neg B \sqcup C) \sqcap (\neg C \sqcup \exists r.D) \sqcap (\neg D \sqcup \neg A) \sqcap (\neg A \sqcup \forall r.A))(c) \}$$

We then apply the tableau and-rule to the last assertion, obtaining

$$\mathcal{A}_3 = \mathcal{A}_2 \cup \{ (\neg B \sqcup C)(a), (\neg C \sqcup \exists r.D)(a), (\neg D \sqcup \neg A)(a), (\neg A \sqcup \forall r.A)(c) \}$$

We then apply the tableau or-rule to the assertion $(\neg D \sqcup \neg A)(c)$, obtaining

$$\begin{aligned} \mathcal{A}_4 &= \mathcal{A}_3 \cup \{ \neg D(c) \} \\ \mathcal{A}_5 &= \mathcal{A}_3 \cup \{ \neg A(c) \} \end{aligned}$$

Now, $\mathcal{A}_4$ contains the clash $\{D(c), \neg D(c)\}$ (since $D(c) \in \mathcal{A}_1$), so it is closed. Moreover, $\mathcal{A}_5$ contains the clash $\{A(c), \neg A(c)\}$ (since $A(c) \in \mathcal{A}_1$), so it is closed too.

Consequently, all the ABoxes (branches) generated by the tableau are closed. We can thus conclude that the knowledge base $\mathcal{K}'$ entails the assertion $\neg A \sqcup \neg D(c)$.

Exercise 3 Given the knowledge base (KB) $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{T}$ is the following TBox:

$$\begin{array}{rcl} A & \sqsubseteq & B \sqcup C \\ B & \sqsubseteq & \exists r.D \\ C & \sqsubseteq & \exists r.E \\ A & \sqsubseteq & \forall r.F \\ D \sqcap F & \sqsubseteq & G \end{array}$$

and $\mathcal{A}$ is the following ABox:

$$A(a)$$

1. using the tableau method, tell whether the concept assertion $\exists r.F(a)$ is entailed by $\mathcal{K}$;
2. using the tableau method, tell whether the concept assertion $\exists r.G(a)$ is entailed by $\mathcal{K}$;
3. using the tableau method, tell whether the concept assertion $\exists r.(D \sqcup E)(a)$ is entailed by $\mathcal{K}$.