1. Stable-matching

1.1 Description

Design a self-reinforcing admission process.

Hospital h and student s form an unstable pair if both:

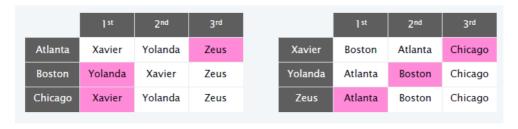
- h prefers s to one of its admitted students.
- s prefers h to assigned hospital.

Stable assignment: assignment with no unstable pairs

Input. A set of n hospitals H and a set of n students S.

- Each hospital $h \in H$ ranks students.
- Each student $s \in S$ ranks hospitals.

Def. A matching M is perfect if |M| = |H| = |S| = n.



Def. unstable pair if both: (improvable by joint action)

- h prefers s to matched student.
- s prefers h to matched hospital.

Stable matching problem. Given the preference lists of n hospitals and n students, find a stable matching (if one exists).

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GALE-SHAPLEY (preference lists for hospitals and students) (O(n^2))

INITIALIZE M to empty matching.

WHILE (some hospital h is unmatched and hasn't proposed to every student)

s - first student on h's list to whom h has not yet proposed.

If (s is unmatched)

Add h-s to matching M.

ELSE IF (s prefers h to current partner h')

Replace h'-s with h-s in matching M.

ELSE

s rejects h.

RETURN stable matching M.
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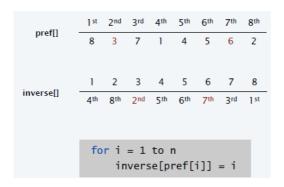
Gale-Shapley terminate, is perfect and stable and at the end produces a matching for every entity.

1.1.1 Efficient implementation

- Maintain a list of free hospitals (in a stack or queue).
- Maintain two arrays student[h] and hospital[s].
 - if h matched to s, then student[h] = s and hospital[s] = h
 - use value 0 to designate that hospital or student is unmatched
- For each hospital, maintain a list of students, ordered by preference.
- · For each hospital, maintain a pointer to students in list for next proposal.

Students rejecting/accepting.

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Def. Student s is a valid partner for hospital h if there exists any stable matching in which h and s are matched.

Claim. All executions of Gale-Shapley yield hospital-optimal assignment.

Corollary. Hospital-optimal assignment is a stable matching!

Claim. Gale-Shapley matching S* is hospital-optimal.

Student-pessimal assignment. Each student receives worst valid partner.

Claim. Gale-Shapley finds student-pessimal stable matching M*.

1.1.2 Extensions

Extension 1. Some participants declare others as unacceptable.

Extension 2. Some hospitals have more than one position.

Extension 3. Unequal number of positions and students.

Def. Matching M is unstable if there is a hospital h and student s such that:

- · h and s are acceptable to each other; and
- · Either s is unmatched, or s prefers h to assigned hospital; and
- Either h does not have all its places filled, or h prefers s to at least one of its assigned students.

1.2 Representative problems

Interval scheduling: find maximum cardinality subset of mutually compatible jobs with start and finished times given.

Bipartite matching: Given a bipartite graph G = (L U R, E), find a max cardinality matching.

Def. A subset of edges $M \subseteq E$ is a **matching** if each node appears in exactly one edge in M.

Independent set: A subset $S \subseteq V$ is independent if for every $(u, v) \in E$, either $u \notin S$ or $v \notin S$ (or both).

Competitive facility location: Input. Graph with weight on each node.

Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbours have been selected.

Goal. Select a maximum weight subset of nodes.

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