Numerische Mathematik Hausaufgaben

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Aufgabe 13.1

Legendre-Polynom:

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Stützstellen:
$\frac{1}{2}(5x^3 - 4x) = 0$

$x_1 = -\sqrt{\frac{3}{5}}$
$x_2 = 0$
$x_3 = \sqrt{\frac{3}{5}}$

$w_i = \int_{-1}^{1} \prod_{j=0, j\neq i}^{n} \frac{x-x_j}{x_i-x_j} dx$

$w_0 = \int_{-1}^{1} \prod_{j=0, j\neq i}^{3} \frac{x-x_j}{0-x_j} dx$
$\prod_{j=0, j\neq i}^{3} \frac{x-x_j}{0-x_j} = \frac{x+\sqrt{\frac{3}{5}}}{0+\sqrt{\frac{3}{5}}} \cdot \frac{x-\sqrt{\frac{3}{5}}}{0-\sqrt{\frac{3}{5}}} = -\frac{5}{3}(x^2 - \frac{3}{5})$
$w_0 = \int_{-1}^{1} -\frac{3}{5}(x^2 - \frac{3}{5}) dx = \frac{8}{9}$

$w_1 = \int_{-1}^{1} \prod_{j=0, j\neq i}^{3} \frac{x-x_j}{-\sqrt{\frac{3}{5}}-x_j} dx$
$\prod_{j=0, j\neq i}^{3} \frac{x-x_j}{-\sqrt{\frac{3}{5}}-x_j} = \frac{x-0}{-\sqrt{\frac{3}{5}}-0} \cdot \frac{x-\sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}}} = \frac{5}{6}x(x - \sqrt{\frac{3}{5}})$

$w_1 = \int_{-1}^{1} \frac{5}{6}x(x - \sqrt{\frac{3}{5}}) dx = \frac{5}{9}$

$w_2 = \int_{-1}^{1} \prod_{j=0, j\neq i}^{3} \frac{x-x_j}{\sqrt{\frac{3}{5}}-x_j} dx$
$\prod_{j=0, j\neq i}^{3} \frac{x-x_j}{\sqrt{\frac{3}{5}}-x_j} = \frac{x-0}{\sqrt{\frac{3}{5}}-0} \cdot \frac{x+\sqrt{\frac{3}{5}}}{\sqrt{\frac{3}{5}}+\sqrt{\frac{3}{5}}} = \frac{5}{6}x(x + \sqrt{\frac{3}{5}})$

$w_2 = \int_{-1}^{1} \frac{5}{6}x(x + \sqrt{\frac{3}{5}}) dx = \frac{5}{9}$

$w_1 = \frac{8}{9}$
$w_0 = w_2 = \frac{5}{9}$

Aufgabe 13.2

a) $\int_0^1 \frac{\sqrt{x}}{1+x}dx = \int_0^1 \frac{\sqrt{t^2}}{1+t^2}dt = \int_0^1 \frac{t}{1+t^2}dt$

Durch diese Substituierung verkleinert sich der Fehler bei Quadraturmethoden.

b) ohne Substituierung
$h := \frac{b-a}{n} = \frac{1}{n}$

$T_n = \frac{h}{2}(f(a) + 2 \cdot \sum_{i=1}^{n-1} f(a+ih) + f(b))$
Für n=1
$h = 1$
$T(1) = 1(0 + \frac{1}{4} + 0) = \frac{1}{4}$

Für n=2
$h = \frac{1}{2}$
$T(\frac{1}{2}) = \frac{1}{2}(0 + \frac{1}{4} + \frac{\frac{\sqrt{2}}{2}}{\frac{3}{2}}) = \frac{1}{24}(3 + 4\sqrt{2})$

Für n=4
$h = \frac{1}{4}$
$T(\frac{1}{4}) = \frac{1}{4}(0 + \frac{1}{4} + (\frac{2}{5} + \frac{\frac{\sqrt{2}}{2}}{\frac{3}{2}} + \frac{\frac{\sqrt{3}}{2}}{\frac{7}{4}})) = \frac{1}{4}(\frac{13}{20} + \frac{\sqrt{2}}{3} + \frac{2\sqrt{3}}{7}) = 0.404069045$

Für n=8
$h = \frac{1}{8}$
$T(\frac{1}{8}) = \frac{1}{8}(0 + \frac{1}{4} + (\frac{\frac{\sqrt{2}}{4}}{\frac{9}{8}} + \frac{2}{5} + \frac{\frac{\sqrt{6}}{4}}{\frac{11}{8}} + \frac{\frac{\sqrt{2}}{2}}{\frac{3}{2}} + \frac{\frac{\sqrt{10}}{4}}{\frac{13}{8}} + \frac{\frac{\sqrt{3}}{2}}{\frac{7}{4}} + \frac{\frac{\sqrt{14}}{4}}{\frac{15}{8}}))$
$= \frac{1}{8}(\frac{13}{20} + \frac{5\sqrt{2}}{9} + \frac{2\sqrt{3}}{7} + \frac{2\sqrt{6}}{11} + \frac{2\sqrt{10}}{13} + \frac{2\sqrt{14}}{15})$
$= 0.42016244$

$P_{i,m} = P_{i,m-1} - \frac{1/k_i^2}{1/k_i^2 - 1/k_i^2}(P_{i,m-1} - P_{i-1,m-1})$

k_i	$P_{i,0}$	$P_{i,1}$	$P_{i,2}$	$P_{i,3}$
1	$\frac{1}{4}$			
2	0.3607022604	0.3976030139		
3	0.404069045	0.4185246399	0.4199194149	
4	0.42016244	0.4255269082	0.42059937261	0.4264443723

$\int_0^1 \frac{\sqrt{x}}{1+x}dx = 2 - \frac{\pi}{2} = 0.4292036732$

$|0.4292036732 - 0.4264443723| = 0.0027593009$

b) mit Substituierung

$f(0) = 0$
$f(1) = \frac{1}{2}$

Für n=1

$h = 1$
$T(1) = 1(0 + \frac{1}{4} + 0) = \frac{1}{4}$

Für n=2

$h = \frac{1}{2}$
$T(\frac{1}{2}) = \frac{1}{2}(0 + \frac{1}{4} + \frac{2}{5}) = \frac{13}{40}$

Für n=4

$h = \frac{1}{4}$
$T(\frac{1}{4}) = \frac{1}{4}\left(0 + \frac{1}{4} + \left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right)\right) = \frac{2321}{6800}$

Für n=8

$h = \frac{1}{8}$
$T(\frac{1}{8}) = \frac{1}{8}\left(0 + \frac{1}{4} + \left(f\left(\frac{1}{8}\right) + f\left(\frac{1}{4}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{7}{8}\right)\right)\right) = 0.3452689484$

$P_{i,m} = P_{i,m-1} - \frac{1/k_i^2}{1/k_i^2 - 1/k_i^2}(P_{i,m-1} - P_{i-1,m-1})$

k_i	$P_{i,0}$	$P_{i,1}$	$P_{i,2}$	$P_{i,3}$
1	$\frac{1}{4}$			
2	$\frac{13}{40}$	$\frac{7}{20}$		
3	$\frac{2321}{6800}$	$\frac{1179}{3400}$	$\frac{8837}{25500}$	
4	0.3452689484	0.344285937	0.3465720469	0.3465724124

$\int_0^1 \frac{\sqrt{t^2}}{1+t^2} dt = \frac{ln(2)}{2} = 0.346573590$

$|0.346573590 - 0.3465724124| = 1.1776 \cdot 10^{-6}$

Aufgabe 13.4

$\int_{-\infty}^{0} te^{2t}dt$ substituiert mit $t = ln(\frac{1}{2}(x+1))$

$\int_{-1}^{1} ln(\frac{1}{2}(x+1))e^{2\cdot(ln(\frac{1}{2}(x+1)))}dx = 0.2222222$

$x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{5}}$
$\gamma_0 = \gamma_2 = \frac{5}{9}$, $\gamma_1 = \frac{8}{9}$

$\sum_{i=1}^{n} f(x_i)\gamma_i$

$\rightarrow f\left(-\sqrt{\frac{3}{5}}\right)\frac{5}{9} + f(0)\frac{8}{9} + f\left(\sqrt{\frac{3}{5}}\right)\frac{5}{9}$
$f\left(-\sqrt{\frac{3}{5}}\right)\frac{5}{9} = -0.0154043757$
$f(0)\frac{8}{9} = -0.154032707$
$f\left(\sqrt{\frac{3}{5}}\right)\frac{5}{9} = -0.05230023$

$= -0.2217373127$
Differenz:
$|-\frac{2}{9} + 0.2217373127| = 0.0004849095$

Zusatz:
$\int_{-\infty}^{0} te^{2t}dt = \frac{1}{2}e^{2t}t\Big|_{-\infty}^{0} \frac{-1}{2}\int_{-\infty}^{0} e^{2t}dt$
$= -\lim_{a\rightarrow-\infty} \frac{1}{2}e^{2a}a - \frac{1}{2}\int_{-\infty}^{0} e^{2t}dt$
$= 0 - \frac{1}{2}\int_{-\infty}^{0} e^{2t}dt = -\frac{1}{4}\int_{-\infty}^{0} e^{u}du$ für $u = 2t$
$= (-\frac{e^u}{4})\Big|_{-\infty}^{0} = (-\frac{e^0}{4} - (-\frac{e^{-\infty}}{4})) = -\frac{1}{4}$
$\int_{-\infty}^{0} te^{2t}dt = -\frac{1}{4}$