Dynamics on Complex Networks and Chaos Synchronization

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Abstract

Dynamical systems sharing information may express synchronization in the sense that they spontaneously establish some relation between their phase-space trajectories. Here we give an overview of the theory behind the synchronization of small networks of oscillators as well as synchronization of more numerous dynamical systems with a complex linking structure. Even chaotic systems, despite the exponential divergence of nearby trajectories, can synchronize when coupled together. We give some examples of physical systems that exhibit synchronization and can be understood within this paradigm.

1 INTRODUCION

We usually have a good understanding of the dynamics of isolated systems. This work is intended to be an overview of the theory behind the mutual synchronization of such systems when they interact with each other. Generally speaking, synchronization is the tendency to express the same dynamical behaviours in a more or less marked way depending on the character of the coupling. Surprisingly, this spontaneous tendency towards order extends to chaotic systems. The chaotic nature in the dynamics, characterized by an exponential divergence of nearby trajectories, seemed incompatible with synchronization. The works of Winfree [1] and Kuramoto [2], and later Fujisaka and Yamada [3], laid the foundations for the study of synchronous chaos. For many interacting systems, the dynamic is strictly dependent on the underlying network structure that represents interactions so that useful results from graph theory can be used to study the most general cases and tackle real-world problems that can be modeled by such networks of interacting elements, spacing from engineering and technology [4–9] to biology and epidemiology [10–13].

2 SYNCHRONIZATION OF COUPLED NON-LINEAR SYSTEMS

Consider two coupled n-dimensional systems:

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1) + \alpha\mathbf{H}(\mathbf{x}_1 - \mathbf{x}_2)\\ \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2) + \alpha\mathbf{H}(\mathbf{x}_2 - \mathbf{x}_1)\end{aligned} \tag{1}$$

With $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ non-linear and $\mathbf{H} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{H}(\mathbf{0}) = \mathbf{0}$. Consider the variable $\mathbf{x}_1 - \mathbf{x}_2$. There exists a critical coupling strength α_c for which the synchronization error

$$E = \frac{1}{T}\int_{t+0}^{T} \|\mathbf{x}_1 - \mathbf{x}_2\| \tag{2}$$

drops to zero for values of $\alpha > \alpha_c$. Fig. 1 shows an example of the shyncronization errors for $\mathbf{H} = \mathbf{I}$. The critical coupling depends on the Lyapunov exponent Λ of the orbit $\mathbf{x}(t)$, e.g., for the simple case of $\mathbf{H} = \mathbf{I}$, we have $\alpha_c = \frac{\Lambda}{2}$. Synchronization of this kind, in coupled systems, is called *complete synchronization*.

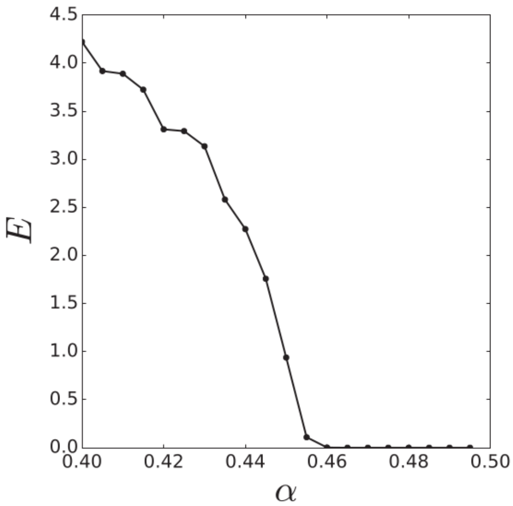

Figure 1: Synchronisation error of two coupled Lorenz systems with coupling matrix $\mathbf{H} = \mathbf{I}$ [14].

3 GENERALIZED SYNCHRONIZATION FOR MASTER-SLAVE CONFIGURATION

When the systems are different, we can still observe a so-called *generalized synchronization*. Consider two systems coupled in a master-slave configuration

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x})\\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{y}, \mathbf{h}(\mathbf{x})),\end{aligned} \tag{3}$$

for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ and some coupling strength in $\mathbf{h}$. Generalized synchronization means that there exists a functional relation ϕ between the two phase spaces so that, after a transitory evolution, all the trajectories will ap-

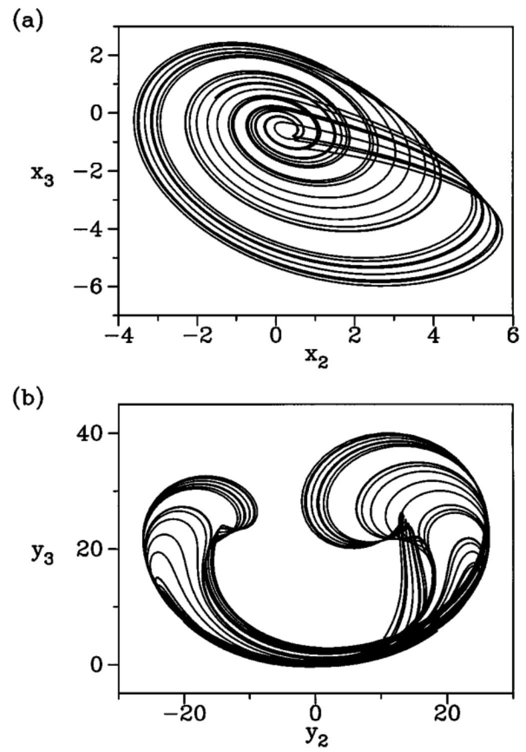

Figure 2: Generalized synchronization of a Lorentz system driven by a Rössler system in a master-slave configuration as in Eq. (5). (b) is the image of the Rössler attractor (a) through the functional relation ϕ [16].

proach to the manifold [15]

$$M = \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} = \phi(\mathbf{x})\}. \tag{4}$$

As an example, consider a Lorentz system driven by a Rössler system in the following way [16]:

$$\begin{aligned} \dot{x}_1 &= 2 + x_1(x_2 - 4) \\ \dot{x}_2 &= -x_1 - x_3 \\ \dot{x}_3 &= x_2 + 0.45x_3 \end{aligned} \qquad \begin{aligned} \dot{y}_1 &= -10(y_1 - y_2) \\ \dot{y}_2 &= 28u(t) - y_2 - u(t)y_3 \\ \dot{y}_3 &= u(t)y_2 - \frac{8}{3}y_3 \end{aligned} \tag{5}$$

where u is an arbitrary scalar function of x_1, x_2, x_3. We have generalized synchronization when two identical slave systems $\mathbf{y}$ and $\mathbf{y}'$, driven in the same way by the same master system, completely synchronize. Using the Lyapunov function for the system $\mathbf{Y} = \mathbf{y} - \mathbf{y}'$

$$L = \frac{(\frac{Y_1}{10} + Y_2 + Y_3)}{2}, \tag{6}$$

we find that the slave system is as asymptotically stable for arbitrary u and arbitrary initial conditions. Fig. 2 shows the non-linear image of the master system attractor through the functional relation of equation (4).

4 COMPLEX NETWORKS SYNCHRONIZATION

The discussion of two coupled n-dimensional systems can be extended to a system of N coupled systems of the form

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \alpha \sum_j^N A_{ij}[\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)] \tag{7}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{H} : \mathbb{R}^n \to \mathbb{R}^n$, α is the coupling strength and $\mathbf{A}$ is the so-called adjacency matrix of the underlying network structure ($A_{ij} = 1$ if the node i recives connection from j and 0 otherwise). Note that the above system represents some dynamics on networks and not the distinct problem of evolving networks.
The coupling term can be rewritten in terms of the Laplacian $L_{ij} = \delta_{ij}k_i - A_{ij}$ of the network so that the system reads as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \alpha \sum_j^N L_{ij}\mathbf{H}(\mathbf{x}_j). \tag{8}$$

The results will depend on the spectral properties of the Laplacian. Note that, as in (1), where the sycrhonization subspace $\mathbf{x}_1 = \mathbf{x}_2$ was invariant under $\mathbf{H}$, here the coupling term vanishes for $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N$ because $\mathbf{L}$ is a zero row sum matrix. An important theorem (see [14] for proof) states that for linear coupling $\mathbf{H}$ the critical coupling strength is

$$\alpha_c(\mathbf{f}, \mathbf{H}, G) = \frac{\Gamma(\mathbf{f}, \mathbf{H})}{\lambda_2(G)}, \tag{9}$$

with $\lambda_2(G)$ the second smallest eigenvalue of $\mathbf{L}$ and $\lambda_2(G) > 0$ for connected graphs. If we relax the assumption of linear coupling and we consider a system of the form

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \alpha \sum_j^N L_{ij}\mathbf{g}(\mathbf{x}_i, \mathbf{x}_j), \tag{10}$$

for a general diffusive function $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the stability condition has the form of an inequality:

$$\frac{\alpha_c^2}{\alpha_c^1} \geq \frac{\lambda_N}{\lambda_2}, \tag{11}$$

for the stability region (α_c^1, α_c^2) of the coupling strength.
As an example, consider an n-dimensional system of the form of Eq. (10) where the isolated oscillators are Rössler systems. The so-called *variational equations* governing the time evolution of the set of infinitesimal vectors about the synchronous solution $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N = \mathbf{s}$ is

$$\delta\dot{\mathbf{x}}_i = \mathbf{Df}(\mathbf{s}) \cdot \delta\mathbf{x}_i - \alpha \sum_{j=1}^N L_{ij}\mathbf{DH}(\mathbf{s}) \cdot \delta\mathbf{x}_i \tag{12}$$

where $\delta\mathbf{x}_i = \mathbf{x}_i(t) - \mathbf{s}(t)$ and $\mathbf{Df}$ and $\mathbf{DH}$ are the Jacobian matrices. This variational equation can be written as N decoupled blocks

$$\delta\dot{\mathbf{y}}_i = [\mathbf{Df}(\mathbf{s}) \cdot \delta\mathbf{x}_i - \kappa_i\mathbf{DH}(\mathbf{s})] \cdot \delta\mathbf{y}_i, \tag{13}$$

where $\delta\mathbf{y}_i = \mathbf{B} \cdot \delta\mathbf{x}_i$ and $\mathbf{B}$ is a matrix whose columns are the set of eigenvectors of $\mathbf{L}$; $\kappa_i = \alpha\lambda_i$. The largest Lyapunov exponent $\Lambda(\kappa)$ determined from this equation is the *Master Stability Function* [17]. For sychronization it is necessary that all the κ_i fall in an intervall (α_c^1, α_c^2) where $\Lambda(\kappa) < 0$. This implies that most synchronizable networks are the ones with a small spread in the κ_i values. Figure 3 shows four typical master stability functions for coupled Rössler oscillators. We can study how the structure of the underlying network affects synchronizability as done in [18]. Fig. 4 compares the eigenratio of a pure random graph, regular ("pristine") graphs, and small-world graphs. In regular graphs each node is linked to the $2k$ nearest neighbors and a small world is obtained by adding random links to a regular graph [19].

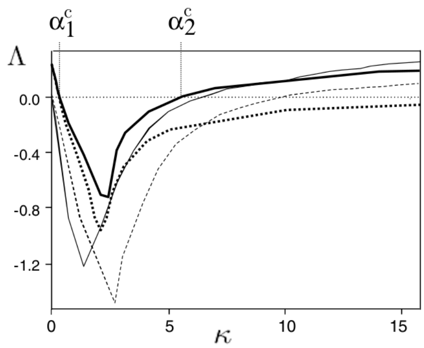

Figure 3: Four master stability functions for coupled Rössler oscillators with y-coupling (dashed) and x-coupling (solid lines) [18].

5 MEAN-FIELD APPROACH

Heterogeneous networks are charactierized by a large amount of low-degree nodes and a few *hubs* with many connections. An example of such a network is the Barabási-Albert model [19, 20]: a preferential attachment model that generates scale-free graphs with a power-law decay of the degree probability distribution ($P(k) \propto k^{-\gamma}$). The mean field reduction gives simplified equations for the dynamics of the hubs on large heterogeneous networks, as done in [21] and [22]. It can be done by replacing the coupling term of the i-th hub with an average $g(x_i) \approx \frac{\alpha}{\Delta}\sum_j^N A_{ij}[\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)]$.

6 OVERVIEW OF THE APPLICATIONS

A wide range of natural phenomena can be modeled as coupled dynamical systems that are often chaotic. In many fields, complex problems have been addressed within the paradigm of chaos synchronization: epidemiology and population dynamics, neural networks and electrical diffusion in heart cells, arrays of Josephson junctions and lasers, to name but a few. Below are the key ideas of some of these applications.

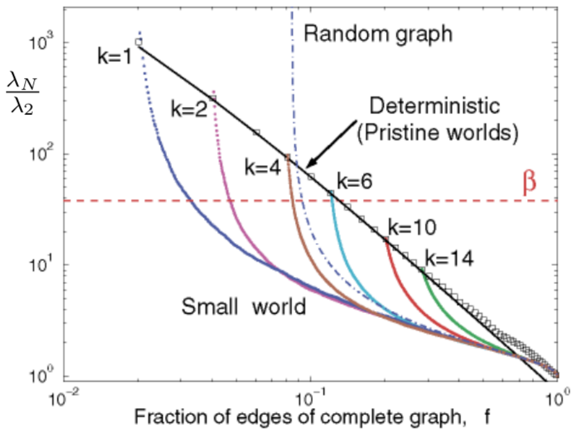

Figure 4: Eigenratio vs. fraction of edges of complete graph. Below the dashed line the network become synchronizable. Note how randomly adding new edges, increases the synchronizability of the regular graphs [18].

Secure Communications

Chaotic synchronization has applications to secure communication: the idea is to overlap a chaotic signal generated from a particular system to the signal to be encrypted. Under low amplitude of noise, the synchronization still occurs so that, if the parameters of the chaotic system are known, the encrypted signal can be obtained by subtracting the synchronized signal [4, 5].

Arrays of Coupled Lasers

Laser arrays can be studied as a system of many coupled limit-cycle oscillators. Their synchronization results in a much larger power expressed [6, 7, 23].

Pathological Diseases

Patients with Parkinsin's desease have shown an excessive synchronization in the neuronal activity. The suppression of such synchronization might be a therapeutic strategy to treat this disease [10, 12].

Arrays of Josephson junctions

A Josephson junction is a device made of two superconductor metals separated by a thin insulating barrier. The dynamic of a single Josephson junction is well-known and depends on the I-V characteristic. An array of Josephson junctions can be modeled as a Kuramoto model and treated analytically [8, 9, 24].

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