STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 8

LAWS OF PROBABILITY

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ADDITION LAW OF PROBABILITY

If A and B are any events then

$$\boxed{P(A\cup B)=P(A)+P(B)-P(A\cap B)}$$

If A, B and C are any events then

$$\boxed{P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)}$$

NOTE THAT

1. If A and B are mutually exclusive events then

$$P(A\cup B)=P(A)+P(B)$$

2. In probability (or, all) means union ie $\cup$.
3. In probability (and, both) means intersection ie $\cap$.
4. To show that events A and B are mutually exclusive prove that $P(A\cap B)=0$
5. To show that events A and B are collectively exhaustive prove that $P(A\cup B)=1$

COMBINED EVENTS

1. $P(A^c\cup B)=P(A^c)+P(B)-P(A^c\cap B)$
2. $P(A\cup B^c)=P(A)+P(B^c)-P(A\cap B^c)$
3. $P(A^c\cup B^c)=P(A^c)+P(B^c)-P(A^c\cap B^c)$
   or
   $P(A^c\cup B^c)=P(A\cap B)^c=1-P(A\cap B)$
4. $P(A\cap B^c)=P(A)-P(A\cap B)$
5. $P(A^c\cap B)=P(B)-P(A\cap B)$

EXAMPLES OF ADDITION LAW OF PROBABILITY

EXAMPLE-1 A card is drawn from a deck of 52 playing cards. What is the probability that selected card is an ace or a diamond card.

SOLUTION $Here\ n(S) = 52$

$$Let\ A\ denote\ the\ event\ that\ selected\ card\ is\ an\ Ace.$$
$$Let\ B\ denote\ the\ event\ that\ selected\ card\ is\ a\ diamond\ card.$$
$$we\ have\ to\ find\ P(A \cup B).$$

$$Using\ Addition\ Law\ of\ Probability$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

EXAMPLE-2 A die is rolled once. What is the probability that number appeared is an even number or a prime number.

SOLUTION $Here\ S = \{1, 2, 3, 4, 5, 6\}\ so\ n(S) = 52$

$$Let\ A\ denote\ the\ event\ that\ number\ appeared\ is\ an\ even\ number.\ So\ A = \{2, 4, 6\}$$
$$Let\ B\ denote\ the\ event\ that\ number\ appeared\ is\ a\ prime\ number.\ So\ B = \{2, 3, 5\}$$
$$Also\ P(A \cap B) = \{2\}$$
$$we\ have\ to\ find\ P(A \cup B).$$

$$Using\ Addition\ Law\ of\ Probability$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6}$$

EXAMPLE-3 Two coins are tossed once. What is the probability that coins show at least one head or at most one tail.

SOLUTION $Here\ S = \{HH, HT, TH, TT\}\ so\ n(S) = 4$

$$Let\ A\ denote\ the\ event\ that\ coin\ shows\ at\ least\ one\ head.\ So\ A = \{HH, HT, HT\}$$
$$Let\ B\ denote\ the\ event\ that\ coin\ shows\ at\ most\ one\ tail.\ So\ B = \{HH, HT, TH\}$$
$$Also\ P(A \cap B) = \{HH, HT, TH\}$$
$$we\ have\ to\ find\ P(A \cup B).$$

$$Using\ Addition\ Law\ of\ Probability$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{4} + \frac{3}{4} - \frac{3}{4} = \frac{3}{4}$$

EXAMPLE-4 Two dice are rolled once. Find the probability that sum of dots appeared is an even number or factor of 30.

SOLUTION *Here sample space is given by*

$$S = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

$So\ n(S) = 52$

Let A denote the event that sum of dots appeared is an even number.

$$\Rightarrow A = \begin{Bmatrix} (1,1),(1,3),(1,5),(2,2),(2,4),(2,6),\ (3,1),(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),((5,3),(5,5), \\ (6,2),(6,4),(6,6) \end{Bmatrix}$$

Let B denote the event that sum of dots appeared is a factor of 30.

$$\Rightarrow B = \{(1,1),(1,2),(1,4),(1,5),(2,1),(2,3),\ (2,4),(3,2),(3,3),(4,1),(4,2),(4,6),(5,5),(6,4)\}$$

$$So\ \ P(A \cap B) = \{(1,1),\ (1,5),\ (2,4),\ (3,3),\ (4,2),(4,6),(5,5),(6,4)\}$$

we have to find $P(A \cup B)$.

Using Addition Law of Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{18}{36} + \frac{14}{36} - \frac{8}{36} = \frac{24}{36} = \frac{2}{3}$$

EXAMPLE-5 There are twenty girls and 10 boys in a class. Half of the boys and half of the girls have blue eyes. A student is selected at random for class representative (CR). What is the probability that selected is a girl or has blue eyes.

SOLUTION *Here* $Boys = 10,\ Girls = 20,\ Total = 30,\ so\ n(S) = 30$

Also for students having blue eyes, $Boys = 5,\ Girls = 10$

Let A denote the event that selected student is a girl.

Let B denote the event that selected student has blue eyes.

we have to find $P(A \cup B)$.

Using Addition Law of Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{20}{30} + \frac{15}{30} - \frac{10}{30} = \frac{25}{30} = \frac{5}{6}$$

MULTIPLICATION LAW OF PROBABILITY

INDEPENDENT AND DEPENDENT EVENTS

INDEPENDENT EVENTS Two events A and Bare said to be independent, if the occurrence of one event does not effect the occurrence of the other event.
For example if we draw balls one by one with replacement from a bag containing balls of different colors then the events are said to be independent.

DEPENDENT EVENTS Two events A and Bare said to be dependent, if the occurrence of one event affects the occurrence of the other event.
For example if we draw balls one by one without replacement from a bag containing balls of different colors then the events are said to be dependent.

MULTIPLICATION LAW OF PROBABILITY FOR INDEPENDENT EVENTS

If A and B are independent events then

$$\boxed{P(A\cap B)=P(A).P(B)}$$

Similarly If A, B and C are independent events then

$$\boxed{P(A\cap B\cap C)=P(A).P(B).P(C)}$$

NOTE THAT

1. If we have to show that events A and B are independent then show that

$$P(A\cap B)=P(A).P(B).$$

2. If A and B are independent events then A and B^c will be also independent.

3. If A and B are independent events then A^c and B will be also independent

4. If A and B are independent events then A^c and B^c will be also independent

5. If A and B are independent events then

   (a) $P(A^c\cap B^c)=P(A^c).P(B^c)$
   (b) $P(A\cap B^c)=P(A).P(B^c)$
   (c) $P(A^c\cap B)=P(A^c).P(B)$

EXAMPLES OF MULTIPLICATION LAW OF PROBABILITY

EXAMPLE-1 If A and B are events two events such that
$P(A)=1/2,\ P(B)=2/3\ \ and\ \ P(A\cap B)=1/3$

(a) *Check whether events A and B are independent ?*

(b) *Check whether events A^c and B are independent ?*

(c) *Check whether events A^c and B^c are independent ?*

SOLUTION *Here* $P(A)=1/2,\ P(B)=2/3\ \ and\ \ P(A\cap B)=1/3$

So $P(A^c)=1-P(A)=1-1/2=1/2\ \ and\ \ P(B^c)=1-P(B)=1-2/3=1/3$

(a) *Events A and B will be independent if* $P(A\cap B)=P(A).\,P(B)$

$$Now\ \frac{1}{3}=\frac{1}{2}\times\frac{2}{3}$$
$$\frac{1}{3}=\frac{1}{3}$$

Since Multiplication Law of independent events is satisfied.
Hence A and B are indenedent events.

(b) *Events A^c and B will be independent if* $P(A^c\cap B)=P(A^c).\,P(B)$

$L.H.S=P(A^c\cap B)=P(B)-P(A\cap B)=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$

$R.H.S=P(A^c).P(B)=\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$

Since Multiplication Law of independent events is satisfied.
Hence A^c and B are indenedent events.

(c) *Events A^c and B^c will be independent if* $P(A^c\cap B^c)=P(A^c).\,P(B^c)$

$$\begin{aligned}L.H.S&=P(A^c\cap B^c)\\&=P(A\cup B)^c\\&=1-P(A\cup B)\\&=1-[P(A)+P(B)-P(A\cap B)]\\&=1-[\frac{1}{2}+\frac{2}{3}-\frac{1}{3}]=1-\frac{5}{6}=\frac{1}{6}\end{aligned}$$

$R.H.S=P(A^c).P(B^c)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

So $\quad L.H.S=R.H.S$

Since Multiplication Law of independent events is satisfied.
Hence A^c and B are indenedent events.

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EXAMPLE-2 Determine the probability of getting two heads in two successive tosses of a balanced coin.

SOLUTION $Here\ S=\{H,T\}\ so\ n(S)=2$

$Let\ A\ denote\ the\ event\ that\ head\ appear\ on\ the\ first\ toss.$

$Let\ B\ denote\ the\ event\ that\ head\ appear\ on\ the\ second\ toss.$

$we\ have\ to\ find\ P(A\cap B).$

$Since\ the\ two\ events\ are\ independent.$

$So\ using\ Multiplication\ Law of\ Probability$

$$P(A\cap B)=P(A).P(B)$$

$$=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$$

EXAMPLE-3 The Probability that a man will be alive in 25 years is $3/5$ and Probability that his wife will be alive in 25 years is $2/3$. Find the probabilities that

$(a)\ both\ will\ be\ alive$

$(b)\ only\ husband\ will\ be\ alive$

$(c)\ at\ least\ one\ of\ them\ will\ be\ alive$

$(d)\ at\ most\ one\ of\ them\ will\ be\ alive$

$(e)\ neither\ will\ be\ alive$

SOLUTION $Here\ S=\{HH,HT,TH,TT\}\ so\ n(S)=4$

$Let\ H\ denote\ the\ event\ that\ man\ will\ be\ alive\ in\ 25\ years.$

$Let\ W\ denote\ the\ event\ that\ his\ wife\ will\ be\ alive\ in\ 25\ years.$

$So\ P(H)=3/5\ \ and\ \ P(H^c)=1-3/5=2/5$

$\&\ P(W)=2/3\ \ and\ \ P(W^c)=1-2/3=1/3$

$Since\ the\ two\ events\ are\ independent.$

$So\ using\ Multiplication\ Law\ of\ Probability$

$(a)\ P(both\ will\ be\ alive)$

$$=P(H\cap W)=P(H).P(W)=3/5\times 2/3=2/5$$

$(b)\ P(only\ husband\ will\ be\ alive)$

$$=P(H\cap W^c)=P(H).P(W^c)=3/5\times 1/3=1/5$$

$(c) P(\ at\ least\ one\ of\ them\ will\ be\ alive)$

$$=P(H\cap W^c)+P(H^c\cap W)+P(H\cap W)=(3/5\times 1/3)+(2/5\times 2/3)+(3/5\times 2/3)=13/15$$

$(d) P(at\ most\ one\ of\ them\ will\ be\ alive)$

$$=P(H\cap W^c)+P(H^c\cap W)+P(H^c\cap W^c)=(3/5\times 1/3)+(2/5\times 2/3)+(2/5\times 1/3)=3/5$$

$(e)\ P(neither\ will\ be\ alive)$

$$=P(H^c\cap W^c)=2/5\times 1/3=2/15$$

EXAMPLE-4 Three shooters **A, B** and **C** hit a target**. A** can hit a target 4 times in 5 shots, **B** can hit a target 3 times in 4 shots, **C** can hit twice in 3 shots. They fire a target. What is the probability that

(a) *all shots hit* (b) *no shot hit* (c) *exactly one shot hit*

(d) *at least one shot hit* (e) *at most one shot hit*

SOLUTION $Here\ P(A)=4/5,\ P(B)=3/4\ \ and\ \ P(C)=2/3$

$So\ \ P(A^c)=1-4/5=1/5\ ,\ P(B^c)=1-3/4=1/4\ \ and\ P(C^c)=1-2/3=1/3$

$(a)\ P(all\ shots\,hit)$

$=\ P(A\cap B\cap C)=P(A).P(B).P(C)=4/5\times 3/4\times 2/3=2/5$

$(b)\ P(no\ shot\ hit)=P(A^c\cap B^c\cap C^c)=P(A^c).P(B^c).P(C^c)=1/5\times 1/4\times 1/3=1/60$

$(c)P(exactly\ one\ shot\ hit)$

$=P(A\cap B^c\cap C^c)+P(A^c\cap B\cap C^c)+P(A^c\cap B^c\cap C)$

$=P(A).P(B^c).P(C^c)+P(A^c).P(B).P(C^c)+P(A^c).P(B^c).P(C)$

$=(4/5\times 1/4\times 1/3)+(1/5\times 3/4\times 1/3)+(1/5\times 1/4\times 2/3)=9/60=3/20$

$(d)\ P(at\ least\,one\ shot\ hit)$

$=1-P(no\ shot\,hit)=1-1/60=59/60$

$(e)\ P(at\ most\,one\ shot\ hit)$

$=P(A\cap B^c\cap C^c)+P(A^c\cap B\cap C^c)+P(A^c\cap B^c\cap C)+P(A^c\cap B^c\cap C^c)$

$=P(A).P(B^c).P(C^c)+P(A^c).P(B).P(C^c)+P(A^c).P(B^c).P(C)+P(A^c).P(B^c).P(C^c)$

$=(4/5\times 1/4\times 1/3)+(1/5\times 3/4\times 1/3)+(1/5\times 1/4\times 2/3)+(1/5\times 1/4\times 1/3)=10/60=1/6$

EXAMPLE-5 Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1 girl and 3 boys. One child is selected from each group. Find the probabilities that the three selected children consists of

(a) *all boys*

(b) 1 *girl and* 2 *boys*

(c) *at least one girl*

SOLUTION $Here\ we\ have\ three\ families$

$Family-1\ contains\ 3\ girls\,and\ 1\ boy$

$Family-2\ contains\ 2\ girls\ and\ 2\ boys$

$Family-3\ contains\ 1\ girl\ and\ 3\ boys$

$(a)\ P(all\ boys)=P(B_1\cap B_2\cap B_3)=P(B_1).P(B_2).P(B_3)=1/4\times 2/4\times 3/4=6/64=3/32$

$(b)\ P(1\ girl\ and\ 2\,boys)=P(G_1\cap B_2\cap B_3)+P(B_1\cap G_2\cap B_3)+P(B_1\cap B_2\cap G_3)$

$=(3/4\times 2/4\times 3/4)+(1/4\times 2/4\times 3/4)+(1/4\times 2/4\times 1/4)=26/64=13/32$

$(c)\ P(at\ least\ one\ girl)=1-P(No\ Girl)=1-3/32=29/32$