Assignment No: 15

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Linear Algebra

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Question No:-1

$$Q(x_1,x_2,x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$$

Solution:-

(a)

Express in Matrix form

$$x^TAx = \begin{bmatrix} x_1, x_2, x_3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b)

Eliminate the cross products

Find eign values:

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 4-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = 0$$

$$= (3-\lambda)\left[(4-\lambda)(5-\lambda) - 4\right] - 2\left[2(5-\lambda)\right] = 0$$

$$(3-\lambda)(20 - 9\lambda + \lambda^2 - 4) - 4(5-\lambda) = 0$$

$$48 - 27\lambda + 3\lambda^2 - 16\lambda + 9\lambda^2 - \lambda^3 - 20 + 4\lambda = 0$$

$$-\lambda^3 + 12\lambda^2 - 39\lambda + 28 = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = 7 \qquad \lambda_3 = 4$$

$Q(x_1, x_2, x_3)$ in terms of new variables

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1$$

$$y_1^2 + 7y_2^2 + 4y_3^2 = 1$$

$$y_1^2 + \frac{y_2^2}{1/7} + \frac{y_3^2}{1/4} = 1$$

(c)

Identify the Quadric Surf.

$y_1^2 + \frac{y_2^2}{1/7} + \frac{y_3^2}{1/4} = 1$ is an equation of ellipsoid.

(d)

Classify the Matrix

As all eign values are +ve therefore this Matrix is positive definite.

(e)

Find the Max, Min

Maximum value of $Q(x) = 7$

Eign vector corresponding to $\lambda = 7$

$$\sim \left(\begin{array}{ccc|c} -4 & 2 & 0 & 0 \\ 2 & -3 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 2 & -3 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \quad \frac{R_1}{2}$$

$$\sim \left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \quad R_2 + R_1$$

$$\sim \left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_3 - R_2$$

$-2x_2 - 2x_3 = 0$

$x_2 = -x_3$

$-2x_1 + x_2 = 0$

$-2x_1 - x_3 = 0$

$x_1 = -\frac{x_3}{2}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3/2 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}$$

normalize $\begin{pmatrix} -1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

Max value will occures when

$x_1 = \frac{-1}{3} \quad x_2 = -\frac{2}{3} \quad x_3 = \frac{2}{3}$ in $Q(x_1)x_1x_2x_3$

Minimum value of $Q(x_1, x_2, x_3)$ is $= 1$

Eign vector corresponding to $\lambda = 1$

$$\sim \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{bmatrix} \quad R_2 - R_1$$

$$\sim \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 + 2R_1$$

$x_2 - 2x_3 = 0$

$x_2 = 2x_3$

$2x_1 + 2x_2 = 0$

$2x_1 + 4x_3 = 0$

$x_1 = -2x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Normalize $= \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

Minimum value will occures when

$x_1 = \frac{-2}{3}$ $x_2 = \frac{2}{3}$ $x_3 = \frac{1}{3}$ in

$Q(x_1, x_2, x_3)$

(f)

Explain the role of ... definitness ..... ensuring the existence of unique Minimum / Maximum

As all eign values of Associated Matrix of $Q(x_1, x_2, x_3)$ are positive therefore $Q(x_1, x_2, x_3)$ is positive definite.

This implies that $Q(x_1, x_2, x_3)$ is strictly convex function. A strictly convex function has a unique global minimum.