Assignment No 10

Muhammad Fahid Imran

Roll No: 23i-0061

Subject: Linear Algebra

Instructor: Dr. Sehrish Hassan

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Fast NUCES Islamabad Campus

Question No:- 1

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

Solution:-

⊙ Eigevalues

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0$$

$$-\lambda\left(-\lambda(4-\lambda)+5\right) - 1(-2) + 0 = 0$$

$$\lambda^2(4-\lambda) - 5\lambda + 2 = 0$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0 \quad \text{—— (i)}$$

⊙ check $\lambda = 1$ is factor or not:

$$1 - 4 + 5 - 2 = 0$$

$$0 = 0$$

⊙ it mean $\lambda = 1$ is on factor of equation (i)

⊙ Use Synthetic division

$$\begin{array}{c|cccc} & 1 & -4 & 5 & -2 \\ 1 & & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \big| \; 0 \end{array}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda^2 - 2\lambda - \lambda + 2 = 0$

$\lambda(\lambda - 2) - 1(\lambda - 2) = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda = 1 \qquad \lambda = 2$

$\lambda = 1 \qquad \lambda = 1 \qquad \lambda = 2$ ← eignvalues

$\lambda_1 = 1 \qquad \lambda_2 = 2$

⊙ Eignvectors

For $\lambda_1 = 1$

$A - \lambda I = \bar{0}$

$$= \left(\begin{array}{ccc|c} 0-1 & 1 & 0 & 0 \\ 0 & 0-1 & 1 & 0 \\ 2 & -5 & 4-1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right) \quad R_3 + 2R_1$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_3 - 3R_2$$

$-V_2 + V_3 = 0 \qquad -V_1 + V_2 = 0 \qquad V_3 = V_3$

$V_2 = V_3 \qquad -V_1 + V_3 = 0$

$V_1 = V_3$

$$\bar{V}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_3 \\ v_3 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

eignvector coorresponding to $\lambda_1 = 1$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 2$

$$= \left[\begin{array}{ccc|c} 0-2 & 1 & 0 & 0 \\ 0 & 0-2 & 1 & 0 \\ 2 & -5 & 4-2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & -5 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right] \quad R_3 + R_1$$

$$\sim \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 - 2R_2$$

$-2v_2 + v_3 = 0$

$2v_2 = v_3$

$v_2 = \frac{v_3}{2}$

$-2v_1 + v_2 = 0$

$-2v_1 + \frac{v_3}{2} = 0$

$v_1 = \frac{v_3}{4}$

$v_3 = v_3$

$$\bar{V}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_3/4 \\ v_3/2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix}$$

⊙ Algebraic Multiplicity

Algebraic Multiplicity of $\lambda_1 = 2$
Algebraic Multiplicity of $\lambda_2 = 1$

⊙ Geometric Multiplicity

Geometric Multiplicity of $\lambda_1 = 1$
Geometric Multiplicity of $\lambda_2 = 1$

⊙ Discuss Diagonalization

As Geometric Multiplicity of λ_1 is less than Algebraic Multiplicity of λ_1 therefore A is not diagonalizable Matrix.

Question No:- 2

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Solution:-

⊙ Eignvalues

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)[-\lambda(-1-\lambda)] + 1(-(-\lambda)) = 0$$

$$(-1-\lambda)(\lambda+\lambda^2)+\lambda = 0$$

$$\cancel{-\lambda} - \lambda^2 - \lambda^2 - \lambda^3 + \cancel{\lambda} = 0$$

$$\lambda^3 + 2\lambda^2 = 0$$

$$\lambda^2(\lambda+2) = 0$$

$\lambda^2 = 0$ $\qquad\qquad$ $\lambda + 2 = 0$

$\lambda = 0 \quad \lambda = 0$ $\qquad\qquad$ $\lambda = -2$

$\lambda_1 = 0 \qquad \lambda_2 = -2$

⊙ **Eignvectors**

For $\lambda_1 = 0$

$A - \lambda_1 I = \bar{0}$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 + 3R_1 \\ R_3 + R_1 \end{array}$$

$-v_1 + v_3 = 0$

$v_1 = v_3 \qquad v_2 = v_2 \qquad v_3 = v_3$

$$= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

eign vectors coorresponding to $\lambda_1 = 0$ are $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\bar{V_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \bar{V_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

for $\lambda_2 = -2$

$A + 2I = \bar{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$2v_2 - 6v_3 = 0 \qquad v_1 + v_3 = 0 \qquad v_3 = v_3$

$v_2 = 3v_3 \qquad v_1 = -v_3$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_3 \\ 3v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

eign vectors coorresponding to $\lambda_2 = -2$ is

$\bar{V_3} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

⊙ Algebraic Multiplicity

A.M of $\lambda_1 = 2$
A.M of $\lambda_2 = 1$

⊙ Geometric Multiplicity

G.M of $\lambda_1 = 2$
G.M of $\lambda_2 = 1$

⊙ Discuss Diagonalization

As G.M and A.M of each eignvalue is equal therefore A is diagonaliza

(b) $A^{10}\begin{pmatrix}5\\1\\2\end{pmatrix}$

write $\begin{pmatrix}5\\1\\2\end{pmatrix}$ as linear combination of $\overline{v_1}, \overline{v_2}, \overline{v_3}$ eign vectors.

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 5 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 1 & 1 & 2 \end{array}\right] \quad \text{swap } R_1, R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 2 & -3 \end{array}\right] \quad R_3 - R_2$$

$2x_3 = -3$ $\qquad$ $x_2 - x_3 = 5$ $\qquad$ $x_1 + 3x_3 = 1$

$x_3 = \frac{-3}{2}$ $\qquad$ $x_2 + \frac{3}{2} = 5$ $\qquad$ $x_1 - \frac{9}{2} = 1$

$x_2 = \frac{7}{2}$ $\qquad$ $x_1 = 1 + \frac{9}{2} = \frac{11}{2}$

$x_1 = \frac{11}{2}$

$$\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \frac{11}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{7}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$A^{10}\bar{V}_1 = \lambda_1^{10}\bar{V}_1 = (0)^{10}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A^{10}\bar{V}_2 = \lambda_2^{10}\bar{V}_2 = (0)^{10}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A^{10}\bar{V}_3 = \lambda_3^{10}\bar{V}_3 = (-2)^{10}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1024 \\ 3072 \\ 1024 \end{pmatrix}$$

$$A^{10}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \frac{11}{2}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \frac{7}{2}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -1024 \\ 3072 \\ 1024 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1536 \\ -4608 \\ -1536 \end{pmatrix}$$

$$A^{10}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1536 \\ -4608 \\ -1536 \end{pmatrix}$$

(c) Diagonalize Matrix A

$$A = PDP^{-1}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{\begin{bmatrix} -3 & -2 & 3 \\ -1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}}{-2} = \begin{bmatrix} 3/2 & 1 & -3/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 3/2 & 1 & -3/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

(d) $A^{10} = PD^{10}P^{-1}$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1024 \end{bmatrix}\begin{bmatrix} 3/2 & 1 & -3/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1024 \\ 0 & 0 & 3072 \\ 0 & 0 & 1024 \end{bmatrix}\begin{bmatrix} 3/2 & 1 & -3/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 512 & 0 & -512 \\ -1536 & 0 & 1536 \\ -512 & 0 & 512 \end{bmatrix}$$