STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 6

INTRODUCTION TO
PROBABILITY

PREPARED BY
HAZBER SAMSON
FAST NUCES ISLAMABAD

INTRODUCTION TO SET THEORY

DEFINITION *The collection of well-defined and distinct objects is called a set. Sets are usually denoted by capital letters while elements of the set are denoted by small letters.*

EXAMPLES $1-A=\{1,2,3,4,5\}$ $2-B-\{a,e,i,o,u\}$

SOME IMPORTANT SETS

1. Set of Natural Numbers

$$N=\{1,2,3,\ldots\ldots\}$$

2. Set of Whole Numbers

$$W=\{0,1,2,3,\ldots\ldots\}$$

3. Set of Integers

$$Z=\{0,\pm 1,\pm 2,\pm 3,\ldots\ldots\}$$

4. Set of Prime Numbers

$$P=\{2,3,5,7,\ldots\ldots\}$$

5. Set of Even Numbers

$$E=\{0,\pm 2,\pm 4,\pm 6\ldots\ldots\}$$

6. Set of Odd Numbers

$$O=\{\pm 1,\pm 3,\pm 5\ldots\ldots\}$$

7. Set of Rational Numbers

$$Q=\left\{x\middle|x\in Q\ \Lambda\ \ x=\frac{p}{q},\ q\neq 0\right\}$$

8. Set of Irrational Numbers

$$Q^c=Non-Ter\min ating\ \ and\ \ Non-\mathrm{Re}curring\ \ Decimals$$

9. Set of Real Numbers

$$R=Q\cup Q^c$$

10. Set of Complex Numbers

$$C=\{A\ Number\ of\ \ the\ \ form\ a+bi\}$$

TYPES OF SETS

EMPTY SET A set which contains no element is called empty set. It is denoted by φ or { }.

SUBSET If A and B are two sets and if all the elements of A are the elements of B then set A is called subset of set B. It is denoted by $A \subseteq B$.

EXAMPLES 1- If A = {1,2} and B = {1,2,3} then $A \subseteq B$.
2- If A = {a,b} and B = {a,b} then $A \subseteq B$.

NOTE Every set is a subset of its self ie $A \subseteq A$
Empty set is a subset of every set ie $\varphi \subseteq A$.

There are two types of subsets **1**- Proper Subset **2**-Improper Subset

PROPER SUBSET If A and B are two sets than set A is called proper subset of set B if $A \subseteq B$ and there is at least one element in B which is not in A. It is denoted by $A \subset B$.

EXAMPLES 1- If A = {1,2} and B = {1,2,3} then $A \subset B$.
2- If A = {A} and B = {a,b} then $A \subset B$.

IMPROPER SUBSET If A and B are two sets than set A is called an improper subset of set B if $A \subseteq B$ and there is no element in B which is not in A. It is denoted by $A \subseteq B$.

EXAMPLES 1- If A = {1,2} and B = {1,2} than $A \subseteq B$.
2- If A = {a} and B = {a} then $A \subseteq B$.

UNIVERSAL SET A set which is superset of all the sets under consideration is called universal set. It is usually denoted by **U**.

POWER SET The power set of a set S is the set which contains all the possible subsets of S. It is denoted by P(A).

EXAMPLES 1- If A = {a,b} than P(A) = { Φ, {a},{b},{a,b} }
2- If B = { } then P(B) = { Φ }

NOTE If set S has n elements than P(S) has 2^n elements.

OPERATIONS ON SETS

UNION OF SETS

If A and B are two sets then their union is denoted by A U B and it contains all the elements of A and B.

EXAMPLES 1- If A = {1,2,3} and B = {2,3,4} than A U B = {1,2,3,4}
2- If A = {a} and B = {b} than A U B = {a,b}

NOTE $1-A\cup A=A,\ 2-A\cup\phi=A,\ 3-A\cup U=U,\ 4-\phi\cup\phi=\phi,\ 5-A\cup B=B\cup A$

INTERSECTION OF SETS

If A and B are two sets than their intersection is denoted by A ∩ B and it contains the common elements of A and B.

EXAMPLES 1- If A = {1,2,3} and B = {2,3,4} than A ∩ B = {2,3}
2- If A = {a} and B= {b} than A ∩ B = { }

NOTE $1-A\cap A=A,\ 2-A\cap\phi=\phi,\ 3-A\cap U=A,\ 4-\phi\cap\phi=\phi,\ 5-A\cap B=B\cap A$

DIFFERENCE OF SETS

If A and B are two sets than their difference is denoted by A – B and it contains elements of A which are not in B.

EXAMPLES 1- If A = {1,2,3} and B = {2,3,4} than A-B = {1} and B-A = {4}
2- If A = {a} and B= {b} than A-B = { } and B-A = { }.

NOTE $1-A-A=\phi,\ 2-A-\phi=A,\ 3-U-A=A^c,\ 4-\phi-\phi=\phi,\ 5-A-B\neq B-A$

COMPLEMENT OF A SET If A is a subset of U than its complement is denoted by A^c or A' and is given by $A'=U-A$

EXAMPLES 1- If U = {1,2,3,4,5} and A = {1,2,3} than A^c= U-A = {4,5}
2- If U = {a,b} and B = {a} than B^c U-B {a}

NOTE $1-A^c=U-A,\ 2-U^c=\varphi,\ 3-\phi^c=U$

LAWS OF SETS

If A, B and C are subsets of a Universal set U then the following laws hold

DEMORGAN LAWS

$$1-(A\cup B)^c=A^c\cap B^c$$
$$2-(A\cap B)^c=A^c\cup B^c$$

DISTRIBUTIVE LAWS

$$1-A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$
$$2-A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$$

ASSOCIATIVE LAWS

$$1-(A\cup B)\cup C=A\cup(B\cup C)$$
$$2-(A\cap B)\cap C=A\cap(B\cap C)$$

BASIC CONCEPTS OF PROBABILITY

EXPERIMENT A well-defined collection or process of obtaining an observation is called an experiment. The performance of an experiment is called trial and the result obtained is called an outcome.

RANDOM EXPERIMENT An experiment which produces different outcomes although it is repeated under similar conditions for a large number of time is called a random experiment. A random experiment has following properties.

1. The experiment can be repeated by any number of times.
2. A random trial consists of at least two possible outcomes.

EXAMPLES

1-Toss a coin,
2-Throw a die etc.

SAMPLE SPACE The set of all possible outcomes of a random experiment is called sample space. It is denoted by S. Each element in a sample space is called sample point.

EXAMPLES

1. When a coin is tossed once
$$S = \{H, T\}$$
2. When a coin is tossed twice or two coins are tossed once
$$S = \{HH, HT, TH, TT\}$$
3. When a coin is tossed thrice or three coins are tossed once
$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$
4. When a die is thrown once
$$S = \{1,2,3,4,5,6\}$$
5. When a die is thrown twice or two dice are thrown once
$$S = \left\{\begin{matrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{matrix}\right\}$$

EVENT AND TYPES OF EVENTS

EVENT A subset of the sample space is called an event. For example

1. When a coin is tossed once then the sets $A = \{H\}$ or $B = \{T\}$ are events.
2. When a die is rolled once then the sets $A = \{2\}$, $B = \{3,4\}$, $C = \{1,3,5\}$ are events.

TYPES OF EVENTS

SIMPLE EVENT An event that contains exactly one sample point is called simple event. It is also called elementary event. For example

1. When two coins are tossed once then the event $A = \{HH\}$ is a simple event.
2. When a die is rolled once then the event $B = \{3\}$ is a simple event.

COMPOUND EVENT An event that contains more then one sample point is called compound event. It is also called composite event. For example

1. When two coins are tossed once then the event $A = \{HH.HT\}$ is a compound event.
2. When a die is rolled once then the event $B = \{2,4,6\}$ is compound event.

IMPOSSIBLE EVENT An event that contains no sample point is called impossible event. It is also called null event. For example

$A = \{\ \}$ is impossible event for every sample space.

SURE EVENT An event that contains all the sample points is called sure event. It is also called certain event. For example

1. When a coin is tossed once then the event $A = \{H,T\}$ is sure event.
2. When a die is rolled once then the event $B = \{1,2,3,4,5,6\}$ is sure event.

MUTUALLY EXCLUSIVE EVENTS Two events A and B are said to be mutually exclusive if they cannot occur at the same time ie $A \cap B = \{\ \}$. For example

1. When a coin is tossed once then the events $A = \{H\}$ and $B = \{T\}$ are mutually exclusive events.
2. When a die is rolled once then the events $A = \{1,3,5\}$ and $B = \{2,4.6\}$ are mutually exclusive events.

COLLECTIVELY EXHAUSTIVE EVENTS Two events A and B are said to be collectively exhaustive events if their union is sample space itself ie $A \cup B = S$. For example

1. When a coin is tossed once then the events $A = \{H\}$ and $B = \{T\}$ are collectively exhaustive events.
2. When a die is rolled once then the events $A = \{1,3,5\}$ and $B = \{2,4.6\}$ are collectively exhaustive events.

EQUALLY LIKELY EVENTS Two events A and b are said to be equally likely, if number of elements in A and B are same. $ie \;\; n(A) = n(B)$. For example

1. When a coin is tossed once then the events $A = \{H\}$ and $B = \{T\}$ are equally likely events.
2. When a die is rolled once then the events $A = \{1,3,5\}$ and $B = \{2,4.6\}$ are equally likely events.

INFORMATION ABOUT CARDS

Total number of cards = 52

<u>COLOURS</u> = 2 (Red and Black)

<u>SETS (SUITS)</u>

There are four sets of Cards (13 in each type)

1. Clubs ♣
2. Hearts ♥
3. Spades ♠
4. Diamonds ♦

<u>ORDER OF CARDS</u>

Ace ***Queen***

A 2 3 4 5 6 7 8 9 10 J Q K

jack ***King***

<u>TYPES</u>

Pictured Cards = 12, Aces = 4

<u>HANDS</u>

Bridge Hand = 13
Poker Hand = 5

PROBABILITY

INTRODUCTION In Real World we are concerned mainly with Natural or Social Sciences but few results in the natural or social sciences are known absolutely. Most are reported in terms of chances or probabilities: the chance of rain tomorrow, the chance of your getting home from school or work safely, the chance of your living past 60 years of age, the chance of contracting (or recovering from) a certain disease, the chance of inheriting a certain trait, the chance of your annual income exceeding $60,000 in two years, the chance of winning an election. *Probability tells us how to measure this chance.* Today's adults must obtain some knowledge of probability and must be able to tie probabilistic concepts to real scientific investigations if they are to understand science and the world around them. Probability provide a strong basis for students who may go on to deeper studies of statistics, mathematics, engineering, business, or the physical and biological sciences; at the same time, it should provide a basis for practical decision making in the face of uncertainty.

CLASSICAL DEFINITION If there are equally likely, mutually exclusive and collectively exhaustive outcomes and m of which are favorable to the occurrence of an event A then the probability of the occurrence of the event A, is denoted by P(A) and is given by

$$P(A)=\frac{no\,of\ \ favourable outcomes}{no\,of\ \ possible\,outcomes}$$

Or $$\boxed{P(A)=\frac{m}{n}}$$

MATHEMATICAL DEFINITION The probability that an event will occur, is the ratio of the number of sample points in A to the total no of sample points in S.

Mathematically $$P(A)=\frac{no\,of\ elements\,in\,A}{no\,of\ elements\,in\,S}$$

Or $$\boxed{P(A)=\frac{n(A)}{n(S)}}$$

AXIOMS OF PROBABILITY

1. Probability of an event cannot be negative $ie\ \ P(A)\geq 0$.
2. Probability of an event lies between 0 and 1 $ie\ \ 0\leq P(A)\leq 1$.
3. Probability of a null event is zero. $ie\ \ P(\phi)=0$
4. Probability of a sure event is 1. $ie\ \ P(S)=1$
5. If A is any event of a sample space S then $P(A^c)=1-P(A)$

EXAMPLES OF BASIC PROBABILITY

EXAMPLE-1 A box contains two white, three red and five blue balls. A ball is selected at random what is the probability that selected ball is a

(a) *white ball*

(b) *red ball*

(c) *not blue ball*

SOLUTION $Here\ White\ balls = 2,\ Red\ balls = 3,\ Blue\ balls = 5$

(a) *Let A denote the event that selected ball is a white ball*

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{10} = \frac{1}{5}$$

(b) *Let B denote the event that selected ball is a red ball*

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{10}$$

(c) *Let C denote the event that selected ball is a Blue ball*

$$P(C^c) = 1 - P(C) = 1 - \frac{n(C)}{n(S)} = 1 - \frac{5}{10} = \frac{5}{10} = \frac{1}{2}$$

EXAMPLE-2 A card is selected from a deck of 52 playing cards. What is the probability that selected card is

(a) *an ace*

(b) *a diamond card*

(c) *a queen of hearts*

(d) *an even numbered card*

(e) *a prime number card of clubs*

SOLUTION $Here\ \ n(S) = 52$

(a) *Let A denote the event that selected card is an ace*

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

(b) *Let B denote the event that selected card is a diamond card*

$$P(B) = \frac{n(B)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

(c) *Let C denote the event that selected card is a queen of hearts*

$$P(C) = \frac{n(C)}{n(S)} = \frac{1}{52}$$

(d) *Let D denote the event that selected card is an even numbered card*

$$P(D) = \frac{n(D)}{n(S)} = \frac{20}{52} = \frac{5}{13}$$

(e) *Let E denote the event that selected card is a prime numbered card of clubs*

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

EXAMPLE-3 A coin is tossed twice. What is the probability that

(a) *head appears*

(b) *exactly one head appears*

(c) *atleast one head appears*

(d) *atmost one head appears*

(e) *atmost two tails appears*

SOLUTION *Here* $S=\{HH, HT, TH, TT\}$

(a) *Let A denote the event that head appears* $\Rightarrow A=\{HH, HT, TH\}$

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{4}$$

(b) *Let B denote the event that exactly one head appears* $\Rightarrow B=\{HT, TH\}$

$$P(B)=\frac{n(B)}{n(S)}=\frac{2}{4}=\frac{1}{2}$$

(c) *Let C denote the event that atleast one head appears* $\Rightarrow C=\{HH, HT, TH\}$

$$P(C)=\frac{n(C)}{n(S)}=\frac{3}{4}$$

(d) *Let D denote the event that atmost one head appears* $\Rightarrow D=\{HT, TH, TT\}$

$$P(D)=\frac{n(D)}{n(S)}=\frac{3}{4}$$

(e) *Let E denote the event that atmost two tails appear* $\Rightarrow E=\{HH, HT, TH, TT\}$

$$P(E)=\frac{n(E)}{n(S)}=\frac{4}{4}=1$$

EXAMPLE-4 A die is rolled once. What is the probability that die shows

(a) *an evenn number*

(b) *an odd number*

(c) *a prime number*

(d) *a number greater than* 2

(e) *a number less than equals to* 2

SOLUTION *Here* $S=\{1, 2, 3, 4, 5, 6\}$

(a) *Let A denote the event that die shows an even number* $\Rightarrow A=\{2, 4, 6\}$

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{6}=\frac{1}{2}$$

(b) *Let B denote the event that die shows an odd number* $\Rightarrow B=\{1, 3, 5\}$

$$P(B)=\frac{n(B)}{n(S)}=\frac{3}{6}=\frac{1}{2}$$

(c) *Let C denote the event that die shows a prime number* $\Rightarrow C = \{2, 3, 5\}$

$$P(C) = \frac{n(C)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

(d) *Let D denote the event that die shows a number greater than* $2 \Rightarrow D = \{3, 4, 5, 6\}$

$$P(D) = \frac{n(D)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

(e) *Let C denote the event that die shows a number less than equals to* $2 \Rightarrow E = \{1, 2\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

EXAMPLE-5 Two dice are rolled once. What is the probability that

(a) *sum of dots appeared is a prime number*

(b) *difference of dots is an even numer*

(c) *product of dots is divisible by* 5

SOLUTION *Here sample space is given by*

$$S = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

(a) *Let A denote the event that sum of dots appeated is a prime number*

$$\Rightarrow A = \{(1,1),(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(4,1),(4,3),(5,2),(5,6),(6,1),(6,5)\}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

(b) *Let B denote the event that difference of dots appeared is an even number*

$$\Rightarrow B = \begin{Bmatrix} (1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),((5,3),(5,5), \\ (6,2),(6,4),(6,6) \end{Bmatrix}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

(c) *Let C denote the event that product of dots is divisible by* 5

$$\Rightarrow C = \{(1,5),(2,5),(3,5),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,5)\}$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{11}{36}$$