STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 19

NORMAL DISTRIBUTION

STANDARD AND GENERAL CASES

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NORMAL DISTRIBUTION

Normal distribution is the most important continuous distribution in the entire field of statistics. It is also called **Gaussian Distribution**. It is widely used in the real world problems as real world phenomena are mostly approximately normally distributed.

EXAMPLES OF NORMAL DISTRIBUTION The following continuous random variables represents approximate normal distribution.

1. Heights and weights of people.
2. Scores on an examination.
3. Amount of milk in a gallon.
4. Time taken to complete a certain job
5. Life of an item (such as a light-bulb or a television set).

DEFINITION AND PDF OF NORMAL DISTRIBUTION

A **normal distribution** is a continuous, symmetric, bell-shaped distribution of a variable.

A random variable X following Normal distribution having mean μ and standard deviation σ is denoted by $X \sim N(\mu, \sigma^2)$. Its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2} \quad -\infty < x < \infty$$

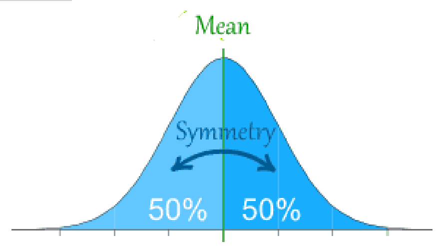
where

$e \approx 2.718$ (≈ means "is approximately equal to")

$\pi \approx 3.14$

μ = population mean

σ = population standard deviation



PROPERTIES OF NORMAL DISTRIBUTION

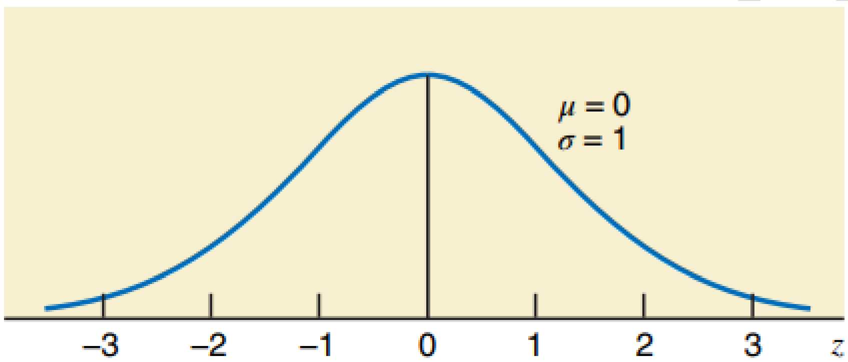
1. Mean of Normal Distribution is $E(X) = \mu$
2. The Variance of Normal Distribution is $V(X) = \sigma^2$
3. It is a bell –shaped distribution.
4. It is symmetric about the mean.
5. It extends from $-\infty$ *to* $+\infty$
6. The total area under the curve is 1.
7. A normal distribution curve is unimodal.
8. In Normal Distribution $Mean = Median = Mode$
9. The curve is continuous; that is, there are no gaps or holes.
10. The curve never touches the *x*-axis.It approaches the x-axis but never crosses it.

STANDARD NORMAL DISTRIBUTION

If Normal distribution have mean 0 and standard deviation 1 than it is denoted by $Z \sim N(0,1)$. In this case the probability density function is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad -\infty < z < \infty$$

The standard normal distribution curve.



FINDING PROBABILITIES USING STANDARD NORMAL DISTRIBUTION

If we have a normal distribution having mean = 0 and variance = 1.
ie $Z \sim N(0,1)$. Then we can find the area using the formulas given below.

- $P(Z < a) = \Phi(a)$
- $P(Z > a) = 1 - P(Z < a) = 1 - \Phi(a)$
- $P(Z < -a) = \Phi(-a) \quad or \quad P(Z < -a) = P(Z > a) = 1 - \Phi(a)$
- $P(Z > -a) = 1 - P(Z < -a) = 1 - \Phi(-a) \quad or \quad P(Z > -a) = P(Z < a) = \Phi(a)$
- $P(a < Z < b) = \Phi(b) - \Phi(a)$
- $P(-a < Z < b) = \Phi(b) + \Phi(-a) = \Phi(a) + \Phi(b) - 1$
- $P(-a < Z < -b) = \Phi(-b) - \Phi(-a) = \Phi(a) - \Phi(b)$
- $P(|z| < a) = P(-a < Z < a) = \Phi(a) - \Phi(-a) = 2\Phi(a) - 1$
- $P(|z| > a) = 1 - P(|z| < a) = 2(1 - \Phi(a))$

EXAMPLES OF STANDARD NORMAL DISTRIBUTION

EXAMPLE-1 If $Z \sim N(0,1)$ then find the following probabilities

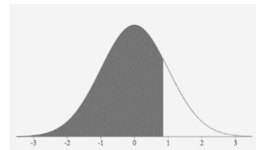
$(a)\ P(Z < 0.85)$ $\qquad$ $(b)\ P(Z > 0.85)$

SOLUTION *Here* $Z \sim N(0,1)$

$(a)\ P(Z < 0.85)$

$= \Phi(0.85)$

$= 0.8023$

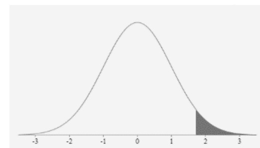


$(b)\ P(Z > 1.72)$

$= 1 - P(Z < 1.72)$

$= 1 - \Phi(1.72)$

$= 1 - 0.9573 = 0.0427$



EXAMPLE-2 If $Z \sim N(0,1)$ then find the following probabilities

$(a)\ P(Z < -1.37)$ $\qquad$ $(b)\ P(Z > -2.14)$

SOLUTION *Here* $Z \sim N(0,1)$

$(a)\ P(Z < -1.37)$

$= \Phi(-1.37)$

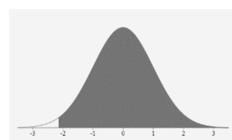
$= 0.0853$

$(b)\ P(Z > -2.14)$

$= 1 - P(Z < -2.14)$

$= 1 - \Phi(-2.14)$

$= 1 - 0.0162 = 0.9838$



EXAMPLE-3 If $Z \sim N(0,1)$ then find the following probabilities

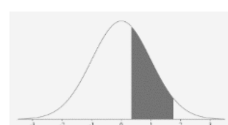
$(a)\ P(0.34 < Z < 1.75)$ $\qquad$ $(b)\ P(-1.4 < Z < -0.6)$ $\qquad$ $(c)\ P(-2.69 < Z < 1.86)$

SOLUTION *Here* $Z \sim N(0,1)$

$(a)\ P(0.34 < Z < 1.75)$

$= \Phi(1.75) - \Phi(0.34)$

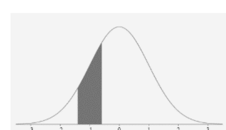
$= 0.9599 - 0.6331 = 0.3268$



$(b)\ P(-1.4 < Z < -0.6)$

$= \Phi(-0.6) - \Phi(-1.4)$

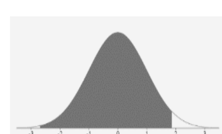
$= 0.2743 - 0.0808 = 0.1935$



$(c)\ P(-2.69 < Z < 1.86)$

$= \Phi(1.86) - \Phi(-2.69)$

$= 0.9686 - 0.0036 = 0.96$



EXAMPLE-4 If $Z \sim N(0,1)$ then find the value of $\boldsymbol{a}$ if

$(a)\ P(Z < a) = 0.9693$

$(b)\ P(Z > a) = 0.3802$

$(c)\ P(Z < a) = 0.0793$

$(d)\ P(Z > a) = 0.7367$

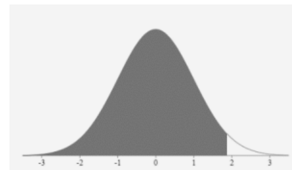
SOLUTION *Here* $Z \sim N(0,1)$

$(a)\ P(Z < a) = 0.9693$

$\Phi(a) = 0.9693$

$a = \Phi^{-1}(0.9693)$

$a = 1.87$



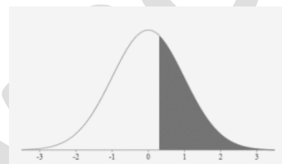
$(b)\ P(Z > a) = 0.3802$

$1 - P(Z < a)) = 0.3802$

$1 - \Phi(a) = 0.3802$

$\Phi(a) = 1 - 0.3802 = 0.6198$

$a = \Phi^{-1}(0.6198) = 0.305$

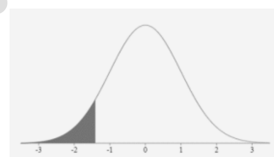


$(c)\ P(Z < a) = 0.0793$

$\Phi(a) = 0.0793$

$a = \Phi^{-1}(0.9693)$

$a = -1.41$



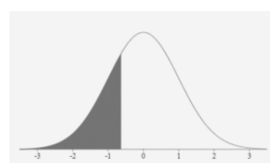
$(d)\ P(Z > a) = 0.7367$

$1 - P(Z < a)) = 0.7367$

$1 - \Phi(a) = 0.7367$

$\Phi(a) = 1 - 0.7367 = 0.2633$

$a = \Phi^{-1}(0.2633) = -0.633$



EXAMPLE-5 If $Z \sim N(0,1)$ then find the value of $\boldsymbol{a}$ if $if\ \ P(|z| < a) = 0.9$

SOLUTION *Here* $Z \sim N(0,1)$

$P(-a < Z < a) = 0.9$

$\Phi(a) - \Phi(-a) = 0.9$

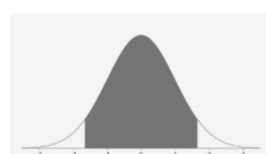
$\Phi(a) - [1 - \Phi(a)] = 0.9$

$\Phi(a) - 1 + \Phi(a) = 0.9$

$2\Phi(a) = 1.9$

$\Phi(a) = 0.95$

$a = \Phi^{-1}(0.95) = 1.645$



GENERAL NORMAL DISTRIBUTION

If Normal distribution have mean μ (any value) and standard deviation σ (any value) than it is denoted by $X \sim N\left(\mu,\sigma^2\right)$. In this case the probability density function is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}(x-\mu)^2} \quad -\infty < x < \infty$$

FINDING PROBABILITIES USING GENERAL NORMAL DISTRIBUTION

If the given normal distribution is not standard. Then first we convert it to standard normal distribution and then solve the problem.

Convert $X \sim N\left(\mu,\sigma^2\right) \to Z \sim N(0,1)$ using the formula

$$Z=\frac{X-\mu}{\sigma}$$

Where $X = Normal Variable$

$\mu = Mean$

$\sigma^2 = Variance$

Probability Calculations for Normal Distributions

If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$$

The random variable Z is known as the "standardized" version of the random variable X. This result implies that the probability values of a general normal distribution can be related to the cumulative distribution function of the standard normal distribution $\Phi(x)$ through the relationship

$$P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

EXAMPLES OF GENERAL NORMAL DISTRIBUTION

EXAMPLE-1 If $X \sim N(-8,12)$ then find the following probabilities

$(a)\ P(X < -9.8)$

$(b)\ P(X > -8.2)$

$(c)\ P(-7 < X < 0.5)$

SOLUTION $Here\ \ X \sim N(-8,12)\ \Rightarrow \mu = -8,\quad \sigma = \sqrt{12}$

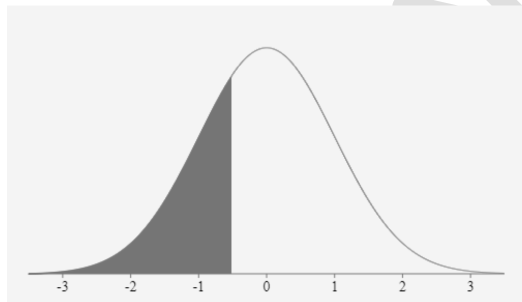
$(a)\ P(X < -9.8) = ?$

$$Z = \frac{x-\mu}{\sigma} = \frac{-9.8+8}{\sqrt{12}} = -0.52$$

$So\ \ P(Z < -0.52)$

$= \Phi(-0.52)$

$= 0.3015$



$(b)\ P(X > -8.2) = ?$

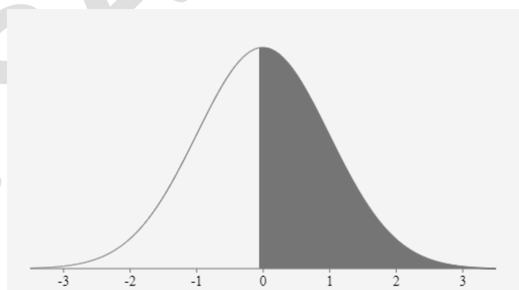
$$Z = \frac{x-\mu}{\sigma} = \frac{-8.2+8}{\sqrt{12}} = -0.06$$

$So\ \ P(Z > -0.06)$

$= 1 - P(Z < -0.06)$

$= 1 - \Phi(-0.06) = 1 - 0.4761$

$= 0.5239$



$(c)\ P(-7 < X > 0.5) = ?$

$$Z_1 = \frac{x-\mu}{\sigma} = \frac{-7+8}{\sqrt{12}} = 0.29$$

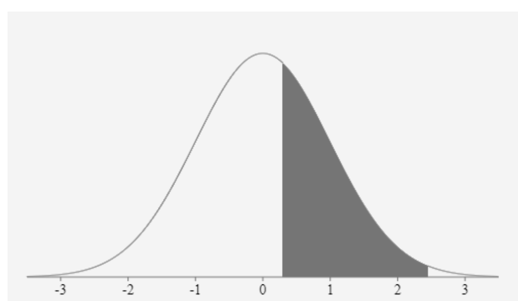
$$Z_2 = \frac{x-\mu}{\sigma} = \frac{0.5+8}{\sqrt{12}} = 2.45$$

$So\ \ P(0.29 < Z > 2.45)$

$= \Phi(2.45) - \Phi(0.29)$

$= 0.9929 - 0.6141$

$= 0.3788$



EXAMPLE-2 If $X \sim N(100,81)$ then find the following probabilities

$(a)\ P(|X-100|<18)$

$(b)\ P(|X-100|>5)$

$(c)\ P(12<X-100<15)$

SOLUTION *Here* $X \sim N(100,81) \Rightarrow \mu=100,\ \sigma=9$

$(a)\ P(|X-100|<18)$

$=P\left(\left|\frac{X-100}{9}\right|<\frac{18}{9}\right)$

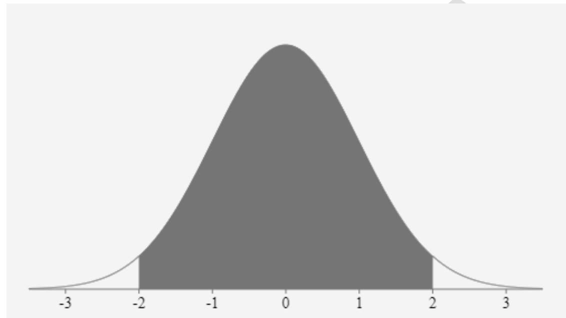
$=P(|Z|<2)$

$=P(-2<Z<2)$

$=\Phi(2)-\Phi(-2)$

$=0.9772-0.0228$

$=0.9544$



$(b)\ P(|X-100|>5)$

$=P\left(\left|\frac{X-100}{9}\right|>\frac{5}{9}\right)$

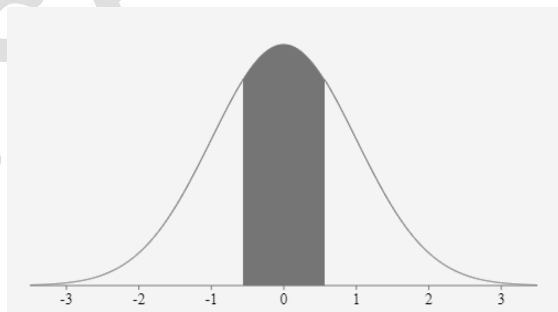
$=P(|Z|>0.56)=1-P(|Z|<0.56)$

$=1-P(-0.56<Z<0.56)$

$=1-[\Phi(0.56)-\Phi(-0.56)]$

$=1-[0.7123-0.2877]$

$=0.5754$



$(c)\ P(12<X-100<15)$

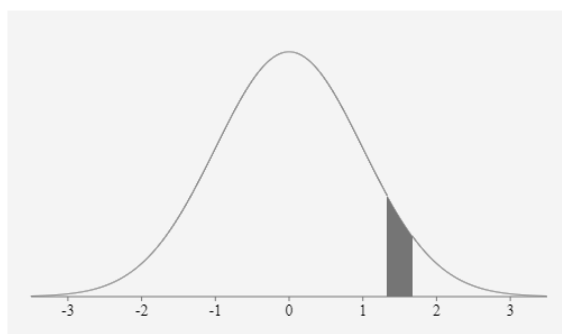
$=P\left(\frac{12}{9}<\frac{X-100}{9}<\frac{15}{9}\right)$

$=P(1.33<Z<1.67)$

$=\Phi(1.67)-\Phi(1.33)]$

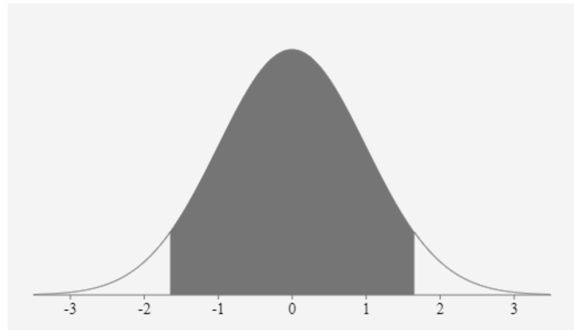
$=0.9525-0.9082$

$=0.0443$



EXAMPLE-3 If $X \sim N(200,36)$ and $if\ \ P(X > a) = 0.9386$ then find the value of $\boldsymbol{a}$.

SOLUTION $Here\ \ X \sim N(200,36)\ \Rightarrow \mu = 200,\quad \sigma = 6$



$$Now\ \ P(X > a) = 0.9386$$

$$Z = \frac{a-200}{6} = b\ (say)$$

$$So\ P(Z > b) = 0.9386$$

$$1 - P(Z < b)) = 0.9386$$

$$1 - \Phi(b) = 0.9386$$

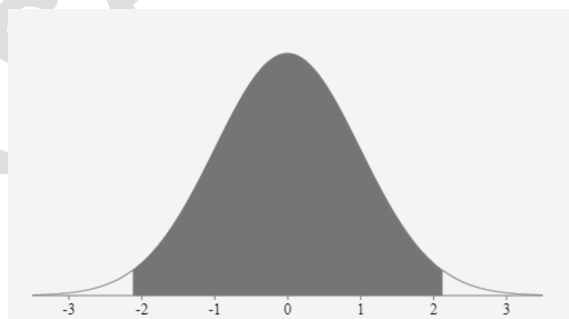
$$\Phi(b) = 1 - 0.9386 = 0.0614$$

$$b = \Phi^{-1}(0.0614) = -1.50$$

$$So\ \frac{a-200}{6} = -1.50 \Rightarrow a = 191$$

EXAMPLE-4 If $X \sim N(10,2)$ and $if\ \ P(|X-10| \le a) = 0.9662$ then find the value of $\boldsymbol{a}$.

SOLUTION $Here\ X \sim N(10,2)\ \Rightarrow \mu = 10,\ \ \sigma = \sqrt{2}$



$$Now\ P(|X-10| \le a) = 0.9662$$

$$P\left(\left|\frac{X-10}{\sqrt{2}}\right| \le \frac{a}{\sqrt{2}}\right) = 0.9662$$

$$P\left(|Z| \le a/\sqrt{2}\right) = 09662$$

$$P\left(|Z| \le b\right) = 0.9662\quad where\ \ b = a/\sqrt{2}$$

$$P(-b \le Z \le b) = 0.9662$$

$$\Phi(b) - \Phi(-b) = 0.9662$$

$$2\Phi(b) - 1 = 0.9662\ \ \Rightarrow \Phi(b) = 0.9831$$

$$b = \Phi^{-1}(0.9831) = 2.12$$

$$So\ \ a/\sqrt{2} = 2.12\quad \Rightarrow \quad a = 3$$

EXAMPLE-5 If $X \sim N\left(4.3, 0.12^2\right)$ then find the following values

(a) *Lower and Upper Quartiles*

(b) 95*th Percentile*

(c) *value of* c *for which* $P(4.3 - c < X < 4.3 + c) = 0.8$

SOLUTION *Here* $X \sim N\left(4.3, 0.12^2\right)$, *So* $\mu = 4.3$ *and* $\sigma = 0.12$

(a) *For Lower Quartile* $= Q_1$,

Solve $P(X \le Q_1) = 0.25$ *for* Q_1

$$Z = \frac{Q_1 - 4.3}{0.12} = a \ (say)$$

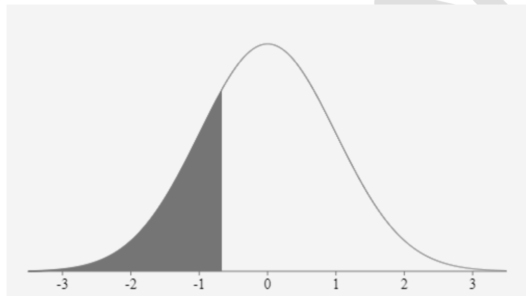
$P(Z \le a) = 0.25$

$\Phi(a) = 0.25$

$a = \Phi^{-1}(0.25) = -0.67$

$$So \ \frac{Q_1 - 4.3}{0.12} = -0.67$$

$\Rightarrow Q_1 = 4.2196$



(a) *For Upper Quartile* $= Q_3$,

Solve $P(X \le Q_3) = 0.75$ *for* Q_3

$$Z = \frac{Q_3 - 4.3}{0.12} = b \ (say)$$

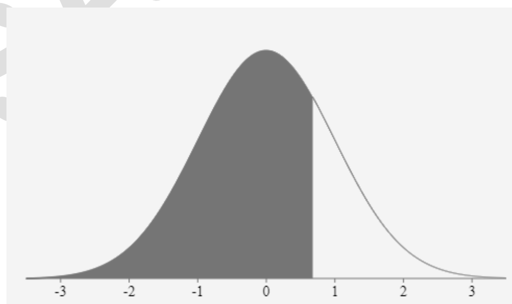
$P(Z \le b) = 0.75$

$\Phi(b) = 0.75$

$b = \Phi^{-1}(0.75) = 0.67$

$$So \ \frac{Q_3 - 4.3}{0.12} = 0.67$$

$\Rightarrow Q_3 = 4.3804$



(b) *For* 95*th Percentile* $= P_{95}$,

Solve $P(X \le P_{95}) = 0.95$ *for* P_{95}

$$Z = \frac{P_{95} - 4.3}{0.12} = c \ (say)$$

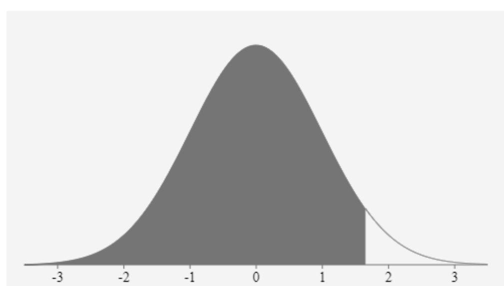
$P(Z \le c) = 0.95$

$\Phi(c) = 0.95$

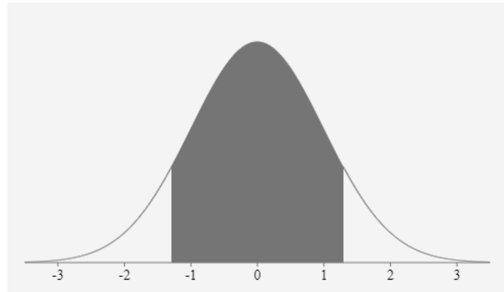
$c = \Phi^{-1}(0.95) = 1.65$

$$So \ \frac{P_{95} - 4.3}{0.12} = 1.65$$

$\Rightarrow P_{95} = 4.498$



(c) *Here* $P(4.3-c<X<4.3+c)=0.8$



Converting to standard version

using $Z=\dfrac{X-\mu}{\sigma}$, *we get*

$$P\left(\frac{(4.3-c)-4.3}{0.12}<Z<\frac{(4.3+c)-4.3}{0.12}\right)=0.8$$

$$P\left(\frac{4.3-c-4.3}{0.12}<Z<\frac{4.3+c-4.3}{0.12}\right)=0.8$$

$$P\left(\frac{-c}{0.12}<Z<\frac{c}{0.12}\right)=0.8$$

$$\Phi\left(\frac{c}{0.12}\right)-\Phi\left(\frac{-c}{0.12}\right)=0.8$$

$$\Phi\left(\frac{c}{0.12}\right)-[1-\Phi\left(\frac{c}{0.12}\right)]=0.8$$

$$\Phi\left(\frac{c}{0.12}\right)-1+\Phi\left(\frac{c}{0.12}\right)=0.8$$

$$2\Phi\left(\frac{c}{0.12}\right)=1.8$$

$$\Phi\left(\frac{c}{0.12}\right)=0.9$$

$$\frac{c}{0.12}=\Phi^{-1}(0.9)$$

$$\frac{c}{0.12}=1.29$$

$$c=0.1548$$