Question No:- 1

$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$. Find

(1) coordinate vector of $\bar{v} = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$ ....

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & -1 & 1 & -3 \\ 1 & -1 & -1 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & 2 & -7 \\ 0 & -2 & 0 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -1 & 7/2 \\ 0 & 1 & 0 & -3/2 \end{array}\right] \begin{array}{l} \\ R_2/-2 \\ R_3/-2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 7/2 \\ 0 & 0 & 1 & -5 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -5 \end{array}\right] \begin{array}{l} R_2 + R_3 \\ \\ \\ \end{array}$$

$C_1 = \frac{1}{2} \qquad C_2 = -\frac{3}{2} \qquad C_3 = -5$

coordinate vector of $\bar{v}$ w.r.to β is:

$$[\bar{v}]_\beta = \begin{bmatrix} 1/2 \\ -3/2 \\ -5 \end{bmatrix}$$

② Find $\bar{u}$ ..... $[\bar{u}]_\beta = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$

$$[u]_\beta = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

Multiply both.

$$\begin{bmatrix} 5-4-1 \\ 5+4+1 \\ 5+4-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 8 \end{bmatrix}$$

$$\bar{u} = \begin{bmatrix} 0 \\ 10 \\ 8 \end{bmatrix}$$

Question No:- 2

Let $C = \{1+2t+t^2, 3-3t, -t+5t^2\}$

① Prove that C is Basis of $\mathbb{P}^2(t)$

$C = \{1+2t+t^2, 3-3t, -t+5t^2\}$

we have to check two conditions:

⊙ Check linearly independency

$$c_1(1+2t+t^2)+c_2(3-3t)+c_3(-t+5t^2)=0$$

t^0 : $c_1+3c_2=0$

t^1 : $2c_1-3c_2-c_3=0$

t^2 : $c_1+5c_3=0$

$$\tilde{A}=\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & -3 & -1 & 0 \\ 1 & 0 & 5 & 0 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -9 & -1 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-R_1 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 1/9 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right] \frac{R_2}{-9}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 1/9 & 0 \\ 0 & 0 & 16/3 & 0 \end{array}\right] \begin{array}{l} R_1-3R_2 \\ R_3+3R_2 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 1/9 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] +\frac{3}{16}R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right] \begin{array}{l} R_1+\frac{1}{3}R_3 \\ R_2-\frac{1}{9}R_3 \end{array}$$

pivot in every column so lin. indep.

$c_1=0$ $c_2=0$ $c_3=0$

Hence C is linearly independent.

$\odot$ C will also span whole $\mathbb{P}^2$.

So, C is a basis for $\mathbb{P}^2$.

Beacause no. of vectors in C is equal to dimension of $\mathbb{P}^2$ (which is 3)

② Find coordinate vector of

$\bar{v} = 2 + 3t - t^2$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 2 & -3 & -1 & 3 \\ 1 & 0 & 5 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & -9 & -1 & -1 \\ 0 & -3 & 5 & -3 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 1/9 & 1/9 \\ 0 & -3 & 5 & -3 \end{array}\right] \frac{R_2}{-9}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 5/3 \\ 0 & 1 & 1/9 & 1/9 \\ 0 & 0 & 16/3 & -8/3 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ R_3 + 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 5/3 \\ 0 & 1 & 1/9 & 1/9 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \frac{3}{16} R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 1/6 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \begin{array}{l} R_1 + \frac{1}{3} R_3 \\ R_2 - \frac{1}{9} R_3 \end{array}$$

$c_1 = 3/2 \quad c_2 = 1/6 \quad c_3 = -1/2$

$$[\bar{v}]_c = \begin{bmatrix} 3/2 \\ 1/6 \\ -1/2 \end{bmatrix}$$

③ Find $\bar{u}$ ...... $[\bar{u}]_C = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & -3 & -1 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

Multiply Both

$$\begin{bmatrix} 5-12+0 \\ 10+12-1 \\ 5+0+5 \end{bmatrix} = \begin{bmatrix} -7 \\ 21 \\ 10 \end{bmatrix}$$

$$\bar{u} = \begin{bmatrix} -7 \\ 21 \\ 10 \end{bmatrix}$$

Question No:- 3

Consider 2 dimensional subspace

$W = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in R \right\}$ of $M^{2\times 2}$

① Prove that $D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

is basis of W.

⊙ Check linearly independent

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} + \begin{pmatrix} 0 & c_2 \\ c_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$c_1 = 0 \qquad c_2 = 0$

Hence D is a linearly independent set.

⊙ As no of elements in D is equal to dimension of W. Therefore, D will span whole W.
Hence D is Basis of W.

② check $E = \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \right\}$ is basis of W.

⊙ check linearly independent.

$$c_1 \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 & 2c_1 \\ 2c_1 & c_1 \end{pmatrix} + \begin{pmatrix} -2c_2 & c_2 \\ c_2 & -2c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - 2c_2 & 2c_1 + c_2 \\ 2c_1 + c_2 & c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$c_1 - 2c_2 = 0$
$2c_1 + c_2 = 0$

$$\tilde{A}\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 5 & 0 \end{array}\right) R_2-2R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right] \frac{R_2}{5}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] R_1+2R_2$$

$C_1 = 0 \qquad C_2 = 0$

Hence E is linearly independent.

ii) As number of vectors in E is equal to dimension of W. Therefore, E will also span whole W.

Hence E is a basis of W.

(3) Find coordinate vector of $\bar{v} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ .....

(i) with respect D

$$C_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_1 \end{bmatrix} + \begin{bmatrix} 0 & C_2 \\ C_2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$C_1 = 5 \qquad C_2 = 3$

$$[\bar{v}]_D = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

(i) with respect to E

$$c_1 \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} C_1 & 2C_1 \\ 2C_1 & C_1 \end{pmatrix} + \begin{pmatrix} -2C_2 & C_2 \\ C_2 & -2C_2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} C_1 - 2C_2 & 2C_1 + C_2 \\ 2C_1 + C_2 & C_1 - 2C_2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$C_1 - 2C_2 = 5$$
$$2C_1 + C_2 = 3$$

$$\tilde{A} = \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 2 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 5 & -7 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -7/5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 11/5 \\ 0 & 1 & -7/5 \end{array}\right]$$

$$C_1 = \frac{11}{5} \qquad C_2 = -\frac{7}{5}$$

$$[\bar{v}]_E = \begin{bmatrix} 11/5 \\ -7/5 \end{bmatrix}$$

(4) Find $\bar{u}$ .... $[\bar{u}]_E = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

$$\left[\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \right] \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

multiply both.

$$= 5 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 4 \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix}$$

$$\bar{u} = \begin{bmatrix} 13 & 6 \\ 6 & 13 \end{bmatrix}$$