Homework #11

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Q#1 Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

a) Find least square solution of the inconsistent system $Ax = b$. (Hint: Use normal equations $A^T A x = A^T b$)

$$A^T A x = A^T b$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 8+1+1 & 3+1+2 \\ 3+1+2 & 1+1+4 \end{bmatrix} \hat{x} = \begin{bmatrix} 3+1+1 \\ 1+1+2 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix} \hat{x} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

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$$\hat{x} = \frac{1}{|A^TA|}\begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\hat{x} = \frac{1}{\cancel{66}-36}\begin{bmatrix} \cancel{6} & -6 \\ -6 & 11 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\hat{x} = \frac{1}{30}\begin{bmatrix} 30-24 \\ -30+44 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} +6 \\ 14 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1/5 \\ 7/15 \end{bmatrix}$$

☆ Alternatively you can find $\hat{x}$ by augmenting $[A^TA]$ with A^Tb and by reducing A^TA to reduced echelon in these cases identity you automatically obtain $\hat{x}$ as augmented form. This is useful for larger matrizes. as it does not requir inversion

b) Using $A = QR$, find the least-squares solution of inconsistent system $Ax = b$ (Hint: Use $Rx = Q^Tb$)

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Using the Gram Schmit process we,

First we find Q: $x_1 = v_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$\|x_1\| = \sqrt{11}$

$x_2 = v_2 - \text{Proj}_{x_1} v_2$

$\text{Proj}_{x_1} v_2 = \frac{v_2 \cdot x_1}{x_1 \cdot x_1} x_1 = \frac{3+1+2}{9+1+1} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 18/11 \\ 6/11 \\ 6/11 \end{bmatrix}$

$x_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 18/11 \\ 6/11 \\ 6/11 \end{bmatrix}$

$x_2 = \begin{bmatrix} 11/11 - 18/11 \\ 11/11 - 6/11 \\ 22/11 - 6/11 \end{bmatrix} = \begin{bmatrix} -7/11 \\ 5/11 \\ 16/11 \end{bmatrix}$, $\|x_2\| = \sqrt{\frac{49}{121} + \frac{25}{121} + \frac{256}{121}}$

$\|x_2\| = \frac{\cancel{330}^{30}}{\cancel{121}^{11}} = \sqrt{\frac{30}{11}}$

$x_1 \Rightarrow \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$, $x_2 \Rightarrow \begin{bmatrix} \frac{-7}{\cancel{11}} \times \frac{\cancel{\sqrt{11}}}{\sqrt{30}} \\ \frac{5}{11} \times \frac{\sqrt{11}}{\sqrt{30}} \\ \frac{16}{11} \times \frac{\sqrt{11}}{\sqrt{30}} \end{bmatrix} = \begin{bmatrix} -7/\sqrt{330} \\ 5/\sqrt{330} \\ 16/\sqrt{330} \end{bmatrix}$

$Q = \begin{bmatrix} 3/\sqrt{11} & -7/\sqrt{330} \\ 1/\sqrt{11} & 5/\sqrt{330} \\ 1/\sqrt{11} & 16/\sqrt{330} \end{bmatrix}$

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To find R:

$$R = Q^T A = \begin{bmatrix} 3/\sqrt{11} & 1/\sqrt{11} & 1/11 \\ -7/\sqrt{330} & 5/\sqrt{330} & 16/\sqrt{330} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{11} & 6/\sqrt{11} \\ 0 & \sqrt{30/11} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{11} & 6/\sqrt{11} \\ 0 & \sqrt{30}/\sqrt{11} \end{bmatrix} \hat{x} = \begin{bmatrix} 3/\sqrt{11} & 1/\sqrt{11} & 1/\sqrt{11} \\ -7/\sqrt{330} & 5/\sqrt{330} & 16/\sqrt{330} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{11} & 6/\sqrt{11} \\ 0 & \sqrt{30}/\sqrt{11} \end{bmatrix} \hat{x} = \begin{bmatrix} 5/\sqrt{11} \\ 14/\sqrt{330} \end{bmatrix}$$

$$\left[\begin{array}{cc:c} \sqrt{11} & 6/\sqrt{11} & 5/\sqrt{11} \\ 0 & \sqrt{30/11} & 14/\sqrt{330} \end{array}\right]$$

$$\sim \left[\begin{array}{cc:c} 1 & 6/11 & 5/11 \\ 0 & 1 & \cancel{14}^{7}/\cancel{30}_{15} \end{array}\right] \begin{array}{l} R_1 \ \sqrt{11} \\ R_2 \times \sqrt{\frac{11}{30}} \end{array}$$

$$\sim \left[\begin{array}{cc:c} 1 & 0 & \frac{75}{165} - \frac{42}{165} \\ 0 & 1 & 7/15 \end{array}\right] R_1 - R_2 \times \frac{6}{11}$$

$$\sim \left[\begin{array}{cc:c} 1 & 0 & 1/5 \\ 0 & 1 & 7/15 \end{array}\right] \qquad \because \ 33/165 = 1/5$$

$$\hat{x} = \begin{bmatrix} 1/5 \\ 7/15 \end{bmatrix}$$

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c) Find the least-squares error vector and least squares error of the inconsistent system $Ax = b$

Error vector $= \epsilon = b - \hat{b} = b - A\hat{x}$

$$\epsilon = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3/15 \\ 7/15 \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 16/15 \\ 10/15 \\ 17/15 \end{bmatrix} = \begin{bmatrix} -1/15 \\ 5/15 \\ -2/15 \end{bmatrix}$$

$$\text{Error} = \|\epsilon\| = \sqrt{\left(\frac{1}{225}\right) + \left(\frac{25}{225}\right) + \left(\frac{4}{225}\right)} = \sqrt{\frac{30}{225}}$$

$$\text{Error} = \sqrt{\frac{2}{15}}$$

d) Without calculation comment on the linear independency of A and invertibility of A^TA

Since there is a unique least-squares solution, it means the columns of A are linearly independent and A^TA must be invertable.

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e) Find the standard Matrix for the orthogonal projection on the column space of Matrix A.

Let P be the orthogonal projection Matrix

$$P = A(A^TA)^{-1}A^T$$

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \left(\frac{1}{30} \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix} \right) \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$P = \frac{1}{30} \begin{bmatrix} 12 & -7 \\ 0 & 5 \\ -6 & 16 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$P = \frac{1}{30} \begin{bmatrix} 36-7 & 12-7 & 12-14 \\ 0+5 & 0+5 & 0+10 \\ -18+16 & -6+16 & -6+32 \end{bmatrix}$$

$$P = \frac{1}{30} \begin{bmatrix} 29 & 5 & -2 \\ 5 & 5 & 10 \\ -2 & 10 & 26 \end{bmatrix}$$

$$P = \begin{bmatrix} 29/30 & 1/6 & -1/15 \\ 1/6 & 1/6 & 1/3 \\ -1/15 & 1/3 & 26/30 \end{bmatrix}$$

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f) Use $A = QR$ to show projection matrix $A(A^TA)^{-1}A^T$ can be written as QQ^T

$P = A(A^TA)^{-1}A^T$

$P = QR((QR)^TQR)^{-1}(QR)^T$

$P = QR(R^TQ^TQR)^{-1}R^TQ^T$ $\because Q$ is orthogonal

$P = QR(R^TR)^{-1}R^TQ^T$ so $Q^TQ = I$

$P = QRR^{-1}(R^T)^{-1}R^TQ^T$

$P = Q\,I\,I\,Q^T$

$P = QQ^T$

g) Find projection of b on column space of A.

$$\hat{b} = Pb = \frac{1}{30}\begin{bmatrix} 29 & 5 & -2 \\ 5 & 5 & 10 \\ -2 & 10 & 26 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{b} = \frac{1}{30}\begin{bmatrix} 29+5-2 \\ 5+5+10 \\ -2+10+26 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 32 \\ 20 \\ 34 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 32/30 \\ 20/30 \\ 34/30 \end{bmatrix}$$

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h) Find left inverse of A (Hint: Use $(A^TA)^{-1}A^T$)

Left inverse be A^+:

$$A^+ = (A^TA)^{-1}A^T = \frac{1}{30}\begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^+ = \frac{1}{30}\begin{bmatrix} 18-6 & 6-6 & 6-12 \\ -18+11 & -6+11 & -6+22 \end{bmatrix}$$

$$A^+ = \frac{1}{30}\begin{bmatrix} 12 & 0 & -6 \\ -7 & 5 & 16 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 12/30 & 0 & -6/30 \\ -7/30 & 5/30 & 16/30 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 2/15 & 0 & -1/5 \\ -7/30 & 1/6 & 8/15 \end{bmatrix}$$