Homework - 13

Syed Ali Haider Naqvi
AI - A
23i-0122

Qn 1.

$$\langle p, q \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3)$$

$x_0 = -3$, $x_1 = -1$, $x_2 = 1$, $x_3 = 3$

a) Find best approx. of $p(x) = x^2$ onto $P_1 = \text{span}\{1, x\}$

$$\hat{p} = \text{Proj}_{P_1} p = \text{Proj}_1 p + \text{Proj}_x p$$

$$= \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} \times 1 + \frac{\langle x^2, x \rangle}{\langle x, x \rangle} \times x$$

$\langle x^2, 1 \rangle = (-3)^2 + (-1)^2 + 1^2 + 3^2$
$= \underline{20}$

$\langle 1, 1 \rangle = 1 + 1 + 1 + 1 = \underline{4}$

$\langle x^2, x \rangle = (-3)^2(-3) + (-1)^2(-1) + 1 + 3^2(3)$
$= -27 - 1 + 1 + 27$
$= \underline{0}$

$\langle x, x \rangle = (-3)(-3) + (-1)(-1) + (1)(1) + (3)(3)$
$= \underline{20}$

$$= \frac{20}{4} + \frac{0}{20} = 5$$

The best approx. of $p(x) = x^2$ onto $P_1 = \text{span}\{1, x\}$ is 5

Qn 1 b

let $\{u_1, u_2, u_3\}$ be $O.B \rightarrow P_2$

$\{1, x, x^2\}$

$$\langle x, 1\rangle = -3 - 1 + 1 + 3 = 0$$

$$x_1 = 1$$

$$x_2 = u_2 - \text{Proj}_{x} u_2$$

$$x_2 = u_2 - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} \times 1$$

$$x_2 = x - \frac{0}{4} \times 1$$

$$x_2 = x$$

$$x_3 = u_3 - \text{Proj}_{x} u_3$$

$$x_3 = u_3 - \frac{\langle x^2, x\rangle}{\langle x, x\rangle} x - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} \times 1$$

$$x_3 = x^2 - \frac{0}{20} x - \frac{20}{4}$$

$$x_3 = x^2 - 5$$

$$O.B = \{1, x, x^2 - 5\}$$

Qn 2

$$\langle A, B\rangle = tr(A^T B)$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$$

$$x_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_2 = u_2 - \mathrm{Proj}_{x_1} u_2 = u_2 - \frac{\langle u_2, x_1\rangle}{\langle x_1, x_1\rangle} x_1$$

$$\langle u_2, x_1\rangle = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

$$= tr\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{0}}$$

$$\langle x_1, x_1\rangle = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \underline{\underline{2}}$$

$$x_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$x_3 = u_3 - \mathrm{Proj}_{x_1} u_3 - \mathrm{Proj}_{x_2} u_3$$

$$x_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{\langle u_3, x_1\rangle}{\langle x_1, x_1\rangle} x_1 - \frac{\langle u_3, x_2\rangle}{\langle x_2, x_2\rangle} x_2$$

$$\langle u_3, x_1\rangle = tr\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= tr\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$= \underline{\underline{2}}$$

$$\langle v_3, x_2 \rangle = tr\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right)$$

$$= \underline{0}$$

$$\langle x_2, x_2 \rangle = tr\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \underline{\underline{2}}$$

$$x_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ //

$$x_4 = v_4 - \mathrm{Proj}_{x_1} v_4 - \mathrm{Proj}_{x_2} v_4 - \mathrm{Proj}_{x_3} v_4$$

$$x_4 = v_4 - \frac{\langle v_4, x_1 \rangle}{\langle x_1, x_1 \rangle} x_1 - \frac{\langle v_4, x_2 \rangle}{\langle x_2, x_2 \rangle} x_2 - \frac{\langle v_4, x_3 \rangle}{\langle x_3, x_3 \rangle} x_3$$

$$\langle v_4, x_1 \rangle = tr\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \underline{0}$$

$$\langle v_4, x_2 \rangle = tr\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\right) = \underline{0}$$

$$\langle v_4, x_2 \rangle = tr\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right) = \underline{2}$$

$$\langle x_3, x_3 \rangle = tr\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \underline{\underline{2}}$$

$$x_4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - 0 - 0 - \frac{2}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$x_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Orthonormal Basis

$$\left\{ \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix} \right\}$$

Qn 3

$f(x) = x - 1$ on the interval $(-\pi, \pi)$

$$f(x) \approx a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x - 1 \, dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx - \cos kx \, dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx - \sin kx \, dx$$

$$a_0 = \frac{1}{2\pi}\left[\frac{x^2}{2} - x\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi}\left(\left(\frac{\pi^2}{2} - \pi\right) - \left(\frac{\pi^2}{2} + \pi\right)\right)$$

$$a_0 = \frac{1}{2\pi}\left(-2\pi\right)$$

$$a_0 = -1$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx - \cos kx \, dx$$

$$a_k = \frac{1}{\pi}\left(\left[\frac{-\sin kx}{k}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} x\cos kx\right)$$

$$a_k = \frac{1}{\pi}\left(\left[\frac{-\sin k\pi}{k}\right] + \left[\frac{\sin k\pi}{k}\right] + \int_{-\pi}^{\pi} x\cos kx \, dx\right)$$

$\sin n\pi$ where n is an integer $= 0$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx \, dx$$

$u = x$ $\quad v' = \cos kx$

$u' = 1$ $\quad v = \frac{\sin kx}{k}$

$$\pi a_k = \frac{x\sin kx}{k} - \int_{-\pi}^{\pi} \frac{\sin kx}{k} \, dx$$

$$\pi a_k = \left[\frac{x\sin kx}{k} + \frac{\cos kx}{k^2}\right]_{-\pi}^{\pi}$$

$$a_k = \frac{1}{\pi}\left(\left(0 + \frac{\cos k\pi}{k^2}\right) - \left(0 + \frac{\cos -k\pi}{k^2}\right)\right)$$

$\cos\theta = \cos -\theta$

hence $a_k = 0$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx - \sin kx \; dx$$

$$\pi b_k = \int_{-\pi}^{\pi} x\sin kx + \left[\frac{\cos kx}{k}\right]_{-\pi}^{\pi}$$

$$\pi b_k = \int_{-\pi}^{\pi} x\sin kx \, dx + \left(\frac{\cos \pi k}{k} - \frac{\cos \pi k}{k}\right)$$

$$\pi b_k = 0 + \int_{-\pi}^{\pi} x \sin kx \; dx$$

$u = x \qquad v' = \sin kx$

$u' = 1 \qquad v = -\frac{\cos kx}{k}$

$$\pi b_k = 0 + \left(-\frac{x\cos kx}{k} - \int_{-\pi}^{\pi} \frac{-\cos kx}{k} \, dx\right)$$

$$\pi b_k = 0 + \left[-\frac{x\cos kx}{k} + \frac{\sin kx}{k^2}\right]_{-\pi}^{\pi}$$

$$\pi b_k = 0 + \left(\frac{-\pi\cos k\pi}{k} + \frac{\sin k\pi}{k^2}\right) - \left(\frac{-\pi\cos(-k\pi)}{k} + \frac{\sin \pi k}{k^2}\right)$$

$$b_k = \frac{-2\cos \pi k}{k} = \frac{-2(-1)^k}{k}$$

$$b_k = -1 + \sum_{k=1}^{n} \frac{-2(-1)^k}{k} \sin kx$$