MT-1004
Linear Algebra

Fall 2023

Week # 8-9

National University of Computer and Emerging Sciences

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Chapter 5

Eigenvalues and Eigenvectors

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Section 5.1

Eigenvectors and Eigenvalues

A Biology Question

Motivation

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have $0, 6, 8$ baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$\begin{aligned} f_n &= \text{first-year rabbits in year } n \\ s_n &= \text{second-year rabbits in year } n \\ t_n &= \text{third-year rabbits in year } n \end{aligned}$$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $\boxed{Av_n = v_{n+1}.}$ $\longleftarrow$ **difference equation**

A Biology Question

Continued

If you know v_0, what is v_{10}?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$

This makes it easy to compute examples by computer:

v_0	v_{10}	v_{11}
$\begin{pmatrix}3\\7\\9\end{pmatrix}$	$\begin{pmatrix}30189\\7761\\1844\end{pmatrix}$	$\begin{pmatrix}61316\\15095\\3881\end{pmatrix}$
$\begin{pmatrix}1\\2\\3\end{pmatrix}$	$\begin{pmatrix}9459\\2434\\577\end{pmatrix}$	$\begin{pmatrix}19222\\4729\\1217\end{pmatrix}$
$\begin{pmatrix}4\\7\\8\end{pmatrix}$	$\begin{pmatrix}28856\\7405\\1765\end{pmatrix}$	$\begin{pmatrix}58550\\14428\\3703\end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.
2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = (\text{scalar}) \cdot \begin{pmatrix}16\\4\\1\end{pmatrix}.$$

Translation: 2 is an eigenvalue, and $\begin{pmatrix}16\\4\\1\end{pmatrix}$ is an eigenvector!

Eigenvectors and Eigenvalues

Definition

Let A be an $n \times n$ (Square)matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of A is a nonzero vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$. In other words, Av is a multiple of v.
2. An **eigenvalue** of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a *nontrivial* solution.

If $Av = \lambda v$ for $v \neq 0$, we say λ is the **eigenvalue for** v, and v is an **eigenvector for** λ.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

Hence v is an eigenvector of A, with eigenvalue $\lambda = 2$.

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence v is an eigenvector of A, with eigenvalue $\lambda = 4$.

Poll

Poll

Which of the vectors

$$\text{A. } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{B. } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{C. } \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{D. } \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{E. } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

What are the eigenvalues?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
eigenvector with eigenvalue 2

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$
not an eigenvector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

In other words, does $Av = 3v$ have a nontrivial solution?

...does $Av - 3v = 0$ have a nontrivial solution?

...does $(A - 3I)v = 0$ have a nontrivial solution?

We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric form: $x = -4y$; parametric vector form: $\begin{pmatrix} x \\ y \end{pmatrix} = y\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Does there exist an eigenvector with eigenvalue $\lambda = 3$? Yes! Any nonzero multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Check:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 1 \end{pmatrix}. \quad ✔$$

Exercise

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

2. Is $\lambda = -3$ an eigenvalue of $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$? Why or why not?

3. Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$? If so, find the eigenvalue.

4. Is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$? If so, find the eigenvalue.

6. Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? If so, find the eigenvalue.

7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

8. Is $\lambda = 1$ an eigenvalue of $\begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$? If so, find one corresponding eigenvector.

Eigenspaces

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned} \lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \mathrm{Nul}(A - \lambda I). \end{aligned}$$

Since the λ-eigenspace is a null space, it is a *subspace* of $\mathbf{R}^n$.

How do you find a basis for the λ-eigenspace? Parametric vector form!

Eigenspaces

Example

Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{\text{parametric} \\ \text{form}}} x = \frac{1}{2}y - 3z$$

$$\xrightarrow{\substack{\text{parametric vector} \\ \text{form}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Exercise

Find a basis for the eigenspace corresponding to listed eigenvalue.

10. $A = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}, \lambda = -5$

11. $A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}, \lambda = -1, 7$

12. $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}, \lambda = 3, 7$

13. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$

14. $A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}, \lambda = 3$

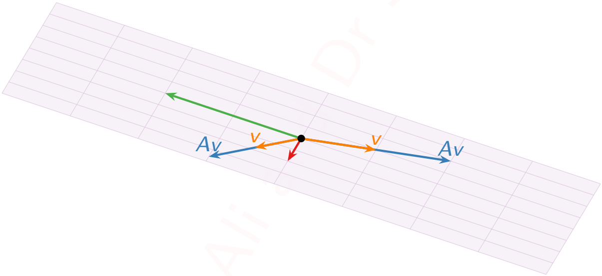
15. $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}, \lambda = -5$

16. $A = \begin{bmatrix} 5 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & -2 & 4 \end{bmatrix}, \lambda = 4$

Eigenspaces

Picture

A basis for the 2-eigenspace of $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$ is $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$. What does this look like?



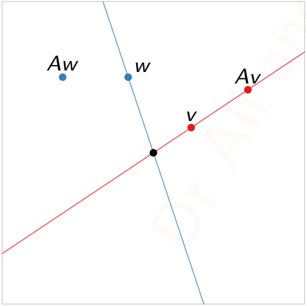
For any v in the 2-eigenspace, $Av = 2v$ by definition. So A acts by *scaling by* 2 on its 2-eigenspace. This is how eigenvalues and eigenvectors make matrices easier to understand.

Eigenspaces
Geometry

Eigenvectors, geometrically

An eigenvector of a matrix A is a nonzero vector v such that:

- Av is a multiple of v, which means
- Av is collinear with v, which means
- Av and v are *on the same line*.



v is an eigenvector

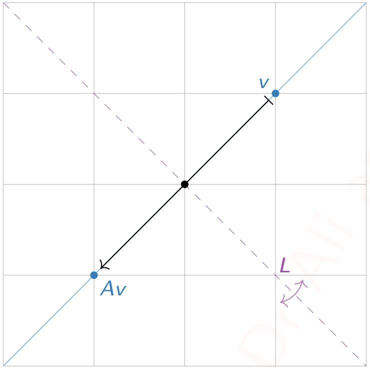
w is not an eigenvector

Eigenspaces

Geometry; example

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

v is an eigenvector with eigenvalue -1.

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

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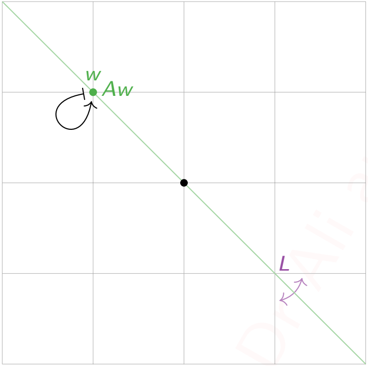

Does anyone see any eigenvectors (vectors that don't move off their line)?

w is an eigenvector with eigenvalue 1.

Eigenspaces

Geometry; example

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

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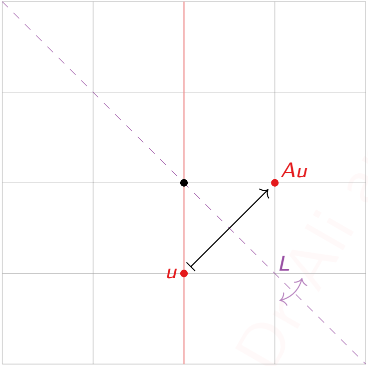

Does anyone see any eigenvectors (vectors that don't move off their line)?

u is *not* an eigenvector.

Eigenspaces

Geometry; example

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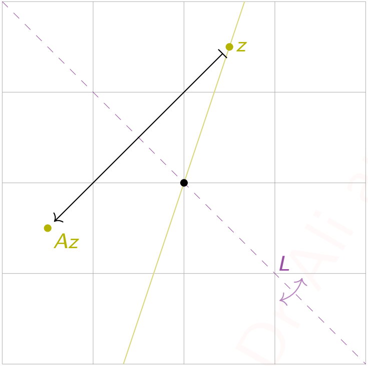

Does anyone see any eigenvectors (vectors that don't move off their line)?

Neither is z.

Eigenspaces

Geometry; example

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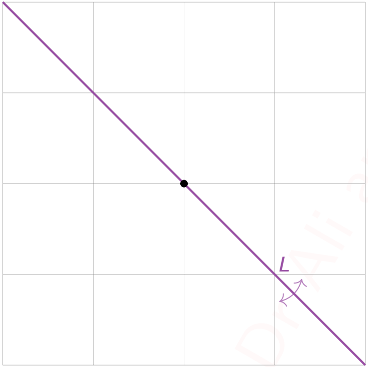


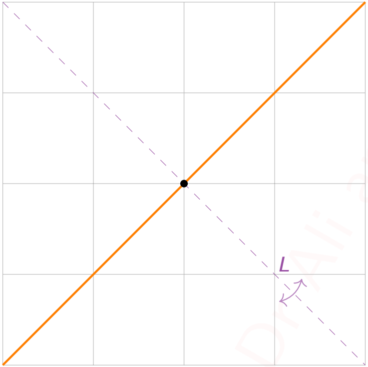
Does anyone see any eigenvectors (vectors that don't move off their line)?

The 1-eigenspace is L
(all the vectors x where $Ax = x$).

Eigenspaces
Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

The (-1)-eigenspace is the line $y = x$ (all the vectors x where $Ax = -x$).

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\mathrm{Nul}(A - \lambda I) \neq \{0\}$.
2. In this case, finding a basis for the λ-eigenspace of A means finding a basis for $\mathrm{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.
3. The eigenvectors with eigenvalue λ are the nonzero elements of $\mathrm{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

The Eigenvalues of a Triangular Matrix are the Diagonal Entries

We've seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix *is not a row reduction problem!* We'll see how to do it in general next time. For now:

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Why? $\text{Nul}(A - \lambda I) \neq \{0\}$ if and only if $A - \lambda I$ is not invertible, if and only if $\det(A - \lambda I) = 0$.

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \lambda I_4 = \begin{pmatrix} 3-\lambda & 4 & 1 & 2 \\ 0 & -1-\lambda & -2 & 7 \\ 0 & 0 & 8-\lambda & 12 \\ 0 & 0 & 0 & -3-\lambda \end{pmatrix}.$$

The determinant is $(3-\lambda)(-1-\lambda)(8-\lambda)(-3-\lambda)$, which is zero exactly when $\lambda = 3, -1, 8$, or -3.

Exercise

Find the eigenvalues of the matrices in Exercises 17 and 18.

17. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$ **18.** $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Why?

$$\begin{aligned} 0 \text{ is an eigenvalue of } A &\iff Ax = 0x \text{ has a nontrivial solution} \\ &\iff Ax = 0 \text{ has a nontrivial solution} \\ &\iff A \text{ is not invertible.} \end{aligned}$$

invertible matrix theorem

Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Why? If $k = 2$, this says v_2 can't lie on the line through v_1.

But the line through v_1 is contained in the λ_1-eigenspace, and v_2 does not have eigenvalue λ_1.

Consequence: An $n \times n$ matrix has at most n distinct eigenvalues.

Difference Equations

Preview

Let A be an $n \times n$ matrix. Suppose we want to solve $Av_n = v_{n+1}$ for all n. In other words, we want vectors $v_0, v_1, v_2, \ldots$, such that

$$Av_0 = v_1 \qquad Av_1 = v_2 \qquad Av_2 = v_3 \qquad \ldots$$

We saw before that $v_n = A^n v_0$. But it is inefficient to multiply by A each time.

If v_0 is an *eigenvector* with eigenvalue λ, then

$$v_1 = Av_0 = \lambda v_0 \qquad v_2 = Av_1 = \lambda v_1 = \lambda^2 v_0 \qquad v_3 = Av_2 = \lambda v_2 = \lambda^3 v_0.$$

In general, $v_n = \lambda^n v_0$. This is *much easier* to compute.

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \qquad Av_0 = 2v_0.$$

So if you start with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit, then the population will exactly double every year. In year n, you will have $2^n \cdot 16$ baby rabbits, $2^n \cdot 4$ first-year rabbits, and 2^n second-year rabbits.

Difference Equations

Preview

Let A be an $n \times n$ matrix. Suppose we want to solve $Av_n = v_{n+1}$ for all n. In other words, we want vectors $v_0, v_1, v_2, \ldots$, such that

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Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \qquad Av_0 = 2v_0.$$

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Section 5.2

The Characteristic Equation

The Invertible Matrix Theorem

Addenda

We have a couple of new ways of saying "A is invertible" now:

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

19\. The determinant of A is *not* equal to zero.

20\. The number 0 is *not* an eigenvalue of A.

The Characteristic Polynomial

Let A be a square matrix.

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff Ax = \lambda x \text{ has a nontrivial solution} \\ &\iff (A - \lambda I)x = 0 \text{ has a nontrivial solution} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff \det(A - \lambda I) = 0. \end{aligned}$$

This gives us a way to compute the eigenvalues of A.

Definition

Let A be a square matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of A is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

Important

The eigenvalues of A are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

The Characteristic Polynomial

Example

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) = \det(A - \lambda I) &= \det \left[\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] = \det \begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \\ &= (5-\lambda)(1-\lambda) - 2 \cdot 2 \\ &= \lambda^2 - 6\lambda + 1. \end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial, which we can find using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}.$$

The Characteristic Polynomial

Example

Question: What is the characteristic polynomial of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}?$$

Answer:

$$f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

What do you notice about $f(\lambda)$?

- The constant term is $\det(A)$, which is zero if and only if $\lambda = 0$ is a root.
- The linear term $-(a+d)$ is the negative of the sum of the diagonal entries of A.

Definition

The **trace** of a square matrix A is $\mathrm{Tr}(A)$ = sum of the diagonal entries of A.

Shortcut

The characteristic polynomial of a 2×2 matrix A is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A).$$

The Characteristic Polynomial

Example

Question: What are the eigenvalues of the rabbit population matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{pmatrix} \\ &= 8\left(\frac{1}{4} - 0 \cdot -\lambda\right) - \lambda\left(\lambda^2 - 6 \cdot \frac{1}{2}\right) \\ &= -\lambda^3 + 3\lambda + 2. \end{aligned}$$

We know from before that one eigenvalue is $\lambda = 2$: indeed, $f(2) = -8 + 6 + 2 = 0$. Doing polynomial long division, we get:

$$\frac{-\lambda^3 + 3\lambda + 2}{\lambda - 2} = -\lambda^2 - 2\lambda - 1 = -(\lambda + 1)^2.$$

Hence $\lambda = -1$ is also an eigenvalue.

Algebraic Multiplicity

Definition

The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

This is not a very interesting notion *yet*. It will become interesting when we also define *geometric* multiplicity later.

Example

In the rabbit population matrix, $f(\lambda) = -(\lambda - 2)(\lambda + 1)^2$, so the algebraic multiplicity of the eigenvalue 2 is 1, and the algebraic multiplicity of the eigenvalue -1 is 2.

Example

In the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, $f(\lambda) = (\lambda - (3 - 2\sqrt{2}))(\lambda - (3 + 2\sqrt{2}))$, so the algebraic multiplicity of $3 + 2\sqrt{2}$ is 1, and the algebraic multiplicity of $3 - 2\sqrt{2}$ is 1.

Fact: If A is an $n \times n$ matrix, the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I)$$

turns out to be a polynomial of degree n, and its roots are the eigenvalues of A:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

Poll

If you count the eigenvalues of A, with their algebraic multiplicities, you will get:

A. Always n.

B. Always at most n, but sometimes less.

C. Always at least n, but sometimes more.

D. None of the above.

The answer depends on whether you allow *complex* eigenvalues. If you only allow real eigenvalues, the answer is B. Otherwise it is A, because any degree-n polynomial has exactly n *complex* roots, counted with multiplicity. Stay tuned.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix C such that

$$A = CBC^{-1}.$$

What does this mean? Say the columns of C are $v_1, v_2, \ldots, v_n$. These form a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ for $\mathbf{R}^n$ because C is invertible. If $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ then

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \implies x = c_1 v_1 + c_2 v_2 + c_n v_n = C[x]_{\mathcal{B}}.$$

Since $x = C[x]_{\mathcal{B}}$ we have $[x]_{\mathcal{B}} = C^{-1}x$.

$$B[x]_{\mathcal{B}} = [y]_{\mathcal{B}} \implies Ax = CBC^{-1}x = CB[x]_{\mathcal{B}} = C[y]_{\mathcal{B}} = y.$$

> A acts on the standard coordinates of x in the same way that B acts on the $\mathcal{B}$-coordinates of x: $B[x]_{\mathcal{B}} = [Ax]_{\mathcal{B}}$.

Similar Matrices Have the Same Characteristic Polynomial

Fact: If A and B are similar, then they have the same characteristic polynomial.

Why? Suppose $A = CBC^{-1}$.

$$\begin{aligned} A - \lambda I &= CBC^{-1} - \lambda I \\ &= CBC^{-1} - C(\lambda I)C^{-1} \\ &= C(B - \lambda I)C^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \det(A - \lambda I) &= \det\big(C(B - \lambda I)C^{-1}\big) \\ &= \det(C)\det(B - \lambda I)\det(C^{-1}) \\ &= \det(B - \lambda I), \end{aligned}$$

because $\det(C^{-1}) = \det(C)^{-1}$.

> Conseqence: similar matrices have the same eigenvalues!
> (But different eigenvectors in general.)

Warning

1. Matrices with the same eigenvalues need not be similar. For instance,

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

   both only have the eigenvalue 2, but they are not similar.
2. Similarity has nothing to do with row equivalence. For instance,

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   are row equivalent, but they have different eigenvalues.

Section 5.3

Diagonalization

Many real-word linear algebra problems have the form:

$$v_1 = Av_0, \quad v_2 = Av_1 = A^2 v_0, \quad v_3 = Av_2 = A^3 v_0, \quad \ldots \quad v_n = Av_{n-1} = A^n v_0.$$

This is called a **difference equation**.

Our toy example about rabbit populations had this form.

The question is, what happens to v_n as $n \to \infty$?

- Taking powers of diagonal matrices is easy!
- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the nth powers of the diagonal entries of D:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix}, \quad \dots \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}.$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad D^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix},$$

$$\dots \quad D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{pmatrix}$$

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. Compute A^n.

In §5.2 lecture we saw that A is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1} \\ A^3 &= (PDP^{-1})(PD^2P^{-1}) = PD(P^{-1}P)D^2P^{-1} = PDID^2P^{-1} = PD^3P^{-1} \\ &\ \vdots \\ A^n &= PD^nP^{-1} \end{aligned}$$

Closed formula in terms of n:
easy to compute

Therefore

$$A^n = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2^{n+1} - 3^n & -2^{n+1} + 2 \cdot 3^n \\ 2^n - 3^n & -2^n + 2 \cdot 3^n \end{pmatrix}.$$

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{for } D \text{ diagonal.}$$

Important

If $A = PDP^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ then

$$A^k = PD^K P^{-1} = P \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{pmatrix} P^{-1}.$$

So diagonalizable matrices are easy to raise to any power.

Diagonalization

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In this case, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \dots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary

a theorem that follows easily from another theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The Corollary is true because eigenvectors with distinct eigenvalues are always linearly independent. We will see later that a diagonalizable matrix need not have n distinct eigenvalues though.

Diagonalization
Example

Problem: Diagonalize $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Therefore the eigenvalues are 2 and 3. Let's compute some eigenvectors:

$$(A - 2I)x = 0 \iff \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} x = 0 \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = 2y$, so $v_1 = \binom{2}{1}$ is an eigenvector with eigenvalue 2.

$$(A - 3I)x = 0 \iff \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} x = 0 \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = y$, so $v_2 = \binom{1}{1}$ is an eigenvector with eigenvalue 3.

The eigenvectors v_1, v_2 are linearly independent, so the Diagonalization Theorem says

$$A = PDP^{-1} \quad \text{for} \quad P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Diagonalization

Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2).$$

Therefore the eigenvalues are 1 and 2, with respective multiplicities 2 and 1. Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric vector form is

$$\begin{aligned} x &= y \\ y &= y \\ z &= \qquad z \end{aligned} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence a basis for the 1-eigenspace is

$$\mathcal{B}_1 = \{v_1, v_2\} \quad \text{where} \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Diagonalization

Another example, continued

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Now let's compute the 2-eigenspace:

$$(A - 2I)x = 0 \iff \begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = 3z, y = 2z$, so an eigenvector with eigenvalue 2 is

$$v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The eigenvectors v_1, v_2, v_3 are linearly independent: v_1, v_2 form a basis for the 1-eigenspace, and v_3 is not contained in the 1-eigenspace. Therefore the Diagonalization Theorem says

$$A = PDP^{-1} \quad \text{for} \quad P = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note: In this case, there are three linearly independent eigenvectors, but only two distinct eigenvalues.

Diagonalization

A non-diagonalizable matrix

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

This is an upper-triangular matrix, so the only eigenvalue is 1. Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = 0.$$

This is row reduced, but has only one free variable x; a basis for the 1-eigenspace is $\{\binom{1}{0}\}$. So *all eigenvectors* of A are multiples of $\binom{1}{0}$.

Conclusion: A has only one linearly independent eigenvector, so by the "only if" part of the diagonalization theorem, A is not diagonalizable.

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Matrix A is not diagonalizable: its only eigenvalue is 1, and its 1-eigenspace is spanned by $\binom{1}{0}$.

Similarly, matrix C is not diagonalizable.

Matrix B is diagonalizable because it is a 2×2 matrix with distinct eigenvalues.

Matrix D is already diagonal!

How to diagonalize a matrix A:

1. Find the eigenvalues of A using the characteristic polynomial.
2. For each eigenvalue λ of A, compute a basis $\mathcal{B}_\lambda$ for the λ-eigenspace.
3. If there are fewer than n total vectors in the union of all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i.

Diagonalization

Proof

Why is the Diagonalization Theorem true?

A diagonalizable implies A has n linearly independent eigenvectors: Suppose $A = PDP^{-1}$, where D is diagonal with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ be the columns of P. They are linearly independent because P is invertible. So $Pe_i = v_i$, hence $P^{-1}v_i = e_i$.

$$Av_i = PDP^{-1}v_i = PDe_i = P(\lambda_i e_i) = \lambda_i Pe_i = \lambda_i v_i.$$

Hence v_i is an eigenvector of A with eigenvalue λ_i. So the columns of P form n linearly independent eigenvectors of A, and the diagonal entries of D are the eigenvalues.

A has n linearly independent eigenvectors implies A is diagonalizable: Suppose A has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let P be the invertible matrix with columns $v_1, v_2, \ldots, v_n$. Let $D = P^{-1}AP$.

$$De_i = P^{-1}APe_i = P^{-1}Av_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1}v_i = \lambda_i e_i.$$

Hence D is diagonal, with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Solving $D = P^{-1}AP$ for A gives $A = PDP^{-1}$.

Non-Distinct Eigenvalues

Definition

Let λ be an eigenvalue of a square matrix A. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem

Let λ be an eigenvalue of a square matrix A. Then

$$1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda).$$

The proof is beyond the scope of this course.

Corollary

Let λ be an eigenvalue of a square matrix A. If the algebraic multiplicity of λ is 1, then the geometric multiplicity is also 1.

The Diagonalization Theorem (Alternate Form)

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and *the geometric multiplicity equals the algebraic multiplicity* of each eigenvalue.

Non-Distinct Eigenvalues

Examples

Example

If A has n distinct eigenvalues, then the algebraic multiplicity of each equals 1, hence so does the geometric multiplicity, and therefore A is diagonalizable.

For example, $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ has eigenvalues 2 and 3, so it is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -(\lambda - 1)^2(\lambda - 2).$$

The algebraic multiplicities of 1 and 2 are 2 and 1, respectively. They sum to 3.

We showed before that the geometric multiplicity of 1 is 2 (the 1-eigenspace has dimension 2). The eigenvalue 2 automatically has geometric multiplicity 1.

Hence the geometric multiplicities add up to 3, so A is diagonalizable.

Non-Distinct Eigenvalues

Another example

Example

The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $f(\lambda) = (\lambda - 1)^2$.

It has one eigenvalue 1 of algebraic multiplicity 2.

We showed before that the geometric multiplicity of 1 is 1 (the 1-eigenspace has dimension 1).

Since the geometric multiplicity is smaller than the algebraic multiplicity, the matrix is *not* diagonalizable.

Non-Distinct Eigenvalues
Example

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

This has eigenvalues 1 and 2, with algebraic multiplicities 2 and 1, respectively.

The geometric multiplicity of 2 is *automatically* 1.

Let's compute the geometric multiplicity of 1:

$$A - I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This has 1 free variable, so the geometric multiplicity of 1 is 1. This is less than the algebraic multiplicity, so the matrix is *not diagonalizable*.

Systems of Linear Differential Equations

First order linear homogenous differential equation

First order differential Equation

$$x' = kx, \quad k \text{ is } \quad \text{a} \quad \text{constant.}$$

Solution: $x(t) = x_0 e^{kt}$.
First order differential system

$$x' = 4x$$
$$y' = 9y$$

In Matrix form of above system can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- Matrix is diagonal
- System is uncoupled

First order linear system of differential equations

General first order linear system

$$x' = ax + by$$
$$y' = cx + dy$$

In Matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We can write the above system as

$$X' = AX,$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

- A is not a diagonal matrix.
- Can we diagonalize A?

Diagonalization

$\boxed{Aim:}$

To solve the system $X' = AX$.

$\boxed{Challenges}$

Matrix is not diagonal.

$\boxed{Possible\ Solution}$

Transform the matrix into a diagonal matrix i.e., diagonalize it.

HOW?

We want to transform

$$X' = AX \xrightarrow{to} Y' = DY.$$

Diagonalization

As $PDP^{-1} = A$, so we can write

$$X' = AX = PDP^{-1}X$$

Pre multiplying by P^{-1} we get

$$P^{-1}X' = DP^{-1}X$$

Since, P is a constant matrix, so

$$\left(P^{-1}X\right)' = D\left(P^{-1}X\right).$$

Put $\left(P^{-1}X\right) = Y$ to get

$$\boxed{Y' = DY}.$$

Uncoupling system of differential equations

Summary

Coupled system of differential equation

$$X' = AX$$

can be transformed (uncoupled) to

$$Y' = DY$$

by using the transformation

$$X = PY.$$

Example

Find a solution to the system

$$x' = x + 3y$$
$$y' = 2x + 2y$$

subject to initial conditions $x(0) = 0,\ y(0) = 5.$

Solution: In Matrix form, we can write it as

$$X' = AX,$$

where $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

We can uncoupled the system by using the transformation

$$X = PY.$$

Example

Eigenvalues:
Characteristic Equation:

$$\lambda^2 - 3\lambda - 4 = 0.$$

Eigenvalues are: $-1,\ 4$.
Corresponding eigenvectors are

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence,

$$P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

By using the transformation $X = PY$, we get

$$Y' = DY,$$

where $Y = \begin{bmatrix} u \\ v \end{bmatrix}$.

Example

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Above equations can be written as

$$u' = -u$$
$$v' = 4v.$$

Solving, we get

$$u = c_1 e^{-t}, \; v = c_2 e^{4t}.$$

As $X = PY$, so, we can write

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

$$x = 3u + v$$
$$y = -2u + v.$$

Substituting values of u and v, we get

$$x = 3c_1 e^{-t} + c_2 e^{4t}$$
$$y = -2c_1 e^{-t} + c_2 e^{4t}.$$

Example

Since, $x(0) = 0$ and $y(0) = 5$, we get

$$\begin{aligned} 0 &= 3c_1 + c_2 \\ 5 &= -2c_1 + c_2. \end{aligned}$$

Solving, above system we get
$c_1 = -1, \; c_2 = 3.$
In matrix form we can the solution as

$$X = -x_1 e^{-t} + 3x_2 e^{4t}$$

where $x_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the eigenvectors corresponding to eigenvalues -1 and 4 respectively.

Example

Find a solution to the system

$$r'(t) = w(t) - 12$$
$$w'(t) = -r(t) + 10$$

Solution:

$\boxed{\textit{Issue}:}$

Presence of -12 and 10.

$\boxed{\textit{How to resolve it}:}$

Put $w(t) - 12 = y(t)$ and $-r(t) + 10 = x(t)$, we get

$$-x'(t) = y(t)$$
$$y'(t) = x(t).$$

In Matrix form, we can write it as

$$X' = AX,$$

where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

By using the substitution $X = PY$ we get

$$Y' = DY$$

where $P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$, $D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$.

So, solution is

$$u = c_1 e^{-it}$$
$$v = c_2 e^{it}$$

By using the relation $X = PY$, we get

$$x = c_1 e^{-it} + c_2 e^{it}$$
$$y = c_1 i e^{-it} - i c_2 e^{it}.$$

Solution of the system is

$$r(t) = 10 - c_1 e^{-it} - c_2 e^{it}$$
$$w(t) = 12 + c_1 i e^{-it} - i c_2 e^{it}$$

- In case of single linear differential equation, we have

$$x' = kx, \; k \text{ is a constant.}$$

Solution of the differential equation is

$$x = ce^{kt}.$$

- In case of system of coupled differential equations, we have

$$X' = AX, \; A \text{ is a constant matrix.}$$

Solution of the linear differential system should be

$$X = c\, e^{At}.$$

Exponential of a Matrix

Compute e^{Dt} where $D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

Since, $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. so,

$$e^{Dt} = I + Dt + \ldots = \sum_{n=0}^{\infty} \frac{D^n t^n}{n!}$$

$$e^{Dt} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 4^n t^n & 0 \\ 0 & t^n \end{bmatrix}$$

$$e^{Dt} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{4^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1^n}{n!} \end{bmatrix}$$

$$e^{Dt} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^t \end{bmatrix}$$

Example

Compute e^{At} where $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

For given matrix, we have

$$P = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

$$e^{At} = \sum_{n=0}^{\infty} \frac{PD^n P^{-1}}{n!}$$

$$e^{At} = P \sum_{n=0}^{\infty} \frac{D^n t^n}{n!} P^{-1}$$

$$e^{At} = Pe^{Dt} P^{-1}$$

$$e^{At} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{4t} \end{bmatrix} \left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \right)^{-1}.$$

Lecture 19

Linear Recurrence Relations

Linear Recurrence Relation

Let $(x_n) = (x_0, x_1, x_2, ...)$ be a sequence of numbers that is defined as follows:

1. $x_0 = a_0, x_1 = a_1, ..., x_{k-l} = a_{k-1}$, where $a_0, a_1, ..., a_{k-l}$ are scalars.

2. For all $n \geq k, x_n = c_1 x_{n-l} + c_2 x_{n-2} + \ + c_k x_{n-k}$ where $c_1, c_2, .., c_k$ are scalars.

If $c_k \neq 0$, the equation in (2) is called a linear recurrence relation of order k. The equations in (1) are referred to as the initial conditions of the

recurrence.

Examples

- $x_{n+2} = x_{n+1} + x_n, \quad x_0 = 1, ; x_1 = 1.$
- $x_{n+1} = 2x_n, \;\; x_0 = 3.$

Linear Recurrence in Matrix Form

I am going to explain it using an example of second order linear recurrence relation
Consider the following linear recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n, \quad x_1 = c_1, \; x_0 = c_0,$$

where c_0 and c_1 are known constants.
We can write it as

$$\begin{aligned} x_{n+2} &= a\, x_{n+1} + b\, x_n \\ x_{n+1} &= x_{n+1}. \end{aligned}$$

In Matrix form, we can write

$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$$

$$\boxed{X_{n+1} = A\, X_n}, \; \forall n \geq 0.$$

where $\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$.

Linear Recurrence in Matrix Form

For $n = 0$, we have

$$X_1 = AX_0,$$

where $X_0 = \begin{bmatrix} c_1 \\ c_0 \end{bmatrix}$

For $n = 1$, we can write

$$X_2 = AX_1 = A(AX_0) = A^2 X_0.$$

$n = 2$, gives us

$$X_3 = AX_2 = A(A^2 X_0) = A^3 X_0.$$

Continuing in the same manner, we have

$$X_{n+1} = A^{n+1} X_0.$$

Examples

Suppose each "Gibonacci" number G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. If $G_0 = 0$ and $G_1 = 1$. Find the *kth* term of the sequence only depending upon k.

$\boxed{Aim:}$

We want to find the general term of the sequence.

$\boxed{Steps:}$

- Matrix Form
- Eigenvalues and Eigenvectors
- Diagonalize

Examples

$$G_{k+2} = \frac{1}{2} G_{k+1} + \frac{1}{2} G_k$$
$$G_{k+1} = G_{k+1}.$$

In Matrix Form

$$\boldsymbol{G}_{k+1} = A \boldsymbol{G}_k, \quad \forall k \geq 0.$$

where $\boldsymbol{G}_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$, $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{G}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\boldsymbol{G}_k = A^k \boldsymbol{G}_0, \quad \forall k \geq 0.$$

Eigenvalues Characteristic Equation

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0.$$

Eigenvalues are: $1, -\frac{1}{2}$.

Eigenvectors $\lambda = 1$

$$(A - 1I)X = 0$$

Augmented matrix

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Eigenvector:All non-zero multiples of $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Eigenvalues $\lambda = -\frac{1}{2}$

$$(A + \frac{1}{2}I)X = 0$$

Augmented matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Eigenvector:All non-zero multiples of $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

As $A^k = PD^kP^{-1}$, so we need to calculate P^{-1}.

$$P^{-1} = \frac{adjP}{detP} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

So,

$$A^k = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{(-1)^k}{2^k} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}.$$

Simplification gives us

$$A^k = \frac{1}{3} \begin{bmatrix} \frac{(-1)^k}{2^k} + 2 & 1 - \frac{(-1)^k}{2^k} \\ 2 - \frac{2(-1)^k}{2^k} & \frac{2(-1)^k}{2^k} + 1 \end{bmatrix}$$

$$\boldsymbol{G}_k = A^k \boldsymbol{G}_0$$

$$\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \dfrac{(-1)^k}{2^k} + 2 \\ 2 - \dfrac{2(-1)^k}{2^k} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$G_k = \frac{2}{3} - \frac{2}{3}(\frac{-1}{2})^k.$$

Theorem

Let $x_n = ax_{n-1} + bx_{n-2}$ be a recurrence relation. Let λ_1 and λ_1 be the eigenvalues of the associated characteristic equation $\lambda^2 - a\lambda - b = 0$.

1. If $\lambda_1 \neq \lambda_2$, then
$$x_n = c_1\lambda_1^n + c_2\lambda_2^n$$
for some scalars c_1 and c_2.
2. If $\lambda_1 = \lambda_2 = \lambda$, then
$$x_n = c_1\lambda_1^n + c_2 n\lambda_2^n$$
for some scalars c_1 and c_2.

Example

Suppose each "Gibonacci" number G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. If $G_0 = 0$ and $G_1 = 1$. Find the *kth* term of the sequence only depending upon k.

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$$
$$G_{k+1} = G_{k+1}.$$

Characteristics equation

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0.$$

Eigenvalues are: 1, $-\frac{1}{2}$. So,

$$G_k = c_1(1)^k + c_2(\frac{-1}{2})^k.$$

As $G_0 = 0,\ G_1 = 1$, so we have

$$0 = G_0 = c_1 + c_2$$
$$1 = G_1 = c_1 - c_2\frac{1}{2}.$$

Solving above system, we get

$$c_1 = \frac{2}{3},\ c_2 = -\frac{2}{3}.$$

Hence, G_k is

$$G_k = \frac{2}{3} - \frac{2}{3}(\frac{-1}{2})^k.$$

Example

Solve the following recurrence relation with the given initial conditions.

$$y_1 = 1,\ y_2 = 6,\ y_k = 4y_{k-1} - 4y_{k-2},\ \mathrm{k} \geq 3.$$

Characteristics equation $\lambda^2 - 4\lambda + 4 = 0$.
Solution of the quadratic equation is Eigenvalues: $\lambda_1 = 2,\ \lambda_1 = 2$. So,

$$y_k = c_1(2)^k + c_2 k 2^k.$$

As, $y_1 = 1$, so, $2c_1 + 2c_2 = 1$,
$y_2 = 6$, so, $4c_2 + 8c_2 = 6$.
Solution of above system is

$$c_1 = -\frac{1}{2},\ c_2 = 1.$$

Hence,

$$y_k = -\frac{1}{2}2^k + k2^k.$$

Practice Problems

1. Solve the recurrence relation with the given initial conditions.
   1.1 $a_0 = 4,\ a_1 = 1,\ a_n = a_{n-1} - a_{n-2}/4$, for $n \geq 2$.
   1.2 $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, subject to $a_0 = 2, a_1 = 2, a_2 = 4$, for $n \geq 3$.
2. Find the limiting values of y_k and z_k, $(k \to \infty)$ if

$$\begin{aligned} y_{k+1} &= .8y_k + .3z_k \quad y_0 = 0 \\ z_{k+1} &= .2y_k + .7z_k, \quad z_0 = 5. \end{aligned}$$

3. Suppose there is an epidemic in which every month half of those who are well become sick, and a quarter of those who are sick become dead. Find the steady state for the corresponding Markov process $\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$.