MT-1004
Linear Algebra

Fall 2023

Week # 1

National University of Computer and Emerging Sciences

September 3, 2023

General Rules:

- Attendance would be marked **P** in the start of lecture, **L** after ten minutes, and **A** after that.
- 100% attendance in mandatory but students with at least 80% attendance would be allowed to appear in final exam if they have genuine issues (verified by academics).
- No retake of missed Quiz or Assignment.
- Retakes of sessional would be on basis of decision of departmental committee.

Tentative Marks Distribution & Text Books

Assessment Type	Tentative number of Assessments	Weightages
Quiz	10-12	9%
Assignment	5-7	3%
Homeworks	16-18	8%
Sessional I	1	15%
Sessional II	1	15%
Final Exam	1	50%

* Absolute grading policy would be followed

Text Books

- Linear Algebra & Its Applications (4th Edition) by David C. Lay

Reference Books

- Linear Algebra a Modern Introduction (4th Edition) by David Poole
- Linear Algebra & Its Applications (4th Edition) by Gilbert Strang
- Elementary Linear Algebra (Applications Version) by Howard Anton, Chris Rorres

Linear. Algebra.

What is Linear Algebra?

Linear

- having to do with lines/planes/etc.
- For example, $x + y + 3z = 7$, not $\sin, \log, x^2$, etc.

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Algebra

- from al-jebr (Arabic), meaning reunion of broken parts
- 9^{th} century Abu Ja'far Muhammad ibn Muso al-Khwarizmi

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$$
\begin{array}{rcrcrcrcrcrcr}
3x_1 & + & 4x_2 & + & 10x_3 & + & 19x_4 & - & 2x_5 & - & 3x_6 & = & 141 \\
7x_1 & + & 2x_2 & - & 13x_3 & - & 7x_4 & + & 21x_5 & + & 8x_6 & = & 2567 \\
-x_1 & + & 9x_2 & + & \frac{3}{2}x_3 & + & x_4 & + & 14x_5 & + & 27x_6 & = & 26 \\
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Often, it's enough to know some information about the set of solutions without having to solve the equations at all!

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- Almost solve the equation $Ax = b$
  - Find best-fit solutions to systems of linear equations that have no actual solution using least squares approximations.

What to Expect This Semester

Your previous math courses probably focused on how to do (sometimes rather involved) computations.

- Compute the derivative of $\sin(\log x)\cos(e^x)$.
- Compute $\int_0^1 (1 - \cos(x))\,dx$.

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So what are we going to do?

- About half the material focuses on how to do linear algebra computations—that is still important.
- The other half is on *conceptual* understanding of linear algebra. This is much more subtle: it's about figuring out *what question* to ask the computer, or whether you actually need to do any computations at all.

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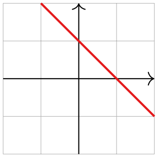
  $\rightsquigarrow$ a line in the plane: $y = 1 - x$



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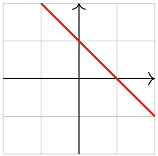

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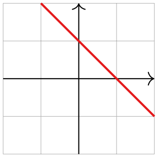

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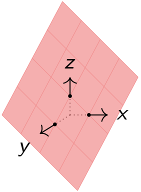

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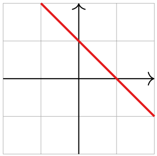

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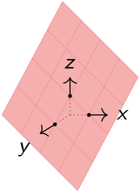

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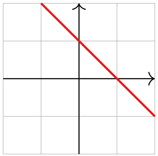

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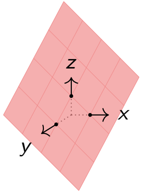

- $x + y + z + w = 1$

  $\rightsquigarrow$ a "3-plane" in "4-space"...

[not pictured here]

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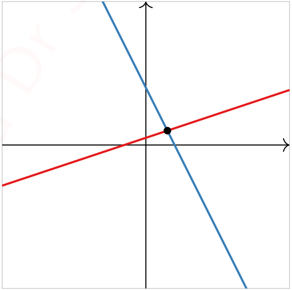

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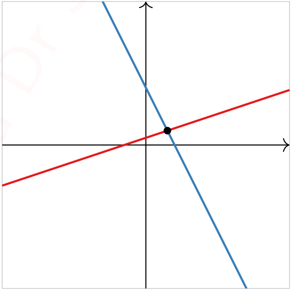

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In general it's an intersection of lines, planes, etc.

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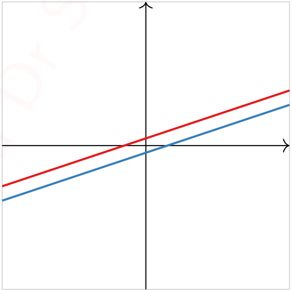

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$$\color{red}{x - 3y = -3}$$
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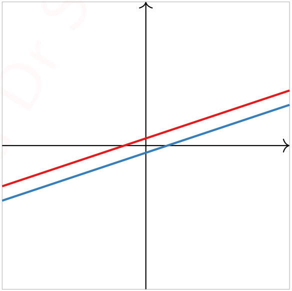

A system of equations with no solutions is called **inconsistent**.

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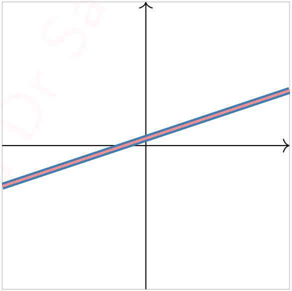

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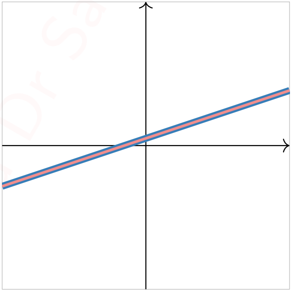

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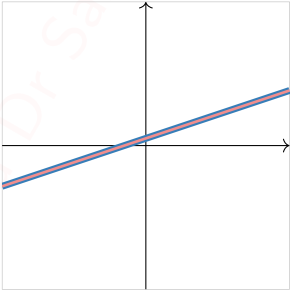

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In other words, they are *equivalent* (systems of) equations.

What about in three variables?

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In how many different ways can three planes intersect in space?

A. One
B. Two
C. Three
D. Four
E. Five
F. Six
G. Seven
H. Eight

Solving Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

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What is a *systematic* way to solve a system of equations?

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What strategies do you know?

- Substitution
- Elimination

Both are perfectly valid, but only elimination scales well to large numbers of equations.

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Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. **(scale)**
- Add a multiple of one equation to another. **(replacement)**
- Swap two equations. **(swap)**

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$$\xrightarrow{\text{Add first to third}} \quad \begin{aligned} -3x - 6y - 9z &= -18 \\ 2x - 3y + 2z &= 14 \\ -5y - 10z &= -20 \end{aligned}$$

Now I've eliminated x from the last equation!

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...but there's a long way to go still. Can we make our lives easier?

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Matrix notation: write just the numbers, in a box, instead!

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- Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**

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This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- Multiply all entries in a row by a nonzero number. **(scale)**
- Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- Swap two rows. **(swap)**

Row Operations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

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Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{aligned} x \qquad\qquad &= A \\ y \qquad &= B \\ z &= C \end{aligned}$$

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Row Operations

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So we need to do row operations that make the start matrix look like the end one.

Row Operations

Example

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So we need to do row operations that make the start matrix look like the end one.

Strategy: fiddle with it so we only have ones and zeros.

Row Operations

Continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Row Operations

Continued

$$\left(\begin{array}{ccc|c} \boxed{\begin{matrix}1\end{matrix}} & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.

Row Operations
Continued

$$\left(\begin{array}{ccc|c} \color{red}{1} & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

Row Operations
Continued

$$\left(\begin{array}{ccc|c} \color{red}{1} & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

Row Operations
Continued

$$\left(\begin{array}{rrr|r} \color{red}{1} & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

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Row Operations

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We want these to be zero.

It would be nice if this were a 1.

Row Operations

Continued

$$\left(\begin{array}{ccc|c} \color{red}{1} & 2 & 3 & 6 \\ \boxed{2} & -3 & 2 & 14 \\ \boxed{3} & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & \boxed{2} & 3 & 6 \\ 0 & \color{green}{-7} & -4 & 2 \\ 0 & \boxed{-5} & -10 & -20 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that
would produce ugly fractions.

Row Operations
Continued

$$\left(\begin{array}{ccc|c} \color{red}{1} & 2 & 3 & 6 \\ \boxed{\begin{matrix}2\\3\end{matrix}} & \begin{matrix}-3\\1\end{matrix} & \begin{matrix}2\\-1\end{matrix} & \begin{matrix}14\\-2\end{matrix} \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & \boxed{2} & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & \boxed{-5} & -10 & -20 \end{array}\right)$$

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We could divide by -7, but that
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Dr Ali and Dr Sara

Row Operations

Continued

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \qquad R_3 = R_3 \div 10 \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$R_1 = R_1 + R_3 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad R_2 = R_2 - 2R_3 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

translates into

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3 \end{aligned}$$

Row Operations

Continued

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translates into
$$\begin{aligned} x \qquad\qquad &= 1 \\ y \qquad &= -2 \\ z &= 3 \end{aligned}$$

Row Operations
Continued

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We want these to be zero.

Let's make this a 1 first.

$$R_1 = R_1 + R_3 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad R_2 = R_2 - 2R_3 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

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translates into
$$\begin{aligned} x \qquad &= 1 \\ y \quad &= -2 \\ z &= 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \xrightarrow{\text{substitute solution}}$$

$$\begin{aligned} 1 + 2\cdot(-2) + 3\cdot 3 &= 6 \\ 2\cdot 1 - 3\cdot(-2) + 2\cdot 3 &= 14 \\ 3\cdot 1 + (-2) - 3 &= -2 \end{aligned} \quad ✔$$

Row Equivalence

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

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Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

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So the linear equations of row-equivalent matrices have the *same solution set*.

A Bad Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

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Let's try doing row operations:

First clear these by subtracting multiples of the first row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right)$$

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First clear these by subtracting multiples of the first row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ \boxed{3} & 4 & 5 \\ \boxed{4} & 5 & 9 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

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$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Let's try doing row operations:

First clear these by subtracting multiples of the first row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right)$$

Now clear this by subtracting the second row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

A Bad Example

Continued

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right) \xrightarrow{\text{translates into}} \begin{aligned} x + y &= 2 \\ y &= -1 \\ {\color{red}0} &{\color{red}=} {\color{red}2} \end{aligned}$$

A Bad Example

Continued

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right) \xrightarrow{\text{translates into}} \begin{aligned} x + y &= 2 \\ y &= -1 \\ {\color{red}0} &{\color{red}=} {\color{red}2} \end{aligned}$$

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

A Bad Example

Continued

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But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

A Bad Example

Continued

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But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

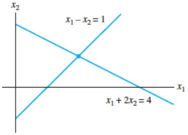
1.1 EXERCISES

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1. $\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$

2. $\begin{aligned} 3x_1 + 6x_2 &= -3 \\ 5x_1 + 7x_2 &= 10 \end{aligned}$

3. Find the point (x_1, x_2) that lies on the line $x_1 + 2x_2 = 4$ and on the line $x_1 - x_2 = 1$. See the figure.



4. Find the point of intersection of the lines $x_1 + 2x_2 = -13$ and $3x_1 - 2x_2 = 1$

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

5. $\begin{bmatrix} 1 & -4 & -3 & 0 & 7 \\ 0 & 1 & 4 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 4 & 1 & 2 \end{bmatrix}$

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

7. $\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 0 & 0 & -5 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 3 & 0 & -2 & -7 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$

Solve the systems in Exercises 11–14.

11. $\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$

12. $\begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ -2x_1 + x_2 + 7x_3 &= -1 \end{aligned}$

13. $\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$

14. $\begin{aligned} 2x_1 - 6x_3 &= -8 \\ x_2 + 2x_3 &= 3 \\ 3x_1 + 6x_2 - 2x_3 &= -4 \end{aligned}$

Determine if the systems in Exercises 15 and 16 are consistent. Do not completely solve the systems.

15. $\begin{aligned} x_1 - 6x_2 &= 5 \\ x_2 - 4x_3 + x_4 &= 0 \\ -x_1 + 6x_2 + x_3 + 5x_4 &= 3 \\ -x_2 + 5x_3 + 4x_4 &= 0 \end{aligned}$

16. $\begin{aligned} 2x_1 - 4x_4 &= -10 \\ 3x_2 + 3x_3 &= 0 \\ x_3 + 4x_4 &= -1 \\ -3x_1 + 2x_2 + 3x_3 + x_4 &= 5 \end{aligned}$

17. Do the three lines $2x_1 + 3x_2 = -1$, $6x_1 + 5x_2 = 0$, and $2x_1 - 5x_2 = 7$ have a common point of intersection? Explain.

18. Do the three planes $2x_1 + 4x_2 + 4x_3 = 4$, $x_2 - 2x_3 = -2$, and $2x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

In Exercises 19–22, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

19. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$

20. $\begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix}$

22. $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix}$

In Exercises 23 and 24, key statements from this section are approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text.

23. a. Every elementary row operation is reversible.

b. A 5×6 matrix has six rows.

c. The solution set of a linear system involving variables $x_1, \ldots, x_n$ is a list of numbers $(s_1, \ldots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

d. Two fundamental questions about a linear system involve existence and uniqueness.

24. a. Two matrices are row equivalent if they have the same number of rows.

b. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

c. Two equivalent linear systems can have different solution sets.

d. A consistent system of linear equations has one or more solutions.

25. Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:
$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

26. Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} 2x_1 + 4x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

27. Suppose a, b, c, and d are constants such that a is not zero and the system below is consistent for all possible values of f and g. What can you say about the numbers a, b, c, and d? Justify your answer.

$$\begin{aligned} ax_1 + bx_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

28. Construct three different augmented matrices for linear systems whose solution set is $x_1 = 3, x_2 = -2, x_3 = -1$.

In Exercises 29–32, find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first.

29. $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & -5 \\ 0 & -2 & 5 \end{bmatrix}$

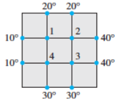
30. $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 10 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{bmatrix}$

31. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 4 & -12 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 15 \end{bmatrix}$

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let $T_1, \ldots, T_4$ denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.[3] For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



33. Write a system of four equations whose solution gives estimates for the temperatures $T_1, \ldots, T_4$.

34. Solve the system of equations from Exercise 33. [*Hint:* To speed up the calculations, interchange rows 1 and 4 before starting "replace" operations.]

[3] See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.

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3. Below a leading entry of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} \star &= \text{any number} \\ \boxed{\star} &= \text{any nonzero number} \end{aligned}$$

1.2: Row Echelon Form

Let's come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. What do we mean by "solved"?

> A matrix is in **row echelon form** if
>
> 1. All zero rows are at the bottom.
> 2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
> 3. Below a leading entry of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} \star &= \text{any number} \\ \boxed{\star} &= \text{any nonzero number} \end{aligned}$$

Definition

A **pivot** $\boxed{\star}$ is the first nonzero entry of a row of a matrix in row echelon form.

Reduced Row Echelon Form

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$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \star = \text{any number} \\ 1 = \text{pivot} \end{array}$$

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Question

Can every matrix be put into reduced row echelon form only using row operations?

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Picture:

$$\begin{pmatrix} \color{red}{1} & 0 & \star & 0 & \star \\ 0 & \color{red}{1} & \star & 0 & \star \\ 0 & 0 & 0 & \color{red}{1} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \star = \text{any number} \\ \color{red}{1} = \text{pivot} \end{array}$$

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Can every matrix be put into reduced row echelon form only using row operations?

Answer: Yes! Stay tuned.

Reduced Row Echelon Form

Continued

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Reduced Row Echelon Form

Continued

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$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

is in reduced row echelon form. It translates into

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3, \end{aligned}$$

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But what happens if there are fewer pivots than rows? . . . parametrized solution set (later).

Poll

Poll

Which of the following matrices are in reduced row echelon form?

A. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ E. $\begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$

F. $\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right)$

Reduced Row Echelon Form

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Maybe you can figure out why it's true!

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Example

$$\left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Row Reduction

Example

$$\left(\begin{array}{ccc|c} \boxed{0} & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Step 1a: Row swap to make this nonzero.

$$\left(\begin{array}{ccc|c} \boxed{2} & 4 & 6 & 12 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Step 1b: Scale to make this 1.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ \boxed{3} & 1 & -1 & -2 \end{array}\right)$$

Step 1c: Subtract a multiple of the first row to clear this.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

Optional: swap rows 2 and 3 to make Step 2b easier later on.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Row Reduction

Example, continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & \boxed{-5} & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Step 2a: This is already nonzero.
Step 2b: Scale to make this 1.

(There are no fractions because of the optional step before.)

$$\left(\begin{array}{ccc|c} 1 & \boxed{2} & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & \boxed{-7} & -4 & 2 \end{array}\right)$$

Step 2c: Add multiples of the second row to clear these.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

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Example, continued

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Note: Step 2 never messes up the first (nonzero) column of the matrix, because it looks like this:

$$\text{“Active” row} \longrightarrow \left(\begin{array}{ccc|c} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{array}\right)$$

Row Reduction

Example, continued

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{10} & 30 \end{array}\right)$$

Step 3a: This is already nonzero.
Step 3b: Scale to make this 1.

$$\left(\begin{array}{ccc|c} 1 & 0 & \boxed{-1} & -2 \\ 0 & 1 & \boxed{2} & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Step 3c: Add multiples of the third row to clear these.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Row Reduction

Example, continued

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{10} & 30 \end{array}\right)$$

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$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Note: Step 3 never messes up the columns to the left.

Success! The reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \Longrightarrow \quad \begin{cases} x & & & = 1 \\ & y & & = -2 \\ & & z & = 3 \end{cases}$$

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Step 4: profit?

Row Reduction

Another example

The linear system

$$\begin{aligned} 2x + 10y &= -1 \\ 3x + 15y &= 2 \end{aligned} \qquad \text{gives rise to the matrix} \qquad \left(\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array}\right).$$

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Let's row reduce it:

The row reduced matrix

$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

corresponds to the *inconsistent* system

$$\begin{aligned} x + 5y &= 0 \\ 0 &= 1. \end{aligned}$$

Inconsistent Matrices

Question

What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

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What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

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An augmented matrix corresponds to an inconsistent system of equations if and only if *the last* (i.e., the augmented) *column is a pivot column.*

Another Example

The linear system

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \quad \text{gives rise to the matrix} \quad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

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Another Example
Continued

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comes from a matrix in reduced row echelon form.

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Yes! Rewrite:

$$\begin{aligned} x &= 1 - 5z \\ y &= -1 - 2z \end{aligned}$$

For any value of z, there is exactly one value of x and y that makes the equations true. But z can be *anything we want*!

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So we have found the solution set: it is all values x, y, z where

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This is called the **parametric form** for the solution.

For instance, $(1, -1, 0)$ and $(-4, -3, 1)$ are solutions.

Free Variables

Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

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1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

In the previous example, z was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

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Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

Important

1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

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$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

In this matrix:

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What happened to x_2? What is it allowed to be? Anything! The general solution is

$$(x_1,\ x_2,\ x_3,\ x_4) = (2 - 3x_4,\ x_2,\ -1 - 4x_4,\ x_4)$$

for any values of x_2 and x_4.
For instance, $(2, 0, -1, 0)$ is a solution ($x_2 = x_4 = 0$), and $(5, 1, 3, -1)$ is a solution ($x_2 = 1,\ x_4 = -1$).

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The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

Poll

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Is it possible for a system of linear equations to have exactly two solutions?

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There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

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3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

1.2 EXERCISES

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

2. a. $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3. $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix}$

5. Describe the possible echelon forms of a nonzero 2×2 matrix. Use the symbols ■, $*$, and 0, as in the first part of Example 1.

6. Repeat Exercise 5 for a nonzero 3×2 matrix.

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & -3 & 4 & -6 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix}$

11. $\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \end{bmatrix}$

16. a. $\begin{bmatrix} ■ & * & * \\ 0 & ■ & * \\ 0 & 0 & ■ \end{bmatrix}$

b. $\begin{bmatrix} ■ & * & * & * & * \\ 0 & 0 & ■ & * & * \\ 0 & 0 & 0 & ■ & * \end{bmatrix}$

In Exercises 17 and 18, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

17. $\begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix}$

In Exercises 19 and 20, choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

19. $\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$ **20.** $\begin{aligned} x_1 - 3x_2 &= 1 \\ 2x_1 + hx_2 &= k \end{aligned}$

In Exercises 21 and 22, mark each statement True or False. Justify each answer.[4]

21. a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

b. The row reduction algorithm applies only to augmented matrices for a linear system.

c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

d. Finding a parametric description of the solution set of a linear system is the same as *solving* the system.

e. If one row in an echelon form of an augmented matrix is $[0 \;\; 0 \;\; 0 \;\; 5 \;\; 0]$, then the associated linear system is inconsistent.

22. a. The reduced echelon form of a matrix is unique.

b. If every column of an augmented matrix contains a pivot, then the corresponding system is consistent.

25. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

26. Suppose a 3×5 *coefficient* matrix for a system has three pivot columns. Is the system consistent? Why or why not?

27. Restate the last sentence in Theorem 2 using the concept of pivot columns: "If a linear system is consistent, then the solution is unique if and only if ________."

28. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

29. A system of linear equations with fewer equations than unknowns is sometimes called an *underdetermined system*. Can such a system have a unique solution? Explain.

30. Give an example of an inconsistent underdetermined system of two equations in three unknowns.

31. A system of linear equations with more equations than unknowns is sometimes called an *overdetermined system*. Can such a system be consistent? Illustrate your answer with a specific system of three equations in two unknowns.

32. Suppose an $n \times (n+1)$ matrix is row reduced to reduced echelon form. Approximately what fraction of the total number of operations (flops) is involved in the backward phase of the reduction when $n = 20$? when $n = 200$?

Suppose experimental data are represented by a set of points in the plane. An **interpolating polynomial** for the data is a polynomial whose graph passes through every point. In scientific work, such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.

WEB

33. Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data $(1, 6)$, $(2, 15)$, $(3, 28)$. That is, find a_0, a_1, and a_2 such that

$$\begin{aligned} a_0 + a_1(1) + a_2(1)^2 &= 6 \\ a_0 + a_1(2) + a_2(2)^2 &= 15 \\ a_0 + a_1(3) + a_2(3)^2 &= 28 \end{aligned}$$

34. [M] In a wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:

Velocity (100 ft/sec)	0	2	4	6	8	10
Force (100 lb)	0	2.90	14.8	39.6	74.3	119

Find an interpolating polynomial for these data and estimate the force on the projectile when the projectile is traveling at 750 ft/sec. Use $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$. What happens if you try to use a polynomial of degree less than 5? (Try a cubic polynomial, for instance.)[5]

[5] Exercises marked with the symbol [M] are designed to be worked with the aid of a "Matrix program" (a computer program, such as MATLAB®, Maple™, Mathematica®, MathCad®, or Derive™, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments or Hewlett-Packard).

With this knowledge you should be able to solve

Exercise 1.1 (1-32) & Exercise 1.2 (1-32)