STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 14

RANDOM VARIABLES

CONTINUOUS RANDOM VARIABLE

PREPARED BY

HAZBER SAMSON

FAST NUCES ISLAMABAD

PROBABILITY DENSITY FNS

THE PROBABILITY DENSITY FUNCTIONS

A random variable 'X' is said to be continuous if there is a function $f(x)$, called the probability density function, such that.

① $f(x) \geq 0$ for all x.

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

③ $P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx$.

Notice that for a continuous random variable X

① $P(x = a) = \int_{a}^{a} f(x)\,dx = 0$

② $P(a \leq x \leq b) = P(a < x \leq b) = P(a \leq x < b) = P(a < x < b)$

EXAMPLES OF PROBABILITY DENSITY FUNCTIONS

EXAMPLE ① let X be a random variable having the function (p.d.f).

$$f(x) = \begin{cases} cx, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find
(a) the value of c.
(b) $P(X < 0.5)$
(c) $P(X > 1)$
(d) $P(\frac{1}{2} < X < 3/2)$
(e) $P(X = 1)$

SOLUTION (a) WKT $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^2 cx\,dx = 1$$

$$c \left| \frac{x^2}{2} \right|_0^2 = 1 \;,\; 2c = 1 \;,\; \boxed{c = \frac{1}{2}}$$

(b) $P(X < 0.5) = \int_0^{0.5} \frac{1}{2}x\,dx = \left| \frac{x^2}{4} \right|_0^{0.5} = 0.0625$

(c) $P(X > 1) = \int_1^2 \frac{1}{2}x\,dx = \left| \frac{x^2}{4} \right|_1^2 = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$

(d) $P(\frac{1}{2} < X < \frac{3}{2})$

$$= \int_{\frac{1}{2}}^{3/2} \frac{1}{2}x\,dx = \left| \frac{x^2}{4} \right|_{1/2}^{3/2} = \frac{1}{2} = 0.5$$

(e) $P(X = 1) = 0$

EXAMPLE-2 X is the delay, in hours, of a flight from chicago, where

$$f(x) = 0.2 - 0.02x \quad , \quad 0 \le x \le 10$$

Find

(a) the probability that the delay will be less than four hours.

(b) the probability that the delay will be between two and six hours.

SOLUTION Here $f(x) = 0.2 - 0.02x \quad , \quad 0 \le x \le 10$

(a) $$P(0 \le X \le 4) = \int_0^4 (0.2 - 0.02x)\,dx$$

$$= \left| 0.2x - 0.02\frac{x^2}{2} \right|_0^4$$

$$= \left((0.2)(4) - (0.01)(4^2)\right) - 0$$

$$= 0.64$$

(b) $$P(2 \le X \le 6) = \int_2^6 (0.2 - 0.02x)\,dx$$

$$= \left| 0.2x - 0.01x^2 \right|_2^6$$

$$= \left[(0.2)(6) - (0.01)(36)\right] - \left[(0.2)(2) - (0.01)(4)\right]$$

$$= (0.84) - (0.36) = 0.48$$

EXAMPLE③ A continuous random variable has p.d.f

$$f(x) = \begin{cases} kx & , \ 0 \le x < 2 \\ k(4-x), & 2 \le x < 4 \\ 0 & , \ \text{otherwise}. \end{cases}$$

Find

(a) $P(X < 1)$ (b) $P(X > 3)$ (c) $P(0.5 \le X \le 2.5)$

First of all we shall find 'k'.

Wkt $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^4 f(x)\,dx = 1$$

$$\int_0^2 kx\,dx + \int_2^4 k(4-x)\,dx = 1$$

$$k\left|\frac{x^2}{2}\right|_0^2 + k\left|4x - \frac{x^2}{2}\right|_2^4 = 1$$

$$\frac{k}{2}(4) + k(8-6) = 1$$

$$2k + 2k = 1 \Rightarrow 4k = 1 \Rightarrow \boxed{k = \frac{1}{4}}$$

Now

(a) $P(x<1) = \int_0^1 \frac{1}{4}x\,dx$

$$= \frac{1}{4}\left|\frac{x^2}{2}\right|_0^1 = \frac{1}{8}(1-0) = \frac{1}{8}$$

(b) $P(x>3) = \int_3^4 \frac{1}{4}(4-x)\,dx$

$$= \frac{1}{4}\left|4x - \frac{x^2}{2}\right|_3^4 = \frac{1}{4}\left(8-12+\frac{9}{2}\right)$$

$$= \frac{1}{4}\left(-4+\frac{9}{2}\right) = \frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$$

(c) $P(0.5 \le x \le 2.5)$

$$= \int_{0.5}^{2.5} f(x)\,dx$$

$$= \int_{0.5}^{2} \frac{1}{4}x\,dx + \int_2^{2.5} \frac{1}{4}(4-x)\,dx$$

$$= \frac{1}{4}\left|\frac{x^2}{2}\right|_{0.5}^2 + \frac{1}{4}\left|4x - \frac{x^2}{2}\right|_2^{2.5}$$

$$= \frac{1}{8}(3.75) + \frac{1}{4}(0.375) = 0.6875.$$

EXAMPLE 4 The continuous random variable x has p.d.f, $f(x)$ where

$$f(x) = \begin{cases} k(x+2)^2 & , -2 \le x < 0 \\ 4k & , 0 \le x \le \frac{4}{3} \\ 0 & , \text{otherwise.} \end{cases}$$

(a) Find the value of k.

(b) $P(-1 \le x \le 1)$

(c) $P(x > 1)$

SOLUTION (a) WKT $\int^{+\infty} f(x)dx = 1$, $\int_0^{\frac{4}{3}} f(x)dx = 1$

$$\Rightarrow \int_{-2}^{0} k(x+2)^2 dx + \int_0^{\frac{4}{3}} 4k\,dx = 1$$

$$k \left| \frac{(x+2)^3}{3} \right|_{-2}^{0} + 4k \left| x \right|_0^{4/3} = 1$$

$$k \left[\frac{8}{3} - 0 \right] + 4k \left[\frac{4}{3} - 0 \right] = 1$$

$$\frac{8k}{3} + \frac{16k}{3} = 1$$

$$\frac{24k}{3} = 1 \Rightarrow \boxed{k = \frac{1}{8}}$$

(b) $P(-1 \le x \le 1) = \int_{-1}^{1} f(x)\,dx$

$$= \int_{-1}^{0} \frac{1}{8}(x+2)^2 dx + \int_0^1 4 \cdot \frac{1}{8}\,dx$$

$$= \frac{1}{8} \cdot \left| \frac{(x+2)^3}{3} \right|_{-1}^{0} + \frac{1}{2} \left| x \right|_0^1$$

$$= \frac{1}{24}(8-1) + \frac{1}{2}(1-0) = \frac{7}{24} + \frac{1}{2} = \frac{19}{24}$$

(c) $P(x > 1) = \int_1^{\frac{4}{3}} 4k\,dx$

$$= \frac{4}{8} \int_1^{\frac{4}{3}} 1\,dx = \frac{1}{2} \left| x \right|_1^{4/3} = \frac{1}{2}\left(\frac{4}{3} - 1\right) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

EXAMPLE (5) A continuous random variable 'x' has the Probability density function.

$$f(x) = \begin{cases} x/2 & , 0 \le x < 1 \\ (3-x)/4 & , 1 \le x < 2 \\ 1/4 & , 2 \le x < 3 \\ (4-x)/4 & , 3 \le x < 4 \\ 0 & , \text{otherwise.} \end{cases}$$

Compute.

(a) $P(x<2)$

(b) $P(x \geqslant 3)$

(c) $P(1<x<3)$

(d) $P(|x|<1.5)$

(e) $P(|x|>1.25)$

SOLUTION

(a) $P(x<2) = \int_0^2 f(x)dx = \int_0^1 \frac{x}{2} dx + \int_1^2 \frac{3-x}{4} dx$

$= \frac{1}{2} \left| \frac{x^2}{2} \right|_0^1 + \frac{1}{4} \left| 3x - \frac{x^2}{2} \right|_1^2$

$= \frac{1}{4}(1-0) + \frac{1}{4}\left[(6-2)-(3-\frac{1}{2})\right]$

$= \frac{1}{4} + \frac{1}{4}\left[4 - \frac{5}{2}\right]$

$= \frac{1}{4} + \frac{1}{4} \times \frac{3}{2} = \frac{1}{4} + \frac{3}{8} = \frac{5}{8}$

(b) $P(x \geqslant 3) = \int_3^4 \frac{4-x}{4} dx$

$= \frac{1}{4} \cdot \left| 4x - \frac{x^2}{2} \right|_3^4$

$= \frac{1}{4}\left[(16-8) - (12 - \frac{9}{2})\right]$

$= \frac{1}{4}\left[8 - \frac{15}{2}\right] = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

(c) $P(1<X<3)$

$$= \int_1^3 f(x)\,dx = \int_1^2 \frac{3-x}{4}\,dx + \int_2^3 \frac{1}{4}\,dx.$$

$$= \frac{1}{4}\left|3x-\frac{x^2}{2}\right|_1^2 + \frac{1}{4}\left|x\right|_2^3$$

$$= \frac{1}{4}\left[(6-2)-(3-\tfrac{1}{2})\right] + \frac{1}{4}\left[3-2\right]$$

$$= \frac{1}{4}\left[4-\frac{5}{2}\right] + \frac{1}{4}$$

$$= \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$$

(d) $P(|X|<1.5) = P(-1.5<X<1.5)$

$$= \int_{-1.5}^{1.5} f(x)\,dx = \int_{-1.5}^{0} 0\,dx + \int_0^1 \frac{x}{2}\,dx + \int_1^{1.5} \frac{3-x}{4}\,dx$$

$$= \frac{1}{4}\left|x^2\right|_0^1 + \frac{1}{4}\left|3x-\frac{x^2}{2}\right|_1^{1.5}$$

$$= \frac{1}{4}(1) + \frac{1}{4}(3.375-2.5)$$

$$= \frac{1}{4} + \frac{1}{4}(0.875) = 0.46875$$

(e) $P(|X|>1.25) = P(X<\cancel{-1.25}) \to 0 + P(X>1.25)$

$$= \int_{1.25}^{4} f(x)\,dx$$

$$= \int_{1.25}^{2}\left(\frac{3-x}{4}\right)dx + \int_2^3 \frac{1}{4}\,dx + \int_3^4\left(\frac{4-x}{4}\right)dx$$

$$= \frac{1}{4}\left|3x-\frac{x^2}{2}\right|_{1.25}^{2} + \frac{1}{4}\left|x\right|_2^3 + \frac{1}{4}\left|4x-\frac{x^2}{2}\right|_3^4$$

$$= \frac{1}{4}(4-2.96875) + \frac{1}{4}(1) + \frac{1}{4}(8-7.5)$$

$$= \frac{1}{4}\left[2.53125\right] = 0.6328125.$$

or $P(|X|>1.25) = 1 - P(|X|<1.25) = 0.6328125.$

DISTRIBUTION FUNCTION FOR CONTINUOUS RANDOM VARIABLE

The dist. function for a random variable 'x' is defined as $F(b) = P(x \le b)$

If 'x' is continuous RV with p.d.f $f(x)$, then.

$$F(b) = \int_{-\infty}^{b} f(x)\,dx$$

IMPORTANT RESULTS

① $f(x) = F'(x)$ ie $p.d.f = \frac{d}{dx}(CDF)$.

② $P(a \le x \le b) = F(b) - F(a)$

③ Median.

If 'm' is the median, then for $f(x)$ defined for $a \le x \le b$. $F(m) = 0.5$, solve and find 'm'.

④ Lower quartile

If 'q_1' is the lower quartile. then to find 'q_1' put

$$F(q_1) = 0.25$$

⑤ Upper quartile

If 'q_3' is the upper quartile, then to find 'q_3', put

$$F(q_3) = 0.75$$

⑥ In general

$$F(n\text{th Percentile}) = \frac{n}{100}$$

⑦ Interquartile Range $= q_3 - q_1$

Scanned with CamScanner

EXAMPLES

EXAMPLE ① 'X' is a continuous RV with p.d.f

$$f(x) = \begin{cases} \frac{1}{8}x, & 0 \le x \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find CDF

(b) $P(0.3 \le X \le 1.8)$

(c) Median 'm'.

(d) Interquartile Range

SOLUTION

(a) $$F(x) = \int_{-\infty}^{x} \frac{1}{8} x \, dx = \int_{0}^{x} \frac{1}{8} x$$

$$= \left| \frac{x^2}{16} \right|_0^x = \frac{x^2}{16}$$

$$\Rightarrow \quad F(x) = \begin{cases} 0, & x \le 0 \\ x^2/16, & 0 \le x \le 4 \\ 1, & x \ge 4 \end{cases}$$

(b) $P(0.3 \le X \le 1.8) = F(1.8) - F(0.3)$

$$= \frac{(1.8)^2}{16} - \frac{(0.3)^2}{16} = 0.005625$$

(c) Median $\quad F(m) = 0.5$

$$\frac{m^2}{16} = 0.5$$

$$m^2 = 8 \Rightarrow \boxed{m = 2.83}$$

Ignore -ve as $0 \le x \le 4$.

(d) Interquartile Range

$IQR = q_3 - q_1$

$F(q_1) = 0.25, \quad \frac{q_1^2}{16} = 0.25 \Rightarrow q_1 = 2$

$F(q_3) = 0.75, \quad \frac{q_3^2}{16} = 0.75 \Rightarrow q_3 = 3.464$

$\Rightarrow IQR = 3.464 - 2 = 1.5$

EXAMPLE ② A continuous random variable 'X' has the density function.

$$f(x) = \begin{cases} \theta e^{-\theta x} , & 0 \le x < \infty \\ 0 , & \text{otherwise.} \end{cases}$$

Find CDF.

SOLUTION $F(x) = \int_0^x \theta e^{-\theta x} dx.$

$$= \theta . \left| \frac{e^{-\theta x}}{-\theta} \right|_0^x$$

$$= - \left| e^{-\theta x} \right|_0^x$$

$$= - \left[e^{-\theta x} - 1 \right]$$

$$F(x) = 1 - e^{-\theta x}$$

So $F(x) = \begin{cases} 1 - e^{-\theta x} , & x > 0 \\ 0 , & x \le 0 \end{cases}$

EXAMPLE ③ 'X' is a continuous random variable with p.d.f $f(x)$ where

$$f(x) = \begin{cases} \frac{x}{3} , & 0 \le x \le 2 \\ -\frac{2x}{3} + 2 , & 2 \le x \le 3 \\ 0 , & \text{otherwise.} \end{cases}$$

Find the following

(a) CDF

(b) $P(1 \le X \le 2.5)$

(c) Median.

SOLUTION (a) CDF

For CDF we shall integrate two times Here.

Now $F_1(x) = \int_0^x \frac{x}{3} dx = \left| \frac{x^2}{6} \right|_0^x = \frac{x^2}{6}$, $0 \le x \le 2$

Now for $2 \le x \le 3$

$$F_2(x) = F_1(2) + \int_2^x \left(-\frac{2x}{3} + 2\right) dx$$

$$= \frac{4}{6} + \left| -\frac{x^2}{3} + 2x \right|_2^x$$

$$= \frac{2}{3} + \left\{ -\frac{x^2}{3} + 2x + \frac{4}{3} - 4 \right\}$$

$$= \frac{2}{3} - \frac{x^2}{3} + 2x - \frac{8}{3}$$

$$F_2(x) = -\frac{x^2}{3} + 2x - 2$$

So $F(x) = \begin{cases} 0 , & x < 0 \\ \frac{x^2}{6} , & 0 \le x \le 2 \\ -\frac{x^2}{3} + 2x - 2 , & 2 \le x \le 3 \\ 1 , & x \ge 3 \end{cases}$

(b) $P(1 \le x \le 2.5) = F(2.5) - F(1)$

$$= \left\{ -\frac{(2.5)^2}{3} + 2(2.5) - 2 \right\} - \left\{ \frac{1}{6} \right\} = \frac{11}{12} - \frac{1}{6} = 0.75$$

(c) For Median $F(m) = 0.5$

$\frac{m^2}{6} = 0.5$ (As Median must be < 2)

$m^2 = 3$

$m = \pm 1.73$

Reject -ve value as $0 \le x \le 3$

So $\boxed{m = 1.73}$

EXAMPLE 4 A continuous RV 'X' has the distribution function as

$$F(x) = \begin{cases} 0 & , \ x \le 0 \\ x^3/27 & , \ 0 < x < 3 \\ 1 & , \ x \geq 3 \end{cases}$$

Find

(a) p.d.f of X

(b) $P(X>2)$ using p.d.f and C.D.F.

SOLUTION

(a) $f(x) = \frac{d}{dx} F(x)$

$$\Rightarrow \quad f(x) = \begin{cases} x^2/9 & , \ 0 \leq x \leq 3 \\ 0 & , \ \text{otherwise.} \end{cases}$$

(b) $P(X>2)$

using p.d.f

$$P(X>2) = \int_2^3 \frac{x^2}{9} dx = \frac{1}{27} \Big| x^3 \Big|_2^3$$

$$= \frac{1}{27} (27-8) = \frac{19}{27}$$

using c.d.f

$$P(X>2) = P(2<X<3)$$

$$= F(3) - F(2)$$

$$= \frac{(3)^3}{27} - \frac{(2)^3}{27}$$

$$= \frac{27}{27} - \frac{8}{27}$$

$$= \frac{19}{27}$$

EXAMPLE (5) The distribution function for a random variable 'x' is given by

$$F(x) = \begin{cases} 0 & , \ x<0 \\ x/4 & , \ 0 \le x < 1 \\ x/2 & , \ 1 \le x < 2 \\ (6x-x^2)/4 & , \ 2 \le x < 3 \\ 1 & , \ x \geq 3 \end{cases}$$

Find

(a) P.d.f of x.

(b) $P(1 \le x \le 2)$ using P.d.f and C.D.F.

SOLUTION

(a) $f(x) = \frac{d}{dx} F(x)$

$$\text{so } f(x) = \begin{cases} 1/4 & , \ 0 \le x < 1 \\ 1/2 & , \ 1 \le x < 2 \\ (3-x)/2 & , \ 2 \le x < 3 \\ 0 & , \ \text{otherwise} \end{cases}$$

(b) $P(1 \le x \le 2)$

using P.d.f

$$P(1 \le x \le 2) = \int_1^2 \frac{1}{2} dx$$

$$= \frac{1}{2} |x|_1^2 = \frac{1}{2}(2-1) = \frac{1}{2}$$

using CDF

$$P(1 \le x \le 2) = F(2) - F(1)$$

$$= \frac{2}{2} - \frac{1}{2}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$