Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvectors and Eigenvalues

Eigenvectors and Eigenvalues

Let $T : V \to V$ be a linear transformation. Suppose for some scalar λ and $0 \neq v \in V$,

$$T(v) = \lambda v$$

Then v is an eigenvector for T and λ is its eigenvalue.

Definition

Let A be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of A is a nonzero vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$.

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Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

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$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

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Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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Multiply:

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Hence v is an eigenvector of A, with eigenvalue $\lambda = 4$.

Poll

Poll

Which of the vectors

$$\text{A. } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{B. } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{C. } \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{D. } \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{E. } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

What are the eigenvalues?

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$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 2

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$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$ not an eigenvector

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not an eigenvector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

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We know how to answer that!

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We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric form: $x = -4y$; parametric vector form: $\begin{pmatrix} x \\ y \end{pmatrix} = y\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Does there exist an eigenvector with eigenvalue $\lambda = 3$? Yes! Any nonzero multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Check:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 1 \end{pmatrix}. \quad ✓$$

Eigenspaces

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\lambda\text{-eigenspace} = \left\{ v \text{ in } \mathbf{R}^n \mid Av = \lambda v \right\}$$

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How do you find a basis for the λ-eigenspace?

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How do you find a basis for the λ-eigenspace? Parametric vector form!

Eigenspaces

Example

Find a basis for the 2-eigenspace of

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

Eigenspaces

Example

Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric form}} x = \frac{1}{2}y - 3z$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Eigenspaces

Picture

A basis for the 2-eigenspace of $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$ is $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

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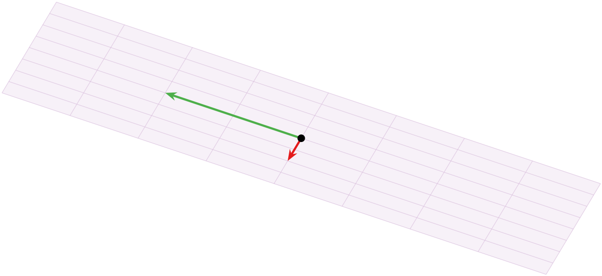

Eigenspaces

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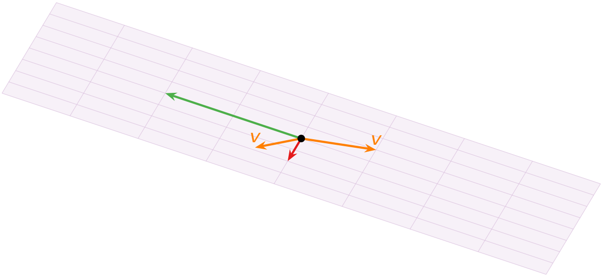

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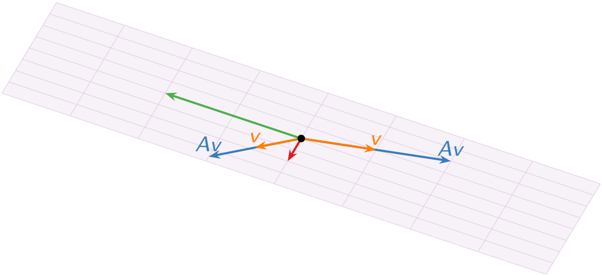

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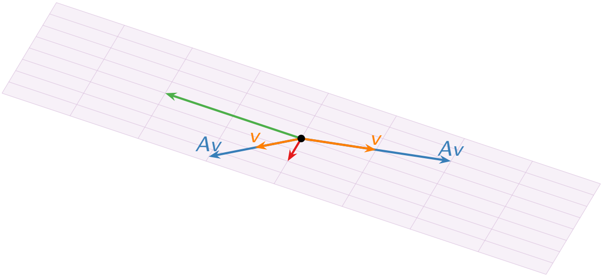

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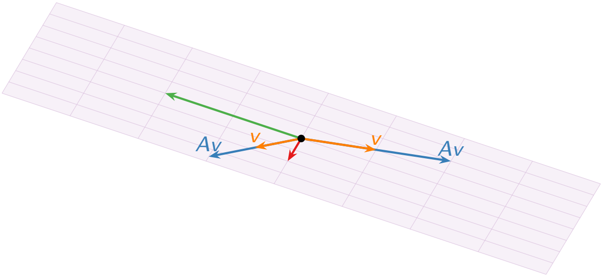

For any v in the 2-eigenspace, $Av = 2v$ by definition. So A acts by *scaling by* 2 on its 2-eigenspace. This is how eigenvalues and eigenvectors make matrices easier to understand.

Eigenvectors, geometrically

An eigenvector of a matrix A is a nonzero vector v such that:

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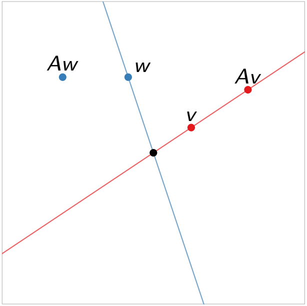

v is an eigenvector

w is not an eigenvector

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

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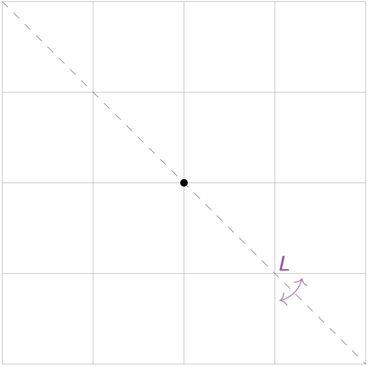

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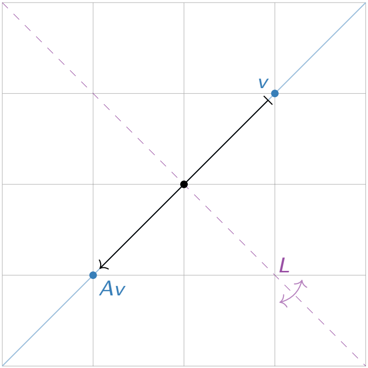

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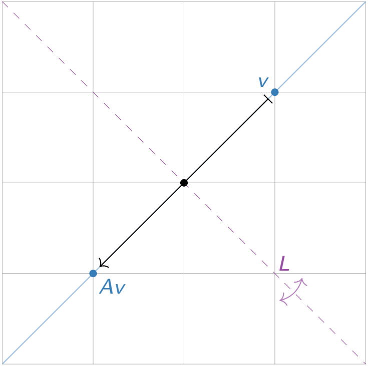

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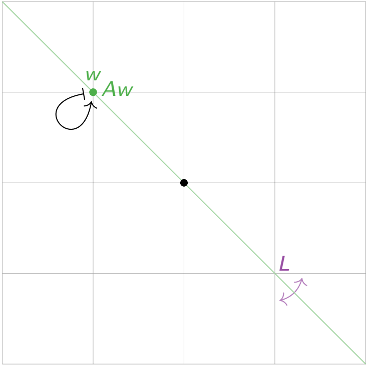

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w is an eigenvector with eigenvalue _.

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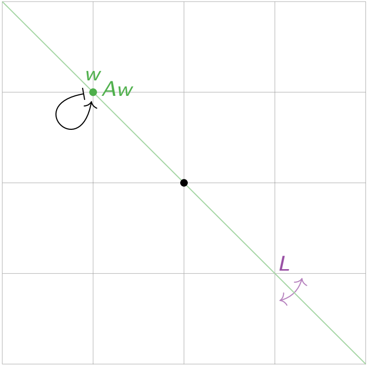

Does anyone see any eigenvectors (vectors that don't move off their line)?

w is an eigenvector with eigenvalue 1.

Eigenspaces

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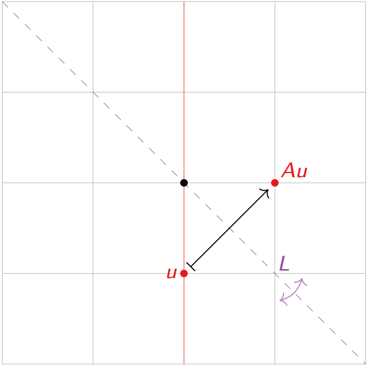

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u is *not* an eigenvector.

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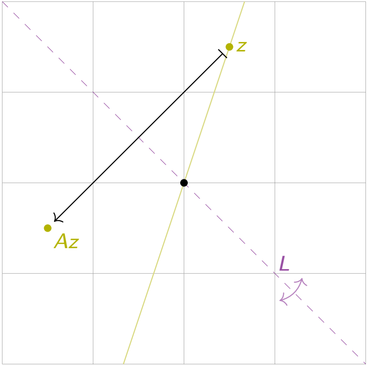

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Neither is z.

Eigenspaces

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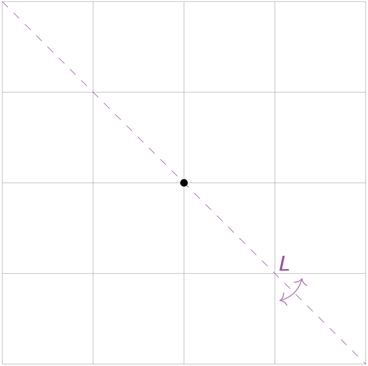

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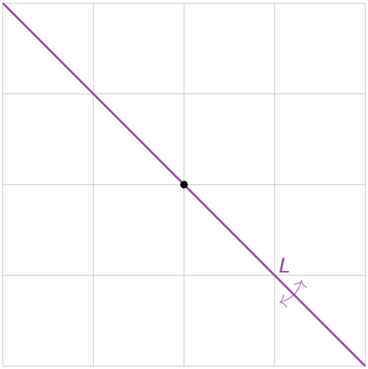

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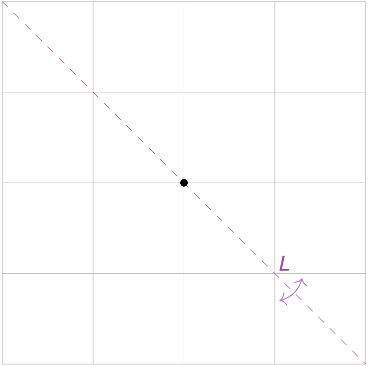

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The (-1)-eigenspace is ____________

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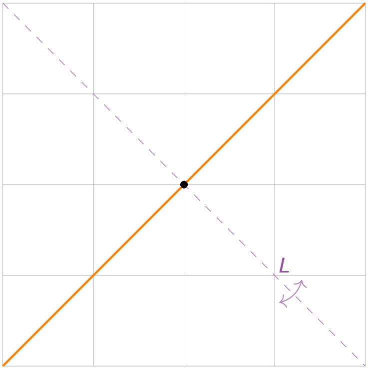

Does anyone see any eigenvectors (vectors that don't move off their line)?

The (-1)-eigenspace is the line $y = x$ (all the vectors x where $Ax = -x$).

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3. The eigenvectors with eigenvalue λ are the nonzero elements of $\mathrm{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

The Eigenvalues of a Triangular Matrix are the Diagonal Entries

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Finding all of the eigenvalues of a matrix *is not a row reduction problem!* We'll see how to do it in general next time. For now:

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Why? $\mathrm{Nul}(A - \lambda I) \neq \{0\}$ if and only if $A - \lambda I$ is not invertible, if and only if $\det(A - \lambda I) = 0$.

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \lambda I_4 = \begin{pmatrix} 3-\lambda & 4 & 1 & 2 \\ 0 & -1-\lambda & -2 & 7 \\ 0 & 0 & 8-\lambda & 12 \\ 0 & 0 & 0 & -3-\lambda \end{pmatrix}.$$

The determinant is $(3-\lambda)(-1-\lambda)(8-\lambda)(-3-\lambda)$, which is zero exactly when $\lambda = 3, -1, 8$, or -3.

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Why?

$$\begin{aligned} 0 \text{ is an eigenvalue of } A &\iff Ax = 0x \text{ has a nontrivial solution} \\ &\iff Ax = 0 \text{ has a nontrivial solution} \\ &\iff A \text{ is not invertible.} \end{aligned}$$

invertible matrix theorem

Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

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Section 5.2

The Characteristic Equation

The Invertible Matrix Theorem

Addenda

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2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
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13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
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20. The number 0 is *not* an eigenvalue of A.

The Characteristic Polynomial

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The **characteristic equation** of A is the equation

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Important

The eigenvalues of A are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

The Characteristic Polynomial

Example

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) = \det(A - \lambda I) &= \det\left[\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \\ &= (5-\lambda)(1-\lambda) - 2 \cdot 2 \\ &= \lambda^2 - 6\lambda + 1. \end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial, which we can find using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}.$$

The Characteristic Polynomial

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Question: What is the characteristic polynomial of

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Answer:

$$f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc$$

$$= \lambda^2 - (a + d)\lambda + (ad - bc)$$

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Shortcut

The characteristic polynomial of a 2×2 matrix A is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A).$$

The Characteristic Polynomial

Example

Question: What are the eigenvalues of the following matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) = \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{pmatrix} \\ &= 8\left(\frac{1}{4} - 0 \cdot -\lambda\right) - \lambda\left(\lambda^2 - 6 \cdot \frac{1}{2}\right) \\ &= -\lambda^3 + 3\lambda + 2. \end{aligned}$$

We know from before that one eigenvalue is $\lambda = 2$: indeed, $f(2) = -8 + 6 + 2 = 0$. Doing polynomial long division, we get:

$$\frac{-\lambda^3 + 3\lambda + 2}{\lambda - 2} = -\lambda^2 - 2\lambda - 1 = -(\lambda + 1)^2.$$

Hence $\lambda = -1$ is also an eigenvalue.

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The Characteristic Polynomial

Poll

Fact: If A is an $n \times n$ matrix, the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I)$$

turns out to be a polynomial of degree n, and its roots are the eigenvalues of A:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

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Section 5.3

Diagonalization

Motivation

Difference equations

Many real-word linear algebra problems have the form:

$$v_1 = Av_0, \quad v_2 = Av_1 = A^2 v_0, \quad v_3 = Av_2 = A^3 v_0, \quad \ldots \quad v_n = Av_{n-1} = A^n v_0.$$

This is called a **difference equation**.

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- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the nth powers of the diagonal entries of D:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix}, \quad \dots \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}.$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad D^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix},$$

$$\dots \quad D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{pmatrix}$$

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Definition

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6. Similar matrices usually do not have the same eigenvectors.

Powers of Matrices that are Similar to Diagonal Ones

What if A is not diagonal?

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Then

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Closed formula in terms of n:
easy to compute

Therefore

$$A^n = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2^{n+1} - 3^n & -2^{n+1} + 2 \cdot 3^n \\ 2^n - 3^n & -2^n + 2 \cdot 3^n \end{pmatrix}.$$

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If $A = PDP^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ then

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So diagonalizable matrices are easy to raise to any power.

Diagonalization

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In this case, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

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An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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The Corollary is true because eigenvectors with distinct eigenvalues are always linearly independent.

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An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In this case, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary

a theorem that follows easily from another theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The Corollary is true because eigenvectors with distinct eigenvalues are always linearly independent. We will see later that a diagonalizable matrix need not have n distinct eigenvalues though.

Diagonalization

Example

Problem: Diagonalize $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Therefore the eigenvalues are 2 and 3. Let's compute some eigenvectors:

$$(A - 2I)x = 0 \iff \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} x = 0 \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = 2y$, so $v_1 = \binom{2}{1}$ is an eigenvector with eigenvalue 2.

$$(A - 3I)x = 0 \iff \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} x = 0 \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = y$, so $v_2 = \binom{1}{1}$ is an eigenvector with eigenvalue 3.

The eigenvectors v_1, v_2 are linearly independent, so the Diagonalization Theorem says

$$A = PDP^{-1} \quad \text{for} \quad P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Diagonalization

Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2).$$

Therefore the eigenvalues are 1 and 2, with respective multiplicities 2 and 1. Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric vector form is

$$\begin{aligned} x &= y \\ y &= y \\ z &= \qquad z \end{aligned} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence a basis for the 1-eigenspace is

$$\mathcal{B}_1 = \{v_1, v_2\} \quad \text{where} \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Diagonalization

Another example, continued

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Now let's compute the 2-eigenspace:

$$(A - 2I)x = 0 \iff \begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = 3z, y = 2z$, so an eigenvector with eigenvalue 2 is

$$v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The eigenvectors v_1, v_2, v_3 are linearly independent: v_1, v_2 form a basis for the 1-eigenspace, and v_3 is not contained in the 1-eigenspace. Therefore the Diagonalization Theorem says

$$A = PDP^{-1} \quad \text{for} \quad P = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

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Note: In this case, there are three linearly independent eigenvectors, but only two distinct eigenvalues.

Diagonalization

A non-diagonalizable matrix

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

This is an upper-triangular matrix, so the only eigenvalue is 1. Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = 0.$$

This is row reduced, but has only one free variable x; a basis for the 1-eigenspace is $\{\binom{1}{0}\}$. So *all eigenvectors* of A are multiples of $\binom{1}{0}$.

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Conclusion: A has only one linearly independent eigenvector, so by the "only if" part of the diagonalization theorem, A is not diagonalizable.

Poll

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

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Matrix A is not diagonalizable: its only eigenvalue is 1, and its 1-eigenspace is spanned by $\binom{1}{0}$.

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Matrix D is already diagonal!

Diagonalization

Procedure

How to diagonalize a matrix A:

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1. Find the eigenvalues of A using the characteristic polynomial.

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3. If there are fewer than n total vectors in the union of all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i.

Diagonalization

Proof

Why is the Diagonalization Theorem true?

A diagonalizable implies A has n linearly independent eigenvectors: Suppose $A = PDP^{-1}$, where D is diagonal with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ be the columns of P. They are linearly independent because P is invertible.

$$AP = A\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_n \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_1 v_2 & \cdots & \lambda_1 v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Hence $AP = PD$.

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$$Av_i = PDP^{-1}v_i = PDe_i = P(\lambda_i e_i) = \lambda_i Pe_i = \lambda_i v_i.$$

Hence v_i is an eigenvector of A with eigenvalue λ_i. So the columns of P form n linearly independent eigenvectors of A, and the diagonal entries of D are the eigenvalues.

A has n linearly independent eigenvectors implies A is diagonalizable: Suppose A has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let P be the invertible matrix with columns $v_1, v_2, \ldots, v_n$. Let $D = P^{-1}AP$.

$$De_i = P^{-1}APe_i = P^{-1}Av_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1}v_i = \lambda_i e_i.$$

Hence D is diagonal, with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Solving $D = P^{-1}AP$ for A gives $A = PDP^{-1}$.

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Definition

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Let A be an $n \times n$ matrix. The following are equivalent:

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2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and *the geometric multiplicity equals the algebraic multiplicity* of each eigenvalue.

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Hence the geometric multiplicities add up to 3, so A is diagonalizable.

Non-Distinct Eigenvalues

Another example

Example

The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $f(\lambda) = (\lambda - 1)^2$.

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The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $f(\lambda) = (\lambda - 1)^2$.

It has one eigenvalue 1 of algebraic multiplicity 2.

We showed before that the geometric multiplicity of 1 is 1 (the 1-eigenspace has dimension 1).

Since the geometric multiplicity is smaller than the algebraic multiplicity, the matrix is *not* diagonalizable.

Linear Recurrence Relation

Linear Recurrence Relation

Let $(x_n) = (x_0, x_1, x_2, \dots)$ be a sequence of numbers that is defined as follows:

1. $x_0 = a_0, x_1 = a_1, \dots, x_{k-1} = a_{k-1}$, where $a_0, a_1, \dots, a_{k-1}$ are scalars.
2. For all $n \geq k, x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$ where $c_1, c_2, \dots, c_k$ are scalars.

If $c_k \neq 0$, the equation in (2) is called a linear recurrence relation of order k. The equations in (1) are referred to as the initial conditions of the recurrence.

Examples

- $x_{n+2} = x_{n+1} + x_n, \quad x_0 = 1,\ x_1 = 1.$
- $x_{n+1} = 2x_n, \quad x_0 = 3.$

Linear Recurrence in Matrix Form

I am going to explain it using an example of second order linear recurrence relation Consider the following linear recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n, \quad x_1 = c_1, x_0 = c_0,$$

where c_0 and c_1 are known constants. We can write it as

$$\begin{aligned} x_{n+2} &= ax_{n+}1 + bx_n \\ x_{n+1} &= x_{n+1}. \end{aligned}$$

In Matrix form, we can write

$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$$

$$\boxed{X_{n+1} = AX_n, \forall n \geq 0.}$$

where $X_{n+1} = \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}$, and $A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$

Linear Recurrence in Matrix Form; contn'd...

For $n = 0$,we have

$$X_1 = AX_0,$$

where $X_0 = \begin{pmatrix} c_1 \\ c_0 \end{pmatrix}$
For $n = 1$, we can write

$$X_2 = AX_1 = A(AX_0) = A^2 X_0.$$

$n = 2$, gives us

$$X_3 = AX_2 = A(A^2 X_0) = A^3 X_0.$$

Continuing in the same manner, we have

$$X_{n+1} = A^{n+1} X_0.$$

Examples

Suppose each "Gibonacci" number G_{k+2} is the average of the two previous numbers G_{k+1} and G_k . If $G_0 = 0$ and $G_1 = 1$. Find the *kth* term of the sequence only depending upon k. $\boxed{Aim:}$ We want to find the general term of the sequence. $\boxed{Steps:}$

- Matrix Form
- Eigenvalues and Eigenvectors
- Diagonalize

Examples; Contn'd...

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$$
$$G_{k+1} = G_{k+1}.$$

In Matrix Form

$$\mathbf{G_{k+1}} = \mathbf{AG_k}, \quad \forall \mathbf{k} \geq \mathbf{0}.$$

where $\mathbf{G_{k+1}} = \begin{pmatrix} G_{k+2} \\ G_{k+1} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$, $\mathbf{G_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\mathbf{G_k} = \mathbf{A^k G_0},, \quad \forall \mathbf{k} \geq \mathbf{0}.$$

Eigenvalues Characteristic Equation

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

Eigenvalues are: $1, -\frac{1}{2}$

Examples; Contn'd...

Eigenvectors $\lambda = 1$

$$(A - 1I)X = 0$$

Augmented matrix

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

Eigenvector: All non-zero multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvectors $\lambda = \frac{-1}{2}$

$$(A + \frac{1}{2}I)X = 0$$

Augmented matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$$

Eigenvector: All non-zero multiples of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Examples; Contn'd...

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}$$

As $A^k = PD^kP^{-1}$, so we need to calculate P^{-1}.

$$P^{-1} = \frac{adjP}{detP} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

So,

$$A^k = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{(-1)^k}{(2)^k} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

Simplification gives us

$$A^k = \frac{1}{3}\begin{pmatrix} \frac{(-1)^k}{(2)^k} + 2 & 1 - \frac{(-1)^k}{(2)^k} \\ 2 - \frac{2(-1)^k}{(2)^k} & \frac{(-1)^k}{(2)^k} + 1 \end{pmatrix}$$

Examples; Contn'd...

$$\mathbf{G_k = A^k G_0}$$

$$\begin{pmatrix} G_{k+2} \\ G_{k+1} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} \frac{(-1)^k}{(2)^k} + 2 & 1 - \frac{(-1)^k}{(2)^k} \\ 2 - \frac{2(-1)^k}{(2)^k} & \frac{(-1)^k}{(2)^k} + 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence

$$G_k = \frac{2}{3} - \frac{2}{3} \frac{(-1)^k}{(2)^k}$$

Important Result

Theorem

Let $x_n = ax_{n-1} + bx_{n-2}$ be a recurrence relation. Let λ_1 and λ_2 be the eigenvalues of the associated characteristic equation $\lambda^2 - a\lambda - b = 0$.

1. If $\lambda_1 \neq \lambda_2$, then
$$x_n = c_1\lambda_1^n + c_2\lambda_2^n$$
for some scalars c_1 and c_2.
2. If $\lambda_1 = \lambda_2$, then
$$x_n = (c_1 + c_2 n)\lambda^n$$
for some scalars c_1 and c_2.

Examples

Solve the following recurrence relation with the given initial conditions.

$$y_1 = 1, y_2 = 6, y_k = 4y_{k-1} - 4y_{k-2}, \quad k \geq 3.$$

Characteristics equation $\lambda^2 - 4\lambda + 4 = 0$.
Solution of the quadratic equation is Eigenvalues: $\lambda_1 = 2, \lambda_1 = 2$. So,

$$y_k = c_1(2)^k + c_2 k 2^k.$$

As, $y_1 = 1$, so, $2c_1 + 2c_2 = 1$, $y_2 = 6$, so, $4c_2 + 8c_2 = 6$.
Solution of above system is

$$c_1 = -\frac{1}{2}, \quad c_2 = 1.$$

Hence,

$$y_k = -\frac{1}{2}2^k + k2^k$$

Practice Problems

1. Solve the recurrence relation with the given initial conditions.
   - $a_0 = 4, a_1 = 1, a_n = a_{n-1} - \frac{a_{n-2}}{4}$, for $n \geq 2$.
   - $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, subject to $a_0 = 2, a_1 = 2, a_2 = 4$, for $n \geq 3$.
2. Find the limiting values of y_k and z_k, $(k \to \infty)$ if

$$\begin{aligned} y_{k+1} &= .8y_k + .3z_k, \quad y_0 = 0 \\ z_{k+1} &= .2y_k + .7z_k, \quad z_0 = 5. \end{aligned}$$

Systems of Linear Differential Equations

First order linear homogenous differential equation

First order differential Equation

$$x' = kx, \quad k \text{ is } \quad \text{a} \quad \text{constant.}$$

Solution: $x(t) = x_0 e^{kt}$.
First order differential system

$$\begin{aligned} x' &= 4x \\ y' &= 9y \end{aligned}$$

In Matrix form of above system can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Matrix is diagonal
- System is uncoupled

First order linear system of differential equations

General first order linear system

$$x' = ax + by$$
$$y' = cx + dy$$

In Matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We can write the above system as

$$X' = AX,$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

- A is not a diagonal matrix.
- Can we diagonalize A?

Diagonalization

$\boxed{\textit{Aim} :}$

To solve the system $X' = AX$.

$\boxed{\textit{Challenges}}$

Matrix is not diagonal.

$\boxed{\textit{Possible Solution}}$

Transform the matrix into a diagonal matrix i.e., diagonalize it.

HOW?

We want to transform

$$X' = AX \xrightarrow{to} Y' = DY.$$

Diagonalization

As $PDP^{-1} = A$, so we can write

$$X' = AX = PDP^{-1}X$$

Pre multiplying by P^{-1} we get

$$P^{-1}X' = DP^{-1}X$$

Since, P is a constant matrix, so

$$\left(P^{-1}X\right)' = D\left(P^{-1}X\right).$$

Put $\left(P^{-1}X\right) = Y$ to get

$$\boxed{Y' = DY}.$$

Uncoupling system of differential equations

Summary

Coupled system of differential equation

$$X' = AX$$

can be transformed (uncoupled) to

$$Y' = DY$$

by using the transformation

$$X = PY.$$

Example

Find a solution to the system

$$x' = x + 3y$$
$$y' = 2x + 2y$$

subject to initial conditions $x(0) = 0,\ y(0) = 5$.

Solution: In Matrix form, we can write it as

$$X' = AX,$$

where $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

We can uncoupled the system by using the transformation

$$X = PY.$$

Example

Eigenvalues:
Characteristic Equation:

$$\lambda^2 - 3\lambda - 4 = 0.$$

Eigenvalues are: -1, 4.
Corresponding eigenvectors are

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence,

$$P = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}, \; D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

By using the transformation $X = PY$, we get

$$Y' = DY,$$

where $Y = \begin{pmatrix} u \\ v \end{pmatrix}$.

Example

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Above equations can be written as

$$\begin{aligned} u' &= -u \\ v' &= 4v. \end{aligned}$$

Solving, we get

$$u = c_1 e^{-t}, \; v = c_2 e^{4t}.$$

As $X = PY$, so, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

$$\begin{aligned} x &= 3u + v \\ y &= -2u + v. \end{aligned}$$

Substituting values of u and v, we get

$$\begin{aligned} x &= 3c_1 e^{-t} + c_2 e^{4t} \\ y &= -2c_1 e^{-t} + c_2 e^{4t}. \end{aligned}$$

Example

Since, $x(0) = 0$ and $y(0) = 5$, we get

$$\begin{aligned} 0 &= 3c_1 + c_2 \\ 5 &= -2c_1 + c_2. \end{aligned}$$

Solving, above system we get
$c_1 = -1, \; c_2 = 3.$
In matrix form we can the solution as

$$X = -x_1 e^{-t} + 3x_2 e^{4t}$$

where $x_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are the eigenvectors corresponding to eigenvalues -1 and 4 respectively.

Example

Find a solution to the system

$$\begin{aligned} r'(t) &= w(t) - 12 \\ w'(t) &= -r(t) + 10 \end{aligned}$$

Solution:

$\boxed{\textit{Issue :}}$

Presence of -12 and 10.

$\boxed{\textit{How to resolve it :}}$

Put $w(t) - 12 = y(t)$ and $-r(t) + 10 = x(t)$, we get

$$\begin{aligned} -x'(t) &= y(t) \\ y'(t) &= x(t). \end{aligned}$$

In Matrix form, we can write it as

$$X' = AX,$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

By using the substitution $X = PY$ we get

$$Y' = DY$$

where $P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \ D = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

So, solution is

$$u = c_1 e^{-it}$$
$$v = c_2 e^{it}$$

By using the relation $X = PY$, we get

$$x = c_1 e^{-it} + c_2 e^{it}$$
$$y = c_1 i e^{-it} - i c_2 e^{it}.$$

Solution of the system is

$$r(t) = 10 - c_1 e^{-it} - c_2 e^{it}$$
$$w(t) = 12 + c_1 i e^{-it} - i c_2 e^{it}$$

- In case of single linear differential equation, we have

$$x' = kx, \ k \text{ is a constant.}$$

Solution of the differential equation is

$$x = ce^{kt}.$$

- In case of system of coupled differential equations, we have

$$X' = AX, \ A \text{ is a constant matrix.}$$

Solution of the linear differential system should be

$$X = c\ e^{At}.$$

Exponential of a Matrix

Compute e^{Dt} where $D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

Since, $e^x = 1 + x + \frac{x^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. so,

$$e^{Dt} = I + Dt + ... = \sum_{n=0}^{\infty} \frac{D^n t^n}{n!}$$

$$e^{Dt} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 4^n t^n & 0 \\ 0 & t^n \end{pmatrix}$$

$$e^{Dt} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{4^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{1^n}{n!} \end{pmatrix}$$

$$e^{Dt} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}$$

Example

Compute e^{At} where $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

For given matrix, we have

$$P = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

$$e^{At} = \sum_{n=0}^{\infty} \frac{PD^n P^{-1}}{n!}$$

$$e^{At} = P \sum_{n=0}^{\infty} \frac{D^n t^n}{n!} P^{-1}$$

$$e^{At} = Pe^{Dt} P^{-1}$$

$$e^{At} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{4t} \end{pmatrix} \left(\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \right)^{-1}.$$