1) $M = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$T(A) = MA = M \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 1 \end{bmatrix}$

Image of $A = (-4, 2, 2)$

$T(B) = MB = M \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -2 \\ -1 \end{pmatrix}$

Image of $B = (-5, 2, 2)$

$T(C) = MC = M \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix}$

Image of $C = (-4, 1, 2)$

$T(D) = MD = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, a point at infinity

2) i) Let $\beta = \{1, x, x^2\}$ and $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be standard basis of $\mathbb{P}_2$ and $\mathbb{R}^3$ respectively. The matrix of transformation w.r.t β and $\mathcal{E}$ is

$$[T]_{\mathcal{E}\leftarrow\beta} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -3 \\ -1 & 0 & 1 \end{bmatrix}$$

Ker T : $T(a+bx+cx^2) = 0$

$$\begin{bmatrix} 2a-b \\ a+b-3c \\ c-a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2a-b=0, \quad a+b-3c=0, \quad c-a=0$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -3 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -3 \\ 0 & -1/2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ 2c \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\text{Ker}(T) = \text{Span}\{1+2x+x^2\} \neq \{\vec{0}\}$

So transformation is not one-to-one and hence not isomorphism.

Since $\dim \mathbb{P}_2 = \text{Nullity}(T) + \text{Rank}(T)$

$3 = 1 + \text{Rank}(T)$

$\Rightarrow \text{Rank}(T) = \dim(\text{Range}(T)) = 2$

So, $\text{Range}(T) \neq \mathbb{R}^3$, T is not onto.

A vector $\vec{b}$ in range is of the form:

$$\vec{b} = \begin{bmatrix} 2a-b \\ a+b-3c \\ c-a \end{bmatrix}$$

$$\vec{b} = a\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

From the above echelon form, we can see that $\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$ is in the span of $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

So, $\vec{b} = \text{Range}(T) = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

ii) Let $\ell = \{E_1, E_2, E_3, E_4\}$ be the standard basis of $M_{2\times 2}$.

$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$[T]_\ell = \left[[T(E_1)]_\ell \quad (T(E_2))_\ell \quad (T(E_3))_\ell \quad (T(E_4))_\ell \right]$$

$$[T(E_1)]_\ell = \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right]_\ell = \left[\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right]_\ell = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_2)]_\ell = \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right]_\ell = \left[\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right]_\ell = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_3)]_\ell = \left[\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right]_\ell = \left[\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right]_\ell = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$[T(E_4)]_\ell = \left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right]_\ell = \left[\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \right]_\ell = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

so $M = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

Ker T: $T(A) = 0$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$AB = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$a-b=0,\ -a+b=0,\ c-d=0,\ -c+d=0$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ b \\ d \\ d \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$\text{Ker}\,T = \text{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} \neq \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

So T is not one-to-one and therefore not isomorphism.

Since $\dim M_{22} = \text{Nullity}(T) + \text{Rank}(T)$

$$4 = 2 + \text{Rank}(T)$$

$\Rightarrow \text{Rank}(T) = \dim(\text{Range}(T)) = 2$

So, $\text{Range}(T) \neq M_{22}$, T is not onto.

An element in the range is of the form $\vec{b} = AB$

$$\vec{b} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

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So, $\vec{b} = \text{Range}(T) = \text{Span}\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$

iii) Let $\beta = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\mathcal{C} = \{E_1, E_2, E_3\}$ be the standard basis of $\mathbb{R}^3$ and W respectively.

$\{$ where $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$

$$[T]_{\mathcal{C} \leftarrow \beta} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

Ker(T): $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0}$

$$\begin{bmatrix} a+b+c & b-2c \\ b-2c & a-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$a+b+c=0$, $b-2c=0$, $b-2c=0$, $a-c=0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{bmatrix}.$$

$\Rightarrow a = b = c = 0$

$\operatorname{Ker} T = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. So T is one-to-one.

Now $\dim \mathbb{R}^3 = \text{Nullity}(T) + \text{Rank}(T)$

$3 = 0 + \text{Rank}(T)$

$\Rightarrow \text{Rank}(T) = \text{Range}(T) = 3 = \dim W$

(Dimension of symmetric 2x2 matrices is 3)

So, Range(T) = W. So T is onto.

Since T is both one-to-one and onto, so T is isomorphism.

Range of T has a vector of the form:

$$\vec{b} = \begin{bmatrix} a+b+c & b-2c \\ b-2c & a-c \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$$

So $\vec{b} = \text{Range}(T) = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}\right\}$

3) Since $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$ forms a basis of $\mathbb{R}^2$. Therefore

$$\begin{bmatrix} -7 \\ 9 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$-7 = c_1 + 3c_2$$

$$9 = c_1 - c_2$$

$$\Rightarrow c_1 = 5, \quad c_2 = -4$$

Since T is linear, ...

$$T\begin{bmatrix} -7 \\ 9 \end{bmatrix} = 5T\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4T\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= 5(1-2x) - 4(x+x^2)$$

$$= 5 - 14x - 4x^2$$

<u>OR</u> $[T]_{\ell \leftarrow \beta} = \left(\left[T \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\ell} \; \left[T \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right]_{\ell} \right)$

where $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$, $\ell = \{1, x, x^2\}$

$$[T]_{\ell \leftarrow \beta} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} -7 \\ 9 \end{bmatrix} \right]_{\beta} = \begin{bmatrix} 5 \\ -4 \end{bmatrix},$$ (Derived above)

$$\left[T \begin{bmatrix} -7 \\ 9 \end{bmatrix} \right]_{\ell} = [T]_{\ell \leftarrow \beta} \left[\begin{bmatrix} -7 \\ 9 \end{bmatrix} \right]_{\beta}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -14 \\ -4 \end{bmatrix}$$

$$\Rightarrow T \begin{bmatrix} -7 \\ 9 \end{bmatrix} = 5 - 14x - 4x^2$$

Similaly,

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

$$a = c_1 + 3c_2$$

$$b = c_1 - c_2$$

$$c_1 = \frac{a+3b}{4}, \quad c_2 = \frac{a-b}{4}$$

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a+3b}{4}(1-2x) + \left(\frac{a-b}{4}\right)(x+x^2)$$

$$= \frac{a+3b}{4} - \left(\frac{a+7b}{4}\right)x + \left(\frac{a-b}{4}\right)x^2.$$