STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 16

PROBABILITY DISTRIBUTIONS

POISSON DISTRIBUTION

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POISSON DISTRIBUTION

The Poisson probability distribution was discovered by a French mathematician S.D Poisson in 1837. Poisson distribution is used for rare events ie events having a very small chance of occurrence hen experiment is repeated is repeated very large number of times. Unit of time may be a minute, an hour, a day or a month, while the region of space may be length, area or volume.

THE POISSON EXPERIMENT A Poisson experiment is a probability experiment that satisfies the following four requirements

1- Trials are not finite ie there is no upper limit of the experiment.
2- Each trial has only two possible outcomes, either success or failure.
3- The trials are independent of each other.
4- The probability of success remains same for each trial.

EXAMPLES OF POISSON EXPERIMENT Some examples of Poisson experiment are as follows

1- Number of road accidents at Motorway per month
2- Number of Twin Babies births in a Hospital per day
3- Number of typing errors on a page
4- Number of students having COVID
5- Number of Amoebas in a Pound Water

THE POISSON PROBABILITY DISTRIBUTION A discrete probability distribution that is useful when n is large and p is small and when the independent variables occur over a period of time is called the Poisson distribution

THE POISSON FORMULA If random variable X follows Poisson distribution then it is denoted by $X \sim P_0(\lambda)$ and the Poisson probability function is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0,1,2,3,\ldots,\infty$$

$$\begin{aligned} where\quad & e = Mathematical\ constant\ approximated\ by\ 2.71828 \\ & \lambda = Average\ Number\ of\ Occurances \\ & x = number\ of\ successes\ in\ t\ segments \end{aligned}$$

PROPERTIES OF BINOMIAL DISTRIBUTION

1- The Mean of Poisson Distribution is $E(X) = \lambda$
2- The variance of Poisson Distribution is $V(X) = \lambda$

EXAMPLES OF POISSON DISTRIBUTION

EXAMPLE-1 If X follows Poisson distribution having $\lambda = 2$ i.e. $X \sim Po(2)$. Find the following

$(a)\ P(X = 0)$

$(b)\ P(X \le 2)$

$(c)\ P(X \ge 3)$

$(d)\ E(X)$

$(e)\ V(X)$

SOLUTION $Here\ \lambda = 2$

$$(a)\ P(X = 0) = \frac{e^{-2}2^0}{0!} = 0.1353$$

$$(b)\ P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!} = 0.6767$$

$$(c)\ P(X \ge 3) = 1 - P(X < 3)$$

$$= 1 - P(X \le 2)$$

$$= 1 - 0.6767 = 0.3233$$

$$(d)\ E(X) = \lambda = 2$$

$$(e)\ V(X) = \lambda = 2$$

EXAMPLE-2 A student finds that the average number of amoebas in 10 ml pond of water from a particular pond is four. Assuming that number of amoebas follows a Poisson distribution find the probability that in a 10 ml sample

(a) *there are exactly five amoebas*

(b) *there are no amoebas*

(c) *there are fewer than three amoebas*

SOLUTION Let X denote number of amoebas in 10ml of pond water. $Here\ \lambda = 4$

$$(a)\ P(X = 5) = \frac{e^{-4}4^5}{5!} = 0.156$$

$$(b)\ P(X = 0) = \frac{e^{-4}4^0}{0!} = 0.0183$$

$$(c)\ P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!}$$

$$= e^{-4}(1 + 4 + 8) = 0.238$$

EXAMPLE-3 Telephone calls are being placed through a telephone exchange at random on average of 6 calls per minute. Assuming Poisson distribution, determine the probability that in 10 second interval, there will be three or more calls.

SOLUTION Let X denote number of calls made in 10 second interval.

Here average number of calls per minute is given i.e. 6. We are interested in in the average for 10 second interval

So $Here\ \lambda = 10 \times \frac{6}{60} = 1$

$$P(there\ will\ be\ three\ or\ more\ calls) = P(X \geq 3)$$

$$\begin{aligned} P(X \geq 3) &= 1 - P(X < 3) \\ &= 1 - [P(X=0) + P(X=1) + P(X=2)] \\ &= 1 - [\frac{e^{-1}1^0}{0!} + \frac{e^{-1}1^1}{1!} + \frac{e^{-1}1^2}{2!}] \\ &= 1 - 0.9193 = 0.0807 \end{aligned}$$

EXAMPLE-4 On average the university photocopier breaks down eight times during the university week (Monday to Friday). Assuming that the number of breakdowns can be modelled by Poisson distribution, find the probability that it breaks down

(a) *five times in a week*

(b) *once on monday*

(c) *eight times in a fortnight*

SOLUTION Let X denote number of breakdowns

(a) Let X denote number of breakdowns in a week $Here\ \lambda = 8$

$$P(X=5) = \frac{e^{-8}8^5}{5!} = 0.0916$$

(b) Let Y denote number of breakdowns in a week $Here\ \lambda = 8/5 = 1.6$

$$P(Y=1) = \frac{e^{-1.6}(1.6)^1}{1!} = 0.323$$

(c) Let Z denote number of breakdowns in a week $Here\ \lambda = 8 \times 2 = 16$

$$P(Z=8) = \frac{e^{-16}(16)^8}{8!} = 0.0120$$

EXAMPLE-5 X follows Poisson distribution with standard deviation 1.5. Find $P(X \geq 3)$.

SOLUTION $Here\ \sqrt{\lambda} = 1.5 \quad \Rightarrow \lambda = (1.5)^2 = 2.25$

$$\begin{aligned} P(X \geq 3) &= 1 - P(X < 3) \\ &= 1 - [P(X=0) + P(X=1) + P(X=2)] \\ &= 1 - [\frac{e^{-2.25}(2.25)^0}{0!} + \frac{e^{-2.25}(2.25)^1}{1!} + \frac{e^{-2.25}(2.25)^2}{2!}] \\ &= 1 - 0.6093 = 0.391 \end{aligned}$$

EXERCISE – 5.2

POISSON DISTRIBUTION

1. Assume a Poisson distribution
(a) If $\lambda = 2.5,\ \ find\ \ P(X = 3)$
(b) If $\lambda = 0.5,\ \ find\ \ P(X \le 1)$
(c) If $\lambda = 8,\ \ find\ \ P(X \ge 3)$
(d) If $\lambda = 3.7,\ \ find\ \ P(X < 1)$
(e) If $\lambda = 4,\ \ find\ \ P(X > 2)$
$(Ans: 0.2138, 0.9098, 0.9862,\ 0.0247,\ 0.9084)$

2. Assume a Poisson distribution with $\lambda = 5$. What is the probability that
(a) $X = 1$
(b) $X < 1$
(c) $X > 1$
(d) $X \le 1$
(e) $X \ge 1$
$(Ans: 0.0337, 0.0067, 0.9596,\ 0.0404,\ 0.9933)$

3. Assume that the number of network errors experienced in a day on a local area network (LAN) is distributed as a Poisson random variable. The mean number of network errors experienced in a day is 2.4. What is the probability that in an given day
(a) zero network errors will occur?
(b) exactly one network error will occur?
(c) two or more network errors will occur?
(d) fewer than three network errors will occur?
$(Ans: 0.0907, 0.2177, 0.6916,\ 0.5697)$

4. The quality control manager of Marilyn's Cookies is inspecting a batch chocolate-chip cookies that has just been baked. If the production process is in control, the mean number of chip parts per cookie is 6.0. What is the probability that in any particular cookie being inspected
(a) fewer than five chip parts will be found?
(b) exactly five chip parts will be found?
(c) five or more chip parts will be found?
(d) either four or five chip parts will be found?
$(Ans: 0.2851, 0.1606, 0.7149,\ 0.2945)$

5. The U.S. Department of Transportation maintains statistics for consumer complaints per 100,000 airline passengers. In the first nine months of 2009, consumer complaints were 0.99 per 100,000 passengers. What is the probability that in the next 100,000 passengers, there will be
(a) no complaints?
(b) at least one complaint?
(c) at least two complaints?
$(Ans: 0.3716, 0.6284,\ 0.2606)$

6. The number of particles emitted by a radioactive source is generally well modeled by the Poisson distribution. If the average number of particles emitted by the source in an hour is four, find the following probabilities.
(a) The number of emitted particles in a given hour is at least 6.
(b) The number of emitted particles in a given hour will be at most 3.
(c) No particles will be emitted in a given 24-hour period.
$(Ans: 0.2149, 0.4335, 0)$

7. Based on past experience, it is assumed that the number of flaws per foot in rolls of grade 2 paper follows a Poisson distribution with a mean of 1 flaw per 5 feet of paper (0.2 flaw per foot). What is the probability that in a
(a) 1-foot roll, there will be at least 2 flaws?
(b) 12-foot roll, there will be at least 1 flaw?
(c) 50-foot roll, there will be more than or equal to 5 flaws and fewer than or equal to 15 flaws?
$(Ans: 0.0176, 0.9093, 0.9220)$

8. The number of students who log in to a randomly selected computer in a college computer lab follows a Poisson probability distribution with a mean of 19 students per day.
(a) Using the Poisson probability distribution formula, determine the probability that exactly 12 students will log in to a randomly selected computer at this lab on a given day.
(b) Using the Poisson probability distribution table, determine the probability that the number of students who will log in to a randomly selected computer at this lab on a given day is i. from 13 to 16 ii. fewer than 8
$(Ans: 0.0259, 0.2314, 0.0015)$

9. Customer arrivals at a checkout counter in a department store have a Poisson distribution with an average of seven per hour. For a given hour, find the probabilities of the following events.
(a) Exactly seven customers arrive.
(b) No more than two customers arrive.
(c) At least two customers arrive.
$(Ans: 0.60, 0.296, 0.993)$

10. The number of fatalities due to shark attack during a year is modeled using a Poisson distribution. The International Shark Attack File (ISAF) investigates shark-human interactions worldwide. Internationally, an average of 4.4 fatalities per year occurred during a 5-year period. Assuming that this mean remains constant for the next 5 years, find the probabilities of the following events.
(a) No shark fatalities will be recorded in a given year.
(b) Sharks will cause at least six human deaths in a given year.
(c) No shark fatalities will occur during the 5-year period.
(d) At most 12 shark fatalities will occur during the 5-year period.
$(Ans: 0.0123, 0.2801, 0,\ 0.0151)$