Homework - 14

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Qn 1

$$u^T = \left[\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right]$$

a) $P = uu^T$

$$= \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{bmatrix}$$

b) $H = I - 2uu^T$

$Q = I - 2P$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/2 & -1/2 & +1/2 & +1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}$$

Since $Q = Q^T$, Q is symmetric

QQ^T should be equal to I if Q is symmetric orthogonal

$$Q^2 = QQ^T = \begin{bmatrix} 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times(1/2)^2 + (-1/2)^2 & -1/4 - 1/4 + 1/4 + 1/4 & \cdots \\ -1/4 - 1/4 + 1/4 + 1/4 & 3(1/2)^2 + (-1/2)^2 & \cdots \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

hence Q is a symmetric orthogonal matrix

c) $P^2 = (uu^T)^2$ $\qquad u^Tu = 1$

$P^2 = (uu^T)(uu^T)$

$P^2 = u\cancel{u^Tu}u^T$

$P^2 = uu^T$

$P^2 = P$ //

$Q = I - 2P$

$Q^2 = (I - 2P)^2$

$Q^2 = I + 4P^2 - 4P$

$Q^2 = I + \cancel{4P} - \cancel{4P}$

$Q^2 = I$ //

d) Since P is projection matrix Eigen values can be 0 or 1

Eigenvalues of P: 1 with eigenvectors in W = Span{$\vec{u}$}
& 0 with eigenvectors in $W^{\perp}$.

$\boxed{\text{For } \lambda = 1}$ $Pv = \lambda v$

$uu^T v = \lambda v$

$$v = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

Eigenvalues of Q: −1 with eigenvector $\vec{u}$
& 1 with eigenvectors orthogonal to $\vec{u}$.

$\boxed{\text{For } \lambda = 0}$

$$\left[\begin{array}{cccc|c} 1/4 & 1/4 & -1/4 & -1/4 & 0 \\ 1/4 & 1/4 & -1/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1/4 & 1/4 & 0 \\ -1/4 & -1/4 & 1/4 & 1/4 & 0 \end{array}\right]$$

$x_1 = -x_2 + x_3 + x_4$

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Qn 2.

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$(A - \lambda I)v = 0$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\big((6-\lambda)(3-\lambda) - 4\big) + 2\big(-2(3-\lambda) - 8\big) + 4\big(-4 - 4(6-\lambda)\big) = 0$$

$$(3-\lambda)(6-\lambda)(3-\lambda) - 4(3-\lambda) - 4(3-\lambda) - 16 - 16 - 16(6-\lambda) = 0$$

$$(9-6\lambda+\lambda^2)(6-\lambda) - 12 + 4\lambda - 12 + 4\lambda - 32 - 96 + 16\lambda = 0$$

$$54 - 9\lambda - 36\lambda + 6\lambda^2 + 6\lambda^2 - \lambda^3 - 152 + 24\lambda = 0$$

$$-\lambda^3 + 12\lambda^2 - 21\lambda - 98 = 0$$

$\lambda_1 = -2$ $\lambda_2 = 7$

$\boxed{A.M = 1}$ $\boxed{A.M = 2}$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$\boxed{\text{for } \lambda = -2}$

$$\left[\begin{array}{ccc|c} 5 & -2 & 4 & 0 \\ -2 & 8 & 2 & 0 \\ 4 & 2 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 10 & -4 & 8 & 0 \\ -10 & 40 & 10 & 0 \\ 10 & 5 & 12.5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 10 & -4 & 8 & 0 \\ 0 & 36 & 18 & 0 \\ 0 & 9 & 4.5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 5 & -2 & 4 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 5 & 0 & 5 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$V_1 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

$\boxed{\text{for } \lambda = 7}$

$$\left[\begin{array}{ccc|c} -4 & -2 & 4 & 0 \\ -2 & -1 & 2 & 0 \\ 4 & 2 & -4 & 0 \end{array}\right]$$

$$-4x_1 - 2x_2 + 4x_3 = 0$$

$$x_1 = -\frac{1}{2}x_2 + x_3$$

$$V = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & -1/2 & 1 \\ -1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} -1 & -1/2 & 1 \\ -1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$x_1 = V_1 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

$$x_2 = V_2 - \mathrm{Proj}_{x_1} V_2$$

$$x_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{(1/2 - 1/2)}{1 + 1/4 + 1} \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} //$$

$$x_3 = v_3 - \mathrm{Proj}_{x_1} v_3 - \mathrm{Proj}_{x_2} v_3$$

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{(-1+1)}{(1 + 1/4 + 1)} \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} - \frac{(-1/2)}{1/4 + 1} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1/2}{5/4} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 4/5 \\ 2/5 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} -2/3 & -\sqrt{5}/5 & \frac{4\sqrt{5}}{15} \\ -1/3 & 2\sqrt{5}/5 & \frac{2\sqrt{5}}{15} \\ 2/3 & 0 & \sqrt{5}/3 \end{bmatrix} //$$

$$A = Q D Q^T$$