STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 15

EXPECTATION AND VARIANCE
OF RANDOM VARIABLES

PREPARED BY
HAZBER SAMSON
FAST NUCES ISLAMABAD

EXPECTATION DRV'S

MATHEMATICAL EXPECTATION.

It is an important concept in Probability and Statistics. If X is a Random variable than expectation of X is denoted by $E(X)$ and it gives an average or Mean value of X.
In general it gives a quick picture of the long-run "Average" result when the experiment is performed over and over again.

PROPERTIES OF EXPECTATION

If X is a random Variable and a, b are constants than the following properties hold.

① $E(a) = a$
② $E(aX) = a \cdot E(X)$
③ $E(aX+b) = aE(X)+b$

EXPECTATION OF DISCRETE RANDOM VARIABLES

If 'X' is a discrete random variable, than the Mean or Expected value of X is given by

$$E(X) = \sum x f(x)$$

If 'x' is a discrete random variable with probability function $f(x)$, than the expectation of the function $g(x)$ is given by

$$E(g(x)) = \sum_{x} g(x) f(x)$$

Note that

① $E(x) = \sum x f(x)$
② $E(x^2) = \sum x^2 f(x)$
③ $E(x^3) = \sum x^3 f(x)$

EXAMPLES

EXAMPLE-① Find the expected Number of Heads when two fair coins are tossed.

SOLUTION Let 'x' denote the Number of Heads when two fair coins are tossed.

x	$p(x)$	$x\,p(x)$
0	1/4	0
1	2/4	2/4
2	1/4	1/2

So $E(x) = \sum x\,p(x)$

$= 0 + \frac{2}{4} + \frac{1}{2}$

$E(x) = 1$

EXAMPLE-② Find the expected Number of Sixes when three fair dice are thrown.

SOLUTION Let 'x' denote the Number of Sixes

So 'x' can take the values 0, 1, 2, 3

Now let D_1 denote six on the 1^{st} die. Also $P(D_1) = \frac{1}{6}$, $P(D_1') = \frac{5}{6}$

D_2 " " " 2^{ND} die.

D_3 " " " 3^{rd} die.

So $P(X=0) = P(D_1' \cap D_2' \cap D_3') = (\frac{5}{6})(\frac{5}{6}).(\frac{5}{6}) = \frac{125}{216}$

$P(X=1) = P(D_1 \cap D_2' \cap D_3') + P(D_1' \cap D_2 \cap D_3') + P(D_1' \cap D_2' \cap D_3)$

$= (\frac{1}{6})(\frac{5}{6})(\frac{5}{6}) + (\frac{5}{6}).(\frac{1}{6}).(\frac{5}{6}) + (\frac{5}{6})(\frac{5}{6})(\frac{1}{6})$

$= \frac{25}{216} \times 3 = \frac{75}{216}$

$P(X=2) = P(D_1 \cap D_2 \cap D_3') + P(D_1 \cap D_2' \cap D_3) + P(D_1' \cap D_2 \cap D_3)$

$= (\frac{1}{6})(\frac{1}{6})(\frac{5}{6}) + (\frac{1}{6})(\frac{5}{6})(\frac{1}{6}) + (\frac{5}{6}).(\frac{1}{6})(\frac{1}{6})$

$= \frac{5}{216} \times 3 = \frac{15}{216}$

$P(X=3) = P(D_1 \cap D_2 \cap D_3) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

Now Expectation is given by the table.

x	0	1	2	3
$P(x)$	$125/216$	$75/216$	$15/216$	$1/216$
$xP(x)$	0	$75/216$	$30/216$	$3/216$

Now $E(x) = \sum x P(x)$

$$E(x) = 0 + \frac{75}{216} + \frac{30}{216} + \frac{3}{216} = \frac{108}{216} = 0.5$$

EXAMPLE-③ The Probability distribution of a random variable 'x' is shown in the table.

x	1	2	3	4	5
$P(X=x)$	0.1	0.3	c	0.2	0.1

Find the following

(a) value of 'c'.

(b) $E(x)$

(c) $E(x^2)$

(d) $E(3)$

(e) $E(2x)$

(f) $E(2x+3)$

SOLUTION

(a) WKT $\sum P(x) = 1$

$$0.1 + 0.3 + c + 0.2 + 0.1 = 1$$

$$c = 1 - 0.7$$

$$\boxed{c = 0.3}$$

(b) For $E(x)$.

X	1	2	3	4	5
$P(x)$	0.1	0.3	0.3	0.2	0.1
$xP(x)$	0.1	0.6	0.9	0.8	0.5

SO $E(x)$

$= 0.1 + 0.6 + 0.9 + 0.8 + 0.5$

$= 2.9$

(c) For $E(x^2)$

$E(x^2) = \sum x^2 P(x)$

$= 9.7$

x	$P(x)$	x^2	$x^2 P(x)$
1	0.1	1	0.1
2	0.3	4	1.2
3	0.3	9	2.7
4	0.2	16	3.2
5	0.1	25	2.5
			9.7

(d) $E(3)$

$= \sum 3 \cdot P(x)$

$= 3 \sum P(x)$

$= 3(1) = 3$

(e) $E(2x)$

$= \sum 2x P(x)$

or

$E(2x) = 2E(x)$

$= 2(2.9)$

$= 5.8$

(f) $E(2x+3)$

$= 2E(x)+3$

$= 2(2.9)+3$

$= 5.8+3$

$= 8.8$

or $E(2x+3) = \sum (2x+3) \cdot P(x) = 8.8.$

VARIANCE AND STANDARD DEVIATION

If 'x' is a random variable than its variance is given by the formula

$$Var(x) = E(X-\mu)^2$$

or $\quad Var(x) = E(x^2) - \mu^2$

or $\quad Var(x) = E(x^2) - (E(x))^2$

PROOF $\quad V(ar x) = E(X-\mu)^2$

$= E(x^2 + \mu^2 - 2\mu E x))$

$= E(x^2) + E(\mu^2) - 2\mu E(x)$

$= E(x^2) + \mu^2 - 2\mu.\mu^2$

$= E(x^2) - \mu^2$

$= E(x^2) - (E(x))^2$

PROPERTIES OF VARIANCE.

If 'x' is a variable and a, b are constants. than

1) $Var(a) = 0$
2) $Var(ax) = a^2 . Var(x)$
3) $Var(ax+b) = a^2 . Var(x)$.

STANDARD DEVIATION

If 'x' is a random variable than Standard deviation of x is denoted by 'σ' and is given by

$$\sigma = \sqrt{Var(x)}$$

EXAMPLES

EXAMPLE-4 The random variable 'x' has probability distribution as shown in the table.

Find (a) $E(x)$
(b) $E(x^2)$
(c) $Var(x)$
(d) $S.D(x)$

x	1	2	3	4	5
$P(X=x)$	0.1	0.3	0.2	0.3	0.1

SOLUTION consider the table.

x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
1	0.1	0.1	1	0.1
2	0.3	0.6	4	1.2
3	0.2	0.6	9	1.8
4	0.3	1.2	16	4.8
5	0.1	0.5	25	2.5
	1	3		10.4

Now

(a) $E(x) = \Sigma x P(x) = 3$

(b) $E(x^2) = \Sigma x^2 P(x) = 10.4$

(c) $Var(x) = E(x^2) - (E(x))^2$
$= 10.4 - (3)^2$
$= 10.4 - 9 = 1.4$

(d) $S.D(x) = \sigma = \sqrt{V(ar(x))} = \sqrt{1.4} = 1.18$

EXAMPLE-⑤ 'x' is the random variable 'the Number on a biased die and the p.d.f of x is as shown

x	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/5	c	1/5	1/6

Find the following

(a) value of 'c'

(b) $E(x)$

(c) $E(x^2)$

(d) $Var(x)$

(e) $S.D(x)$

(f) $E(2x+3)$

(g) $V(3x+4)$

(h) $S.D(4x+5)$

SOLUTION (a) WKT $\Sigma P(x) = 1$

$\frac{1}{6} + \frac{1}{6} + \frac{1}{5} + c + \frac{1}{5} + \frac{1}{6} = 1$ so $\boxed{c = \frac{1}{10}}$

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SOLUTION. consider the table.

x	$P(x)$	$x \cdot P(x)$	x^2	$x^2 \cdot P(x)$
1	1/6	1/6	1	1/6
2	1/6	2/6	4	4/6
3	1/5	3/5	9	9/5
4	1/10	4/10	16	16/10
5	1/5	5/5	25	25/5
6	1/6	6/6	36	36/6
	1	3.5		457/30

(b) $E(x) = \Sigma x P(x)$

$= 3.5$

(c) $E(x^2) = \Sigma x^2 P(x)$

$= \frac{457}{30}$

(d) $Var(x) = E(x^2) - (E(x))^2$

$= \frac{457}{30} - (3.5)^2 = \frac{457}{30} - \frac{49}{4} = \frac{179}{60} = 2.983$

(e) $S.D(x) = \sqrt{Var(x)} = 1.7272$

(f) $E(2x+3)$

$= 2E(x)+3$

$= 2(3.5)+3 = 10$

(g) $V(3x+4)$

$= 9 \cdot V(x) = 9 \times \frac{179}{60} = \frac{1611}{60}$

(h) $S.D(4x+5)$

For S.D we shall first find $var(4x+5)$

$Var(4x+5) = 16 \cdot V(x) = 16 \cdot \frac{179}{60} = \frac{716}{15}$

Now $S.D(4x+5) = \sqrt{Var(4x+5)} = \sqrt{\frac{716}{15}} = 6.91$

EXPECTATION CRV'S

EXPECTATION OF CONTINUOUS RANDOM VARIABLES

If 'x' is a continuous random variable, then the Mean or expected value of 'x' is given by

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx.$$

If 'x' is a continuous random variable with density function $f(x)$, then expectation of the function $g(x)$ is given by $E(g(x)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx.$

Note that

① $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

② $E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$

PROPERTIES OF EXPECTATION

If 'x' is a CRV and a, b are constants. then

(a) $E(a) = a$

(b) $E(ax) = a \cdot E(x)$

(c) $E(ax+b) = a \cdot E(x) + b$

VARIANCE AND STANDARD DEVIATION.

If x is a CRV then

1) $Var(x) = E(x^2) - (E(x))^2$

2) $S.D(x) = \sqrt{Var(x)}$

PROPERTIES OF VARIANCE.

If x is a CRV and a, b are constants then

1) $Var(a) = 0$

2) $Var(ax) = a^2 \cdot V(x)$

3) $Var(ax+b) = a^2 \cdot Var(x)$

EXAMPLES

EXAMPLE-① The continuous random variable x has p.d.f given by

$$f(x) = \begin{cases} k(x+3), & 0 \le x \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

Find the following

(a) the value of k.

(b) $P(X<1)$

(c) $E(X)$

(d) $E(X^2)$

(e) $E(2X+5)$

SOLUTION (a) $c=?$

WKT $\int_0^4 k(x+3)\,dx = 1$

$$k\left|\frac{x^2}{2}+3x\right|_0^4 = 1$$

$$k\left(\frac{\cancel{16}^{8}}{\cancel{2}}+12\right) = 1$$

$$20k = 1$$

$$\boxed{k = \frac{1}{20}}$$

(b) $P(X<1)$

$$= \int_0^1 \frac{1}{20}(x+3)\,dx = \frac{1}{20}\left|\frac{x^2}{2}+3x\right|_1^4$$

$$= \frac{1}{20}\left[(8+12)-\left(\frac{1}{2}+3\right)\right] = \frac{1}{20}\left[20-\frac{7}{2}\right] = \frac{33}{40}$$

(c) $E(x) = \int_0^4 x\cdot\frac{1}{20}(x+3)\,dx = \frac{1}{20}\int_0^4 (x^2+3x)\,dx$

$$= \frac{1}{20}\left|\frac{x^3}{3}+\frac{3}{2}x^2\right|_0^4 = \frac{1}{20}\left(\frac{64}{3}+\frac{48}{2}\right) = \frac{34}{15} = 2.267$$

(d) $E(x^2)$

$$= \int_0^4 x^2 \cdot \frac{1}{20} \cdot (x+3)\,dx = 6.4$$

(e) $E(2x+5)$

$$= 2E(x)+5 = 2\left(\frac{34}{15}\right)+5 = \frac{143}{15} = 9.53.$$

EXAMPLE-② The continuous random variable x has p.d.f given below

$$f(x) = \begin{cases} x/8, & 0 \le x \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

Find the following

(a) $E(x)$

(b) $E(x^2)$ (c) $Var(x)$ (d) $S.D(x)$ (e) $Var(3x+2)$.

SOLUTION

(a) $E(x) = \int_0^4 \frac{x}{8} \cdot x \cdot dx = \frac{1}{8}\int_0^4 x^2 dx = \frac{64}{24} = \frac{8}{3} = 2.67.$

(b) $E(x^2)$

$$= \frac{1}{8}\int_0^4 x^2 \cdot x\,dx = \frac{1}{8}\int_0^4 x^3 dx = 8.$$

(c) $Var(x)$

$$= E(x^2) - (E(x))^2 = 8 - \left(\frac{8}{3}\right)^2 = 8 - \frac{64}{9} = \frac{8}{9} = 0.89$$

(d) $S.D(x)$

$$= \sqrt{Var(x)} = \sqrt{\frac{8}{9}} = 0.942.$$

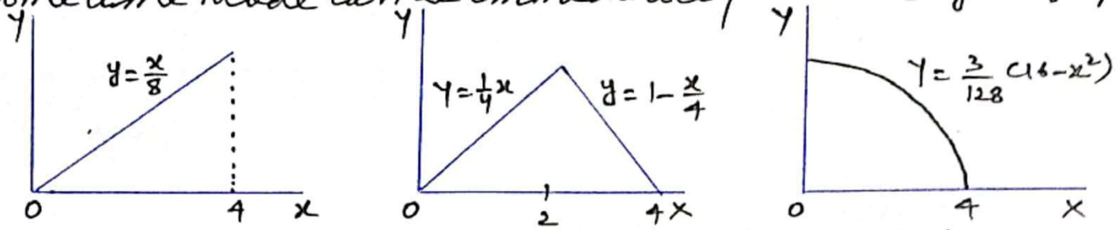
(e) $Var(3x+2)$

$$= 9 \cdot Var(x) = 9 \times \frac{8}{9} = 8$$

THE MODE

Mode is the value of X for which $f(x)$ is greatest in the given interval of X.

① Some time Mode can be immediately deduced by the graph.



Mode = 4   Mode = 2   Mode = 0

② Some times for Mode we use second derivative test. Stationary point (ie $x = a$) will be the Mode.

EXAMPLE ③ If 'X' is a continuous RV having p.d.f

$$f(x) = \begin{cases} \frac{1}{80}(2+x)(4-x) & , 0 \le x \le 4 \\ 0 & , \text{otherwise.} \end{cases}$$

Find Mode.

SOLUTION we apply second derivative test for Mode.

Here $f(x) = \frac{1}{80}(8 - 2x + 4x - x^2)$

$$= \frac{1}{80}(-x^2 + 2x + 8)$$

F.O.C. $f'(x) = 0$

$$\frac{1}{80}(-2x + 2) = 0$$

$$-2x + 2 = 0$$

$$2x = 2 \Rightarrow \boxed{x = 1}$$

S.O.C $f''(x) = -\frac{2}{80} = -\frac{1}{40} = -ve$

$\Rightarrow f(x)$ has Maximum value at $x = 1$

Hence $\boxed{\text{Mode} = 1}$

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EXAMPLE-④ A random variable x has p.d.f

$$f(x) = \begin{cases} Ax(6-x)^2, & 0 \le x \le 6 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the following
(a) Value of A
(b) Mean
(c) Mode (d) Variance (e) standard deviation

SOLUTION

(a) For the value of A. $\int_0^6 Ax(6-x)^2 dx = 1$

$108 \cdot A = 1$

$$\boxed{A = \frac{1}{108}}$$

(b) Mean

$$\text{Mean} = E(x) = \int_0^6 x \cdot x(6-x)^2 dx$$

$$= \int_0^6 x^2(6-x)^2 dx = 2.4$$

(c) Mode. Here $f(x) = \frac{1}{108} x(36+x^2-12x) = \frac{1}{108}(36x+x^3-12x^2)$

Now F.O.C $f'(x) = 0 \Rightarrow \frac{1}{108}(36+3x^2-24x) = 0$

$3x^2-24x+36=0$, $x^2-8x+12=0 \Rightarrow x=2, x=6$

Now S.O.C $f''(x) = \frac{1}{108}(6x-24)$

at $x=2$, $f''(x) = \frac{1}{108}(-12) = -ve \Rightarrow f(x)$ has Max value.

at $x=6$, $f''(x) = \frac{1}{108}(12) = +ve \Rightarrow f(x)$ has Min value.

So $\boxed{\text{Mode} = 2}$

(d) Variance. $Var(x) = E(x^2) - (E(x))^2$, Now

$$E(x^2) = \frac{1}{108}\int_0^6 x^2 \cdot x(6-x)^2 dx = \frac{1}{108}\int_0^6 x^3(6-x)^2 dx = 7.2$$

So $Var(x) = 7.2 - (2.4)^2 = 1.44$

(e) $S.D(x) = \sqrt{Var(x)} = \sqrt{1.44} = 1.2$

EXAMPLE-⑤ The time taken to perform a particular task, t hours, has the probability density function.

$$f(t) = \begin{cases} 10ct^2 & , 0 \le t \le 0.6 \\ 9c(1-t) & , 0.6 \le t \le 1.0 \\ 0 & , \text{otherwise.} \end{cases}$$

where 'c' is a constant.

(a) Find the value of c.

(b) write down the most likely time.

(c) Find the expected time.

(d) Find the probability that time will be more than 48 mins.

(e) " " " " " " b/w 24 and 48 mins.

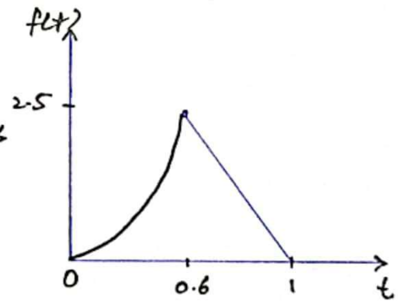
SOLUTION (a) For the value of 'c'. $\int_0^1 f(t)dt = 1$

or $\int_0^{0.6} 10ct^2 dt + \int_{0.6}^1 9c(1-t)dt = 1 \Rightarrow c = \frac{25}{36}$

(b) Mode = ? For Mode Sketch the fn.

From the graph Max value of t = 0.6 hrs

$\Rightarrow t = 0.6 \times 60 = 36$ Minutes.



(c) Expected time is

$E(t) = \int_0^1 t \cdot f(t) dt$

$= \int_0^{0.6} \frac{125}{18} t \cdot t^2 dt + \int_{0.6}^1 \frac{25}{4} \cdot t \cdot (1-t) dt = 0.591 \text{ hrs} = 35.5 \text{ mins}$

(d) $P(t > 48 \text{mins}) = P(t > 0.8 \text{hrs}) = \int_{0.8}^1 9c(1-t)dt$

$= \frac{25}{4} \int_{0.8}^1 (1-t)dt = 0.125$

(e) $P(\underset{\text{Mins}}{24 < t < 48}) = P(\underset{\text{HRS}}{0.4 < t < 0.8})$

$= \int_{0.4}^{0.6} 10ct^2 dt + \int_{0.6}^{0.8} 9c(1-t)dt = \frac{125}{18} \int_{0.4}^{0.6} t^2 dt + \frac{25}{4} \int_{0.6}^{0.8} (1-t)dt$

$= 0.727$