National University of Computer & Emerging Sciences Islamabad

FAST School of Computing | Fall-2024 | Islamabad Campus

MT1004 – Linear Algebra

Homework # 2

Question # 1

Determine if the vector $\boldsymbol{b} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$ is in the span of the columns of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

$Ax = b$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right] \begin{array}{l} R_2 - 4R_1 \\ R_3 - 7R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 - 2R_2$$

The last row is all zeros, so the system is consistent, free variable x_3 shows system has many solutions and as system is consistent, vector b is in the span of the columns of A.

vector $b = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$ is in the span of the columns of the matrix A.

Question # 2

Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$. For what value(s) of h is $\boldsymbol{w}$ in the plane generated by $\boldsymbol{u}$ and $\boldsymbol{v}$?

$$u \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + v \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = w \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{array}\right] \quad \text{Augmented matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & -5+2h \end{bmatrix} \quad R_3 + 2R_1$$

$$\sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 4+2h \end{bmatrix} \quad R_3 - 3R_2$$

~~for sol~~ If w is in plane generated by u and v (is in span of u and v span{u, v}) then system must be consistent. So

$4+2h=0$

$\boxed{h=-2}$

Question # 3

Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$$

(i) Do the columns of A span $\mathbb{R}^3$?

(ii) Do the columns of A span $\mathbb{R}^4$?

(iii) Does the equation $A\boldsymbol{x} = \boldsymbol{b}$ have a solution for each $\boldsymbol{b}$ in $\mathbb{R}^4$?

$$A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 3 & -11 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -7 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 + 7R_3 \end{array}$$

The columns of A do not span $\mathbb{R}^3$ since each column of A is in $\mathbb{R}^4$, not in $\mathbb{R}^3$.
Not every row of A contains a pivot position, therefore, columns of A do not span $\mathbb{R}^4$.
Again, since not every row of A contains a pivot position (or columns of A do not span $\mathbb{R}^4$), therefore, the equation $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution for each $\boldsymbol{b}$ in $\mathbb{R}^4$.

Question # 4

Let

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 6 \\ -12 \\ 0 \end{bmatrix}$$

(i) Describe all solutions of $A\boldsymbol{x} = \boldsymbol{b}$. Express the solutions in parametric form.

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 6 \\ -4 & -4 & -8 & -12 \\ 0 & -3 & -3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R1/2 \\ -R2/4 \\ -R3/3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R2-R1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Swap } R2, R3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 + 2x_3 = 3 \quad (i)$$
$$x_2 + x_3 = 0 \quad (ii)$$

x_3 free variable

$$x_3 = -x_2$$
$$x_1 - x_3 + 2x_3 = 3$$
$$\boxed{x_1 + x_3 = 3}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(ii) Describe all solutions of $A\boldsymbol{x} = \mathbf{0}$. Express the solutions in parametric form.

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Sol: $\text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(iii) Describe the geometric interpretation of the solution sets obtained in part **(i)** and **(ii)**.

1- Solution set for $Ax=b$: The solution set describes a line in R^3 that passes through the point $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ and , vector $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

2- Solution set for $Ax=0$: The solution set describes a line in R^3 that passes through the origin and vector $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.