MT-1004
Linear Algebra

Fall 2023

Week # 4

National University of Computer and Emerging Sciences

September 14, 2023

Section 1.8

Introduction to Linear Transformations

Motivation

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It's a special kind of transformation called a *linear transformation*.

This is also a way to understand the *geometry* of *matrices*.

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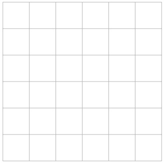

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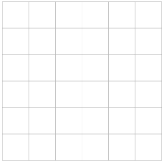

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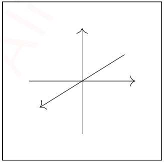

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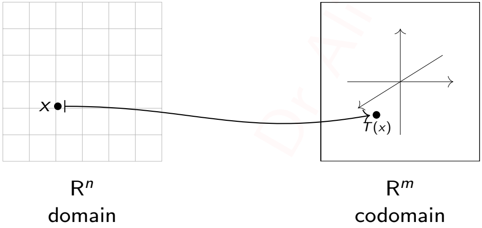
  Notation: $x \mapsto T(x)$.



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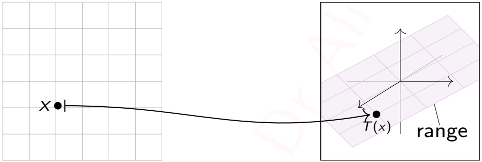

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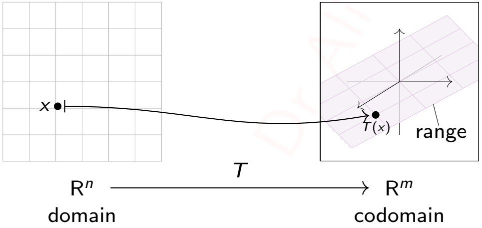

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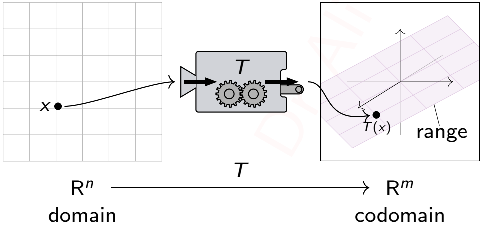

It may help to think of T as a "machine" that takes x as an input, and gives you $T(x)$ as the output.

Functions from Calculus

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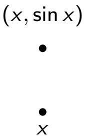

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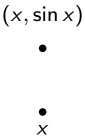

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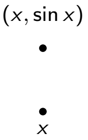

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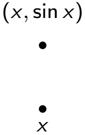

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Your life will be much easier if you just remember these.

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Matrix Transformations

Example

$$\text{Let } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and let } T(x) = Ax, \text{ so } T : \mathrm{R}^{-} \to \mathrm{R}^{-}.$$

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- Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in R^2 such that $T(v) = b$. Is there more than one?

  We want to find v such that $T(v) = Av = b$. We know how to do that:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} v = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \xrightarrow{\text{augmented matrix}} \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right).$$

  This gives $x = 2$ and $y = 5$, or $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ (unique). In other words,

$$T(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}.$$

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Matrix Transformations

Example, continued

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathrm{R}^2 \to \mathrm{R}^3$.

- Is there any c in R^3 such that there is more than one v in R^2 with $T(v) = c$?

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  Translation: Find c not in the column span of A (i.e., the range of T).

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  We could draw a picture, or notice: $a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ b \\ a+b \end{pmatrix}$.

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Matrix Transformations

Geometric example

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and let } T(x) = Ax, \text{ so } T : \mathrm{R}\text{–} \rightarrow \mathrm{R}\text{–}.$$

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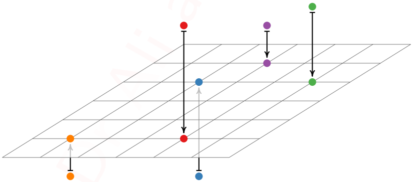
This is *projection onto the xy-axis*. Picture:



Matrix Transformations

Geometric example

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathrm{R}^{-} \to \mathrm{R}^{-}$.

Matrix Transformations

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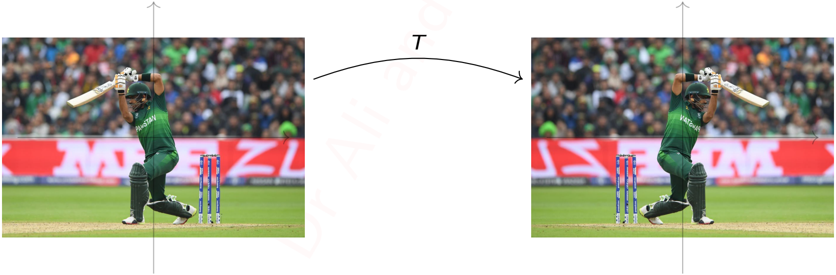
This is *reflection over the y-axis*. Picture:



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This is *reflection over the x-axis*.

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This is *reflection about origin*.

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathrm{R}^2 \to \mathrm{R}^2$. Then

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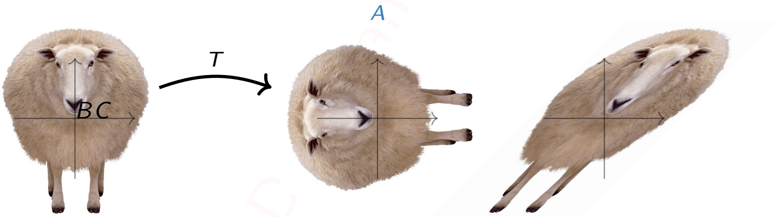
This is *reflection about the line* $y = x$.

Poll

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathrm{R}^2 \to \mathrm{R}^2$. ($T$ is called a **shear**.)

> **Poll**
>
> What does T do to this sheep?
>
> Hint: first draw a picture what it does to the box *around* the sheep.



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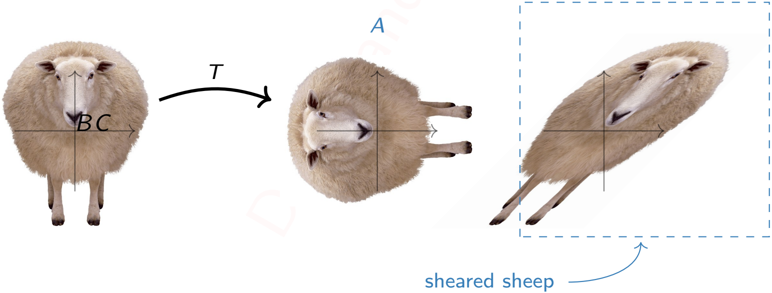

Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

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Check: if T is linear, then

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

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In engineering this is called **superposition**.

Define $T \colon \mathrm{R}^2 \to \mathrm{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

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$$\begin{aligned} T(u+v) &= 1.5(u+v) = 1.5u + 1.5v = T(u) + T(v) \\ T(cv) &= 1.5(cv) = c(1.5v) = c(Tv). \end{aligned}$$

Linear Transformations
Dilation

Define $T : \mathrm{R}^2 \to \mathrm{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

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So T satisfies the two equations, hence T is linear.

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This is called **dilation** or **scaling** (by a factor of 1.5). Picture:

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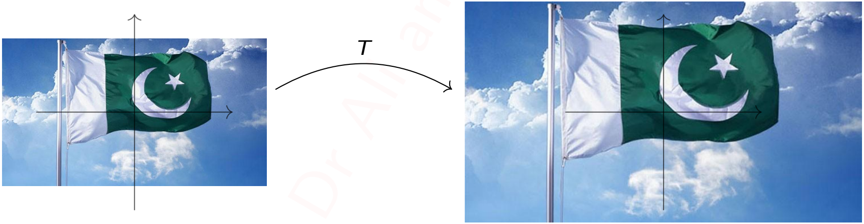

Linear Transformations
Rotation

Define $T\colon \mathrm{R}^2 \to \mathrm{R}^2$ by

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Is T linear? Check:

$$T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} + \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} -(u_2 + v_2) \\ (u_1 + v_1) \end{pmatrix} = T\begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix}$$

$$T\left(c\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix} = \begin{pmatrix} -cv_2 \\ cv_1 \end{pmatrix} = c\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = cT\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

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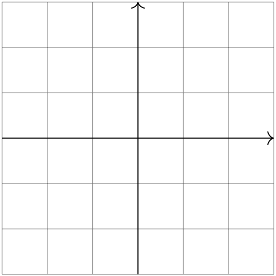

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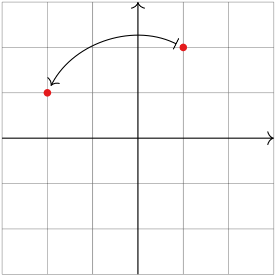

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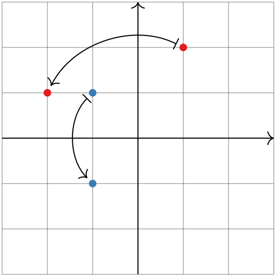

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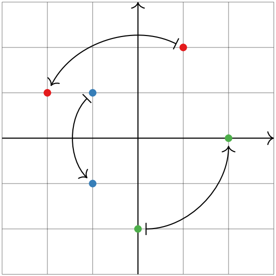

Section 1.9

The Matrix of a Linear Transformation

Unit Coordinate Vectors

Definition

The **unit coordinate vectors** in $\mathbb{R}^n$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

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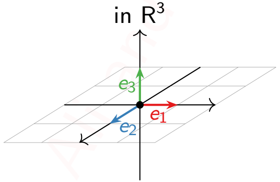

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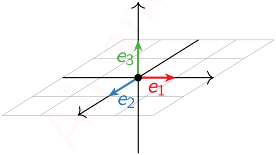

Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$:

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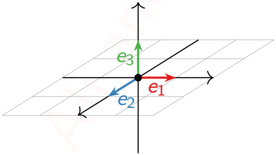

Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by e_i gives you the ith column.

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\4\\7\end{pmatrix} \qquad \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\5\\8\end{pmatrix} \qquad \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\6\\9\end{pmatrix}$$

Linear Transformations are Matrix Transformations

Recall: A matrix A defines a linear transformation T by $T(x) = Ax$.

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Theorem

Let $T \colon \mathrm{R}^n \to \mathrm{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an _____ matrix,

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Linear transformations are the same as matrix transformations.

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Linear Transformations are Matrix Transformations

Continued

Why is a linear transformation a matrix transformation?

Linear Transformations are Matrix Transformations

Continued

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Suppose for simplicity that $T\colon \mathrm{R}^3 \to \mathrm{R}^2$.

$$
\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\
&= T\big(xe_1 + ye_2 + ze_3\big) \\
&= xT(e_1) + yT(e_2) + zT(e_3) \\
&= \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}
$$

Linear Transformations are Matrix Transformations

Example

Before, we defined a **dilation** transformation $T : \mathrm{R}^2 \to \mathrm{R}^2$ by $T(x) = 1.5x$. What is its standard matrix?

$$\left.\begin{aligned} T(e_1) &= 1.5e_1 = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \\ T(e_2) &= 1.5e_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

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Check:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5 \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T: \mathrm{R}^3 \to \mathrm{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

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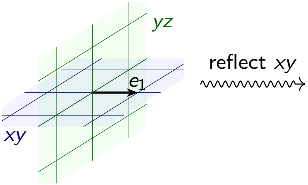

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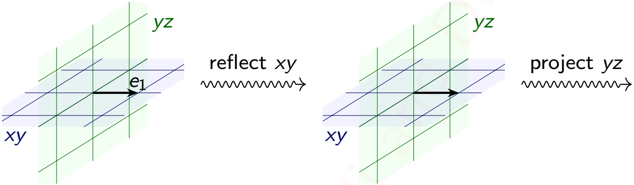

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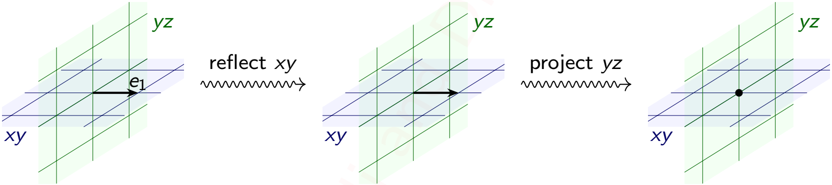

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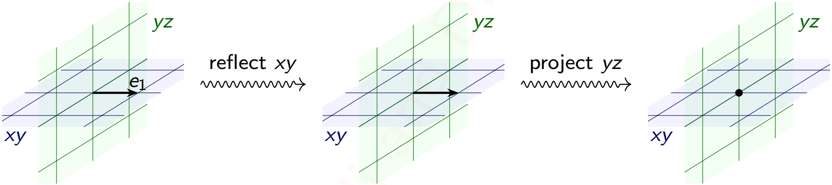

$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

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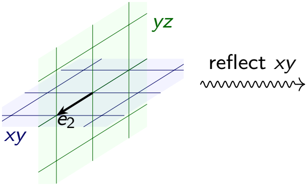

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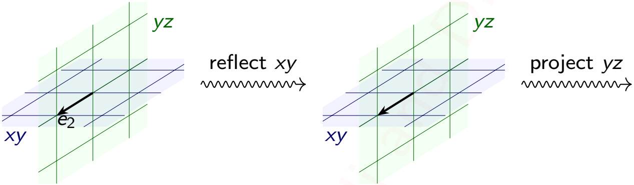

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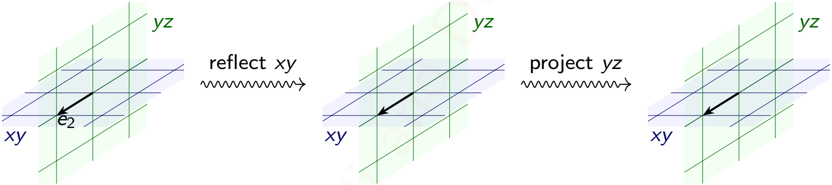

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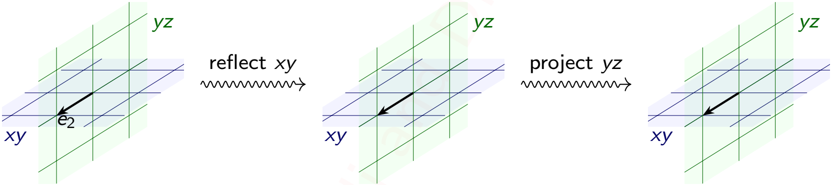

$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Linear Transformations are Matrix Transformations

Example, continued

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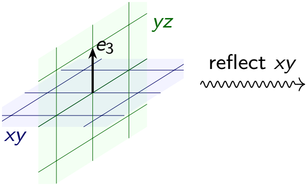

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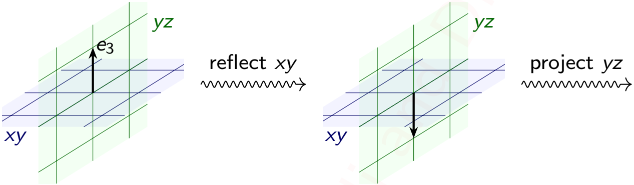

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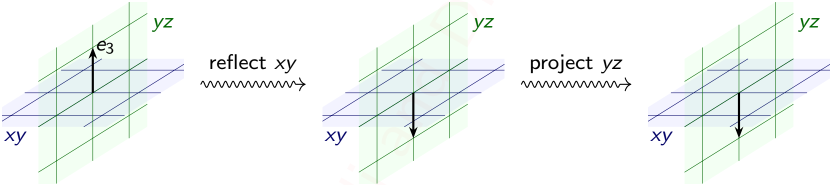

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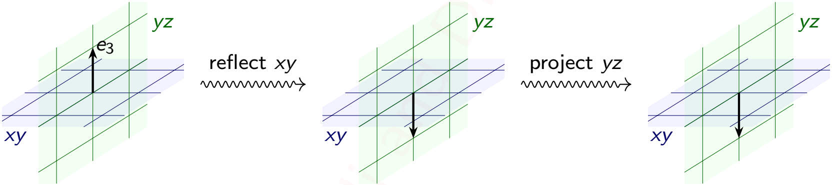

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Linear Transformations are Matrix Transformations

Example, continued

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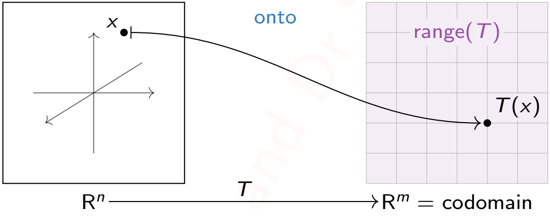

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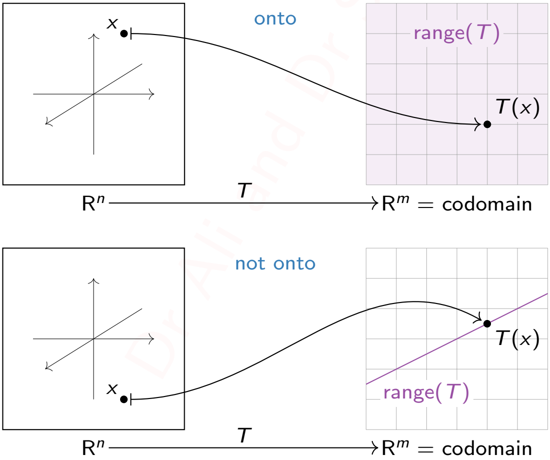

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For instance, $\mathbf{R}^2$ is "too small" to map *onto* $\mathbf{R}^3$.

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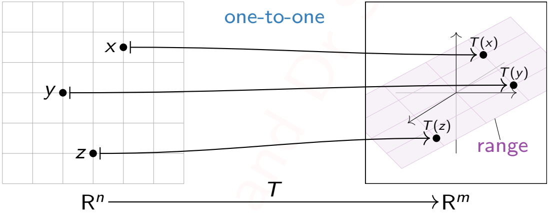

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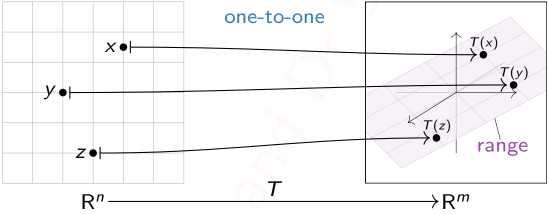

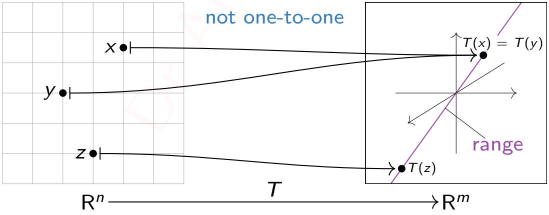

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For instance, $\mathbb{R}^3$ is "too big" to map *into* $\mathbb{R}^2$.

Questions

- What matrix transforms $(1, 0)$ into $(2, 5)$ and transforms $(0, 1)$ to $(1, 3)$?
- What matrix transforms $(2, 5)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$? Why does no matrix transform $(2, 6)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?
- What transformation take x_1 to Ax_1 and x_2 to Ax_2.

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to Ax_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to Ax_2 = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}.$$

  Means whenever we have transformation of two (independent) vectors in $\mathbb{R}^2$ we would be able to find transformation matrix.
- What matrix has the effect of rotating every vector through 90° and then project- ing the result onto the x -axis?
- What matrix represents projection onto the x-axis followed by projection onto the y-axis?

Questions

- he matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ produces a stretching in the x direction. Draw the circle $x^2 + y^2 = 1$ and sketch around it the points (2x, y) that result from multiplication by A. What shape is that curve?
- What 3×3 matrices represent the transformations that
  - (a) project every vector onto the x-y plane?
  - (b) reflect every vector through the x-y plane?
  - (c) rotate the x-y plane through 90, leaving the z-axis alone?
  - (d) rotate the x-y plane, then x-z, then y-z, through 90?
  - (e) carry out the same three rotations, but each one through 180?
- Every straight line remains straight after a linear transformation. If z is halfway between x and y, show that Az is halfway between Ax and Ay.