STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 7

PERMUTATIONS AND COMBINATIONS

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ARRANGEMENTS

THE BASIC COUNTING PRINCIPLE

If task A can be performed in *m* ways, followed by task B which can be performed in *n* ways, then task A followed by task B can be performed in $(m\times n)$ ways.

FACTORIAL FUNCTION Let ***n*** be a positive integer then factorial of ***n*** read as n factorial is denoted by ***n!*** and is given by

$$\boxed{n!=n\times(n-1)\times(n-2)\times\ldots\ldots\times3\times2\times1}$$

For example

- 1! = 1
- 2! = 2×1= 2
- 3! = $3\times2\times1$= 6

Note that

- $n!=n(n-1)!$
- 0! = 1

RESULTS

1. The number of ways of arranging *n* unlike objects in a line is given by $\boxed{n!.}$

2. The number of ways of arranging n objects in a line, of which p are alike, is given by $\boxed{\dfrac{n!}{p!}}$.

3. The number of ways of arranging n objects in a line, of which p of one type are alike, q of a second type are alike, r of a third type are alike, and so on, is given by

$$\boxed{\frac{n!}{p!\times q!\times r!\ldots}}$$

4. The number of ways of arranging n unlike objects in a ring, when clockwise and anticlockwise arrangements are different is given by

$$\boxed{(n-1)!.}$$

5. The number of ways of arranging n unlike objects in a ring, when clockwise and anticlockwise arrangements are the same, is given by

$$\boxed{\frac{(n-1)!}{2}}$$

EXAMPLES OF FACTORIAL FUNCTION

EXAMPLE-1 **(a)** Without using calculator evaluate the following

$$(a)\ \frac{8!}{6!} \qquad (b)\ \frac{n!}{(n-3)!}$$

(b) Write in Factorial Notation

$$(a)\ \frac{6.\,5.\,4}{4.\,3} \qquad (b)\ \frac{53.52.\,51.\,50}{21.\,20.\,19}$$

SOLUTION **(a)** Without using calculator evaluate the following

$$(a)\ \frac{8!}{6!} = \frac{8.\,7.\,6!}{6!} = 56$$

$$(b)\ \frac{n!}{(n-3)!} = \frac{n.(n-1)(n-2)(n-3)!}{(n-3)!} = n^3 - 3n^2 + 2n$$

(b) Write in Factorial Notation

$$(a)\ \frac{6.\,5.\,4}{4.\,3} = \frac{6.\,5.\,4.\,3!.\,2!}{3!.\,4.\,3.\,2!} = \frac{6!.\,2!}{3!.\,4!}$$

$$(b)\ \frac{53.52.\,51.\,50}{21.\,20.\,19} = \frac{53.\,52.\,51.\,50.\,49!.\,18!}{49!.\,21.\,20.\,19.\,18!} = \frac{53!.\,18!}{49!.\,21!}$$

EXAMPLE-2 In how many ways Ayesha, Saima, Sana and Zara arrange themselves in a Queue.

SOLUTION

$$Possible\ Arrangements = 4! = 24$$

EXAMPLE-3 **(a)** How many new arrangements can be formed from the letters of the word **EGO**.
(b) How many new arrangements can be formed from the letters of the word **EGG**.
(c) How many new arrangements can be formed from the letters of the word **MISSISSIPPI**.

SOLUTION

(a) $Here\ total\ number\ of\ letters = 3$

$$Possible\ Arrangements = 3! = 6$$

(b) $Here\ total\ number\ of\ letters = 3$

$G\ is\ repeated\ 2\ times$

$$Possible\ Arrangements = \frac{3!}{2!} = 3$$

(c) $Here\ total\ number\ of\ letters = 11$

$I\ is\ repeated\ 4\ times$

$S\ is\ repeated\ 4\ times$

$P\ is\ repeated\ 2\ times$

$$Possible\ Arrangements = \frac{11!}{4! \times 4! \times 2!} = 34650\ ways$$

EXAMPLE-4 In how many different ways can a set of four different mathematics books and 5 different physics books be placed on a shelf with a space of 9 books.

(a) *If books are placed in any order*

(b) *If books on same subject are kept together*

SOLUTION

(a) *If books are placed in any order*

$$Possible\ Arrangements = 9! = 362880\ ways$$

(b) *If books on the same subject are kept together*

$$Possible\ Arrangements = 4! \times 5! \times 2! = 5760\ ways$$

EXAMPLE-5

(a) *In how many ways can* 5 *persons be seated at a round table.*

(b) *In how many ways can a necklace of* 8 *beads of different colours be made.*

SOLUTION

(a) *If* 5 *persons can be seated at a round table than*

$$Possible\ ways = (5-1)! = 4! = 24\ ways$$

(b) *If a necklace of* 8 *beads of different colours be made than*

$$Possible\ ways = \frac{1}{2}(8-1)! = \frac{1}{2} \times 7! == 2520\ ways$$

PERMUTATIONS AND COMBINATIONS

PERMUTATION A permutation is an arrangement of objects in a definite order. The number of permutations of n different objects taken r at a time is denoted by $^{n}P_{r}\ \ or\ \ P(n,r)$ and is given by

$$\boxed{^{n}P_{r}=\frac{n!}{(n-r)!}}$$

For example

- $^{5}P_{3}=60$
- $^{8}P_{2}=56$

Note that

- $^{n}P_{n}=n!$
- $^{n}P_{n-1}=n!$
- $^{n}P_{1}=n$
- $^{n}P_{0}=1$

COMBINATION A combination is an arrangement of objects without regard for the order in which they are arranged. The number of combinations of n different objects taken r at a time is denoted by $^{n}C_{r}\ \ or\ \ C(n,r)$ and is given by

$$\boxed{^{n}C_{r}=\frac{n!}{r!\times(n-r)!}}$$

For example

- $^{5}C_{3}=10$
- $^{8}C_{2}=28$

Note that

- $^{n}C_{n}=1$
- $^{n}C_{n-1}=n$
- $^{n}C_{1}=n$
- $^{n}C_{0}=1$
- $^{n}C_{r}={}^{n}C_{n-r}$

RELATION BETWEEN PERMUTATIONS AND COMBINATIONS

$$\boxed{^{n}P_{r}={}^{n}C_{r}\times r!}$$

EXAMPLES OF PERMUTATIONS

EXAMPLE-1 **(a)** Evaluate the following without using calculator

$(a)\ {}^7P_3 \quad (b)\ {}^6P_0 \quad (c)\ {}^5P_1 \quad (d)\ {}^5P_4 \quad (e)\ {}^5P_5$

(b) Find the value of n if ${}^nP_2 = 20$

SOLUTION **(a)** Evaluate the following without using calculator

$$(a)\ {}^7P_3 = \frac{7!}{4!} = \frac{7.\,6.\,5.\,4!}{4!} = 210$$

$$(b)\ {}^6P_0 = 1$$

$$(c)\ {}^5P_1 = 5$$

$$(d)\ {}^5P_4 = 5! = 120$$

$$(e)\ {}^5P_5 = 5! = 120$$

(b) We have to find the value of n if ${}^nP_2 = 20$

$$Here\ {}^nP_2 = 20$$

$$\frac{n!}{(n-2)!} = 20$$

$$\frac{n.\,(n-1).\,(n-2)!}{(n-2)!} = 20$$

$$n^2 - n = 20$$

$$n^2 - n - 20 = 0$$

$$n = 5,\ -4$$

$$So\ n = 5$$

EXAMPLE-2 **(a)** How many different 4-digit numbers can be formed out of the digits $1, 2, 3, 4, 5, 6$ when no digit is repeated.

(b) How many signals can be made using with 4 different flags when any number of them are to be used at a time.

SOLUTION

(a) We have to find 4-digit numbers out of the digits $1, 2, 3, 4, 5, 6$ when no digit is repeated. So the required number is ${}^6P_4 = 360$

(b) We can use one, two, three or four flags at a time from the four flags. So total number of signals in this case will be

$$Total\ number\ of\ Signals = {}^4P_1 + {}^4P_2 + {}^4P_3 + {}^4P_4 = 64$$

EXAMPLE-3 Calculate how many different 5-digit numbers can be formed from the eight digits $1, 3, 4, 5, 6, 7, 8, 9$ used without repetition. How many of these 5-digit numbers are
(a) less than 40,000.
(b) Even

SOLUTION $Total\ different\ 5-digit\ numbers = {}^8P_5 = 6720$

(a) For a number to be less than 40,000, first digit must be 1 or 3.

$$So\ Total\ 5-digit\ numbers\ less\ than\ 40{,}000 = {}^2P_1 \times {}^7P_4 = 1680$$

(b) For a number to be even last digit must be 4,6 or 8.

$$So\ Total\ even\ 5-digit\ numbers = {}^3P_1 \times {}^7P_4 = 2520$$

EXAMPLE-4 Find the number of permutations of all the letters in the word **HISTORY.** Find the number of these permutations in which
(a) *the letters O and R are together*
(b) *the letters O and R are not together*

SOLUTION $Total\ number\ of\ Permutations = {}^7P_7 = 7! = 5040$

(a) *Number of Permutations when letters O and R are together*
$$Possible\ Arrangements = {}^6P_6 \times {}^2P_2 = 6! \times 2! = 1440$$

(b) *Number of Permutations when letters O and R are not together*
$$Possible\ Arrangements = 5040 - 1440 = 6! \times 2! = 3600$$

EXAMPLE-5 4 boys 5 girls are to form a line. In how many ways can this be done. Find also the number of permutations in which
(a) *the first two are girls*
(b) *the first is a boy and last is a girl*
(c) *the boys are together*
(d) *No two girls stand next to each other*

SOLUTION $Total\ number\ of\ Arrangements = {}^9P_9 = 9! = 362880$

(a) *Number of Permutations that first two are girls* $= {}^5P_2 \times {}^7P_7 = 100{,}800$ *ways*

(b) *Number of Permutations that first is a boy and last is a girl* $= {}^4P_1 \times {}^5P_1 \times {}^7P_7 = 100{,}800,$ *ways*

(c) *Number of Permutations that all boys are together* $= {}^6P_6 \times {}^4P_4 = 6! \times 4! = 17280$ *ways*

(d) *Number of Permutations that No two girls stand next to each other* $= {}^5P_5 \times {}^4P_4 = 2880$ *ways*
$(Hint: GBGBGBGBG)$

EXAMPLE-6 Ten students placed at random in a line. What is the probability that the two youngest students are separated.

SOLUTION $If\ S\ denote\ the\ sample\ space\ then\ n(S) = {}^{10}P_{10} = 10!$

$Let\ A\ denote\ the\ event\ that\ two\ youngest\ students\ are\ together\ then\ n(A) = {}^{10}P_{10} = 9! \times 2!$

$Now\ \Pr obability\ that\ two\ youngest\ students\ are\ together\ is\ P(A) = \frac{n(A)}{n(S)} = \frac{9! \times 2!}{10!} = \frac{1}{5}$

$Now\ \Pr obability\ that\ two\ youngest\ students\ are\ seperated\ is\ P(A^c) = 1 - P(A) = 1 - \frac{1}{5} = \frac{4}{5}$

EXAMPLE-7 If a four digit number is formed from the digits $1, 2, 3\ and\ 5$ and repetitions are not allowed, find the probability that number is divisible by 5.

SOLUTION $If\ S\ denote\ the\ sample\ space\ then\ n(S) = {}^{4}P_{4} = 4! = 24$

$Let\ A\ denote\ the\ event\ that\ four\ digit\ number\ is\ divisible\ by\ 5\ then\ n(A) = {}^{1}P_{1} \times {}^{3}P_{3} = 1! \times 3! = 6$

$Now\ \Pr obability\ that\ four\ digit\ number\ is\ divisible\ by\ 5\ is\ P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$

EXAMPLE-8 The letters of the word MATHEMATICS are written, one on each of 11 separate cards. The cards are laid out in a line, determine the probability that the vowels are placed together.

SOLUTION $If\ S\ denote\ the\ sample\ space\ then\ n(S) = \frac{11!}{2! \times 2! \times 2!} = 4989600$

$Let\ A\ denote\ the\ event\ that\ vowels\ are\ placed\ together\ then\ n(A) = \frac{8!}{2! \times 2!} \times \frac{4!}{2!} = 120960$

$Now\ \Pr obability\ that\ vowels\ are\ placed\ together\ is\ given\ by\ P(A) = \frac{n(A)}{n(S)} = \frac{120960}{4989600} = 0.024$

EXAMPLE-9 How many different numbers can be formed by taking one, two, three and four digits from the digits $1, 2, 7\ and\ 8$ if repetitions are not allowed, One of these numbers is chosen at random, find the probability that number is greater than 200.

SOLUTION $If\ S\ denote\ the\ sample\ space\ then\ n(S) = {}^{4}P_{1} + {}^{4}P_{2} + {}^{4}P_{3} + {}^{4}P_{4} = 64$

$Let\ A\ denote\ the\ event\ that\ required\ number\ is\ greater\ than\ 200\ then\ n(A) = ({}^{3}P_{1} \times {}^{3}P_{2}) + {}^{4}P_{4} = 42$

$Now\ \Pr obability\ that\ required\ number\ is\ greater\ than\ 200\ is\ given\ by\ P(A) = \frac{n(A)}{n(S)} = \frac{42}{64} = \frac{21}{32}$

EXAMPLE-10 One white, one blue, one red and two yellow beads are threaded on a ring to make a bracelet. Find the probability that red and white beads are next to each other.

SOLUTION $If\ S\ denote\ the\ sample\ space\ then\ n(S) = \frac{1}{2} \times \frac{4!}{2!} = 6$

$Let\ A\ denote\ the\ event\ that\ red\ and\ white\ beads\ are\ next\ to\ each\ other\ then\ n(A) = \frac{1}{2} \times \frac{3!}{2!} \times 2! = 3$

$Now\ \Pr obability\ that\ red\ and\ white\ beads\ are\ next\ to\ each\ other\ is\ given\ by\ P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

EXAMPLES OF COMBINATIONS

EXAMPLE-1 **(a)** Evaluate the following without using calculator

$(a)\ {}^5C_3 \quad (b)\ {}^3C_2 \quad (c)\ {}^4C_1 \quad (d)\ {}^6C_0 \quad (e)\ {}^5C_5$

(b) Find the value of n *if* ${}^nC_8 = {}^nC_{12}$

SOLUTION **(a)** Evaluate the following without using calculator

$$(a)\ {}^5C_3 = \frac{5!}{3!\times 2!} = 10 \quad (b)\ {}^3C_2 = 3 \quad (c)\ {}^4C_1 = 4 \quad (d)\ {}^6C_0 = 1 \quad (e)\ {}^5C_5 = 1$$

(b) We have to find the value of n *if* ${}^nC_8 = {}^nC_{12}$

$$We\ know\ that\ {}^nC_r = {}^nC_{n-r} \ \Rightarrow {}^nC_8 = {}^nC_{n-8}$$

$$Hence\ using\ given\ equation\ {}^nC_{n-8} = {}^nC_{12}$$

$$n-8 = 12 \ \Rightarrow n = 12+8 = 20$$

EXAMPLE-2 How many diagonals can be formed by joining the vertices of the polygon having $(a)\ 5\ Sides \quad (b)\ n\ Sides$

SOLUTION **(a)** For 5 sided polygon

$$Number\ of\ Diagonals = {}^5C_2 - 5 = 10 - 5 = 5$$

(b) For n sided polygon

$$Number\ of\ Diagonals = {}^nC_2 - n = \frac{n(n-3)}{2}$$

EXAMPLE-3 A group of 5 members is to be selected from 6 boys and 4 girl. Find the number of ways in which this can be done. If

(a) *there are no restrictions*

(b) *Group has exactly* 3 *boys*

(c) *Group has atleast* 1 *girl*

(d) *Group has atmost* 2 *boys*

(e) *Group has more boys than girls*

SOLUTION $Here\ Boys = 6,\ Girls = 4,\ Total = 10,\ Selected = 5$

(a) *Number of ways if there are no restrictions* $= {}^{10}C_5 = 252$ *ways*

(b) *Number of ways if group has exactly* 3 *boys* $= {}^6C_3 \times {}^4C_2 = 120$ *ways*

(c) *Number of ways if group has atleast* 1 *girl*

$$= ({}^4C_1 \times {}^6C_4) + ({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1) = 246$$

(d) *Number of ways if group has atmost* 2 *boys*

$$= ({}^6C_2 \times {}^4C_3) + ({}^6C_1 \times {}^4C_4) = 66$$

(e) *Number of ways if group has more boys than girls*

$$= ({}^6C_5 \times {}^4C_0) + ({}^6C_4 \times {}^4C_1) + ({}^6C_3 \times {}^4C_2) = 186$$

EXAMPLE-4 A delegation of 4 people is to be selected from 5 women and 6 men. Find the number of possible delegations if

(a) *there are no restrictions*

(b) *there is atleast one woman*

(c) *there are atmost two men*

One of the man cannot get along with one of the woman. Find the number of delegations which include this particular man or woman but not both.

SOLUTION *Here* $Men = 6,\ Women = 5,\ Total = 11,\ Selected = 4$

(a) *Number of delegations if there are no restrictions* $= {}^{11}C_4 = 330$ *ways*

(b) *Number of delegations if there is atleast one woman*

$$= ({}^5C_1 \times {}^6C_3) + ({}^5C_2 \times {}^6C_2) + ({}^5C_3 \times {}^6C_1) + ({}^5C_4 \times {}^6C_0) = 315$$

(c) *Number of delegations if there are atmost two men*

$$= ({}^6C_2 \times {}^5C_2) + ({}^6C_1 \times {}^5C_3) + ({}^6C_0 \times {}^5C_4) = 215$$

If one of the man cannot get along with one of the woman. Then the number of delegations which include this particular man or woman but not both.

Here Number of delegations if there are no restrictions $= {}^2C_1 \times {}^9C_3 = 168$

EXAMPLE-5 A School club has members from 3 different year-groups. Year-1, Year-2 and Year-3. There are seven members from Year-1, 2 members from Year-2 and 2 members from Year-3. Five members of the club are selected. Find the

(a) Number of possible selections that include 2 members of Year-1,1 member of Year-2 and 2 members of Year-3.

(b) Number of possible selections that include at least one member from each group.

SOLUTIO *Here* $Year-1 = 7,\ Year-2 = 2,\ Year-3 = 2,\ Selected = 5$

(a) *Number of possible selections that include 2 members of Year*-1, *1 member of Year*-2 *and 2 members of Year*-3 $= {}^7C_2 \times {}^2C_1 \times {}^2C_2 = 42$

(b) *Number of possible selections that include at least one member from each group*

$$= ({}^7C_1 \times {}^2C_2 \times {}^2C_2) + ({}^7C_2 \times {}^2C_2 \times {}^2C_1) + ({}^7C_2 \times {}^2C_1 \times {}^2C_2) + ({}^7C_3 \times {}^2C_1 \times {}^2C_2) = 231$$

EXAMPLE-6 A Box contains 2 white, 3 red and 5 blue balls. Three balls are selected at random from the box. What is the probability that selected balls contain

(a) only blue balls (b) balls of every color (c) balls of the same color

SOLUTION *Here* $White\ Balls = 2,\ Red\ Balls = 3,\ Blue\ Balls = 5,\ Total = 10,\ Selected = 3$

If S denote the sample space then $n(S) = {}^{10}C_3 = 120$

(a) *If A denote the event that selected balls contains blue balls then* $P(A) = \dfrac{{}^5C_3 \times {}^2C_0 \times {}^3C_0}{{}^{10}C_3} = \dfrac{10}{120} = \dfrac{1}{12}$

(b) *If B denote the event that selected balls of every color then* $P(B) = \dfrac{{}^2C_1 \times {}^3C_1 \times {}^5C_1}{{}^{10}C_3} = \dfrac{30}{120} = \dfrac{1}{4}$

(c) *If C denote the event that selected balls of Same color then* $P(C) = \dfrac{({}^5C_3 \times {}^2C_0 \times {}^3C_0) + ({}^5C_0 \times {}^2C_0 \times {}^3C_3)}{{}^{10}C_3} = \dfrac{11}{120}$

EXAMPLE-7 Two cards are selected from the deck of 52 playing cards, what is the probability that

(a) both are aces
(b) one of them is a King
(c) at least one of them is of black color

SOLUTION *If S denote the sample space then* $n(S) = {}^{52}C_2 = 1326$

$$(a)\ \text{If } A \text{ denote the event that both cards are aces then } P(A) = \frac{{}^{4}C_2 \times {}^{48}C_0}{{}^{52}C_2} = \frac{6}{1326} = \frac{1}{221}$$

$$(b)\ \text{If } B \text{ denote the event that one of them is a king then } P(B) = \frac{{}^{4}C_1 \times {}^{48}C_1}{{}^{52}C_2} = \frac{192}{1326} = \frac{32}{221}$$

(c) *If C denote the event that at least one of them is black color then*

$$P(C) = \frac{({}^{26}C_1 \times {}^{26}C_1) + ({}^{26}C_2 \times {}^{26}C_0)}{{}^{10}C_3} = \frac{1001}{1326} = \frac{77}{102}$$

EXAMPLE-8 Four letters are picked from the word BREAKDOWN. What is the probability that there is at least one vowel among the letters.

SOLUTION *If S denote the sample space then* $n(S) = {}^{9}C_4 = 126$

Let A denote the event that there is one vowel among the letters

$$\text{then } P(A) = \frac{({}^{3}C_1 \times {}^{6}C_3) + ({}^{3}C_2 \times {}^{6}C_2) + ({}^{3}C_3 \times {}^{6}C_1)}{{}^{9}C_4} = \frac{111}{126} = \frac{37}{42}$$

EXAMPLE-9 From a group of ten people, four are to be chosen to serve on a committee. Among the ten people there is one married couple, Find the probability that both husband and wife will be chosen.

SOLUTION *If S denote the sample space then* $n(S) = {}^{10}C_4 = 210$

$$\text{Let } A \text{ denote the event that both husband and wife will be chosen then } P(A) = \frac{{}^{2}C_2 \times {}^{8}C_2}{{}^{10}C_4} = \frac{2}{15}$$

EXAMPLE-10 A company has to select three persons from the candidates including three men and four women. What is the probability that selected persons contain

(a) one man and two woman **(b)** at least one man **(c)** at most two woman

SOLUTION *Here* $Men = 3,\ Women = 4,\ Total = 7,\ Selected = 3$

If S denote the sample space then $n(S) = {}^{7}C_3 = 35$

$$\text{Let } A \text{ denote the event that selection contains one man and two woman then } P(A) = \frac{{}^{3}C_1 \times {}^{4}C_2}{{}^{7}C_3} = \frac{18}{35}$$

$$\text{Let } B \text{ denote the event that selection contains at least one man then } P(B) = \frac{{}^{3}C_1 \times {}^{4}C_2 + {}^{3}C_2 \times {}^{4}C_1 + {}^{3}C_3 \times {}^{4}C_0}{{}^{7}C_3} = \frac{31}{35}$$

$$\text{Let } C \text{ denote the event that selection contains at most two women then } P(C) = \frac{{}^{4}C_2 \times {}^{3}C_1 + {}^{4}C_1 \times {}^{3}C_2 + {}^{4}C_0 \times {}^{3}C_3}{{}^{7}C_3} = \frac{31}{35}$$