STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE # 27

HYPOTHESIS TESTING

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HYPOTHESIS TESTING

BASIC TERMS AND CONCEPTS

HYPOTHESIS TESTING It is a procedure to determine whether to accept or reject a statement or an assumption about the value of a population parameter on the basis of observed random sample.

STATISTICAL HYPOTHESIS It is a statement about a characteristic of one or more populations. This statement may or may not be true. Its validity is tested on the basis of an observed random sample.

NULL HYPOTHESIS Any hypothesis which is to be tested for possible rejection under the assumption that it is true is called a null hypothesis. It is denoted by H_o.
For example $H_0 = 25,\quad H_0 \leq 25,\quad H_0 \geq 25$

Null Hypothesis A **null hypothesis** is a claim (or statement) about a population parameter that is assumed to be true until it is declared false.

ALTERNATE HYPOTHESIS Any hypothesis which we accept when null hypothesis is rejected is called alternate hypothesis. It is denoted by H_1.
For example $H_1 \neq 25,\quad H_1 > 25,\quad H_1 < 25$

Alternative Hypothesis An **alternative hypothesis** is a claim about a population parameter that will be declared true if the null hypothesis is declared to be false.

KEY POINTS ABOUT NULL AND ALTERNATIVE HYPOTHESIS

The following key points summarize the null and alternative hypotheses:

- The null hypothesis, H_0 , represents the current belief in a situation.
- The alternative hypothesis, H_1 , is the opposite of the null hypothesis and represents a research claim or specific inference you would like to prove.
- If you reject the null hypothesis, you have statistical proof that the alternative hypothesis is correct.
- If you do not reject the null hypothesis, you have failed to prove the alternative hypothesis. The failure to prove the alternative hypothesis, however, does not mean that you have proven the null hypothesis.
- The null hypothesis, H_0 , always refers to a specified value of the population parameter (such as μ), not a sample statistic (such as $\bar{x}$).
- The statement of the null hypothesis always contains an equal sign regarding the specified value of the population parameter. $(e.g\ \ \mu = 368\ gms)$.
- The statement of the alternative hypothesis never contains an equal sign regarding the specified value of the population parameter $(e.g\ \ \mu \neq 368\ gms)$

TYPES OF ERRORS

If we either reject a true hypothesis or accept a false hypothesis, we have made an incorrect decision. These two types of errors are called the Type 1 error and Type 2 error respectively. Thus

Type 1 error: Rejection of $H_{\circ}$ when $H_{\circ}$ is true.

Type 2 Error: Acceptance of $H_{\circ}$ when H_1 is true.

Type I Error A **Type I error** occurs when a true null hypothesis is rejected. The value of α represents the probability of committing this type of error; that is,

$$\alpha = P(H_0 \text{ is rejected} \mid H_0 \text{ is true})$$

The value of α represents the **significance level** of the test.

EXAMPLE If a person is declared guilty at the end of a trial, there are two possibilities.
1. The person has *not* committed the crime but is declared guilty
2. The person *has* committed the crime and is rightfully declared guilty.
In the first case, the court has made an error by punishing an innocent person. In statistics, this kind of error is called a **Type I** or an ***α*** (*alpha*) **error**. In the second case, because the guilty person has been punished, the court has made the correct decision.

Type II Error A **Type II error** occurs when a false null hypothesis is not rejected. The value of β represents the probability of committing a Type II error; that is,

$$\beta = P(H_0 \text{ is not rejected} \mid H_0 \text{ is false})$$

The value of $1 - \beta$ is called the **power of the test**. It represents the probability of rejecting H_0 when it is false.

EXAMPLE Suppose that in the court trial case the person is declared not guilty at the end of the trial. in this situation there are again two possibilities.
1. The person has *not* committed the crime and is declared not guilty.
2. The person *has* committed the crime but, *because of the lack of enough evidence*, is declared not guilty
In the first case, the court's decision is correct. In the second case, however, the court has committed an error by setting a guilty person free. In statistics, this type of error is called a **Type II** or a ***β*** (the Greek letter *beta*) **error**.

	ACTUAL SITUATION	
STATISTICAL DECISION	H_0 True	H_0 False
Do not reject H_0	Correct decision Confidence coefficient $= (1 - \alpha)$	Type II error $P(\text{Type II error}) = \beta$
Reject H_0	Type I error $P(\text{Type I error}) = \alpha$	Correct decision Power $= (1 - \beta)$

LEVEL OF SIGNIFICANCE

DEFINITION The **significance level** of a hypothesis test is the maximum acceptable probability of rejecting a true null hypothesis.

The **level of significance** α is the probability of rejecting H_0 when it is true. This is the probability of a type I error.

The reason for specifying α (rather than β) for a hypothesis test is based on the premise that the type I error is of prime concern. For this reason the hypothesis statement must be set up in such a manner that the type I error is indeed the more costly. The significance level is then chosen considering the cost of making that error.

We usually select $\alpha = 1\% = 0.01,\ 5\% = 0.05\ and\ 10\% = 0.1$

TEST STATISTIC

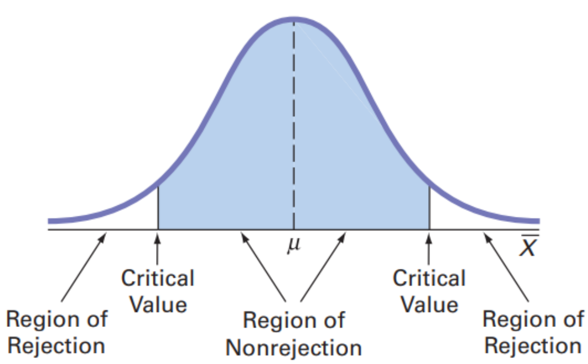
The **test statistic** is a sample statistic whose sampling distribution can be specified for both the null and alternative hypothesis case (although the sampling distribution when the alternative hypothesis is true may often be quite complex). After specifying the appropriate significance level of α, the sampling distribution of this statistic is used to define the rejection region.

REJECTION AND NON REJECTION REGIONS

CRITICAL VALUES The values of the test statistic which separates their rejection and non-rejection rejoin for the test are called critical values.

CRITICAL REGION A region which specifies a set of values of the test statistic for which the null hypothesis Ho is rejected (and for which alternate hypothesis H_1 is accepted) is called critical region. It is also called **rejection rejoin**.

NON-CRITICAL REGION A region which specifies a set of values of the test statistic for which null hypothesis is not rejected is called non-critical region or **non-rejection region**.



TYPES OF STATISTICAL TESTS

A test with two rejection regions is called a **two-tailed test**, and a test with one rejection region is called a **one-tailed test**. The one-tailed test is called a **left-tailed test** if the rejection region is in the left tail of the distribution curve, and it is called a **right-tailed test** if the rejection region is in the right tail of the distribution curve.

Tails of the Test A **two-tailed test** has rejection regions in both tails, a **left-tailed test** has the rejection region in the left tail, and a **right-tailed test** has the rejection region in the right tail of the distribution curve.

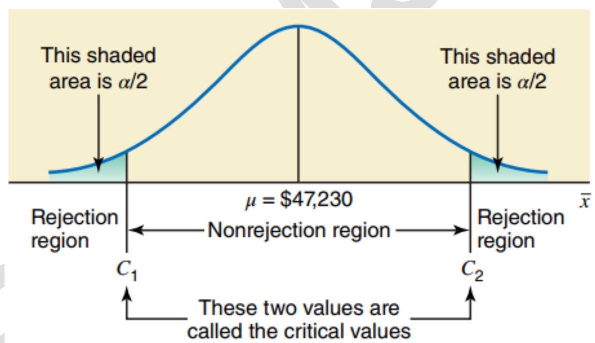
EXAMPLES OF TAILS OF A TEST

A Two-Tailed Test

Here

$$H_0: \mu = \$47{,}230$$

$$H_1: \mu \neq \$47{,}230$$

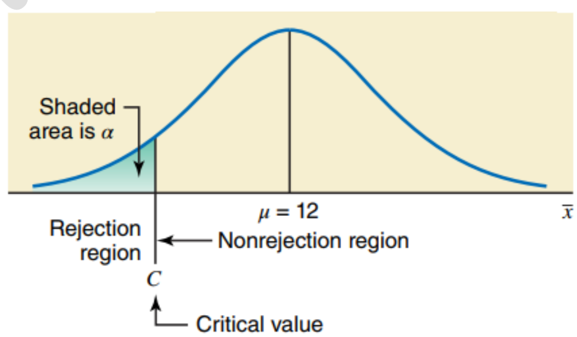


A Left-Tailed Test

Here

$$H_0: \mu = 12 \text{ ounces}$$

$$H_1: \mu < 12 \text{ ounces}$$

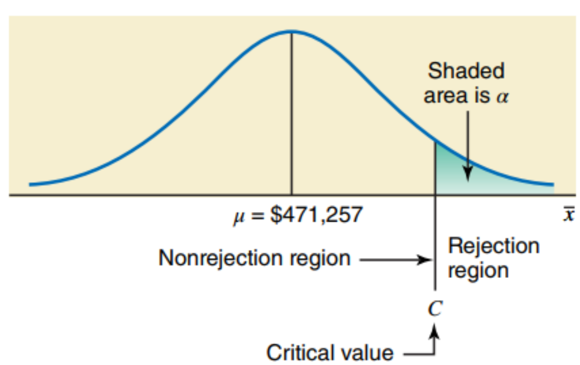


A Right-Tailed Test

Here

$$H_0: \mu = \$471{,}257$$

$$H_1: \mu > \$471{,}257$$



SIGNS IN NULL AND ALTERNATIVE HYPOTHESIS

Table below summarizes the foregoing discussion about the relationship between the signs in *H*0 and *H*1 and the tails of a test.

Signs in H_0 and H_1 and Tails of a Test

	Two-Tailed Test	Left-Tailed Test	Right-Tailed Test
Sign in the null hypothesis H_0	$=$	$=$ or $\geq$	$=$ or $\leq$
Sign in the alternative hypothesis H_1	$\neq$	$<$	$>$
Rejection region	In both tails	In the left tail	In the right tail

Note that the null hypothesis always has an *equal to* ($=$) or a *greater than or equal to* ($\geq$) or a *less than or equal to* ($\leq$) sign, and the alternative hypothesis always has a *not equal to* ($\neq$) or a *less than* ($<$) or a *greater than* ($>$) sign.

Table Hypothesis-Testing Common Phrases

$>$	$<$
Is greater than	Is less than
Is above	Is below
Is higher than	Is lower than
Is longer than	Is shorter than
Is bigger than	Is smaller than
Is increased	Is decreased or reduced from
$=$	$\neq$
Is equal to	Is not equal to
Is the same as	Is different from
Has not changed from	Has changed from
Is the same as	Is not the same as

THE P-VALUE OF A STATISTICAL TEST

Statisticians usually test hypotheses at the common a levels of 0.05 or 0.01 and sometimes at 0.10. Recall that the choice of the level depends on the seriousness of the type I error. Besides listing an alpha value, many computer statistical packages give a *P*-value for hypothesis tests.

P-values indicate the strength of the sample evidence against the null hypothesis. In simpler terms, p-values tell you how strongly your sample data contradict the null. Lower p-values represent stronger evidence against the null.

DEFINITION The ***p*-value** is the smallest significance level at which the null hypothesis is rejected.

DEFINITION The ***p*-value** is the probability of getting a test statistic equal to or more extreme than the sample result, given that the null hypothesis, *H*0, is true. The *p*-value is also known as the ***observed level of significance*** or ***exact level of significance.***

If the p-value is less than or equal to the significance level, you reject the null hypothesis and your results are statistically significant. The data support the alternative hypothesis that the effect exists in the population. When the p-value is greater than the significance level, your sample data don't provide enough evidence to conclude that the effect exists.

Decision Rule When Using a *P*-Value

If P-value $\leq \alpha$, reject the null hypothesis.

If P-value $> \alpha$, do not reject the null hypothesis.

If you need help remembering this general rule about comparing p-values to significance levels, here are two mnemonic phrases:

- When the p-value is low, the null must go.
- If the p-value is high, the null will fly.

STATISTICAL SIGNIFICANCE

When your p-value is less than the significance level, your results are statistically significant. This condition indicates the strength of the evidence in your sample (p-value) exceeds the evidentiary standard you defined (significance level). Your sample evidence provides sufficient evidence to conclude that the effect exists in the population.

DIFFRERENT APPROACHES OF HYPOTRHESIS TESTING

The three methods used to test the statistical hypothesis are

1. The Critical Value Approach
2. The Confidence Interval Approach
3. The P-Value Approach

THE CRITICAL VALUE APPROACH

STEP-1 STATISTICAL HYPOTHESIS
State the null hypothesis, H_0 and the alternative hypothesis, H_1.
STEP-2 LEVEL OF SIGNIFICANCE
Choose the level of significance, α and the sample size, n.
STEP-3 TEST STATISTIC
Determine the appropriate test statistic and sampling distribution. Also collect the sample data, organize the results and compute the value of the test statistic.
STEP-4 CRITICAL REGION
Determine the critical values that divide the rejection and non-rejection regions.
STEP-5 CONCLUSION
If the test statistic falls into the rejection region, you reject the null hypothesis, If the test statistic does not fall into the rejection region, you do not reject the null hypothesis

THE P-VALUE APPROACH

STEP-1 STATISTICAL HYPOTHESIS
State the null hypothesis, H_0 and the alternative hypothesis, H_1.
STEP-2 LEVEL OF SIGNIFICANCE
Choose the level of significance, α and the sample size, n.
STEP-3 TEST STATISTIC
Determine the appropriate test statistic and sampling distribution. Also collect the sample data, organize the results and compute the value of the test statistic.
STEP-4 P-VALUE
Compute the P-value using appropriate formula.
STEP-5 CONCLUSION
If the $p-value < \alpha$, reject the null hypothesis, If the $p-value \geq \alpha$,do not reject the null hypothesis.

THE CONFIDENCE INTERVAL APPROACH

STEP-1 CONFIDENCE INTERVAL
Compute the confidence interval using appropriate formula.
STEP-2 CONCLUSION
If the value of parameter in H_0 falls into the confidence interval, do not reject the null hypothesis.
If the value of parameter in H_0 does not fall into the confidence interval, reject the null hypothesis.

Z-TEST

If $H_\circ : (=, \leq, \geq)$ then $H_1 : (\neq, >, <)$

CRITICAL REGIONS IN Z-TEST

- If H_1 *contains* $\neq$ then C.R is $|z| \geq Z_{\frac{\alpha}{2}}$ (Two Sided)
- If H_1 *contains* $>$ then C.R is $Z > Z_\alpha$ (One Sided)
- If H_1 *contains* $<$ then C.R is $Z < -Z_\alpha$ (One Sided)

FREQUENTLY USED CRITICAL VALUES OF Z

Significance Level (α)	Confidence Level $(1-\alpha)$	One-Tailed Test	Two-Tailed Test
$\alpha = 0.01$	99%	$Z_\alpha = 2.33$	$Z_{\alpha/2} = 2.58$
$\alpha = 0.02$	98%	$Z_\alpha = 2.05$	$Z_{\alpha/2} = 2.33$
$\alpha = 0.03$	97%	$Z_\alpha = 1.88$	$Z_{\alpha/2} = 2.17$
$\alpha = 0.04$	96%	$Z_\alpha = 1.75$	$Z_{\alpha/2} = 2.05$
$\alpha = 0.05$	95%	$Z_\alpha = 1.65$	$Z_{\alpha/2} = 1.96$
$\alpha = 0.06$	94%	$Z_\alpha = 1.55$	$Z_{\alpha/2} = 1.88$
$\alpha = 0.07$	93%	$Z_\alpha = 1.48$	$Z_{\alpha/2} = 1.81$
$\alpha = 0.08$	92%	$Z_\alpha = 1.41$	$Z_{\alpha/2} = 1.75$
$\alpha = 0.09$	91%	$Z_\alpha = 1.34$	$Z_{\alpha/2} = 1.70$
$\alpha = 0.10$	90%	$Z_\alpha = 1.28$	$Z_{\alpha/2} = 1.65$

P-VALUE FORMULAS IN Z-TEST

$$P = \begin{cases} 2[1-\Phi(|z_0|)] & if \quad H_1 : \mu \neq \mu_0 \\ 1-\Phi(z_0) & if \quad H_1 : \mu > \mu_0 \\ \Phi(z_0) & if \quad H_1 : \mu < \mu_0 \end{cases}$$

T-TEST

If $H_{\circ} : (=, \leq, \geq)$ then $H_1 : (\neq, >, <)$

Now for Critical Region

- If H_1 *contains* $\neq$ then C.R is $|t| \geq t_{\frac{\alpha}{2}(n-1)}$ (Two Sided)
- If H_1 *contains* $>$ then C.R is $t > t_{\alpha(n-1)}$ (One Sided)
- If H_1 *contains* $<$ then C.R is $t < -t_{\alpha(n-1)}$ (One Sided)

STUDENTS T-DISTRIBUTION TABLE

df	α = 0.1	0.05	0.025	0.01	0.005	0.001	0.0005
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578
2	1.886	2.92	4.303	6.965	9.925	22.328	31.6
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.61
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869
6	1.44	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.86	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.25	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.93	4.318
13	1.35	1.771	2.16	2.65	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.14
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.12	2.583	2.921	3.686	4.015
17	1.333	1.74	2.11	2.567	2.898	3.646	3.965
18	1.33	1.734	2.101	2.552	2.878	3.61	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.85
21	1.323	1.721	2.08	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.5	2.807	3.485	3.768
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.06	2.485	2.787	3.45	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.689
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.66
30	1.31	1.697	2.042	2.457	2.75	3.385	3.646
60	1.296	1.671	2	2.39	2.66	3.232	3.46
120	1.289	1.658	1.98	2.358	2.617	3.16	3.373
∞	1.282	1.645	1.96	2.326	2.576	3.091	3.291

STATISTICAL INFERENCE TOPICS

Inference about Mean when Sigma is known

Inference about Mean when Sigma is unknown

Inference about Difference of Means when Sigma is known

Inference about Difference of Means when Sigma is unknown

Inference about Paired Observations

BASIC SYMBOLS AND FORMULAS

1. $\mu =$ Population Mean
2. $\bar{x} = \frac{\sum x}{n} =$ Sample Mean
3. $\sigma =$ Population Standard Deviation
4. $s = \sqrt{\frac{\sum (x - \bar{x})^2}{n-1}} =$ Sample Standard Deviation
5. $n =$ sample size
6. $\alpha =$ Level of Significance
7. $1 - \alpha =$ Confidence Coefficient
8. $d.f =$ degree of freedom
9. At most, As Low, As Small means $\leq$
10. At Least, as High, As Large means $\geq$