MT-1004
Linear Algebra

Fall 2023

Week # 3

National University of Computer and Emerging Sciences

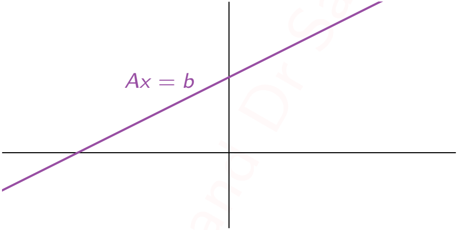
September 6, 2023

Section 1.3

Vector Equations

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.



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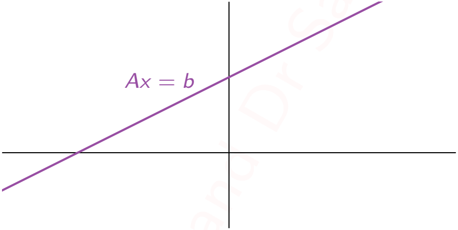

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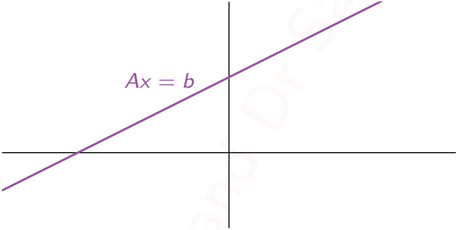

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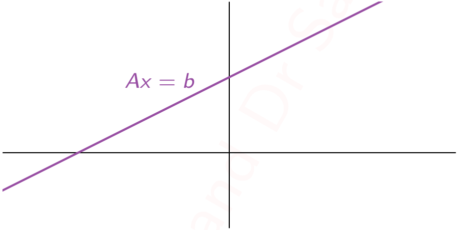

Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Last time we discussed the set of vectors b for which $Ax = b$ has a solution.

We also described this set using spans, but it was a *different problem*.

Homogeneous Systems

Everything is easier when $b = 0$, so we start with this case.

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$Ax = 0$ has a nontrivial solution

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$Ax = 0$ has a nontrivial solution

$\iff$ there is a free variable

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Observation

$Ax = 0$ has a nontrivial solution

$\iff$ there is a free variable

$\iff$ A has a column with no pivot.

Homogeneous Systems

Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

Homogeneous Systems

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What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

We know how to do this: first form an augmented matrix and row reduce.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

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Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero!

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Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

Homogeneous Systems

Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} x_1 - 3x_2 = 0$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 3x_2 \\ x_2 = x_2 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

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This last equation is called the **parametric vector form** of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Homogeneous Systems

Example, continued

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Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in R.

Homogeneous Systems

Example, continued

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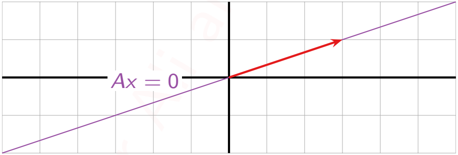

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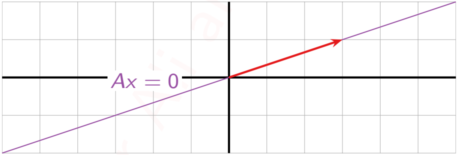

Note: *one* free variable means the solution set is a *line* in R^2 ($2 = \#$ variables $= \#$ columns).

Homogeneous Systems

Example

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What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 \phantom{{}+x_2} - 2x_3 = 0 \\ \phantom{x_1 {}} x_2 + x_3 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases}$$

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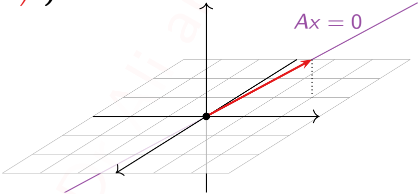

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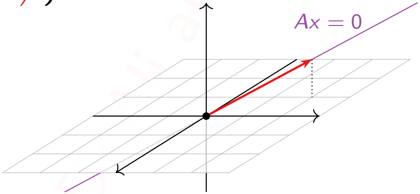

Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

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What is the solution set of $Ax = 0$, where

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Homogeneous Systems

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$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer: Span $\left\{ \vphantom{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}} \right.$

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Homogeneous Systems

Example, continued

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[not pictured here]

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Answer: $\text{Span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

[not pictured here]

Note: *two* free variables means the solution set is a *plane* in R^4 ($4 = \#$ variables $= \#$ columns).

Parametric Vector Form

Homogeneous systems

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Then the solutions to $Ax = 0$ can be written in the form

$$x = x_i v_i + x_j v_j + x_k v_k + \cdots$$

for some vectors $v_i, v_j, v_k, \dots$ in R–,

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The solution set is

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The equation above is called the **parametric vector form** of the solution.

Poll

Poll

How many solutions can there be to a homogeneous system with more equations than variables?

A. 0

B. 1

C. ∞

Poll

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This matrix has only one solution to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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This matrix has infinitely many solutions to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Nonhomogeneous Systems

Example

Question

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$$\xrightarrow{\text{equation}} x_1 - 3x_2 = -3$$

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Note that p is itself a solution: take $x_2 = 0$.

Nonhomogeneous Systems

Example, continued

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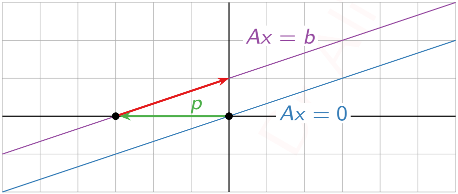

Nonhomogeneous Systems

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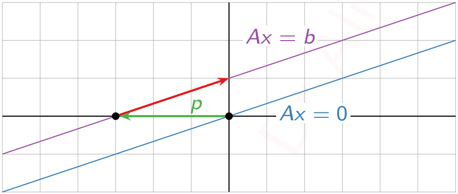

It can be written

$$\mathrm{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Nonhomogeneous Systems

Example

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & -5 \\ 2 & -1 & -5 & -3 \\ 1 & 0 & -2 & -2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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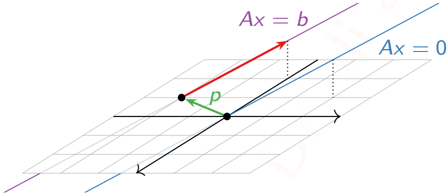

The solution set is a *translate* of

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it is the parallel line through

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Homogeneous vs. Nonhomogeneous Systems

Key Observation

The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

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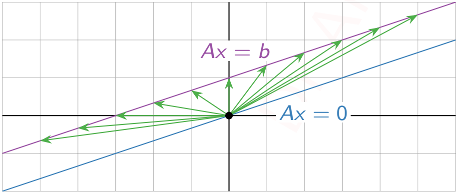

This works for *any* specific solution p: it doesn't have to be the one produced by finding the parametric vector form and setting the free variables all to zero, as we did before.

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Varying b

If we understand the solution set of $Ax = 0$, then we understand the solution set of $Ax = b$ for all b: they are all translates (or empty).

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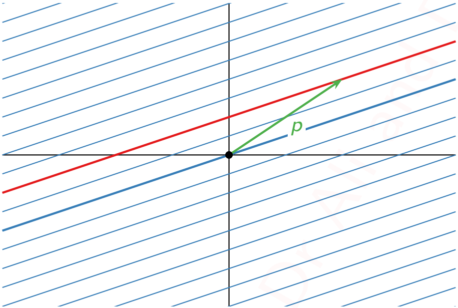

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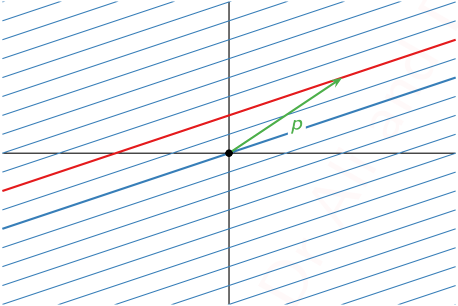

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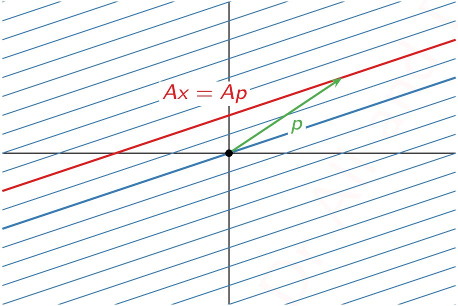

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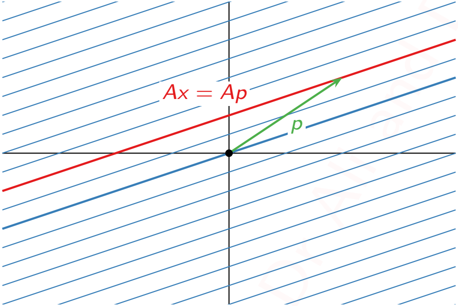

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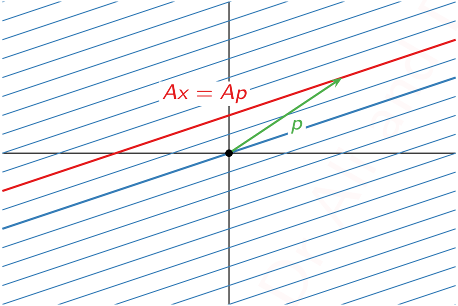

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For a matrix equation $Ax = b$, you now know how to find which b's are possible, and what the solution set looks like for all b, both using spans.

Section 1.7

Linear Independence

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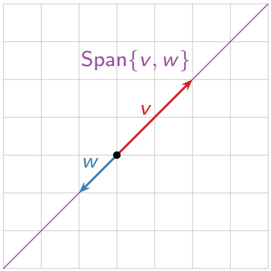

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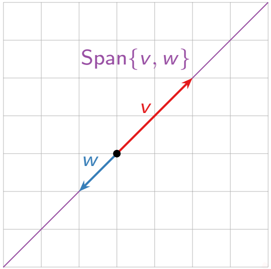

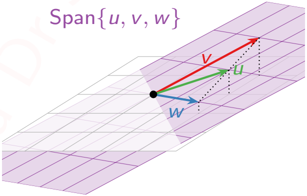

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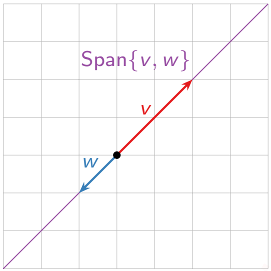

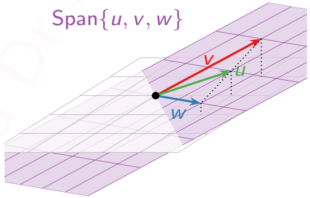

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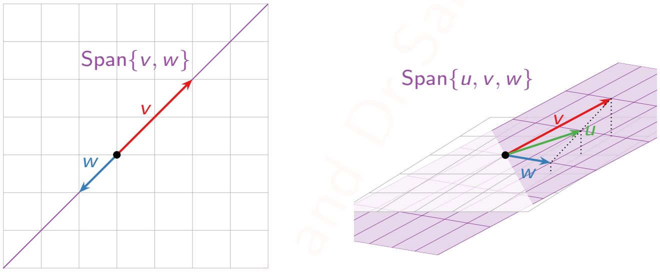

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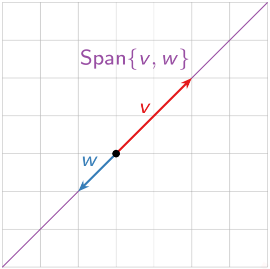

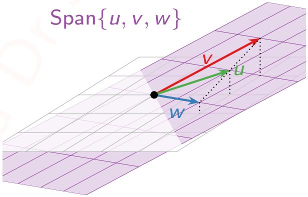

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Notice in each case that one vector in the set is already in the span of the others—so it doesn’t make the span bigger.

Today we will formalize this idea in the concept of *linear (in)dependence*.

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in R^n is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

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This is called a **linear dependence relation**.

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Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

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Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$-2\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

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The trivial solution $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ is the unique solution. So the vectors are linearly *in*dependent.

Linear Independence and Matrix Columns

In general, $\{v_1, v_2, \ldots, v_p\}$ is linearly independent

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Important

- The vectors $v_1, v_2, \ldots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \ldots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \ldots, v_p$ of A are linearly independent, or will produce a linear dependence relation.

Linear Independence
Criterion

Suppose that one of the vectors $\{v_1, v_2, \ldots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

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$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Then the vectors are linearly *de*pendent:

$$2v_1 - \frac{1}{2}v_2 - v_3 + 6v_4 = 0.$$

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Conversely, if the vectors are linearly dependent

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then one vector is a linear combination of (in the span of) the other ones:

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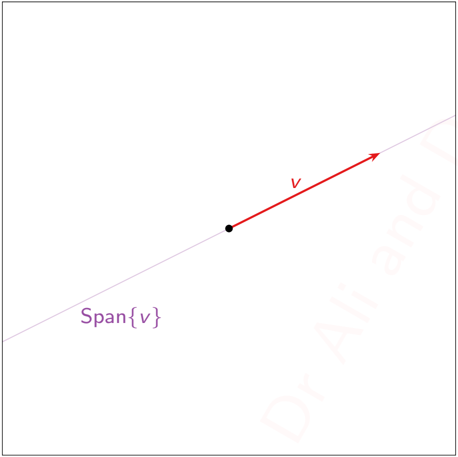
Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Linear Independence

Pictures in $\mathbf{R}^2$

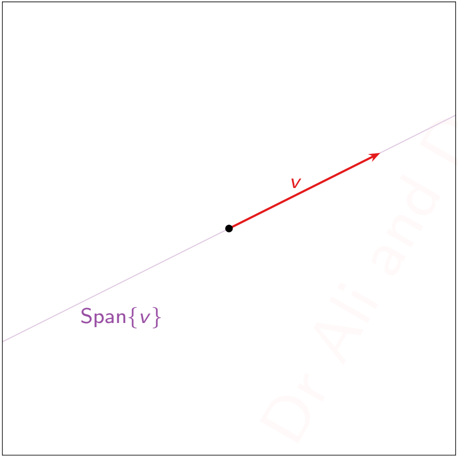


In this picture

One vector $\{v\}$:

Linear Independence

Pictures in R^2



In this picture

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Linear Independence

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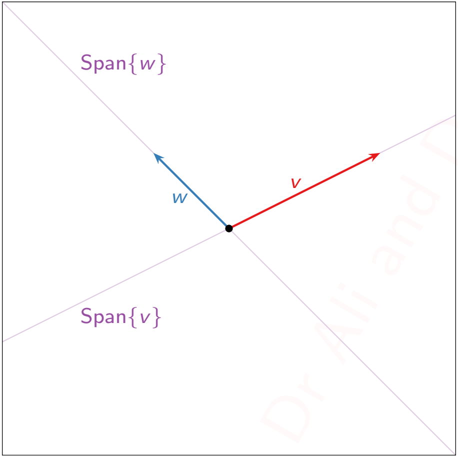

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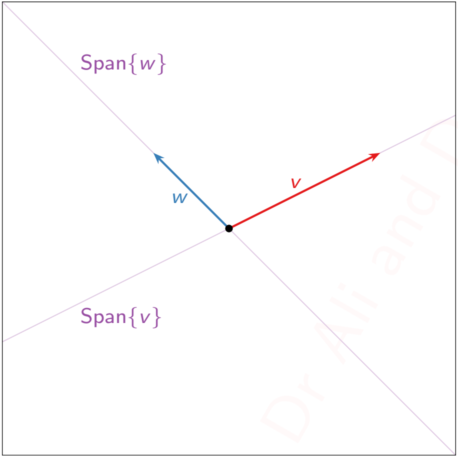
Two vectors $\{v, w\}$:



In this picture

One vector $\{v\}$:
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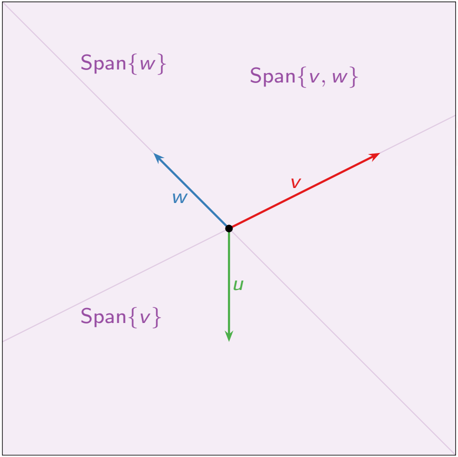
Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Linear Independence

Pictures in $\mathbb{R}^2$



In this picture

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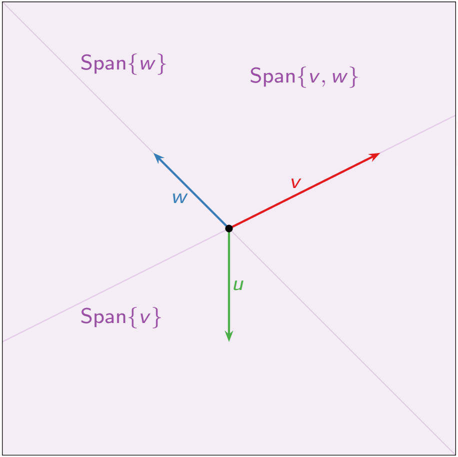
Three vectors $\{v, w, u\}$:

Linear Independence

Pictures in $\mathbb{R}^2$



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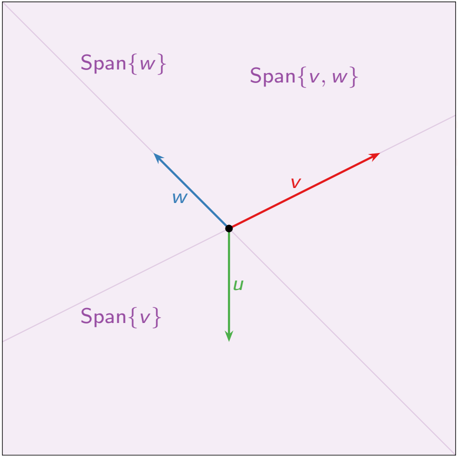
Linearly dependent: u is in $\text{Span}\{v, w\}$.



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Three vectors $\{v, w, u\}$:
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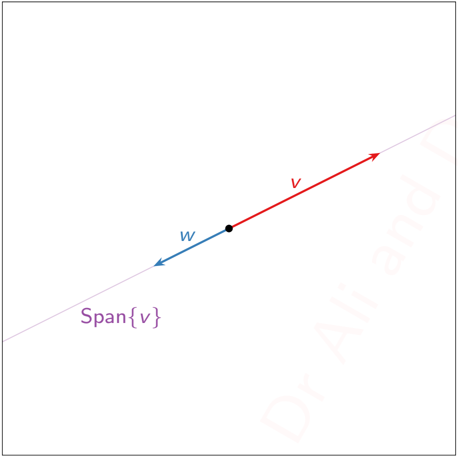
Also v is in $\mathrm{Span}\{u, w\}$ and w is in $\mathrm{Span}\{u, v\}$.

Linear Independence

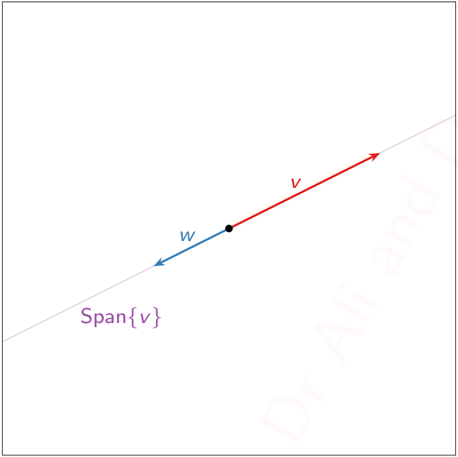
Pictures in $\mathbf{R}^2$



Two collinear vectors $\{v, w\}$:

Linear Independence

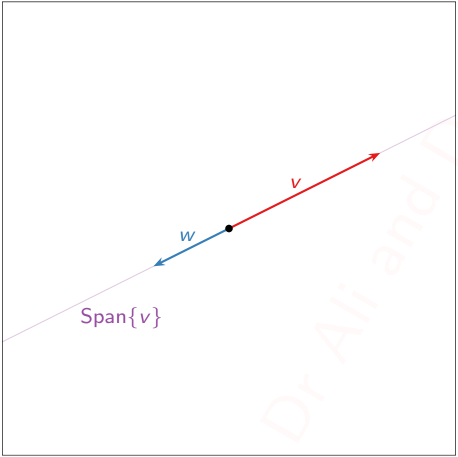
Pictures in $\mathbb{R}^2$



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Linear Independence

Pictures in $\mathbf{R}^2$



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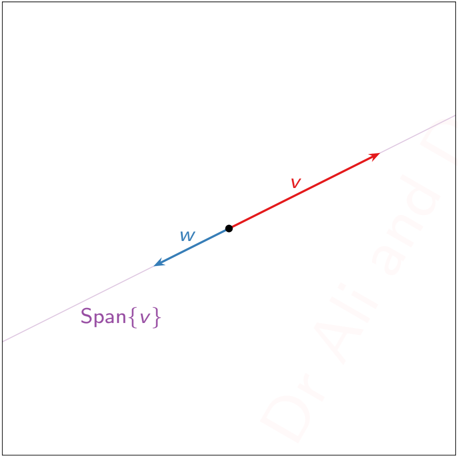
Observe: *Two* vectors are linearly *de*pendent if and only if



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Observe: *Two* vectors are linearly *de*pendent if and only if they are *collinear*.

Linear Independence

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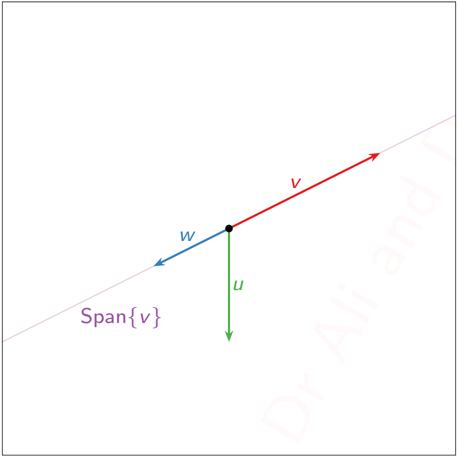

Two collinear vectors $\{v, w\}$:
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Three vectors $\{v, w, u\}$:

Linear Independence

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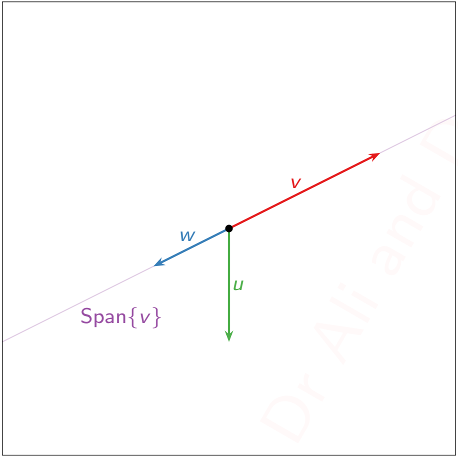

Two collinear vectors $\{v, w\}$:
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Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Three vectors $\{v, w, u\}$:
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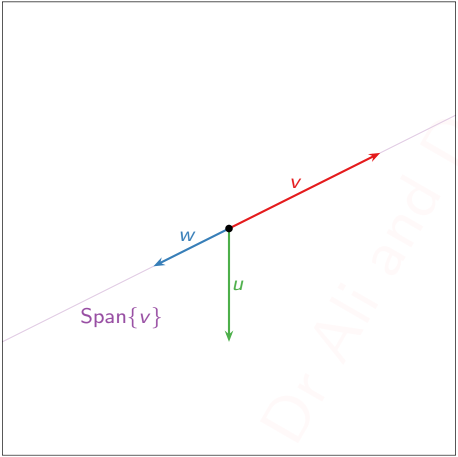
Linear Independence

Pictures in R^2



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Three vectors $\{v, w, u\}$:
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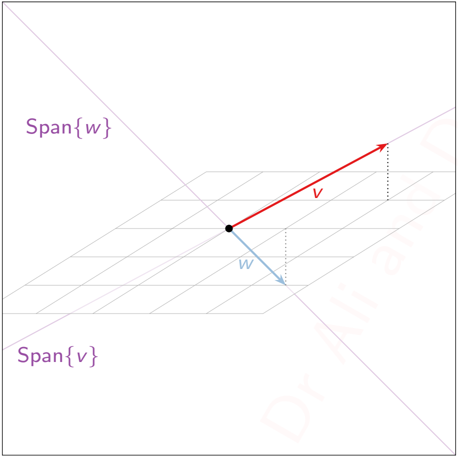
Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!

Linear Independence

Pictures in $\mathbb{R}^3$

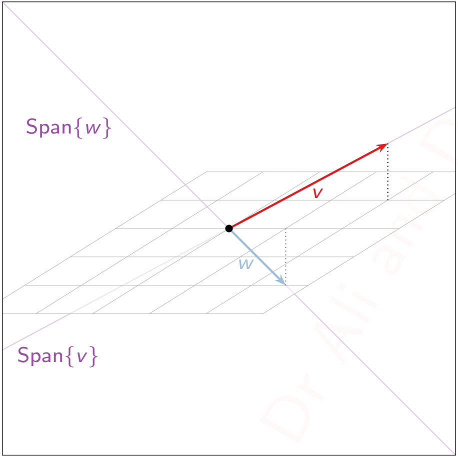


In this picture

Two vectors $\{v, w\}$:

Linear Independence

Pictures in $\mathbb{R}^3$

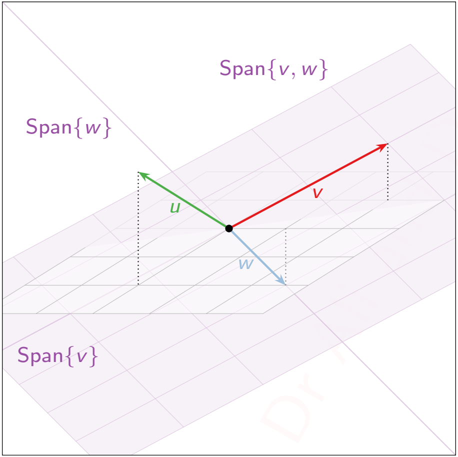


In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Linear Independence

Pictures in $\mathbb{R}^3$



In this picture

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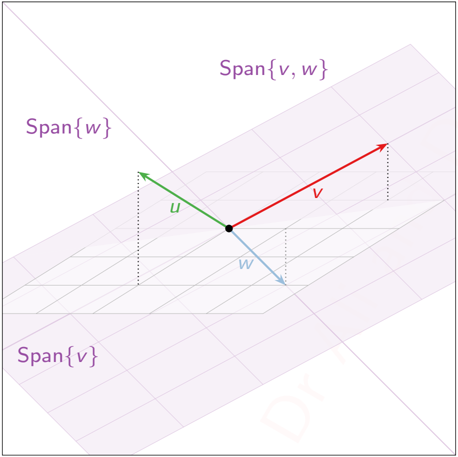
Linear Independence

Pictures in $\mathbf{R}^3$



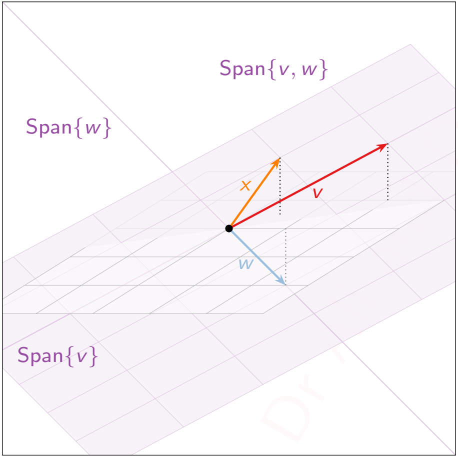
In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly independent: no one is in the span of the other two.

Linear Independence

Pictures in $\mathbb{R}^3$



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Three vectors $\{v, w, x\}$:

Linear Independence

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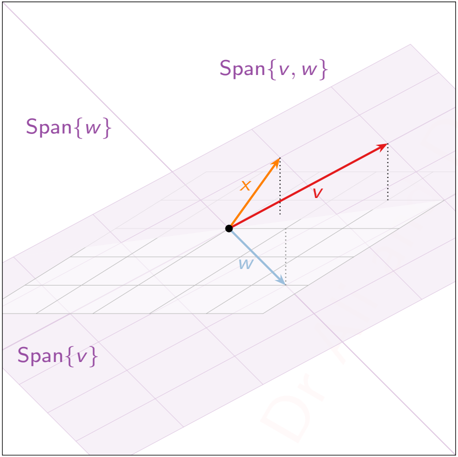

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Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: x is in Span$\{v, w\}$.

Poll

Poll

Are there four vectors u, v, w, x in R^3 which are linearly dependent, but such that u is *not* a linear combination of v, w, x? If so, draw a picture; if not, give an argument.

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Are there four vectors u, v, w, x in $\mathbb{R}^3$ which are linearly dependent, but such that u is *not* a linear combination of v, w, x? If so, draw a picture; if not, give an argument.

Yes: actually the pictures on the previous slides provide such an example.

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Linear dependence of $\{v_1, \ldots, v_p\}$ means *some* v_i is a linear combination of the others, not *any*.

Linear Independence

Stronger criterion

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Linear Independence

Stronger criterion

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A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Take the largest j such that v_j is in the span of the others.

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Stronger criterion

Theorem

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Take the largest j such that v_j is in the span of the others. Then v_j is in the span of $v_1, v_2, \ldots, v_{j-1}$. Why?

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Rearrange:

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so v_4 works as well, but v_3 was supposed to be the last one that was in the span of the others.

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Better Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

Theorem

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Equivalently, $\{v_1, v_2, \ldots, v_p\}$ is linearly *in*dependent if for every j, the vector v_j is not in $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in Span$\{v_1, v_2, \ldots, v_{j-1}\}$.

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This means Span$\{v_1, v_2, \ldots, v_j\}$ is *bigger* than Span$\{v_1, v_2, \ldots, v_{j-1}\}$.

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Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if, for every j, the span of $v_1, v_2, \ldots, v_j$ is strictly larger than the span of $v_1, v_2, \ldots, v_{j-1}$.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in Span$\{v_1, v_2, \ldots, v_{j-1}\}$.

Equivalently, $\{v_1, v_2, \ldots, v_p\}$ is linearly *in*dependent if for every j, the vector v_j is not in Span$\{v_1, v_2, \ldots, v_{j-1}\}$.

This means Span$\{v_1, v_2, \ldots, v_j\}$ is *bigger* than Span$\{v_1, v_2, \ldots, v_{j-1}\}$.

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Translation

A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

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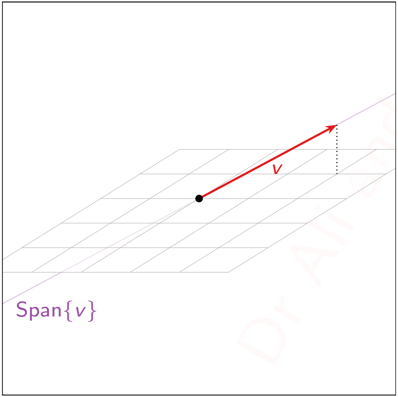

One vector $\{v\}$:
Linearly independent: span got bigger (than $\{0\}$).

Linear Independence

Increasing span criterion: pictures

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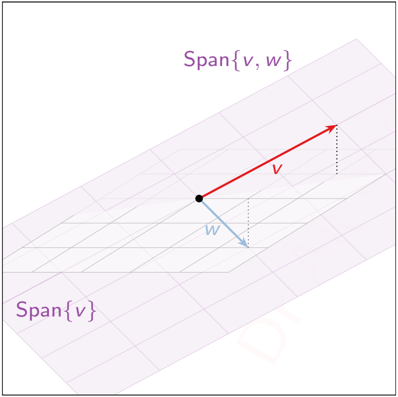

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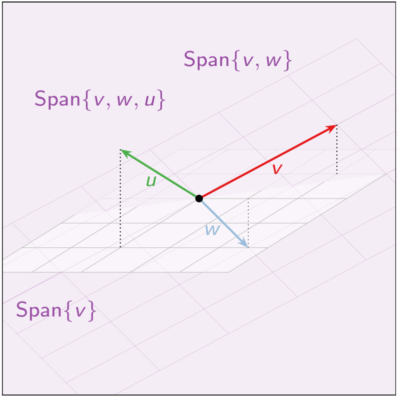

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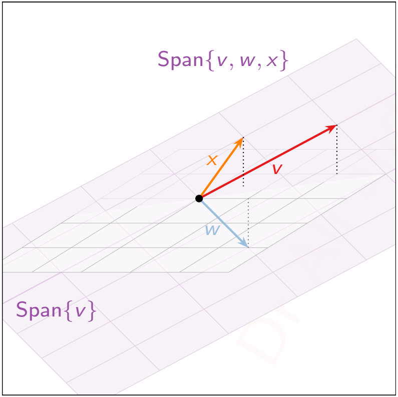

One vector $\{v\}$:
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Two vectors $\{v, w\}$:
Linearly independent: span got bigger.

Three vectors $\{v, w, x\}$:
Linearly dependent: span didn't get bigger.

Linear Independence

Two more facts

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A set containing the zero vector is linearly dependent.

- The columns of a matrix A are linearly independent if the equation $Ax = 0$ has the trivial solution.
- If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.
- The columns of any 4×5 matrix are linearly dependent. If x and y are linearly independent, and if $\{x; y; z\}$ is linearly dependent, then z is in $Span\{x; y\}$
- If $\{v_1, ..., v_5\}$re in R^5 and $v_3 = 0$, then $\{v_1, ..., v_5\}$ is linearly dependent.
- Suppose A is an $m \times n$ matrix with the property that for all b in R^m the equation $Ax = b$ has at most one solution. Explain why the columns of A must be linearly independent.

(a). Find the value(s) of h for which the vectors are linearly dependent. Is $\left\{ \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ h \end{pmatrix} \right\}$ linearly independent?

(b). (i) For what values of h is v_3 in $Span\{v_1; v_2\}$, and (ii) for what values of h is $\{v_1; v_2; v_3\}$ linearly dependent?
$\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 9 \\ -6 \end{pmatrix}, \begin{pmatrix} 5 \\ -7 \\ h \end{pmatrix} \right\}$

(c). Determine by inspection whether the vectors are linearly independent

(i). $\left\{ \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\}$

(ii). $\left\{ \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 2 \\ 4 \end{pmatrix} \right\}$

(iii). $\left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \end{pmatrix} \right\}$

Test your understanding:

(i) The columns of a matrix A are linearly independent if the equation $Ax = 0$ has the trivial solution.

(ii) If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

(iii) The columns of any 4×5 matrix are linearly dependent. If x and y are linearly independent, and if $\{x; y; z\}$ is linearly dependent, then z is in $Span\{x; y\}$

(iv) How many pivot columns must a 6×4 matrix have if its columns are linearly independent? Why?

(v) How many pivot columns must a 4×6 matrix have if its columns span R^4? Why?

(vi) If $\{v_1, ..., v_5\}$re in R^5 and $v_3 = 0$, then $\{v_1, ..., v_5\}$ is linearly dependent.

(vii) Suppose A is an m n matrix with the property that for all b in R^m the equation $Ax = b$ has at most one solution. Explain why the columns of A must be linearly independent.

Section: 2.8

Subspaces

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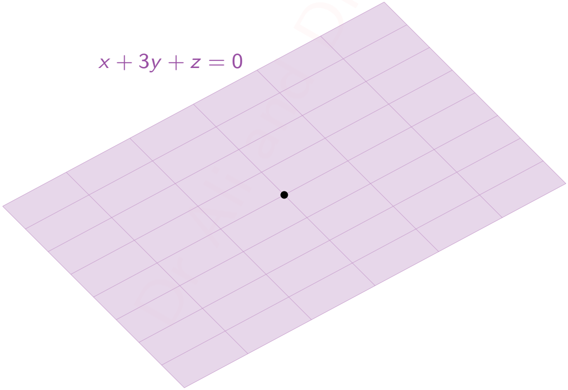
A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbb{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.



Definition of Subspace

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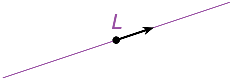
$\boxed{\text{A subspace } V \text{ contains the span of any set of vectors in } V.}$

Examples

Example

A line L through the origin: this contains the span of any vector in L.



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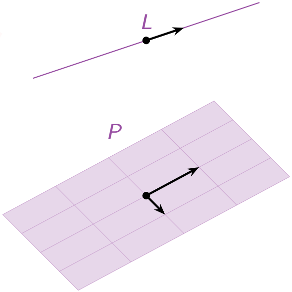
Example

A plane P through the origin: this contains the span of any vectors in P.



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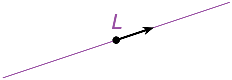

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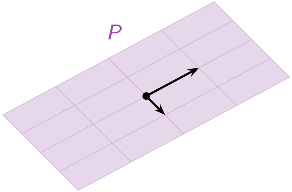

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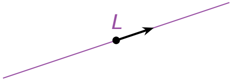

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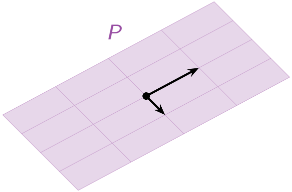

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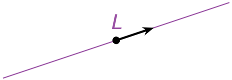

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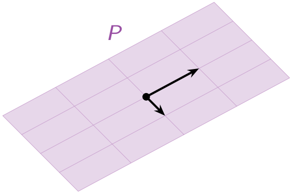

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Note these are all pictures of spans! (Line, plane, space, etc.)

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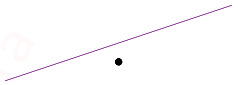

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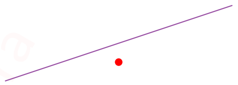

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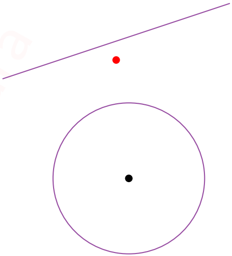

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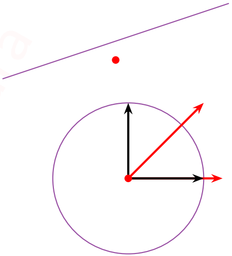

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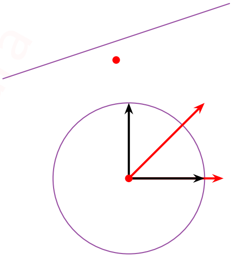

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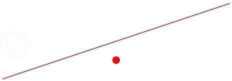

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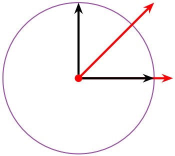

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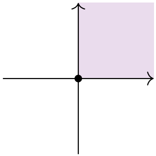
The first quadrant in $\mathbb{R}^2$ is not a subspace. Fails:



Non-Examples

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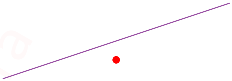

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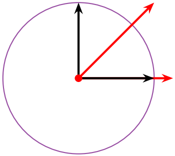

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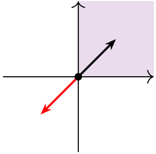

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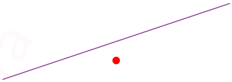

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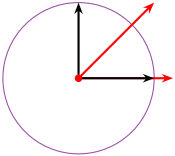

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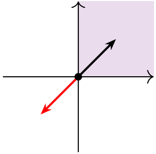

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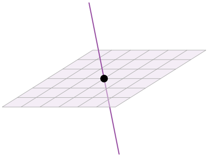

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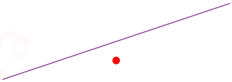

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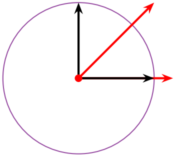

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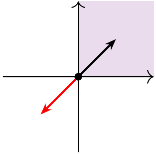

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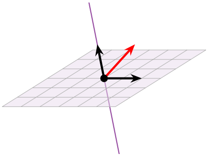

Subsets and Subspaces

They aren't the same thing

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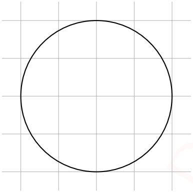

Subset: *yes*
Subspace: *no*

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Any Span$\{v_1, v_2, \ldots, v_n\}$ is a subspace.

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Definition

If $V = \text{Span}\{v_1, v_2, \ldots, v_n\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_n$.

Check:

1. $0 = 0v_1 + 0v_2 + \cdots + 0v_n$ is in the span.
2. If, say, $u = 3v_1 + 4v_2$ and $v = -v_1 - 2v_2$, then

$$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

   is also in the span.
3. Similarly, if u is in the span, then so is cu for any scalar c.

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Question: What is the difference between $\{\}$ and $\{0\}$?

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathrm{R}^2 \;\middle|\; ab = 0 \right\}$. Let's check if V is a subspace or not.

1. Does V contain the zero vector? $\binom{a}{b} = \binom{0}{0} \implies ab = 0$ ✔

3. Is V closed under scalar multiplication?
   - Let $\binom{a}{b}$ be in V.
   - *This means:* a and b are numbers such that $ab = 0$.
   - Let c be a scalar. Is $c\binom{a}{b} = \binom{ca}{cb}$ in V?
   - *This means:* $(ca)(cb) = 0$.
   - Well, $(ca)(cb) = c^2(ab) = c^2(0) = 0$ ✔

2. Is V closed under addition?
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   - Is $\binom{a}{b} + \binom{a'}{b'} = \binom{a+a'}{b+b'}$ in V?
   - *This means:* $(a + a')(b + b') = 0$.
   - This is not true for all such a, a', b, b': for instance, $\binom{1}{0}$ and $\binom{0}{1}$ are in V, but their sum $\binom{1}{0} + \binom{0}{1} = \binom{1}{1}$ is not in V, because $1 \cdot 1 \neq 0$. ✗

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We conclude that V is *not* a subspace.

Subspaces
Verification

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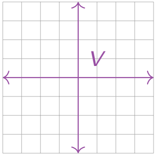

We conclude that V is *not* a subspace. A picture is above. (It doesn't look like a span.)

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Check that the null space is a subspace:

1. 0 is in $\operatorname{Nul} A$ because $A0 = 0$.
2. If u and v are in $\operatorname{Nul} A$, then $Au = 0$ and $Av = 0$. Hence
   $$A(u + v) = Au + Av = 0,$$
   so $u + v$ is in $\operatorname{Nul} A$.
3. If u is in $\operatorname{Nul} A$, then $Au = 0$. For any scalar c, $A(cu) = cAu = 0$. So cu is in $\operatorname{Nul} A$.

Column Space and Null Space

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the column space:

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

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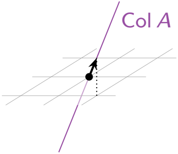

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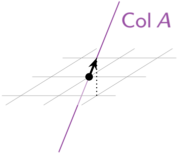

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$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix}.$$

This zero if and only if $x = -y$. So

$$\text{Nul}\, A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbb{R}^2 \mid y = -x \right\}.$$

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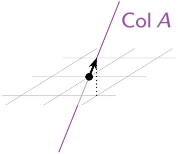

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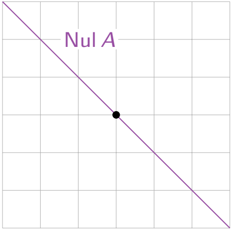

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$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, x_3 and x_4 are the free variables.

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Refer back to the slides for §1.5 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

The Null Space is a Span

Example, revisited

Find vector(s) that span the null space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

The reduced row echelon form is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives the equation $x + y = 0$, or

$$\begin{aligned} x &= -y \\ y &= y \end{aligned} \quad \xrightarrow{\text{parametric vector form}} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

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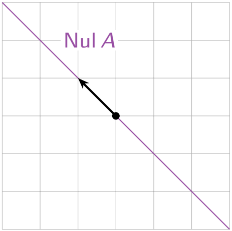

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- Is it all of $\mathbb{R}^n$ or the zero subspace $\{0\}$?

Subspaces
Summary

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Important

A subspace has *many different* bases, but they all have the same number of vectors (see the exercises in §2.9).

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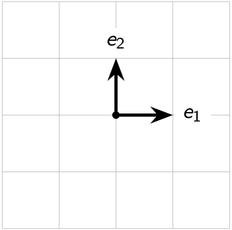

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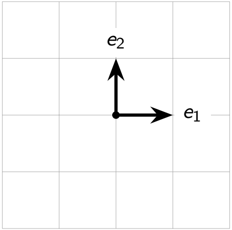

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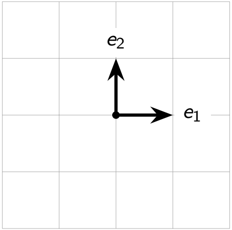

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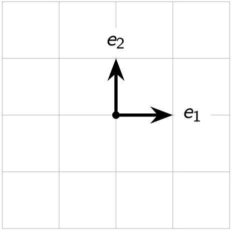

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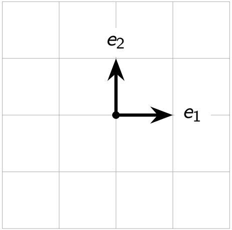

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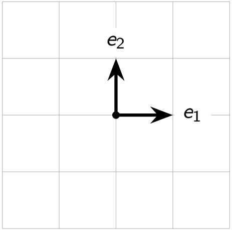

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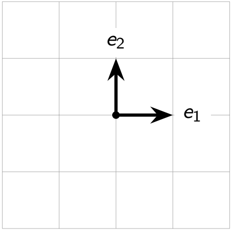

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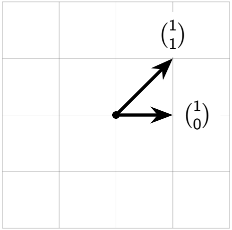

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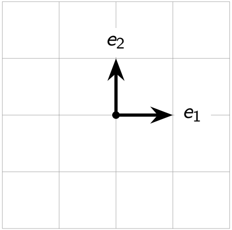

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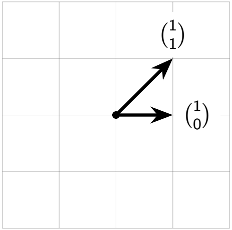

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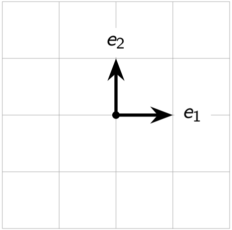

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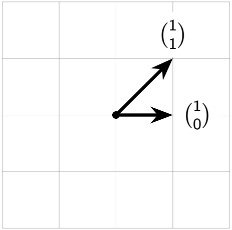

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Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathrm{R}^3 \;\middle|\; x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that $\mathcal{B}$ is a basis for V.

0. In V: both vectors are in V because
$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$
1. Span: If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V, then $y = -\frac{1}{3}(x + z)$, so
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$
2. Linearly independent:
$$c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

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$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{\text{parametric} \\ \text{vector} \\ \text{form}}} x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\substack{\text{basis of} \\ \text{Nul}\, A}} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. The vectors span Nul A by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)

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Why?

Subspaces: Dimension & Ranks

Ex 2.8 & 2.9

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Check that the null space is a subspace:

1. 0 is in $\operatorname{Nul} A$ because $A0 = 0$.
2. If u and v are in $\operatorname{Nul} A$, then $Au = 0$ and $Av = 0$. Hence
$$A(u + v) = Au + Av = 0,$$
   so $u + v$ is in $\operatorname{Nul} A$.
3. If u is in $\operatorname{Nul} A$, then $Au = 0$. For any scalar c, $A(cu) = cAu = 0$. So cu is in $\operatorname{Nul} A$.

Column Space and Null Space

Example

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Let's compute the column space:

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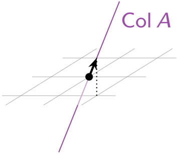

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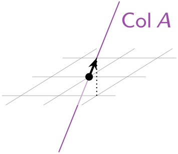

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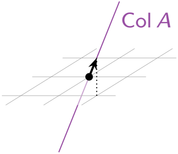

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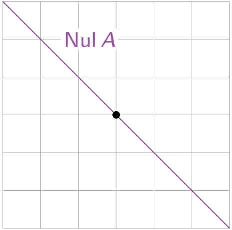

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Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

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Refer back to the slides for §1.5 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

The Null Space is a Span

Example, revisited

Find vector(s) that span the null space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

The reduced row echelon form is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives the equation $x + y = 0$, or

$$\begin{aligned} x &= -y \\ y &= y \end{aligned} \quad \xrightarrow{\text{parametric vector form}} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

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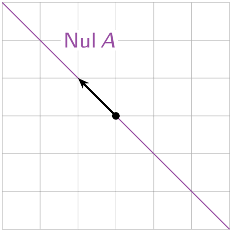

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- Can you verify directly that it satisfies the three defining properties?

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Important

A subspace has *many different* bases, but they all have the same number of vectors (see the exercises in §2.9).

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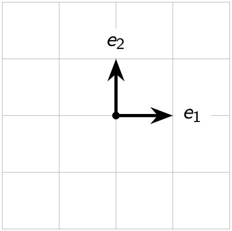

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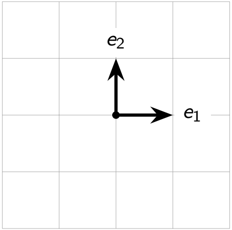

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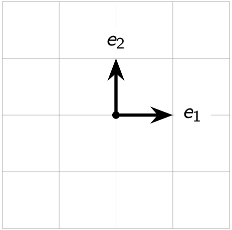

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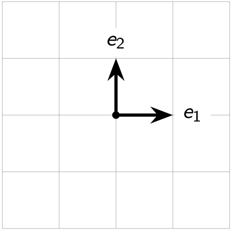

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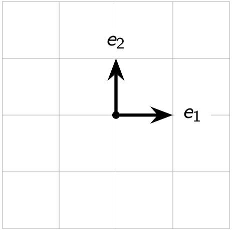

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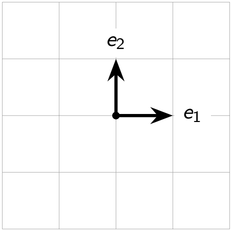

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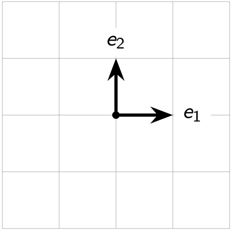

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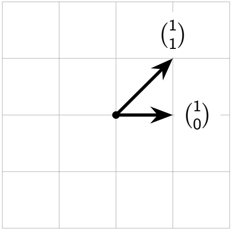

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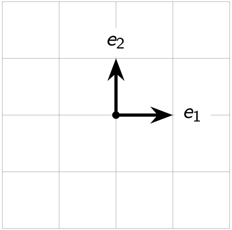

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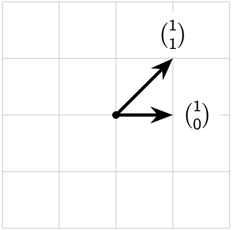

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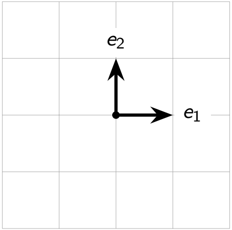

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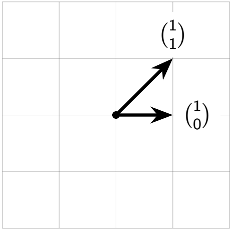

Bases of R^n

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

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$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for $\mathbb{R}^n$. The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

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has a pivot in every row and every column, i.e. if A is *invertible*.

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathrm{R}^3 \;\middle|\; x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that $\mathcal{B}$ is a basis for V.

0. In V: both vectors are in V because
$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$
1. Span: If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V, then $y = -\frac{1}{3}(x+z)$, so
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3}\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3}\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$
2. Linearly independent:
$$c_1\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

Basis for Nul A

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

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Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{\text{parametric} \\ \text{vector} \\ \text{form}}} x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\substack{\text{basis of} \\ \text{Nul}\,A}} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. The vectors span Nul A by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)

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The *pivot columns* of A always form a basis for Col A.

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pivot columns = basis ⇜ pivot columns in rref

So a basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

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Why?

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Recall: a **basis** of a subspace V is a set of vectors that *spans* V and is *linearly independent*.

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If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V, then any vector x in V can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for *unique* coefficients $c_1, c_2, \ldots, c_m$.

We know x is a linear combination of the v_i because they span V. Suppose that we can write x as a linear combination with different coefficients:

$$\begin{aligned} x &= c_1 v_1 + c_2 v_2 + \cdots + c_m v_m \\ x &= c_1' v_1 + c_2' v_2 + \cdots + c_m' v_m \end{aligned}$$

Subtracting:

$$0 = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \cdots + (c_m - c_m')v_m$$

Since $v_1, v_2, \ldots, v_m$ are linearly independent, they only have the trivial linear dependence relation. That means each $c_i - c_i' = 0$, or $c_i = c_i'$.

Bases as Coordinate Systems

The unit coordinate vectors $e_1, e_2, \ldots, e_n$ form a basis for R^n. Any vector is a unique linear combination of the e_i:

$$v = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2 - 2e_3.$$

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Definition

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace V. Any vector x in V can be written uniquely as a linear combination $x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$. The coefficients $c_1, c_2, \ldots, c_m$ are the **coordinates of** x **with respect to** $\mathcal{B}$. The $\mathcal{B}$**-coordinate vector of** x is the vector

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

Bases as Coordinate Systems

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$$\text{Let } v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathcal{B} = \{v_1, v_2\}, \quad V = \text{Span}\{v_1, v_2\}.$$

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Question: If $[x]_{\mathcal{B}} = \binom{5}{2}$, then what is x?

$$[x]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{means} \quad x = 5v_1 + 2v_2 = 5\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 7 \end{pmatrix}.$$

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Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$.

We have to solve the vector equation $x = c_1 v_1 + c_2 v_2$ in the unknowns c_1, c_2.

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

So $c_1 = 2$ and $c_2 = 3$, so $x = 2v_1 + 3v_2$ and $[x]_{\mathcal{B}} = \binom{2}{3}$.

Bases as Coordinate Systems

Example 2

$$\text{Let } v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \; v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}, \quad V = \text{Span}\{v_1, v_2, v_3\}.$$

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Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$.

We have to solve $x = c_1 v_1 + c_2 v_2$.

$$\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 3 & 1 & 11 \\ 2 & 1 & 8 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

So $x = 3v_1 + 2v_2$ and $[x]_{\mathcal{B}} = \binom{3}{2}$.

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V and x is in V, then

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1v_1 + c_2v_2 + \cdots + c_mv_m.$$

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Summary

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Finding the $\mathcal{B}$-coordinates for x means solving the vector equation

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Question: What happens if you try to find the $\mathcal{B}$-coordinates of x *not* in V?
You end up with an inconsistent system: V is the span of $v_1, v_2, \ldots, v_m$, and if x is not in the span, then $x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$ has no solution.

Bases as Coordinate Systems

Picture

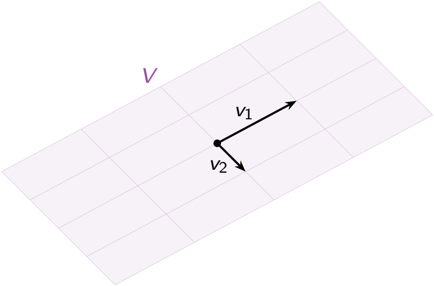
Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

These form a basis $\mathcal{B}$ for the plane

$$V = \mathrm{Span}\{v_1, v_2\}$$

in $\mathbf{R}^3$.



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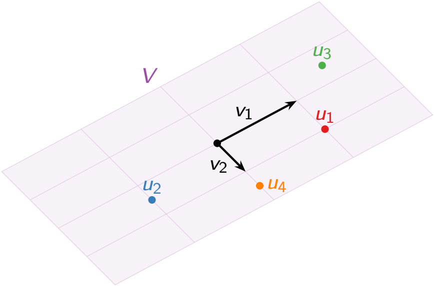

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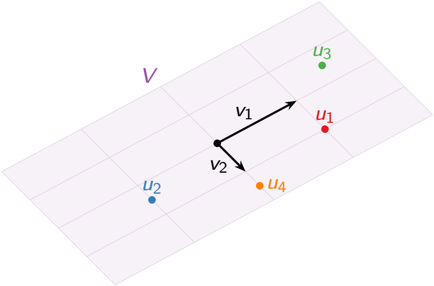

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Remark

Many of you want to think of a plane in $\mathbf{R}^3$ as "being" $\mathbf{R}^2$. Choosing a basis $\mathcal{B}$ and using $\mathcal{B}$-coordinates is one way to make sense of that. But remember that the coordinates are the coefficients of a linear combination of the basis vectors.

The Rank Theorem

Recall:

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The **rank** of a matrix A, written $\operatorname{rank} A$, is the dimension of the column space $\operatorname{Col} A$.

Observe:

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If A is an $m \times n$ matrix, then

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$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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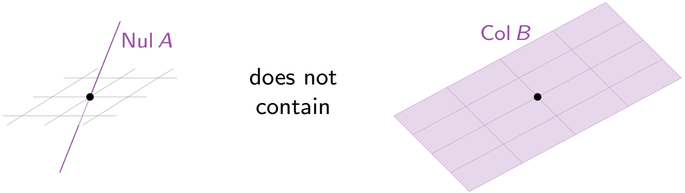

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