Linear Algebra

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Name: Abdul Haseeb Ahmed
Roll No: 23i-0132
Section: BAI-A.

(Question No 01)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

a) By looking our matrix, it 2 rows and three columns. So, there is at most 2 pivots in rows & columns of A! So,

Columns are linearly dependent
Rows are linearly independent
A^TA is not investible
AA^T is investible

b) Right inverse of $A = A^T (AA^T)^{-1}$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right]^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$$

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c) Orthogonal Projection on row space

$$P = A^T(AA^T)^{-1}A$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & \cancel{4}1 & \cancel{0}-1 \\ 1 & 2 & +1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

(Question No 02)

a) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

$$\hat{x} = \begin{bmatrix} \frac{u_1 \cdot b}{u_1 \cdot u_1} \\ \frac{u_2 \cdot b}{u_2 \cdot u_2} \end{bmatrix}$$ for orthogonal columns of A

$$\hat{x} = \begin{bmatrix} 9/3 \\ 12/24 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$$

b) Least squares approximating line for points $(2,3)$, $(-4,1)$, & $(2,5)$

In general $y = \alpha + \beta x$

$$3 = \alpha + 2\beta$$
$$1 = \alpha + (-4)\beta$$
$$5 = \alpha + 2\beta$$

$$\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$b = A \quad x$$

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As the columns of A are orthogonal

So $\hat{x} = \begin{bmatrix} \frac{v_1 \cdot b}{v_1 \cdot v_1} \\ \frac{v_2 \cdot b}{v_2 \cdot v_2} \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 9/3 \\ 12/24 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$ i.e. $y = 3 + \frac{x}{2}$ (line) closest to all points

$$\text{error} = \|b - \hat{b}\|$$
$$= \|b - A\hat{x}\|$$
$$= \left\| \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1/2 \end{bmatrix} \right\|$$
$$= \left\| \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ +1 \\ 4 \end{bmatrix} \right\|$$
$$= \left\| \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\|$$
$$= \sqrt{2} \quad \text{(Total error)}$$

(2,3) lies 1 unit below the line, (2,5) lies 1 unit above the line and (-4,1) has no error

(Question No 03)

a) In general $y = \alpha + \beta x + \gamma x^2$

(1,1), (2,-2), (3,3), (4,4)

$$1 = \alpha + \beta + \gamma$$
$$-2 = \alpha + 2\beta + 4\gamma$$
$$3 = \alpha + 3\beta + 9\gamma$$
$$4 = \alpha + 4\beta + 16\gamma$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$
$$b = A \quad x$$

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b) In general; $y = \alpha + \beta x + \gamma z$

$(1,1,1), (2,-2,2), (3,3,3), (4,4,4)$

$$1 = \alpha + \beta + \gamma$$
$$-2 = \alpha + 2\beta + 2\gamma$$
$$3 = \alpha + 3\beta + 3\gamma$$
$$4 = \alpha + 4\beta + 4\gamma$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$b = A \quad x$$

OR, in general

$$z = \alpha + \beta x + \gamma y$$
$$1 = \alpha + \beta + \gamma$$
$$2 = \alpha + 2\beta - 2\gamma$$
$$3 = \alpha + 3\beta + 3\gamma$$
$$4 = \alpha + 4\beta + 4\gamma$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$b = A \quad x$$