STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 09

CONDITIONAL PROBABILITY

MULTIPLICATION LAW FOR DEPENDENT EVENTS

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CONDITIONAL PROBABILITY

If A and B are two events, then the conditional probability that A occurs, given that B has already occurred, is written by P(A given B) or $P(A|B)$ and is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

IMPORTANT FORMULAS

If A and B are any events then

1. $P(A|B) = \frac{P(A \cap B)}{P(B)}$

2. $P(B|A) = \frac{P(A \cap B)}{P(A)}$

3. $P(A^c|B) = \frac{P(A^c \cap B)}{P(B)}$

4. $P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)}$

5. $P(A^c|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{P(A \cup B)^c}{P(B^c)} = \frac{1 - P(A \cup B)}{1 - P(B)}$

NOTE THAT

$P(A|B) \times P(B) = P(B|A) \times P(A)$

EXAMPLES OF CONDITIONAL PROBABILITY

EXAMPLE-1 When a die was rolled once the score was an odd number what is the probability that it was a prime number.

SOLUTION *Here* $S=\{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ *so* $n(S)=52$

Let O *denote the event that number appeared was an odd number. So* $O=\{1,\ 3,\ 5\}$

Let P *denote the event that number appeared was a Prime number So* $P=\{2,\ 3,\ 5)$

Also $P\cap B=\{3,\ 5\}$

we have to find $P(P|A)$

Using definition of Conditional Probability

$$P(P|O)=\frac{P(P\cap O)}{P(O)}=\frac{2/6}{3/6}=\frac{2}{3}$$

EXAMPLE-2 Two coins are tossed once. What is the probability that two heads result given that there is at least one head.

SOLUTION *Here* $S=\{HH,HT,TH,TT\}$ *so* $n(S)=4$

Let A *denote the event that two heads result. So* $A=\{HH\}$

Let B *denote the event that there is at least one head. So* $B=\{HH,\ HT,\ TH\}$

Also $A\cap B=\{HH\}$

we have to find $P(A|B)$

Using definition of Conditional Probability

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{1/4}{3/4}=\frac{1}{3}$$

EXAMPLE-3 A pair of fair dice is rolled once. What is the probability that the sum of dots appeared will be 7, given that sum of dots appeared is greater than 6.

SOLUTION *Here sample space is given by*

$$S=\left\{\begin{matrix}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{matrix}\right\}$$

So $n(S)=52$

Let A denote the event that sum of dots appeared will be 7.

$So\ A = \{(1,6),\ (2,5),\ (3,4),\ (4,3),\ (5,2),\ (6,1)\}$

Let B denote the event that that sum of dots appeared will be greater than 6.

$$So\ B = \begin{Bmatrix} (1,6),(2,5),(2,6),(3,4),(3,5),(3,6),(4,3),(4,4),(4,5),(4,6),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

$Also\ A \cap B = \{(1,6),\ (2,5),\ (3,4),\ (4,3),\ (5,2),\ (6,1)\}$

we have to find $P(A|B)$

Using definition of Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{6/36}{21/36} = \frac{6}{21} = \frac{2}{7}$$

EXAMPLE-4 A group of girls at a School is entered for Mathematics Course. Eight Girls takes only module M_1 or only module M_2 or both M_1 *and* M_2. The probability that a girl is taking M_2 given that she is taking M_1 is $1/5$. The probability that a girl is taking M_1 given that she is taking M_2 is $1/3$. Find the probability that

(a) A girl selected at random is taking both M_1 *and* M_2.

(b) A girl selected at random is taking only M_1.

SOLUTION *Lets define the events first*

Let M_1 *denote the event that girl takes module* M_1.

Let M_2 *denote the event that girl takes module* M_2.

It is given that $P(M_2|M_1) = 1/5,\ P(M_1|M_2) = 1/3$ *also* $P(M_1 \cup M_2) = 1$

(a) $P(M_1 \cap M_2) = ?$ *also let* $P(M_1 \cap M_2) = x$

$$Now\ P(M_2|M_1) = 1/5 \Rightarrow \frac{x}{P(M_1)} = \frac{1}{5} \Rightarrow P(M_1) = 5x$$

$$Also\ P(M_1|M_2) = 1/3 \Rightarrow \frac{x}{P(M_2)} = \frac{1}{3} \Rightarrow P(M_2) = 3x$$

Now using Addition Law of Probability

$$P(M_1 \cup M_2) = P(M_1) + P(M_2) - P(M_1 \cap M_2)$$

$$1 = 5x + 3x - x$$

$$\Rightarrow x = \frac{1}{7}\ So\ P(M_1 \cap M_2) = \frac{1}{7}$$

(b) $P(M_1 \cap M_2^c) = ?$

$$Now\ P(M_1 \cap M_2^c) = P(M_1) - P(M_1 \cap M_2)$$

$$= 5\left(\frac{1}{7}\right) - \frac{1}{7}$$

$$= \frac{4}{7}$$

EXAMPLE-5 The events A and B are such that

$P(A|B) = 0.4,\ P(B|A) = 0.25\ and\ P(A \cap B) = 0.12$,calculate the following probabilities

$(a)\ P(B)$

$(b)\ P(A \cup B)$

$(c)\ P(A|B^c)$

$(d)\ P(A^c|B)$

$(e)\ P(A^c|B^c)$

SOLUTION $Here\ P(A|B) = 0.4,\ P(B|A) = 0.25\ and\ P(A \cap B) = 0.12$

$(a)\ P(A|B) = 0.4$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = 0.4$$

$$\Rightarrow \frac{0.12}{P(B)} = 0.4 \Rightarrow P(B) = 0.3$$

$(b)\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$As\ P(B|A) = 0.25$

$$\Rightarrow \frac{P(A \cap B)}{P(A)} = 0.25$$

$$\Rightarrow \frac{0.12}{P(B)} = 0.25 \Rightarrow P(B) = 0.48$$

$$Now\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.48 + 0.3 - 0.12 = 0.66$$

$$(c)\ P(A|B^c) = \frac{P(A \cap B^C)}{P(B^c)}$$

$$= \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.48 - 0.12}{0.7} = 0.51$$

$$(d)\ P(A^c|B) = \frac{P(A^c \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)} = \frac{0.3 - 0.12}{0.3} = 0.60$$

$$(e)\ P(A^c|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)}$$

$$= \frac{P(A \cup B)^c}{1 - P(B)} = \frac{1 - P(A \cup B)}{1 - P(B)} = \frac{1 - 0.66}{1 - 0.3} = \frac{0.34}{0.7} = 0.4857$$

MULTIPLICATION LAW OF PROBABILITY FOR DEPENDENT EVENTS

If A and B are dependent events then $\boxed{P(A\cap B)=P(A).P(B|A)}$

If A, B and C are dependent events then $\boxed{P(A\cap B\cap C)=P(A).P(B|A).P(C|A\cap B)}$

EXAMPLES OF MULTIPLICATION LAW OF PROBABILITY

EXAMPLE-1 Two cards are drawn without replacement from an ordinary deck of 52 playing cards

(a) what is the probability that both cards drawn are aces.

(b) what is the probability that second card is an Ace.

SOLUTION *Here* $n(S)=52$

Let A_1 *denote the event that first drawn card is an Ace.*

Let A_2 *denote the event that second drawn card is an Ace.*

(a) *we have to find* $P(A_1\cap A_2)$

$$So\ P(A_1\cap A_2)=P(A_1).P(A_2|A_1)=\frac{4}{52}\times\frac{3}{51}=\frac{1}{221}$$

(b) *we have to find* $P(A_2)$

$$So\ P(A_2)=P(A_1\cap A_2)+P(A_1^c\cap A_2)=P(A_1).P(A_2|A_1)+P(A_1^c).P(A_2|A_1^c)=\frac{4}{52}\times\frac{3}{51}+\frac{48}{52}\times\frac{4}{51}=$$

EXAMPLE-2 Three cards are drawn in succession without replacement from an ordinary deck of 52 playing cards. Find the probability that first card is an Ace, second card is a King and third card is a Queen.

SOLUTION *Here* $n(S)=52$

Let A denote the event that first drawn card is an Ace.

Let B denote the event that second drawn card is a King.

Let C denote the event that third drawn card is an Queen.

we have to find $P(A\cap B\cap C)$

$$So\ P(A\cap B\cap C)=P(A).P(B|A).P(C|A\cap B)=\frac{4}{52}\times\frac{4}{51}\times\frac{4}{50}=\frac{64}{132600}=-$$