An Introduction to a Tensor

Definition 0.1: Tensors

A tensor is a multi-dimensional array, generalizing the concept of scalars (0th-order tensors), vectors (1st-order tensors), and matrices (2nd-order tensors). A tensor of order N has N modes or dimensions.

Notation:

- A tensor is denoted as $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, where I_n represents the size along the n-th mode.
- A specific entry of $\mathcal{A}$ is represented as $a_{i_1 i_2 \cdots i_N}$.

Examples of Tensors by Order

- **Order 0 (Scalar):**
  A single number:
  $$\mathcal{A} = 5$$
- **Order 1 (Vector):**
  A 1-dimensional array:
  $$\mathcal{A} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathcal{A} \in \mathbb{R}^3.$$
- **Order 2 (Matrix):**
  A 2-dimensional array:
  $$\mathcal{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathcal{A} \in \mathbb{R}^{2\times 2}.$$

Examples of Tensors by Order Contn'd...

- **Order 3 (Tensor):**
  A 3-dimensional array:

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{A}(:,:,2) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix},$$

$$\mathcal{A} \in \mathbb{R}^{2\times 2\times 2}.$$

Example: Tensor Addition

Given Tensors:
Two tensors of the same dimensions $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{2\times 2\times 2}$:

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathcal{A}(:,:,2) = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix},$$

$$\mathcal{B}(:,:,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{B}(:,:,2) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Addition:

$$\mathcal{C} = \mathcal{A} + \mathcal{B}, \quad \mathcal{C}(:,:,k) = \mathcal{A}(:,:,k) + \mathcal{B}(:,:,k).$$

Result:

$$\mathcal{C}(:,:,1) = \begin{bmatrix} 1 & 3 \\ 4 & 4 \end{bmatrix}, \quad \mathcal{C}(:,:,2) = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}.$$

Example: Scalar Multiplication

Given Tensor:
Tensor $\mathcal{A} \in \mathbb{R}^{2\times 2\times 2}$:

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathcal{A}(:,:,2) = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Scalar Multiplication:
Multiply $\mathcal{A}$ by a scalar $\alpha = 3$:

$$\mathcal{B} = \alpha \cdot \mathcal{A}, \quad \mathcal{B}(:,:,k) = \alpha \cdot \mathcal{A}(:,:,k).$$

Result:

$$\mathcal{B}(:,:,1) = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}, \quad \mathcal{B}(:,:,2) = \begin{bmatrix} 15 & 18 \\ 21 & 24 \end{bmatrix}.$$

Introduction to Mode-n Product

- The **Mode-n product** is a generalization of matrix multiplication for tensors.
- Denoted as: $\mathcal{B} = \mathcal{A} \times_n M$.
- Involves multiplication of a matrix M with the n-th mode of tensor $\mathcal{A}$.
- Examples: **Mode-1**, **Mode-2**, and **Mode-3** products.

Tensor-Times-Matrix (TTM) Operation

Definition 0.2: Mode-n Product

The n-mode product of a tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ and a matrix $M \in \mathbb{R}^{J \times I_n}$ is denoted by:

$$\mathcal{B} = \mathcal{A} \times_n M,$$

where $\mathcal{B} \in \mathbb{R}^{I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \cdots \times I_N}$.

Steps:

1. **Unfold the tensor:** Convert $\mathcal{A}$ into its mode-n unfolding $A_{(n)} \in \mathbb{R}^{I_n \times (I_1 \cdots I_{n-1} I_{n+1} \cdots I_N)}$.
2. **Matrix multiplication:** Multiply M with $A_{(n)}$:

$$B_{(n)} = M \cdot A_{(n)}.$$

3. **Fold the result:** Transform $B_{(n)}$ back into the tensor $\mathcal{B}$ using the fold operation.

Example 1: Mode-1 Product

Question 0.1:

Given Tensor: $\mathcal{A} \in \mathbb{R}^{2\times 3\times 2}$

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \mathcal{A}(:,:,2) = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

Given Matrix: $M \in \mathbb{R}^{4\times 2}$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Goal: Compute $\mathcal{B} = \mathcal{A} \times_1 M$, where $\mathcal{B} \in \mathbb{R}^{4\times 3\times 2}$.

Mode-1 Product: Solution Steps

Step 1: Unfold the Tensor

$$A_{(1)} = \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \end{bmatrix}, \quad A_{(1)} \in \mathbb{R}^{2\times 6}.$$

Step 2: Matrix Multiplication

$$B_{(1)} = M \cdot A_{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \end{bmatrix}.$$

Result:

$$B_{(1)} = \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \\ 5 & 7 & 9 & 17 & 19 & 21 \\ 4 & 5 & 6 & 10 & 11 & 12 \end{bmatrix}.$$

Mode-1 Product: Reshape

Step 3: Reshape Back into Tensor

- Reshape $B_{(1)} \in \mathbb{R}^{4\times 6}$ into $\mathcal{B} \in \mathbb{R}^{4\times 3\times 2}$.
- Slices:

$$\mathcal{B}(:,:,1) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \\ 4 & 5 & 6 \end{bmatrix}, \quad \mathcal{B}(:,:,2) = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \\ 17 & 19 & 21 \\ 10 & 11 & 12 \end{bmatrix}.$$

Example 2: Mode-2 Product

Question 0.2:

Given Tensor: $\mathcal{A} \in \mathbb{R}^{2\times 3\times 2}$

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \mathcal{A}(:,:,2) = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

Given Matrix: $M \in \mathbb{R}^{5\times 3}$

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Goal: Compute $\mathcal{B} = \mathcal{A} \times_2 M$, where $\mathcal{B} \in \mathbb{R}^{2\times 5\times 2}$.

Mode-2 Product: Solution Steps

Step 1: Unfold the Tensor

$$A_{(2)} = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \quad A_{(2)} \in \mathbb{R}^{3\times 4}.$$

Step 2: Matrix Multiplication

$$B_{(2)} = M \cdot A_{(2)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}.$$

Result:

$$B_{(2)} = \begin{bmatrix} 4 & 10 & 16 & 22 \\ 2 & 5 & 8 & 11 \\ 6 & 15 & 24 & 33 \\ 3 & 6 & 9 & 12 \\ 1 & 4 & 7 & 10 \end{bmatrix}.$$

Mode-2 Product: Reshape

Step 3: Reshape Back into a Tensor

- Reshape $B_{(2)} \in \mathbb{R}^{5\times4}$ into $\mathcal{B} \in \mathbb{R}^{2\times5\times2}$.
- Slices:

$$\mathcal{B}(:,:,1) = \begin{bmatrix} 4 & 2 & 6 & 3 & 1 \\ 10 & 5 & 15 & 6 & 4 \end{bmatrix},$$

$$\mathcal{B}(:,:,2) = \begin{bmatrix} 16 & 8 & 24 & 9 & 7 \\ 22 & 11 & 33 & 12 & 10 \end{bmatrix}.$$

Example 3: Mode-3 Product

Question 0.3:

Given Tensor: $\mathcal{A} \in \mathbb{R}^{2\times3\times2}$

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \mathcal{A}(:,:,2) = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

Given Matrix: $M \in \mathbb{R}^{3\times2}$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Goal: Compute $\mathcal{B} = \mathcal{A} \times_3 M$, where $\mathcal{B} \in \mathbb{R}^{2\times3\times3}$.

Mode-3 Product: Solution

Step 1: Unfold the Tensor

$$A_{(3)} = \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \end{bmatrix}, \quad A_{(3)} \in \mathbb{R}^{2\times 6}.$$

Step 2: Matrix Multiplication

$$B_{(3)} = MA_{(3)}$$

Result:

$$B_{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \end{bmatrix}$$

Mode-3 Product: Reshape Back

Step 3: Reshape Back into Tensor

- Reshape $B_{(3)} \in \mathbb{R}^{3\times 6}$ into $\mathcal{B} \in \mathbb{R}^{2\times 3\times 3}$.
- Final Tensor $\mathcal{B}$ has updated slices.

Higher Order Singular Value Decomposition(HOSVD)

Simple Example: Just to Give an Idea

Question 0.4:

Given Tensor: $\mathcal{A} \in \mathbb{R}^{2\times 1\times 2}$

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathcal{A}(:,:,2) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Goal: compute the Tensor SVD (T-SVD). Find:

- Core tensor $\mathcal{S}$,
- Factor matrices $U^{(1)}$, $U^{(2)}$, and $U^{(3)}$.

Step 1: Mode-1,2,3 Unfolding

- **Mode-1 unfolding:**

$$A_{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},$$

- **Mode-2 unfolding:**

$$A_{(2)} = \begin{bmatrix} 1 & 0 & 0 & 4 \end{bmatrix},$$

- **Mode-3 unfolding:**

$$A_{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Step 2: Perform SVD (Mode-1 Unfolding)

The SVD of $A_{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$:

$$A_{(1)} = U_1 \Sigma_1 V_1^T,$$

where:

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 2: Perform SVD (Mode-2 Unfolding)

The SVD of $A_{(2)} = \begin{bmatrix} 1 & 0 & 0 & 4 \end{bmatrix}$:

$$A_{(2)} = U_2 \Sigma_2 V_2^T,$$

where:

$$U_2 = \begin{bmatrix} 1 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} \sqrt{17} \end{bmatrix}, \quad V_2 = \begin{bmatrix} \frac{1}{\sqrt{17}} & 0 & 0 & \frac{4}{\sqrt{17}} \end{bmatrix}.$$

Step 2: Perform SVD (Mode-3 Unfolding)

The SVD of $A_{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$:

$$A_{(3)} = U_3 \Sigma_3 V_3^T,$$

where:

$$U_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 3: Compute the Core Tensor

The core tensor is computed as:

$$\mathcal{S} = \mathcal{A} \times_1 U_1^T \times_2 U_2^T \times_3 U_3^T.$$

Here, since U_1, U_2, U_3 are identity matrices, $\mathcal{S} = \mathcal{A}$:

$$\mathcal{A} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right].$$

Step 4: Reconstruction

The original tensor can be reconstructed as:

$$\mathcal{A} = \mathcal{S} \times_1 U_1 \times_2 U_2 \times_3 U_3.$$

Substituting the values of U_1, U_2, U_3:

$$\mathcal{A} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right].$$

- Core tensor is the same as the original tensor, it indicates that the tensor is already a low-rank representation and cannot be further decomposed into simpler components without losing information.

HOSVD of Tensor of Oder $\mathbb{R}^{2\times3\times2}$

Question 0.5:

Given Tensor: $\mathcal{A} \in \mathbb{R}^{2\times3\times2}$:

$$\mathcal{A}(:,:,1) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \mathcal{A}(:,:,2) = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

Goal: HOSVD to decompose $\mathcal{A}$ into:

$$\mathcal{A} = \mathcal{S} \times_1 U_1 \times_2 U_2 \times_3 U_3,$$

where:

- $\mathcal{S}$ is the core tensor,
- U_1, U_2, U_3 are orthogonal matrices(of left singular vectors) for each mode.

Step 1: Unfolding the Tensor

Unfold the tensor $\mathcal{A}$ along each mode:

- **Mode-1 unfolding:**

$$A_{(1)} = \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \end{bmatrix}.$$

- **Mode-2 unfolding:**

$$A_{(2)} = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}.$$

- **Mode-3 unfolding:**

$$A_{(3)} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{bmatrix}.$$

Step 2: SVD for Each Mode

Perform SVD on the unfolded matrices.

- **Mode-1 unfolding:**

$$A_{(1)} = U_1 \Sigma_1 V_1^T,$$

  where:

$$U_1 = \begin{bmatrix} -0.37 & -0.93 \\ -0.93 & 0.37 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} 25.46 & 0 \\ 0 & 1.29 \end{bmatrix}.$$

- **Mode-2 unfolding:**

$$A_{(2)} = U_2 \Sigma_2 V_2^T,$$

  where:

$$U_2 = \begin{bmatrix} -0.43 & 0.81 & 0.39 \\ -0.56 & -0.58 & 0.59 \\ -0.71 & -0.04 & -0.70 \end{bmatrix}.$$

Step 2: SVD for Each Mode Contn'd...

- **Mode-3 unfolding:**

$$A_{(3)} = U_3 \Sigma_3 V_3^T,$$

where:

$$U_3 = \begin{bmatrix} -0.55 & -0.83 \\ -0.83 & 0.55 \end{bmatrix}.$$

Step 3: Compute the Core Tensor

The core tensor $\mathcal{S}$ is computed as:

$$\mathcal{S} = \mathcal{A} \times_1 U_1^T \times_2 U_2^T \times_3 U_3^T.$$

This results in a compact representation of the original tensor.

Final Decomposition

The tensor $\mathcal{A}$ is now represented as:

$$\mathcal{A} = \mathcal{S} \times_1 U_1 \times_2 U_2 \times_3 U_3,$$

where:

- U_1, U_2, U_3 are the orthogonal factor matrices for each mode.
- $\mathcal{S}$ is the core tensor, capturing the essence of $\mathcal{A}$.

This decomposition helps in dimensionality reduction, compression, and feature extraction.