

Homework -13

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AI-A

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Qn 1.

$$\langle p, q \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3)$$

$$x_0 = -3, x_1 = -1, x_2 = 1, x_3 = 3$$

a) Find best approx. of $p(x) = x^2$ onto $P_1 = \text{span}\{1, x\}$

$$\hat{p} = \text{Proj}_{P_1} p = \text{Proj}_1 p + \text{Proj}_x p$$

$$= \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} x_1 + \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x$$

$$\begin{aligned} \langle x^2, 1 \rangle &= (-3)^2 + (-1)^2 + 1^2 + 3^2 \\ &= 20 \end{aligned}$$

$$\langle 1, 1 \rangle = 1 + 1 + 1 + 1 = 4$$

$$\begin{aligned} \langle x^2, x \rangle &= (-3)^2(-3) + (-1)^2(-1) + 1 + 3^2(3) \\ &= -27 - 1 + 1 + 27 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \langle x, x \rangle &= (-3)(-3) + (-1)(-1) + (1)(1) + (3)(3) \\ &= 20 \end{aligned}$$

$$= \frac{20}{4} + \frac{0}{20} = 5$$

The best approx. of $p(x) = x^2$ onto $P_1 = \text{span}\{1, x\}$ is 5

Qn 1 b

$$(B^T A) v = (B^T A) v$$

let $\{u_1, u_2, u_3\}$ be O.B. \mathbb{R}^3

$$\{1, x, x^2\} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = 1$$

$$\langle x, 1 \rangle = -3 - 1 + 1 + 3$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = u_2 - \text{Proj}_{x_1} u_2$$

$$\text{in } \mathbb{R}^3, x_1 = u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \langle x_1, 1 \rangle x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} x_1 - 0 \frac{\langle x_1, x_1 \rangle}{\langle x_1, x_1 \rangle} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x_2 = x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} u_3 - \text{Proj}_{x_1} u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = u_3 - \frac{\langle x_2, x_2 \rangle}{\langle x_2, x_2 \rangle} x_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = x^2 = \frac{0}{20} x = \frac{20}{4}$$

$$x_3 = x^2 - 5$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{O.B.} = \{1, x, x^2 - 5\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Qn 2

$$\langle A, B \rangle = \text{tr}(A^T B)$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$$

$$x_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_2 = u_2 - \text{Proj}_{x_1} u_2 = u_2 - \frac{\langle u_2, x_1 \rangle}{\langle x_1, x_1 \rangle} x_1$$

$$\begin{aligned} \langle u_2, x_1 \rangle &= \text{tr} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 \end{aligned}$$

$$\langle x_1, x_1 \rangle = \text{tr} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$x_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$x_3 = u_3 - \text{Proj}_{x_1} u_3 - \text{Proj}_{x_2} u_3$$

$$x_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{\langle u_3, x_1 \rangle}{\langle x_1, x_1 \rangle} x_1 - \frac{\langle u_3, x_2 \rangle}{\langle x_2, x_2 \rangle} x_2$$

$$\begin{aligned} \langle u_3, x_1 \rangle &= \text{tr} \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$= 2$$

$$\begin{aligned}\langle u_3, x_2 \rangle &= \frac{1}{2} \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right) \\ &= 0\end{aligned}$$

$$\begin{aligned}\langle x_2, x_2 \rangle &= \frac{1}{2} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \frac{2}{2} = 1\end{aligned}$$

$$\begin{aligned}x_3 &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{0}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ x_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\end{aligned}$$

$$x_4 = u_4 - \text{proj}_{x_1} u_4 - \text{proj}_{x_2} u_4 - \text{proj}_{x_3} u_4$$

$$x_4 = u_4 - \frac{\langle u_4, x_1 \rangle}{\langle x_1, x_1 \rangle} x_1 - \frac{\langle u_4, x_2 \rangle}{\langle x_2, x_2 \rangle} x_2 - \frac{\langle u_4, x_3 \rangle}{\langle x_3, x_3 \rangle} x_3$$

$$\begin{aligned}\langle u_4, x_1 \rangle &= \frac{1}{2} \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) = 0\end{aligned}$$

$$\begin{aligned}\langle u_4, x_2 \rangle &= \frac{1}{2} \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \right) = 0\end{aligned}$$

$$\begin{aligned}\langle u_4, x_3 \rangle &= \frac{1}{2} \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right) = 0\end{aligned}$$

$$\begin{aligned} \langle x_3, x_3 \rangle &= \frac{1}{2} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{2} \cdot 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} x_4 &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ x_4 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

Orthonormal Basis

$$\left\{ \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix} \right\}$$

Qn 3

$f(x) = x-1$ on the interval $(-\pi, \pi)$

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x-1) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-1) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-1) \sin kx dx$$

$$a_0 = \frac{1}{2\pi} \left[\frac{x^2}{2} - x\pi \right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi} \left(\left(\frac{\pi^2}{2} - x\pi \right) - \left(-\frac{\pi^2}{2} + \pi \right) \right)$$

$$a_0 = \frac{1}{2\pi} (-2\pi)$$

$$a_0 = -1$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx = \cos kx \, dx$$

$$a_k = \frac{1}{\pi} \left(\left[\frac{-\sin kx}{k} \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} x \cos kx \, dx \right)$$

$$a_k = \frac{1}{\pi} \left(\left[\frac{-\sin k\pi}{k} \right] + \left[\frac{\sin k\pi}{k} \right] + \int_{-\pi}^{\pi} x \cos kx \, dx \right)$$

$\sin n\pi$ where n is an integer = 0

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx$$

$$u = x \quad u' = \cos kx$$

$$u' = 1 \quad v = \frac{\sin kx}{k}$$

$$\pi a_k = \frac{x \sin kx}{k} - \int_{-\pi}^{\pi} \frac{\sin kx}{k} \, dx$$

$$\pi a_k = \left[\frac{x \sin kx}{k} + \frac{\cos kx}{k^2} \right]_{-\pi}^{\pi}$$

$$a_k = \frac{1}{\pi} \left(\left(0 + \frac{\cos k\pi}{k^2} \right) - \left(0 + \frac{\cos -k\pi}{k^2} \right) \right)$$

$$\cos \theta = \cos -\theta$$

$$\text{hence } a_k = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx - \sin kx \, dx$$

$$\pi b_k = \int_{-\pi}^{\pi} x \sin kx \, dx + \left[\frac{\cos kx}{k} \right]_{-\pi}^{\pi}$$

$$\pi b_k = \int_{-\pi}^{\pi} x \sin kx \, dx + \left(\frac{\cos \pi k}{k} - \frac{\cos -\pi k}{k} \right)$$

$$\pi b_k = \cancel{\frac{2 \cos \pi k}{k}} + \int_{-\pi}^{\pi} x \sin kx \, dx$$

$$u = x \quad v' = \sin kx$$

$$u' = 1 \quad v = -\frac{\cos kx}{k}$$

$$\pi b_k = \cancel{\frac{2 \cos \pi k}{k}} + \left(-\frac{x \cos kx}{k} + \int_{-\pi}^{\pi} \frac{-\cos kx}{k} \, dx \right)$$

$$\pi b_k = \cancel{\frac{2 \cos \pi k}{k}} + \left[-\frac{x \cos kx}{k} + \frac{\sin kx}{k^2} \right]_{-\pi}^{\pi}$$

$$\pi b_k = \cancel{\frac{2 \cos \pi k}{k}} + \left(\frac{-\pi \cos \pi k + \sin \pi k}{k^2} \right) - \left(\frac{-\pi \cos (-\pi k) + \sin (-\pi k)}{k^2} \right)$$

$$b_k = \frac{-2 \cos \pi k}{k} = \frac{-2(-1)^k}{k}$$

$$b_k = \frac{-2(-1)^k}{k}$$

$$b_k = -1 + \sum_{k=1}^{\infty} \frac{-2(-1)^k}{k} \sin kx$$