# MT-1004
# Linear Algebra

Fall 2023

**Week # 13-14**

National University of Computer and Emerging Sciences



November 23, 2023

# Orthogonal Diagonalization of Symmetric Matrices

# Properties of Symmetric Matrices



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If $A$ is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of $A$ are orthogonal.

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### Theorem (The Spectral Theorem)

Let $A$ be an $n \times n$ real matrix. Then $A$ is symmetric if and only if it is orthogonally diagonalizable.

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Orthogonally diagonalize the following matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

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Calculate

**Step-I** eigenvalues (verify they are real)

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Eigenvalues: $\lambda = 0, 5$.

Eigenvectors: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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Hence,

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How can we make them orthogonal?

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**Gram-Schmidt Process**

$$\left\{ v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

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Set of eigenvectors can be written as

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$$. \quad \left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{2}, \quad \left\| \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \right\| = \sqrt{18}, \quad \left\| \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \right\| = 3.$$

Hence,

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{pmatrix}.$$

$$D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

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Solution

$$A = 0 \begin{pmatrix} -2 \\ 1 \end{pmatrix} (-2 \ \ 1) + 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} (1 \ \ 2).$$

# Applications of Orthogonal Diagonalization

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- A quadratic form in $n$ variables is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form where $A$ is a symmetric $n \times n$ matrix and $x$ is in $\mathbb{R}^n$. We refer to $A$ as the matrix associated with $f$

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where $Y = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $D$ is a diagonal matrix whose diagonal entries are the eigenvalues of $A$.

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$$D = \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

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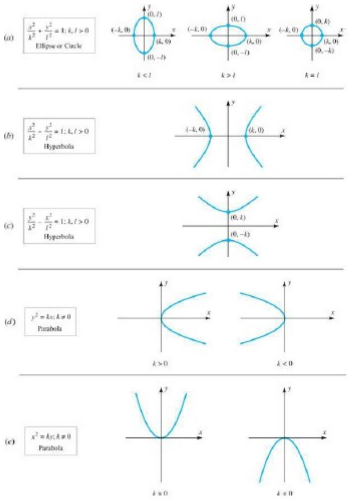
Hence, we have

$$3x_1^2 + 7y_1^2 = 48.$$


(a)
Ellipse or Circle
k < l
k > l
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(b)
Hyperbola
(c)
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(d)
Parabola
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In matrix form we can write

$$x^T AX + KX + 4 = 0,$$

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Put $X = QY$ to get

$$Y^T DY + KQY + 4 = 0,$$

where

$$Y = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2\sqrt{5} & -1\sqrt{5} \\ 1\sqrt{5} & 2\sqrt{5} \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

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$$4x_1^2 + 9y_1^2 - 8x_1 - 36y_1 + 4 = 0.$$

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$$4(x_1 - 1)^2 + 9(y_1 - 2)^2 = 36.$$

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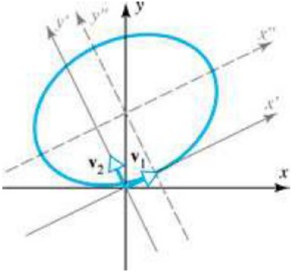
Put $x''=x_1-1, \quad y''=y_1-2$, we get

$$\frac{x''^2}{9}+\frac{y''^2}{4}=1.$$



## Definition

A quadratic form $f(x) = x^T Ax$ is classified as one of the following:

1. positive definite if $f(x) > 0$ for all $x \neq 0$
2. positive semidefinite if $f(x) \geq 0$ for all $x$
3. negative definite if $f(x) < 0$ for all $x \neq 0$
4. negative semidefinite if $f(x) \leq 0$ for all $x$
5. indefinite if $f(x)$ takes on both positive and negative values

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- indefinite if and only if $A$ has both positive and negative eigenvalues.

Find the maximum and minimum values of $Q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $x^T x = 1$.

**Solution**

$$\begin{aligned} Q(\mathbf{x}) &= 9x_1^2 + 4x_2^2 + 3x_3^2 \\ &\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \\ &= 9(x_1^2 + x_2^2 + x_3^2) \\ &= 1 \end{aligned}$$

So the maximum value of $Q(\mathbf{x})$ cannot exceed 9 when $\mathbf{x}$ is a unit vector. Thus 9 is the maximum value of $Q(\mathbf{x})$.

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# CONSTRAINED OPTIMIZATION

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To find the minimum value of $Q(\mathbf{x})$, observe that

$$Q(\mathbf{x}) \geq 3x_1^2 + 3x_2^2 + 3x_3^2 = 3$$

$Q(\mathbf{x}) = 3$ is the minimum value

**Theorem**

Let $f(\mathbf{x}) = x^T Ax$ be a quadratic form with associated $n \times n$ symmetric matrix $A$. Let the eigenvalues of $A$ be $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. Then the following are true, subject to the constraint $\|\mathbf{x}\| = 1$

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- $\lambda_1 \geq f(\mathbf{x}) \geq \lambda_n$

### Theorem

Let $f(\mathbf{x}) = x^T A x$ be a quadratic form with associated $n \times n$ symmetric matrix $A$. Let the eigenvalues of $A$ be $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. Then the following are true, subject to the constraint $\|\mathbf{x}\| = 1$

- $\lambda_1 \geq f(\mathbf{x}) \geq \lambda_n$
- The maximum value of $f(\mathbf{x})$ is $\lambda_1$, and it occurs when $x$ is a unit eigenvector corresponding to $\lambda_1$.

**Theorem**

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- $\lambda_1 \geq f(\mathbf{x}) \geq \lambda_n$
- The maximum value of $f(\mathbf{x})$ is $\lambda_1$, and it occurs when $x$ is a unit eigenvector corresponding to $\lambda_1$.
- The minimum value of $f(\mathbf{x})$ is $\lambda_n$, and it occurs when $x$ is a unit eigenvector corresponding to $\lambda_n$.

# Example

Find the maximum and minimum values of $Q(\mathbf{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $x^T x = 1$.

**Solution**

Matrix corresponding to given quadratic form is

$$A = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Eigenvalues of $A$ are $9,\ 4,\ 3.$

Eigenvector corresponding to 9 is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Eigenvector corresponding to 3 is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

# Singular Value Decomposition

# Matrix Factorization

- Diagonalization of a square matrix $A$

$$A = PDP^{-1}.$$

- Orthogonal Diagonalization of symmetric matrix $A$

$$A = QDQ^T.$$

- What will if matrix is not symmetric?
- What if even matrix is not square?

# Motivation

- The absolute values of the eigenvalues of a symmetric matrix $A$ measure the amounts that $A$ stretches or shrinks certain vectors (the eigenvectors). If $Ax = \lambda x$ and $\|x\| = 1$, then

$$\|Ax\| = \|\lambda x\| = |\lambda|\|x\| = |\lambda|.$$

- If

$$A = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix},$$

then the linear transformation $x \to Ax$ maps the unit sphere $\{x : \|x| = 1\}$ in $\mathbb{R}^3$ onto an ellipse in $\mathbb{R}^2$. Find a unit vector $x$ at which the length $\|Ax\|$ is maximized, and compute this maximum length.

$$\|Ax\|^2 = Ax \cdot Ax = (Ax)^T(Ax) = x^T(A^TA)x.$$

Problem is to maximize the quadratic form $x^T(A^TA)x$ subject to the constraint $\|x\| = 1$.

How to solve this problem?

# Motivation

- The maximum value is attained at a unit eigenvector of $A^T A$ corresponding to $\lambda_1$.

$$A^T A = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix}.$$

The eigenvalues are $360, \ 90, \ 0$. Maximum eigenvalue is 360 and corresponding eigenvector is

$$v_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

Hence, the maximum value of $\|Ax\|^2$ is 360, attained when $x$ is the unit vector $v_1$.

For $\|x\| = 1$, the maximum value of $\|Ax\|$ is

$$\|Av_1\| = \sqrt{360}$$

.

# Singular Values

- For any matrix $A$ of size $m \times n$

$$A^T A \text{ is symmetric.}$$

- Let $\{v_1, v_2, ..., v_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$ and let $\lambda_1,\ \lambda_2, ..., \lambda_n$ be the associated eigenvalues.

- $\|Av_1\|^2 = Av_1 \cdot Av_1 = (Av_1)^T (Av_1) = v_1^T (A^T A) v_1 = v_1^T (\lambda_1 v_1) = \lambda_1.$

- All the eigenvalues of $A^T A$ are all nonnegative.

- The singular values of $A$ are the square roots of the eigenvalues of $A^T A$.

# Singular Values

Find the singular values of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solution

$$A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

has eigenvalues 3 and 1.

So, singular values of $A$ are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3},$$

and

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

**Theorem**

Suppose $\{v_1, ..., v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^TA$, arranged so that the corresponding eigenvalues of $A^TA$ satisfy $\lambda_1 \geq ... \geq \lambda_n$, and suppose $A$ has $r$ nonzero singular values. Then $\{Av_1, ..., Av_r\}$ is an orthogonal basis for $\mathrm{Col}\ A$, and $\mathrm{rank}\ A = r$.

**Proof**

$$Av_i \cdot Av_j = (Av_j)^T(Av_i) = v_j^TA^TAv_i = v_j^T\lambda_i v_i = 0.$$

Hence, $\{Av_1, ..., Av_r\}$ is an orthogonal set.

Let $y \in \mathrm{Col}\ A$ then $y = Ax = A(c_1v_1 + ...c_rv_r + c_{r+1}v_{r+1} + ... + c_nv_n)$.

Above, equation can be written as

$$y = Ax = c_1Av_1 + ...c_rAv_r + 0 + 0... + 0.$$

So,

$$y \in \mathrm{Span}\ \{Av_1, ..., Av_r\}.$$

# Singular Value Decomposition

## Theorem

Let $A$ be an $m \times n$ matrix with $\mathrm{rank} r$. Then there exists an $m \times n$ matrix $\Sigma$ of the form

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

for which the diagonal entries in $D$ are the first $r$ singular values of $A$, $\sigma_1 \geq ... \geq \sigma_r > 0$ and there exist an $m \times n$ orthogonal matrix $U$ and an $n \times n$ orthogonal matrix $V$ such that

$$A = U\Sigma V^T$$

# Singular Value Decomposition

**Theorem**

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# Singular Value Decomposition

- For positive definite matrices, *Sigma* is $D$ and $U\Sigma V^T$ is identical to $QDQ^T$.

- For other symmetric matrices, any negative eigenvalues in $D$ become positive in $\Sigma$.

- $U$ and $V$ give orthonormal bases for all four fundamental subspaces: first $r$ columns of $U$: column space of $A$

  last $m - r$ columns of $U$: left nullspace of $A$

  first $r$ columns of $V$: row space of $A$

  last $n - r$ columns of $V$: nullspace of $A$

- $AV = U\Sigma$

# Singular Value Decomposition

- Eigenvectors of $AA^T$ and $A^TA$ must go into the columns of $U$ and $V$:
  $AA^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T$ and similarly, $A^TA = V\Sigma^T\Sigma V^T$.

- $A^TAv_j = \sigma_j^2 v_j$. Multiply by $A$, we get

$$AA^TAv_j = \sigma_j^2 Av_j$$

  This is eigenvalue equation

$$AA^T(Av_j) = \sigma_j^2(Av_j).$$

  Hence, $Av_j$ is the eigenvector of $AA^T$ and $\sigma_j^2$ is the eigenvalue.

  So, the unit eigenvector is $Av_j/\sigma_j = u_j$.

  In other words,

$$AV = U\Sigma.$$

## Example

Find a singular value decomposition of

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Eigenvalues are $2, 1, 0$ and corresponding normalized eigenvectors are

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

Hence,

$$V = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In order to find $U$, we compute $u_1 = \frac{1}{\sigma_1} A v_1, \; u_2 = \frac{1}{\sigma_2} A v_2$.
So,

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

## Example

Find a singular value decomposition of

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}.$$

The eigenvalues of $A^T A$ are 18 and 0 with corresponding unit eigenvectors

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \; v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

Hence,

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

and

$$\Sigma = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

# Example

To construct $U$, first construct $Av_1$ and $Av_2$ :

$$Av_1 = \begin{pmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{pmatrix}, \quad Av_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$u_1 = \frac{1}{3\sqrt{2}} Av_1$$

In order to write $U$, we need to extend the set $\{u_1\}$ to orthonormal basis for $\mathbb{R}^3$.

$$U = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{pmatrix}.$$

**Theorem**

Let $A$ be an $m \times n$rix with singular vales $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$. Let $u_1, \ u_2, \ , ..., u_r$ be left singular vectors and let $v_1, \ v_2, \ , ..., v_r$ be right singular vectors of $A$ corresponding to these singular vales. The,

$$A = \sigma_1 u_1 v_1^T + ... + \sigma_r u_r v_r^T.$$

# The Pseudoinverse of a Matrix

## Two Sided Inverse

Any matrix $A^{-1}$ is said to be **inverse** of $A$ of size $m \times n$ if

$$A^{-1} A = I = AA^{-1},$$

- $\mathrm{rank}(A) = m = n$.
- Rows and Columns are independent.
- $\mathrm{Nullspace}(A) = \{0\} = \mathrm{Nullspace}(A^T)$.
- $Ax = b$ possess unique solution.

# Left Inverse

- Consider $A$ of order $m \times n$ such that $m > n$.
- Matrix $A$ has full column rank i.e., $\mathrm{rank}(A) = n$.
- Columns are independent.
- $\mathrm{Nullspace}(A) = \{0\}$.
- $Ax = b$ either possess unique solution or no solution.

# Left Inverse

- Consider $A$ of order $m \times n$ such that $m > n$.
- Matrix $A$ has full column rank i.e., $\mathrm{rank}(A) = n$.
- Columns are independent.
- $\mathrm{Nullspace}(A) = \{0\}$.
- $Ax = b$ either possess unique solution or no solution.

Any matrix $A_{\text{left}}^{-1}$ is said to be **left inverse** of $A$ if

$$A_{\text{left}^{-1}} \; A = I_n,$$

where

$$A_{\text{left}}^{-1} = (A^T A)^{-1} A^T.$$

# Right Inverse

- Consider $A$ of order $m \times n$ such that $m < n$.
- Matrix $A$ has full row rank i.e., $\mathrm{rank}(A) = m$.
- Rows are independent.
- $\mathrm{Nullspace}(A^T) = \{0\}$.
- $Ax = b$ always possess solution.

## Right Inverse

- Consider $A$ of order $m \times n$ such that $m < n$.
- Matrix $A$ has full row rank i.e., $\mathrm{rank}(A) = m$.
- Rows are independent.
- $\mathrm{Nullspace}(A^T) = \{0\}$.
- $Ax = b$ always possess solution.

Any matrix $A_{\mathrm{right}}^{-1}$ is said to be **right inverse** of $A$ if

$$A\, A_{\mathrm{right}}^{-1} = I,$$

where

$$A_{\mathrm{right}}^{-1} = A^T(AA^T)^{-1}.$$

# Left/Right Inverse

- $A_{\text{left}}^{-1}\, A = I_n$

- What about

$$A\, A_{\text{left}}^{-1} = A(A^T A)^{-1} A^T = \text{Projection on Column Space}$$

- $A\, A_{\text{right}}^{-1} = I_m$

- What about

$$A_{\text{right}}^{-1}\, A = A^T (AA^T)^{-1} A = \text{Projection on Row Space}$$

# Lecture 34

# The Pseudoinverse of a Matrix

# Left/Right Inverse

- $A_{\text{left}}^{-1}\, A = I_n$
- What about

$$A\, A_{\text{left}}^{-1} = A(A^T A)^{-1} A^T = \text{Projection on Column Space}$$

- $A\, A_{\text{right}}^{-1} = I_m$
- What about

$$A_{\text{right}}^{-1}\, A = A^T (A A^T)^{-1} A = \text{Projection on Row Space}$$

# Properties of Pseudoinverse

Let $A$ be a real $m \times n$ matrix. The pseudoinverse of $A$ is the unique $n \times m$ matrix $A^+$ such that $A$ and $A^+$ satisfy the following conditions

- $AA^+A = A$ and $A^+AA^+ = A^+$.
- $AA^+$ and $A^+A$ are symmetric

# Pseudoinverse

Let $A$ be a real $m \times n$ matrix, and let $A = U\Sigma V^T$ is any SVD for $A$. Then,

$$A^+ = V\Sigma^+ U^T.$$

# Pseudoinverse

Let $A$ be a real $m \times n$ matrix, and let $A = U\Sigma V^T$ is any SVD for $A$. Then,

$$A^+ = V\Sigma^+ U^T.$$

$$\Sigma^+ = \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{n\times m}.$$

- $\Sigma\Sigma^+\Sigma = \Sigma$
- $\Sigma^+\Sigma\Sigma^+ = \Sigma^+$
- $\Sigma\Sigma^+ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m\times m}$
- $\Sigma^+\Sigma = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n\times n}$

# Example

Find the pseudoinverse of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

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# Example

Find the pseudoinverse of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

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So, pesudoinverse of $A$ is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

# Example

Find the pseudoinverse of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

$$A^+ = (A^T A)^{-1} A^T = \begin{pmatrix} \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

In this case

$$A^+ A = I$$

# Least Squares and Pseudoinverse

The least squares problem $Ax = b$ has a unique least squares solution $x$ of minimal length that is given by

$$\overline{x} = A^{+}b.$$

# Example

Find the least squares solutions to $Ax = b$ where:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.$$

Least square solution

$$\overline{x} = A^+ b.$$

In order to find $A^+$, we will find SVD of $A$.

$$U = \begin{pmatrix} -0.2185 & 0.8863 & 0.4082 \\ -0.5216 & 0.2475 & -0.8165 \\ -0.8247 & -0.3913 & 0.4082 \end{pmatrix}, \quad S = \begin{pmatrix} 2.6762 & 0 \\ 0 & 0.9513 \\ 0 & 0 \end{pmatrix}.$$

$$V = \begin{pmatrix} -0.5847 & 0.8112 \\ -0.8112 & -0.5847 \end{pmatrix}.$$

Pseudoinverse of $A$ is

$$A^+ = V\Sigma^+ U^+ = \begin{pmatrix} 0.8333 & 0.3333 & -0.1667 \\ -0.5000 & -0.0000 & 0.5000 \end{pmatrix}.$$

Hence, $\overline{x} = A^+ b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$