STATISTICS IS THE GRAMMAR OF SCIENCE

PROBABILITY AND STATISTICS

LECTURE – 16

PROBABILITY DISTRIBUTIONS

BINOMIAL DISTRIBUTION

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BINIOMIAL DISTRIBUTION

The binomial probability distribution is one of the most widely used discrete probability distribution. The Binomial Distribution is applied to experiments that satisfy the conditions of a Binomial Experiment, Lets study about Binomial Experiment first.

THE BINOMIAL EXPERIMENT A binomial experiment is a probability experiment that satisfies the following four requirements

1- There must be a fixed number of trials. ie there are n identical trials.
2- Each trial has only two possible outcomes, either success or failure.
3- The trials are independent of each other.
4- The probability of success remains same for each trial.

EXAMPLES OF BINOMIAL EXPERIMENT Some examples of Binomial experiment are as follows

1- Tossing a coin 100 times and observing the probability of number of heads appeared.
2- Throwing a Die 50 times and observing the probability of number of sixes appeared.
3- Selecting a customer from a sample of ten customers who use credit card.

THE BINOMIAL PROBABILITY DISTRIBUTION A binomial experiment and its results give rise to a special probability distribution called the binomial distribution.

> The outcomes of a binomial experiment and the corresponding probabilities of these outcomes are called a **binomial distribution.**

THE BINOMIAL FORMULA If random variable X follows binomial distribution then it is denoted by $X \sim B(n, p)$ and the binomial probability function is given by

$$P(X = x) = {}^nC_x p^x q^{n-x}, \qquad x = 0,1,2,3,\dots,n$$

$$\begin{aligned} Where \quad n &= number\ of\ trials \\ p &= probability\ of\ success \\ q &= probability\ of\ failure \\ x &= number\ of\ successes\ in\ n\ trials \\ n - x &= number\ of\ failures\ in\ n\ trials \end{aligned}$$

PROPERTIES OF BINOMIAL DISTRIBUTION

1- The Mean of Binomial Distribution is $E(X) = np$
2- The variance of Binomial Distribution is $V(X) = npq$

EXAMPLES OF BINOMIAL DISTRIBUTION

EXAMPLE-1 The random variable X is distributed as $B(6, 0.42)$. Find the following

$(a)\ P(X=6)$

$(b)\ P(X \le 2)$

$(c)\ P(X > 4)$

$(d)\ E(X)$

$(e)\ V(X)$

SOLUTION $Here \quad n=6, \quad p=0.42, \quad q=1-p=1-0.42=0.58$

$(a)\ P(X=6)={}^6C_6(0.42)^6(0.58)^0=0.0055$

$(b)\ P(X \le 2)=P(X=0)+P(X=1)+P(X=2)$

$$={}^6C_0(0.42)^0(0.58)^6+{}^6C_1(0.42)^1(0.58)^5+{}^6C_2(0.42)^2(0.58)^4=0.5029$$

$(c)\ P(X>4)=P(X=5)+P(X=6)$

$$={}^6C_5(0.42)^5(0.58)^1+{}^6C_6(0.42)^6(0.58)^0=0.0510$$

$(d)\ E(X)=np=(6)(0.42)=2.52$

$(e)\ V(X)=npq=(6)(0.42)(0.58)=1.4616$

EXAMPLE-2 The random variable X is distributed as $B(7, 0.2)$. Find the following

$(a)\ P(X=3)$

$(b)\ P(X>1)$

$(c)\ P(X \le 6)$

$(d)\ P(1 < X \le 4)$

$(e)\ P(2 \le X \le 5)$

SOLUTION $Here \quad n=7, \quad p=0.2, \quad q=1-p=1-0.2=0.8$

$(a)\ P(X=3)={}^7C_3(0.2)^3(0.8)^4=0.1147$

$(b)\ P(X>1)=P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)$

$or\ \ P(X>1)=1-P(X \le 1)=1-[P(X=0)+P(X=1)]$

$$=1-[\,{}^7C_0(0.2)^0(0.8)^7+{}^7C_1(0.2)^1(0.8)^6\,]=0.4233$$

$(c)\ P(X \le 6)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)$

$or\ \ P(X \le 6)=1-P(X>6)=1-[P(X=7)]$

$$={}^7C_7(0.2)^5(0.8)^0=1$$

$(d)\ P(1<X \le 4)=P(X=2)+P(X=3)+P(X=4)$

$or \qquad ={}^7C_2(0.2)^2(0.8)^5+{}^7C_3(0.2)^3(0.8)^4+{}^7C_4(0.2)^4(0.8)^3=0.4186$

$(e)\ P(2 \le X \le 5)=P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$or \qquad ={}^7C_2(0.2)^2(0.8)^5+{}^7C_3(0.2)^3(0.8)^4+{}^7C_4(0.2)^4(0.8)^3+{}^7C_5(0.2)^5(0.8)^2=0.4229$

EXAMPLE-3 At Islamabad Supermarket 60% of customers pay by credit card. Find the probability that in a randomly selected sample of ten customers
(a) exactly two pay by credit card
(b) more than seven pay by credit card
(c) less than three pay by credit card

SOLUTION Let X denote number of customers who pay by credit card

$$Here \quad n=10, \quad p=0.6, \quad q=1-p=1-0.6=0.4$$
$$(a)\ P(X=2)={}^{10}C_2(0.6)^2(0.4)^8=0.0106$$

$$(b)\ P(X>7)=P(X=8)+P(X=9)+P(X=10)$$
$$={}^{10}C_8(0.6)^8(0.4)^2+{}^{10}C_9(0.6)^9(0.4)^1+{}^{10}C_{10}(0.6)^{10}(0.4)^0=0.1673$$

$$(b)\ P(X<3)=P(X=0)+P(X=1)+P(X=2)$$
$$={}^{10}C_0(0.6)^0(0.4)^{10}+{}^{10}C_1(0.6)^1(0.4)^9+{}^{10}C_2(0.6)^2(0.4)^8=0.0123$$

EXAMPLE-4 The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that
(a) at least ten will survive
(b) at most two will die

SOLUTION Let X denote number of patients who will survive

$$Here \quad n=15, \quad p=0.4, \quad q=1-p=1-0.4=0.6$$

$$(a)\ P(X\geq 10)=P(X=10)+P(X=11)+P(X=12)+P(X=13)+P(X=14)+P(X=15)$$
$$={}^{15}C_{10}(0.4)^{10}(0.6)^5+{}^{15}C_{11}(0.4)^{11}(0.6)^4+{}^{15}C_{12}(0.4)^{12}(0.6)^3+{}^{15}C_{13}(0.4)^{13}(0.6)^2$$
$$+{}^{15}C_{14}(0.4)^{14}(0.6)^1+{}^{15}C_{15}(0.4)^{15}(0.6)^0=0.0338$$

Let Y denote number of patients who will die

$$Here \quad n=15, \quad p=0.6, \quad q=1-p=1-0.6=0.4$$

$$(b)\ P(X\leq 2)=P(X=0)+P(X=1)+P(X=2)$$
$$={}^{15}C_0(0.6)^0(0.4)^{15}+{}^{15}C_1(0.6)^1(0.4)^{14}+{}^{15}C_2(0.6)^2(0.4)^{13}=0.0003$$

EXAMPLE-5 In a group of people the expected number who wear glasses is two and the variance is 1.6. find the probability that
(a) a person chosen at random from the group wears glasses.
(b) six people in the group wear glasses.

SOLUTION Let X denote number of people who wear glasses

$$Here \quad E(X)=2 \qquad and \qquad V(X)=1.6$$
$$So \quad np=2 \qquad and \qquad npq=1.6$$

$\Rightarrow \quad q = 0.8 \quad and \quad p = 0.2 \quad also \quad n = 10$

$(a)\ P(X = 1) = {}^{10}C_1 (0.2)^1 (0.8)^9 = 0.2684$

$(b)\ P(X = 6) = {}^{10}C_6 (0.2)^6 (0.8)^4 = 0.0055$

EXAMPLE-6 A box contains large number of pens. The probability that a pen is faulty is 0.1. How many pens would you need to be more than 95% certain of picking at least one faulty pen.

SOLUTION Let X denote number of faulty pens and n denote number of pens you need to select

$Here \quad n = n, \quad p = 0.1, \quad q = 1 - p = 1 - 0.1 = 0.9$

According to given condition

$$P(X \geq 1) > 0.95$$
$$1 - P(X < 1) > 0.95$$
$$1 - P(X = 0) > 0.95$$
$$1 - {}^{n}C_0 (0.1)^0 (0.9)^n > 0.95$$
$$1 - (0.9)^n > 0.95$$
$$1 - 0.95 > (0.9)^n$$
$$(0.9)^n < 0.05$$
$$Taking\ \ln\ on\ both\ sides$$
$$\ln(0.9)^n < \ln(0.05)$$
$$n\ .\ \ln(0.9) < \ln(0.05)$$
$$n > \frac{\ln(0.05)}{\ln(0.9)} \quad (As\ \ln(0.9)\ \ is\ \ negative$$
$$n > 28.43$$
$$n = 29$$
$$So\ \ least\ number\ of\ \ pens\ will\ be\ 29.$$

EXERCISE – 5.1

BINOMIAL DISTRIBUTION

1. (a) Given that $X \sim B(6, 0.2)$. Calculate
(a) $P(X = 3)$ (b) $P(X = 0)$ (c) $P(X = 6)$ (d) $P(X < 2)$ (e) $P(X \geq 4)$
(f) $P(X = 5\ or\ 6)$ (g) $P(atleast\,3)$ (h) $P(atmost\,2)$
$(Ans: 0.0819, 0.2621, 0.0001, 0.6554,\ 0.0170,\ 0.0016,\ 0.0989,\ 0.9011)$

2. Given that $X \sim B(4, p)$ and $P(X = 4) = 0.0256$ then find
(a) $P(X = 2)$ (b) $P(X < 2)$ (c) $P(X \geq 3)$ (d) $E(X)$ (e) $V(X)$
$(Ans: 0.3456, 0.4752, 0.1792, 1.6, 0.96)$

3. (a) A fair coin is tossed six times. Find the probability of throwing at least four heads.
(b) An unbiased die is thrown seven times. Find the probability of throwing at least 5 sixes.
$(Ans: 0.344, 0.002)$

4. A bag contains counters of which 40% are red and the rest yellow. A counter is taken from the bag, its color noted and then replaced. This is performed eight times in all. Calculate the probability that
(a) exactly three will be red (b) at least one will be red (c) more than four will be yellow
$(Ans: 0.279, 0.983, 0.594)$

5. The probability that it will rain on any given day in September is 0.3. Calculate the probability that in a given week in September, it will rain on
(a) exactly two days (b) at least two days (c) at most two days
$(Ans: 0.318, 0.671, 0.647)$

6. A coin is biased so that it is twice as likely to show heads as tails. The coin is tossed five times, calculate the probability that
(a) exactly three heads are obtained
(b) more than three are obtained
$(Ans: 0.0329, 0.461)$

7. (a) Given that $X \approx B(n, 0.3)$. Find the least possible value of n such that $P(X \geq 1) = 0.8$.
(b) The probability that a target is hit is 0.3. find the least number of shots which should be fired if the probability that the target is hit at least once is greater than 0.95.
$(Ans: 5, 9)$

8. 10% of the articles from a certain production line are defective. A sample of 25 articles is taken. Find the expected number of defective items and the standard deviation.
$(Ans: 2.5, 1.5)$

9. The random variable X is $B(n, 0.3)$ and $E(X) = 2.4$. Find n and the standard deviation of X.
$(Ans: 8, 1.30)$

10. The random variable X is $B(10, p)$ where $p < 0.5$ The variance of X is 1.875. Find
(a) the value of p (b) $E(X)$ (c) $P(X = 2)$
$(Ans: 0.25, 2.5, 0.282)$