Vector Spaces and Subspaces

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Definition of a Vector Space

Vector Space

A **vector space** (or linear space) over a field $\mathbb{F}$ is a set V together with two operations:

1. **Addition:** $+ : V \times V \to V$
2. **Scalar Multiplication:** $\cdot : \mathbb{F} \times V \to V$

such that the following properties hold for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in V$ and $c, d \in \mathbb{F}$:

1. **Commutativity:** $\mathrm{u} + \mathrm{v} = \mathrm{v} + \mathrm{u}$
2. **Associativity:** $(\mathrm{u} + \mathrm{v}) + \mathrm{w} = \mathrm{u} + (\mathrm{v} + \mathrm{w})$
3. **Identity Element:** There exists an element $0 \in V$ such that $\mathrm{u} + 0 = \mathrm{u}$
4. **Inverse Elements:** For every $\mathrm{u} \in V$, there exists $-\mathrm{u} \in V$ such that $\mathrm{u} + (-\mathrm{u}) = 0$
5. **Distributivity of Scalar Multiplication over Vector Addition:** $c(\mathrm{u} + \mathrm{v}) = c\mathrm{u} + c\mathrm{v}$
6. **Distributivity of Scalar Multiplication over Field Addition:** $(c + d)\mathrm{u} = c\mathrm{u} + d\mathrm{u}$

Example 1: Euclidean Space $\mathbb{R}^n$

The set $\mathbb{R}^n$ with standard vector addition and scalar multiplication is a vector space.

$$\mathbb{R}^n = \{\mathrm{x} = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$$

Proof Outline

1. **Closure under Addition:** Let $\mathrm{u} = (u_1, u_2, \ldots, u_n)$ and $\mathrm{v} = (v_1, v_2, \ldots, v_n)$ be in $\mathbb{R}^n$. Then $\mathrm{u} + \mathrm{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \in \mathbb{R}^n$.
2. **Closure under Scalar Multiplication:** Let $\mathrm{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then $c\mathrm{u} = (cu_1, cu_2, \ldots, cu_n) \in \mathbb{R}^n$.
3. **Zero Vector:** The vector $0 = (0, 0, \ldots, 0) \in \mathbb{R}^n$.
4. **Additive Inverse:** For any $\mathrm{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, the vector $-\mathrm{u} = (-u_1, -u_2, \ldots, -u_n) \in \mathbb{R}^n$.

Conclusion

Since $\mathbb{R}^n$ satisfies all vector space axioms, it is a vector space.

Example 2: The Set of Polynomials $\mathbb{P}_n(\mathbb{R})$

The set of all polynomials of degree at most n with coefficients in $\mathbb{R}$ is a vector space.

$$\mathbb{P}_n(\mathbb{R}) = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}$$

Proof Outline

1. **Closure under Addition:** If $p(x)$ and $q(x)$ are in $\mathbb{P}_n(\mathbb{R})$, then $p(x) + q(x)$ is also a polynomial of degree at most n.
2. **Closure under Scalar Multiplication:** If $p(x) \in \mathbb{P}_n(\mathbb{R})$ and $c \in \mathbb{R}$, then $cp(x)$ is also in $\mathbb{P}_n(\mathbb{R})$.
3. **Zero Polynomial:** The polynomial $p(x) = 0$ is in $\mathbb{P}_n(\mathbb{R})$.
4. **Additive Inverse:** For any $p(x) \in \mathbb{P}_n(\mathbb{R})$, the polynomial $-p(x)$ is also in $\mathbb{P}_n(\mathbb{R})$.

Conclusion

Since $\mathbb{P}_n(\mathbb{R})$ satisfies all vector space axioms, it is a vector space.

Example 3: The Set of Continuous Functions $C([a, b])$

The set of all continuous functions on the interval $[a, b]$ with pointwise addition and scalar multiplication is a vector space.

$$C([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}$$

Proof Outline

1. **Closure under Addition:** If $f(x)$ and $g(x)$ are continuous on $[a, b]$, then $f(x) + g(x)$ is also continuous on $[a, b]$.
2. **Closure under Scalar Multiplication:** If $f(x)$ is continuous on $[a, b]$ and $c \in \mathbb{R}$, then $cf(x)$ is continuous on $[a, b]$.
3. **Zero Function:** The zero function $f(x) = 0$ is in $C([a, b])$.
4. **Additive Inverse:** For any $f(x) \in C([a, b])$, the function $-f(x)$ is also in $C([a, b])$.

Conclusion

Since $C([a, b])$ satisfies all vector space axioms, it is a vector space.

Definition of a Subspace

Subspace

A subset W of a vector space V over a field $\mathbb{F}$ is called a **subspace** of V if W is itself a vector space under the operations of V.

Conditions for a Subspace

A non-empty subset $W \subseteq V$ is a subspace of V if:

1. $0 \in W$ (contains the zero vector)
2. $\mathrm{u}, \mathrm{v} \in W$ implies $\mathrm{u} + \mathrm{v} \in W$ (closed under addition)
3. $\mathrm{u} \in W$ and $c \in \mathbb{F}$ implies $c\mathrm{u} \in W$ (closed under scalar multiplication)

Example 1: The Zero Subspace

The set containing only the zero vector, $W = \{0\}$, is a subspace of any vector space V.

Proof Outline

1. **Zero Vector:** By definition, 0 is in W.
2. **Closure under Addition:** Since $0 + 0 = 0$, W is closed under addition.
3. **Closure under Scalar Multiplication:** For any scalar $c \in \mathbb{F}$, $c0 = 0$, so W is closed under scalar multiplication.

Conclusion

The zero subspace $\{0\}$ is a subspace of V.

Example 2: The Set of All Vectors Parallel to a Given Vector

In $\mathbb{R}^3$, the set $W = \{t\mathrm{v} \mid t \in \mathbb{R}\}$ for a fixed vector $\mathrm{v} \in \mathbb{R}^3$ is a subspace.

Proof Outline

Let v be a vector in $\mathbb{R}^3$ and consider the set $W = \{t\mathrm{v} \mid t \in \mathbb{R}\}$.

1. **Zero Vector:** If $t = 0$, then $t\mathrm{v} = 0 \in W$.
2. **Closure under Addition:** If $\mathrm{u} = t_1\mathrm{v}$ and $\mathrm{w} = t_2\mathrm{v}$ are in W, then $\mathrm{u} + \mathrm{w} = (t_1 + t_2)\mathrm{v} \in W$.
3. **Closure under Scalar Multiplication:** If $\mathrm{u} = t\mathrm{v} \in W$ and $c \in \mathbb{R}$, then $c\mathrm{u} = (ct)\mathrm{v} \in W$.

Conclusion

The set W is a subspace of $\mathbb{R}^3$.

Example 3: The Set of Solutions to a Homogeneous Linear System

Given a matrix A of size $m \times n$, the set $W = \{\mathrm{x} \in \mathbb{R}^n \mid A\mathrm{x} = 0\}$ is a subspace of $\mathbb{R}^n$.

Proof Outline

Let $A\mathrm{x} = 0$ be a homogeneous linear system.

1. **Zero Vector:** $A0 = 0$, so 0 is in the solution set W.
2. **Closure under Addition:** If $\mathrm{u}, \mathrm{v} \in W$, then $A\mathrm{u} = 0$ and $A\mathrm{v} = 0$. Hence, $A(\mathrm{u} + \mathrm{v}) = 0$, so $\mathrm{u} + \mathrm{v} \in W$.
3. **Closure under Scalar Multiplication:** If $\mathrm{u} \in W$ and $c \in \mathbb{R}$, then $A(c\mathrm{u}) = 0$, so $c\mathrm{u} \in W$.

Conclusion

The set $W = \{\mathrm{x} \in \mathbb{R}^n \mid A\mathrm{x} = 0\}$ is a subspace of $\mathbb{R}^n$.