Question 1

Let S be the parallelopiped determined by the vectors

$$\mathbf{b_1} = \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}, \mathbf{b_3} = \begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}$$

and let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

Compute the volume of the image of S under the mapping $\mathbf{x} \to \mathbf{A}\mathbf{x}$.

Question 2

By what factor does the following transformation change the size of the box?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ 2x + y \end{bmatrix}$$

Question 3

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 5 & 4 \\ 8 & 7 & 10 \end{bmatrix}$$

by using:

1. Cofactor Expansion method
2. Row operation
3. Also write A and A^{-1} as product of elementary matrices (Note that you can use elementary row operations done in last part to construct these elementary matrices)

Solution of Q # 01

(Question No 01)

$b_1 = \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}$, So $B = \begin{bmatrix} -5 & -2 & -2 \\ 1 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$

Expanding B by R_3:

$|\det B| = |4(-25+2)| \Rightarrow |-23 \times 4| \Rightarrow |-92|$

$|\det B| = 92$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

$\det A = 1 \cdot 2 \cdot 3 \Rightarrow 6$

So, volume of the image is: $6 \times 92 \Rightarrow 552$

$\therefore$ Area of $T(S)$ = area of $S \times \det A$

Solution of Q # 02

the transformation matrix is

$$T = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Changing factor $= \det(T) = 1+2 = 3$

So, by factor of 3, size of box changes.

Solution of Q # 03

7. Cofactor Expansion method

$$|A| = 1(50-28) - 2(70-32) + 3(49-40)$$

$$= 22 - 76 + 27$$

$$|A| = -27$$

$$Adj\,A = \begin{bmatrix} (50-28) & -(70-32) & 49-40 \\ -(70-21) & (10-24) & -(7-16) \\ (8-15) & -(4-21) & (5-14) \end{bmatrix}^t$$

$$Adj\,A = \begin{bmatrix} 22 & -38 & 9 \\ 1 & -14 & 9 \\ -7 & 17 & -9 \end{bmatrix}^t$$

$$Adj\,A = \begin{pmatrix} 22 & 1 & -7 \\ -38 & -14 & 17 \\ 9 & 9 & -9 \end{pmatrix}$$

$$A^{-1} = \frac{Adj\,A}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix} 22 & 1 & -7 \\ -38 & -14 & 17 \\ 9 & 9 & -9 \end{bmatrix}}{-27}$$

$$A^{-1} = \begin{bmatrix} \frac{-22}{27} & \frac{-1}{27} & \frac{7}{27} \\ \frac{38}{27} & \frac{14}{27} & \frac{-17}{27} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix}$$

2. Using Row Operations

$$\tilde{A} = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 7 & 5 & 4 & 0 & 1 & 0 \\ 8 & 7 & 10 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -9 & -17 & -7 & 1 & 0 \\ 0 & -9 & -14 & -8 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 - 7R_1 \\ R_3 - 8R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 17/9 & 7/9 & -1/9 & 0 \\ 0 & -9 & -14 & -8 & 0 & 1 \end{array}\right) \frac{R_2}{-9}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -7/9 & -5/9 & 2/9 & 0 \\ 0 & 1 & 17/9 & 7/9 & -1/9 & 0 \\ 0 & 0 & 3 & -1 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ R_3 + 9R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -7/9 & -5/9 & 2/9 & 0 \\ 0 & 1 & 17/9 & 7/9 & -1/9 & 0 \\ 0 & 0 & 1 & -1/3 & -1/3 & 1/3 \end{array}\right) \frac{R_3}{3}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -22/27 & -1/27 & 7/27 \\ 0 & 1 & 0 & 38/27 & 14/27 & -17/27 \\ 0 & 0 & 1 & -1/3 & -1/3 & 1/3 \end{array}\right) \begin{array}{l} R_1 + \frac{7}{9}R_3 \\ \\ R_2 - \frac{17}{9}R_3 \end{array}$$

$$A^{-1} = \begin{pmatrix} \frac{-22}{27} & \frac{-1}{27} & \frac{7}{27} \\ \frac{38}{27} & \frac{14}{27} & \frac{-17}{27} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{1}{3} \end{pmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 - 7R_1$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -8 & 0 & 1 \end{bmatrix} \quad R_3 - 8R_1$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \frac{R_2}{-9}$$

$$E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 - 2R_2$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 9 & 1 \end{bmatrix} \quad R_3 + 9R_2$$

$$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \frac{R_3}{3}$$

$$E_7 = \begin{bmatrix} 1 & 0 & 7/9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 + \frac{7}{9} R_3$$

$$E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -17/9 \\ 0 & 0 & 1 \end{bmatrix} R_2 - \frac{17}{9} R_3$$

$$E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I$$

$$A = (E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1)^{-1} I$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 + 7R_1$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix} R_3 + 8R_1$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 1 \end{bmatrix} -9R_2$$

$$E_4^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 + 2R_2$$

$$E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -9 & 1 \end{bmatrix} \quad R_3 - 9R_2$$

$$E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad 3R_3$$

$$E_7^{-1} = \begin{bmatrix} 1 & 0 & -7/9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 - \frac{7}{9} R_3$$

$$E_8^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 17/9 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 + \frac{17}{9} R_3$$

$A^{-1} = E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1$

A in terms of multiplication of elementry Mat..

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -9 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & -7/9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 17/9 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -17/9 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 7/9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 9 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -8 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$