Chapter # 8 (Exercise 8.1-8.3)

Sara Aziz

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Affine Combination

Definition: An **affine combination** of points $\{x_1, x_2, \ldots, x_k\}$ in $\mathbb{R}^n$ is:

$$x = \sum_{i=1}^{k} c_i x_i, \quad \text{where } \sum_{i=1}^{k} c_i = 1.$$

Properties:

- The coefficients c_i must sum to 1, ensuring "balance."
- Represents points that preserve the relative positioning of $x_1, \ldots, x_k$.

Affine Combination Examples

Case 1: One Vector

- If $x_1 \in \mathbb{R}^n$, the affine combination is simply x_1, since $c_1 = 1$.

Case 2: Two Vectors

- For $x_1, x_2 \in \mathbb{R}^n$:

$$x = c_1x_1 + c_2x_2, \quad c_1 + c_2 = 1.$$

- Example: $x_1 = (1, 0), x_2 = (0, 1), c_1 = 0.7, c_2 = 0.3$:

$$x = 0.7(1, 0) + 0.3(0, 1) = (0.7, 0.3).$$

Case 3: n Vectors

- For $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n$:

$$x = \sum_{i=1}^{n} c_i x_i, \quad \sum_{i=1}^{n} c_i = 1.$$

- Example:
  $x_1 = (1,0), x_2 = (0,1), x_3 = (1,1), c_1 = 0.2, c_2 = 0.3, c_3 = 0.5$:

$$x = 0.2(1,0) + 0.3(0,1) + 0.5(1,1) = (0.7, 0.8).$$

Convex Combination

Definition: A **convex combination** of points $\{x_1, x_2, \ldots, x_k\}$ is an affine combination where $c_i \geq 0$:

$$x = \sum_{i=1}^{k} c_i x_i, \quad \sum_{i=1}^{k} c_i = 1, \quad c_i \geq 0 \, \forall i.$$

Properties:

- Convex combinations lie inside or on the boundary of the convex region formed by $\{x_i\}$.

Convex Combination Examples

Case 1: One Vector

- If $x_1 \in \mathbb{R}^n$, the convex combination is simply x_1, since $c_1 = 1$ and $c_1 \geq 0$.

Case 2: Two Vectors

- For $x_1, x_2 \in \mathbb{R}^n$:

$$x = c_1 x_1 + c_2 x_2, \quad c_1 + c_2 = 1,\ c_1, c_2 \geq 0.$$

- Example: $x_1 = (1, 0), x_2 = (0, 1), c_1 = 0.4, c_2 = 0.6$:

$$x = 0.4(1, 0) + 0.6(0, 1) = (0.4, 0.6).$$

Case 3: n Vectors

- For $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n$:

$$x = \sum_{i=1}^{n} c_i x_i, \quad \sum_{i=1}^{n} c_i = 1,\ c_i \geq 0.$$

- Example:
  $x_1 = (1, 0), x_2 = (0, 1), x_3 = (1, 1), c_1 = 0.1, c_2 = 0.2, c_3 = 0.7$:

$$x = 0.1(1, 0) + 0.2(0, 1) + 0.7(1, 1) = (0.8, 0.9).$$

Independence and Combinations

Affine Combination:

- If the vectors $x_1, x_2, \ldots, x_k$ are **affinely independent**, no vector in the set can be written as an affine combination of the others.
- Example: The points $(1, 0), (0, 1), (1, 1)$ are affinely independent because their affine hull forms a plane in $\mathbb{R}^2$.

Convex Combination:

- Convex independence implies that no point in the set lies in the convex hull of the others.
- Example: The points $(1, 0), (0, 1), (0.5, 0.5)$ are not convexly independent because $(0.5, 0.5)$ lies in the convex hull of $(1, 0)$ and $(0, 1)$.

Summary

Affine Combinations:

- General form: $x = \sum c_i x_i, \sum c_i = 1.$
- Includes points "balanced" relative to the given set.

Convex Combinations:

- General form: $x = \sum c_i x_i, \sum c_i = 1, c_i \geq 0.$
- Includes points within the convex hull of the set.