Assignment: Parameter Estimation

Name: Riddhi Garg
Roll No: 102103282
Section: 3C010

Q1) Density function for normal distribution with mean μ & variance σ^2

$$\text{pmf}(x_i) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \times e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$\mu = \theta_1, \quad \sigma^2 = \theta_2$$

$$\therefore f(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$f(x_i \mid \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_1}} e^{-\frac{(x_i-\theta_1)^2}{\theta_2^2}} \qquad \theta_1 = \mu, \quad \theta_2 = \sigma^2$$

Likelihood function

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i \mid \theta_1, \theta_2)$$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} (\theta_2)^{-1/2} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

$$\ln(L(\theta_1, \theta_2)) = \ln\left[(\theta_2)^{-n/2} \cdot (2\pi)^{-n/2} \cdot e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}\right]$$

$$\boxed{\ln(L(\theta_1,\theta_2)) = -\frac{n}{2}\ln\theta_2 - \frac{n}{2}\ln 2\pi - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2} \quad (1)$$

Differentiate w.r.t θ_1

$$\frac{\partial(\ln(L(\theta_1,\theta_2)))}{\partial\theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)$$

Now, $\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = 0$

$$\frac{1}{\theta_2}\left(\sum_{i=1}^{n}(x_i-\theta_1)\right) = 0$$

$$\sum_{i=1}^{n}x_i - n\theta_1 = 0$$

$$\theta_1 = \frac{\sum_{i=1}^{n}x_i}{n} = \bar{x}_n$$

$$\therefore \quad \boxed{\theta_1 = \bar{x}_n} \rightarrow (2)$$

MLE

Differentiating (1) w.r.t (2)

$$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_2} = \frac{-n}{2\theta_2} - \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

now $\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_2} = 0$

$$\therefore \quad -\frac{n}{2\theta_2} - \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 = 0$$

$$\frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 = \frac{-n}{2\theta_2}$$

$$\boxed{\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

from (2)

$$\boxed{\theta_{2\,MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2}$$

Q2)

For Binomial Distribution, Density function is :-

~~PDF~~ $PMF(x_i) = {}^{n}C_{x_i}\, p^{x_i}(1-p)^{n-x_i}$

$n = m, \quad p = \theta$

$$L(P) = \prod_{i=1}^{n} P(x_i \mid m, \theta)$$

$$= \prod_{i=1}^{n}\left({}^{m}C_{x_i}\cdot\theta^{x_i}(1-\theta)^{m-x_i}\right)$$

$$= \prod_{i=1}^{n}{}^{m}C_{x_i}\prod_{i=1}^{n}\theta^{x_i}\prod_{i=1}^{n}(1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n}{}^{m}C_{x_i}\;\theta^{\sum_{i=1}^{n}x_i}\;(1-\theta)^{nm-\sum_{i=1}^{n}x_i}$$

$$\ln(L(P)) = \ln\left[\prod_{i=1}^{n}{}^{m}C_{x_i}\;\theta^{\sum_{i=1}^{n}x_i}\;(1-\theta)^{nm-\sum_{i=1}^{n}x_i}\right]$$

$$= \ln\left(\prod_{i=1}^{n}{}^{m}C_{x_i}\right) + \ln\left(\theta^{\sum_{i=1}^{n}x_i}\right) + \text{—}$$

(P.T.O)

$$\ldots + \ln\left((1-\theta)^{nm - \sum_{i=1}^{n} x_i}\right)$$

$$= \ln\left(\prod_{i=1}^{n} {}^{m}(x_i)\right) + \ln(\theta) \cdot \sum_{i=1}^{n} x_i + \ln(1-\theta)\left(nm - \sum_{i=1}^{n} x_i\right)$$

<u>diff. w.r.t θ.</u>

$$\frac{\partial L(P)}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{-1}{1-\theta}\right)\left(nm - \sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{\theta}\sum_{i=1}^{n} x_i - \left(\frac{1}{1-\theta}\right)\left(nm - \sum_{i=1}^{n} x_i\right)$$

now $\dfrac{\partial L(P)}{\partial \theta} = 0$

$$\frac{1}{\theta}\sum_{i=1}^{n} x_i = \left(\frac{1}{1-\theta}\right)\left(nm - \sum_{i=1}^{n} x_i\right)$$

$$\frac{1-\theta}{\theta} = \frac{nm - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}$$

$$\frac{nm}{\sum_{i=1}^{n} x_i} - 1 = \frac{1}{\theta} - 1$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{nm} = \frac{\bar{x}_n}{nm}$$

$$\boxed{\hat{\theta}_{MLE} \in (0,1) = \frac{\bar{x}_n}{m}}$$