Theoretical and Performance Comparisions of Single-Mode Codes

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Introduction

The fundamental objective of quantum error correction entails identifying two logical code words ($|W_\sigma\rangle$), representing a qubit within a vast Hilbert space, that exhibit *robustness* and satisfies Knill-Laflamme Condition:

$$\langle W_\sigma | E_l^\dagger E_k | W_\sigma\rangle = \alpha_{l,k}\delta_{\sigma,\sigma'}$$

for all ($E_{l,k} \in \mathcal{E}$), where ($\alpha_{l,k}$) denote the elements of a Hermitian matrix that are independent of the logical code words.

- non-Hermitian creation/annihilation operators: $a^\dagger, a$
- $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle$
- $b = a^\dagger a, \quad n|n\rangle = n|n\rangle$
- $[a, a^\dagger] = 1, \quad [n, a^\dagger] = a^\dagger, \quad [n, a] = -a,$
- Coherent states: eigenfunctions of annihilation operator

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^\dagger}|0\rangle\ ,$$

- Note: notation $|\alpha\rangle$ does not refer to a Fock/number state.
- Expression $|\alpha\rangle$ with $\alpha = 2$ represents a Poisson distribution of number states $|n\rangle$ with a mean photon number of two.
- Errors generated by action of $\hat{a}$:"loss" errors, by $\hat{a}^\dagger$: "gain" errors, and by $\hat{n}$: "dephasing" errors.

Single Mode Codes

- Simple encoding of M qubits: 2^M Fock states cover photon numbers $0, 1, \dots, (2^{M-1})$.
- Use binary representation: $|n\rangle = |b_{M-1}b_{M-2}\cdots b_0\rangle$
- The jth binary digit represents the eigenvalue $(1 + Z_j)/2$ for the corresponding physical qubit.
- Photon loss occurs.
- QEC schemes based on models of independent single qubit errors cannot be easily transferred to this problem.

Simple Code

- Reminder from quantum optics: mode = frequency + spatial distribution + polarization
- Protect against $\mathcal{E} = \{I, a\}$
- $|W_\uparrow\rangle = \frac{|0\rangle+|4\rangle}{2}$, $|W_\downarrow\rangle = |2\rangle$
- Hence, $|E_1\rangle = |3\rangle$ and $|E_2\rangle = |1\rangle$

Simple Code

- Distinguish states by measuring number and checking mod 4.
- Same mean photon number i.e. $\langle W_\sigma | n | W_\sigma\rangle = 2$
  - So, $a : \alpha|W_\uparrow\rangle + \beta|W_\downarrow\rangle \mapsto \alpha|E_1\rangle + \beta|E_2\rangle$ (no deformation)
- Generalize
  - Greater spacing between states: can detect higher order loss errors or alternatively gain errors
  - Action by n "dephases" (see how it can shift relative phases?). This leads to a superposition of codewords and error words. Project onto word basis to recover (efficient).

Cat Codes

- Superposition of 'well-separated coherent' states ("legs")
- $2(L+1)$ legs protects L photon losses. Compare to binomial code with $S = L$
- E.g. $L = 1$

$$\left|C^\alpha_{\uparrow/\downarrow}\right\rangle = |\alpha\rangle \pm |i\alpha\rangle + |-\alpha\rangle \pm |-i\alpha\rangle$$

up to a normalization factor.

- As $\alpha \to \infty$, $\left\langle C^\alpha_\uparrow\middle| N^p \middle| C^\alpha_\uparrow\right\rangle = \left\langle C^\alpha_\downarrow\middle| N^p \middle| C^\alpha_\downarrow\right\rangle$ so potentially immune from unlimited order dephasing.
- Fock states are distributed as Poisson and for large N, Binomial and Poisson approach normal distribution

Binomial Codes

- Protect against $\mathcal{E} = \{I, a, a^2, \cdots, a^L, a^\dagger, \cdots, (a^\dagger)^G, N, \cdots, N^D\}$
- Consider

$$\left|W_{\uparrow/\downarrow}\right\rangle = \frac{1}{\sqrt{2^N}}\sum_{p \text{ even/odd}}^{[0,N+1]}\sqrt{\binom{N+1}{p}}\,|p(S+1)\rangle$$

with $S = L + G$, $N = \max\{L, G, 2D\}$.

- Mean photon numbers equal (no deformation) and QEC condition holds
- Can be shown by writing difference in lth moment of photon number of codewords as lth derivative of $(1+x)^{N+1}|_{x=-1}$ with $l \leq \max\{L, G\}$ up to a factor
- Measure photon number mod $S + 1$

GKP Codes

- Constructed based on the continuous basis of non-normalizable eigenstates of the position operator $\hat{x}$. Ideal GKP encompasses both infinite mean photon number and infinite number of states that are perfectly squeezed within the $\hat{x}$ lattice.
- Ideal qubit GKP states can be expressed

$$\left|GKP_{\uparrow/\downarrow}\right\rangle \sim \sum_{p \text{ even/odd}}^{(-\infty,\infty)}\hat{D}\left(p\sqrt{\frac{\pi}{2}}\right)|\hat{x} = 0\rangle$$

where, $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$

- Correctable Errors themselves form a continuous set.
- Protection against displacement error ($\hat{D}(\alpha)$).
- The GKP code achieves approximate error correction through the utilization of ancillae states prepared in an equal superposition of logical basis states, homodyne measurements, and incoherent $\chi^{(2)}$ interactions.

Theoretical Comparisons

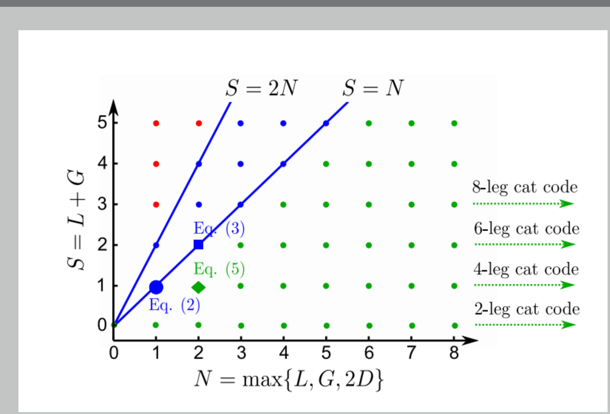

Figure: Figure comparing binomial codes to cat codes.[1]

- When considering the average photon number $[\bar{n}]$ for single-loss correction, we find binomial codes outperform cat codes, followed by GKP codes.

binomial > cat > GKP

This sequence was deduced by examining the photon number pump needed to satisfy the approximate quantum error correct conditions upto $\kappa\delta t$ order.

- Single mode binomial codes and GKP codes necessitate explicit correction gates at each time step, irrespective of whether a photon jump has occurred.
- Binomial codes operate within a constant Hilbert space which is advantageous when dealing with errors involving $\hat{a}^\dagger$ operators for error detection.
- The fock- -state distributions of binomial and cat codes are binomial and Poissonian respectively. So, as the average Photon count increases [larger N], Binomial and cat codes converges to normal distribution.

Performance Comparisons

- Channel Fidelity (F_ε) is a feasible measure to assess the efficacy of single-mode bosonic code protection.
- If we assume that our noise is entirely characterized by the lossy bosonic channel, the F_ε measures the degree of overlap between the initial state and final state in the presence of noisy bosonic channel.
- Numerical findings demonstrate codes designed to effectively counter predominant errors at low loss rate (γ) might not necessarily translate into equally effective performance at higher (γ) values.

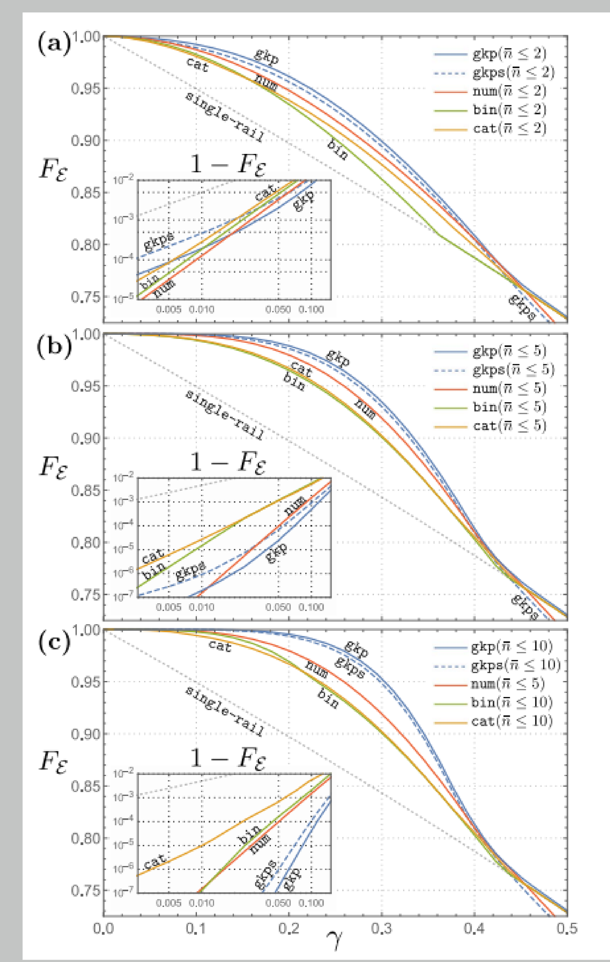

Figure: Performance evaluation of the optimal code under the constraint of mean photon number, while varying γ

- GKP codes achieve the quantum capacity of Gaussian loss channels upto a constant gap relative to an upper bound of the quantum capacity.

Reference

The poster is based on the paper 'A Comprehensive Review of Bosonic Quantum Error Correcting Codes' by Riddhiman Bhattacharya.

[1] Marios H Michael, Matti Silveri, RT Brierley, Victor V Albert, Juha Salmilehto, Liang Jiang, and Steven M Girvin.
New class of quantum error-correcting codes for a bosonic mode.
Physical Review X, 6(3):031006, 2016.