Jacobians and Hessians of Mean Value Coordinates for Closed Triangular Meshes

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***Abstract*—In this technical note, we present the formulae of the derivatives of the Mean Value Coordinates [2] based transformations using an enclosing triangle mesh, acting as a cage for the deformation of an interior object.**

I. Background

A. Mean Value Coordinates for Closed Triangular Meshes

Mean Value Coordinates for closed triangular meshes were introduced in [2]. In this section, we briefly review this work and describe the notations we will use in the rest of this note.

As written in [2], a 3D point η can be expressed as a linear sum of the 3D positions p_i of the vertices of a triangular mesh M by: $\eta = \frac{\sum_i w_i \cdot p_i}{\sum_i w_i} = \sum_i \lambda_i \cdot p_i$.

For a point x onto the surface (a two-dimensional parameter), we note as usual $\phi_i[x]$ the linear function on M that takes value 1 on vertex i and 0 on other vertices, and $p[x]$ its 3D position. The definition of the weights λ_i should guarantee *linear precision* (i.e. $\eta = \sum_i \lambda_i(\eta) p_i$).

Since $\int_{B_\eta(M)} \frac{p[x]-\eta}{|p[x]-\eta|} dS_\eta(x) = 0$ (the integral of the unit outward normal onto the unit sphere is 0), we have

$$\eta = \frac{\int_{B_\eta(M)} \frac{p[x]}{|p[x]-\eta|} dS_\eta(x)}{\int_{B_\eta(M)} \frac{1}{|p[x]-\eta|} dS_\eta(x)} \tag{1}$$

$B_\eta(M)$ being the projection of the manifold M onto the unit sphere centered in η.

Writing that $\forall x \; p[x] = \sum_i \phi_i[x] p_i$, with $\sum_i \phi_i[x] = 1$, we obtain

$$\eta = \frac{\sum_i \int_{B_\eta(M)} \frac{\phi_i[x]}{|p[x]-\eta|} dS_\eta(x) p_i}{\int_{B_\eta(M)} \frac{1}{|p[x]-\eta|} dS_\eta(x)} \tag{2}$$

The weights λ_i are given by

$$\lambda_i = \frac{\int_{B_\eta(M)} \frac{\phi_i[x]}{|p[x]-\eta|} dS_\eta(x)}{\int_{B_\eta(M)} \frac{1}{|p[x]-\eta|} dS_\eta(x)} \tag{3}$$

And the weights w_i such that $\lambda_i = \frac{w_i}{\sum_j w_j}$ are given by

$$w_i = \int_{B_\eta(M)} \frac{\phi_i[x]}{|p[x]-\eta|} dS_\eta(x) \tag{4}$$

This definition guarantees linear precision; it gives a linear interpolation of the function onto the triangles of the cage; and it extends it in a regular way to the entire 3D space.

Computing the weights w_i: The support of the function $\phi_i[x]$ is only composed of the adjacent triangles to the vertex i. Then, we can rewrite Eq. 4 as $w_i = \sum_{T \in N1(i)} w_i^T$, with

$$w_i^T = \int_{B_\eta(T)} \frac{\phi_i[x]}{|p[x]-\eta|} \cdot d\overline{T} \tag{5}$$

Given a triangle T with vertices t_1, t_2, t_3, we see that

$$\begin{aligned} \sum_j w_{t_j}^T \cdot (p_{t_j} - \eta) &= \int_{B_\eta(T)} \frac{\sum_j \phi_{t_j}[x] \cdot (p_{t_j} - \eta)}{|p[x]-\eta|} \cdot d\overline{T} \\ &= \int_{B_\eta(T)} \frac{p[x]-\eta}{|p[x]-\eta|} \cdot d\overline{T} \triangleq m^T \end{aligned} \tag{6}$$

This last integral is simply the integral of the unit outward normal on the spherical triangle $\overline{T}$.

By noting $n_i^T = \frac{N_i^T}{|N_i^T|}$, with $N_i^T \triangleq (p_{t_{i+1}} - \eta) \wedge (p_{t_{i+2}} - \eta)$ (see Fig.1), it can be easily expressed as

$$m^T = \sum_i \frac{1}{2} \theta_i^T n_i^T \tag{7}$$

This comes from the fact that the integral of the unit outward normal on a closed surface is always 0.

Finally, we obtain

$$\sum_j w_{t_j}^T \cdot (p_{t_j} - \eta) = m^T \tag{8}$$

This point was discussed in [2]. As the authors pointed out, by noting A^T the 3 by 3 matrix $\{p_{t_1} - \eta, p_{t_2} - \eta, p_{t_3} - \eta\}$, we can derive the weights $w_{t_j}^T$ by

$$\{w_{t_1}^T, w_{t_2}^T, w_{t_3}^T\}^t = {A^T}^{-1} \cdot m^T \tag{9}$$

Since ${N_i^T}^t \cdot (p_{t_j} - \eta) = 0 \;\; \forall i \neq j$, we have from Eq. 8 that

$$w_{t_i}^T = \frac{{N_i^T}^t \cdot m^T}{{N_i^T}^t \cdot (p_{t_i} - \eta)} = \frac{{N_i^T}^t \cdot m^T}{\det(A^T)} \quad \forall \eta \notin Support(T) \tag{10}$$

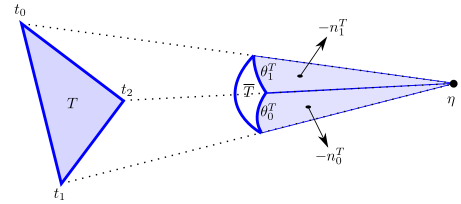

Fig. 1. Triangle T projected on the spherical triangle $\bar{\text{T}}$.

II. MVC DERIVATIVES

We now present the derivatives of the Mean Value Coordinates. Deforming the cage mesh with $f(p_i) = \overline{p_i}$ induces a deformation of the 3D space by $f = \sum_i \lambda_i \cdot \overline{p_i}$. In the rest of the document, for any function $h : E \rightarrow F$, we note $\partial_x h, \partial_y h, \partial_z h$ its derivative by x, y, and z, $\overrightarrow{\bigtriangledown} h$ its gradient, Jh its jacobian, and Hh its hessian.

The deformation function f as defined acts now on $\mathbb{R}^3$ entirely. The derivatives of f can be expressed as a linear sum of positions $\overline{p_i} = \{\overline{x_i}, \overline{y_i}, \overline{z_i}\}$:

$$\begin{cases} Jf & = \sum_i \overline{p_i} \cdot \overrightarrow{\bigtriangledown}\lambda_i{}^t \\ H(f_x) & = \sum_i \overline{x_i} \cdot H\lambda_i \\ H(f_y) & = \sum_i \overline{y_i} \cdot H\lambda_i \\ H(f_z) & = \sum_i \overline{z_i} \cdot H\lambda_i \end{cases} \tag{11}$$

Consequently, it allows to specify implicit equations on the cage in a linear system by giving specified rotations and scales on 3D locations, or to minimize the norm of the hessian to force rigidity, as done in the case of Green Coordinates in [1].

Since $\lambda_i = \frac{w_i}{\sum_j w_j}$,

$$\overrightarrow{\bigtriangledown}\lambda_i = \frac{\overrightarrow{\bigtriangledown} w_i}{\sum_j w_j} - \frac{w_i \cdot \sum_j \overrightarrow{\bigtriangledown} w_j}{(\sum_j w_j)^2} \tag{12}$$

We also have $\forall c = x,\ y,\ z$

$$\begin{aligned} \partial_c(\overrightarrow{\bigtriangledown}\lambda_i) = & \frac{\partial_c(\overrightarrow{\bigtriangledown} w_i)}{\sum_j w_j} - \frac{\overrightarrow{\bigtriangledown} w_i \cdot \sum_j \partial_c(w_j)}{(\sum_j w_j)^2} \\ & - \frac{\partial_c(w_i) \cdot \sum_j \overrightarrow{\bigtriangledown} w_j + w_i \cdot \sum_j \partial_c(\overrightarrow{\bigtriangledown} w_j)}{(\sum_j w_j)^2} \\ & + \frac{2w_i \cdot (\sum_j \overrightarrow{\bigtriangledown} w_j) \cdot (\sum_k \partial_c(w_k))}{(\sum_j w_j)^3} \end{aligned} \tag{13}$$

or

$$\begin{aligned} H\lambda_i = & \frac{Hw_i}{\sum_j w_j} - \frac{w_i \sum_j Hw_j}{(\sum_j w_j)^2} \\ & - \frac{\overrightarrow{\bigtriangledown} w_i \cdot \sum_j \overrightarrow{\bigtriangledown}(w_j)^t + \sum_j \overrightarrow{\bigtriangledown}(w_j) \cdot \overrightarrow{\bigtriangledown} w_i{}^t}{(\sum_j w_j)^2} \\ & + \frac{2w_i (\sum_j \overrightarrow{\bigtriangledown} w_j) \cdot (\sum_j \overrightarrow{\bigtriangledown} w_j)^t}{(\sum_j w_j)^3} \end{aligned} \tag{14}$$

From these expressions, we see that, in order to get $\overrightarrow{\bigtriangledown}\lambda_i(\eta)$ and $H\lambda_i(\eta)$, we first need to obtain $\overrightarrow{\bigtriangledown} w_i(\eta)$ and $Hw_i(\eta)$ for each vertex i of the cage.

Special case: η lies on the surface of the cage

Mean Value Coordinates define an *interpolation* process. The function represented onto the vertices of the cage (in our case, a space transformation) is extended to the interior of the triangles with linear interpolation on each triangle. Then it is extended to the space by means of a surfacic integration of the function (see Eq. 1).

Since we represent the cage as triangle mesh in the 3D case, **the deformation function cannot be anything more than continuous onto the edges of the cage in 3D**. Therefore Jacobians and Hessians of the deformation cannot be evaluated everywhere on the surface of the cage, and we do not provide any formula for Jacobians and Hessians of the deformation onto the surface of the cage.

A. Expression of the Jacobians

In the general case where $\det(A^T) = (p_{e_i} - \eta)^t \cdot N_i^T \neq 0$, we have
$\overrightarrow{\bigtriangledown} w_{t_i}^T = \frac{B^{T\,t} \cdot N_i^T}{\det(A^T)}$
with

$$\begin{aligned} B^T = & \sum_j \frac{eq_1(\theta_j^T) N_j^T \cdot N_j^{T\,t} \cdot JN_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} \\ & + \sum_j \frac{N_j^T \cdot (p_{t_{j+1}} + p_{t_{j+2}})^t}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2} \\ & + \sum_j \frac{eq_2(\theta_j^T) JN_j^T}{2|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|} + \sum_j w_{t_j}^T \cdot I_3 \end{aligned}$$

and $eq_1(x) = \frac{\cos(x)\sin(x) - x}{\sin(x)^3}$ and $eq_2(x) = \frac{x}{\sin(x)}$ two well defined functions on $]0, \pi[$ that admit well controlled Taylor expansion around 0, $JN_j^T = (p_{t_{j+2}} - p_{t_{j+1}})_{[\wedge]}$, $k_{[\wedge]}$ being the skew 3 by 3 matrix (i.e. $k_{[\wedge]}{}^t = -k_{[\wedge]}$) such that $k_{[\wedge]} \cdot u = k \wedge u \quad \forall k, u \in \mathbb{R}^3$.

Special case: $\eta \in Support(T), \notin T$:

$$\begin{aligned} -2|T|\overrightarrow{\bigtriangledown} w_i^T = & \sum_j \frac{eq_2(\theta_j^T)(p_{t_{i+2}} - p_{t_{i+1}})^t \cdot (p_{t_{j+2}} - p_{t_{j+1}})}{2|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|} n_T \\ & + \sum_j \frac{eq_1(\theta_j^T)|p_{t_{j+2}} - p_{t_{j+1}}|^2 N_i^{T\,t} \cdot N_j^T}{4(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^3} n_T \\ & + \sum_j \frac{\cos(\theta_j^T) eq_3(\theta_j^T) N_i^{T\,t} \cdot N_j^T}{2(|p_{t_{j+2}} - \eta||p_{t_{j+1}} - \eta|)^2} n_T \end{aligned} \tag{15}$$

with $eq_2(x) = \frac{x}{\sin(x)}, eq_1(x) = \frac{\cos(x)\sin(x) - x}{\sin(x)^3}$, and $eq_3(x) = \frac{\cos(x) - 1}{\sin(x)^2}$ being functions well defined on $]0, \pi[$ and that admit controlable Taylor expansion around 0.

B. Expression of the Hessians

We note $\delta^x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \delta^y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \delta^z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$\begin{aligned} Hw_i^T = & \frac{1}{\det(A^T)} \begin{pmatrix} N_i^{T\,t} \cdot \partial_x(Jm^T) \\ N_i^{T\,t} \cdot \partial_y(Jm^T) \\ N_i^{T\,t} \cdot \partial_z(Jm^T) \end{pmatrix} \\ & + \frac{1}{\det(A^T)} (N_i^T \cdot (\sum_j \overrightarrow{\bigtriangledown} w_j^T)^t + \sum_j \overrightarrow{\bigtriangledown} w_j^T \cdot N_i^{T\,t}) \end{aligned} \tag{16}$$

with

$$
\begin{aligned}
\partial_c(Jm^T) &= \sum_j \frac{eq_6(\theta_j^T)(J{N_j^T}^t \cdot N_j^T)_{(c)}N_j^T \cdot {N_j^T}^t \cdot JN_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^5} \\
&+ \sum_j \frac{eq_7(\theta_j^T)(p_{t_{j+1}}+p_{t_{j+2}})_{(c)}N_j^T \cdot {N_j^T}^t \cdot JN_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^4} \\
&+ \sum_j \frac{eq_1(\theta_j^T)\partial_c(N_j^T) \cdot {N_j^T}^t \cdot JN_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^3} \\
&+ \sum_j \frac{eq_1(\theta_j^T)N_j^T \cdot \partial_c(N_j^T)^t \cdot JN_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^3} \\
&- \sum_j \frac{3eq_1(\theta_j^T)(\eta-p_{t_{j+1}})_{(c)}N_j^T \cdot {N_j^T}^t \cdot JN_j^T}{2|p_{t_{j+2}}-\eta|^3|p_{t_{j+1}}-\eta|^5} \\
&- \sum_j \frac{3eq_1(\theta_j^T)(\eta-p_{t_{j+2}})_{(c)}N_j^T \cdot {N_j^T}^t \cdot JN_j^T}{2|p_{t_{j+2}}-\eta|^5|p_{t_{j+1}}-\eta|^3} \\
&+ \sum_j \frac{\partial_c(N_j^T) \cdot (p_{t_{j+1}}+p_{t_{j+2}})^t}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^2} \\
&- \sum_j \frac{(\eta-p_{t_{j+1}})_{(c)}N_j^T \cdot (p_{t_{j+1}}+p_{t_{j+2}})^t}{|p_{t_{j+2}}-\eta|^2|p_{t_{j+1}}-\eta|^4} \\
&- \sum_j \frac{(\eta-p_{t_{j+2}})_{(c)}N_j^T \cdot (p_{t_{j+1}}+p_{t_{j+2}})^t}{|p_{t_{j+2}}-\eta|^4|p_{t_{j+1}}-\eta|^2} \\
&+ \sum_j \frac{eq_8(\theta_j^T)(J{N_j^T}^t \cdot N_j^T)_{(c)}JN_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^3} \\
&+ \sum_j \frac{eq_9(\theta_j^T)(p_{t_{j+1}}+p_{t_{j+2}})_{(c)}JN_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^2} \\
&- \sum_j \frac{(\eta-p_{t_{j+1}})_{(c)}eq_2(\theta_j^T)JN_j^T}{2|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|^3} \\
&- \sum_j \frac{(\eta-p_{t_{j+2}})_{(c)}eq_2(\theta_j^T)JN_j^T}{2|p_{t_{j+2}}-\eta|^3|p_{t_{j+1}}-\eta|}
\end{aligned}
\tag{17}
$$

and $eq_6(x) = \frac{d(eq_1)}{dx}(x)cos(x)/sin(x), eq_7(x) = \frac{d(eq_1)}{dx}(x)sin(x), eq_8(x) = \frac{d(eq_2)}{dx}(x)cos(x)/sin(x)$, and $eq_9(x) = \frac{d(eq_2)}{dx}(x)sin(x)$ being functions well defined on $]0,\pi[$ and that admit controllable Taylor expansion around 0.

Special case: $\eta \in Support(T), \notin T$ *:*

$$
H(w_i^T)(\eta) = \overrightarrow{\bigtriangledown} dw_i^T(\eta) \cdot n_T^t \tag{18}
$$

with

$$
\begin{aligned}
&-2|T|\overrightarrow{\bigtriangledown} dw_i^T = \\
&\sum_j \frac{((p_{t_{i+2}}-p_{t_{i+1}})^t \cdot (p_{t_{j+2}}-p_{t_{j+1}}))(p_{t_{j+2}}+p_{t_{j+1}})}{(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^2} \\
&+ \sum_j \frac{eq_1(\theta_j^T)((p_{t_{i+2}}-p_{t_{i+1}})^t \cdot (p_{t_{j+2}}-p_{t_{j+1}}))J{N_j^T}^t \cdot N_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^3} \\
&- \sum_j \frac{|p_{t_{j+2}}-p_{t_{j+1}}|^2({N_i^T}^t \cdot N_j^T)(p_{t_{j+2}}+p_{t_{j+1}})}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^4} \\
&+ \sum_j \frac{eq_1(\theta_j^T)|p_{t_{j+2}}-p_{t_{j+1}}|^2(J{N_j^T}^t \cdot N_i^T + J{N_i^T}^t \cdot N_j^T)}{4(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^3} \\
&- \sum_j \frac{eq_4(\theta_j^T)|p_{t_{j+2}}-p_{t_{j+1}}|^2({N_i^T}^t \cdot N_j^T)J{N_j^T}^t \cdot N_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^5} \\
&+ \sum_j \frac{\cos(\theta_j^T)eq_3(\theta_j^T)(J{N_j^T}^t \cdot N_i^T + J{N_i^T}^t \cdot N_j^T)}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^2} \\
&+ \sum_j \frac{(1-2\cos(\theta_j^T))({N_i^T}^t \cdot N_j^T)(p_{t_{j+2}}+p_{t_{j+1}})}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^3} \\
&+ \sum_j \frac{eq_5(\theta_j^T)({N_i^T}^t \cdot N_j^T)J{N_j^T}^t \cdot N_j^T}{2(|p_{t_{j+2}}-\eta||p_{t_{j+1}}-\eta|)^4}
\end{aligned}
\tag{19}
$$

and $eq_1(x) = \frac{\cos(x)\sin(x)-x}{\sin(x)^3}, eq_4(x) = \frac{2\cos(x)\sin(x)^3+3(\sin(x)\cos(x)-x)}{\sin(x)^5}, eq_3(x) = \frac{\cos(x)-1}{\sin(x)^2}$, and $eq_5(x) = \frac{\cos(x)\sin(x)^2(1-2\cos(x))-2\cos(x)^2+2\cos(x)}{\sin(x)^4}$ being functions well defined on $]0,\pi[$ and that admit controllable Taylor expansion formula around 0.

References

[1] M. Ben-Chen, O. Weber, and C. Gotsman, *Variational harmonic maps for space deformation*, ACM Transactions on Graphics (Proc. of ACM SIGGRAPH) (2009), 1–11.

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