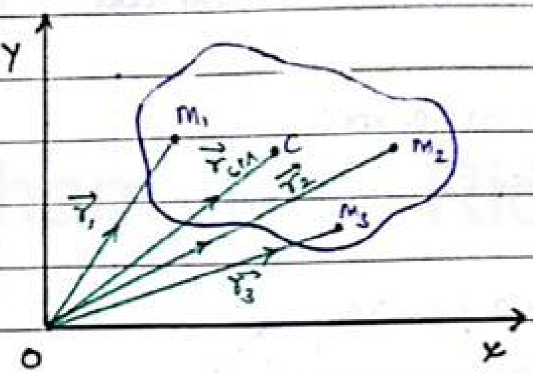
COM

1. COM of discrete particles

$$\vec{r}_{cm} = \frac{1}{M} \sum_{i=1}^{n} \underbrace{m_i r_i}_{}$$

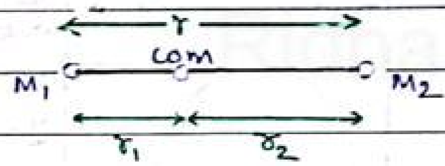
Mass of whole body.

Moment of inertia W.R.T. O of particle



2. COM of 2 particle system.

$r \propto 1/m$



$$\frac{r_1}{r_2} = \frac{m_2}{m_1} \quad \& \quad r_1 = \frac{m_2}{M_{sys}} r.$$

[i.e. pos vector of 1 particle = $\frac{\text{Mass of other part}}{\text{mass of system}}$ × distance between them]

3. COM of Continuous Mass Distribution.

$$\vec{r}_{cm} = \frac{1}{M} \int \vec{r}\, dm \quad \text{(with limits)}$$

If mass is distributed in length;

$$dm = \frac{M}{L} \times dx \quad \left[\text{i.e. } \frac{\text{Total Mass}}{\text{Total length}} \times \text{length of element}\right]$$

mass of element.

$= \lambda \times dx$ where $\lambda \rightarrow$ density in length

If mass is distributed in area;

$$dm = \frac{M}{A} \times dA \quad \left[\text{i.e. } \frac{\text{Total Mass}}{\text{Total Area}} \times \text{Area of element}\right]$$

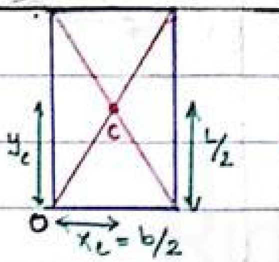
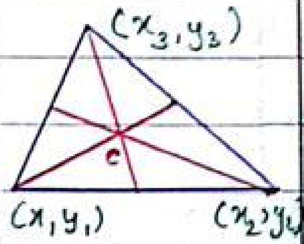
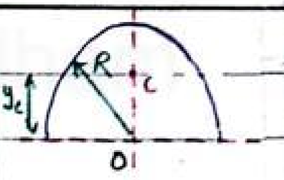
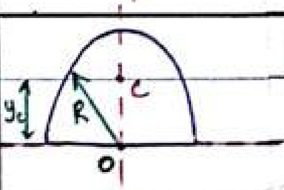
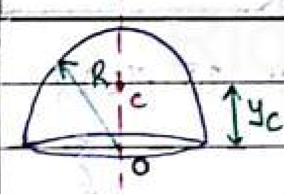
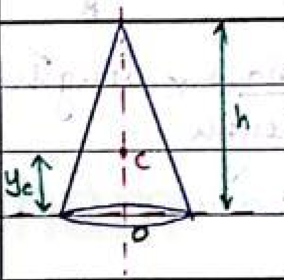
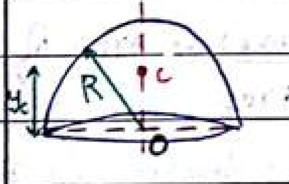
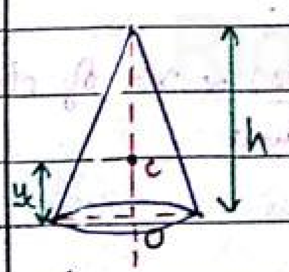
$= \sigma \times dA$ where $\sigma \rightarrow$ density in area

If mass is distributed in volume;

$$dm = \frac{M}{V} \times dV \quad \left[\text{i.e. } \frac{\text{Total mass}}{\text{Total volume}} \times \text{vol. of element}\right]$$

$= \rho \times dV$ where $\rho \rightarrow$ density in vol.

4. COM of some common systems (Continuous Mass Distribution)

Serial No.	System	Diagram	Co-ordinates of COM	Method / element chosen
1.	Rectangular plate		$x_c = b/2$ $y_c = l/2$	mass distributed in area Method: By symmetry
2.	Triangular plate.		at centroid. $x_c = \frac{x_1+x_2+x_3}{3}$ Similarly for y	mass distributed in area Method: By
3.	Semicircular ring		$y_c = \frac{2R}{\pi}$; $x_c = 0$	Mass distributed in length method: By choosing a part of ring as element*
4.	semicircular disc		$y_c = \frac{4R}{3\pi}$; $x_c = 0$	mass distributed in area method: By choosing a semicircular ring as element
5.	hemispherical shell		$y_c = \frac{R}{2}$; $x_c = 0$	mass distributed in area Method: By choosing a ring of the shell as element*
6.	hollow cone (circular)		$y_c = \frac{h}{3}$; $x_c = 0$	mass distributed in area method: —
7.	solid hemi-sphere		$y_c = \frac{3R}{8}$; $x_c = 0$	mass distributed in vol Method: —
8.	Solid cone (circular).		$y_c = \frac{h}{4}$; $x_c = 0$	Mass distributed in Vol method: —

* while taking elements, donot consider extreme cases. Only gen

5. Motion of COM

vel: $\vec{v}_{cm} = \sum_1^n P_i / M$ momentum; ↳ mass of whole body (sys.) $\vec{P}_{sys} = M\vec{v}_{cm}$

acc: $\vec{a}_{cm} = \sum_0^n m_i a_i / M$

$= \frac{\text{Net force on sys}}{\text{mass of sys}} = \frac{\text{External force (net) on sys}}{\text{Mass of sys}}$

(By vector summation; net internal force = 0)

$$\Sigma F_{ext} = M\vec{a}_{cm} \quad \text{(valid for rigid \& non rigid bodies)}$$

⇒ If $a_c = 0$; v_c = const. ∵ Total momentum = const. $F_{ext} = dP_{sys}/dt$

This verifies law of conservation of momentum in absence of ext. forces

∴ If $F_{net} = 0$; P_{net} = const (for system) [If $F_{net} = 0$, P is cons.]

. Notes on determination of COM.

i) COM of Symmetrical / Regular objects

Com of Regularly shaped objects is at pt. of intersection of diagonals and for symmetrical / Regular objects; pt. of intersection of lines of symmetry / planes of symmetry is COM.

ii) COM of 2-D. bodies with negligible thickness (uniform).

$\vec{r}_c = \frac{\sum_1^\infty M_i r_i}{\sum_1^\infty M_i}$ In 2-D body; mass distributed in area. but here thickness is negligible; not zero. ∴

for an element; the negligible thickness matters. Hence; mass = $\rho \times$ vol $= \rho \times A \times t$ ∴ Mass = $\rho A t$ → thickness of element [Same for each element as thickness is uniform throughout]

ρ ← density in vol. Same for each element as mass distribution is same throughout the body.

A ↓ Area of element

$$\therefore \vec{r}_c = \frac{\sum_1^\infty \rho A_i t r_i}{\sum_1^\infty A_i t \rho} = \frac{\cancel{\rho t}\sum_1^\infty A_i r_i}{\cancel{\rho t}\sum_1^\infty A_i} \Rightarrow \vec{r}_c = \frac{\sum_1^\infty A_i r_i}{\sum_1^\infty A_i}$$

iii) COM of body (uniform) with cavity

$$\vec{r}_c = \frac{M_1\vec{r}_1 - M_2\vec{r}_2}{M_1 - M_2} \Leftrightarrow \frac{A_1\vec{r}_1 - A_2\vec{r}_2}{A_1 - A_2}$$

M_1: Mass of object without cavity

M_2: mass of object with cavity*

r_1: co-ord. of com of whole body*

r_2: co-ord. of com of cavity

A_1: Area of whole body*

A_2: Area of cavity.

for x_c, y_c, z_c; replace r with x/y/z

* without cavity

7. Motion of com in a sys. of particles / bodies

1) COM at rest. $F_{ext} = 0$; $v_{cm} = 0$ [Net p of sys = 0]

* internal components move with non-zero p, a due to internal eg. When a bomb explodes [in equilibrium†] then its fragments move in diff. dirn.s. but the com. of them remains at rest. explosive forces were internal [F = 0 at equilibrium] hence their net momentum = 0.

2) COM moving with uniform velocity : $F_{ext} = 0$.
net momentum is conserved ; v_c = const.

* Again internal components may have acc; p, etc. due to interna eg. When a bomb moves in dynamic equilibrium and explodes during motion.

3) Com moving with uniform acc. F_{ext} = const.
eg. Projectile motion. R_{com} / H_{com} = Range/ ht*. of proj
Circular motion. $F_{com} = m\omega^2 R_{com}$

8. Impulse

$$\vec{I} = \int_{t_i}^{t_f} \vec{F} dt \Rightarrow \vec{I} = \int m d\vec{v} \quad \therefore \vec{I} = m(v_2 - v_1) = \Delta\vec{p}$$

$$\therefore \vec{I}_{res} = \int_{t_i}^{t_f} \vec{F}_{res} dt = \Delta\vec{p} \quad \text{(impulse-momentum theorem)}$$

also gives the change in mom.

Impulse = area under curve of F - t graph in magn.
Instantaneous impulse: $I_{ins.} = P_f - P_i$
- not prop. of particle; a measure of degree of Δp by F_{ext} on part.
- Dirn. is along change in momentum.

$$I = \int \vec{F} dt = \vec{F}_{av} \int dt = \vec{F}_{av} \Delta t$$

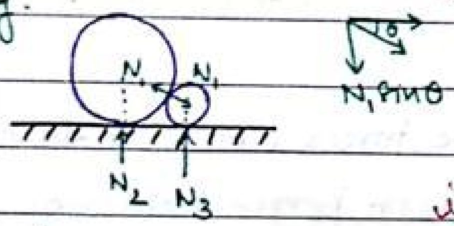
9. Impulsive forces.
- Forces having large magn; acting for a small time
- Rel. term. Usually colliding forces are impulsive

† static * max.

- as application time → 0 ; negligible motion takes place
- Gravitational force & spring force are always non-impulsive.
- Normal, friction, Tension are case dependent.
- Only 2 'impulsive' forces can balance each other

i) Impulsive Normal: During collisions; Normals at surface of collisions are always impulsive in nature

eg.

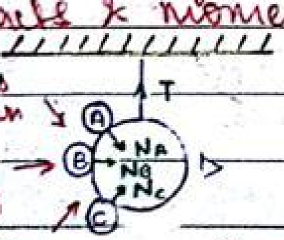


component of impulsive force is also impulsive.
$\therefore N_1 \sin\theta \Rightarrow$ impulsive. a_y for both balls $= 0$
$\therefore N_3 = N_1 \sin\theta$. N_1 is impulsive as it is // to line of collision. $\therefore$ To balance an impulsive force, another impulsive force is req. $\therefore N_3$ will also be impulsive.
$N_2 \rightarrow$ Non impulsive.

ii) Impulsive friction: If Normal between two objects is impulsive, then friction between them is also impulsive.
eg. In above eg; friction due to N_2 is non impulsive and due to N_1, N_3 is impulsive.

iii) Impulsive Tension: When a string jerks; equal and opp. tension-s act at each end. Consequently equal and opp. impulses act on the bodies attached with the string in the dirn. of the string. There are 2 cases:
⇒ One end fixed: impulsive tension at other end cannot change p of fixed object while the free one undergoes it in dirn. of string. Momentum is unchanged in dirn.s ⊥ to string where no impul. forces act.
⇒ Both ends movable: equal & opp. impulses act on both free bodies, producing equal & opp. Δp.* $\therefore \Delta p_{sys} = 0$. ⊥ to string no impulse acts & momentum of each particle is unchanged in this dirn.

(Balls thrown one by one)



All 3 Normals are impulsive. But only component of N_A opposes Tension. Hence only when the ball A strikes the ball D; Tension ⇒ impulsive. Otherwise Non impulsive

* in dirn. of string

Tension in rod is always impulsive while in Spring it is non-impulsive for the previous example; if String is replaced by a rod or spring.

10. Collision / Impact.

i) In a collision, particles may or may not come in physical con[tact]

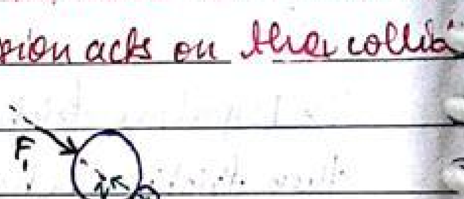
ii) duration of collision is negligible compared to usual time inte[rval] of motion observations

iii) During duration of collision; impulsive forces dominate the motion of the system. Hence other non-impulsive forces are neglected.

iv) Collision is redistribution of total momentum of particles.

v) Line of impact: Line along which common normal (i.e. Normal of collision) acts. Net force during collision acts on the collid[ing] object along this line

(If this doesnot happen; objects will not collide).



dim. of line of impact = dim. of Δp.

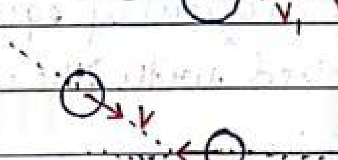
11. Types of Collisions.

On Basis of Line of impact:

i) Head on

Vel. along same line before & after collision

ii) Oblique



vel. along diff. lines before, after collision.

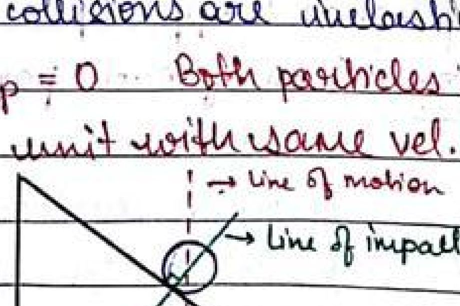
On basis of Energy:

i) Elastic: deformation pot. energy = 0. K.E. is conserved. momentum is conserved.

ii) Inelastic: deformation P.E $\neq 0$. momentum is conserved; K.E is unconserved. All natural collisions are inelastic.

iii) perfectly inelastic: $V_{sep} = 0$. Both particles stick together and travel as a single unit with same vel.

* Ball falling on wedge

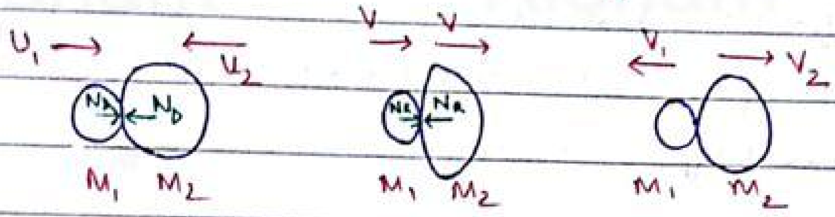


12. Co-efficient of Restitution (e) OR RESILIENCE.

$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_R\,dt}{\int F_D\,dt}$$

$$= \frac{V_{sep} \text{ of pt.s of contact along LoI}}{V_{app} \text{ of pt.s of contact along LoI}} = \frac{V_{sep} \text{ along LoI}}{V_{app} \text{ along LoI}}$$ (in most of cases.)

ex. (of)



deformation phase Reformation phase

Max deformation when both particles have same velocity.

$M_1 u_1 + M_2 v_2 = (M_1 + M_2) v = M_1 v_1 + m_2 v_2$ $F_{ext} = 0 \therefore P_{sys}$ is cons.

$\therefore v = \frac{M_1 u_1 + M_2 u_2}{M_1 + m_2} = \frac{M_1 v_1 + M_2 v_2}{M_1 + m_2}$ (mom. cons.)

$\therefore$ impulse of deformation : Δp on 1 object during deformation

$J_D = m_2 (v - u_2)$ for m_2 & $m_1(-v + u_1)$ for m_1

Similarly impulse of Reformation: Δp on 1 obj during Reformation.

$J_R = m_2 (v_2 - v)$ for m_2 & $m_1 (v - v_1)$ for m_1

Δp (defor.)

$\therefore e = \frac{J_D}{J_R} = \frac{v_2 - v_1}{u_1 - u_2} = \frac{\text{vel. of sep. along LoI}}{\text{vel. of app. along LoI}}$

Δp (refor.)

e depends on material if e for a pair of objects which underwent collision $\Rightarrow e = 0 \Rightarrow$ Collision perfectly inelastic

$\Rightarrow e = 1 \Rightarrow$ collision elastic

$\Rightarrow e \in (0,1) \Rightarrow$ collision inelastic.

13. Imp. pt.s regarding collision.

* Mom. can always be conserved. KE. can be conserved only for elastic collision, else there is a loss of Energy which is max. when collision is perfectly inelastic.
* Vel. $\perp$ to the LOI remains unchanged after collision.
* For head on collision; e, p cons and KE cons. are used to get equs. For inelastic collisions; KE cons. can be applied as $KE_i = KE_f + \Delta Q$

ΔQ : Heat loss

Collision in two dimension (oblique)

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles do change along common normal direction. If mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.

2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remain unchanged along this direction.

3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remain conserved before and after collision in any direction.

4. Definition of coefficient of restitution can be applied along common normal direction, i.e., along common normal direction we can apply
Relative speed of separation = e (relative speed of approach)

Special cases:

★ $e=0 \Rightarrow v_1 = v_2$ ∴ For perfectly inelastic collision; both bodies move together after colliding.

★ $e=1$, $m_1 = m_2 (= m) \Rightarrow v_1 = u_2$, $u_1 = v_2$ ∴ Vel. get exchanged

★ $m_1 >> m_2$ ∴ $m_1 + m_2 \approx m_1$, $\frac{m_2}{m_1} \approx 0$. ∴ It can be assumed that

$v_2 = u_1 + e(u_1 - u_2)$ $\left(\because \frac{v_2 - v_1}{u_1 - u_2} = e, v_1 = u_1\right)$ heavy obj's vel ≈ const $(v_1 = u_1)$

If after n collisions with ground, the body rebounds with a velocity v_n and rises to a height h_n, then –

$$e^n = \frac{v_n}{v_0} = \sqrt{\frac{h_n}{h_0}}$$

VARIABLE MASS SYSTEM :

If a mass is added or ejected from a system, at rate μ kg/s and relative velocity $\vec{v}_{rel}$ (w.r.t. the system), then the force exerted by this mass on the system has magnitude $\mu|\vec{v}_{rel}|$.

Thrust Force ($\vec{F}_t$)

$$\vec{F}_t = \vec{v}_{rel}\left(\frac{dm}{dt}\right)$$

$\mu = \left|\frac{dM}{dt}\right|$ kg/s $\therefore \mu = -\frac{dM}{dt}$ (mass ejected) and $\frac{dM}{dt}$ (mass added)

Suppose at some moment t = t mass of a body is m and its velocity is $\vec{v}$. After some time at t = t + dt its mass becomes (m – dm) and velocity becomes $\vec{v}+d\vec{v}$. The mass dm is ejected with relative velocity $\vec{v}_r$. Absolute velocity of mass 'dm' is therefore ($\vec{v}$ + $\vec{v}_r$). If no external forces acting on the system, the linear momentum of the system will remain conserved, or

$$\vec{P}_i = \vec{P}_f$$

or $$m\vec{v} = (m - dm)(\vec{v} + d\vec{v}) + dm(\vec{v} + \vec{v}_r)$$

or $$\cancel{m\vec{v}} = \cancel{m\vec{v}} + md\vec{v} - \cancel{(dm)\vec{v}} - \cancel{(dm)(d\vec{v})} + \cancel{(dm)\vec{v}} + \vec{v}_r\,dm$$

The term (dm) ($d\vec{v}$) is too small and can be neglected.

$\therefore$ $$md\vec{v} = -\vec{v}_r\,dm$$

or $$m\left(\frac{d\vec{v}}{dt}\right) = \vec{v}_r\left(-\frac{dm}{dt}\right)$$

$|V_{rel}|$: Rel. vel. of added/ejected mass.

Here, $$m\left(-\frac{d\vec{v}}{dt}\right) = \text{thrust force } (\vec{F}_t)$$

Dim. of $\vec{F}_t$ (thrust force) is opp. to dim. of V_{rel} when mass is ejected from sys. and vice versa

and $$-\frac{dm}{dt} = \text{rate at which mass is ejecting}$$

or $$\vec{F}_t = \vec{v}_r\left(\frac{dm}{dt}\right)$$ $\therefore \vec{F}_T = \vec{V}_{rel}\,\mu$

Problems related to variable mass can be solved in following four steps

1. Make a list of all the forces acting on the main mass and apply them on it.
2. Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{v}_r$ in case the mass is increasing and otherwise the direction of $-\vec{v}_r$ if it is decreasing.
3. Find net force on the mass and apply

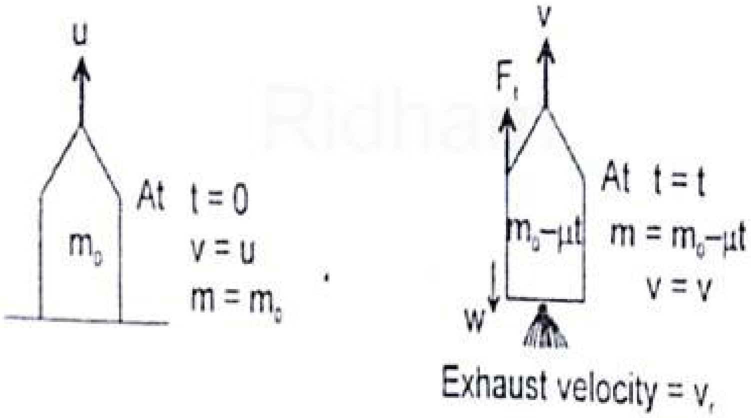
$$\vec{F}_{net} = m\frac{d\vec{v}}{dt} \qquad \text{(m = mass at the particular instant)}$$

4. Integrate it with proper limits to find velocity at any time t.

Problems of one-dimensional motion (which are mostly asked in JEE) can be solved in easier manner just by assigning positive and negative signs to all vector quantities. Here are few example in support of the above theory.

Rocket propulsion :

Let m_0 be the mass of the rocket at time $t = 0$. m its mass at any time t and v its velocity at that moment. Initially, let us suppose that the velocity of the rocket is u.



Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases with respect to rocket. Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time $t = t$,

1. Thrust force on the rocket $\quad F_t = v_r\left(\frac{-dm}{dt}\right)$ (upwards)
2. Weight of the rocket $\quad W = mg$ (downwards)
3. Net force on the rocket $\quad F_{net} = F_t - W$ (upwards)

or $\quad F_{net} = v_r\left(\frac{-dm}{dt}\right) - mg$

4. Net acceleration of the rocket $\quad a = \frac{F}{m}$

or $\quad \frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$

or $\quad dv = \frac{v_r}{m}(-dm) - g\,dt$

or $\quad \int_u^v dv = v_r \int_{m_0}^{m} \frac{-dm}{m} - g\int_0^t dt$

Thus $v = u - gt + v_r \ln\left(\frac{M_0}{M}\right)$ — (i)

Note: If $F_t = v_r\left(-\frac{dM}{dt}\right)$ is up; v_r is down then $\frac{dM}{dt}$ is −ve.

If gravity is ignored & $(v_i)_{rocket} = 0$, then eqn (i) becomes $v = v_r \ln\left(\frac{M_0}{M}\right)$