IMC Practice

Day 1 - Question 1

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Problem:

let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two Sequences of positive numbers. Show that the following statements are equivalent:

① There is a sequence $(C_n)_{n=1}^{\infty}$ of positive numbers such that

$$\sum_{n=1}^{\infty} \frac{a_n}{c_n} \text{ and } \sum_{n=1}^{\infty} \frac{c_n}{b_n} \text{ both converge}$$

② $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

Solution:

Let

$$(C_n)_{n=1}^{\infty} = (a_n^2)_{n=1}^{\infty}$$

$$(b_n)_{n=1}^{\infty} = (a_n^3)_{n=1}^{\infty}$$

From ①

$$\sum_{n=1}^{\infty} \frac{a_n}{c_n} = \sum_{n=1}^{\infty} \frac{a_n}{a_n^2} = \sum_{n=1}^{\infty} \frac{1}{a_n} \text{ converges}$$

From ①

$$\sum_{n=1}^{\infty} \frac{c_n}{b_n} \text{ Converges}$$

$$= \sum_{n=1}^{\infty} \frac{a_n^2}{a_n^3} \text{ Converges}$$

$$= \sum_{n=1}^{\infty} \frac{1}{a_n} \text{ Converges}$$

From 2

$$\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}} \text{ Converges}$$

$$= \sum_{n=1}^{\infty} \sqrt{\frac{a_n}{a_n^3}} \text{ Converges}$$

$$= \sum_{n=1}^{\infty} \sqrt{\frac{1}{a_n^2}} \text{ Converges}$$

$$= \sum_{n=1}^{\infty} \frac{1}{a_n} \text{ Converges}$$

$\Rightarrow$ 1 and 2 are equivelent in saying that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ Converges for $(c_n)_{n=1}^{\infty} = (a_n^2)_{n=1}^{\infty}$

$(b_n)_{n=1}^{\infty} = (a_n^3)_{n=1}^{\infty}$

Reflection:

My original solution was good for a specific case where $(c_n)_{n=1}^{\infty} = (a_n^2)_{n=1}^{\infty}$

$(b_n)_{n=1}^{\infty} = (a_n^3)_{n=1}^{\infty}$

but I should have carried on to find c_n in terms of any a_n and b_n.

find c_n such that

$$\frac{a_n}{c_n} = \frac{c_n}{b_n} = \sqrt{\frac{a_n}{b_n}}$$

$$c_n = \sqrt{\frac{a_n}{b_n}} \cdot b_n$$

$$= \sqrt{a_n \cdot b_n}$$

A choice of $c_n = \sqrt{a_n \cdot b_n}$ shows equivalence of 1 and 2 for any two positive sequences a_n and b_n.