PH 3120 – Computational Physics Laboratory I

Ordinary Differential Equations

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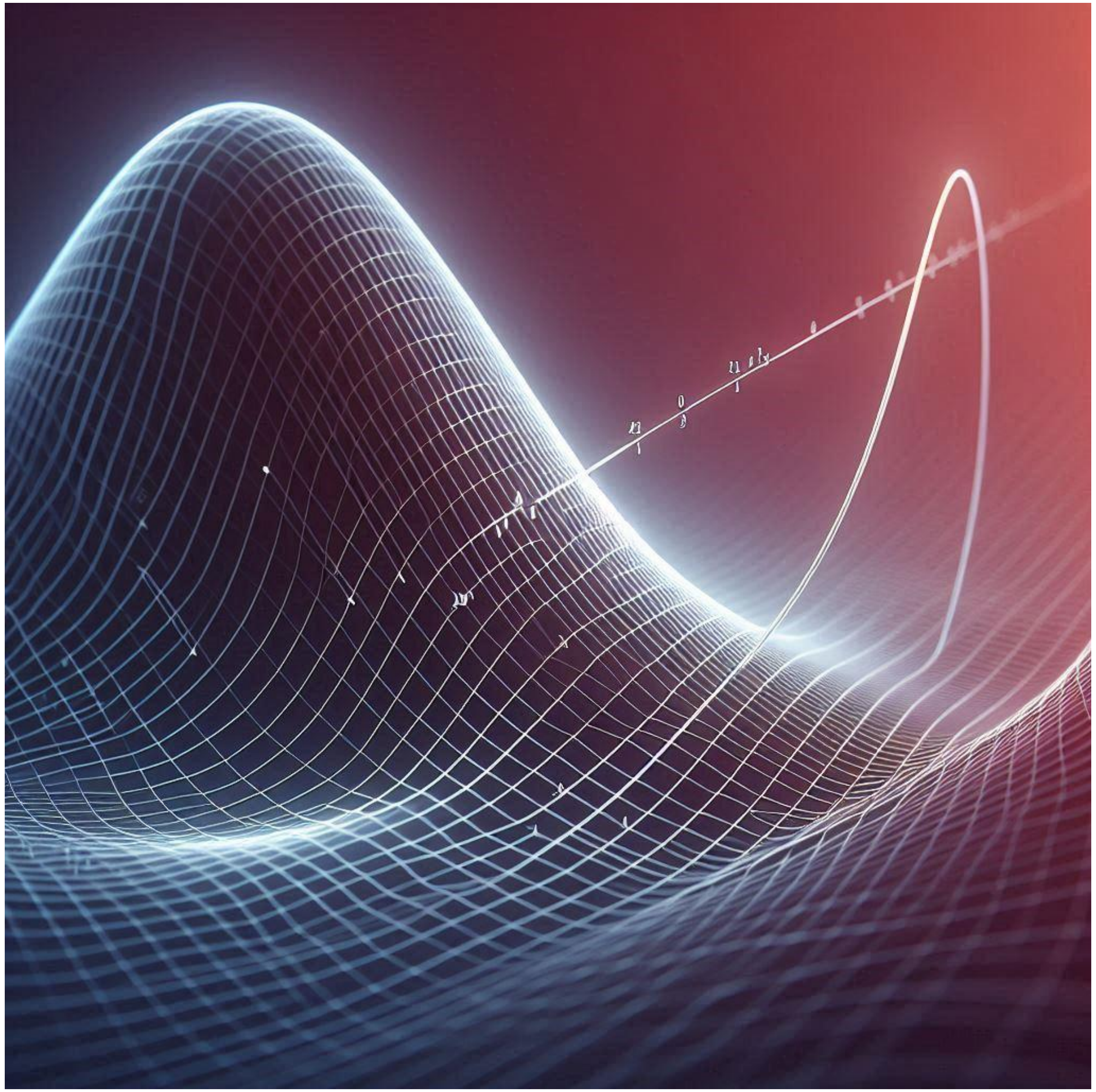


Ordinary Differential Equations

- Ordinary Differential Equations (ODEs) are equations that involve functions and their derivatives.
- They describe the relationship between a function and its rate of change.
- ODEs are used extensively in various fields such as physics, engineering, biology, and economics to model dynamic systems.
- ODEs in General Form:

$$\boxed{F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0}$$

$$y' = \frac{dy}{dx} \qquad y'' = \frac{d^2y}{dx^2} \qquad y^{(n)} = \frac{d^ny}{dt^n}$$

Order of ODEs

- First-order ODEs: Involves the first derivative of the unknown function.
- Second-order ODEs: Involves up to the second derivative.
- n-th order ODEs: Involves up to the n-th derivative.

Examples

First-order ODEs

$$F(x, y, y') = 0$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Second-order ODEs

$$F(x, y, y', y'') = 0$$

$$\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + L(x)y = S(x)$$

ODEs in Physics

Example: Radioactive decay

Radioactive decay describes how the number of unstable nuclei in a sample decreases over time. The rate of decay is proportional to the number of nuclei present.

$$\frac{d\lambda}{dt} = -N\lambda$$

- N is the number of radioactive nuclei.
- λ is the decay constant.

The solution for the above ODE is given by

$$\lambda = N_0 e^{-Nt}$$

ODEs in Physics

Example: Damped Harmonic Oscillator (under Damping Force)

For an underdamped harmonic oscillator, the rate of change of velocity is proportional to the force applied (Newton's second law), and the force due to damping is proportional to velocity.

ODE (velocity form)

$$m\frac{dv}{dt} = -bv - kx$$

First order ODE

ODE (displacement form)

$$m\frac{d^2x}{d^2t} = -b\frac{dx}{dt} - kx$$

Second order ODE

Initial Value Problem

An Initial Value Problem (IVP) is a type of differential equation along with specified values for the unknown function at a given point in its domain.

The goal is to find a function that satisfies the differential equation and meets the initial conditions.

An Initial Value Problem for a first-order differential equation is typically written as:

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

where:

$\frac{dy}{dx} = f(x, y)$ is the differential equation.

$y(x_0) = y_0$ is the initial condition, specifying the value of the function y at the point $x = x_0$.

Initial Value Problem

Example

Consider the following initial value problem:

$$\frac{dy}{dx} = x + y, \qquad y(0) = 1$$

Here:

The differential equation is $\frac{dy}{dx} = x + y$.

The initial condition is $y(0) = 1$.

Euler Method

Euler's Method is derived from the concept of using the tangent line to approximate the solution of an Ordinary Differential Equation (ODE) at discrete points.

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

Euler's Method Algorithm

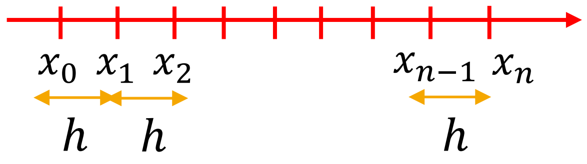
Starting from the initial condition (x_0, y_0) the algorithm proceeds as follows:

1. Initialize: $x = x_0, y = y_0$.

2. Iterate for n steps:

$$x_{i+1} = x_i + h$$

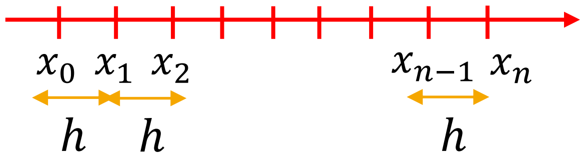
$$y_{i+1} = y_i + hf(x_i, y_i)$$



Euler Method Derivation

Consider the Taylor series expansion of $f(x+h)$ around x:

$$F(x_i + h) = F(x_i) + F'(x_i)(h) + \frac{F''(x_i)}{2!}h^2 + \frac{F'''(x_i)}{3!}h^3 + \cdots$$



It is given that

$$x_{i+1} = x_i + h$$

$$F(x_{i+1}) = F(x_i) + F'(x_i)(h) + \frac{F''(x_i)}{2!}h^2 + \frac{F'''(x_i)}{3!}h^3 + \cdots$$

Euler Method Derivation

$$F(x_{i+1}) = F(x_i) + F'(x_i)(h) + \frac{F''(x_i)}{2!}h^2 + \frac{F'''(x_i)}{3!}h^3 + \cdots$$

h is very small. Therefore, higher order terms (h^2, h^3, ...) are negligible.

$$F(x_{i+1}) = F(x_i) + F'(x_i)(h)$$ ----------- Equation A

$$y_i = F(x_i) \qquad F'(x_i) = \frac{dy}{dx}$$

Therefore, we can rewrite Equation A as,

$$y_{i+1} = y_i + hf(x_i, y_i)$$

What is the local truncation error of the above algorithm?

What is the global truncation error of the above algorithm?

Example

Let's solve the initial value problem using Euler's Method:

$$\frac{dy}{dx} = x + y, \qquad y(0) = 1$$

with step size $h = 0.1$ for 10 steps.

Initialization

$$x_0 = 0, \qquad y_0 = 1$$

$$\frac{dy}{dx} = f(x, y) = x + y$$

Step size: $h = 0.1$

Number of steps: $n = 10$

Iteration:

Step 1

$$x_1 = x_0 + 0.1 = 0 + 0.1 = 0.1$$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

Step 2

$$x_2 = x_1 + 0.1 = 0.1 + 0.1 = 0.2$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1)=1.22$$