16.3.

$\boxed{16.3.1.}$ C/k integral finite type of dim 1

then $\exists$ regular integral projective C'/k of dim 1 birational to C

proof rational map defines on open sets.

WMA C affine. By Noether Normalization.

$\exists x \in K(C)$ s.t. $K(C)/k(x)$ finite. ($C = \operatorname{Spec} A$, $A/k[x]$ finite)

Recall. 10.7.I.

X integral $\tilde{X} \to X$ normalization w.r.t. $L/K(X)$

locally it is $A \hookrightarrow B$ where

B is integral closure of A in L.

Now we take $C' := \widetilde{\mathbb{P}^1_k} \to \mathbb{P}^1_k$ w.r.t. $K(C)/k(x)$

by 10.7.M. $K(C)/k(x)$ finite $\Rightarrow$ $C' \to \mathbb{P}^1_k$ finite.

by 12.1.G. $\dim C' = \dim \mathbb{P}^1_k = 1$. C' integral.

C' Noetherian since $C'/\mathbb{P}^1_k$ finite. of finite type.

By 13.5.F. $\forall$ closed point x

$\mathcal{O}_{C',x}$ Noetherian local normal of dim 1. $\Rightarrow$ $\mathcal{O}_{C',p}$ regular

on generic point $K(C')$ is regular.

then C' regular, C' projective by 16.2.G.

C/k separated reduced of finite type $\Rightarrow$ C is k-variety.

$C' \to \mathbb{P}^1_k$ separated (local on the target) then C' is k-variety

$K(C') = K(C) \Rightarrow C' \hookrightarrow C$ birational by 7.5.E.

$C' \to \mathbb{P}^1_k$ finite $\mathbb{P}^1_k \to k$ projective $\Rightarrow C'$ projective by 16.2.G.

$\boxed{16.3.2.}$ C irreducible regular k-variety of dim 1.

then $\exists\, C \hookrightarrow C'$ birational. C' projective regular integral curve

proof Simple case: C affine.

then
$$\begin{tikzcd} C \arrow[r, dashed, "\text{birational}"] \arrow[d, "\pi\ \text{finite}"] & C' \arrow[d, "\pi'\ \text{finite}"] \\ \mathbb{A}^1_k \arrow[r, hook] & \mathbb{P}^1_k \end{tikzcd}$$
since Normalization is local

C already normalization of $\mathbb{A}^1_k$

$\pi'^{-1}(\mathbb{A}^1_k) \xrightarrow{\sim} C$ i.e. $C \hookrightarrow C'$

In general $\forall C_1 \hookrightarrow C$ affine, we have $C_1 \hookrightarrow \overline{\widetilde{C_1}}$

where $\overline{\widetilde{C_1}}$ regular projective curve.

then by 15.3.1. $C_1 \hookrightarrow \overline{\widetilde{C_1}}$ extends to $C \to \overline{\widetilde{C_1}}$ over finite points.

$C \to \overline{\widetilde{C_1}}$ birational since $C_1 \hookrightarrow \overline{\widetilde{C_1}}$ birational.

On topology, since $\{\text{points of } C\} \hookrightarrow \{\text{valuation of } K(C)\} = \{\text{points of } \overline{\widetilde{C_1}}\}$

we have $C \hookrightarrow \overline{\widetilde{C_1}}$ injective. $C_1 \hookrightarrow \overline{\widetilde{C_1}} \Rightarrow C \hookrightarrow \overline{\widetilde{C_1}}$

it suffices to show $C \hookrightarrow \overline{\widetilde{C_1}}$ open immersion

$\forall p \in C$ C_2 affine open neighborhood

$$\begin{tikzcd} C_1 \arrow[r, hook] \arrow[d] \arrow[drr, red] & C \arrow[dl] \arrow[dr] & C_2 \arrow[l, hook'] \arrow[d] \\ \overline{\widetilde{C_1}} & & \overline{\widetilde{C_2}} \end{tikzcd}$$

$C_1 \to \overline{\widetilde{C_2}}$ extends to $\pi_{12}: \overline{\widetilde{C_1}} \to \overline{\widetilde{C_2}}$

also we have $\pi_{21}: \overline{\widetilde{C_2}} \to \overline{\widetilde{C_1}}$

then $\pi_{12} = \pi_{21}^{-1}$ on $C_1 \cap C_2$

then $\pi_{12} = \pi_{21}^{-1}$ by 11.3.2 (agree locus), we have

$$\begin{tikzcd} C_1 \arrow[d, hook] \arrow[dr, hook] & & C_2 \arrow[d, hook] \arrow[dl, hook'] \\ & C \arrow[dl] \arrow[dr] & \\ \overline{C_1} \arrow[rr, hook] & & \overline{C_2} \end{tikzcd}$$

then $C_2 \to \overline{C_1}$ open immersion

$C \to \overline{C_1}$ open immersion near p.

then $C \to \overline{C_1}$ open immersion (local on the target)

$\boxed{16.3.A}$ we take $C \hookrightarrow \overline{C}$ C/k proper $\xRightarrow{\text{(cancellation)}}$ $C/\overline{C}$ proper

then $\overline{C}$ integral $\Rightarrow$ $C = \overline{C}$ C projective

$\boxed{16.3.3.}$ TFAE

① integral smooth projective curves with surjective morphism

② integral smooth pro— — . with dominant — —

③ — — — dominant rational.

④ integral curve dominant rational map.

⑤ opposite category of function fields$/k$.

proof. we have inclusion of category ① → ② → ③ → ④

from ④ to ①: given $C_1 \dashrightarrow C_2$ we have $U \xrightarrow{f} C_2$

$\widetilde{U} \to \widetilde{C_2}$ normalization

$\widetilde{U} \hookrightarrow \overline{\widetilde{U}}$ $\widetilde{C_2} \hookrightarrow \overline{\widetilde{C_2}}$ then $\widetilde{U} \to \overline{\widetilde{C_2}}$ extends to $\overline{\widetilde{U}} \xrightarrow{g} \overline{\widetilde{C_2}}$

$f(\eta_U) = \eta_{C_2} \Rightarrow \widetilde{f}(\eta_{\widetilde{U}}) = \eta_{\widetilde{C_2}}$ then $\widetilde{f}(\eta_{\widetilde{U}}) = \eta_{\overline{\widetilde{C_2}}}$

$g(\eta_{\overline{\widetilde{U}}}) = \eta_{\overline{\widetilde{C_2}}}$ i.e. g dominant, g surjective since g proper

$\boxed{16.3.B}$ check. ?

$\boxed{16.3.4}$ $\pi : C \to C'$ surjective map of regular integral projective curves

① First we show π is finite.

Let C'' Normalization of C' w.r.t. $K(C)/K(C')$

then $K(C) = K(C'')$ i.e. $C \dashrightarrow C''$ which extends to $C \rightleftarrows C''$

then $C \xrightarrow{\sim} C''$, but $C'' \to C'$ finite.

② Next we show $\pi_* \mathcal{O}_C$ locally of finite rank.

WMA C' affine, then C affine, $\pi_* \mathcal{O}_C$ of finite type

since C' reduced by 14.3.K

$\pi_* \mathcal{O}_C$ locally free $\Leftrightarrow$ $\pi_* \mathcal{O}_C$ constant rank

We show that rank on closed point = rank on generic point.

assume $C' = \operatorname{Spec} A'$ $p = [m]$ closed point $C = \operatorname{Spec} A$

$\dim \pi_* \mathcal{O}_C \otimes k(p) = \dim_{\underset{\text{Frac} A'/m}{\overset{\|}{A'/m}}} \underset{A \otimes_{A'} A'_m/mA'_m = A \otimes_{A'} A'/m}{\overset{\|}{A/mA}}$ $\dim \pi_* \mathcal{O}_C \otimes k(\eta) = \dim_{\underset{A'_0/0}{\overset{\|}{\text{Frac} A'}}} \underset{A \otimes_{A'} A'_0/0}{\overset{\|}{(A' \setminus 0)^{-1} A}}$

Claim. $\underset{(A' \setminus m)^{-1} A}{\overset{\|}{A_m}}$ is finite rank free A'_m-mod

if Claim is true $A_m \otimes \text{Frac} A'$ is free $\text{Frac} A'$-mod of dim n

$A_m \otimes_{A'_m} A'_m/mA'_m = A \otimes_{A'} A'_m \otimes A'_m/mA'_m = A/mA$ is free A'/m-mod of dim n

proof of Claim: A_m' is DVR $\Rightarrow$ A_m' is PID t uniformizer

A_m is f.g. A_m'-mod, then $A_m = \bigoplus_1^k A_m'/(t^{n_i}) \oplus (A_m')^r$

If $Tor(A_m) \neq 0$ t is zero divisor of A_m but A_m integral

$\square$

$\boxed{16.3.G}$ Def rank $\pi_* \mathcal{O}_C =: \deg \pi$.

$\boxed{16.3.C}$ $\deg \pi = [K(C) : K(C')]$ since $\deg \pi = \dim_{\operatorname{Frac} A'} (A \setminus 0)^{-1} A$

$\forall x \in A \setminus 0$ $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$

then $x^{-1} = -\left(\frac{1}{a_0} x^{n-1} + \frac{a_{n-1}}{a_0} x^{n-2} + \cdots \frac{a_1}{a_0}\right)$ i.e. $(A \setminus 0)^{-1} A = \operatorname{Frac} A$

$\deg \pi = \dim_{\operatorname{Frac} A'} \operatorname{Frac} A = [K(C) : K(C')]$

$\boxed{16.3.D}$

(a) Recall $\deg p = [k(p) : k]$ WMA C, C' affine.

then assume $U = \operatorname{Spec} A$ $V = \operatorname{Spec} B$

$\pi^{-1}(p) = \operatorname{Spec} B \otimes_B k(p) = \operatorname{Spec} B \otimes_B A_m \otimes_{A_m} A_m/mA_m = \operatorname{Spec} B/mB$

$\Gamma(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}) = B/mB$ $\dim_k B/mB = \dim_{k(p)} B/mB \times \dim_k k(p) = \deg \pi \deg p$.

(b) $\deg(\pi) = \dim_{k(p)} B/mB$ $\pi^{-1}(p) = \{ p_i \in \operatorname{Spec} B \mid p_i \cap A = m \}$

B/mB finite $k(p)$-algebra $\Rightarrow$ $B/mB = \prod (B/mB)_{p_i}$

then $\deg \pi = \dim_{k(p)} B/mB = \sum \dim_{k(p)} (B/mB)_{p_i} = \sum \dim_{k(p)} (B_{p_i}/mB_{p_i})$

now $B_{\mathfrak{p}_i}/A_m$ extension of DVR. $\dim_{\kappa(\mathfrak{p})} B_{\mathfrak{p}_i}/m B_{\mathfrak{p}_i} = \dim_{\kappa(\mathfrak{p})} B_{\mathfrak{p}_i}/\mathfrak{p}_i B_{\mathfrak{p}_i} \cdot \dim_{\kappa(\mathfrak{p})} \mathfrak{p}_i B_{\mathfrak{p}_i}/m B_{\mathfrak{p}_i}$

along exact sequence $0 \to \mathfrak{p}_i B_{\mathfrak{p}_i}/m B_{\mathfrak{p}_i} \to B_{\mathfrak{p}_i}/m B_{\mathfrak{p}_i} \to B_{\mathfrak{p}_i}/\mathfrak{p}_i B_{\mathfrak{p}_i} \to 0$

then $\dim_{\kappa(\mathfrak{p})} B_{\mathfrak{p}_i}/m B_{\mathfrak{p}_i} = \mathrm{val}_{\mathfrak{p}_i}(\pi^* t)\ \deg(\kappa(\mathfrak{p}_i)/\kappa(\mathfrak{p}))$ $\square$

Remark: $\mathcal{O}_L/\mathcal{O}_K$ dimension formula.

$\boxed{16.3.E}$ s ration function on C $\quad s: C \to \mathbb{P}^1_k$

then $\deg s = \sum_{\text{poles}} \mathrm{val}_{p_i}(\pi^* t_\infty) = \sum_{\text{zeros}} \mathrm{val}_{q_i}(\pi^* t_0)$

$\boxed{16.3.8}$ deg of line bundle on smooth proj curve.

C irreducible smooth curve $\mathcal{L}/C$ line bundle.

then $\forall s$ s.t nonzero rational setion of $\mathcal{L}$

s/t rational function $\Rightarrow$ zeros of s − poles of s well-defined

$\forall \phi, \psi$ local trivialization. $\phi(s)/\phi(t) = \psi(s)/\psi(t)$

since $\psi(t)\phi(s) = \phi(t)\psi(s) = u \cdot \phi(t)\phi(s)$