§15.5.

15.5.A $\mathrm{Aut}\,\mathbb{P}^n_k \cong PGL_{n+1}(k) = GL_{n+1}(k)/\sim_{\text{scalar}}$ $\quad M\sim aM,\ a\in k^*$

Pf. $\pi: \mathbb{P}^n_k \to \mathbb{P}^n_k$ automorphism, inverse ϖ.

pull back of a line bundle is still a line bundle

(reason: $(f^*\mathcal{L})_x \cong \mathcal{L}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \cong \mathcal{O}_{Y,f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \cong \mathcal{O}_{X,x}$) $\quad f: X \to Y$

$\because \mathrm{Pic}\,\mathbb{P}^n_k \cong \mathbb{Z}$. $\therefore \exists d\in\mathbb{Z}$ s.t. $\pi^*\mathcal{O}(1)\cong\mathcal{O}(d)$.

$\mathcal{O}(1) = (\varpi\pi)^*\mathcal{O}(1) = \varpi^*\pi^*\mathcal{O}(1) \cong \varpi^*\mathcal{O}(d) \cong \mathcal{O}(d'(d))$

$\mathcal{O}(d') \cong \varpi^*\mathcal{O}(\frac{d}{(d)})$

$\therefore d = \pm 1$.

$\because \pi$ isom. of schemes so.

$\Gamma(\mathbb{P}^n_k, \pi^*\mathcal{O}(1)) \cong \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$.

$\therefore$ It must be $d=1$. $\quad \pi^*\mathcal{O}(1)\cong\mathcal{O}(1)$.

($f: X\to Y$ isom. of sch. By def. $f^{-1}\mathcal{O}_Y \xrightarrow{\cong} \mathcal{O}_X$

$\forall \mathcal{F}: \mathrm{Mod}_{\mathcal{O}_Y}$. $\quad f^*\mathcal{F} = f^{-1}\mathcal{F}\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X \cong f^{-1}\mathcal{F}$.

$\Gamma(X, f^*\mathcal{F}) \underset{\varphi}{\cong} \Gamma(X, f^{-1}\mathcal{F}) \underset{\varphi}{=} \Gamma(Y,\mathcal{F})$.) $\quad \because$ isom.

题干中φ本质上是从$\pi^*\mathcal{O}_P \xrightarrow{\cong} \mathcal{O}_P$来的. ▨

15.5.B given (p_1, p_2, p_3) 互不同, (q_1, q_2, q_3) 互不同. $\quad p_i, q_i$ 为 $\mathbb{P}^1_k$ 上 k 值点..

$\exists!\, \pi \in \mathrm{Aut}\,\mathbb{P}^1_k$ s.t. $\pi(p_i) = q_i$.

Pf. 补: 如何将15.5.A "函数层面"上的PGL 转化成"闭点层面"上的PGL.

想法: 一个点可通过函数在这个点上取值来知道信息.

比如: $p = [x_0(p), x_1(p), x_2(p), \ldots\ldots, x_n(p)]$ $\quad x_i$看成section

对 $\pi(p) = q$. $\quad q = [x_0(q) \ldots\ldots\ldots x_n(q)]$

$= [\pi^*(x_0)(p) \ldots\ldots \pi^*(x_n)(p)]$

$= \left[\sum_{i=0}^n a_{0i}x_i(p) \ldots\ldots \sum_{i=0}^n a_{ni}x_i(p)\right]$ (这里用15.5.A)

$= [x_0(p) \ldots x_n(p)]\cdot$ 一个阵.

$\therefore$ 现在可以在闭点PGL上考虑上述问题.

只要记 $p_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $p_2 = \begin{bmatrix}0\\1\end{bmatrix}$ $p_3 = \begin{bmatrix}1\\1\end{bmatrix}$ 情况 即可.

$q_1 = \begin{bmatrix}a\\b\end{bmatrix}$ $q_2 = \begin{bmatrix}c\\d\end{bmatrix}$ $q_3 = \begin{bmatrix}\alpha a+\beta c\\ \alpha b+\beta d\end{bmatrix}$ $\quad T = \begin{bmatrix}\alpha a & \beta c\\ \alpha b & \beta d\end{bmatrix}$ 即可. ▨

(q_1, q_2) 线性无关 不然由P. $q_1 = q_2$.

15.5.C (a). 找 linear transformation $f \in PGL(2)$. has order precisely 3. in $PGL(2)$.

Sol. $\begin{pmatrix}0&1\\-1&1\end{pmatrix}^3 = \begin{pmatrix}0&1\\-1&1\end{pmatrix}\begin{pmatrix}-1&1\\-1&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$

在PGL(2)中是3阶元.

(b). PGL(2)中任何两个三阶元共轭.

Pf. σ为PGL(2)的3阶元. $\because \sigma\neq id$. $\therefore \exists p\in\mathbb{P}^1_k$ s.t. $p\neq\sigma(p)$.

考虑 $\{p, \sigma(p), \sigma^2(p)\}$ (若$\sigma^2p=p$. 由$\sigma^3 p = p$. 矛盾) $\therefore$ 互不同.

τ 为另一个3阶元. 相同地. $\exists q\in\mathbb{P}^1_k$. s.t. $q\neq\tau(q)$

$\leadsto \{q, \tau(q), \tau^2(q)\}$

由15.5.B. $\exists!\, g\in PGL(2)$. s.t. $g(p) = q$. $g\sigma(p) = \tau(q)$

$g\sigma^2(p) = \tau^2(q)$.

断言: $g\sigma g^{-1} = \tau$.

原因: $\tau g\sigma^{-1}(p) = \tau g\sigma^2(p) = \tau\tau^2(q) = q$.

$\tau g\sigma^{-1}\cdot\sigma(p) = \tau(q)$.

$\tau g\sigma^{-1}\cdot\sigma^2(p) = \tau g\sigma(p) = \tau\tau(q) = \tau^2(q)$. 比较

由g的唯一性. $\tau g\sigma^{-1} = g$. ▨

15.5.D $p_0 \ldots\ldots p_{n+1}$. $n+2$ 不同 k-value pts. of $\mathbb{P}^n_k$.

任 $n+1$ 个点不落在某 hyperplane 上.

求证: $\exists!$ proj. transformation. 把 p_i 打到 $[0 \ldots 1 \ldots 0]$ 上 (1 在第 i 位, $0\le i\le n$)

$p_{n+1} \mapsto [1\ 1 \ldots\ldots\ldots 1]$.

Pf omitted. same as 15.5.B

15.5.E X q-proj. k-scheme. $\pi: \mathbb{P}^n_k \to X$. morphism over k.

Show: 要么 $\dim \mathrm{Im}\,\pi = n$.

要么. π 把 $\mathbb{P}^n_k$ 打到某一点..

Pf. Step1: 化到 $k=\bar{k}$ 情形. 特别地. k infinite.

原因: 考虑 base change

$$\begin{tikzcd} \mathbb{P}^n_{\bar{k}} \arrow[r, "\pi_{\bar{k}}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X_{\bar{k}} \arrow[d] \\ \mathbb{P}^n_k \arrow[r, "\pi"] & X \end{tikzcd}$$

若对 $\bar{k}$ 成立

$$\begin{tikzcd} \mathbb{P}^n_k \arrow[rr, "\pi"] \arrow[dr, "\text{proper}"'] & & X \arrow[dl, "\text{q-proj. by 11.2.15 sep.}"] \\ & k & \end{tikzcd}$$

(CANC) $\Rightarrow \pi$ proper. $\quad \downarrow\downarrow$ from $\bar{k}/k$ alg. extn.

$Y \hookrightarrow X \quad \leadsto 0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$.

$\bar{k}\to k$ flat. $\leadsto 0 \to \mathcal{I}\otimes\bar{k} \to \mathcal{O}_X\otimes\bar{k} \to \mathcal{O}_Y\otimes\bar{k} \to 0$

$\mathcal{O}_X\otimes\bar{k} = \mathcal{O}_{X_{\bar{k}}}$, $\mathcal{O}_Y\otimes\bar{k} = \mathcal{O}_{Y_{\bar{k}}}$

代表 $Y_{\bar{k}}$ 为 $X_{\bar{k}}$ 的闭子概形

$$\begin{tikzcd} \mathbb{P}^n_{\bar{k}} \arrow[r, two heads, "\because \text{surj. (BC)}"'] \arrow[d] & Y_{\bar{k}} \arrow[r, hook] \arrow[d] & X_{\bar{k}} \arrow[d] \\ \mathbb{P}^n_k \arrow[r, two heads] & Y \arrow[r, hook] & X \end{tikzcd}$$

Y 闭 $\because \pi$ proper.

再利用 $\dim Y_{\bar{k}} = \dim Y$.

(可用12.1.H(a)).

Y 的仿射覆盖 $\{U_i\}$

$\dim Y = \max(\dim U_i)$

$= \max(\dim U_{i\bar{k}}) = \dim Y_{\bar{k}}$

Step2 $\quad \pi: \mathbb{P}^n \to X$. $\quad \iota: X \hookrightarrow \mathbb{P}^N$.

考虑 $(\pi\iota)^*\mathcal{O}_{\mathbb{P}^N}(1)$. 为 $\mathbb{P}^n$ 上线丛. $\therefore \cong \mathcal{O}_{\mathbb{P}^n}(d)$.

断言: $d=0$.

原因: 回顾12.3.D(a): $X \hookrightarrow \mathbb{P}^n_k$ $\quad \dim X\geq 1$. H: nonempty hypersurface.

Then $H\cap X\neq\emptyset$.

对 $\mathbb{P}^n_k$ 上 m 个超曲面 $H_1 \ldots\ldots\ldots H_m$. $\quad m\leq n$.

$\mathbb{P}^n_k = \bigcup_{i=0}^n U_i$ $\quad U_i$ 为基本 affine open.

$H_i\cap U_j =: V_j(f_{ij})$. $\quad$ 12.3.D(a) $\dim H_1 = n-1\geq 1 \leadsto H_1\cap H_2\neq\emptyset$

对 $H_1\cap H_2$. $\quad \forall j\in[0,n]$. $H_1\cap H_2\cap U_j = V_j(f_{1j}, f_{2j})$

Krull's Height Theorem: $\forall$ irr. comp of $H_1\cap H_2\cap U_j$ Z in U_j.

$\mathrm{codim}\, Z\leq 2$.

U_j irr. $\Rightarrow \mathrm{codim}\, Z + \dim Z = \dim U_j = n$

$\therefore \dim Z\geq n-2$. $\therefore \dim H_1\cap H_2\cap U_j \geq n-2$.

$\therefore \dim H_1\cap H_2\geq n-2$.

12.3.D(a) $\Rightarrow H_1\cap H_2\cap H_3\neq\emptyset$

以此类推. $\Rightarrow H_1\cap H_2\cap\ldots\cap H_m\neq\emptyset$ for $m\leq n$. $\quad (*)$

Suppose. $\dim \mathrm{Im}\,\pi < n$. 由 π proper $\mathrm{Im}\,\pi =: Y \hookrightarrow X$. $\quad \dim Y = r$.

可找到 $r+1$ 个 hyperplanes $\quad L_1 \ldots L_{r+1} \subseteq \mathbb{P}^N$. $\quad$ s.t.

$Y\cap L_1\cap L_2\cap \ldots L_{r+1} = \emptyset$. $\quad$ (12.3.D(e) 这里 hyperplane 用到 k-infinite.

$\emptyset = \tilde{\pi}^{-1}(\emptyset) = \tilde{\pi}^{-1}(Y\cap L_1\cap\ldots\ldots\cap L_{r+1})$

$= \tilde{\pi}^{-1}(Y)\cap\tilde{\pi}^{-1}(L_1)\ldots\cap\tilde{\pi}^{-1}(L_{r+1})$ $\quad \tilde{\pi} = \iota\circ\pi$

$= \tilde{\pi}^{-1}(L_1)\cap\ldots\cap\tilde{\pi}^{-1}(L_{r+1})$.

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, "\pi"] \arrow[rr, bend right, "\tilde{\pi}"'] & X \arrow[r, hook, "\iota"] & \mathbb{P}^N \end{tikzcd}$$

hyperplane $\Rightarrow L_1 \ldots L_{r+1}$ 由 $\Gamma(\mathbb{P}^N, \mathcal{O}(1))$ 的元素定义 $S_1 \ldots\ldots S_{r+1}$.

考虑 $\tilde{\pi}^*S_1 \ldots\ldots \tilde{\pi}^*S_{r+1}$ 分别记为 $\sigma_1 \ldots\ldots \sigma_{r+1}\in\Gamma(\mathbb{P}^n, \mathcal{O}(d))$

$\mathcal{O}(d) \cong \tilde{\pi}^*\mathcal{O}(1)$

断言: $\tilde{\pi}^{-1}(L_i) = V_+(\sigma_i)$.

(原因: $x\in V_+(\sigma_i) \Leftrightarrow \sigma_i(x) = 0 \in (\tilde{\pi}^*\mathcal{O}(1))_x\otimes\kappa(x) = k$. $\quad \kappa(x) = k$ (k-pt)

$\Leftrightarrow S_i(\tilde{\pi}(x)) = 0 \in \mathcal{O}(1)_{\tilde{\pi}(x)}\otimes\kappa(\tilde{\pi}(x)) = k$.

$\Leftrightarrow \tilde{\pi}(x)\in V_+(S_i) = L_i \Leftrightarrow x\in\tilde{\pi}^{-1}(L_i)$.) $\quad$ line bundle

假设 $d>0$. $\quad \emptyset = V_+(\sigma_1)\cap\ldots\cap V_+(\sigma_{r+1})$.

由 $r<n$. $r+1\leq n$. 和12.3.D(a)推论. 上述不为空. 矛盾.

(补: 上面 $\sigma_i\neq 0$. 否则 $\tilde{\pi}^{-1}(L_i) = \mathbb{P}^n$ $\quad Y\subseteq L_i$.

$\mathbb{P}^n$ 不可约 + π proper $\Rightarrow Y$ 不可约. 取 Y 的光滑开集 Y_{reg}.

Bertini. $\Rightarrow Y_{reg}\not\subseteq L_i$. 矛盾!)

由补内容还可知 $d<0$ 也不行. 故 $d=0$. $\quad \tilde{\pi}^*\mathcal{O}(1)\cong\mathcal{O}_{\mathbb{P}^n}$

$\forall p\in\mathbb{P}^n$. $\quad \tilde{\pi}(p) = q$. $\quad x_0 \ldots\ldots x_N$ 为 section $\in\Gamma(\mathbb{P}^N, \mathcal{O}(1))$

$q = [x_0(q) \ldots\ldots x_N(q)]$

$= [\underbrace{\tilde{\pi}^*x_0(p)}_{\in\,\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k} \ldots\ldots \tilde{\pi}^*x_N(p)]$ constant. ▨

15.5.4

15.5.G $Y \subseteq \mathbb{P}^n_k$ hypersurface. $Y = V_+(f)$. $\deg f = d$.

Show that $\mathrm{Pic}(\mathbb{P}^n_k - Y) \cong \mathbb{Z}/d\mathbb{Z}$.

Pf. $\mathbb{Z} \longrightarrow \mathrm{Cl}(\mathbb{P}^n_k) \longrightarrow \mathrm{Cl}(\mathbb{P}^n_k - Y) \longrightarrow 0$ exact.

$1 \longmapsto [Y] = d[H]$

$\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$. 生成元为$[H]$

$\underset{\text{开}}{\underline{\mathbb{P}^n_k - Y}}$ $\underbrace{\text{Noetherian}}$ + $\underbrace{\text{factorial}}_{\text{inherited from } \mathbb{P}^n_k}$ $\Rightarrow \mathrm{Cl} \cong \mathrm{Pic}$ ▨

15.5.H. 和15.5.G相同记号. $d > 1$. (So $\mathrm{Pic}(\mathbb{P}^n - Y) \neq 0$)

$H_0, H_1, \ldots, H_n$ $n+1$ coordinate hyperplanes on $\mathbb{P}^n$.

- Show that $\mathbb{P}^n - Y$ affine, $\mathbb{P}^n - Y - H_i$ distinguished open subset of it, covers $\mathbb{P}^n - Y$.
- Show $\mathrm{Pic}(\mathbb{P}^n - Y - H_i) = 0$.
- Show each $\mathbb{P}^n - Y - H_i$ is Spec of a UFD, but $\mathbb{P}^n - Y$ not.

($\therefore$ UFD is not aff.-loc. property).

Pf. $\mathbb{P}^n - Y \cong \mathrm{Spec}\left(k[x_0 \ldots x_n]_{(f)}\right)_{\deg 0}$.

$\mathbb{P}^n - H_i = U_i$. $\mathbb{P}^n - Y - H_i = U_i \cap (\mathbb{P}^n - Y)$

$= D\left(\frac{x_i^d}{f}\right) \subseteq \mathbb{P}^n - Y$.

$= D\left(\frac{f}{x_i^d}\right) \subseteq \mathbb{P}^n - H_i$.

$\mathbb{P}^n - H_i = \mathrm{Spec}\, \underbrace{k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]}_{\text{UFD}}$

UFD局部化还是UFD.

$\mathrm{Pic}(\mathbb{P}^n - H_i - Y) \underset{\text{Noe + factorial}}{=} \mathrm{Cl}(\mathbb{P}^n - H_i - Y) = \mathrm{Cl}(\mathrm{Spec} A)$ A: UFD.

$\mathrm{Spec} A$ 素除子 $\longleftrightarrow$ $\mathrm{ht}\, p = 1$ 但UFD+整闭+Noether. 有 $\mathrm{ht}\, p = 1 \Rightarrow p = (f)$.

$\therefore \mathrm{Cl}(\mathrm{Spec} A) = 0$.

若 $\left(k[x_0 \ldots x_n]_{(f)}\right)_{\deg 0}$ UFD. 则 $\mathrm{Cl}(\mathbb{P}^n - Y) = 0$

而 $= \mathbb{Z}/d\mathbb{Z}$ 矛盾!

$\therefore$ 不为UFD. ▨

15.5.I 和之前记号一样. 求证: 在$\mathbb{P}^n - Y$上. $H_i \cap (\mathbb{P}^n - Y)$ is an eff. Cartier divisor that is not cut out by a single equation.

Pf. 对应 Cartier 除子是 $\left\{\left(U_j \cap \mathbb{P}^n - Y, \frac{x_i}{x_j}\right)\right\}_{j = 0, \ldots, n}$.

若可被单个方程定义. 则 $H_i \cap (\mathbb{P}^n - Y)$ principal.

考虑 $\mathbb{Z} \longrightarrow \mathrm{Cl}(\mathbb{P}^n - Y) \xrightarrow{\neq 0} \mathrm{Cl}(\mathbb{P}^n - Y - H_i) \longrightarrow 0$

$1 \longmapsto [H_i] = 0$, $\mathrm{Cl}(\mathbb{P}^n - Y - H_i) = 0$

矛盾! ▨

15.5.J Show that $A := \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is not UFD. but $A \otimes \mathbb{C}$ is.

Pf. Veronese subring of $T_\bullet = \mathbb{R}[u, v]$

Consider $T_{2\bullet}$. well-known that $\mathrm{Proj}\, T_{2\bullet} \cong \mathrm{Proj}\, T_\bullet$.

Claim: $T_{2\bullet} \cong \mathbb{R}[x, y, z]/(x^2 + y^2 - z^2)$

$\mathbb{R}[x, y, z] \xrightarrow{\varphi} T_{2\bullet}$

$x \longmapsto u^2 - v^2$

$y \longmapsto 2uv$

$z \longmapsto u^2 + v^2$

Surj. v. ker: $(x^2 + y^2 - z^2) \subseteq \ker \varphi$.

若 $\ker \varphi \supsetneq (x^2 + y^2 - z^2)$ $T_{2\bullet}$ prime $\longleftrightarrow$ $\mathbb{R}[x, y, z]$ 中 kernel φ 含 $\cancel{x^2 + y^2 - z^2}$ 的 prime

则 $\underset{3}{\dim \mathbb{R}[x, y, z]} \geq \underset{\dim T_\bullet}{\dim T_{2\bullet}} + 2 = 4$ 矛盾!

$\bar{C}$: 代表 $\mathrm{Proj}\, \mathbb{R}[x, y, z]/(x^2 + y^2 - z^2) \cong \mathbb{P}^1_{\mathbb{R}}$ $\mathrm{Cl}\left(\mathrm{Proj}\, \mathbb{R}[x, y, z]/(x^2 + y^2 - z^2)\right) \cong \mathbb{Z}$.

$\left(\mathbb{R}[x, y, z]/(x^2 + y^2 - z^2)\right)_{(z)\, \deg 0} \cong A$.

$\mathbb{Z} \longrightarrow \mathrm{Cl}(\bar{C}) \longrightarrow \mathrm{Cl}(\mathrm{Spec} A) \longrightarrow 0$ exact.

$\mathrm{Proj}\left(\mathbb{R}[x, y, z]/(x^2 + y^2 - z^2)\right)$ 中 $V_+(z)$ $\therefore \mathbb{Z} \longrightarrow \mathrm{Cl}(\bar{C}) \cong \mathbb{Z}$

$1 \longmapsto 2$

$\Leftrightarrow$ $\mathrm{Proj}\, T_{2\bullet}$ 中 $V_+(u^2 + v^2)$

$\Leftrightarrow$ $\mathrm{Proj}\, T_\bullet$ 中 $V_+(u^2 + v^2)$. 2次

$\mathbb{C}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{C}[u, v]/(uv - 1) \cong \mathbb{C}[u]_u$.

$x^2 + y^2 = (x + iy)(x - iy)$ 是UFD ▨

15.5.K Consider $X=\mathbb{P}^1_k\times\mathbb{P}^1_k \cong \operatorname{Proj} k[w,x,y,z]/(wx-zy)$.
Show that $\operatorname{Pic} X\cong \mathbb{Z}\oplus\mathbb{Z}$.

Pf. $\mathbb{P}^1\setminus\{\infty\}\cong \mathbb{A}^1$. $\quad L=pr_1^{-1}(\{\infty\})$. $\quad M=pr_2^{-1}(\{\infty\})$.

$$\begin{tikzcd} \mathbb{P}^1\times\mathbb{P}^1 \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & \mathbb{P}^1 \arrow[d] \\ \mathbb{P}^1 \arrow[r] & k \end{tikzcd}$$

$X-L-M=pr_1^{-1}(\mathbb{A}^1)\cap pr_2^{-1}(\mathbb{A}^1)\cong \mathbb{A}^1\times_k\mathbb{A}^1\cong\mathbb{A}^2_k$.

locally: $\{\infty\}\in\mathbb{A}^1=\operatorname{Spec} k[\frac{x}{y}]\subseteq\mathbb{P}^1$.

$k[\frac{x}{y}]\xrightarrow{\varphi} k[\frac{x}{y}]\otimes_k k[\frac{z}{w}]$

$pr_1^{-1}(\{\infty\}) \leftrightsquigarrow p\in \operatorname{Spec} k[\frac{x}{y},\frac{z}{w}]$

with. $\varphi^{-1}(p)=(\frac{x}{y})$.

∴ $pr_1^{-1}(\{\infty\})$ 局部上同构于 $\mathbb{A}^1$. 为1维. 且同

且 codim=1. ∴ L, M 为素除子.

考虑 $\mathbb{Z}\oplus\mathbb{Z}\xrightarrow{\alpha}\operatorname{Cl}(\mathbb{P}^1\times\mathbb{P}^1)\xrightarrow{\bar\beta}\operatorname{Cl}(\underset{\substack{\|\\ U}}{\mathbb{A}^1\times\mathbb{A}^1})\longrightarrow 0$.

$(1,0)\longmapsto[L]$

$(0,1)\longmapsto[M]$

正合性: $\operatorname{Im}\alpha\subseteq\ker\bar\beta$ 自然. 对 $\ker\beta\subseteq\operatorname{Im}\alpha$.

若 $[D]\in\operatorname{Cl}(X)$ 有 $\bar\beta([D])=0$. 则 $D|_U$ 为 U 上主除子.

∴存在 U 上有理函数 f s.t. $D|_U=\operatorname{div} f$.

$\underline{K(U)=K(X)}$. f 可看成 $K(X)$ 中元素.

$\beta(D-\operatorname{div} f)=0$. 代表 $D=\sum n_iD_i$ 中 D_i 素除子. $D_i\subseteq L\cup M$.

(拓扑命题: X 有不可约分解 $Z_1\cup\ldots\cup Z_n$ 若还有 $Z_i\not\subseteq Z_j$ $i\neq j$. 则分解唯一) 于是 $D_i=L$ 或 M.

∴正合性成立 ∴ $\mathbb{Z}\oplus\mathbb{Z}\longrightarrow\operatorname{Cl}(X)\cong\operatorname{Pic}(X)$ surj.

Inj. 有空再补. 累了........

15.15.L Show that irr. smooth proj. surfaces over k can be birational but not isomorphic.

Pf. $\begin{matrix}\mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1\times\mathbb{P}^1\\ \cup & & \cup\\ \mathbb{A}^2 & & \mathbb{A}^1\times\mathbb{A}^1=\mathbb{A}^2\end{matrix}$ $\quad$ birational

$\operatorname{Cl}(\mathbb{P}^2)=\mathbb{Z}$. $\quad\operatorname{Cl}(\mathbb{P}^1\times\mathbb{P}^1)=\mathbb{Z}^2$. not isom. ∴ $\mathbb{P}^2\not\cong\mathbb{P}^1\times\mathbb{P}^1$ ▨

但对 curve birational $\Longrightarrow$ isom.

15.5.M $X=\operatorname{Spec} k[x,y,z]/(xy-z^2)$. cone $\quad$ char $k\neq 2$.
Show that $\operatorname{Pic}X=0$. $\quad\operatorname{Cl}X\cong\mathbb{Z}/2\mathbb{Z}$

Pf. $\bigoplus_i\mathbb{Z}\cdot[Z_i]\longrightarrow\operatorname{Cl}(X)\longrightarrow\operatorname{Cl}(X-V(x))\longrightarrow 0$. $\quad =0$

$D(x)\cong\operatorname{Spec}\underline{k[x,z]_x}$ UFD $\quad V(x)=\bigcup_i Z_i$ irr. comp.

$k[x,y,z]/(xy-z^2)$ 为整环.

$V(x)=V(x,z)$. $\quad (x,z)\in\operatorname{Spec}k[x,y,z]/(xy-z^2)$

$\dim V(x,z)=\dim k[y]=1$. ∴ $\operatorname{codim}V(x,z)=1$
为素除子.

若 $V(x,z)$ 为主除子.

$V(x,z)=\operatorname{div} f$.

$V(x,z)$ 的 gen. pt 是 $[p]$ $\quad p=(x,z)$

则 f_p 在 A_p 的赋值恰为1. x 赋值为2 z 赋值为1 (∵ $x=\frac{1}{y}z^2$)

f 局部上只和 z 差一个单位元.

再看 $\operatorname{div} z$: $\quad V(z)=V(x,z)\cup V(z,y)$. 多出来了 矛盾!

$\operatorname{div} x=2V(x,z)$. ∴ $\operatorname{Cl}X\cong\mathbb{Z}/2\mathbb{Z}$.

$D=V(x,z)$. $\underline{15.4.K}$ D not locally principal

$\underline{15.4.8}$ $\mathcal{O}(D)$ not line bundle

∴ $\operatorname{Pic}X=0$. ▨

15.5.N $X = \operatorname{Spec} k[w,x,y,z]/(wz-xy)$.

Z: Weil divisor cut out by $V(w,x)$

否记 Z 不可以写成 $V(f)$ 形式. (13.1.D)

Show: if $n \neq 0$, then $n[Z]$ is not locally principal.

Hence Z is ~~even~~ not cut out loc. set-theo. by a single equation.

@ Cartier 指存在一个子集：某上面可对应一个 line. bundle.

pf. Recall 8.3.4 $\pi: X \to Y$ affine. if $\exists$ cover of Y by aff. open U s.t. $\pi^{-1}(U)$ affine

Step 1: By definition. D: eff. Cartier divisor. of $S = \operatorname{Spec} A$.

$D \hookrightarrow S$. $\exists S = \bigcup U_i$ affine open covering $f_i \in \Gamma(U_i, \mathcal{O}_S)$. nonzero divisor.

s.t. $D|_{U_i} = V(f_i)$

$(S-D) \cap U_i = \underbrace{D(f_i)}_{\text{affine open}} \subseteq U_i$

$j: S - D \hookrightarrow S$. $\underbrace{j^{-1}(U_i)}_{\text{aff.}} = \underbrace{(S-D) \cap U_i}_{\text{aff.}}$ $\therefore j$ affine.

Step 2: Suppose $\exists n \in \mathbb{Z}^*$ s.t. $n[Z]$ loc. principal.

(general) $\exists X$ 的 aff. open covering $X = \bigcup U_i$ s.t.

$D|_{U_i} = \operatorname{div}_{U_i} f_i$ $\quad f_i \in K(U_i) = K(X)$. $\quad U_i \cong \operatorname{Spec} A_i$

If D is effective i.e. $\forall p \subseteq A_i$ of $ht = 1$.

$v_p(f_i) \geq 0$. i.e. $f_i \in A_p$.

A_i normal. Noetherian. Hartogs $\Rightarrow f_i \in A_i$

$\Rightarrow$ eff. Cartier.

In this case X is smooth.

若是 regular 必是 normal.

$\therefore n[Z]$ eff. Cartier. Support 就是 Z.

故 $X \setminus Z$ affine by step 1.

Step 3. Consider $J = (y,z)$ R. $\quad R = k[x,y,z,w]/(wz-xy)$

$Y := \operatorname{Spec}(R/J) \cong \mathbb{A}^2$.

Consider $U := X \setminus Z$. $U \cap Y$ closed in U.

是 $\underbrace{\mathbb{A}^2 \setminus (0,0)}_{\text{not aff.}}$

but U aff.

cl. must aff. 矛盾!

$\therefore n[Z]$ can't be loc. principal.

Step 4 If Z loc cut out by one eq.

$[Z]$ eff. Cart. by above. 矛盾! ∎

***15.5.P** A Noetherian integral.

A UFD $\Leftrightarrow$ A integral closed and $\operatorname{Cl} \operatorname{Spec} A = 0$

pf. "$\Rightarrow$" ✓

"$\Leftarrow$". (12.3.7) A UFD $\Leftrightarrow$ A integral cl. + every prime ideal of 1 is principal.

$\forall p \subseteq A$ of $ht\, p = 1$.

$V(p)$ prime divisor. $\therefore V(p) = \operatorname{div} f$ for some $f \in \operatorname{Frac} A^*$.

By $\operatorname{Cl} \operatorname{Spec} A = 0$.

$\forall q$ of ht 1. $\operatorname{val}_q f \geq 0$. i.e. $f \in A_q$

Hartogs $\Rightarrow f \in A^*$

$(f) \subseteq p$. $\because \operatorname{val}_p(f) = 1$. $\therefore f \in p$.

$p \subseteq (f)$. $\because \forall g \in p$. $\operatorname{val}_p(\frac{g}{f}) \geq 1 - 1 = 0$ $\quad \therefore \frac{g}{f} \in A$

$\operatorname{val}_q(\frac{g}{f}) = \operatorname{val}_q(g) \geq 0$ $\quad$ by Hartogs

$\therefore g \in (f)$.

$\therefore (f) = p$. ∎