Def: X ringed space

Free sheaf: $\cong \mathcal{O}_X^{\oplus I}$. $\operatorname{rank}(\mathcal{O}_X^{\oplus I}) := |I|$.

Locally free sheaf: $\cong \mathcal{O}_X^{\oplus I}$ locally

The local isom is called trivialization.

Phylosophy: mod — free mod

quasicoherent - locally free

coherent: fin rank.

Def: Invertible sheaf: $\cong \mathcal{O}_X$ locally.

Fact: $\mathcal{F}$ invertible $\Leftrightarrow \exists \mathcal{G}$ s.t $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X$.

14.1.B (Pulling back vector bundles)

$\pi: X \to Y$. $\in \operatorname{Mor}(\text{Ringed spaces})$, $n \in \mathbb{Z}_{\geq 0}$.

(a) $\pi^*(\mathcal{O}_Y^{\oplus n}) \cong \mathcal{O}_X^{\oplus n}$ canonically.

(Def. $\pi^*(\mathcal{F}) := \pi^{-1}(\mathcal{F}) \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X$).

(b) $\mathcal{F}$ locally free on Y. rank $= n$

$\Rightarrow \pi^*\mathcal{F}$ - — on X, — —.

(c) $\mathcal{F}$ — — on Y — — ,

$Y = \bigcup_i U_i$, $\varphi_i : \mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$.

$$\begin{tikzcd}
 & \mathcal{O}_{U_i \cap U_j}^{\oplus n} \arrow[dd, "S.\ T."] \\
\mathcal{F}|_{U_i \cap U_j} \arrow[ur, "\varphi_i|_{U_i \cap U_j}", "\sim"'] \arrow[dr, "\sim", "\varphi_j|_{U_i \cap U_j}"'] \arrow[r, phantom, "\circlearrowright"] & \\
 & \mathcal{O}_{U_i \cap U_j}^{\oplus n}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \vec{s}^{\,i} = \begin{pmatrix} s_1^i \\ \vdots \\ s_n^i \end{pmatrix} \arrow[dd, "T_{ij}"] \\
s \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & \vec{s}^{\,j} = \begin{pmatrix} s_1^j \\ \vdots \\ s_n^j \end{pmatrix}
\end{tikzcd}$$

$$\Big\downarrow \pi^*$$

$$\begin{tikzcd}
 & \vec{s}^{\,i} \otimes 1 \arrow[dd, "\pi^* T_{ij}"] \\
s \otimes 1 \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & \vec{s}^{\,j} \otimes 1
\end{tikzcd}$$

$\mathcal{O}_X$-mods $\supseteq$ Quasicoherent sheaves $\supseteq$ locally free sheaves

Ab-cat Ab-cat not Ab-cat.

no cokernel

coherent sheaves

Def: An A-mod M is coherent if

(1) M f.g

(2) $\forall$ map $A^{\oplus p} \to M$ has a f.g kernel.

Prop 6.4.3: coherent mods is an ab-cat

Assume locally Noe:

Quasicoherent $\supseteq$ coherent $\supseteq$ locally free.

Construct locally free sheaves: Hom, dual, $\otimes$.

14.1.C. $\mathcal{F}, \mathcal{G}$ locally free of rank m, n resp.

$\Rightarrow$ $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ locally free of rank mn

$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ sheaf).

pf: $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^m, \mathcal{O}_X^n) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)^{mn}$

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X(X).$$

$$f \longmapsto \{f(U)(1) \mid U \subseteq X\}.$$

$$f(U)(1) = a|_U \longleftarrow a \qquad \square$$

14.1 D. $\mathcal{F}$ locally free sheaf of rank n,

$\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ - ——————, called the dual of $\mathcal{F}$.

① Given transition funcs of $\mathcal{F}$, describe the ... of $\mathcal{F}^\vee$.

② Show $\mathcal{F} \cong \mathcal{F}^{\vee\vee}$ canonically.

pf: ① $\mathcal{F}$ $[T_{ij}] \rightsquigarrow \mathcal{F}^\vee$ $[T_{ji}]^{-1}$.

② Lemma: $\mathcal{O}_X \cong \mathcal{O}_X^\vee$ canonically

Pf of lemma: $\forall U \hookrightarrow X$,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)(U)$$
$$\|$$
$$\varphi_U: \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) \to \mathcal{O}_X(U).$$
$$g \longmapsto g(U)(1). \qquad \text{surj.}$$

$$g(U)(1)=0 \Rightarrow \forall V \hookrightarrow U.\ g(V)(1) = g(U)(1)|_V = 0.$$
$$\Rightarrow g=0.$$

$\Rightarrow \varphi_U$ inj.

$\forall V \hookrightarrow U$,

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) \arrow[r, "\varphi_U"] \arrow[d, "Res"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}_X(U) \arrow[d, "Res"] \\
\operatorname{Hom}_{\mathcal{O}_V}(\mathcal{O}_V, \mathcal{O}_V) \arrow[r, "\varphi_V"'] & \mathcal{O}_X(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, mapsto] \arrow[d, mapsto] & g(U)(1) \arrow[d, mapsto] \\
g|_V \arrow[r, mapsto] & g(V)(1) = g(U)(1)|_V
\end{tikzcd}$$

$\rightsquigarrow$ isom $\varphi: \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{O}_X$.

lemma ✓.

$\Rightarrow \exists$ canonical isom $\mathcal{O}_X \to \mathcal{O}_X^{\vee\vee}$.

Generally case:

$$\begin{tikzcd}
 & \mathcal{O}_X^n \arrow[r, "\sim"] \arrow[dd, "i_2 \circ i_1^{-1}"] & (\mathcal{O}_X^{\vee\vee})^n \arrow[r, "\sim"] \arrow[dd] & (\mathcal{O}_X^n)^{\vee\vee} \arrow[dr, "i_1^{\vee\vee}"] \arrow[dd, "i_2^{\vee\vee} \circ (i_1^{\vee\vee})^{-1}"] & \\
\mathcal{F} \arrow[ur, "i_1", "\sim"'] \arrow[dr, "i_2"', "\sim"] & & & & \mathcal{F}^{\vee\vee}. \\
 & \mathcal{O}_X^n \arrow[r, "\sim"] & (\mathcal{O}_X^{\vee\vee})^n \arrow[r, "\sim"] & (\mathcal{O}_X^n)^{\vee\vee} \arrow[ur, "i_2^{\vee\vee}", "\sim"'] &
\end{tikzcd}$$

$\square$

14.1.E. ① $\mathcal{F}, \mathcal{G}$ locally free sheaves

$\Rightarrow \mathcal{F}\otimes\mathcal{G}$ locally free sheaves

② If $\mathcal{F}$ invertible, then $\mathcal{F}\otimes\mathcal{F}^{\vee} \cong \mathcal{O}_X$.

($\widetilde{\mathcal{F}\otimes\mathcal{G}}(U) := \mathcal{F}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{G}(U)$. $\widetilde{\mathcal{F}\otimes\mathcal{G}}$ presheaf

$\mathcal{F}\otimes\mathcal{G} = \widetilde{\mathcal{F}\otimes\mathcal{G}}^{\#}$) (ex. 2.6.K)

Pf: ① wlog let $\mathcal{F} = \mathcal{G} = \mathcal{O}_X$.

$\widetilde{\mathcal{O}_X \otimes \mathcal{O}_X}(U) := \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \cong \mathcal{O}_X(U)$.

$\Rightarrow \mathcal{O}_X \otimes \mathcal{O}_X = \widetilde{\mathcal{O}_X \otimes \mathcal{O}_X} \cong \mathcal{O}_X$.

② Wlog let $\mathcal{F} = \mathcal{O}_X$, then $\mathcal{F}^\vee \cong \mathcal{O}_X \Rightarrow \mathcal{F}^\vee \otimes \mathcal{F} \cong \mathcal{O}_X$. □.

14.1.F: $\mathcal{F}$ locally free, $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ exact seq of $\mathcal{O}_X$-mods.

$\Rightarrow \mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$ exact.

pf: Wlog let $\mathcal{F} = \mathcal{O}_X$, $\forall p \in X$.

$$(\mathcal{G}' \otimes \mathcal{O}_X)_p = \mathcal{G}'_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p} = \mathcal{G}'_p.$$

✓. □

14.1.G. $\mathcal{E}$ locally free of fin rank, $\mathcal{F}, \mathcal{G} \in \mathcal{O}_X$-mod

then $\mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}) \cong \mathcal{H}om(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G}) \cong \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}$

In parti, $\mathcal{D}$ locally free of fin rank $\Rightarrow \mathcal{H}om(\mathcal{D},\mathcal{G}) \cong \mathcal{D}^\vee \otimes \mathcal{G}$

Pf: ① Wlog let $\mathcal{E} = \mathcal{O}_X$. ✓.

② Wlog let $\mathcal{D} = \mathcal{O}_X$.

$\mathcal{H}om(\mathcal{O}_X, \mathcal{G}) \xrightarrow{\sim} \mathcal{O}_X^\vee \otimes \mathcal{G} = \mathcal{G}$.

$\forall U \hookrightarrow X$, $f|_U \longmapsto f(U)(1)$. □.

The Picard Group

14.1.H. $\mathrm{Pic}\, X := \{\text{invertible sheaves on } X\}/_{\text{isom}}$

is an ab gp under tensor product.

14.1.I. Pic is a contravariant functor

$X \longrightarrow Y$.

$\rightsquigarrow \mathrm{Pic}\, Y \xrightarrow{\pi^*} \mathrm{Pic}\, X$.

(Verify on the stalks $\Rightarrow \pi^*$ is a hom of ab gps).

14.1.J I have no complex-analytic background.

14.1.K. K number field. $X = \operatorname{Spec} \mathcal{O}_K$,

$$\{\text{Fractional ideals of } \mathcal{O}_K\} \longrightarrow \operatorname{Pic} X.$$

$$\mathfrak{a} \longmapsto [\tilde{\mathfrak{a}}]$$

$$\mathfrak{a}\mathfrak{a}^{-1} = \mathfrak{a} \otimes_{\mathcal{O}_K} \mathfrak{a}^{-1} \longmapsto \widetilde{\mathfrak{a} \otimes \mathfrak{a}^{-1}} = \tilde{\mathcal{O}}_K = [\mathcal{O}_X]$$

principal ideal $(a) \mapsto [\widetilde{(a)}] = [\tilde{\mathcal{O}}_K] = [\mathcal{O}_X]$

$\rightsquigarrow \psi : Cl(K) \longrightarrow \operatorname{Pic} X$. Show ψ isom.

pf: ψ inj ✓.

ψ surj: $\mathcal{F}$ invertible sheaf on X,

$\mathcal{F}$ quasi coherent $\Rightarrow \mathcal{F} = \tilde{M}$, M is an invertible

$\mathcal{O}_K$-mod. $\exists\, \mathcal{O}_K$-mod N s.t. $M \otimes_{\mathcal{O}_K} N = \mathcal{O}_K$.

$$0 \to \ker \to \underbrace{F}_{\text{Free } \mathcal{O}_K\text{-mod}} \twoheadrightarrow M \to 0$$

free $\mathcal{O}_k$ mod.

$$\otimes_{\mathcal{O}_k} N \rightsquigarrow 0 \to \ker \otimes_{\mathcal{O}_k} N \to F \otimes_{\mathcal{O}_k} N \twoheadrightarrow M \otimes_{\mathcal{O}_k} N \to 0$$

$$\begin{tikzcd} F \otimes_{\mathcal{O}_k} N \arrow[r, two heads] & M \otimes_{\mathcal{O}_k} N \\ & \mathcal{O}_k \arrow[u, phantom, "\cup"] \arrow[ul, bend left] \end{tikzcd}$$

$$\otimes_{\mathcal{O}_k} M \rightsquigarrow \begin{tikzcd} 0 \arrow[r] & \ker \arrow[r] & F \arrow[r, two heads] & M \arrow[l, bend left] \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow M$ projective $\Rightarrow M$ torsion-free.

$M \hookrightarrow M_k = K \otimes_{\mathcal{O}_k} M$. inj

$m \mapsto 1 \otimes m$. $\quad (M \otimes_{\mathcal{O}_k} N = \mathcal{O}_k \hookrightarrow K = K \otimes_{\mathcal{O}_k} M \otimes_{\mathcal{O}_k} N)$.

$$K = \mathcal{O}_k \otimes_{\mathcal{O}_k} K = M \otimes_{\mathcal{O}_k} N \otimes_{\mathcal{O}_k} K$$

$$= (M \otimes_k K) \otimes_k (K \otimes_k N)$$

$$= M_k \otimes N_k$$

$\Rightarrow \dim M_k = 1$.

$\rightsquigarrow M \hookrightarrow K$ inj.

$$M \otimes N \cong \mathcal{O}_k.$$

$$\sum_{i=1}^{p} m_i \otimes n_i \quad \mapsto 1$$

$M' = (m_1, \ldots, m_p) \subseteq M$.

$$0 \to M' \to M \to M/M' \to 0.$$

$$\rightsquigarrow 0 \to \underset{\substack{\cong \\ O_K}}{M' \otimes_{O_K} N} \hookrightarrow \underset{\substack{\cong \\ O_K}}{M \otimes_{O_K} N} \to \underset{\substack{\cong \\ 0}}{(M/M') \otimes_K N} \to 0.$$

$\Rightarrow M/M' = 0 \Rightarrow M$ f.g. $\Rightarrow M \cong$ an fractional ideal.

$\Leftarrow$) surj ✓. $\square$