

16.3.

[16.3.1.]

C/k integral finite type of dim 1

then \Rightarrow regular integral projective C/k of dim 1 birational to C

proof rational map defines on open sets.

wrt C affine. By Noether Normalization.

$\exists x \in K(C)$ s.t. $K(C)/k(x)$ finite. ($C = \text{Spec } A/k(x)$ finite)

Recall. [0.7.1].

X integral $X \xrightarrow{\text{normalization}} X'$ wrt. $L/k(x)$

locally it is $A \hookrightarrow B$ where

B is integral closure of A in L .

Now we take $C' := \widehat{A_k} \rightarrow A_k$ wrt. $K(C)/k(x)$

by (0.7.1). $K(C)/k(x)$ finite $\Rightarrow C' \rightarrow A_k$ finite.

by (2.1.G.) $\dim C' = \dim A_k = 1$. C' integral.

C' Noetherian since C'/A_k finite. of finite type.

By 13.5. f. \forall closed point x

$O_{C,x}$ Noetherian local normal of dim 1. $\Rightarrow O_{C,p}$ regular

on generic point $K(C')$ is regular.

then C' regular, C' projective by 16.2.G.

C/k separated reduced of finite type $\Rightarrow C$ is k -variety.

$C \rightarrow A_k$ separated (local on the target) then C' is k -variety

$K(C) = K(C) \Rightarrow C \xrightarrow{\sim} C$ birational by 7.5.E.

$C \rightarrow \mathbb{P}^n$ finite $\mathbb{A}^n \rightarrow k$ projective $\Rightarrow C$ projective by 16.2.G.

[16.3.2]

C irreducible regular k -variety of dim 1.

then $\exists C \xrightarrow{\sim} C'$ birational. C' projective regular integral curve

proof Simple case: C affine.

then $C \xrightarrow{\text{normalize}} C'$ since Normalization is local

$\pi_1 \text{finc}$ $\pi'_1 \text{finc}$
 $A_k \hookrightarrow \mathbb{A}'_k$

C already normalization of A_k

$\pi_1(A_k) \hookrightarrow C$ i.e. $C \xrightarrow{\sim} C'$

In general $\forall C, C \xrightarrow{\sim} C'$ affine, we have $C \xrightarrow{\sim} \widetilde{C}$

where \widetilde{C} regular projective curve.

then by 15.3.1. $C \xrightarrow{\sim} \widetilde{C}$ extends to $C \rightarrow \widetilde{C}$ over finite points.

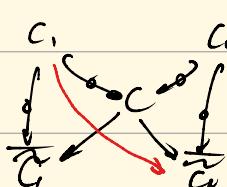
$C \rightarrow \widetilde{C}$ birational since $C \xrightarrow{\sim} \widetilde{C}$ birational.

In topology, since [points of $C \hookrightarrow$ (valuation of $K(C)$)] = [points of \widetilde{C}]

we have $C \hookrightarrow \widetilde{C}$ projective. $C \xrightarrow{\sim} \widetilde{C} \Rightarrow C \xrightarrow{\sim} \widetilde{C}$

it suffices to show $C \hookrightarrow \widetilde{C}$ open immersion

$\forall p \in C$ C_p affine open neighborhood

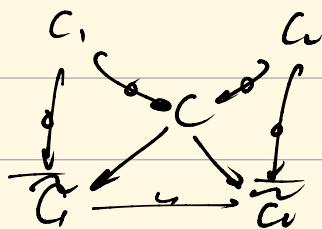


$C \rightarrow \widetilde{C}$ extends to $\pi_{12}: \widetilde{C}_1 \rightarrow \widetilde{C}_2$

also we have $\pi_{21}: \widetilde{C}_2 \rightarrow \widetilde{C}_1$

then $\pi_{12} = \pi_{21}^{-1}$ on $C_1 \cap C_2$

then $\pi_{12} \circ \pi_{21}^{-1}$ by 11.3.2 (agree locus), we have



then $C_1 \rightarrow \bar{C}$ open immersion

$C \rightarrow \bar{C}$ open immersion near P .

then $C \rightarrow \bar{C}$ open immersion (local on the target)

[16.3.A]

we take $C \hookrightarrow \bar{C}$ \mathcal{O}/k proper $\xrightarrow{\text{(cancellation)}}$ \mathcal{O}/\bar{C} proper

then \bar{C} integral $\Rightarrow C - \bar{C}$ C projective

[16.3.3.]

TFAE

① integral smooth projective curves with surjective morphism

② integral smooth pro-— with dominant —

③ — — — dominant rational.

④ integral curve dominant rational map.

⑤ opposite category of function fields/ k .

proof. we have inclusion of category $① \rightarrow ② \rightarrow ③ \rightarrow ④$

from ④ to ① : given $C_1 \rightarrow C_2$ we have $U \xrightarrow{f} C_2$

$\bar{U} \rightarrow \bar{C}_2$ normalization.

$\bar{U} \rightarrow \bar{U}' \quad \bar{C}_2 \rightarrow \bar{C}'_2$ then $\bar{U}' \rightarrow \bar{C}'_2$ extends to $\bar{U} \xrightarrow{g} \bar{C}_2$

$f(\eta_{\bar{U}}) = \eta_{\bar{C}_2} \Rightarrow f(\tilde{\eta}_{\bar{U}'}) = \eta_{\bar{C}'_2}$ then $\tilde{f}(\eta_{\bar{U}'}) = \eta_{\bar{C}_2}$

$\tilde{f}(\eta_{\bar{U}'}) = \eta_{\bar{C}_2}$ i.e. \tilde{f} dominant, \tilde{f} surjective since f proper

16.3.B

check

(?)

16.3.4

$\pi: C \rightarrow C'$ surjective map of regular integral projective curves

① First we show π is finite.

Let C'' Normalization of C w.r.t. $K(C)/K(C')$

then $K(C) = K(C'')$ i.e. $C \hookrightarrow C''$ which extends to $C \supseteq C'$

then $C \supseteq C'$, but $C'' \rightarrow C'$ finite.

② Next we show $\pi_* \mathcal{O}_C$ locally of finite rank.

WMA C' affine, then C affine, $\pi_* \mathcal{O}_C$ of finite type
since C' reduced by 16.3.K

$\pi_* \mathcal{O}_C$ locally free $\Leftrightarrow \pi_* \mathcal{O}_C$ constant rank

We show that rank on closed point = rank on generic point.

assume $C' = \text{Spec } A'$ $p = [n]$ closed point $C = \text{Spec } A$

$$\dim \pi_* \mathcal{O}_C \otimes K(p) = \dim_{A/m} A/m^n \quad \dim \pi_* \mathcal{O}_C \otimes K(\eta) = \dim_{\text{Frac } A} (A \setminus 0)^{-1} A$$

$\frac{\text{Frac } A}{A}$ $\frac{A/m}{A} = A/m$

Claim. A_m is finite rank free A'_m -mod
 $\frac{\text{Frac } A'}{(A'_m)^{-1} A}$

If claim is true $A_m \otimes \text{Frac } A'$ is free $\text{Frac } A'$ -mod of dim n

$$A_m \otimes_{A_m} \frac{A_m}{A_m} = A'_m \otimes_{A'_m} \frac{A'_m}{A'_m} = A'_m \text{ is free } A'_m \text{-mod of dim n}$$

Proof of claim: $A_{n'}$ is DVR $\Rightarrow A_{n'}$ is PID + uniformizer

A_n is P_S . $A_{n'}$ -mod, then $A_n = \bigoplus_{i=1}^k A_{n'}/(t^{n_i}) \oplus (A_{n'})^r$

If $\text{Tor}(A_n) \neq 0$ t is zero divisor of A_n but $A_{n'}$ integral

□

1b.3.b Def rank $\pi \otimes_C =: \deg \pi$.

1b.3.c $\deg \pi = [K(C) : K(C')]$ since $\deg \pi = \dim_{\text{Frac } A} (A[0])^\wedge$

$$\forall x \in A[0] \quad x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$$

$$\text{then } x^{-1} = -\left(\frac{a_0}{a_1} x^{1-1} + \frac{a_1}{a_2} x^{2-1} + \dots + \frac{a_{n-1}}{a_n}\right) \text{ i.e. } (A[0])^\wedge = \text{Frac } A$$

$$\deg \pi = \dim_{\text{Frac } A} \text{Frac } A = [K(C) : K(C')]$$

1b.3.d

(a) Recall $\deg p = [k(p) : k]$ when C, C' affine.

then assume $U = \text{Spec } A$ $V = \text{Spec } B$

$$\pi^{-1}(p) = \text{Spec } B \otimes_{k(p)} K(p) = \text{Spec } B \otimes_{A[0]} A_{n'}/m_{A_{n'}} = \text{Spec } B/mB$$

$$T(\pi^{-1}(p), \mathcal{O}_{V,p}) = B/mB \quad \dim_{k(p)} B/mB = \dim_{k(p)} B/mB \times \dim_k K(p) = \deg \pi \deg p.$$

$$(b) \quad \deg(\pi) = \dim_{k(p)} B/mB \quad \pi^{-1}(p) = \{\text{pt Spec } B \mid p \cap A = m\}$$

B/mB finite $k(p)$ -algebra $\Rightarrow B/mB = T(B/mB)_p$

$$\text{then } \deg \pi = \dim_{k(p)} B/mB = \sum \dim_{k(p)} (B/mB)_p = \sum \dim_{k(p)} (B_p/mB_p)$$

now B_P/\mathfrak{m} extension of DVR. $\dim_{k(P)} B_P/\mathfrak{m} B_P = \dim_{k(P)} B_P/\mathfrak{p}_P B_P \cdot \dim_{k(P)} \frac{\mathfrak{p}_P B_P}{\mathfrak{m} B_P}$
 along exact sequence $0 \rightarrow \frac{\mathfrak{p}_P B_P}{\mathfrak{m} B_P} \rightarrow \frac{B_P}{\mathfrak{m} B_P} \rightarrow \frac{B_P}{\mathfrak{p}_P B_P} \rightarrow 0$
 then $\dim_{k(P)} B_P/\mathfrak{m} B_P = \text{val}_{p_i}(z^*t) \deg(K(P)/k(P))$ D

Remark: O_2/O_k dimension formula.

(16.3.E)

s rational function on C $s: C \rightarrow \mathbb{P}^1$

$$\text{then } \deg s = \sum_{\text{poles}} \text{val}_{p_i}(z^*t_w) - \sum_{\text{zeros}} \text{val}_{p_i}(z^*t_o)$$

(16.3.8)

deg of line bundle on smooth proj curve.

C irreducible smooth curve \mathcal{L}/C line bundle.

then $\forall s, t$ nonzero rational section of \mathcal{L}

s/rational function \Rightarrow zeros of s-poles of s well-defined

$$\forall \phi, \psi \text{ local trivialization. } \frac{\phi(s)}{\phi(t)} = \frac{\psi(s)}{\psi(t)}$$

$$\text{since } \psi(t)\phi(s) = \phi(t)\psi(s) = u \cdot \phi(t)\phi(s)$$