Topics of Fibered Product.

We will start from 10.4.   08 24 sep FOAG

10.4.A. omitted.

10.4.B.

(a) base change

(b) intersection of two subschemes $X, Y \hookrightarrow Z$ is defined as $X \times_Z Y$ then $X \times_Z Y \hookrightarrow X$ $\underline{|X \times_Z Y| = |X| \cap |Y|}$

$$\begin{tikzcd}
X\cap Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y\cap U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\
X\cap V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U\cap V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\
X \arrow[r, hook] & U \arrow[r, hook] & Z
\end{tikzcd}$$

$$|U \times_Z V| = U \cap V$$

$$|(U \times_Z V) \times_V Y| = (U\cap V) \cap Y = U \cap Y$$

$$|(U \times_Z V) \times_U X| = (U \cap V) \cap X = V \cap X$$

$$|((U \times_Z V) \times_U X) \times_{U \times_Z V} ((U \times_Z V) \times_V Y)| = (U \cap Y) \cap (V \cap X) = X \cap Y$$

10.4.D / 10.4.E

Reasonable or not ?

① qc ② affine ③ finite ④ quasi-finite ⑤ Integral

⑥ locally of finite type ⑦ of finite type ⑧ separated ⑨ qs

⑩ Surjective / <u>injective</u> / <u>bijective</u> / <u>homeomorphism</u> / <u>open</u> / <u>closed</u>

not stable under base change.

Ref of proof.

Δ General Cancellation Law [Gortz] Remark 9.11.

[Gortz]

① prop. 10.3

② prop. 12.3

③ prop. 12.11

④ prop. 12.17.

⑤ prop. 12.11

⑥⑦ prop. 10.7

⑧ prop 9.13

⑨ prop. 10.25 (Lemma 10.26 Affine Communication?)

⑩ prop. 4.32

10.4.G

surjective stable under base change.

Ref. [Gortz] Lemma 4.28.

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

$\exists\, z \in X \times_S Y$ s.t. $p(z) = x$, $q(z) = y$ iff $f(x) = g(y)$

Proof. $\operatorname{Spec} k(x) \longrightarrow X$ $\quad \operatorname{Spec} k(y) \longrightarrow Y$

then $\operatorname{Spec} k(x) \times_S Y = p^{-1}(x)$ $\quad \operatorname{Spec} k(y) \times_S Y = q^{-1}(y)$

$a \longmapsto$

$$\begin{tikzcd}
p^{-1}(x)\times_{X\times_S Y} g^{-1}(y) \arrow[r] \arrow[d] & g^{-1}(y) \arrow[r] \arrow[d] & \mathrm{Spec}\,k(y) \arrow[d] \\
p^{-1}(x) \arrow[r] \arrow[d] & X\times_S Y \arrow[r, "g"] \arrow[d, "p"] & Y \arrow[d, "g"] \\
\mathrm{Spec}\,k(x) \arrow[r] & X \arrow[r, "f"] & S
\end{tikzcd}$$

$p^{-1}(x)\cap g^{-1}(y)$

$p^{-1}(x)\times g^{-1}(y) = \mathrm{Spec}\,k(x)\times_S \mathrm{Spec}\,k(y)$

$(\Longleftarrow)$

since $\mathrm{Spec}\,k(y)$, $\mathrm{Spec}\,k(x)$ factor through $\underline{\mathrm{Spec}\,k(s)}$ $s=f(x)=g(y)$

$p^{-1}(x)\times g^{-1}(y) = \mathrm{Spec}\,k(x)\otimes_{k(s)} k(y) \neq \emptyset$
Zorn's Lemma

Assume $X\to S$ surjective then $\forall y\in Y$ take $x\in f^{-1}(g(y))$
then $\exists z$ $p(z)=x$ $g(z)=y$, then g surjective

10.4.1

Injective not stable under base change.

proof. $\mathbb{R}\hookrightarrow\mathbb{C}$ $\mathrm{Spec}\,\mathbb{C}\to\mathrm{Spec}\,\mathbb{R}$ bijective

by base change $\mathrm{Spec}\,\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}\to\mathrm{Spec}\,\mathbb{C}$ not injective

if injective $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ field. $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}\cong \mathbb{R}[x]/(x^2+1)\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}^2$

consider $\mathbb{C}\hookrightarrow\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ then $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}/\mathbb{C}$ algebraic, then $\mathbb{C}=\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$
$x\mapsto x\otimes x$ $(\forall x\otimes y\in\mathbb{C}$ $m_x(x\otimes y)=0\otimes m_x(y)=0)$

contradiction.

Remark: if not stable under base change, use "universally xxx"

10.4.H.

X, Y Integral of finite type $/\bar{k}$ then $X \times_{\bar{k}} Y$ integral of finite type

proof it suffices to show integral

WMA X, Y affine $X = \operatorname{Spec} A$ $Y = \operatorname{Spec} B$ A, B integral.

suppose $(\sum a_i \otimes b_i)(\sum a_j' \otimes b_j') = 0$

WMA $\{b_i\}$ $\{b_j'\}$ linearly independent $/\bar{k}$

WMA $a_i, b_i, a_j', b_j' \neq 0$ then $D(a_1 a_1') \neq \emptyset$

(prime avoidance)

By 3.2.5 $\exists m \in \max\operatorname{Spec} A$ $m \in D(a_1 a_1')$ $A/m = \bar{k}$

then $A \otimes_{\bar{k}} B \in A\text{-Mod-}B$ then in $A\otimes B / m A\otimes B$

$$0 \to m \hookrightarrow A \to \bar{k} \to 0$$

$$\begin{tikzcd}
m\otimes B \arrow[r] \arrow[dr] & A\otimes B \arrow[r] & A\otimes B/m(A\otimes B) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
 & m(A\otimes B) \arrow[u] & \bar{k}\otimes B \arrow[r, hook] & B
\end{tikzcd}$$

$\sum(\bar{a}_i \otimes b_i) \sum(\bar{a}_j' \otimes b_j') = 0$ in B then $(\sum \bar{a}_i b_i) \cdot (\sum \bar{a}_j' b_j') = 0$

WMA $\sum \bar{a}_i b_i = 0$ then since $\bar{a}_1 \neq 0$ (otherwise $\bar{a}_1 \neq 0$)

and b_i linear independant contradiction. $\square$

$$\begin{tikzcd}
X\times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] & g^{-1}(s) \arrow[l] \arrow[d] \\
X \arrow[r, "f"] & S \arrow[ur, phantom, "\square"] & \operatorname{Spec} k(s) \arrow[l]
\end{tikzcd}$$

10.5.1. pull back of fibers ($g^{-1}(s) \to k(s)$)

if $f(x)=s$ we have
$$\begin{tikzcd} k(x)\times_{k(s)} g^{-1}(y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & g^{-1}(s) \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r] & \operatorname{Spec} k(s) \end{tikzcd}$$

$k(x)\times_{k(s)} g^{-1}(y) \longrightarrow \operatorname{Spec} k(x)$ if fiber of $X\times_S Y \longrightarrow X$

10.5.A.
$$\begin{tikzcd} \operatorname{Spec} k(u)\otimes_{k(u^p)} k(u) \arrow[r] \arrow[d] & \operatorname{Spec} k(u) \arrow[d] \\ \operatorname{Spec} k(u) \arrow[r] & \operatorname{Spec} k(u^p) \end{tikzcd}$$
$\operatorname{Spec} k(u)$ ← reduced

show $\operatorname{Spec} k(u)\otimes_{k(u^p)} k(u)$ not reduced.

Consider $x = u\otimes 1 - 1\otimes u$, then $x^p = u^p\otimes 1 - 1\otimes u^p = 0$

□

Inspired by 10.4.H & 10.5.A

△ Want to define Connectness / irre-- / integral / reduce on morphism of schemes.

10.5.2. (Geometric Point)

① A Geometric point is $\operatorname{Spec} k \to X$ where k algebraically closed.

geometric fiber of $X\to Y$ is $X\times_Y K \to \operatorname{Spec} K$
$$\begin{tikzcd} X\times_Y K \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
$(k(y) \to \overline{k(y)} \to K)$

or Rising Sea 10.5.F
Stacks Project Section 33.6/7/8/9

here
$$\begin{tikzcd} X \times_k K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d] \\ \widetilde{f^{-1}(y)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \overline{k(y)} \arrow[d] \\ f^{-1}(y) \arrow[r] & \operatorname{Spec} k(y) \end{tikzcd}$$

[since $X \times_k \bar{k}$ connected/—— - -
$\Rightarrow X \times_k L$ connected/—— - $\forall L/k$]

it suffices that $\widetilde{f^{-1}(y)}$ connected ——.

(different K is different geo points)

② A morphism has geometrically connected/irreducible/integral/reduced fibers if every geometric fiber is - —— - -

③ X is geometrically - — if X/k geometrically - — -

10.5.B geometrically connected/irreducible/integral/reduced fibers stable under base change

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "q"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

$X \to S$ has geometrically connected/irreducible/integral/reduced fibers

if $g(y) = s$

then

$\forall \operatorname{Spec} k(y) \to Y \quad s := g(y) \qquad k(s) \hookrightarrow k(y) \rightsquigarrow \overline{k(s)} \hookrightarrow \overline{k(y)}$

(geometric)
$$\begin{tikzcd} \widetilde{f^{-1}(y)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \overline{k(y)} \arrow[d] \\ X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}
\qquad
\begin{tikzcd} \widetilde{f^{-1}(y)} \arrow[r] \arrow[d] \arrow[dd, bend right, dashed] & \operatorname{Spec} \overline{k(y)} \arrow[d] \arrow[dd, bend left] \\ X \arrow[r] \arrow[dr, phantom, "\square"] & S \\ \widetilde{f^{-1}(s)} \arrow[r] \arrow[u] & \operatorname{Spec} \overline{k(s)} \arrow[u] \end{tikzcd}$$

$$\Longrightarrow \begin{tikzcd} \widetilde{f^{-1}(y)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}\,\widetilde{k(y)} \arrow[d, "s"] \\ \widetilde{f^{-1}(s)} \arrow[r, "f"] & \mathrm{Spec}\,\widetilde{k(s)} \end{tikzcd}$$ where $\widetilde{f^{-1}(s)}$ —— -

it suffices to prove if $X/_{k=\bar{k}}$ connected / irreducible / integral / reduced, L/k $L=\bar{L}$

then $X \times_k L$ is ·· —— -

$X \times_k \bar{k}$ reduced / —— iff $X \times_k k'$ reduced / —— for every extension

take $X = f^{-1}(s)$ here $\square$

Theorem. having geometrically connected / irreducible / integral / reduced fibers is reasonable.

base change ✓

local on the target: if $f^{-1}(U_i) \to U_i$ geometry —— .

then
$$\begin{tikzcd} f^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & f^{-1}(U_i) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ \mathrm{Spec}\,k(x) \arrow[r] & U_i \arrow[r, hook] & X \end{tikzcd}$$
then $f^{-1}(x)$ ——

composition: if $X \to Y$ $Y \to Z$ geometry P

then $\forall$ $\mathrm{Spec}\,\widetilde{k(x)}$ $\widetilde{g f^{-1}(x)} \to Z$, $\widetilde{f^{-1}(x)}$ is P
$$\begin{tikzcd} \widetilde{gf^{-1}(x)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d, "g"] \\ f^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ \mathrm{Spec}\,k(x) \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd}
g^{-1}(y) \arrow[r] \arrow[d] \arrow[rr, bend left, red] \arrow[dr, phantom, "\square"] & Z \arrow[d] \arrow[dr, phantom, "\square"] & (g\circ f)^{-1}(x) = Z \times_X \operatorname{Spec} k(x) \arrow[l] \arrow[d] \\
\operatorname{Spec} k(y) \arrow[r] \arrow[rr, bend right, red] \arrow[dr, bend right] & Y \arrow[dr, bend left] & f^{-1}(x) = Y \times_X \operatorname{Spec} k(x) \arrow[l] \\
& \operatorname{Spec} k(x) \arrow[r] & X
\end{tikzcd}$$

then $\forall y \in f^{-1}(x)$

then
$$\begin{tikzcd}
g^{-1}(y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (gf)^{-1}(x) \arrow[d, "g"] \\
\operatorname{Spec} k(y) \arrow[r] & f^{-1}(x)
\end{tikzcd}$$

then
$$\begin{tikzcd}
g^{-1}(y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (gf)^{-1}(x) \arrow[d] \\
\operatorname{Spec} k(y) \arrow[r] & \operatorname{Spec} k(x)
\end{tikzcd}$$

$g^{-1}(y)$ integral / — — — .

$\Rightarrow$ $(gf)^{-1}(x)$ integral / — —

$\square$

10.5.C/D/E/F/G omitted.

Cancellation:

$$\begin{tikzcd}
\operatorname{Spec} k(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k(x) \arrow[d] \\
U \arrow[r, hook] & X
\end{tikzcd}$$

$(x \in U)$ then open immersion is · —

$$\begin{tikzcd}
f^{-1}(x) \times_k k(x) \arrow[r, hook] \arrow[d] & \operatorname{Spec} k(x) \arrow[d] \\
Z \arrow[r, hook, "f"] & X
\end{tikzcd}$$

$(x \in Z)$

Locally
$$\begin{tikzcd}
A/I \otimes_A k(x) \arrow[r] \arrow[d] & \operatorname{Spec} k(x) \arrow[d] \\
\operatorname{Spec} A/I \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

$(\mathfrak{p} \in V(I) \quad I \subseteq \mathfrak{p})$

$A/I \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \dfrac{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}{I A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}} = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ then closed —— geo——

Immersion is geometrically P $\Rightarrow$
$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & Z \arrow[d] \\
& Y
\end{tikzcd}$$
X/Y · — $\Rightarrow$ X/Z - -

then if
$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & k \\
& Z \arrow[u]
\end{tikzcd}$$
X/Z geo - — fiber then X/k - - -

Which is $X_{\bar{k}}$ — , goes back

10.5.H.

$\operatorname{Spec} E \longrightarrow \operatorname{Spec} F \qquad E/F$

① if $\operatorname{tr.deg}_F E \geq 1$ then φ is not universally injective.

② if $E\backslash F$ contains a separable element φ is not ······

③ if E/F purely inseparable, $|\operatorname{Spec} E \otimes_F E'| = 1$.

proof ①
$$\begin{tikzcd}
E \otimes_F F(y) \arrow[r, two heads] \arrow[d] & F(x) \otimes_F F(y) \arrow[r] \arrow[d] & F(y) \arrow[d] \\
E \arrow[r, two heads] & F(x) \arrow[r] & F
\end{tikzcd}$$
where x transcen— $F(y)/F$ purely tr…

$$F(x) \otimes_F F(y) = S^{-1}F[x] \otimes S'^{-1}F[y] \cong (S_x \otimes S_y)^{-1}(k[x,y])$$

$$S_x \otimes S_y \cong \{ pg \mid p \in F[x]\backslash 0,\ g \in F[y]\backslash 0 \} \quad \text{then}$$

$$\operatorname{Spec} F(x) \otimes_F F(y) = \{ p \in \operatorname{Spec} F[x,y] \mid p \cap S = \emptyset \} \supseteq \{ f(x,y) \text{ prime} \mid f \notin F[x], F[y] \}$$

then x transcen-- $\Rightarrow |\operatorname{Spec} F(x) \otimes_F F(y)| \geq 2 \Rightarrow |\operatorname{Spec} E \otimes F(y)| \geq 2$

Since $\operatorname{Spec} F(y) \longrightarrow \operatorname{Spec} F$ injective, $\operatorname{Spec} E \otimes_F F(y) \longrightarrow \operatorname{Spec} E$ not

② $x \in E\backslash F$ m separable then $F(x) \cong F[x]/m$

$$\begin{tikzcd}
\operatorname{Spec} E \otimes \overline{F} \arrow[r] \arrow[d] & \operatorname{Spec}(F[t]/m) \otimes_F \overline{F} \arrow[r] \arrow[d] & \operatorname{Spec} \overline{F} \arrow[d] \\
\operatorname{Spec} E \arrow[r] & \operatorname{Spec}(F[t]/m) \arrow[r] & \operatorname{Spec} F
\end{tikzcd}$$
$m = \prod_1^n (t - x_i)$ in $\overline{F}$

$$(F[t]/m) \otimes_F \overline{F} \cong ((F[t]/m) \otimes_{F[t]} F[t]) \otimes_F \overline{F} \cong F[t]/m \otimes_{F[t]} \overline{F}[t]$$
$$\cong \overline{F}[t]/m \cong \prod_1^n \overline{F}$$

$\mathrm{Spec}(F[t]/m) \otimes_F \bar{F} = \coprod \mathrm{Spec}\,\bar{F}$

$\mathrm{Spec}\,E \longrightarrow \mathrm{Spec}(F[t]/m) \Rightarrow \mathrm{Spec}\,E\otimes\bar{F} \longrightarrow \mathrm{Spec}(F[t]/m)\otimes_F \bar{F}$

then $|\mathrm{Spec}(E\otimes\bar{F})| \geq n \geq 2$

$\mathrm{Spec}\,E\otimes\bar{F} \longrightarrow \mathrm{Spec}\,E$ not injective

③ $\mathrm{char}\,F = p > 0$ $\quad x = \sum_1^n a_i \otimes b_i \in E \otimes_F E'$

then $\exists m_i$ $\quad a_i^{p^{m_i}} \in F$ $\quad m = \max\{m_1, \cdots m_n\}$

$x^{p^m} = \sum a_i^{p^m} \otimes b_i^{p^m} = 1 \otimes \sum a_i^{p^m} b_i^{p^m} \in E'$ then

$x^{p^m} = 0$ or unit hence x is unit or nilpotent.

$E \otimes_F E'$ is a local ring. $\quad (J(-) = nil(-))$

then $\mathrm{Spec}\,E\otimes_F E' = \{m\}$ as desired.

10.5.I.

λ is universally injective $\Leftrightarrow$ λ injective and $k(p)/k(\lambda(p))$ purely insep

proof ($\Rightarrow$) if $p \in X$ $\quad k(p)/k(\lambda(p))$ not purely separable

by
$$\begin{tikzcd} Z\times_Y X \arrow[r, "t"] \arrow[d, "\lambda"'] & X \arrow[d, "f"] \\ \mathrm{Spec}\,L = Z \arrow[r, "s"] & Y \end{tikzcd} \quad \text{take } L, \quad \begin{tikzcd} k(p)\otimes L \arrow[r] \arrow[d, "\text{not injective}"] & \mathrm{Spec}\,k(p) \arrow[d] \\ L \arrow[r] & \mathrm{Spec}\,k(\lambda(p)) \end{tikzcd}$$

we have $Z\times_Y X \longrightarrow Z$ injective, then $|Z\times_Y X| = 1$

but $\mathrm{Spec}\,k(p)\otimes L \hookrightarrow \lambda^{-1}(*) \cap t^{-1}(p)$ then contradiction

$(\Leftarrow)$ $\forall\, p_1', p_2' \in X \times_Y Z$ $\quad \pi(p_1') = \pi(p_2') = \mathfrak{q} \in Z$ then

$p_1 := t(p_1')$ $\quad p_2 = t(p_2') \in X$

then $\pi(p_1) = \pi(p_2) = s(\mathfrak{q}) \xrightarrow{\pi \text{ injective}} p_1 = p_2 = p$

$$\begin{tikzcd} X \times_Y Z \arrow[r, "t"] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\ Z \arrow[r, "s"] & Y \end{tikzcd}$$

then $\operatorname{Spec} k(\mathfrak{q}) \otimes_{k(s(\mathfrak{q}))} k(p) \xrightarrow{\sim} t^{-1}(p) \cap \pi^{-1}(\mathfrak{q}) \supseteq \{p_1', p_2'\}$

then $|\operatorname{Spec} k(\mathfrak{q}) \otimes_{k(s(\mathfrak{q}))} k(p)| = 1 \Rightarrow p_1' = p_2'$

$\square$

10.5.5/6/7/J omitted.

10.5.K $\phi: X \to Y$ open nonempty connected fibers

$\Rightarrow \Phi: \underbrace{\pi_0(X) \xrightarrow[\text{set}]{\sim} \pi_0(Y)}_{\text{connected components}}$.

10.5.L X is disconnected iff $\exists e \in \Gamma(X, \mathcal{O}_X)$ $e^2 = e$ $e \neq 0, 1$.

10.5.10.

k separably closed, $A \in k\text{-Alg}$ $\operatorname{Spec} A$ connected

then $\operatorname{Spec} A$ geo-- connected over k.