17.1. Relative spec

Given a scheme $X \in Sch$, let's consider Sch_X, the category of X-schemes.

Recall: Sch_X. Obs: $W \xrightarrow{\mu} X$

Mors
$$\begin{tikzcd} W_1 \arrow[rr, "\varphi"] \arrow[dr, "\mu_1"'] & \arrow[d, phantom, "\circlearrowleft"] & W_2 \arrow[dl, "\mu_2"] \\ & X & \end{tikzcd} \Rightarrow \begin{tikzcd} \mathcal{O}_{W_2} \arrow[r, "\varphi^\#"] & \varphi_* \mathcal{O}_{W_1} \arrow[d, Rightarrow] \\ \mu_{2*}\mathcal{O}_{W_2} \arrow[r, "\mu_{2*}\varphi^\#"] & \mu_{1*}\mathcal{O}_{W_1} \end{tikzcd}$$

Let $\mathcal{R}$ be a quasi-coherent sheaf of algebras on X, i.e. $\mathcal{R}$ is a sheaf of $\mathcal{O}_X$-algebras and $\forall U = \operatorname{Spec} A \hookrightarrow X$, $\mathcal{R}|_U = \widetilde{R}$ for some A-algebra R.

Def. $\operatorname{Spec} \mathcal{R}$ is a functor from Sch_X to Set.

$$(\mu : W \to X) \longmapsto \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{R}, \mu_* \mathcal{O}_W) \cong \operatorname{Hom}_{\mathcal{O}_W}(\mu^* \mathcal{R}, \mathcal{O}_W)$$

$\left(\begin{array}{ll} \mathcal{O}_X \to \mu_*\mathcal{O}_W & \text{is determined by } \mu \\ \mathcal{O}_X \to \mathcal{R} & \text{is the structure morphism of } \mathcal{R} \end{array}\right)$

Fact: Pullback and Pushforward adjointness still holds for sheaf of algebras.

Lemma. $\operatorname{Spec} \mathcal{R}$ is a Zariski-sheaf.

p.f. $\forall W \in Sch_X$, $W = \bigcup_i W_i$, $W_{ij} := W_i \cap W_j$.

$\mu : W \to X$, $W_i \overset{\iota_i}{\hookrightarrow} W$, $W_{ij} \overset{\iota_{ij}}{\hookrightarrow} W$. $\mu_i = \mu \circ \iota_i$, $\mu_{ij} = \mu \circ \iota_{ij}$.

$$\begin{array}{ccccc} (\operatorname{Spec}\mathcal{R})(W) & \longrightarrow & \prod_i (\operatorname{Spec}\mathcal{R})(W_i) & \rightrightarrows & \prod_{i,j} (\operatorname{Spec}\mathcal{R})(W_{ij}) \\ \| & & \| & & \| \\ \operatorname{Hom}_{\mathcal{O}_W}(\mu^*\mathcal{R}, \mathcal{O}_W) & \longrightarrow & \prod_i \operatorname{Hom}_{\mathcal{O}_W}(\mu^*\mathcal{R}|_{W_i}, \mathcal{O}_W|_{W_i}) & \rightrightarrows & \prod_{i,j} \operatorname{Hom}_{\mathcal{O}_W}(\mu^*\mathcal{R}|_{W_{ij}}, \mathcal{O}_W|_{W_{ij}}) \end{array}$$

It's an equalizer by gluing of morphisms of sheaves. □

17.1.A. If $X = \operatorname{Spec} A$, $\mathcal{R} = \widetilde{R}$, then $\operatorname{Spec} \mathcal{R}$ is represented by $(\operatorname{Spec} R \to \operatorname{Spec} A) \in Sch_X$

p.f. $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{R}, \mu_*\mathcal{O}_W) = \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{R}, \mu_*\mathcal{O}_W) \xrightarrow{\sim} \operatorname{Hom}_A(R, \Gamma(W, \mathcal{O}_W)) = \operatorname{Hom}_X(W, \operatorname{Spec} R)$

$\longrightarrow$ take global sections

It remains to find the inverse of $\rightarrow$

Check on $D(f)$'s ! $\forall \alpha$ s.t. $\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & \operatorname{Spec} R \arrow[d, "\nu"] \\ & X = \operatorname{Spec} A \end{tikzcd}$ $\rightsquigarrow \alpha^\#: \mathcal{O}_R \to \alpha_* \mathcal{O}_W \rightsquigarrow \tilde{R} \to \mu_* \mathcal{O}_W$

ν_*

Inverse to each other. $\square$

17.1.3. Prop If $S \xrightarrow{\beta} X$ represents $\operatorname{Spec} R$, $U \hookrightarrow X$, then $S \times_X U \xrightarrow{\beta|} U$ represents $\operatorname{Spec}(R|_U) \in \operatorname{Fun}(\mathrm{Sch}_U)$

pf. $\forall (W \xrightarrow{\nu} U) \in \mathrm{Sch}_U$

$$\begin{tikzcd} W \arrow[r, "\nu'"] \arrow[rr, bend left] \arrow[dr, bend right, "\nu"'] & S \times_X U \arrow[r] \arrow[d] & S \arrow[d, "\beta"] \\ & U \arrow[r, "i"] & X \end{tikzcd} \qquad \mu = i \circ \nu$$

$\operatorname{Hom}_U(W, S \times_X U) = \operatorname{Hom}_X(W, S) = \operatorname{Hom}_{\mathcal{O}_X}(R, \mu_* \mathcal{O}_W)$

$= \operatorname{Hom}_{\mathcal{O}_W}(\mu^* R, \mathcal{O}_W) = \operatorname{Hom}_{\mathcal{O}_W}(\nu^* i^* R, \mathcal{O}_W) = \operatorname{Hom}_{\mathcal{O}_U}(R|_U, \nu_* \mathcal{O}_W)$ $\square$

17.1.B In general, $\operatorname{Spec} R$ is representable

pf. Write $X = \bigcup_i U_i$ s.t. $R|_{U_i} = \tilde{R}_i$ where $U_i = \operatorname{Spec} A_i$ and R_i is an A_i-algebra

Let $S_i = \operatorname{Spec} R_i \ (\xrightarrow{\beta_i} \operatorname{Spec} A_i = U_i)$ $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$

Let $S_{ij} = S_i \times_{U_i} U_{ij} \Rightarrow S_{ij} \xrightarrow[\sim]{\exists! f_{ij}} S_{ji}$ (Since they both represent $\operatorname{Spec}(R|_{U_{ij}})$)

Check cocycle condition: Let $S_{ijk} = S_i \times_{U_i} U_{ijk} = S_{ij} \times_{U_{ij}} U_{ijk} \hookrightarrow S_{ij} \hookrightarrow S_i$. $S_{ijk} = S_{ikj}$

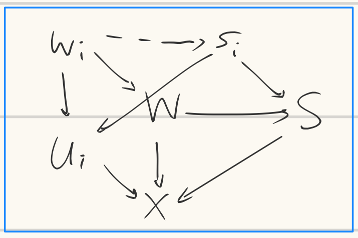
$f_{ij}(S_{ij} \cap S_{ik}) = f_{ij}(S_{ijk}) = S_{ji} \times_{U_{ij}} U_{ijk} = S_{jik} = S_{ji} \cap S_{jk}$

$f_{ik}: S_{ijk} \longrightarrow S_{kij}$ $\qquad f_{ik} = f_{jk} \circ f_{ij}$ $\qquad$ Since "unique up to a unique isom."

$f_{jk} \circ f_{ij}: S_{ijk} \to S_{jik} \to S_{kij}$

$\Rightarrow$ Glue up $S = \bigcup_i S_i$ via S_{ij}.

Then we'll show S represents $\operatorname{Spec} R$.



$\forall W \in \mathrm{Sch}_X$, Let $W_i = W \times_X U_i$, $W_{ij} = W_i \cap W_j = W \times_X U_{ij}$

Then $\operatorname{eq}(\operatorname{Hom}_X(W, S) \longrightarrow \prod_i \operatorname{Hom}_{U_i}(W_i, S_i) \rightrightarrows \prod_{i,j} \operatorname{Hom}_{U_{ij}}(W_{ij}, S_{ij}))$ gluing of morphism

$\qquad\qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \|$

By lemma, $\operatorname{eq}(\operatorname{Spec} R(W) \to \prod_i \operatorname{Spec} R(W_i) \rightrightarrows \prod_{i,j} \operatorname{Spec} R(W_{ij})) \Rightarrow \operatorname{Spec} R(W) = \operatorname{Hom}_X(W, S)$ $\square$

17.1.4. (Rmk) $\operatorname{Spec} \mathcal{R} \times_X U_i = S_i = \operatorname{Spec} \mathcal{R}_i = \operatorname{Spec}(\mathcal{R}|_{U_i})$

$\beta : \operatorname{Spec} \mathcal{R} \to X$, $(\beta_* \mathcal{O}_{\operatorname{Spec}\mathcal{R}})|_{U_i} \cong (\beta|_{S_i})_* \mathcal{O}_{S_i} \cong \widetilde{\mathcal{R}_i} = \mathcal{R}|_{U_i} \Rightarrow \mathcal{R} \xrightarrow[\simeq]{\Phi} \beta_* \mathcal{O}_{\operatorname{Spec}\mathcal{R}}$.

17.1.C. Given
$$\begin{tikzcd} W \arrow[rr, "\gamma"] \arrow[dr, "\mu"'] & & \operatorname{Spec}\mathcal{R} \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$
, then
$$\begin{array}{ccc} \gamma & \in & \operatorname{Hom}_X(W, \operatorname{Spec}\mathcal{R}) \\ \downarrow & & \downarrow 1:1 \\ \alpha & & \operatorname{Spec}\mathcal{R}(W) \end{array}$$

$\alpha : \mathcal{R} \xrightarrow{\Phi} \beta_* \mathcal{O}_{\operatorname{Spec}\mathcal{R}} \longrightarrow \beta_* \gamma_* \mathcal{O}_W = \mu_* \mathcal{O}_W$ (an explicit formula)

p.f. Unwind the definition and check locally.

Note: $\operatorname{Hom}_X(W, \operatorname{Spec}\mathcal{R})$
① $\downarrow 1:1$ is given by $\left[\begin{array}{c} eq\left(\operatorname{Hom}_X(W, S) \to \prod_i \operatorname{Hom}_{U_i}(W_i, S_i) \rightrightarrows \prod_{i,j} \operatorname{Hom}_{U_{ij}}(W_{ij}, S_{ij})\right) \\ \| \qquad\qquad \| \\ eq\left(\operatorname{Spec}\mathcal{R}(W) \to \prod_i \operatorname{Spec}\mathcal{R}(W_i) \rightrightarrows \prod_{i,j} \operatorname{Spec}\mathcal{R}(W_{ij})\right) \end{array}\right]$
$\operatorname{Spec}\mathcal{R}(W)$

② Φ is given by gluing locals

③ When $X = \operatorname{Spec} A$, $\mathcal{R} = \widetilde{R}$, the assertion follows immediately by 17.1.A □

17.1.D (1) $\beta : \operatorname{Spec}\mathcal{R} \to X$ is affine.

(2) $\forall \mu : Z \to X$ affine, Z represents $\operatorname{Spec}(\mu_* \mathcal{O}_Z)$

p.f. (1) is clear.

(2) $\mu_* \mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras since μ is affine.

The assertion follows by the construction and gluing morphisms.

Write $X = \bigcup_i U_i$, $U_i = \operatorname{Spec} A_i$.

Then $Z_i := Z \times_X U_i = \operatorname{Spec} B_i$ affine and $\widetilde{B_i} = \mu_* \mathcal{O}_Z|_{U_i}$

$\Rightarrow \exists!\ Z_i \xrightarrow[\simeq]{f_i} \operatorname{Spec}(\mu_* \mathcal{O}_Z) \times_X U_i$

$f_i|_{Z_i \cap Z_j} = f_j|_{Z_i \cap Z_j}$ since "unique by unique isom."

$\Rightarrow$ Gluing morphism $\rightsquigarrow Z \cong \operatorname{Spec}(\mu_* \mathcal{O}_Z)$ □

17.1.E Give $\mu : \operatorname{Spec} \mathcal{R} \to X$.

(1) $QCoh_{\operatorname{Spec}\mathcal{R}} \cong$ category of quasi-coherent $\mathcal{R}$-modules on X $= C$

(2) $QCoh_{\operatorname{Spec}\mathcal{R}}$ of finite type $\cong$ category of finite type quasi-coherent $\mathcal{R}$-modules on X $= C'$

pf. (1) ① fully-faithfulness

$\forall \mathcal{F} \in QCoh_{\operatorname{Spec}\mathcal{R}}$, $\forall \operatorname{Spec} A \hookrightarrow X$, $\mu^{-1}(\operatorname{Spec} A) = \operatorname{Spec} \tilde{R}$ where $\tilde{R} = \mathcal{R}|_{\operatorname{Spec} A}$, $\mathcal{F}|_{\operatorname{Spec} R} = \tilde{M}$

Then $(\mu_* \mathcal{F})|_{\operatorname{Spec} A} = (\mu|_{\operatorname{Spec} R})_* (\mathcal{F}|_{\operatorname{Spec} R}) = (\mu|_{\operatorname{Spec} R})_* (\tilde{M}) = \tilde{M}$

$M \in Mod_R$

$\Rightarrow \mu_* \mathcal{F} \in C$

$\forall \mathcal{F}, \mathcal{G} \in QCoh_{\operatorname{Spec}\mathcal{R}}$. $X = \bigcup_i U_i$, $U_i = \operatorname{Spec} A_i$. $\beta^{-1}(U_i) = \operatorname{Spec} R_i$, $\mathcal{F}|_{\operatorname{Spec} R_i} = \tilde{M}_i$, $\mathcal{G}|_{\operatorname{Spec} R_i} = \tilde{N}_i$

$$\begin{tikzcd}
\operatorname{Hom}_C(\mu_*\mathcal{F}, \mu_*\mathcal{G}) \arrow[r] & \prod_i \operatorname{Hom}_{R_i}(M_i, N_i) \arrow[r, shift left] \arrow[r, shift right] & * \\
\operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec}\mathcal{R}}}(\mathcal{F}, \mathcal{G}) \arrow[u, "1:1"] \arrow[r] & \prod_i \operatorname{Hom}_{R_i}(M_i, N_i) \arrow[u, equal] \arrow[r, shift left] \arrow[r, shift right] & * \arrow[u, equal]
\end{tikzcd}$$

$U_i \cap U_j = \bigcup_k U_{ijk}$ affines

(Gluing morphisms of sheaves) $\Rightarrow$ fully faithful ✓

② essentially full.

$\forall \mathcal{M} \in C$, Define $\tilde{\mathcal{M}}|_{\operatorname{Spec} R_i} = \widetilde{\mathcal{M}(U_i)}$ (gluing sheaves)

$\Rightarrow \mu_* \tilde{\mathcal{M}} \cong \mathcal{M}$ $\Rightarrow$ essentially full. ✓

(2): ① fully faithful: 1' $\mu_*|_{f.t\ QCoh_{\operatorname{Spec}\mathcal{R}}}$ factors through C'

2' f.t $QCoh_{\operatorname{Spec}\mathcal{R}}$ is a full subcategory of $QCoh_{\operatorname{Spec}\mathcal{R}}$

C' is a full subcategory of C

② essentially full: same to (1). □

17.1.F Spec commutes with base change

$\mu : Z \to X$, then $\operatorname{Spec}\mathcal{R} \times_X Z = \operatorname{Spec} \mu^* \mathcal{R}$

pf.

$$\begin{tikzcd}
\forall\ W \arrow[rd, bend right, "\nu"] & \operatorname{Spec}\mathcal{R} \times_X Z \arrow[r] \arrow[d] & \operatorname{Spec}\mathcal{R} \arrow[d, "\beta"] \\
& Z \arrow[r, "\mu"] & X
\end{tikzcd}$$

$$\mathrm{Hom}_Z(W, \mathrm{Spec}\,\mathcal{R}\times_X Z) \overset{1:1}{=} \mathrm{Hom}_X(W, \mathrm{Spec}\,\mathcal{R}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{R}, (\mu\circ\nu)_*\mathcal{O}_W)$$
$$= \mathrm{Hom}_{\mathcal{O}_Z}(\mu^*\mathcal{R}, \nu_*\mathcal{O}_W) = \mathrm{Spec}(\mu^*\mathcal{R})(W) \qquad \square.$$

17.1.5. Def. If $\mathcal{F}$ is a finite rank locally free sheaf $\mathcal{F}$, then we define the total space of $\mathcal{F}$ to be $\mathrm{Spec}(\mathrm{Sym}^\bullet\mathcal{F}^\vee)$.

17.1.6 (1) $\forall p\in X$, $\exists\, U$ open nbd of p s.t. $\mathrm{Spec}(\mathrm{Sym}^\bullet\mathcal{F}^\vee|_U) \cong \mathbb{A}^n_U$.

(2) And $\mathcal{F} \cong \mathrm{Hom}_X(-, \mathrm{Spec}(\mathrm{Sym}^\bullet\mathcal{F}^\vee))$

p.f. (1). Let U be a small nbd of p s.t. $\mathcal{F}|_U$ is free and U is affine

$(\mathrm{Sym}^\bullet\mathcal{F}^\vee)|_U = \mathrm{Sym}^\bullet(\mathcal{F}|_U)^\vee \qquad \mathrm{Sym}^\bullet A^n \cong A[x_1,\dots x_n]$

(2). $\forall U \hookrightarrow X$,

$$\mathrm{Hom}_X(U, \mathrm{Spec}(\mathrm{Sym}^\bullet\mathcal{F}^\vee)) = \mathrm{Hom}_U(U, \mathrm{Spec}(\mathrm{Sym}^\bullet\mathcal{F}^\vee)\times_X U) = \mathrm{Hom}_U(U, \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathcal{F}|_U)^\vee))$$
$$= \mathrm{Hom}_{\mathcal{O}_U}(\mathrm{Sym}^\bullet(\mathcal{F}|_U)^\vee, \mathcal{O}_U) \overset{\triangle}{=} \mathrm{Hom}_{\mathcal{O}_U}((\mathcal{F}|_U)^\vee, \mathcal{O}_U) = \Gamma(U, ((\mathcal{F}|_U)^\vee)^\vee) = \Gamma(U, \mathcal{F}|_U)$$
$$= \Gamma(U, \mathcal{F}) \quad \Rightarrow \mathcal{F} \cong \mathrm{Hom}_X(-, \mathrm{Spec}(\mathrm{Sym}^\bullet\mathcal{F}^\vee))$$

$\triangle$: $\mathrm{Hom}_{\mathcal{O}_X\text{-modules}}(\mathcal{G}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X\text{-algebras}}(\mathrm{Sym}^\bullet\mathcal{G}, \mathcal{O}_X)$. $\quad \mathcal{G}$ is locally free.

$$\varphi \longmapsto \mathrm{Sym}^\bullet\varphi.$$

Inj: check locally. surj: gluing local morphisms of sheaves

Notation. $\mathrm{Spec}(\mathrm{Sym}^\bullet\mathcal{F}^\vee)$ is the vector bundle corresponding to the locally free sheaf $\mathcal{F}$.
line bundle ... invertible sheaf.

17.2. Relative Proj.

Universal property is ugly

Definition 27.16.7. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over S*, or the *homogeneous spectrum of $\mathcal{A}$ over S*, or the *relative Proj of $\mathcal{A}$ over S* is the scheme constructed in Lemma 27.15.4 which represents the functor F (27.16.4.1), see Lemma 27.16.6. We denote it $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$.

Lemma 27.16.11. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. In this case the scheme $X = \underline{\mathrm{Proj}}_S(\mathcal{A})$ represents the functor F_1 which associates to a scheme $f : T \to S$ over S the set of pairs $(\mathcal{L}, \psi)$, where*

(1) *$\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and*
(2) *$\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a graded $\mathcal{O}_T$-algebra homomorphism such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective*

up to strict equivalence as above. Moreover, in this case all the quasi-coherent sheaves $\mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}(n)$ are invertible $\mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}$-modules and the multiplication maps induce isomorphisms $\mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}(n) \otimes_{\mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}} \mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}(m) = \mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}(n+m)$.

Stacks tag 01NS

So we use another style to introduce relative proj.

17.2.A. $A \to B$ rings, $S_\bullet$, a $\mathbb{Z}_{\geq 0}$-graded ring over A.

(a) $\exists\ \alpha: \mathrm{Proj}_B(S_\bullet \otimes_A B) \xrightarrow{\sim} (\mathrm{Proj}_A S_\bullet) \times_{\mathrm{Spec} A} \mathrm{Spec} B$.

(b) If X is a projective A-scheme, then $X \times_{\mathrm{Spec} A} \mathrm{Spec} B$ is a projective B-scheme.

(c) Suppose $S_\bullet$ is generated by degree 1, then $\mathcal{O}_{\mathrm{Proj}_B(S_\bullet \otimes_A B)}(1) \xleftrightarrow{\sim} \alpha^* \gamma^* \mathcal{O}_{\mathrm{Proj}_A S_\bullet}(1)$

where γ is $\to$

$$\begin{tikzcd} \mathrm{Proj}_A S_\bullet \times_A \mathrm{Spec} B \arrow[r, "\gamma"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Proj}_A S_\bullet \arrow[d] \\ \mathrm{Spec} B \arrow[r] & \mathrm{Spec} A \end{tikzcd}$$

pf. (1) see Liu Qing Prop 3.1.9.

(2) easy by (1)

(3) Check on $D_+(f_i)$'s where $f_i \in S_1$ and $\{f_i\}_i$ generates $S_\bullet$. $\square$

17.2.B. EXERCISE. Suppose we are given a scheme X, and the following data:

(i) For each affine open subset $U \subset X$, we are given some morphism $\pi_U : Z_U \to U$ (a "scheme over U").

(ii) For each (open) inclusion of affine open subsets $V \subset U \subset X$, we are given an open embedding $\rho_V^U : Z_V \hookrightarrow Z_U$.

Assume this data satisfies:

(a) for each $V \subset U \subset X$, ρ_V^U induces an isomorphism $Z_V \xrightarrow{\sim} \pi_U^{-1}(V)$ of schemes over V, and

(b) whenever $W \subset V \subset U \subset X$ are three nested affine open subsets, $\rho_W^U = \rho_V^U \circ \rho_W^V$.

Show that there exists an X-scheme $\pi : Z \to X$, and isomorphisms $i_U : \pi^{-1}(U) \xrightarrow{\sim} Z_U$ over each affine open set U, such that for nested affine open sets $V \subset U$, ρ_V^U agrees with the composition

$$Z_V \xrightarrow{i_V^{-1}} \pi^{-1}(V) \hookrightarrow \pi^{-1}(U) \xrightarrow{i_U} Z_U$$

p.f. For affine-diagonal X, $\begin{cases} 1^\circ \text{ gluing of schemes} \Rightarrow Z, \\ 2^\circ \text{ gluing of morphisms} \Rightarrow Z \to X. \end{cases}$

For general X, by the first arguement, we may replace "affine" by "affine-diagonal" in "17.2.B". then $\begin{cases} 1^\circ \text{ gluing of schemes} \Rightarrow Z, \\ 2^\circ \text{ gluing of morphisms} \Rightarrow Z \to X. \end{cases}$

X is affine-diagonal $\Leftrightarrow$ $\forall$ affine opens $U, V \subseteq X$, $U \cap V$ is still affine. $\square$

17.2.B′ (we only admit $U \subseteq V$ (affines) when U is distinguished open in V.)

p.f. First, prove this in the case when X is affine.

Then, reduce to 17.2.B. $\square$

17.2.C. IMPORTANT EXERCISE AND DEFINITION (**relative** $\mathcal{P}roj$). Suppose $\mathcal{S}_\bullet = \oplus_{n \geq 0} \mathcal{S}_n$ is a quasicoherent sheaf of $\mathbb{Z}^{\geq 0}$-graded algebras on a scheme X. Over each affine open subset $\operatorname{Spec} A \cong U \subset X$, we have an U-scheme $\operatorname{Proj}_A \mathcal{S}_\bullet(U) \to U$. Show that these can be glued together to form an X-scheme, which we call $\mathcal{P}roj_X \mathcal{S}_\bullet$; we have a "structure morphism" $\beta : \mathcal{P}roj_X \mathcal{S}_\bullet \to X$. (The structure morphism β is part of the definition.) Hint/Warning: This problem would be easier if Exercise 17.2.B required considering only *distinguished* affine inclusions. But that would make Exercise 17.2.B notably harder. Instead, use Exercise 17.2.A.

p.f. Clearly, it satisfies the condition in 17.2.B'.

If we only admit 17.2.B, then define $Z_U := \mathrm{Proj}(\mathcal{S}_\cdot(U))$,

$V \subset U \subset X$, $\rho^U_V : Z_V \longrightarrow Z_U$ given by $\mathcal{S}_\cdot(U) \xrightarrow{\rho^U_V} \mathcal{S}_\cdot(V)$

(a). $Z_V \xrightarrow{\sim} \pi_U^{-1}(V) \Leftrightarrow$
$$\begin{tikzcd} Z_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ Z_U \arrow[r] & U \end{tikzcd}$$

By 17.2.A, it suffices to show $\mathcal{S}_\cdot(V) = \mathcal{S}_\cdot(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$. It follows by definitions.

$\mathcal{S}_\cdot(V) = \Gamma(V, \mathcal{S}_\cdot) = \Gamma(V, (\mathcal{S}_\cdot|_U)|_V) \underset{\uparrow}{=} \mathcal{S}_\cdot|_U(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$ $\quad i: V \hookrightarrow U$

$(\mathcal{S}_\cdot|_U)|_V = i^*(\mathcal{S}_\cdot|_U) = \mathcal{S}_\cdot|_U \otimes_{\mathcal{O}_U} \mathcal{O}_V$

(b) $\rho^U_W = \rho^U_V \circ \rho^V_W$ is clear since $\mathcal{S}_\cdot$ is a sheaf. $\square$

Now, we always assume that $\mathcal{S}_\cdot$ satisfying:

(i) $\mathcal{S}_0 = \mathcal{O}_X$, (ii) $\mathcal{S}_\cdot$ is generated in degree 1. (iii) $\mathcal{S}_1$ is of finite type.

$\Updownarrow$

$\mathrm{Sym}^\cdot_{\mathcal{O}_X} \mathcal{S}_1 \twoheadrightarrow \mathcal{S}_\cdot$ (surj) $\quad$ (Then we call $\mathcal{S}_\cdot$ is good.)

17.2.D. We'll define $\mathcal{O}(1)$ on $\mathrm{Proj}\,\mathcal{S}_\cdot$

Glue $\mathcal{O}_{\mathrm{Proj}_A \mathcal{S}_\cdot(\mathrm{Spec}A)}(1)$ on $\mathrm{Proj}_A(\mathcal{S}_\cdot(\mathrm{Spec}A))$ for each $\mathrm{Spec}A \hookrightarrow X$.

$\underset{D(f)}{\overset{\shortparallel}{\mathrm{Spec}(A_f)}} \subseteq \underset{U}{\overset{\shortparallel}{\mathrm{Spec}A}}$ $\quad \mathrm{Proj}(\mathcal{S}_\cdot(D(f))) = \mathrm{Proj}(\mathcal{S}_\cdot(U) \otimes_A A_f) = \mathrm{Proj}(\mathcal{S}_\cdot(U)) \times_A D(f) \hookrightarrow \mathrm{Proj}(\mathcal{S}_\cdot(U))$

$\mathcal{O}_{\mathrm{Proj}\,\mathcal{S}_\cdot(U)}(1)(\underbrace{D_+(s) \times_A D(f)}_{= D_+(f \cdot s)}) = ((\mathcal{S}_\cdot(U))_{f \cdot s})_0 = (((\mathcal{S}_\cdot(U))_f)_{s/1})_0$

$s \in \mathcal{S}_1(U)$ $\quad = \mathcal{O}_{\mathrm{Proj}\,\mathcal{S}_\cdot(D(f))}(1)(D_+(s/1))$

17.2.E. Proj commutes with base change.

$\rho: Z \to X$, then $(\mathrm{Proj}\,\rho^*\mathcal{S}_\cdot, \mathcal{O}_{\mathrm{Proj}\,\rho^*\mathcal{S}_\cdot}(1)) \xleftrightarrow{\sim} (Z \times_X \mathrm{Proj}\,\mathcal{S}_\cdot, \psi^* \mathcal{O}_{\mathrm{Proj}\,\mathcal{S}_\cdot}(1))$

$$\begin{tikzcd} Z \times_X \mathrm{Proj}\,\mathcal{S}_\cdot \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Proj}\,\mathcal{S}_\cdot \arrow[d, "\beta"] \\ Z \arrow[r, "\rho"] & X \end{tikzcd}$$

pf. omit. (Use the constructions)

17.2.2. Definition (π- very ample)

$\pi: X \longrightarrow Y$ proper. If $\mathcal{L}$ invertible on X, then we say $\mathcal{L}$ is π-very ample if we can write $X = \mathrm{Proj}_Y \mathcal{S}.$ with $\mathcal{L} \cong \mathcal{O}(1)$, where $\mathcal{S}.$ is good.

17.2.F omit (see 15.7)

17.2.G

$\mathcal{S}.$ is good. $\mathcal{L}$ is invertible. Define $\mathcal{S}.' = \bigoplus_{n=0}^{\infty} (\mathcal{S}_n \otimes \mathcal{L}^{\otimes n})$. Then $\mathcal{S}.'$ is still a good quasi-coherent sheaf of graded algebras

Then $(\mathrm{Proj}\, \mathcal{S}.', \mathcal{O}_{\mathrm{Proj}\, \mathcal{S}.'}(1)) \xleftrightarrow{\sim} (\mathrm{Proj}\, \mathcal{S}., \mathcal{O}_{\mathrm{Proj}\, \mathcal{S}.}(1) \otimes \beta^* \mathcal{L})$

where $\beta: \mathrm{Proj}\, \mathcal{S}. \longrightarrow X$.

pf. $(\mathcal{S}_n \otimes \mathcal{L}^{\otimes n}) \otimes (\mathcal{S}_m \otimes \mathcal{L}^{\otimes m}) \longrightarrow \mathcal{S}_{m+n} \otimes \mathcal{L}^{\otimes (m+n)} \Rightarrow \mathcal{S}.'$ is graded.

Define $\phi: \mathcal{S}.' \longrightarrow \mathcal{S}.$

$\forall U$ s.t. $\mathcal{L}|_U$ is trivial, choose $t \in \mathcal{L}(U)$ generating $\mathcal{L}(U)$. Then we'll construct $\phi_t: \mathcal{S}.' \longrightarrow \mathcal{S}.$.

$$\begin{tikzcd} \phi_t \quad \mathcal{S}_n(U) \otimes \mathcal{L}(U)^{\otimes n} \arrow[r, "\sim"] \arrow[d, equal] & \mathcal{S}_n(U) \\ \mathcal{S}_n(U) \cdot t^{\otimes n} \arrow[ur, "/t^{\otimes n}"'] & \end{tikzcd} \Longrightarrow \mathrm{Proj}\, \mathcal{S}. \xrightarrow{\tilde{\phi}} \mathrm{Proj}\, \mathcal{S}.'$$

We'll show ϕ_t's are compatible and independent of t

$\forall (U, t), (V, s)$. Let $W = U \cap V$, then $t|_W$ generates $\mathcal{L}(W)$, $s|_W$ generates $\mathcal{L}(W)$

$\exists\, g_W \in \mathcal{O}_X(W)^*$ s.t. $t|_W = g_W \cdot s|_W$.

$$\begin{tikzcd} & \mathrm{Proj}(\mathcal{S}.(U)) \arrow[r, "\phi_t"] & \mathrm{Proj}(\mathcal{S}.'(U)) \\ \mathrm{Proj}(\mathcal{S}.(V)) \arrow[rr, "\phi_s"] & & \mathrm{Proj}(\mathcal{S}.'(V)) \\ \mathrm{Proj}(\mathcal{S}.(W)) \arrow[u, hook] \arrow[uur, dashed, hook] \arrow[rr, "\phi_t|_{\cdot} = \phi_s|_{\cdot}"] & & \mathrm{Proj}(\mathcal{S}.'(W)) \arrow[u, hook] \arrow[uu, dashed, hook, bend right] \end{tikzcd}$$

$\forall f \in \mathcal{S}_n(W)$,

$$\begin{tikzcd} ((\mathcal{S}.'(W))_{f \otimes t^n}) \arrow[dr, "\phi_t"] \arrow[d, equal] & \\ ((\mathcal{S}.'(W))_{f \otimes s^n}) \arrow[r, "\phi_s"'] & ((\mathcal{S}.(W))_f)_0 \end{tikzcd}$$

$h \in \mathcal{S}_{nk}(W)$

$$\begin{tikzcd} \dfrac{h \otimes t^{nk}}{(f \otimes t^n)^k} \arrow[dr] \arrow[d, equal, "\text{Cancel } g_W^{nk}"'] & \\ \dfrac{h \otimes s^{nk}}{(f \otimes s^n)^k} \arrow[r] & \dfrac{h}{f^k} \end{tikzcd}$$

$\square$

Appendix.

17.1.H. (Example) Let's study the line bundle $\mathcal{O}(-1)$ on $\mathbb{P}_k^n$.

$$L = \operatorname{Spec}(\operatorname{Sym}^\bullet \mathcal{O}(-1)^\vee) = \operatorname{Spec}(\operatorname{Sym}^\bullet \mathcal{O}(1)) \xrightarrow{\pi} \mathbb{P}_k^n$$

We'll show it is a closed subscheme of $\mathbb{A}_k^{n+1} \times \mathbb{P}_k^n$

$\mathbb{P}_k^n = \bigcup_{i=0}^n U_i$, $U_i = D_+(X_i)$

$\pi^{-1}(U_i) \hookrightarrow L$, construct $\pi^{-1}(U_i) \xrightarrow{\varphi} \mathbb{A}_k^{n+1} \times \mathbb{P}_k^n$

$$\operatorname{Spec}(k[\tfrac{X_0}{X_i}, \cdots, \tfrac{X_n}{X_i}, X_i]) \longrightarrow \mathbb{A}_k^{n+1} \times U_i = \operatorname{Spec}(k[\tfrac{X_0}{X_i}, \cdots, \tfrac{X_n}{X_i}, T_0, \cdots, T_n])$$

$$k[\tfrac{X_0}{X_i}, \cdots, \tfrac{X_n}{X_i}, X_i] \twoheadleftarrow k[\tfrac{X_0}{X_i}, \cdots, \tfrac{X_n}{X_i}, T_0, \cdots, T_n]$$

$$T_k \longmapsto X_k := X_i (\tfrac{X_k}{X_i}) \quad \left(k[\tfrac{X_0}{X_i}, \cdots, \tfrac{X_n}{X_i}, T_0, \cdots, T_n] / (T_k - X_i (\tfrac{X_k}{X_i})) \right)$$

Compatibility is natural. $\Rightarrow$

$$\begin{tikzcd} L \arrow[r, hook, "\varphi"] \arrow[d, "\pi"'] & \mathbb{A}_k^{n+1} \times \mathbb{P}_k^n \arrow[dl, "p"] \\ \mathbb{P}_k^n & \end{tikzcd}$$

WMA $x_0 \neq 0$

Then $\forall\, l = [x_0, \cdots, x_n] \in \mathbb{P}_k^n(\bar{k})$ consider $\pi^{-1}(l) \subseteq L(\bar{k}) \Leftrightarrow \varphi^{-1}(p^{-1}(l)) \subseteq L(\bar{k})$

$$p^{-1}(l) = \mathbb{A}_k^{n+1}(\bar{k}) \times \{l\} \qquad \varphi^{-1}(p^{-1}(l)) = \{(t, \tfrac{x_1}{x_0} t, \cdots, \tfrac{x_n}{x_0} t) \mid t \in \bar{k}\}$$