15.2 Line bundles and maps to projective space

Recall that the functor
$$\mathfrak{F} : (Sch/S)^{op} \longrightarrow Sets$$
$$X \longmapsto \{(f_1, \dots, f_n) \mid f_i \in \Gamma(X, \mathcal{O}_X)\}$$
is representable by $\mathbb{A}^n_S$.

Pf. for $S = \operatorname{Spec} A$, $\mathbb{A}^n_S = \operatorname{Spec} A[x_1, \dots, x_n]$.

By the universal property of affine space,

$\operatorname{Hom}_S(X, \mathbb{A}^n_S) = \operatorname{Hom}_A(A[x_1, \dots, x_n], \Gamma(X, \mathcal{O}_X)) \cong \Gamma(X, \mathcal{O}_X)^n$.

For general S, gluing.

Example.

$\mathbb{P}^2_k = \operatorname{Proj} k[X_0, X_1, X_2]$, $\mathbb{P}^3_k = \operatorname{Proj} k[Y_0, Y_1, Y_2, Y_3]$

$[X_0, X_1, X_2] \longmapsto [X_0^3, X_0^2 X_2, X_0 X_1 X_2 + X_1^3, X_2^3]$

$\rightsquigarrow \mathbb{P}^2_k \to \mathbb{P}^3_k \quad (Y_0 \mapsto X_0^3, \dots)$

$[X_0, X_1, X_2] \longmapsto [X_0^3, X_0^2 X_2, X_0 X_1 X_2 + X_1^3, X_1^3]$

$\rightsquigarrow \mathbb{P}^2_k \setminus [0,0,1] \longrightarrow \mathbb{P}^3_k$.

Thm. 15.2.2 ([Har] II 7.1)

1. Suppose $s_0, \dots, s_n$ are $n+1$ global sections of an invertible sheaf $\mathcal{L}$ on a scheme X/S, with no common zero. ($\mathcal{L}$ is gen. by $s_0, \dots, s_n$)
Then there is a unique S-morphism
$$\varphi : X \xrightarrow{[s_0, \dots, s_n]} \mathbb{P}^n_S$$
s.t. $\mathcal{L} \cong \varphi^*(\mathcal{O}(1))$, and $s_i = \varphi^*(x_i)$ under this isom.

2. The functor

$$F: (Sch/S)^{op} \to Sets$$
$$X \mapsto \{(\mathcal{L}, s_0, \dots, s_n) \mid \mathcal{L}/X \text{ is an invertible sheaf gen. by } s_0, \dots, s_n\}$$

is representable by $\mathbb{P}^n_S$. ($Grass(\mathcal{O}_S^n, 1) = \mathbb{P}^n_S$)

Pf. 1. If $S = \operatorname{Spec} A$, Assume $\mathbb{P}^n_S = \operatorname{Proj} A[x_0, \dots, x_n]$.

Let $X_i = \{p \in X \mid (s_i)_p \notin m_p \mathcal{L}_p\}$ ($s_i(p) \neq 0$), open

$U_i = V_+(x_i)^c = \operatorname{Spec} A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$.

We define $\varphi_i : X_i \to U_i$ by

$$A[\tfrac{x_0}{x_i}, \dots, \tfrac{x_n}{x_i}] \to \Gamma(X_i, \mathcal{O}_X)$$
$$\frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}$$

(Here if $V \hookrightarrow X_i$, $\mathcal{L}|_U \cong s_i \mathcal{O}_V$,

$\frac{s_j}{s_i}$ is the element s.t. $s_j = s_i(\frac{s_j}{s_i}) \in \Gamma(V, \mathcal{L})$.)

It's easy to check $\frac{s_j}{s_i}$ glue, φ_i glue.

So we get $\varphi : X \to \mathbb{P}^n_S$.

Moreover, $\mathcal{L}$ is trivial on $X_i = \varphi^{-1}(U_i)$.

On $X_i \cap X_j$, the transition function of $\mathcal{L}$ is

precisely $\frac{s_i}{s_j} = \varphi^*(\frac{x_i}{x_j}) =$

So we can construct $\mathcal{L} \cong \varphi^*(\mathcal{O}(1))$ s.t. $s_i = \varphi^*(x_i)$.

By the construction, φ is unique.

2. Given $\varphi : X \to \mathbb{P}^n_S$,

It's clear that $\varphi^*\mathcal{O}(1)$ is a invertible sheaf.

which gen. by $\varphi^*(x_i)$.

So there is a bijection between $Mor_S(X, \mathbb{P}^n_S)$ and $F(X)$.

Rmk.

1. view s_i as functions on X, φ is given by
$$\varphi(P) = [s_0(P), \dots, s_n(P)].$$
2. when consider invertible sheaf, it's useful to consider transition functions. (like bundle)

Def. 15.2.1.

1. If $\mathcal{L}$ is an invertible bundle on X, then those pts where all global sections of $\mathcal{L}$ vanish are called the base points of $\mathcal{L}$, and the set of base points is called the base locus of $\mathcal{L}$.
   It is a closed subset of X.
   (taking the scheme-theoretic intersection $\rightsquigarrow$ scheme-theoretic base locus.)
2. A linear series (or linear system) on a k-scheme X is a k-vector space V (usually fin. dim), an invertible sheaf $\mathcal{L}$, and a linear map $\lambda: V \to \Gamma(X, \mathcal{L})$.
   If λ is an isom., it is called a complete linear series, and is often written $|\mathcal{L}|$.
   The language of base points readily applies to linear series.

Prop. 15.2.B

If $\mathcal{L}$ and $\mathcal{M}$ are base-pt-free invertible sheaves, Then $\mathcal{L} \otimes \mathcal{M}$ as well.

Pf. $\forall\ P \in X$, $\exists\ l \in \Gamma(X, \mathcal{L})$, $m \in \Gamma(X, \mathcal{M})$, s.t. l, m don't vanish at P.
Then $l \otimes m \in \Gamma(X, \mathcal{L} \otimes \mathcal{M})$ don't vanish at P.

Example. the veronese embedding is $|\mathcal{O}_{\mathbb{P}^n_k}(d)|$ 15.2.7

Consider the line bundle $\mathcal{O}_{\mathbb{P}^n_k}(d)$ on $\mathbb{P}^n_k$

$\dim_k \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{n+d}{d}$,

with a basis corresponding to homogenous degree d polynomial.

They have no common zeros,

$\rightsquigarrow V_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$.

Prop. 15.2.G

1. Suppose X is a k-scheme, a fin-dim base-pt-free linear series V on X corresponding to $\mathcal{L}$ induce a morphism to projective space

$$\phi_V : X \longrightarrow \mathbb{P}V^\vee$$

2. $\mathbb{P}V^\vee$ represents the functor $\underline{\text{Grass}}(V \otimes \mathcal{O}_X, 1)$

Pf. $\mathbb{P}V^\vee = \text{Proj Sym} V$.

Choose a basis of V, then use Thm 15.2.2.

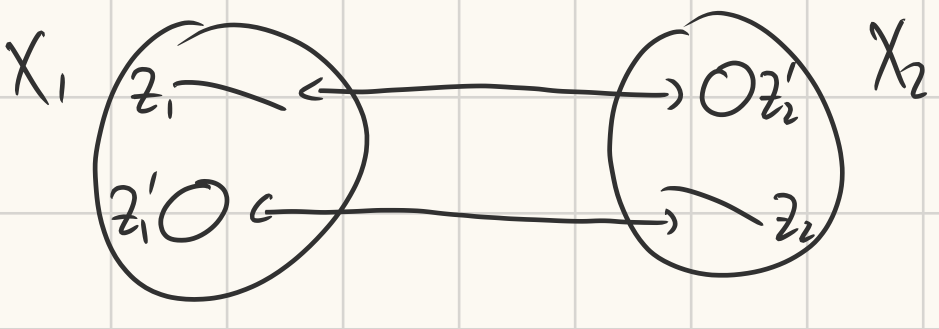
Example. (A proper nonprojective k-variety) 15.2.11

For $i = 1, 2$, let $X_i \cong \mathbb{P}^3_k$. Z_i be a line in X_i,

Z_i' be a regular conic in X_i disjoint from Z_i.

$Z_i, Z_i' \cong \mathbb{P}^1_k$. (fiber product)

Glue X_1 to X_2 by identifying Z_1 and Z_2', Z_1' and Z_2.



The result, called X, is proper.

X is not projective. For if it were, $X \hookrightarrow \mathbb{P}^N_k \rightsquigarrow \mathcal{L}/X$

$\mathcal{L}|_{X_1} \cong \mathcal{O}_{X_1}(n_1)$, $\mathcal{L}|_{X_2} \cong \mathcal{O}_{X_2}(n_2)$, where $n_1 > 0$.

Check that $\mathcal{L}|_{Z_1} \cong \mathcal{O}_{Z_1}(n_1)$, $\mathcal{L}|_{Z_1'} \cong \mathcal{O}_{Z_1'}(2n_1)$

$\mathcal{L}|_{Z_2} \cong \mathcal{O}_{Z_2}(n_1)$, $\mathcal{L}|_{Z_2'} \cong \mathcal{O}_{Z_2}(2n_1)$

$\Rightarrow n_1 = 2n_2$, $n_2 = 2n_1$, impossible!

(For example, consider the veronese embedding $\mathbb{P}^1_k \to \mathbb{P}^2_k$ correspond to $\mathcal{O}_{\mathbb{P}^1_k}(2)$)

Prop.

$\mathbb{P}^n_S$ is proper over S.

Pf. We use DVR criterion: let R be a DVR, $K = \mathrm{Frac}(R)$.

$$\begin{tikzcd}
\{\eta\} = \operatorname{Spec} K \arrow[r] \arrow[d, hook] & \mathbb{P}^n_S \arrow[d] \\
\{\eta, p\} = \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

By the universal property of $\mathbb{P}^n_S = \mathrm{Grass}(\mathcal{O}_S^n, 1)$,

It suffice to show that:

(up to isom. of quot. bundle)

For any $f: K^n \twoheadrightarrow K$, $\exists!$ $g: R^n \twoheadrightarrow R$, s.t. $f = g \otimes_R K$.

Let f is given by $(a_1, \dots, a_n) \mapsto \sum_{i=1}^n c_i a_i$,

let π be a uniformizer of R, $N = \min_i \{v(c_i)\}$,

Then $g: R^n \to R$ satisfies the condition.

$(a_1, \dots, a_n) \mapsto \sum_{i=1}^n \pi^{-N} c_i a_i$

15.3 The Curve-to-Projective Extension Thm

Example.

Consider $f: C_1 = \{Y^2Z = X^3 - XZ^2\} \subseteq \mathbb{P}^2 \to C_2 = \mathbb{P}^1$

$[X, Y, Z] \mapsto [X, Z]$.

f is not well defined at $[0, 1, 0]$,

but we can extend f by use $[X, Y, Z] \mapsto [Y^2, X^2 - Z^2]$.

Thm. (The Curve-to-Projective Extension Thm) 15.3.1

Suppose C is a pure dim 1 Noetherian scheme over $S = \operatorname{Spec} A$.

$p \in C$ is a regular closed pt of it.

Suppose Y is a projective S-scheme,

Then any morphism $C \setminus \{p\} \to Y$ (of S-scheme) extends to all of C.

Pf 1. By p is regular, $\mathcal{O}_{C,p}$ is an DVR.

So there is only one irreducible component contains p.

By replacing C by an open nbd of p,

W.M.A C is integral and affine.

We next reduce to the case where $Y = \mathbb{P}^n_A$.

$Y \hookrightarrow \mathbb{P}^n_A$, If the result holds for $\mathbb{P}^n_A$ and we have $C \to \mathbb{P}^n_A$

with $C \setminus \{p\} \to Y$, then C must map to Y as well.

(Assume $p \mapsto \mathbb{A}^n_A \hookrightarrow \mathbb{P}^n_A$, then the functions vanishing on $Y \cap \mathbb{A}^n_A$

pull back to functions that vanish at the generic pt of C.

By C is integral, these function vanish everywhere on C

i.e. C maps to Y.)

$\mathcal{O}_{C,p}$ is a DVR, choose a uniformizer $t_p \in \mathfrak{m}_p \setminus \mathfrak{m}_p^2$.

W.M.A $t \in \Gamma(C, \mathcal{O}_C)$, clear $V(t) \neq C$.
By replacing C by an affine open nbd of p in $C \setminus V(t) \cup \{P\} \subset C$
($C \setminus V(t) \cup \{P\}$ is open because $V(t)$ contains finite closed pts).
W.M.A p is the only zero of the function t.
We have a map $C \setminus \{P\} \longrightarrow \mathbb{P}^n_A$, which by Thm 15.2.2
corresponds to a line bundle $\mathcal{L}$ on $C \setminus \{P\}$ and $n+1$ sections
of it with no common zeros in $C \setminus \{P\}$.
Let $U \hookrightarrow C \setminus \{P\}$ s.t. $\mathcal{L}|_U \cong \mathcal{O}_U$. Then replace C by $U \cup \{P\}$,
we interpete the map to $\mathbb{P}^n$ as $n+1$ rational functions $f_0, \cdots, f_n$, defined
on U with no common zero on U. Let $N = \min_i (\mathrm{val}_p f_i)$,
Then $t^{-N} f_0, \cdots, t^{-N} f_n$ are $n+1$ functions with no common zeros
Thus they determine a morphism $C \to \mathbb{P}^n_A$ extending $C \setminus \{P\} \to \mathbb{P}^n_A$
as desired

Prop. 15.3.B

Suppose X is a Noe k-scheme, Z is a irreducible codim 1 subvariety
whose generic pt is a regular pt of X ($\mathcal{O}_{X,Z}$ is a DVR).
Suppose $\pi : X \dashrightarrow Y$ is a rational map to a projective k-scheme,
Then the domain of definition of the rational map induce a dence open
subset of Z.
In other words, rational maps from Noe k-scheme to projective k-scheme
can be extended over regular codim 1 sets.

Pf. As above, W.M.A X is integral and affine.

If $\pi: X \dashrightarrow Y$ is defined on an affine open subset U,

$\pi|_U \rightsquigarrow (\mathcal{L}, s_0, \ldots, s_n)$, assume $U' \hookrightarrow U$, $\mathcal{L}|_{U'} \cong \mathcal{O}_{U'}$,

Then $\pi|_{U'} \rightsquigarrow (\mathcal{L}|_{U'}, f_0, \ldots, f_n)$

where $f_0, \ldots, f_n$ are rational functions.

let η be the generic pt of Z, t be a uniformizer of $\mathcal{O}_{X,Z}$

$N = \min_i \{ \mathrm{val}_{\mathcal{O}_{X,\eta}} f_i \}$, $t^{-N} f_0, \ldots, t^{-N} f_n$ extend on η.

Example. tangents to regular plane curves are limits of secants.

Let k be a field, $C = \{ f(x,y) = 0 \} \subseteq \mathbb{A}_k^2$ is a plane curve.

Fix a pt $p = (x_0, y_0)$ of C, given $q = (x_1, y_1)$ of C,

the slope of the line joining them is $m = \frac{y_1 - y_0}{x_1 - x_0}$.

$(m = [x_1 - x_0, y_1 - y_0] \in \mathbb{P}_k^1)$.

We thus have a map $C \setminus p \to \mathbb{P}_k^1$, $q \mapsto m$.

Thm 15.3.1 state that when p is a smooth pt of C,

this will extend over p.

15.4 Line bundles and Weil divisor

Recall. (Algebraic Hartogs's lemma) 13.5.19

Suppose A is a integrally closed Noe int. domain,

Then $A = \bigcap_{p \text{ of codim } 1} A_p$.

For the rest of this section, we consider only Noe. normal scheme.

Def. 15.4.1

1. A Weil divisor is a formal $\mathbb{Z}$-linear combination of codim 1 irreducible closed subset of X. It is of the form

$$\sum_{Y \text{ codim } 1} n_Y [Y]$$

where $n_Y \in \mathbb{Z}$. Weil divisor form an ab. gp., denoted $\mathrm{Weil}\, X$.

2. We say that $[Y]$ is an irreducible divisor.

A Weil divisor $D = \sum n_Y [Y]$ is said to be effective if $n_Y \geq 0, \forall Y$.

In this case we say $D \geq 0$, and by $D_1 \geq D_2$ we mean $D_1 - D_2 \geq 0$.

The support of D, denoted $\mathrm{Supp}\, D$, is the subset $\bigcup_{n_Y \neq 0} Y$.

If $U \hookrightarrow X$, we define the restriction map

$$\mathrm{Weil}\, X \longrightarrow \mathrm{Weil}\, U$$

$$\sum n_Y [Y] \longmapsto \sum_{Y \cap U \neq \emptyset} n_Y [Y \cap U]$$

Prop. 15.4A

The irreducible divisor on $\mathbb{P}^n_k$ correspond to irreducible homogeneous polynomial in $k[X_0, \cdots, X_n]$, up to scalar multiplication.

Pf. We admit 15.2.I: Each closed subscheme of $\mathbb{P}^n_k = \mathrm{Proj}\, k[X_0, \cdots, X_n]$ arises from homogeneous ideal $I \subseteq k[X_0, \cdots, X_n]$.

Notice that $k[X_0, \cdots, X_n]$ is a UFD, so all of ht 1 prime ideal is principle. If I correspond to a irreducible closed subset Y of codim 1 of $\mathbb{P}^n_k$, I is gen. by a homogeneous irreducible ideal.

15.4.2

Suppose now that X is also regular in codim 1.
Assume also that X is reduced. (avoiding embedding pts)
Suppose that $\mathcal{L}$ is an invertible sheaf on X, s is a rational section not vanishing on any irreducible component of X.
(rational section are given by section over a dense open subset of X).

We define the valuation of s along an irreducible Weil divisor Y (denote $val_Y(s)$) as follows.
Take $U \hookrightarrow X$ s.t. U contains the generic pt of Y, $\mathcal{L}|_U \cong \mathcal{O}_U$
under this trivialization, s is a nonzero rational function on U, which thus has a valuation. ($\mathcal{O}_{X,Y}$ is a DVR)
Any two such trivialization differ by an invertible function, thus well-defined.
$val_Y(s) = 0$ for all but many Y by 13.5.G.
Thus $s \rightsquigarrow div(s) := \sum_Y val_Y(s)[Y]$.
We call $div(s)$ the divisor of zeros and poles of the rational section s.

15.4.3

Group of "line bundle with rational section
Consider set $\{(\mathcal{L}, s)\}$ of pairs of line bundle $\mathcal{L}$ with rational section s, not the zero section on any irreducible component of X $\}$
$(\mathcal{L}, s) \cong (\mathcal{L}', s')$ iff $\exists \varphi: \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$, $s \mapsto s'$ under φ.
$(\{(\mathcal{L}, s)\}/\sim, \otimes)$ is an ab. gp. with identity $(\mathcal{O}_X, 1)$
$(\mathcal{L}, s)^{-1} = (\mathcal{L}^{-1}, s^{-1})$

Notice that if t is an invertible function on X, $\mathcal{L} \xrightarrow{\times t} \mathcal{L}$

$(\mathcal{L}, s) \cong (\mathcal{L}, st)$

Similarly, $(\mathcal{L}, s)/(\mathcal{L}, u) = (\mathcal{O}_X, s/u)$.

Where s/u is a rational function.

$$\mathrm{div} : \{(\mathcal{L}, s)\}/\sim \longrightarrow \mathrm{Weil}\, X$$

The homomorphism will be the key to determine all the line bundles on many X. (Any invertible sheaf will have such a rational section.)

Prop. 15.4.4

If X is normal and Noetherian, div is injective.

Pf. Suppose $\mathrm{div}(\mathcal{L}, s) = 0$, s has no poles.

By Exercise 14.2.F, s is a regular section

Claim: $\times s : \mathcal{O}_X \to \mathcal{L}$ is an isom.

$\forall$ affine open $U \hookrightarrow X$ s.t. $\mathcal{L}|_U \cong \mathcal{O}_U$,

$$\mathcal{O}_U \xrightarrow{\times s} \mathcal{L} \xrightarrow{i} \mathcal{O}_U \rightsquigarrow \mathcal{O}_U \xrightarrow{\times s'} \mathcal{O}_U.$$

$s' = i(s)$ has no zeros and pole $\Rightarrow$ s' is invertible

So $\times s'$ is an isom, $\times s$ is an isom over U.

We try to find an inverse of div, or at least its image.

Def. $\mathcal{O}_X(D)$ 15.4.5

Assume X is normal and irreducible

(normal scheme is a disjoint union of irreducible normal scheme)

Suppose $D \in \mathrm{Weil}\, X$. Define the sheaf $\mathcal{O}_X(D)$ by

$$\Gamma(U, \mathcal{O}_X(D)) := \{t \in K(X)^\times \mid \mathrm{div}|_U(t) + D|_U \geq 0\} \cup \{0\}$$

Here $D|_U$ is the restriction of D to U, $\mathrm{div}|_U(t) = \mathrm{div}(t|_U)$

The section of $\mathcal{O}_X(D)$ over U are the rational functions on U that have poles zeros "constrained by D".

Away from the support of D, this is isom. to the structure sheaf by Algebraic Hartogs's lemma.

Rmk. 15.4.6

$\mathcal{O}_X(D)$ comes along with a canonical nonzero "rational section" correspond to $1 \in K(X)^\times$ (defined on $X \setminus \mathrm{Supp}(D)$)

Prop. 15.4.D

$\mathcal{O}_X(D)$ is a quasi-coherent sheaf

Pf. $\forall\, U = \mathrm{Spec} A \hookrightarrow X$, $f \in A$, W.M.A A integral, normal, Noe.

$\Gamma(U, \mathcal{O}_X(D)) = \{t \in K(X)^\times \mid \mathrm{div}|_U(t) + D|_U \geq 0\} \cup \{0\} \subseteq K(X)$

$\Gamma(U_f, \mathcal{O}_X(D)) = \{t \in K(X)^\times \mid \mathrm{div}|_{U_f}(t) + D|_{U_f} \geq 0\} \cup \{0\} \subseteq K(X)$

Consider them as subgp of $K(X) = \mathrm{Frac}(A)$,

$\Gamma(U, \mathcal{O}_X(D))_f = \{\frac{t}{f^r} \mid \mathrm{div}|_U(t) + D|_U \geq 0\} \cup \{0\} \subseteq \Gamma(U_f, \mathcal{O}_X(D))$

On the other hand, $\mathrm{div}|_U(t) = \mathrm{div}|_{U_f}(t) + \sum_{Y \in V(f)} \mathrm{val}_Y(f)[Y]$

let $N > \max_{Y \in V(f)} \{\mathrm{val}_Y(f) + n_Y\}$, $t^N f \in \Gamma(U, \mathcal{O}_X(D))$. $(D = \sum n_Y [Y])$

So $\Gamma(U, \mathcal{O}_X(D))_f = \Gamma(U_f, \mathcal{O}_X(D))$.

We next show that in good circumstances, $\mathcal{O}_X(D)$ is a line bundle.

Prop. 15.4.G

Suppose $\mathcal{L}$ is an invertible sheaf, s is a nonzero rational section of $\mathcal{L}$.

1. $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$
2. Let σ be the map from $K(X)$ to the rational sections of $\mathcal{L}$, where $\sigma(t)$ is the rational section defined via $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$. Then $\sigma(1) = s$.

Pf. 1. W.M.A X is integral, normal, Noe.

$\forall$ affine open $U = \operatorname{Spec} A \hookrightarrow X$ s.t. $\exists\ \mathcal{L}|_U \cong \mathcal{O}_U$,

Consider
$$\phi_U : \Gamma(U, \mathcal{O}(\operatorname{div} s)) \longrightarrow \Gamma(U, \mathcal{L}),$$
$$t \longmapsto ts$$

$\operatorname{div}(ts) = \operatorname{div}(t) + \operatorname{div}(s) \geq 0$ by definition of $\mathcal{O}(\operatorname{div} s)$

So ts is a regular section on U. (identity $\mathcal{L}|_U$ with $\mathcal{O}_U$), ϕ_U is well-defined.

Clear ϕ_U is an isom. by consider them as subgp of $K(X)$.

It's easy to check ϕ_U glue. ϕ: So $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$.

2. Follows from the construction of ϕ.

In conclusion, image of div are those Weil divisor s.t. $\mathcal{O}_X(D)$ is a invertible sheaf. $(\mathcal{O}(D), \sigma(1)) \xmapsto{\operatorname{div}} D$.

Lemma. 15.4.H

Suppose $X = \mathbb{P}^n_k$, $\mathcal{L} = \mathcal{O}(1)$, s is the section of $\mathcal{O}(1)$ corresponding to x_0, $D = \operatorname{div} s$. Then $\mathcal{O}(mD) \cong \mathcal{O}(m)$, and the canonical rational section of mD is s^m.

(x_0 can be replaced by any linear form)

Pf. x_0^m is a regular section of $\mathcal{O}(m)$.

By last prop, $\mathcal{O}(mD) \cong \mathcal{O}(\operatorname{div}(x_0^m)) \cong \mathcal{O}(m)$

Cor. 15.4.I

$\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$

Pf. For any line bundle $\mathcal{L}/\mathbb{P}^n_k$, let s be a rational section of $\mathcal{L}$.

Then $\mathcal{L} \cong \mathcal{O}(\operatorname{div} s) = \mathcal{O}(\sum_Y \operatorname{val}_Y(s)[Y])$

$$= \bigotimes_Y \mathcal{O}([Y])^{\operatorname{val}_Y(s)} \qquad (15.4.M)$$

But by 15.4.A, for any Weil divisor $[Y]$, $[Y] = V_+(f)$ for some homogeneous polynomial f of some deg d.

Consider f as a section of $\mathcal{O}(d)$,

Similarly $\mathcal{O}(d) \cong \mathcal{O}(\operatorname{div} f) = \mathcal{O}([Y])$.

So $\mathcal{L} \cong \mathcal{O}(\sum_Y \operatorname{val}_Y(s) \cdot \deg Y)$

That is, $\mathbb{Z} \to \operatorname{Pic}(\mathbb{P}^n_k)$ is surj.

$m \mapsto \mathcal{O}(m)$

Clearly it is inj.

Def. 15.4.7

Let X be a Noe. normal irreducible scheme, $D \in \mathrm{Weil}\, X$.

1. If $D = \mathrm{div} f$ for some rational function f, we say that D is principle.
   Principle divisor clearly form a subgp of $\mathrm{Weil}\, X$, denote $\mathrm{Prin}\, X$. $\mathrm{div} : K(X)^\times \longrightarrow \mathrm{Prin}\, X$
2. If $X = \bigcup_i U_i$, $U_i \hookrightarrow X$, s.t. $D|_{U_i}$ is principle, Then we say D is locally principle.
   Locally principle divisor form a subgp of $\mathrm{Weil}\, X$, which we denoted $\mathrm{LocPrin}\, X$.

Rmk. [Har] II.6.11; 15.4.8

1. $\mathrm{LocPrin}\, X = \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$ (group of Cartier divisor)
2. If $D \in \mathrm{Prin}\, X$, $\mathcal{O}(D) \cong \mathcal{O}$
   If $D \in \mathrm{LocPrin}\, X$, $\mathcal{O}(D)$ is a line bundle.

Prop. 15.4.J

If $\mathcal{O}_X(D)$ is a line bundle, D is locally principle.

Pf. $\forall$ Affine open $U \hookrightarrow X$ s.t. $\mathcal{O}_X(D)|_U \cong \mathcal{O}_U$,

Consider $\Gamma(U, \mathcal{O}_U)$ and $\Gamma(U, \mathcal{O}_X(D))$ both as subgp of $K(X)$,

$$\begin{array}{ccc} \Gamma(U, \mathcal{O}_U) & \xrightarrow{\sim} & \Gamma(U, \mathcal{O}_X(D)) \\ 1 & \longmapsto & t \end{array}$$

Then $D|_U = \mathrm{div}|_U(t^{-1})$ $\left(s + D|_U \geq 0 \iff s \cdot \mathrm{div}|_U t \geq 0 \right)$.
$\forall s \in K(X)^\times$

Prop. 15.4.L [Har] 2.6.11

If X is Noe. and factorial (hence normal),
then for all $D \in \operatorname{Weil} X$, $\mathcal{O}(D)$ is an invertible sheaf.

Pf. W.M.A X irreducible.

$\forall\, x \in X$, $D \rightsquigarrow D_x \in \operatorname{Weil} \operatorname{Spec} \mathcal{O}_{X,x}$.

$\mathcal{O}_{X,x}$ is a UFD. By 12.1.?, the prime corresponding to Y_x for any irreducible Y passing through x of codim 1 is principle, so D_x is a principle divisor in $\operatorname{Spec} \mathcal{O}_{X,x}$.

Let $D_x = \operatorname{div}|_x(f_x)$, where $f_x \in K(X)^\times$.

There are finite Y s.t. $\operatorname{val}_Y(f_x) \neq 0$ or $n_Y \neq 0$
Y doesn't passing through x.

Let Z_x be the union of these Y. $U_x = X \setminus Z_x$.

Then $D|_{U_x} = \operatorname{div}|_{U_x}(f_x)$. i.e. $D|_{U_x}$ is principle.

$X = \bigcup_x U_x$. So D is locally principle, $\mathcal{O}(D)$ is a line bundle.

The class gp 15.4.10

Define the class group $Cl(X) := \operatorname{Weil} X / \operatorname{Prin} X$.

We have the inclusion $\operatorname{Pic}(X) \hookrightarrow Cl(X)$

$$\begin{tikzcd}
\{(\mathcal{L}, s)\}/\sim \arrow[r, "\sim", "\operatorname{div}"'] \arrow[d] & \operatorname{LocPrin} X \arrow[r, hook, "= \text{ if } X \text{ factorial}"] \arrow[d, "\operatorname{Prin} X"] \arrow[l, bend right, "(\mathcal{O}(D),\, s(1)) \mapsfrom D"'] & \operatorname{Weil} X \arrow[d, "/\operatorname{Prin} X"] \\
\operatorname{Pic} X = \{\mathcal{L}\}/\sim \arrow[r, "\sim"] & \operatorname{LocPrin} X / \operatorname{Prin} X \arrow[r, hook] \arrow[l, bend left, "\mathcal{O}(D) \mapsfrom D"] & Cl\, X
\end{tikzcd}$$

Rem.

If X is regular (smooth), X is factorial.

Example. 15.4.11

If A is a UFD, then all Weil divisor on Spec A are principal by 12.1.7.

So $Cl(\mathrm{Spec} A) = 0$. $\mathrm{Pic}(\mathrm{Spec} A) = 0$.

As $k[X_1, \dots, X_n]$ is UFD, $Cl(\mathbb{A}_k^n) = 0$, $\mathrm{Pic}(\mathbb{A}_k^n) = 0$.
"$\mathbb{C}^n$ is contractible complex manifold, and hence should have no nontrivial line bundles."

$\mathbb{A}_k^n$ also has no nontrivial vector bundles.
This is the Quillen-Suslin Thm, formerly known as Serre's Conj.

Removing a closed subset of X of codim ≥ 2 doesn't change the class group. (But it can affect Pic gp)

Suppose Z is an irreducible codim 1 subset of X, we have an exact seq:

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \mathrm{Weil}\, X \longrightarrow \mathrm{Weil}(X \setminus Z) \to 0.$$

Take quotient,

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} Cl(X) \to Cl(X \setminus Z) \to 0.$$

15.4.12

If $U \hookrightarrow \mathbb{A}^n_k$, $\mathrm{Pic}\, U = 0$.

Pf. $\mathrm{codim}\, \mathbb{A}^n_k \setminus U \geq 1$.

15.4.13, 15.4.14

$X = \mathbb{P}^n_k$, $Z = \{x_0 = 0\}$,

$$\mathbb{Z} \longrightarrow Cl(\mathbb{P}^n_k) \longrightarrow Cl(\mathbb{A}^n_k) \to 0$$

From which $Cl(\mathbb{P}^n_k)$ is gen. by the class $[Z]$.

$\mathrm{Pic}(\mathbb{P}^n_k) \cong Cl(\mathbb{P}^n_k) \cong \mathbb{Z}$ ($\mathbb{P}^n_k$ factorial)

The degree of an invertible sheaf on $\mathbb{P}^n$ is defined using this: define $\deg \mathcal{O}(d)$ to be d.

Twisting line bundle by divisors 15.4.16

X normal Noe scheme. $\mathcal{L}$ is a line bundle on X.

$\mathcal{L}(D) := \mathcal{O}_X(D) \otimes \mathcal{L}$.

If D is locally principle, $\mathcal{L}(D)$ is a line bundle.

Prop. 15.4.M

1. Assume for convenient that X is irreducible.

$$\Gamma(U, \mathcal{L}(D)) := \{t \text{ nonzero rational section of } \mathcal{L} \mid \mathrm{div}|_U t + D|_U \geq 0\} \cup \{0\}.$$

2. Suppose D_1 and D_2 are locally principle,

$$(\mathcal{O}(D_1))(D_2) = \mathcal{O}(D_1 + D_2).$$

Pf. It suffice to verify for affine open U s.t. $\mathcal{L}|_U$ trivial.

...

Prop.

Suppose X is a QCQS scheme, $\mathcal{F} \in QCoh(X)$.
$\mathcal{L}$ is an invertible sheaf of X with section s,
X_s is the open subset of X where s doesn't vanish.
We interpret s as a deg 1 element of the graded ring
$$R(\mathcal{L})_\cdot := \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$
Note that $\bigoplus_{n\geq 0} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ is a graded $R(\mathcal{L})_\cdot$ mod.
Then $((\bigoplus_{n\geq 0} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}))_s)_0 \to \Gamma(X_s, \mathcal{F})$
is an isom.

Rmk:
1. If $\mathcal{L} = \mathcal{O}_X$, this is the QCQS lemma.
2. Any section of $\mathcal{F}$ over X_s can be extended to a section over X after multiplying by some appropriate powers of s.
And if we have two such extensions, they become equal after multiplying by another appropriate powers of s.

Pf. Using the same strategy of the pf of QCQS by choosing an affine open cover of X s.t. $\mathcal{L}$ locally trivial, we only need to prove for $X = \operatorname{Spec} A$.
In this situation, $\mathcal{F} = \widetilde{M}$, $\mathcal{L} = \widetilde{tA}$ for some $t \in \Gamma(X, \mathcal{L})$.
Assume $s = ft$ for $f \in A$, then $X_s = D(f)$,
$$\Gamma(X_s, \mathcal{F}) = \Gamma(D(f), \widetilde{M}) = M_f.$$
$$R(\mathcal{L})_\cdot = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n}) = \bigoplus_{n\geq 0} t^n A$$

$$\bigoplus_{n\geq 0}\Gamma(X, \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}) = \bigoplus_{n\geq 0} t^n M.$$

$$\left(\left(\bigoplus_{n\geq 0}\Gamma(X, \mathcal{F}\otimes_{\mathcal{O}_S}\mathcal{L}^{\otimes n})\right)_s\right)_0 = \left\{\frac{t^n m}{s^n} \,\middle|\, m\in M, n\geq 0\right\}$$

$$= \left\{\frac{m}{f^n} \,\middle|\, m\in M, n\geq 0\right\}$$

$$= M_f.$$

So they are isom.