P is a class of morphisms that preserve by base change and composition

PS is the class of morphism s.t $\forall \pi: X \to Y \in PS$, $\delta_\pi: X \to X \times_Y X \in P$.

11.1.1 The cancellation thm:

Suppose
$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z & \end{tikzcd}$$
then $\rho \in PS$, $\tau \in P \Rightarrow \pi \in P$.

pf: $X = X \times_Y Y \xrightarrow{\in P} X \times_Y Y \times_Z Y = X \times_Z Y \xrightarrow{\in P} Z \times_Z Y = Y$. □

11.1.2 Thm: P is reasonable $\Rightarrow$ PS is reasonable

pf: ① Composition: $X \to Y$, $Y \to Z \in PS$

$X \xrightarrow{\in P} X \times_Y X = X \times_Y Y \times_Y X \xrightarrow{\in P} X \times_Y Y \times_Z Y \times_Y X = X \times_Z X$

② Base change:
$$\begin{tikzcd} X \times_S Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \times_S Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ X \arrow[r, "\in PS"] & Y \arrow[r] & S \end{tikzcd}$$

$X\times_S Z = X\times_Y (Y\times_S Z)$.

$$X\times_S Z \xrightarrow{\in P} X\times_Y (X\times_S Z) = X\times_Y (Y\times_S Z)\times_{Y\times_S Z}(X\times_S Z) = (X\times_S Z)\times_{Y\times_S Z}(X\times_S Z)$$

③ Local on the target: $X\xrightarrow{\pi} Y \in P$,

$Y=\bigcup U_i$, $X=\bigcup \pi^{-1}(U_i)$, $X\times_Y X = \bigcup \pi^{-1}(U_i)\times_{U_i}\pi^{-1}(U_i)$

$\pi^{-1}(U_i)\xrightarrow{\in P_S} U_i \Rightarrow \pi^{-1}(U_i)\to \pi^{-1}(U_i)\times_{U_i}\pi^{-1}(U_i)\in P$

$\Rightarrow X\to X\times_Y X\in P$. □

11.1 E: P reasonable, S-sch mor $X\xrightarrow{\rho} Y$, $X\xrightarrow{\sigma} Z\in P$,

$X\to S\in P_S$. Show $(\rho,\sigma): X\to Y\times_S Z\in P$.

pf: $X\xrightarrow{\in P} X\times_S X \xrightarrow{\in P} Y\times_S X \xrightarrow{\in P} Y\times_S Z$ □

e.g Monomorphism:

$\pi: X\to Y$ monomorphism $\Leftrightarrow \delta_\pi: X\to X\times_Y X$ isom.

So monomorphisms are reasonable

pf: "$\Leftarrow$" $\forall f,g: Z\to X$ s.t.
$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

$$Z \xrightarrow[\substack{\shortparallel \\ (f,f)}]{(f,g)} X\times_Y X = X \Longrightarrow f=g.$$

"$\Rightarrow$"

$$\begin{tikzcd}
X\times_Y X \arrow[r, "\beta_1"] \arrow[d, "\beta_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}
\Longrightarrow \beta_1=\beta_2=:\beta.$$

$$\begin{tikzcd}
X\times_Y X \arrow[dr, "\beta"] \arrow[ddrrr, bend left, "\beta"] \arrow[dddrr, bend right, "\beta"'] & & & \\
& X \arrow[dr, "\alpha"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& & X\times_Y X \arrow[r, "\beta"'] \arrow[d, "\beta"] & X \arrow[d] \\
& & X \arrow[r] & Y
\end{tikzcd}$$

$\Rightarrow \beta\circ\alpha = id,\ \alpha\circ\beta = id \Rightarrow X \cong X\times_Y X.$ □

Def: $\pi: X\to Y$ is qs if $\delta_\pi: X\to X\times_Y X$ qc.

11.1.G: $\pi: X\to Y$ qs $\Leftrightarrow$ $\forall\, \operatorname{Spec} A \hookrightarrow Y$, $\forall$ aff $U,V\hookrightarrow X$ s.t. $U,V\subseteq \pi^{-1}(\operatorname{Spec}A)$, $U\cap V$ is a fin union of aff.

pf: Wlog let $Y=\operatorname{Spec}A$

"$\Rightarrow$" $X\to Y$ qs $\Rightarrow \delta_\pi: X\to X\times_Y X$ qc $\Rightarrow U\cap V = \delta_\pi^{-1}(\underbrace{U\times_Y V}_{\text{aff}})$ qc

$\Rightarrow U\cap V$ a fin union of aff

"$\Leftarrow$" $X\times_Y X = \bigcup_i U_i \times_Y V_i$, $U_i \cap V_i = \delta_\pi^{-1}(U_i \times_Y V_i)$ qc $\Rightarrow \delta_\pi$ qc

$\Rightarrow \pi$ qs. $\square$.

11.1.5 qc reasonable $\Rightarrow$ qs reasonable.

11.2 Separatedness and varieties

11.2.1 Prop: $X \xrightarrow{\pi} Y$

(a) X, Y affine $\Rightarrow \delta_\pi : X \to X\times_Y X$ closed emb.

(b) In general, $\delta_\pi : X \to X\times_Y X$ locally closed emb.

pf: (a) $A\otimes_B A \twoheadrightarrow A$.

(b) $Y = \bigcup_i V_i$, $\pi^{-1}(V_i) = \bigcup_j U_{ij}$ (aff cover)

$$U_{ij} \hookrightarrow U_{ij} \times_{V_i} U_{ij}$$ $\square$.

11.2.A: $X \to Y$, $X \to Z$ locally closed emb.

$\Rightarrow X \to Y\times Z$ locally closed emb.

pf: By 11.1.E, It is suffices to show locally closed embs reasonable

① Composition: ✓

② Base change: ✓

③ Locally on target: ✓ ☐.

Def: $\pi: X \to Y$ separated: $\delta_\pi: X \to X \times_Y X$ closed emb.

A-sch X separated: $X \to \operatorname{Spec} A$ separated

By 11.2.1(b) δ_π is locally closed emb.

So π separated $\Leftrightarrow \Delta = \operatorname{Im}(\delta_\pi)$ closed (Like Haus).

11.2.3 Prop: Sep reasonable

pf: closed emb reasonable + Thm 11.1.2 ☐.

11.2.C. monomorphism $\Rightarrow$ sep.

pf: $X \to Y$ monomorphism $\Rightarrow X \xrightarrow{\sim} X \times_Y X$ ☐

11.2.D. sep $\Rightarrow$ qs.

pf: closed emb is qc. ☐.

11.2.4: Aff $\Rightarrow$ sep, fin $\Rightarrow$ sep pf: Prop 11.2.1 (a) + local on the target $\square$.

11.2.E: $X \in A$-sch, X sep over $A \Leftrightarrow X$ sep over $\mathbb{Z}$.

(e.g. $A = \mathbb{C}$).

pf:
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \operatorname{Spec} A \arrow[d] \\ & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

$\operatorname{Spec} A \to \operatorname{Spec} A \otimes_{\mathbb{Z}} A$ sep.

$\square$.

11.2.F: $X \xrightarrow{\pi} Y \xrightarrow{\rho} Z$.

(a) $\rho \circ \pi$ locally closed emb (resp. locally of finite type, sep),

then so is π.

(b) $\rho \circ \pi$ qc, Y Noe $\Rightarrow \pi$ qc

(c) $\rho \circ \pi$ qs $\Rightarrow \pi$ qs.

pf: (a) ① locally closed emb:

$\delta_\rho : Y \to Y \times_Z Y$ locally closed emb $\Rightarrow \checkmark$.

② locally of finite type:

(b) Y Noe $\Longrightarrow Y \xrightarrow{\delta_\rho} Y \times_Z Y$ qc $\Longrightarrow \pi$ qc.

(c) Wlog let Z affine, $X \xrightarrow{\rho \circ \pi_1} Z \longrightarrow \operatorname{Spec} \mathbb{Z}$ qs

$\Longrightarrow X$ qs

By 8.3.B(b), π qs.

11.2.G. $\mu: Z \longrightarrow X$, $\sigma: X \longrightarrow Z$. $\mu \circ \sigma = id_X$.

$$\begin{tikzcd} Z \arrow[d, shift right, "\mu"'] \\ X \arrow[u, shift right, "\sigma"'] \end{tikzcd}$$

① σ locally closed emb

② μ sep. $\Longrightarrow \sigma$ closed emb

③ μ qs $\Longrightarrow \sigma$ qc.

④ σ may not be closed emb

pf: ① $$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "id"'] & Z \arrow[d, "\mu"] \\ & X \end{tikzcd}$$ $\left\{ \begin{array}{l} id \\ \delta_\mu \end{array} \right.$ locally closed emb

$\Longrightarrow \sigma$ locally closed emb.

② $\left. \begin{array}{l} \mu \text{ sep} \Longrightarrow \delta_\mu \text{ closed emb} \\ id \text{ closed emb} \end{array} \right\} \Longrightarrow \sigma$ closed emb id closed emb

③ — — —

④ $X = \operatorname{Spec} k[x]$, $Z = \operatorname{Spec} k[y] \cup \operatorname{Spec} k[z]$.

—— : ——

$$X \overset{\sigma}{\hookrightarrow} Z \xrightarrow{\mu} X.$$

$$\operatorname{Spec} k[x] \xrightarrow{\sim} \operatorname{Spec} k[y] \xrightarrow{\sim} \operatorname{Spec} k[x]$$

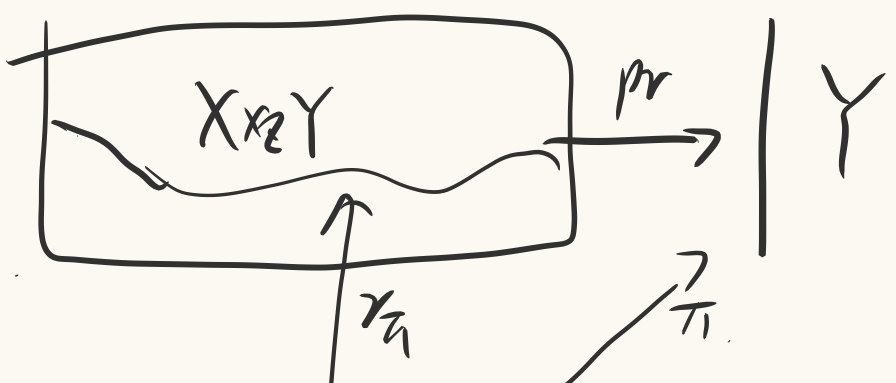
$$\operatorname{Spec} k[z] \xrightarrow{\sim} \operatorname{Spec} k[x]$$

σ not closed emb.

11.2.5. $\pi : X \to Y$ morphism of Z-sch.

$$\rightsquigarrow \begin{tikzcd} X \arrow[dr, "\gamma_\pi"] \arrow[drr, bend left] \arrow[ddr, bend right, "\pi"'] & & \\ & X \times_Z Y \arrow[r] \arrow[d, "pr"] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\gamma_\pi"] \arrow[dr, "\pi"'] & X \times_Z Y \arrow[d, "pr"] \\ & Y \end{tikzcd}$$

γ_π is called the graph morphism, $\Gamma_\pi := \operatorname{Im} \gamma_\pi$.



X

11.2.6: γ_π locally closed emb. If Y sep Z-sch, then γ_π closed emb. If Y qs Z-sch, then γ_π qc.

pf: Y sep $\Leftrightarrow$ $Y \to Y\times_Z Y$ closed emb $\Rightarrow$ $X \to X\times_Z Y$ closed emb

qs qc qs $\square$.

11.2.H, P preserved by composition and base change, closed embs $\subseteq P$.

$\forall \pi: X \longrightarrow Y$, $\rightsquigarrow$ $\exists! \, \pi^{red}: X^{red} \longrightarrow Y^{red}$.

Show $\pi \in P \Rightarrow \pi^{red} \in P$.

pf:

$$\begin{tikzcd}
X^{red} \arrow[r, "\pi^{red}"] \arrow[d, "\in P"'] \arrow[dr, "\in P"] & Y^{red} \arrow[d, "\in P"] \\
X \arrow[r, "\pi \in P"'] & Y
\end{tikzcd}$$

$X^{red} \longrightarrow X$, $X \longrightarrow Y \in P \Rightarrow X^{red} \longrightarrow Y \in P$.

$Y^{red} \to Y$ closed emb $\Rightarrow$ $Y^{red} \to Y^{red} \times_Y Y^{red}$ closed emb. $\Rightarrow$ $Y^{red} \to Y \in PS$

$\Rightarrow \pi^{red}: X^{red} \to Y^{red} \in PS$. 12.

Important examples:

11.2.8 Prop: $\mathbb{P}^n_A \xrightarrow{\pi} \operatorname{Spec} A$ sep

proof 1: Direct calculation:

$U_i = \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}]$, $(x_{i/i} = 1)$.

$\mathbb{P}^n_A = \bigcup_{i=0}^{n} U_i$, $\mathbb{P}^n_A \times_A \mathbb{P}^n_A = \bigcup_{i,j} U_i \times_A U_j$. $\delta_\pi: \mathbb{P}^n_A \to \mathbb{P}^n_A \times_A \mathbb{P}^n_A$.

Show $\delta_\pi^{-1}(U_i \times_A U_j) \longrightarrow U_i \times_A U_j$ closed emb.

$i=j$: $U_i \longrightarrow U_i \times_A U_i$, aff ✓.

$i \neq j$: $U_i \times_A U_j \cong \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}] \otimes_A A[y_{0/j}, \dots, y_{n/j}]$

$\cong \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, y_{0/j}, \dots, y_{n/j}]$.

$(\delta_\pi)^{-1}(U_i \times_A U_j) = U_i \cap U_j \cong \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}]/(x_{j/i} - 1)$.

$U_i \cap U_j \longrightarrow U_i \times_A U_j$

$$A[x_{0/i},\dots,x_{n/i}]/(x_{j/i}-1) \longleftarrow A[x_{0/i},\dots,x_{n/i},y_{0/j},\dots,y_{n/j}]$$

$$x_{k/i} \longleftarrow\!\shortmid\ x_{k/i}$$

$$x_{k/i} \longleftarrow\!\shortmid\ y_{k/i}.$$

$$\Longrightarrow (\delta\pi)^{-1}(U_i \times_A U_j) = U_i \cap U_j \hookrightarrow U_i \times_A U_j. \quad ✓ \qquad \square$$

Proof 2: Segre emb: $s: \mathbb{P}^n_A \times_A \mathbb{P}^n_A \hookrightarrow \mathbb{P}^{n^2+2n}$

The second Veronese emb: $v_2: \mathbb{P}^n_A \hookrightarrow \mathbb{P}_A^{\binom{n+2}{2}-1}$

$$S_\cdot = A[x_0,\dots,x_n].$$

$$\mathbb{P}^n_A = \operatorname{Proj} S. \qquad S_{2\cdot} = A[x_i x_j]_{i,j=0,\dots,n}$$

$$S_{2\cdot} \hookrightarrow S.$$

$$\rightsquigarrow v_2: \mathbb{P}^n_A \hookrightarrow \mathbb{P}_A^{\binom{n+2}{2}-1}.$$

$$\begin{tikzcd}
 & \mathbb{P}^n_A \times_A \mathbb{P}^n_A \arrow[dr, "s"] & \\
\mathbb{P}^n_A \arrow[ur, "\delta\pi"] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\circlearrowleft"] & & \mathbb{P}^{n^2+2n}_A \\
 & \mathbb{P}_A^{\binom{n+2}{2}-1} \arrow[ur, "L"'] &
\end{tikzcd}$$

$(n+2)(n+1)$

$$
\begin{tikzcd}
& ([x_0,\dots,x_n],[x_0,\dots,x_n]) \arrow[dr, mapsto] & \\
{[x_0,\dots;x_n]} \arrow[ur, mapsto] \arrow[dr, mapsto] \arrow[rr, phantom, "\circlearrowleft"] & & \begin{bmatrix} x_0^2 & x_0x_1 & \cdots & x_0x_n \\ x_1x_0 & & & \vdots \\ \vdots & & & \vdots \\ x_nx_0 & \cdots & \cdots & x_n^2 \end{bmatrix} \\
& {[x_0^2, x_0x_1, \dots, x_{n-1}x_n, x_n^2]} \arrow[ur, mapsto] &
\end{tikzcd}
$$

$\Delta \circ v_2$, S closed emb $\Rightarrow \delta_\pi$ closed emb. □

11.2.9 Def (Important):

A variety over k (or k-variety), is a reduced, sep sch of fin type over k.

11.2.J (Easy): Mor of k-varieties are finite type and sep.

Pf.

$$
\begin{tikzcd}
X \arrow[r, "\checkmark"] \arrow[dr, "\text{fin type (or sep)}"'] & Y \arrow[d, "\text{sep}"] \\
& \operatorname{Spec} k
\end{tikzcd}
$$

□

Def: subvariety of a variety X is a reduced locally closed subsch of X.

An open subvariety is an open subsch of X.
closed reduced closed.

11.2.11.

(a) k perfect. X, Y k-variety $\Rightarrow X \times_k Y$ k-variety.

(b) k alg closed, X, Y irreducible k-variety $\Rightarrow X \times_k Y$ irreducible k-variety

(c) k alg closed, X, Y connected k-variety $\Rightarrow X \times_k Y$ connected k-variety.

Pf: (a) $X \times_k Y$ fin type, sep.

Cor 10.5.23: k perfect, A, B reduced k-alg $\Rightarrow A \otimes_k B$ reduced

Wlog let X, Y affine. $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $X \times_k Y = \operatorname{Spec} A \otimes_k B$.
✓

(b). alg closed $\Rightarrow$ perfect $\Rightarrow X \times_k Y$ k-variety.

10.5.N: X irr $\Leftarrow$ $\exists$ open cover $X = \bigcup_i U_i$, U_i irr, $U_i \cap U_j \neq \emptyset$. $(\forall i,j)$

$X = \bigcup_i U_i$, $Y = \bigcup_j V_j$, $X \times_k Y = \bigcup_{i,j} U_i \times_k V_j$.

$\forall (i,j), (i',j')$, $(U_i \times_k V_j) \cap (U_{i'} \times_k V_{j'})$

$= (U_i \cap U_{i'}) \times_k (V_j \cap V_{j'})$.

Suffices to show $U_i \cap V_j$ irr. $\forall i, j$

Wlog let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $X \times_k Y = \operatorname{Spec} A \otimes_k B$.

10.5.M k separably closed, A, B k-alg. with irr Spec

$\Rightarrow A \otimes_k B$ has irr Spec.

✓.

(c) Cor 10.5.12: k separably closed, X connected k-sch

$\Rightarrow X$ geometrically connected

10.5.9 lem: X geometrically connected over k, $\forall Y/k$,

$X \times_k Y \to Y \rightsquigarrow$ a bij of connected components.

$\Rightarrow X \times_k Y$ connected. □.

Back to sep:

11.2.13 Prop: $X \to \operatorname{Spec} A$ sep, aff $U, V \subseteq X \Rightarrow U \cap V$ aff.

(sep $\Rightarrow$ affine-diagonal) ($\not\Leftarrow$, e.g. —:—)

11.2.M: (Easy): $U, V \overset{o}{\hookrightarrow} X$, $X \in A\text{-sch}$,

$\Delta := \operatorname{Im}(X \hookrightarrow X \times_A X)$.

Show $\Delta \cap (U \times_A V) \cong U \cap V$

$$\begin{array}{ccc} \Delta \cap (U \times_A V) & \cong & U \cap V \\ \| & & \| \\ X \times_{X \times_A X} (U \times_A V) & & U \times_X V \end{array}$$

pf: (1.2.5)

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_A V \arrow[d] \\ X \arrow[r] & X \times_A X \end{tikzcd}$$

$\square$.

$$\begin{array}{c} X \times_{(X \times_A X)} U \times_A V \\ \| \\ \left[X \times_{(X \times_A X)} U \times_A X \right] \times_X V \\ \| \\ U \times_X V. \end{array}$$

pf of 11.2.13: $U \cap V \cong (U \times_A V) \cap \Delta \hookrightarrow \underbrace{U \times_A V}_{\text{affine}}$ $\square$.

Rem:

11.2.15 Cor: $\forall$ quasiprojective A-sch sep over A,
Every reduced quasiprojective k-sch is a k-variety
pf: ✓. $\square$.

11.2.IV $X \xrightarrow{\pi} \operatorname{Spec} A$ sep $\Leftrightarrow$ $\forall$ affine $U, V \hookrightarrow X$,

(i) $U \cap V$ affine

(ii) $\mathcal{O}(U) \otimes_A \mathcal{O}(V) \longrightarrow \mathcal{O}(U \cap V)$ surj.

pf: $\delta_\pi : X \longrightarrow X \times_A X$,

$$(\delta_\pi)^{-1}(U \times_A V) = U \cap V.$$

π sep $\Rightarrow$ $U \cap V$ aff.

$$U \cap V \hookrightarrow U \times_A V \Rightarrow \underset{\mathcal{O}(U \otimes_A V)}{\mathcal{O}(U) \otimes_A \mathcal{O}(V)} \twoheadrightarrow \mathcal{O}(U \cap V). \quad ✓$$

(i)(ii) $\Rightarrow$ $U \cap V = (\delta_\pi)^{-1}(U \times_A V) \hookrightarrow U \times_A V$ closed emb

$$X \times_A X = \bigcup U \times_A V$$

$\Rightarrow \pi$ sep $\square$.

Rem: We can change "$\forall$ affine $U, V \hookrightarrow X$" by "$\forall i \in I, U_i$ ($X = \bigcup_i U_i$)"

Universally inj :

11.20 $\pi : X \to Y$. π universally inj $\Longleftrightarrow$ $\delta_\pi : X \to X \times_Y X$ surj

pf: "$\Rightarrow$"

$$\begin{tikzcd}
X \arrow[dr, "\delta_\pi"'] \arrow[drr, bend left, "id"] & & \\
& X \times_Y X \arrow[r, "inj"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
& X \arrow[r, "inj"] & Y
\end{tikzcd}$$

$\Rightarrow$ δ_π is surj.

"$\Leftarrow$"

surj morphisms reasonable $\Rightarrow$ universally injective morphisms reasonable.

11.2.P:
$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "p\circ\pi"'] & Y \arrow[d, "p"] \\ & Z \end{tikzcd}$$
$p\circ\pi$ universally inj $\Rightarrow \pi$ universally inj.

pf: $p\circ\pi$ universally inj $\Rightarrow X \to X\times_Z X$ surj.

$\Rightarrow X \to X\times_Y X$ surj.

$$\begin{tikzcd} X \arrow[dr] \arrow[drr, bend left, two heads] \arrow[ddr, bend right, "\pi"'] & & \\ & X\times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X\times_Z X \arrow[d] \\ & Y \arrow[r] & Y\times_Z Y \end{tikzcd}$$

11.2.Q:

(a) universally inj $\Rightarrow$ sep

(b) X, Y finite type over alg closed $\bar{k}$, $\pi: X\to Y$

then π universally inj $\Leftrightarrow \pi$ is inj on closed points.

pf: (a) ✓

(b) - - -

Def: $\pi, \pi' : X \to Y \in \mathrm{Mor}(Z\text{-sch})$, $\mu: W \to X$

We say π, π' agree on μ if $\pi \circ \mu = \pi' \circ \mu$

11.3. A: $\exists$ a locally closed subsch $i: V \hookrightarrow X$ s.t π, π' agree on i and $\forall \mu: W \to X$ s.t π, π' agree on μ, $\exists !\, j: W \to V$

s.t
$$\begin{tikzcd} W \arrow[r, "j"] \arrow[dr, "\mu"'] & V \arrow[d, "i"] \\ & X \end{tikzcd} \quad ?$$

If $Y \to Z$ sep, then i closed emb

pf: Define
$$\begin{tikzcd} V \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "\delta"] \\ X \arrow[r, "(\pi, \pi')"] & Y \times_Z Y \end{tikzcd}$$

δ locally closed emb $\Rightarrow$ i locally closed emb

$Y \to Z$ sep $\Rightarrow$ δ closed emb $\Rightarrow$ i closed emb.

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists"] \arrow[rrd, bend left, "\pi\circ\mu=\pi'\circ\mu"] \arrow[ddr, "\mu"'] & & \\
& V \arrow[r] \arrow[d, "i"] & Y \arrow[d, "\delta"] \\
& X \arrow[r, "(\pi,\pi')"] & Y\times_Z Y
\end{tikzcd}$$

$\square$

Rem: V is called the locus where π, π' agree.

As a set, $V = \{x\in X \mid \pi(x)=\pi'(x)=y,\ \pi^\# = (\pi')^\#: k_y \to k_x\}$.

pf: "$\supseteq$" $\forall x\in$, let $W=\operatorname{Spec} k_x$,

$$\begin{tikzcd}
W \arrow[r, hook, "\mu"] & X \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\pi'"'] & Y \\
x \arrow[r, mapsto] & x \arrow[r, mapsto] & y \\
k_x & & k_y \arrow[ll]
\end{tikzcd}$$

$\Rightarrow \pi\circ\mu = \pi'\circ\mu \Rightarrow x\in V$.

"$\subseteq$" $\forall x\in V$, $W=\operatorname{Spec} k_x$.

$$\begin{tikzcd}
W \arrow[r, "j"] & V \arrow[r, "i"] & X \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\pi'"'] & Y \\
x \arrow[r, mapsto] & x & & \\
k_x \arrow[r, equal] & k_x \arrow[r, equal] & k_x & k_y \arrow[l]
\end{tikzcd}$$

$\pi\circ i\circ j = \pi'\circ i\circ j \Rightarrow \pi^\# = (\pi')^\#: k_y \to k_x$.

$\square$

11.3.B $\pi, \pi' : X \to Y \in \mathrm{Mor}(\bar{k}\text{-varieties})$.

π, π' are the same at the level of closed points

(i.e $\forall$ closed point $x \in X$, $\pi(x) = \pi'(x)$)

$\Rightarrow \pi = \pi'$.

pf: Wlog let $X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$.

$$A = \bar{k}[x_1, \dots, x_n]/I, \quad B = \bar{k}[y_1, \dots, y_m].$$

$(\mathrm{Spec}\, \bar{k}[y_1, \dots, y_m]/J \hookrightarrow \mathrm{Spec}\, \bar{k}[y_1 \dots y_m])$

(n is minimal)

$$\pi^\#: y_i \longmapsto \varphi_i(x_1, \dots, x_n) = \sum_I e_I x^I$$

$$(\pi')^\#: y_i \longmapsto \varphi_i'(x_1, \dots, x_n) = \sum_I e_I' x^I.$$

The closed points of X has the form

$p = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$.

$$\pi(p) = \pi'(p) \Rightarrow \text{“} \varphi_i(\alpha_1, \dots, \alpha_n) = 0 \Leftrightarrow \varphi_i'(\alpha_1, \dots, \alpha_n) = 0 \text{”}$$

$$\Rightarrow e_I \equiv e_I', \forall I$$

$$\Rightarrow \varphi_i = \varphi_i'$$

$$\Rightarrow \pi = \pi'.$$

$\square$

11.3.2. (Reduced-to-Sep thm)

$\pi: U \to Z$, $\pi': U \to Z \in Mor(S\text{-}sch)$, U reduced, Z sep over S,

If $\exists$ dense open $W \hookrightarrow U$ s.t π, π' agree on W, then $\pi = \pi'$.

pf: Let V be the locus where π, π' agree. then $W \subseteq V$.

By 11.3.A, $Z \to S$ sep $\Rightarrow$ V closed and dense in U.

U is reduced $\Rightarrow V = U \Rightarrow \pi = \pi'$. $\square$

11.3.D Reduced and sep are necessary in 11.3.2.

pf: Skipped. $\square$

11.3.4 Graphs of rational maps: X reduced, Y sep,

$\pi: X \dashrightarrow Y$ rational map $\quad (X \to X \times_Z Y$

Let (U, π') be any representative of π, $\Gamma_{\pi'} \hookrightarrow U \times Y \hookrightarrow X \times Y$.

$\Gamma_\pi :=$ the scheme-theoretic closure of $\Gamma_{\pi'} \hookrightarrow X \times Y$.

11.3.E: Γ_π is independent of the choice of (U, π').

pf: Let (V, π'') be a rep of π, s.t $\forall$ rep (U, π'), $U \subseteq V$

Wlog let $V = X$, $\pi'' = \pi$,

$$\Rightarrow \begin{tikzcd} X \arrow[r, "\pi"] & Y \arrow[d, equal] \\ U \arrow[u] \arrow[r, "\pi'"] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook, "\gamma_\pi", "\text{Locally closed emb}"'] \arrow[rr, bend right, "id_X"'] & X \times Y \arrow[r, "pr"] & X \end{tikzcd}$$

$\Rightarrow \Gamma_\pi = \operatorname{Im} \gamma_\pi \cong X$. , Likewise, $\Gamma_{\pi'} \cong$ Scheme-theoretic closure of U in X $\overset{\substack{U \text{ dense} \\ X \text{ reduced}}}{=} X$. ✓. □

$$\begin{tikzcd} \Gamma_\pi \arrow[r, hook, "\text{closed emb}"] & X \times Y \arrow[dl, "pr_1"] \arrow[dr, "pr_2"] & \\ X \arrow[u, dashed, "\gamma_\pi"] \arrow[rr, dashed, "\pi"] & & Y \end{tikzcd}$$

11.3.F $\mathbb{A}^2_k \overset{\pi}{\dashrightarrow} \mathbb{P}^1_k$, $(x,y) \mapsto [x,y]$.

① Show π cannot be extended over the origin

② The graph of π is the morphism described in 10.3.F pf: ① Assume $(0,0) \mapsto [x,y]$. Wlog let $y \neq 0$,

Let $U_0 = \operatorname{Spec} k[x_0/x_1] \hookrightarrow \mathbb{P}^1_k$,

$\exists f \in k[x_0, x_1]$ s.t. $D(f) \subseteq \pi^{-1}(U_0)$ and $(0,0) \in D(f)$.

$\rightsquigarrow k[x_0/x_1] \longrightarrow k[x_0, x_1, f^{-1}]$.

11.4 Proper morphisms

Def: universally closed: $\pi: X \to Y$, $\forall Z \to Y$,

$X \times_Y Z \to Z$ closed.

11.4.A (easy): universally cl mors reasonable

pf: ① composition ✓

② base change ✓.

③ local on the target ✓

□.

Def: $\pi: X \to Y$ is proper if π sep, fin type, universally cl.

X is proper if $X \to \operatorname{Spec} \mathbb{Z}$ proper

X is proper over S if $X \to S$ proper

11.4.B (Easy) $\pi : \mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ not universally closed.

11.4.2. cl emb $\Rightarrow$ proper

11.4.3. Fin $\Rightarrow$ proper.

pf: Fin $\Rightarrow$ sep, fin type

Fin is preserved by base change $+ \underbrace{(\text{Fin} \Rightarrow \text{cl.})}_{8.3.\text{C (going-up)}}$.

$\Rightarrow$ (Fin $\Rightarrow$ universally cl.)

12.

11.4.4 Prop

(a)-(c) Proper reasonable.

(d) $\pi : X \to Y$, $\pi' : X' \to Y' \in \operatorname{Mor}(\mathbb{Z}\text{-sch})$ proper.

$\Rightarrow \pi \times \pi' : X \times_{\mathbb{Z}} X' \to Y \times_{\mathbb{Z}} Y'$ proper

(e) $\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z & \end{tikzcd}$, τ proper, ρ sep $\Rightarrow \pi$ proper.

pf. (a)-(c) ✓.

(d) $X \times_{\mathbb{Z}} X' \xrightarrow{\text{proper}} Y \times_{\mathbb{Z}} X' \xrightarrow{\text{proper}} Y \times_{\mathbb{Z}} Y'$ ✓.

(e) δ_ρ cl emb $\Rightarrow$ δ_ρ proper $\Rightarrow$ π proper □.

11.4.C
$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\tau"'] & Y \arrow[d, "\rho"] \\ & Z \end{tikzcd}$$
τ proper, ρ sep. fin type

$\Rightarrow$ The scheme theoretic image of X under π is a proper Z-scheme

pf:
$$\begin{tikzcd} & \bar{X} \arrow[d, hook] \\ X \arrow[ur] \arrow[r] \arrow[dr, "\tau"'] & Y \arrow[d, "\rho"] \\ & Z \end{tikzcd}$$
???

11.4.5 Thm: Projective A-schs are proper over A.

pf: It is suffices to show $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ universally closed,

i.e $\forall$ sch X, $\mathbb{P}^n_X \to X$ cl.

i.e $\forall$ aff sch $\operatorname{Spec}B$, $\mathbb{P}^n_B \to \operatorname{Spec}B$ cl., By 8.4.10 ✓. □.

Rem: Proper $\not\Rightarrow$ Projective

11.4.7 X connected reduced proper k-sch. $k=\bar{k}$ then $\Gamma(X,\mathcal{O}_X)=k$

Pf: $\forall f\in\Gamma(X,\mathcal{O}_X)$, $\rightsquigarrow \pi: X\longrightarrow \mathbb{A}_k^1$
$$f \leftarrow\!\shortmid x$$

$\pi': X\xrightarrow{\pi}\mathbb{A}_k^1\hookrightarrow\mathbb{P}_k^1$,
$$\begin{tikzcd} X \arrow[r, "\pi'"] \arrow[dr] & \mathbb{P}_k^1 \arrow[d] \\ & \mathrm{Spec}\,k \end{tikzcd}$$
$\Rightarrow \pi'$ proper $\Rightarrow \pi'$ cl.

X connected $\Rightarrow$ The set-theoretic image of π' is a closed point p or all of $\mathbb{P}^1$ (impossible for $\mathbb{A}_k^1\subsetneq\mathbb{P}_k^1$).

By 9.4.5 Cor. the scheme theoretic image of π' is closure$(p)=p$.

By 9.4.A. X reduced $\Rightarrow$ $\underbrace{\text{the scheme theoretic image of } \pi'}_{\text{"}=\text{" } p \text{ with reduced structure}}$

$\Rightarrow \pi: X\longrightarrow \mathrm{Spec}\,k\longrightarrow\mathbb{A}_k^1$

$f=\alpha \leftarrow\!\shortmid \alpha \leftarrow\!\shortmid x$ ✓. □
$(p=(x-\alpha))$.

Facts: A morphism of locally Noe sch is finite $\Leftrightarrow$ proper and aff $\Leftrightarrow$ proper and quasifinite.

Application: $\underbrace{X \text{ proper}, Y \text{ sep}}_{\text{Locally Noe}}$, $\pi : X \to Y$ quasifinite

$\Longrightarrow \pi$ finite.

e.g.: $\pi : \mathbb{P}^1_k \to \mathbb{P}^1_k$, $[x,y] \mapsto [f(x,y), g(x,y)]$

fin fibers + π proper $\Longrightarrow \pi$ finite.

$$\left(\begin{tikzcd} \mathbb{P}^1_k \arrow[r, "\pi"] \arrow[dr] & \mathbb{P}^1_k \arrow[d] \\ & k \end{tikzcd} \right)$$