§1 (10.1.17) Discribing the existence of fibred products using the fancy language of representable functors

Preliminaries

"functor": contravariant functor $F: Sch \to Set$

"h_X": $X \in Sch$, $h_X = Hom(\_, X)$ (Yoneda embedding)

"Fun": the category of all functors ($Sch \to Set$)

morphisms: natural transformations.

Then $h: Sch \to Fun$ making Sch a full subcategory of Fun.

$X \longmapsto h_X$

$\forall F \in Fun$, we say F is representable if $\exists X \in Sch$ s.t. $F = h_X$

And for $h, h', h'' \in Fun$ $\left(\begin{tikzcd}[column sep=small] h \arrow[dr] & & h' \arrow[dl] \\ & h'' & \end{tikzcd}\right)$, $h \times_{h''} h'$ always exists

(Def $h \times_{h''} h'(T) \triangleq h(T) \times_{h''(T)} h'(T)$, the fibre product in Set)

Then "$X \times_Z Y$ exists" $\Leftarrow$ "$h_X \times_{h_Z} h_Y$ is representable"

($\because$ $Sch \xhookrightarrow{\text{full sub}} Fun$)

The sketch of the proof

Step 1 (Zariski sheaves)

$F \in Fun$ is called a Zariski sheaf if

$\forall Y \in Sch$, $\forall \{U_i\}_i$ open covering of Y, we have the following equalizer.

$$\cdot \longrightarrow F(Y) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

Since we could glue the morphisms, it's clear that h_X is a Zariski sheaf for each $X \in Sch$. Then being a Zariski sheaf is a necessary condition for being a representable functor.

Ex (10.1.C) Show that $h_X \times_{h_Z} h_Y$ is a Zariski sheaf.

Step 2 (subfunctor, open subfunctor, the cover of functors)

h' is called a subfunctor if $h'(X) \subseteq h(X)$ $\forall X \in Sch$

h' is called an open subfunctor if $h' \times_h h_X$ is represented by h_U where $U \overset{\circ}{\hookrightarrow} X$, $\forall X \in Sch$

$\{h_i\}_i$, a collection of open subfunctors of h, is called a cover of h if $\{U_i\}_i$ forms an open covering for X where $h_{U_i} = h_i \times_h h_X$, $\forall X \in Sch$.

Ex (10.1.E(b)) Define the intersection of two open subfunctors.

Step 3 (A sufficient condition for a Zariski sheaf to be representable)

10.1.H. KEY EXERCISE. If a functor h is a Zariski sheaf that has an open cover by representable functors ("is covered by schemes"), then h is representable.

Ex (10.1.G) Show that $h_X \times_{h_Z} h_Y$ is representable

§2. (10.5) Geometrically connected, irreducible, reduced

First, we'll introduce some statements Let $X \in Sch$

· C_K, $C_{K(=\bar{K})}$, $C_{\bar{k}}$, C_{k^s}	Notation explanation:
· I_K, $I_{K(=\bar{K})}$, $I_{\bar{k}}$, I_{k^s}	$P_L := X_L = X \times_k \operatorname{Spec} L$ is [P] for all fields [L]
· R_K, $R_{K(=\bar{K})}$, $R_{\bar{k}}$, R_{k^p}	C ~ connected, I - irreducible, R - reduced

geometrically ***

Thm (10.5.3): C_K, $\boxed{C_{K(=\bar{K})}}$, $C_{\bar{k}}$, C_{k^s} are equivalent.

I_K, $\boxed{I_{K(=\bar{K})}}$, $I_{\bar{k}}$, I_{k^s} are equivalent.

R_K, $\boxed{R_{K(=\bar{K})}}$, $R_{\bar{k}}$, R_{k^p} are equivalent.

(We will show it in the following contents.)

Fact☆(10.5.6) $X \in Sch_k$, then $X \to \operatorname{Spec} k$ is universally open i.e. remains open after any base change. (to be proved in Thm 24.5.11).

Lemma (10.5.7) E/F purely insep $X \in Sch_F$. Then $\phi_E : X_E \to X$ is a homeo.

p.f.
$$\begin{tikzcd} \phi_E^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_E \arrow[r] \arrow[d, "\phi_E"] \arrow[dr, phantom, "\square"] & \operatorname{Spec} E \arrow[d, "\phi"] \\ \operatorname{Spec} k(x) = \{x\} \arrow[r, hook] & X \arrow[r] & \operatorname{Spec} F \end{tikzcd}$$

Since E/F is purely inseparable, then ϕ is universally injective by 10.5.I

As sets, $X_E = \bigsqcup_{x \in X} \phi_E^{-1}(x) \xleftrightarrow{1:1} \bigsqcup_{x\in X} \{x\} = X$. $(\#\phi_E^{-1}(x) = 1, \forall x \in X)$

And ϕ_E is continuous naturally.

Moreover, ϕ_E is open, which comes from the Fact☆.

Thus, ϕ_E is a homeomorphism $\square$

§2.1 (10.5.8) Connectedness

Recall (10.5.J): $X \in Top$, $X = \bigsqcup_{i \in I} X_i$ where X_i's are connected components of X and X_i's are closed

10.5.K. EASY TOPOLOGICAL EXERCISE. Suppose $\phi: X \to Y$ is open, and has nonempty connected fibers. Show that ϕ induces a bijection of connected components.

p.f. Write $X = \bigsqcup_{i\in I} X_i$, $Y = \bigsqcup_{j \in J} Y_j$ We'll define a map $\varphi: I \to J$

Then $\phi(X_i) \subseteq Y_j$ for some (unique) $j \in J$, let $\varphi(i) = j$.

1° φ is surj. clear, since ϕ has non-empty fibres $\Rightarrow \phi$ is surj.

2° φ is inj. First, $\phi(X_i) \cap \phi(X_{i'}) = \phi$ If not, $\exists y \in \phi(X_i) \cap \phi(X_{i'})$

$\Rightarrow \phi^{-1}(y) \cap X_i \neq \phi$, $\phi^{-1}(y) \cap X_{i'} \neq \phi \Rightarrow \phi^{-1}(y) \subseteq X_i, X_{i'}$, since $\phi^{-1}(y)$ is connected.

$\Rightarrow$ a contradiction $(X_i \cap X_{i'} = \phi)$

$\Rightarrow Y_j = \bigsqcup_{i \in \varphi^{-1}(j)} \phi(X_i) = \phi(\bigsqcup_{i \in \varphi^{-1}(j)} X_i) \Rightarrow \#\varphi^{-1}(j) = 1$, $\because \bigsqcup_{i\in\varphi^{-1}(j)} X_i$ should be connected.

If $\phi^{-1}(Y_j) = (U \cup U') \cap \phi^{-1}(Y_j)$, then $\phi(U) \cup \phi(U') \supseteq Y_j \Rightarrow \phi(U')$ or $\phi(U) \supseteq Y_j$ $\square$

10.5.9. Lemma. — *Suppose X is geometrically connected over k. Then for any scheme Y/k, $X \times_k Y \to Y$ induces a bijection of connected components.*

Proof. Combine Fact 10.5.6 and Exercise 10.5.K. $\square$

$$\begin{tikzcd} X\times_k Y \arrow[r, "open"', "\phi"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ X \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

and ϕ has nonempty connected fibres

$\phi^{-1}(y) = X \times_k \operatorname{Spec}(k(y))$

nonempty : surjectiveness is stable under base change.

connected : X is C_K

rmk. $C_{K(=\bar{K})} \Rightarrow C_K$

pf. If X is $C_{K(=\bar{K})}$, then for any K/k, $X_{\bar{K}}$ is connected.

$$\begin{tikzcd} X_{\bar{K}} \arrow[r] \arrow[d, "p"', "surj"] \arrow[dr, phantom, "\square"] & \operatorname{Spec}\bar{K} \arrow[d, "surj", Leftarrow] \\ X_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d] \\ X \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

p is surjective and continuous $\Rightarrow X_K = P(X_{\bar{K}})$ is connected.

(By the same reason, $I_{K(=\bar{K})} \Rightarrow I_K$, and $R_{K(=\bar{K})} \Rightarrow R_K$ is easy) $\square$

Prop (10.5.L) $X \in Sch$. X is disconnected $\Leftrightarrow \exists\, 0,1 \neq e \in \Gamma(X, \mathcal{O}_X)$ s.t. $e^2 = e$.

pf. "$\Rightarrow$" $X = U \sqcup V$ opens $\Rightarrow \Gamma(X, \mathcal{O}_X) \cong \mathcal{O}_X(U) \times \mathcal{O}_X(V)$. $e = (1,0)$ ✓

"$\Leftarrow$" $X_0 = X_e := \{x \in X \mid e_x \neq 0 \text{ in } \mathcal{O}_{X,x}\}$, $X_1 = X_{1-e}$

Then $X = X_0 \sqcup X_1$, ($0 = e_x(1-e_x) \in \mathfrak{m}_x \Rightarrow e_x$ or $1-e_x \in \mathfrak{m}_x \Rightarrow e_x$ or $1-e_x \in \mathcal{O}_{X,x}^*$)

And X_0, X_1 are open $\Rightarrow X$ is disconnected $\square$

Prop (10.5.10) Suppose $k = k^s$ and $A \in k\text{-alg}$ and $\operatorname{Spec} A$ is connected. Then $\operatorname{Spec} A$ is geometrically connected.

pf. It suffices to show $\operatorname{Spec} A$ satisfies $C_{K(=\bar{K})}$. Thus WMA $K = \bar{K}$. $\Rightarrow k \to \bar{k} \to K$

Consider
$$\begin{tikzcd} \operatorname{Spec} A \otimes_k K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \otimes_k \bar{k} \arrow[r, "\text{homeo.}", "\text{by Lem. 10.5.7}"'] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} \bar{k} \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

It suffices to assume $k=\bar{k}$.

If $\mathrm{Spec}(A\otimes_k K)$ is disconnected, then $\exists\, e \triangleq \sum_i^N a_i\otimes l_i \in \mathrm{Idem}(A\otimes_k K)\setminus\{0,1\}$.

Let $B=k[l_1,\dots,l_N]\subseteq K$. Then B is integral and f.t. over k. Then $\mathrm{Frac}(B)$ is a f.g. field ext of k and $\mathrm{Spec}\,B$ is connected

$B\hookrightarrow K \Rightarrow A\otimes_k B\hookrightarrow A\otimes_k K \Rightarrow e\in \mathrm{Idem}(A\otimes_k B)\setminus\{0,1\} \Rightarrow \mathrm{Spec}(A\otimes_k B)$ is disconnected. Write $\mathrm{Spec}(A\otimes_k B)=U\sqcup V$

$$\begin{tikzcd}
\mathrm{Spec}\,A\otimes_k K \arrow[r] \arrow[d] & \mathrm{Spec}\,A\otimes_k B \arrow[r] \arrow[d, "\phi"', "\text{open}"] & \mathrm{Spec}\,A \arrow[d, "\text{universally open}"] \\
\mathrm{Spec}\,K \arrow[r] & \mathrm{Spec}\,B \arrow[r] & \mathrm{Spec}\,k
\end{tikzcd}$$

$\Rightarrow \{\phi(U),\phi(V)\}$ is an open cover of $\mathrm{Spec}\,B \Rightarrow \phi(U)\cap\phi(V)\neq\emptyset$.

$\Rightarrow \exists$ clsd pt $p\in\phi(U)\cap\phi(V) \Rightarrow k(p)=k$ (Noe. normalization lemma)

$\Rightarrow \mathrm{Spec}\,A\simeq\phi^{-1}(p)\subseteq U\sqcup V$ and $U\cap\mathrm{Spec}\,A\neq\emptyset$, $V\cap\mathrm{Spec}\,A\neq\emptyset$

$\Rightarrow \mathrm{Spec}\,A$ is disconnected, a contradiction $\square$

(tensor-finiteness trick)

Coro (10.5.12) If $k=k^s$ and Y is a connected k-scheme, then Y is geometrically connected.

pf. $\mathrm{Spec}\,K$ is geometrically connected by the above prop.

Then $Y_K\longrightarrow Y$ induces a bijection of connected components by taking $X=\mathrm{Spec}\,K$ in Lemma (10.5.9) $\Rightarrow Y_K$ is connected $\square$

Rmk: $C_{k^s}\Rightarrow C_K$

§2.2 (10.5.13) Irreducibility

Prop (10.5.14) Suppose $k=k^s$, $A\in k\text{-alg}$ with $\operatorname{Spec} A$ irr, and K/k field ext, then $\operatorname{Spec}(A\otimes_k K)$ is irr.

pf. Lemma: $I_{K(=\bar K)} \Rightarrow I_K$.

proof of the lemma: $X_{\bar K} \twoheadrightarrow X_K$ and $X_{\bar K}$ is irr $\Rightarrow X_K$ is irr.

Thus, WMA $K=\bar K$. And $k\hookrightarrow \bar k \hookrightarrow K$. Since $\bar k/k$ is purely inseparable, we may assume $k=\bar k$.

If $\operatorname{Spec}(A\otimes_k K)$ is not irr, then $\exists$ proper closed subset $V(x)$, $V(y)$ covering $\operatorname{Spec}(A\otimes_k K)$. Write $x=\sum a_i\otimes l_i$, $y=\sum b_j\otimes s_j$ ($\Rightarrow x\cdot y$ is nilp)

Let $B=k[l_i,s_j]$. $\Rightarrow B$ is integral. f.t. $A\otimes_k B\hookrightarrow A\otimes_k K \Rightarrow x\cdot y$ is still nilp.

$$\begin{tikzcd}
\operatorname{Spec} A\otimes_k K \arrow[r] \arrow[d] & \operatorname{Spec} A\otimes_k B \arrow[r] \arrow[d, "\phi"', "\text{open}"] & \operatorname{Spec} A \arrow[d, "\text{universally open}"] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} B \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

$\Rightarrow \phi(D(x))\cap\phi(D(y))\neq\emptyset$ $\because$ $\operatorname{Spec} B$ is irreducible

$\Rightarrow \exists$ closed pt $p\in D(x)\cap D(y) \Rightarrow \phi^{-1}(p)\cong \operatorname{Spec} A\subseteq V(x)\cup V(y)$, $\operatorname{Spec} A\nsubseteq V(x), V(y)$

$\Rightarrow \operatorname{Spec} A$ is not irr, a contradiction. $\square$

Prop (10.5.M) $k=k^s$. $A,B\in k\text{-alg}$, $\operatorname{Spec} A$, $\operatorname{Spec} B$ irr. Then $\operatorname{Spec}(A\otimes_k B)$ is irr.

pf. Firstly, we may assume A, B are reduced and hence integral.

Since $\operatorname{Spec} A_{red}\simeq \operatorname{Spec} A$, $\operatorname{Spec} B_{red}\simeq \operatorname{Spec} A_{red}$

$$\operatorname{Spec}(A_{red}\otimes_k B_{red})\simeq \operatorname{Spec}(A_{red}\otimes_k B_{red})_{red}\underset{\Delta}{\simeq}\operatorname{Spec}(A\otimes_k B)_{red}\simeq \operatorname{Spec}(A\otimes_k B)$$

Δ: Write $\mathfrak{a}=\{a\in A\mid a^n=0 \text{ for some } n\}$, $\mathfrak{b}=\cdots$, then $A/\mathfrak{a}\otimes_k B/\mathfrak{b}=A\otimes_k B/(\mathfrak{a}\otimes_k B+A\otimes_k \mathfrak{b}$

And $\mathfrak{a}\otimes_k B+A\otimes_k\mathfrak{b}\subseteq \operatorname{nil}(A\otimes_k B)$, then we've done.

$$\begin{tikzcd}
\mathrm{Spec}(A\otimes_k B) \arrow[r] \arrow[d] & \mathrm{Spec}(A\otimes_k B') \arrow[r] \arrow[d, "\text{open}"] & \mathrm{Spec}\,A \arrow[d] \\
\mathrm{Spec}\,B \arrow[r] & \mathrm{Spec}\,B' \arrow[r] & \mathrm{Spec}\,k
\end{tikzcd}$$

$\mathrm{Spec}(A\otimes_k B) = V(x)\cup V(y)$

$\Rightarrow B'\subseteq B$, $\mathrm{Spec}(A\otimes_k B') = V(x)\cup V(y)$

Similar to the above prop, B' is f.t. and $B'\subseteq B \Rightarrow B'$ is Integral.

$\mathrm{Spec}(A\otimes_k B)$ is not irr $\Rightarrow \mathrm{Spec}\,A$ is not irr, a contradiction $\square$

10.5.N. EASY EXERCISE. Show that a scheme X is irreducible if and only if there exists an open cover $X = \cup U_i$ with U_i irreducible for all i, and $U_i \cap U_j \neq \varnothing$ for all i, j.

p.f. ref. Lin Qing.

Proposition 4.5. *Let X be a topological space.*

(a) *If X is irreducible, then any non-empty open subset of X is dense in X and is irreducible.*

(b) *Let U be an open subset of X. Then the irreducible components of U are the $\{X_i \cap U\}_i$, where the X_i are the irreducible components of X which meet U.*

Then "$\Rightarrow$" is clear.

"$\Leftarrow$": By (b), $U_k \subseteq X_i \setminus \bigcup_{j\neq i} X_j$ for some i, $\forall k$.

$\Rightarrow$ If #(the irr components of X) $\geq 2 \Rightarrow \exists\, U_1 \subseteq X_1\setminus X_2$,

$U_2 \subseteq X_2 \setminus X_1 \Rightarrow \phi \neq U_1\cap U_2 \subseteq (X_1\setminus X_2)\cap(X_2\setminus X_1) = \phi$, a contradiction $\square$

Coro (10.5.15) If $k = k^s$ and Y is a irreducible k-scheme, then Y is geometrically irreducible ($\exists K$).

p.f. $Y = \bigcup_i U_i$ where U_i's are affine open Then $U_i \cap U_j \neq \phi$ and U_i's are irr $\Rightarrow Y_K = \bigcup_i (U_i)_K$ and $(U_i)_K \overset{\circ}{\hookrightarrow} Y_K$ and $(U_i)_K \cap (U_j)_K \neq \phi$ ($\because (U_i\cap U_j)_K \neq \phi$)

And $(U_i)_K$'s are irr by the first prop Then Y_K is irr by 10.5.N. $\square$

§2.3. (10.5.16) Reducedness

Fact (10.5.17) Any f.g. field ext E/F over a perfect field F can be factored into a finite separable part and a purely transendental part:

$$\begin{array}{c} E \\ | \\ F(t_1,\dots,t_n) \\ | \\ F \end{array}$$

(? To show $\dim_E \Omega^1_{E/F} = n$ ref QingLiu 6.1.15.)

Prop (10.5.20) Suppose A is a reduced k-alg Then

(a) $A \otimes_k k(t)$ *is reduced.*

(b) *If* E/k *is a finite separable extension, then* $A \otimes_k E$ *is reduced.*

p.f. (a) Since localization preserves reducedness, it suffices to show $A \otimes_k k[t] \cong A[t]$ is reduced and that's clearly true

(b) By induction, WMA $E = k[t]/(p(t))$.

If $A \otimes_k E$ is not reduced $\Rightarrow \exists\, x = \sum_i^N a_i \otimes e_i \in \mathrm{nilp}(A \otimes_k E) \Rightarrow A' = k[a_1, \dots, a_N] \subseteq A$ is reduced. And $A' \otimes_k E$ is not, since $x \in \mathrm{nilp}(A' \otimes_k E)$. Since A' is a f.g. k-alg $\Rightarrow$ $A' \hookrightarrow \prod_{\mathfrak{p}} A'/\mathfrak{p} \hookrightarrow \prod_{\mathfrak{p}} \mathrm{Frac}(A'/\mathfrak{p}) \triangleq \tilde{A}$ where $\mathfrak{p}$'s are the minimal prime ideals of A The injection comes from the reducedness of A'. And $\tilde{A}$ is a finite product of fields since A' is Noe. $\Rightarrow$ It suffices to assume A is a field

And $A \otimes_k E = A[t]/(p(t))$ is reduced by the separableness of $p(t)$. □

Lemma (10.5.21) Suppose E/k is a field ext of a perfect field k Then $A \otimes_k E$ is reduced.

p.f. If $x = \sum_i^N a_i \otimes e_i \in \mathrm{Nilp}(A \otimes_k E) \Rightarrow x \in \sum_i^N a_i \otimes e_i \in \mathrm{Nilp}(A \otimes_k E')$. $E' = k(e_1, \dots, e_N)$

Thus, we may assume E is f.g. Then, by the fact (10.5.17) and induction, we've actually done by using the above prop □

Coro (10.5.23) If $k=k^p$ and Y is a reduced k-scheme, then Y is geometrically reduced (Rk)

pf. Since reducedness is a local property, we've done by the above lemma.

Then we've proved the main theorem

§2.4 (Appendix)

10.4.H. EXERCISE (CF. EXERCISE 10.2.E). Suppose X and Y are integral finite type $\overline{k}$-schemes. Show that $X \times_{\overline{k}} Y$ is an integral finite type $\overline{k}$-scheme.

10.5.O. EXERCISE (GEOMETRICALLY INTEGRAL × INTEGRAL = INTEGRAL). Suppose B is a k-algebra such that $B \otimes_k \overline{k}$ is an integral domain (Spec B is geometrically integral), and A is a k-algebra that is an integral domain (Spec A is integral). Show that $A \otimes_k B$ is an integral domain (Spec $A \otimes_k B$ is integral).

10.5.19. Proposition (geometrically reduced × reduced = reduced). — *Suppose B is a geometrically reduced k-algebra, and A is a reduced k-algebra. Then $A \otimes_k B$ is reduced.*

10.5.23. Corollary. — *Suppose k is perfect, and A and B are reduced k-algebras. Then $A \otimes_k B$ is reduced.*

10.5.R. EXERCISE. Suppose that A and B are two integral domains that are $\overline{k}$-algebras. Show that $A \otimes_{\overline{k}} B$ is an integral domain. (Compare this to Exercise 10.4.H, which had finite type hypotheses.)

§3 (10.6) The Serge embedding.

Classical case $\mathbb{P}^m_k \times \mathbb{P}^n_k \hookrightarrow \mathbb{P}^{mn+m+n}_k$

$$([X_0, \cdots, X_m], [Y_0, \cdots, Y_n]) \longmapsto \begin{bmatrix} X_0 \\ \vdots \\ X_m \end{bmatrix} [Y_0 \cdots Y_n]$$

Scheme case $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \hookrightarrow \mathbb{P}^{mn+m+n}_A$

① Glueing of morphism

② Show the image of this map is cut out by the equations

$$\mathrm{rank}\begin{pmatrix} Z_{00} & - & Z_{0n} \\ \vdots & \ddots & \\ Z_{m0} & & Z_{mn} \end{pmatrix} = 1$$

§4. (10.7) Normalization

Recall: **7.5.1.** *Definition.* A **rational map** π of reduced schemes, from X to Y, denoted $\pi: X \dashrightarrow Y$, is the data of a morphism $\alpha: U \to Y$ from a dense open set $U \subset X$, with the equivalence relation $(\alpha: U \to Y) \sim (\beta: V \to Y)$ if there is a dense open set $Z \subset U \cap V$ such that $\alpha|_Z = \beta|_Z$. (In §11.3.3, we will improve this to: if $\alpha|_{U\cap V} = \beta|_{U \cap V}$ in good circumstances — when Y is separated.)

A rational map $\pi: X \dashrightarrow Y$ is **dominant** (or in some sources, *dominating*) if for some (and hence every) representative $U \to Y$, the image is dense in Y. A morphism is a **dominant morphism** (or *dominating morphism*) if it is dominant as a rational map.

For an integral scheme X, its normalization $(\widetilde{X}, \nu)$ exists and it is characterized by the following properties:

① $\widetilde{X}$ is normal and irreducible. (normal: $\mathcal{O}_{\widetilde{X},x}$ is normal for all $x \in \widetilde{X}$.)

② $\widetilde{X} \xrightarrow{\nu} X$ is a dominant morphism

③ (universal property) $\forall$ irr normal

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, dashed, "\exists!"'] & X \\ & \widetilde{X} \arrow[u, "\nu"'] \end{tikzcd}$$

Firstly, let's deal with the affine case.

Prop (10.7.A) If A is an integral domain, and $\widetilde{A}$ is its integral closure, then $\operatorname{Spec}\widetilde{A} \to \operatorname{Spec}A$ is the normalization of $\operatorname{Spec}A$.

p.f. It's clear from the commutative algebras and the following easy fact.

If X is an integral normal scheme, then X admits an affine open covering $\{\operatorname{Spec}A_i\}_i$ s.t. A_i's are integral and integral closed (normal).

Prop. (10.7.B) For integral schemes, the normalization exists

p.f. Routine arguement

Affine case $\longrightarrow$ Separated case $\longrightarrow$ General case.

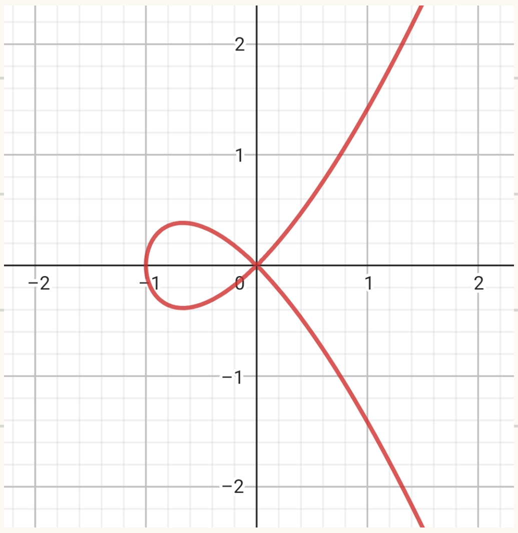
10.7.C. EASY EXERCISE. Show that normalizations are integral and surjective.

More general normalization see Stacks, tag 035Q.

Then we'll show some examples.

Example (10.7.E) $\mathrm{Spec}(k[t]) \longrightarrow \mathrm{Spec}(k[x,y]/(y^2-x^2(x+1)))$ given by $x \mapsto t^2-1$, $y \mapsto t(t^2-1)$

is a normalization. (char $k \neq 2$)



p.f. First, it's well-defined, since $(t(t^2-1))^2 = (t^2-1)^2(t^2-1+1)$ ✓

Then, since they're affine, the assertion follows from the following observations:

$$\begin{tikzcd} \underset{\text{① int domain}}{k[x,y]/(y^2-x^2(x+1))} \arrow[r, hook, "③"] \arrow[rr, bend right, "② \text{ Frac field}"'] & \overset{\text{④ int closed}}{k[t]} \arrow[r, hook, "\text{Frac field}", "⑤"'] & k(t) \end{tikzcd}$$

④, ⑤ Obvious ② $\because \frac{y}{x} = t$

① routine check ③ $\Leftarrow$ ① + ② + ⑤

Example (10.7.F) The normalization of $y^2 = x^3$ i.e. $\mathrm{Spec}\, k[x,y]/(y^2-x^3)$ is $\mathrm{Spec}\, k[t]$ with the morphism defined by $(x,y) \longmapsto (t^2, t^3)$.

$$\begin{tikzcd} \underset{\text{Int dom}}{k[x,y]/(y^2-x^3)} \arrow[r, hook] \arrow[rr, bend right, "\text{Frac}"'] & k[t] \arrow[r, hook, "\text{Frac}"] & k(t) \end{tikzcd}$$

$\because \frac{y}{x} = t$

Example (10.7.H) $\mathrm{Spec}(\mathbb{Z}[i]) \longrightarrow \mathrm{Spec}(\mathbb{Z}[15i])$ is a normalization

Then we'll give a generalized type of normalization.

Def X, int scheme. The normalization $\nu: \widetilde{X} \longrightarrow X$ of X in a given algebraic field extension L of the function field $K(X)$ of X is a dominant morphism from a normal integral scheme $\widetilde{X}$ with $K(\widetilde{X}) = L$, such that

ν induces the inclusion $K(X) \hookrightarrow L$, and that is universal with this property.

$$
\begin{tikzcd}
\operatorname{Spec} L \arrow[r, equal] \arrow[dr, equal] & \operatorname{Spec} K(Y) \arrow[r] \arrow[d, "\simeq"'] & Y \arrow[d, dashed, "\exists!"'] \arrow[dd, bend left] & \text{normal} \\
& \operatorname{Spec} K(\tilde{X}) \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\nu"'] & \text{normal} \\
& \operatorname{Spec} K(X) \arrow[r] & X &
\end{tikzcd}
$$

And the existence follows by the routine arguement.

The most classical example is the rings of integers in number fields. We admit that readers are familier this.

Example (6.7.L)

(a) $\operatorname{Spec} k[x,y]/(y^2-x^2-x)$ is the normalization of $X = \operatorname{Spec} k[x]$ in the field extension $k(x)(y)$ where $y^2 = x^2+x$ i.e. $\operatorname{Frac}(k(x)[y]/(y^2-x^2-x))$.

$$k[x] \xhookrightarrow{\text{normal}} k[x,y]/(y^2-x^2-x) \xhookrightarrow{\text{Frac}} k(x)(y)$$

Thus, all we need to do is to show $k[x,y]/(y^2-x^2-x)$ is normal.

It follows from 5.4.H

5.4.H. Handy Exercise (yielding many enlightening examples later). Suppose A is a unique factorization domain with 2 invertible, and $z^2 - f$ is irreducible in $A[z]$.
(a) Show that if $f \in A$ has no repeated prime factors, then $\operatorname{Spec} A[z]/(z^2 - f)$ is normal. Hint: $B := A[z]/(z^2 - f)$ is an integral domain, as $(z^2 - f)$ is prime in $A[z]$. Suppose we have monic $F(T) \in B[T]$ so that $F(T) = 0$ has a solution α in $K(B) \setminus K(A)$. Then by replacing $F(T)$ by $\overline{F}(T)F(T)$, we can assume $F(T) \in A[T]$. Also, $\alpha = g + hz$ where $g, h \in K(A)$. Now α is the solution of $Q(T) = 0$ for monic $Q(T) = T^2 - 2gT + (g^2 - h^2 f) \in K(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K(A)[T]$. By Gauss's lemma, $2g, g^2 - h^2 f \in A$. Say $g = r/2$, $h = s/t$ (s and t have no common factors, $r, s, t \in A$). Then $g^2 - h^2 f = (r^2t^2 - 4s^2 f)/4t^2$. Then t is invertible.
(b) Show that if $f \in A$ has repeated prime factors, then $\operatorname{Spec} A[z]/(z^2 - f)$ is *not* normal.

(b) Now, we could figure out the normalization of $\mathbb{P}^1_k$ in the field extension $k(X)(Y)$ where $Y^2 = X^2 + X$

$$
\begin{tikzcd}[column sep=small]
\mathbb{P}^1_k = \mathrm{Proj}(k[X_0, X_1]) = \mathrm{Spec}\, k[\frac{X_0}{X_1}] & \cup & \mathrm{Spec}\, k[\frac{X_1}{X_0}] \\
\mathrm{Spec}\, k[Y_1, \frac{X_0}{X_1}]/(Y_1^2 - (\frac{X_0}{X_1})^2 - \frac{X_0}{X_1}) \arrow[u] & \cup & \mathrm{Spec}\, k[Y_0, \frac{X_1}{X_0}]/(Y_0^2 - (\frac{X_1}{X_0})^2 - \frac{X_1}{X_0}) \arrow[u] \\
& \| & \\
& \mathrm{Proj}(k[X_0, X_1, Y]/(Y^2 - X_0(X_0 + X_1))) &
\end{tikzcd}
$$