Def 17.2.4

$\mathcal{F}$ finite rank locally free sheaf on X,
$\mathbb{P}\mathcal{F} := \mathrm{Proj}(\mathrm{Sym}^{\cdot}\mathcal{F}^{\vee})$ is called its projectivization.

Prop.

1. $p: Z \to X$,

$$\begin{array}{ccc} \mathbb{P}(p^*\mathcal{F}) & \longrightarrow & Z \\ \downarrow & \square & \downarrow \\ \mathbb{P}\mathcal{F} & \longrightarrow & X \end{array}$$

i.e. $\mathbb{P}(p^*\mathcal{F}) \cong \mathbb{P}(\mathcal{F}) \times_X Z$.

Pf 17.2.4

2. $\mathbb{P}_X := \mathbb{P}(\mathcal{O}_X^{n+1})$ agree with earlier definition.

Pf. $\mathbb{P}(\mathcal{O}_X^{n+1}) \cong \mathbb{P}_{\mathbb{Z}} \times_{\mathbb{Z}} X$.

3. $\mathcal{L}$ line bundle on X,
$\mathbb{P}(\mathcal{F}) \cong \mathbb{P}(\mathcal{L} \otimes \mathcal{F})$.

Rmk. 17.2.5.

There is violent disagreement on $\mathbb{P}\mathcal{F}$ should be defined as
$\mathrm{Proj}\,\mathrm{Sym}^{\cdot}\mathcal{F}^{\vee}$ or $\mathrm{Proj}\,\mathrm{Sym}^{\cdot}\mathcal{F}$.
We try to avoid the notation $\mathbb{P}\mathcal{F}$.

Example. Ruled surface 17.2.6

If C is a regular curve, $\mathcal{F}/C$ locally free of rk 2.

$\mathbb{P}\mathcal{F}$ is called a ruled surface over C.

If $C \cong \mathbb{P}^1$, $\mathbb{P}\mathcal{F}$ is called a Hirzebruch surface

$\mathcal{F} \cong \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$, $\mathbb{P}\mathcal{F} \cong \mathbb{P}(\mathcal{O}(n_1) \oplus \mathcal{O}(n_2))$

only depends on $n_2 - n_1$. $\mathbb{F}_n := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$.

Prop. 17.2.H

If $\mathcal{S}_\bullet$ is fin. gen. in deg 1, we have a canonical closed embedding

$$\begin{tikzcd} \mathcal{P}roj\, \mathcal{S}_\bullet \arrow[rr, hook, "i"] \arrow[dr, "\beta"'] & & \mathcal{P}roj_X \mathrm{Sym}^\bullet \mathcal{S}_1 \arrow[dl, "\alpha"] \\ & X & \end{tikzcd}$$

and an isom. $\mathcal{O}_{\mathcal{P}roj\, \mathcal{S}_\bullet}(1) \xleftrightarrow{\sim} i^* \mathcal{O}_{\mathcal{P}roj_X \mathrm{Sym}^\bullet \mathcal{S}_1}(1)$

arising from $\mathrm{Sym}^\bullet \mathcal{S}_1 \twoheadrightarrow \mathcal{S}_\bullet$

In particular, if $\mathcal{S}_1$ locally free, $\mathcal{P}roj\, \mathcal{S}_\bullet \hookrightarrow \mathbb{P}\mathcal{S}^\vee$.

Pf. Let $X = \bigcup_i U_i$ affine cover, $U_i = \mathrm{Spec}\, A_i$

Then $\mathcal{P}roj\, \mathcal{S}_\bullet = \bigcup_i \beta^{-1}(U_i) = \bigcup_i \mathrm{Proj}\, \mathcal{S}_\bullet(U_i)$

$\mathcal{P}roj_X \mathrm{Sym}^\bullet \mathcal{S}_1 = \bigcup_i \alpha^{-1}(U_i) = \bigcup_i \mathcal{P}roj_X \mathrm{Sym}^\bullet \mathcal{S}_1(U_i)$.

Over U_i, $\mathrm{Proj}\, \mathcal{S}_\bullet(U_i) \hookrightarrow \mathcal{P}roj_X \mathrm{Sym}^\bullet \mathcal{S}_1(U_i)$.

- - - -

Prop.

$\mathcal{F}/X$ locally free of rk $n+1$,

$\mathrm{Grass}(\mathcal{F},1) \cong \mathbb{P}\mathcal{F}$.

Pf. It suffice to check that

$\mathrm{Mor}_X(X, \mathbb{P}\mathcal{F}) \xleftrightarrow{\sim} \{\mathcal{F} \twoheadrightarrow \mathcal{L} \mid \mathcal{L}/X \text{ invertible bundle}\}$.

$\varphi \longmapsto (\mathcal{F} \twoheadrightarrow \varphi^* \mathcal{O}_{\mathbb{P}\mathcal{F}}(1))$

is bijective.

$X = \bigcup_i U_i$, $U_i = \mathrm{Spec}\, A_i$ s.t. $\mathcal{F}|_{U_i}$ trivial,

$\mathbb{P}\mathcal{F}|_{U_i} = \mathrm{Proj}\, \mathrm{Sym}^\bullet \mathcal{F}|_{U_i}^\vee$

$= \mathrm{Proj}\, \mathrm{Sym}^\bullet \mathcal{F}^\vee(U_i)$.

Assume $\mathcal{F}(U_i) \cong t_{i0}\mathcal{O}_{U_i} \oplus \cdots \oplus t_{in}\mathcal{O}_{U_i}$,

$\mathrm{Proj}\, \mathrm{Sym}^\bullet \mathcal{F}^\vee(U_i) \cong \mathrm{Proj}\, A_i[t_{i0}^\vee, \cdots, t_{in}^\vee] \cong \mathbb{P}^n_{A_i}$.

So by the universal property of $\mathbb{P}^n_{A_i}$,

$\mathrm{Mor}_X(U_i, \mathbb{P}\mathcal{F}|_{U_i}) \xleftrightarrow{1:1} \{\mathcal{F}|_{U_i} \twoheadrightarrow \mathcal{L}_i\}$.

(check that the morphism is independent with the choice of $t_{i0}, \cdots, t_{in}$)

Then gluing morphism.

17.3 Projective morphisms

Recall

In §12.1,

affine morphisms $X \to Y \longleftrightarrow$ isom. $X \xrightarrow{\sim} \mathrm{Spec}\, \mathcal{B}$

for some QCoh sheaf of algebras on Y.

Def. 17.3.1

A morphism $\pi: X \to Y$ is projective if there is a isom.

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[dr, "\pi"'] & \mathrm{Proj}\, \mathcal{S}. \arrow[d] \\ & Y \end{tikzcd}$$

for "good" $\mathcal{S}$. (fin. gen. in deg 1, 17.2.1)

We say X is a projective Y-scheme

or X is projective over Y.

Prop. 17.3.A

1. $\pi: X \to Y$ is proj. iff $\exists$ fin. type qcoh sheaf $\mathcal{S}_1$ on Y, and a closed embedding $i: X \hookrightarrow \mathrm{Proj}_Y \mathrm{Sym}^\bullet \mathcal{S}_1$ over Y.
2. $\mathcal{L}$ invertible sheaf on X, $\pi: X \to Y$ proj. with $\mathcal{O}(1) \cong \mathcal{L}$ iff $\exists\, i: X \hookrightarrow \mathrm{Proj}_Y \mathrm{Sym}^\bullet \mathcal{S}_1$ over Y and $i^* \mathcal{O}_{\mathrm{Proj}_Y \mathrm{Sym}^\bullet \mathcal{S}_1}(1) \xrightarrow{\sim} \mathcal{L}$.
3. Suppose Y admits an ample line bundle, as is the case whenever Y is projective, affine or quasi-projective, π is proj. iff π factor through $X \hookrightarrow \mathbb{P}^n_Y \to Y$ for some n.

Here Y is a quasi-compact scheme, we say $\mathcal{M}/Y$ is ample if $\{X_f \mid f \in \Gamma(X, \mathcal{M}^{\otimes n}), n > 0\}$ form a base for the top of X.

Pf. 1. "$\Rightarrow$": By 17.2.H, $X \simeq \mathrm{Proj}\, \mathcal{S}. \rightsquigarrow X \hookrightarrow \mathrm{Proj}_Y \mathrm{Sym}^\bullet \mathcal{S}_1$.

"$\Leftarrow$" $X \xrightarrow{\sim} \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{S}_1)/\mathcal{J}$ for some ideal sheaf of $\mathrm{Sym}^\bullet \mathcal{S}_1$.

2. "$\Rightarrow$" Similarly By 17.2.H

"$\Leftarrow$" $X \xrightarrow{\sim} \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{S}_1/\mathcal{J})$. ...

3. "$\Leftarrow$" triv. by 1. $(\mathbb{P}^n_Y = \mathbb{P}(\mathcal{O}_Y^{n+1}))$

"$\Rightarrow$" $X \xrightarrow{\sim} \mathrm{Proj}\, \mathcal{S}_\cdot \hookrightarrow \mathrm{Proj}_Y \mathrm{Sym}^\cdot \mathcal{S}_1 \to Y$ (by 1)

$\mathcal{M}$ is a invertible bundle on Y. by Thm 16.2.6,

$\mathcal{S}_1 \otimes \mathcal{M}^{\otimes N}$ is gen. by global section for $N >> 0$

So we have $\mathcal{O}_Y^{(n+1)} \twoheadrightarrow \mathcal{S}_1 \otimes \mathcal{M}^{\otimes N}$ for some n.

By 17.2.I, $\mathcal{O}_Y^{(n+1)} \twoheadrightarrow \mathcal{S}_1 \otimes \mathcal{M}^{\otimes N}$

$\rightsquigarrow \mathrm{Proj}_Y \mathrm{Sym}^\cdot \mathcal{S}_1 \otimes \mathcal{M}^{\otimes N} \hookrightarrow \mathrm{Proj}\, \mathrm{Sym}^\cdot \mathcal{O}_Y^{(n+1)} = \mathbb{P}^n_Y$.

By 17.2.G, $X \hookrightarrow \mathrm{Proj}_Y \mathrm{Sym}^\cdot \mathcal{S}_1 \xrightarrow{\sim} \mathrm{Proj}_Y \mathrm{Sym}^\cdot \mathcal{S}_1 \otimes \mathcal{M}^{\otimes N} \hookrightarrow \mathbb{P}^n_Y$.

Prop. 17.3.2

projective morphism is preserved by base change.

projective $\Rightarrow$ proper

Prop.

finite $\Rightarrow$ projective

Pf. Assume $Z \to X$ is finite, $Z \cong \mathrm{Spec}\, \mathcal{B}$

for a fin. type qcoh sheaf of alg. $\mathcal{B}$ on X.

Consider a sheaf of graded algebra $\mathcal{S}_\cdot$: $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_n = \mathcal{B}$, $\forall n > 0$.

Clearly $\mathcal{S}_\cdot$ is "good".

Claim: $Z \xrightarrow{\sim} \mathrm{Proj}\, \mathcal{S}_\cdot$ over X.

$\forall\, U = \mathrm{Spec}\, A \hookrightarrow X$, $\mathcal{B}(U) := B$, $\mathrm{Spec}\, \mathcal{B}|_U = \mathrm{Spec}\, B$.

$\mathrm{Proj}\, \mathcal{S}_\cdot(U) = \mathrm{Proj}\, S_\cdot$, where $S_0 = A$, $S_n = B$, $\forall n > 0$.

$\mathrm{Proj}\, S_\cdot = D_+(1 \in S_1 = B) = \mathrm{Spec}\, B$.

closed embedding $\Rightarrow$ finite $\Rightarrow$ projective $\Rightarrow$ proper.

Warning 12.3.4

1. [Har]: $\pi: X \to Y$ is projective iff π factor through $X \hookrightarrow \mathbb{P}^n_Y \to Y$.
   This is the same as our def in most circumstances (12.3.A(c))
   But finite $\not\Rightarrow$ projective under this def.
2. projective morphisms are NOT local on the target.
   so it is not reasonable in the sense of § 8.1.
   (it is local on the target with additional data of $\mathcal{O}(1)$ and Noe. condition)
3. projective morphisms are preserved by base change
   only when the final target is affine or Noe.

Prop.

1. $\pi: X \hookrightarrow Y$ closed emb., $\rho: Y \to Z$ projective
   $\Rightarrow \rho \circ \pi: X \hookrightarrow Y \to Z$ proj.
2. $\rho: Y \to Z$ has projective diagonal iff ρ is seperated.
3. (Cancellation thm)

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\text{proj.}"'] & & Y \arrow[dl, "\text{sep.}"] \\ & Z & \end{tikzcd} \Rightarrow \pi \text{ proj.}$$

1. $Y \simeq \operatorname{Proj} S_\bullet$, $X \simeq \operatorname{Proj} S_\bullet / I$. - -
2. $\delta: Y \to Y \times_Z Y$
   "$\Leftarrow$": closed emb. $\Rightarrow$ proj.
   "$\Rightarrow$": proj. $\Rightarrow$ proper $\Rightarrow$ universally closed
   but δ is a locally closed emb. $\Rightarrow \delta$ is a closed emb.

3. $$\begin{tikzcd} X = X\times_Y Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X\times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z\times_Z Y = Y \arrow[d] \\ Y \arrow[r, "\text{proj.}"] & Y\times_Z Y & \\ & X \arrow[r, "\text{proj.}"] & Z \end{tikzcd}$$

Prop. 17.3.D

A morphism from a proj. k-scheme to a sep. k-scheme is projective.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\text{proj.}"'] & Y \arrow[d, "\text{sep.}"] \\ & \operatorname{Spec} k \end{tikzcd}$$

cancellation thm.

Prop. 17.3.5

$\pi: X \to Y$ proj, $\pi': X' \to Y'$ proj

$\Rightarrow \pi\times\pi_1\ X\times_S X' \to Y\times_S Y'$ proj.

Pf. $X \cong \operatorname{Proj} \mathcal{S}_\bullet \to Y$, $X' \cong \operatorname{Proj} \mathcal{S}'_\bullet \to Y'$.

If $S = \operatorname{Spec} A$,

$X\times_A X' = \operatorname{Proj} \bigoplus_{n=0}^\infty (\mathcal{S}_n \otimes_A T_n) \to Y\times_A Y'$.

For general S, gluing.

$(\operatorname{Proj} S_\bullet)\times_A(\operatorname{Proj} T_\bullet) \xleftrightarrow{\sim} \operatorname{Proj} \bigoplus_{n=0}^\infty (S_n \otimes_A T_n)$ in affine case)

Def. 17.3.5

$\pi: X \to Y$ is QCQS. $\mathcal{F} \in QCoh(X)$,

We say $\mathcal{F}$ is globally generated w.r.t. π

if $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ surj.

QCQS $\Rightarrow \pi_*\mathcal{F} \in QCoh(Y)$.

Suppose $\mathcal{L}/X$ locally free, say $\mathcal{L}$ is base-pt-free w.r.t. π if it is globally generated w.r.t. π.

Exercise. 17.3.F

$\mathcal{L}$ relatively base-pt-free line bundle on X, $\pi: X \to Y$ QC, sep., $\pi_* \mathcal{L}$ is finite type on Y.

There is a canonical morphism $\psi: X \longrightarrow \mathrm{Proj}_Y \mathrm{Sym}^\cdot(\pi_*\mathcal{L})$.

Pf. $\pi_*\mathcal{L} \in QCoh(Y)$ is finite type,

Case 1. $Y = \mathrm{Spec}\, A$ affine, $\pi_*\mathcal{L} \cong \widetilde{\Gamma(Y, \pi_*\mathcal{L})} \cong \underbrace{\Gamma(X, \mathcal{L})}_{\text{finite type}}$.

$\pi^*\pi_*\mathcal{L} = \Gamma(X,\mathcal{L}) \otimes_A \mathcal{O}_X \twoheadrightarrow \mathcal{L}$,

$\mathrm{Proj}_Y \mathrm{Sym}^\cdot(\pi_*\mathcal{L}) = \mathrm{Proj}\ \mathrm{Sym}^\cdot \Gamma(X,\mathcal{L})$

$\forall\, s \in \Gamma(X,\mathcal{L})$,

Consider $\psi_s: X_s \longrightarrow D_+(m) = \mathrm{Spec}((\mathrm{Sym}^\cdot \Gamma(X,\mathcal{L}))_s)_0$

induced by $((\mathrm{Sym}^\cdot \Gamma(X,\mathcal{L}))_s)_0 \longrightarrow ((\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^n))_s)_0$

$$\downarrow s \quad (15.4.N)$$

$$\Gamma(X_s, \mathcal{O}_X)$$

given by $\frac{s'}{s} \longmapsto \frac{s'}{s}$.

check ψ_s glues.

Case 2 General Y

glueing morphism.

We say $\mathcal{L}$ is relatively ample or π-ample
if $\forall \operatorname{Spec} B \hookrightarrow Y$, $\mathcal{L}|_{\pi^{-1}(\operatorname{Spec} B)}$ is ample on $\pi^{-1}(\operatorname{Spec} B)$.

17.3.6

Many statement of § 16.1 carry over without change.

Thm.

Suppose $\pi: X \to Y$ is proper, $\mathcal{L}$ is an invertible sheaf on X, Y is quasi-compact, TFAE:

(a) $\exists N > 0$, $\mathcal{L}^{\otimes N}$ is π-very ample

(a') $\forall n \gg 0$, $\mathcal{L}^{\otimes n}$ is π-very ample

(b) $\forall$ finite type $\mathcal{F} \in QCoh(X)$, $\exists n_0$ s.t. $\forall n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is relatively globally gen.

(c) $\mathcal{L}$ is π-ample.

Pf. Y is quasi-compact. use 16.2.2 on an finitely affine cover of Y.

Prop. 17.3.H

$\pi: X \to Y$ proper, Y locally Noetherian (hence X is too)
$\mathcal{L}/X$ invertible bundle on X,
Suppose we know that $\pi_* \mathcal{L}$ is finite type, (proved later, 18.9.1)
Show that $\mathcal{L}$ is π-very ample iff
(i) $\mathcal{L}$ is relatively base-pt-free, and
(ii) the canonical morphism $\psi: X \to \operatorname{Proj}_Y \operatorname{Sym}^\cdot(\pi_*\mathcal{L})$ of 17.3.F is a closed embedding.
In particular, relatively very ampleness is affine-local on Y, if π proper.

Pf "$\Leftarrow$": clearly $\pi: X \to Y$ is projective.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & \mathrm{Proj}_Y \mathrm{Sym}^{\cdot} \pi_* \mathcal{L} \arrow[d] \\ & Y \end{tikzcd}$$

$\mathcal{O}_X(1) \cong \mathcal{L}$ by 17.3.A.(b).

"$\Rightarrow$" $\mathcal{L}$ is π-very ample,

$X \cong \mathrm{Proj}\, \mathcal{S}_{\cdot} \xrightarrow{\pi} Y$,

$\forall\, U \hookrightarrow Y$ affine, $\mathrm{Proj}_Y \mathcal{S}_{\cdot}(U) = \pi^{-1}(U) \xrightarrow{\pi} U$

$\pi^* \pi_* (\mathcal{O}_X(1)|_{\pi^{-1}(U)}) \twoheadrightarrow \mathcal{O}_X(1)|_{\pi^{-1}(U)}$ by direct calculation over $U = \mathrm{Spec} A$.

$\Rightarrow \mathcal{L} \cong \mathcal{O}_X(1)$ is relatively base pt free

recall $\varphi_U : X|_U \to \mathrm{Proj}_Y \mathrm{Sym}^{\cdot} \pi_* \mathcal{L}|_U$ is induced by

$$((\mathrm{Sym}^{\cdot}\, \Gamma(U, \pi_*\mathcal{L}))_s)_0 \longrightarrow ((\bigoplus_{n \geq 0} \Gamma(X|_U, \mathcal{L}|_U^{\otimes n}))_s)_0 \cong \Gamma((X|_U)_s, \mathcal{L}|_U)$$

$$\frac{s'}{s} \longmapsto \frac{s'}{s}$$

We need to check that this map is surjective.

i.e. $\forall\, s' \in \Gamma(X|_U, \mathcal{L}|_U^{\otimes n})$, $\dfrac{s'}{s^n} = \dfrac{s_1 s_2 \cdots s_n}{s^n}$

for some $s_i \in \Gamma(X|_U, \mathcal{L}|_U)$.

Notice that $\Gamma(X|_U, \mathcal{L}|_U) = \Gamma(X|_U, \mathcal{O}_X(1))$

$\Gamma(X|_U, \mathcal{L}|_U^{\otimes n}) = \Gamma(X|_U, \mathcal{O}_X(n))$.

Then the surjectivity follows from $\mathcal{S}_{\cdot}$ is gen. by $\mathcal{S}_1$.

Prop.

$\pi: X \to Y$ and $\rho: Y \to Z$ proj.

If Z is affine or Noetherian, $\rho \circ \pi$ is proj.

Pf. Let $X \cong \mathrm{Proj}_Y \mathcal{S}^\bullet$, $Y \cong \mathrm{Proj}_Z \mathcal{T}^\bullet$.

$$X \hookrightarrow \mathrm{Proj}_Y \mathrm{Sym}^\bullet \mathcal{S}_1 \overset{(+)}{\hookrightarrow} \mathbb{P}_Y^{n-1} = \mathbb{P}_Z^{n-1} \times_Z Y \hookrightarrow \mathbb{P}_Z^{n-1} \times_Z \mathrm{Proj}_Z \mathrm{Sym}^\bullet \mathcal{T}_1$$

$$= \mathrm{Proj}_Z \mathrm{Sym}^\bullet \mathcal{O}_Z^n \times_Z \mathrm{Proj}_Z \mathrm{Sym}^\bullet \mathcal{T}_1 = \mathrm{Proj}_Z \bigoplus_{k \geq 0} (\mathrm{Sym}^k \mathcal{O}_Z^n \otimes \mathrm{Sym}^k \mathcal{T}_1).$$

It suffices to construct $(+)$.

Case 1. Z affine

Suppose $\mathcal{M}/Y$ is the very ample w.r.t. $Y \to Z$.

by thm 16.2.2, for $m \gg 0$, $\mathcal{S}_1 \otimes \mathcal{M}^m$ is gen by global section.

$$\mathcal{O}_Y^n \twoheadrightarrow \mathcal{S}_1 \otimes \mathcal{M}^m$$

$\rightsquigarrow \mathrm{Proj}_Y \mathrm{Sym}^\bullet \mathcal{S}_1 \cong \mathrm{Proj}_Y \mathrm{Sym}\, \mathcal{S}_1 \otimes \mathcal{M}^m \hookrightarrow \mathbb{P}_Y^{n-1}$.

in this case, if $\mathcal{L}$ is π-very ample, $\mathcal{M}$ is ρ-very ample,

under the above sequence of closed embedding,

one can deduce that $\mathcal{L} \otimes \pi^* \mathcal{M}^m$ is $(\rho \circ \pi)$-very ample.

Case 2. Z Noetherian

claim: $\mathcal{L} \otimes \pi^* \mathcal{M}^m$ is $(\rho \circ \pi)$-very ample.

By 17.3.H, this can be check affine-locally.