Chap 14. 15.

14.2. C rank n v.b. on $\mathbb{A}^1_k$ is trivial bundle.

proof note that if M v.b. $M \in k[x]$-Mod

$\mathbb{A}^1_k$ is compact $\Rightarrow \exists\, U_i = D(f_i)$ principal open set

$M|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^n$ then $A := k[x]$ $A_{f_i}^n \xrightarrow{\sim} M_{f_i}$

assume $s_1^i \cdots s_n^i \in M_{f}$ s.t. $M_{f_i} = \bigoplus_{j=1}^{n} A_{f_i} s_j^i$

WMA $s_j \in M$ then $\bigoplus_{i,j} A s_j^i \longrightarrow M$

the morphism is surjective after $\otimes A_{f_i}$ $\forall f_i$

then surjective after $\otimes A_p$ $\forall p$.

thus surjective, M f.g. over PID

M is torsion free since. if $\exists\, 0 \neq x \in M$ $f \in A$ $fx = 0$

then $M = A^m \oplus T(M)$ $M_p = A_p^m \oplus T(M)_p$

first take $p = (0)$ then $T(M)_p = 0$ $m = n$

then take $m \supseteq \mathrm{Ann}_M(x) \trianglelefteq A$, then

$\frac{x}{1} \in T(M)_m$ $\frac{x}{1} \neq 0$ (if so $\exists\, k \in A \setminus m$ $kx = 0$)

by $M_m = A_m^n \oplus T(M)_m \xrightarrow{\sim} A_m^n$ Contradiction.

then $T(M) = 0$ $M = A^n$ trivial bundle.

14.2.D. Zero Locus of a section. $V(s) \hookrightarrow X$

① $\mathcal{F}$ locally free $s \in \Gamma(X, \mathcal{F}) \leftrightarrow \mathcal{O}_X \to \mathcal{F}$

$V(s) :=$ closed scheme corresponding to $\mathrm{Im}(\mathcal{F}^\vee \to \mathcal{O}_X)$

② Set theoretically, we want $V(s)|_{\mathrm{Spec} A} = \{ p \in \mathrm{Spec} A \mid s(p) = 0 \}$.

check locally. $A \to \mathcal{F}|_{\mathrm{Spec} A} \in A\text{-Mod}$. $\quad \mathcal{F}|_{\mathrm{Spec} A}^\vee \xrightarrow{ev} A$

$1 \mapsto s|_{\mathrm{Spec} A}$ $\quad\quad \| \quad M^\vee \quad f \mapsto f(s)$

$\forall p \quad s(p) = 0 \Leftrightarrow s_p \in p\mathcal{F}_p \Leftrightarrow s_p \in pM_p \overset{M/pM \text{ is torsion free } A/p\text{-mod}}{\Longleftrightarrow} s \in pM.$

$\Rightarrow \forall f \in M^\vee \quad f(pM) \subseteq p \quad f(s) \in p \Rightarrow \mathrm{Im}\, ev \subseteq p$

Conversely if $\mathrm{Im}\, ev \subseteq p$ but $s \notin pM$

$M \hookrightarrow A^{\oplus I} \quad s = a_1e_1 + \cdots + a_ne_n \quad \exists k \ a_k \notin p$.

then take $f: M \to A \quad f(\Sigma b_i e_i) = b_k \quad f(s) \notin p$ contradiction

then $V(s)|_{\mathrm{Spec} A} = \{ p \in \mathrm{Spec} A \mid s(p) = 0 \}$.

Also if $M \hookrightarrow A^n \quad s \leftrightarrow (f_i)_I$ then

$A^I \hookleftarrow M^\vee \to A$ the Image is exactly $((f_i)_I)$

$(a_i)_I \mapsto \Sigma a_i f_i$

③ Another View (finite rank case)

$\mathrm{Spec}\, \mathrm{Sym}^\bullet \mathcal{F} \rightsquigarrow \begin{array}{c} E \\ \downarrow p \\ X \end{array}$ $\quad s$ corresponds to $\begin{array}{c} E \\ \uparrow s \\ X \end{array}$

since $p s = \mathrm{id}$, P is separated (locally $\begin{array}{c} \mathbb{A}^n_B \\ \downarrow \\ \operatorname{Spec} B \end{array}$)

by [Stack Lemma 26.21.11], s is closed immersion.

now $0 \in \Gamma(X, \mathcal{F}) \Rightarrow$
$$\begin{tikzcd} E \\ X \arrow[u, "0"] \end{tikzcd}$$

$$\begin{tikzcd} \text{“}X_0 \cap X_s\text{''} = X \times_E X \arrow[r, hook, "U(s)"] \arrow[d] & X \arrow[d, "0"] \\ X \arrow[r, "s"] & E \end{tikzcd}$$

$s: X \longrightarrow \coprod_{x \in X} \mathbb{A}^n_{k(x)}$

$x \longmapsto s(x)$

$0: X \longrightarrow \coprod_{x \in X} \mathbb{A}^n_{k(x)}$

$x \longmapsto 0$

14.2.1. Rational Section.

rational section is $\{(U, s) \mid s \in \Gamma(U, \mathcal{F})\}/\sim$

$(U, s) \sim (V, t)$ iff $s|_{U \cap V} = t|_{U \cap V}$

not serious example. $(\mathbb{R}^2, C^\infty)$ differential form $\Omega^1_{\mathbb{R}^2}$

a rational section $[(\mathbb{R}^2 \setminus 0, \frac{y\,dx + x\,dy}{x^2+y^2})]$

14.2.E.

$s \in \Gamma(U, \mathcal{F})$ if X affine integral & $\mathcal{F}$ trivial

$s \longleftrightarrow (f_i)_i$ $\quad f_i \in \Gamma(U, \mathcal{F})$ $\quad \operatorname{codim}_X X \setminus U \geq 2$.

then $f_i \in \Gamma(U, \mathcal{O}_X) \xrightarrow[\text{view in } K(X)]{} f_i \in \Gamma(X, \mathcal{O}_X)$

$\Rightarrow s \in \Gamma(X, \mathcal{O}_X)$ by 13.5.19.

Noetherian

generally take $\operatorname{Spec} A \subseteq X$ $\mathcal{F}|_{\operatorname{Spec} A}$ free, WMA $\operatorname{Spec} A$ connected

since if not $\operatorname{Spec} A = U_1 \amalg U_n$ U_k clopen

take U_1 if U_1 disconnected $U_1 = U_{11} \amalg U_{1n}$ U_{1k} clopen

then $U_1 \supsetneq U_{11} \supsetneq U_{111} \cdots$ descending chain of closed set

the $\exists N$ U_N connected. closed $\Rightarrow$ U_N affine

since U_1 clopen in U, U_{11} clopen in U_1 — U_N open

since X integral on stalk.

$\operatorname{Spec} A$ connected $\Rightarrow$ $\operatorname{Spec} A$ integral. done.

14.2.F same proof as 14.2.E

14.2.H.

(a) $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

if $\mathcal{F}$, $\mathcal{F}''$ v.b. then $\mathcal{F}'$ is v.b.

Theorem. M is f.g. A-mod TFAE:

① M projective.

② M is finitely presented & $M_p \xrightarrow{\sim} R_p^{r(p)}$

③ $\exists f_1 \cdots f_n \in R$ $\operatorname{Spec} R = \bigcup D(f_i)$ M_{f_i} is free f_i-mod.

Now check locally for $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

$\mathcal{H}om(M,N)|_U = \mathcal{H}om_U(M|_U, N|_U)$

Theorem (Scheme Ver).

X locally Noetherian $M \in Coh(X)$ (M f.g.)

(Def) M is projective if $\mathcal{H}om(M,-): QCoh(X) \longrightarrow QCoh(X)$ exact

then TFAE: ① M projective $\Leftrightarrow$ M locally projective. proof later

② $M_x \simeq \mathcal{O}_{X,x}^{r(x)}$ $\forall x \in X$

③ M is finite locally free

(b) if $\mathcal{F}, \mathcal{F}'$ v.b. $\mathcal{F}''$ need not be v.b.

14.2.J locally or on stalk.

14.2.4. (map of vector bundles)

$\mathcal{E}, \mathcal{F}$ v.b. then $\phi: \mathcal{E} \longrightarrow \mathcal{F}$ map of v.b. if $\exists a$

$\forall p \in X$ $\exists U \ni p$

$$\begin{tikzcd}
\mathcal{E}|_U \arrow[r, "\phi"] \arrow[d, "\wr"] & \mathcal{F}|_U \arrow[d, "\wr"] \\
\mathcal{O}_U^{r_1} \arrow[dr] & \mathcal{O}_U^{r_2} \\
& \mathcal{O}_U^{\oplus a} \arrow[u]
\end{tikzcd}$$

(constant rank).

☆ $\phi: \mathcal{E} \longrightarrow \mathcal{F}$ map of v.b. iff $\operatorname{Coker}\phi$ v.b.

in which case $\operatorname{Ker}\phi$ $\operatorname{Im}\phi$ v.b.

sketch of proof

$(\Rightarrow)$ ✓

$(\Leftarrow)$ $0 \to \operatorname{Im}\phi \to \mathcal{F} \to \operatorname{Coker}\phi \to 0$ then $\operatorname{Im}\phi$ v.b.

$0 \to \operatorname{Ker}\phi \to \mathcal{E} \to \operatorname{Im}\phi \to 0$ $\operatorname{Ker}\phi$ v.b.

locally $0 \to A^{\oplus n} \to \mathcal{F}_U \leftrightarrows A^{\oplus m} \to 0$

$\mathcal{F}_U = A^{\oplus n+m}$

$0 \to A^{\oplus p} \to \mathcal{E} \leftrightarrows A^{\oplus n} \to \mathcal{F}$

$\mathcal{E} = A^{\oplus n+p}$

14.3.K.

Prop. X reduced

$\phi : \mathcal{E} \to \mathcal{F}$ map of v.b. iff rank ϕ_p constant.

proof $(\Rightarrow)$ ✓

$(\Leftarrow)$ affine locally. $R^{r_1} \to R^{r_2} \Leftrightarrow \Phi \in M_{r_2\times r_1}(R)$

Assume $K(p)^{r_1} \to K(p)^{r_2}$ of rk a.

first. $\forall$ $a+1$ 阶子阵 $(\det \Phi_{a+1})(p) = 0$ $\forall p$

X reduced $\Rightarrow \det \Phi_{a+1} = 0$

WMA $\exists$子阵 $\det \Phi\begin{pmatrix} 1 \cdots a \\ 1 \cdots a\end{pmatrix}(p) \neq 0$

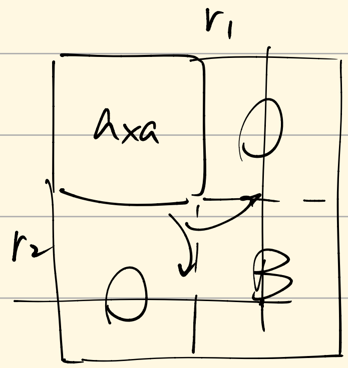
$f := \det \Phi\begin{pmatrix} 1 \cdots a \\ 1 \cdots a\end{pmatrix}$ $p \in D(f)$

$\Rightarrow \det \Phi\begin{pmatrix} 1 \cdots a \\ 1 \cdots a\end{pmatrix}\Big|_{D(f)}$ invertible

then $\Phi\begin{pmatrix} 1 \cdots a \\ 1 \cdots a\end{pmatrix}\Big|_{D(f)}$ invertible.

$\forall b_{ij}$ $b_{ij} \cdot \det \Phi\begin{pmatrix} 1 \cdots a \\ 1 \cdots a\end{pmatrix}\Big|_{D(f)} = 0$

$\Rightarrow b_{ij} = 0$, then $B = 0$.



$R_f^{r_1} \to R_f^{a} \to R_f^{r_2}$ □

14.2.K. omitted.

14.2.6. (tensor algebra, sym and $\wedge$)

① X Ringed Space $\mathcal{F}\in\mathcal{O}_X\text{-Mod}$ $\mathcal{F}^{\otimes n} := \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{F}\cdots\otimes_{\mathcal{O}_X}\mathcal{F}$. $\mathcal{F}^{\otimes 0} := \mathcal{O}_X$

② $\mathcal{F}\in\mathcal{O}_X\text{-Mod}$

$$\mathrm{Sym}^n\mathcal{F} := \left(U\longmapsto \mathrm{Sym}^n(\mathcal{F}(U))\right)^+$$
$$\wedge^n\mathcal{F} := \left(U\longmapsto \wedge^n(\mathcal{F}(U))\right)^+$$

if $\mathcal{F}$ v.b. of rank m, $\mathcal{F}^{\otimes n}$ of rank m^n

$\mathrm{Sym}^n\mathcal{F}$ of rank $\binom{n+m-1}{n}$

$\wedge^n\mathcal{F}$ of rank $\binom{m}{n}$

Caution: $\wedge M$ is $M\otimes M/(a\otimes a)_{a\in M}$.

※ Alternating is better then skew-symmetric.

③ $\mathcal{A}\in\mathcal{O}_X\text{-Mod}$ $\mathcal{A}$ is $\mathcal{O}_X$-Alg if,

$\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}$

$\mathcal{A}(U)\times\mathcal{A}(U)\longrightarrow\mathcal{A}(U)$ gives $\mathcal{A}(U)$ structure of $\mathcal{O}_X(U)$-alg.
$(a,a')\longmapsto aa'$

iff $\exists\ \mathcal{O}_X\xrightarrow{\alpha}\mathcal{A}\in Sh_{CRing}(X)$

$\mathcal{F}\in\mathcal{O}_X\text{-Mod}$ $\mathrm{Sym}\,\mathcal{F} := \bigoplus_{0}^{+\infty}\mathrm{Sym}^n\mathcal{F}$ identified by Universal Property.

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathrm{Sym}\,\mathcal{F} \arrow[dl, dashed] \\
\mathcal{A} &
\end{tikzcd}$$

then $\mathrm{Sym}\,\mathcal{F} = \left(U\longmapsto \mathrm{Sym}\,\mathcal{F}(U)\right)^+$

14.2.M.O omitted.

14.2.N.

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact sequence of locally free sheaves.

$\Rightarrow \wedge^r \mathcal{F} = G^0 \supseteq G^1 \supseteq \cdots \supseteq G^{r+1} = 0$ where $G^p / G^{p+1} = \wedge^p \mathcal{F}' \otimes \wedge^{r-p} \mathcal{F}''$

proof. locally $\mathcal{F}|_U = \mathcal{F}'|_U \oplus \mathcal{F}''|_U$ then $\wedge^r \mathcal{F}|_U = \bigoplus_{p+q=r} (\wedge^p \mathcal{F}'|_U \otimes \wedge^q \mathcal{F}''|_U)$

Induction: Assume $G^{j+1} \cdots G^{r+1}$ done.

Consider

$$\begin{tikzcd} \varphi : \wedge^j \mathcal{F}'|_U \otimes \wedge^{r-j} \mathcal{F}''|_U \arrow[r] & \wedge^r \mathcal{F}|_U / G^{j+1}|_U \\ & \wedge^r \mathcal{F}|_U \arrow[u, "\pi"'] \end{tikzcd}$$

Claim: $\pi^{-1}(\mathrm{Im}\,\varphi)$ is independent of choice of U.

pick basis $x_1 \cdots x_s$ $\mathcal{F}'|_U \subset \mathcal{F}|_U$ $\quad y_1 \cdots y_t$ $\mathcal{F}''|_U \subset \mathcal{F}|_U$

this image of φ is $(x_{p_1} \wedge \cdots \wedge x_{p_j} \wedge y_{q_1} \wedge \cdots \wedge y_{q_{r-j}})$

For another splitting $0 \to \mathcal{F}'|_U \to \mathcal{F}|_U \rightleftarrows \mathcal{F}''|_U \to 0$

we have another basis of $\mathcal{F}''|_U \subset \mathcal{F}|_U$ $\quad y_1 + c_1 \cdots y_t + c_t$

where $c_1 \cdots c_t \in \mathcal{F}'|_U$

image is $(x_{p_1} \wedge \cdots \wedge x_{p_j} \wedge (y_{q_1} + c_{q_1}) \wedge \cdots \wedge (y_{q_{r-j}} + c_{q_{r-j}}))$

the difference of generators has at least $j+1$ x's

$(x_{p_1} \wedge \cdots \wedge x_{p_j} \wedge y_{q_1} \wedge \cdots \wedge y_{q_{r-j}}) + F^{j+1} =$

$(x_{p_1} \wedge \cdots \wedge x_{p_j} \wedge (y_{q_1} + c_{q_1}) \wedge \cdots \wedge (y_{q_{r-j}} + c_{q_{r-j}})) + F^{j+1}$

then $G^j := \pi^{-1}(\operatorname{Im}\varphi)$ then $G^j / G^{j+1} = \operatorname{Im}\varphi$

φ is injective (check on stalk) then

$$G^j / G^{j+1} = \operatorname{Im}\varphi = \bigwedge^j \mathcal{F}'|_U \otimes \bigwedge^{r-j} \mathcal{F}''|_U \qquad \square$$

14.2.P

$$\bigwedge^{\text{rank}} \mathcal{F} \;\; \| \;\; \det \mathcal{F}$$

① $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ v.b. $\Rightarrow$ $\det \mathcal{F} = \det \mathcal{F}' \otimes \det \mathcal{F}''$

② $0 \to \mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_2 - - \to \mathcal{F}_n \to 0$ v.b. then

$$\det \mathcal{F}_1 \otimes \det \mathcal{F}_2^\vee \otimes \text{——} \otimes (\det \mathcal{F}_n)^{(-1)^n} \xrightarrow{\sim} \mathcal{O}_X .$$

proof. ① take $r = \operatorname{rank} \mathcal{F}$ $\det \mathcal{F} = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_{r+1}$.

$$G^p / G^{p+1} = \bigwedge^p \mathcal{F}' \otimes \bigwedge^{r-p} \mathcal{F}'' = \begin{cases} \bigwedge^p \mathcal{F}' \otimes 0 & p < rk(\mathcal{F}') \\ 0 \otimes \bigwedge^{r-p} \mathcal{F}' & p > rk(\mathcal{F}') \\ \det \mathcal{F}' \otimes \det \mathcal{F}'' & p = rk(\mathcal{F}') \end{cases}$$

②
$$\begin{tikzcd} \mathcal{F}_k \arrow[dr] & & & \operatorname{Im} f_{k+1} \arrow[d, phantom, "\|"] & \\ 0 \arrow[r] & \operatorname{Ker} f_{k+1} \arrow[r] \arrow[d, phantom, "\|"] & \mathcal{F}_{k+1} \arrow[r] & \operatorname{Coker} f_k \arrow[r] \arrow[dr] & 0 \\ & \operatorname{Im} f_k & & & \mathcal{F}_{k+2} \end{tikzcd}$$ v.b.

$$\det \mathcal{F}_{k+1} = \det \operatorname{Im} f_k \otimes \det \operatorname{Im} f_{k+1}$$

$$\det \mathcal{F}_k = \det \operatorname{Im} f_{k-1} \otimes \det \operatorname{Im} f_k$$

$f_0 = f_n = 0$ $\quad LHS = \det \operatorname{Im} f_1 \otimes \det \operatorname{Im} f_1^\vee \otimes \det \operatorname{Im} f_2 \otimes - \text{—} \otimes \det \operatorname{Im} f_{n-1}^{(-1)^n} = \mathcal{O}_X$

$\square$

14.2.Q.

$\mathcal{F}\in \mathrm{Qcoh}(X) \Longrightarrow \mathcal{F}^{\otimes n}\quad \mathrm{Sym}^n\mathcal{F}\quad \bigwedge^n\mathcal{F}\in \mathrm{Qcoh}(X)$

Recall:

Def 16.4.6. (finite type and finite presentation)

$(X,\mathcal{O}_X)$ Ringed Space.

① An $\mathcal{O}_X$-module $\mathcal{F}$ is called of finite type

if $\forall x\in X\ \exists U\in \mathcal{U}_x$ and $n\in\mathbb{Z}^+$ s.t. $\mathcal{O}_X|_U^{\oplus n}\twoheadrightarrow \mathcal{F}|_U$

② $\mathcal{F}$ is of finite presentation if $\forall x\in X\ \exists U\in\mathcal{U}_x$ $m,n\in\mathbb{Z}^+$ and exact sequence

$$\mathcal{O}_X|_U^{\oplus m}\longrightarrow \mathcal{O}_X|_U^{\oplus n}\twoheadrightarrow \mathcal{F}|_U\longrightarrow 0$$

14.3.A

(a) $\mathcal{H}om(\text{fpr}, \text{Qcoh}) = \text{Qcoh}$

(b) $\mathcal{H}om(\text{fpr}, \text{coh}) = \text{coh}$

(c) $\mathcal{F}$ fpr $\quad -\otimes_{\mathcal{O}_X}\mathcal{F}: \mathrm{Qcoh}(X)\longrightarrow \mathrm{Qcoh}(X) : \mathcal{H}om(\mathcal{F},-)$ adjoint.

proof (a) **Prop 16.4.9.**

① $X\in \mathrm{Sch}$ $\quad \mathcal{F}\in\mathcal{O}_X$-Mod of finite presentation, $\mathcal{G}\in\mathrm{Qcoh}(X)$

then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})\in \mathrm{Qcoh}(X)$

② If $X=\mathrm{Spec}A$ affine $\quad M,N\in A$-Mod $\quad M$ finite presentation

then $\widetilde{\mathrm{Hom}_A(M,N)}\cong \mathcal{H}om_A(\widetilde{M},\widetilde{N})$

proof [Görtz Wedhorn AG I] prop. 7.29.

14.3.1 (Vakil being wrong?)

① X Locally Noetherian $\Longrightarrow$ $\mathcal{O}_X$ coherent.

② X Locally Noetherian $\Longrightarrow$ $\mathcal{F} \in Coh(X) \Rightarrow \mathcal{F}^\vee \in Coh(X)$

proof ① ($\Longrightarrow$) $\forall U$ affine $\mathcal{O}_X|_U$ Noetherian

$\forall V$ $\mathcal{O}_X^n|_V \xrightarrow{\varphi} \mathcal{O}_X|_V$

14.3.B

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \quad \in QCoh(X)$$

$\mathcal{H}$ is locally free, then $\forall \mathcal{E} \in Qcoh(X)$

$Hom(-, \mathcal{E})$ exact on this sequence.

proof. locally split.

14.3.C. ⊗ Not Involving Nakayama.

Prop. 16.4.11. (generator lift from stalk to local)

$(X, \mathcal{O}_X)$ Ringed Space, $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ of finite type.

Fix $x \in U \subseteq X$, if $\exists \{S_i\}_1^n \subseteq \Gamma(U, \mathcal{F})$ s.t. $\{(S_i)_x\}_1^n$ generate $\mathcal{F}_x$

then $\exists V \subseteq U$ $x \in V$ s.t. $\{S_i|_V\}$ genarate $\mathcal{F}|_V$.

proof of finite type $\Rightarrow \exists U' \subseteq U$ $x \in U'$ $m \in \mathbb{Z}^+$ s.t. $\mathcal{O}_{U'}^m \twoheadrightarrow \mathcal{F}|_{U'}$

then $\exists t_1 - t_m \in \Gamma(U', \mathcal{F})$ s.t. $\{t_j\}$ generates $\mathcal{F}|_{U'}$

since $(S_i)_x$ generate $\mathcal{F}_x$ $\exists (a_{ij})_{m\times n} \in M_{m\times n}(\Gamma(U'', \mathcal{F}))$ s.t. $(t_j)_x = \sum_i (a_{ij})_x (S_i)_x$

then $\exists V \subseteq U'' \subseteq U' \subseteq U$ $x \in V$ s.t. $t_j|_V = \sum a_{ij}|_V S_i|_V$

thus $\{S_i|_V\}$ generate $\mathcal{F}|_V$

Corollary. 16.4.12.

Let $(X, \mathcal{O}_X)$ be a ringed space $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ of finite type,

then $\mathrm{Supp}(\mathcal{F})$ is closed in X.

proof. $\forall x \in X \setminus \mathrm{Supp}(\mathcal{F})$ $\mathcal{F}_x$ is generated by ϕ then $\exists V \in \mathcal{U}_x$

$\mathcal{F}|_V$ is generated by ϕ i.e. $\mathcal{F}|_V = 0$ $V \subseteq X \setminus \mathrm{Supp}\,\mathcal{F}$.

14.3.D (Geometric Nakayama)

14.3.E. omitted.

Lemma 16.4.21.

$(X, \mathcal{O}_X)$ Ringed space $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$.

① $\mathcal{F}$ of finite type $\Rightarrow$ $(\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}))_x = \mathrm{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$

② $\mathcal{I} \subseteq \mathcal{O}_X$ ideal sheaf $\mathcal{I} \subseteq \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}) \Rightarrow \mathcal{F} \in \mathcal{O}_X/\mathcal{I}\text{-Mod}$

sheaf ring

proof by Prop 16.4.8 ①

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_x \hookrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{F}_x)$

$\forall f_x \in (\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})_x)$ $\exists U \ni x$ $f \in \mathrm{Ann}(\mathcal{F})(U)$

then $\forall V \subseteq U$ $f|_V \cdot \mathcal{F}V = 0$ thus $f_x \cdot \mathcal{F}_x = 0$ $f_x \in \mathrm{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$

Conversely if $f_x \in \mathrm{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$ then $f_x \in (\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}))_x$

$$\begin{tikzcd} & \mathcal{O}_{X,x} \arrow[dl] \arrow[dr] & \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})_x \arrow[rr, hook] & & \mathcal{H}om_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{F}_x) \end{tikzcd}$$

thus $\exists U$ s.t. $f|_U \in \mathcal{A}nn_{\mathcal{O}_X}(\mathcal{F})(U)$

②
$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \\ \mathcal{O}_X/\mathcal{I} \arrow[ur, dashed] & \end{tikzcd}$$

$\mathcal{O}_X/\mathcal{I} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$

induces $\mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ thus

$\mathcal{F}$ admit an $\mathcal{O}_X/\mathcal{I}$ structure

Prop. 16.4.22.

$X \in Sch$ $\mathcal{F} \in Qcoh(X)$ of finite type

then ① $\mathcal{A}nn(\mathcal{F})$ is quasi-coherent ideal of $\mathcal{O}_X$

$\Gamma(U, \mathcal{A}nn\,\mathcal{F}) = \mathrm{Ann}\,\mathcal{F}(U,\mathcal{F})$ $\forall U \subseteq X$ affine open.

② $V(\mathcal{A}nn(\mathcal{F})) = \mathrm{Supp}\,\mathcal{F}$.

③ $\forall \mathcal{I} \subseteq \mathcal{A}nn(\mathcal{F})$ $V(\mathcal{I}) \overset{i}{\hookrightarrow} X$ then $\mathcal{F} \xrightarrow{u} i_*(i^*\mathcal{F})$