2025 FALL SIMIS AG I NOTE

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Contents

1 Preface

This course was taught in the Shanghai Institute for Mathematics and Interdisciplinary Sciences (referred to as “SIMIS”) over the Fall semester of 2025, by Yehao Zhou.

The content presented herein aims to remain faithful to the structure and flow of the classroom lectures, serving primarily as a companion for review and consolidation of the material.

While every effort has been made to capture the essence of the course, these notes are a product of personal study. Consequently, the reader is kindly asked to bear with any potential omissions of technical details, lack of rigor in certain proofs, or minor inaccuracies that may have occurred during the AI (Using Gemini) transcription process.

Please note that the chapters covering the foundational theory of **Varieties**, **Sheaves**, and **Schemes** have been omitted from this compilation because of lack of time.

2 2025.9.15-2025.9.22

Lecture 1: Varieties

omitted.

Lecture 2: Regular functions and Morphisms I

Let Y be a quasi-affine variety, i.e., an open subset of an affine variety in $\mathbb{A}^n$.

Definition 2.1. A function $f : Y \to k$ is called **regular at a point** $P \in Y$ if there exists an open neighborhood U of P in Y (i.e., $P \in U \subseteq Y$) and polynomials $g, h \in k[X_1, \ldots, X_n]$ such that:

1. The denominator does not vanish on U, i.e., $Z(h) \cap U = \emptyset$.
2. The function f agrees with the rational function on U, i.e., $f|_U = \frac{g}{h}$.

The function f is called **regular** on Y if it is regular at all points $P \in Y$.

Lemma 2.2. Identifying the field k with the affine line $\mathbb{A}^1$, a regular function $f : Y \to \mathbb{A}^1$ is continuous with respect to the Zariski topology.

Proof. It suffices to show that for any point $t \in \mathbb{A}^1$, the preimage $f^{-1}(t)$ is a closed subset of Y. Since f is regular, we can cover Y by open sets U_i such that on each U_i, the function is given by a quotient of polynomials:

$$Y = \bigcup_i U_i, \quad \text{with } f|_{U_i} = \frac{g_i}{h_i},$$

where $g_i, h_i \in k[X_1, \ldots, X_n]$ and $Z(h_i) \cap U_i = \emptyset$.

Consider the intersection of the preimage with the open set U_i:

$$f^{-1}(t) \cap U_i = \left\{ x \in U_i \;\middle|\; \frac{g_i(x)}{h_i(x)} = t \right\} = \{x \in U_i \mid g_i(x) - th_i(x) = 0\}.$$

This set can be written as $Z(g_i - th_i) \cap U_i$. Since $Z(g_i - th_i)$ is a closed set in $\mathbb{A}^n$, its intersection with U_i is closed in the relative topology of U_i. Since being closed is a local property (specifically, a subset $Z \subseteq Y$ is closed if and only if $Z \cap U_i$ is closed in U_i for an open cover $\{U_i\}$), we conclude that $f^{-1}(t)$ is closed in Y. $\square$

Definition 2.3. Let $Y \subseteq \mathbb{P}^n$ be a quasi-projective variety. Let $f : Y \to k$ be a function and let $P \in Y$ be a point. The following conditions are equivalent (used to define regularity):

(1) There exists an open neighborhood U of P in Y ($P \in U \subseteq Y$) and homogeneous polynomials $g, h \in S_d$ of the same degree $d \in \mathbb{Z}_{\geq 0}$ such that:

$$f|_U = \frac{g}{h} \quad \text{and} \quad Z(h) \cap U = \emptyset.$$

(2) For any standard affine open set $U_i = \mathbb{P}^n \setminus H_i$ (where $H_i = Z(X_i)$ is the hyperplane at infinity) such that $P \in U_i$, the restriction $f|_{U_i \cap Y}$ is regular at P when viewed as a function on the quasi-affine variety $U_i \cap Y$.

If one of the above conditions holds, then we say f is **regular at** P. We say f is **regular** on Y if it is regular at every point of Y.

Proof of Equivalence. This is left as an exercise. (Hint: Use the dehomogenization isomorphism between U_i and $\mathbb{A}^n$). □

Definition 2.4. A **variety** over k is any affine, quasi-affine, projective, or quasi-projective variety.
A **morphism** $\varphi : X \to Y$ between two varieties is a continuous map (with respect to the Zariski topology) such that for every open set $U \subseteq Y$ and any regular function $f : U \to k$, the composition $f \circ \varphi$ is a regular function on the preimage $\varphi^{-1}(U)$.

Remark. *It is easy to verify that the composition of morphisms is a morphism. If $\psi : X \to Y$ and $\varphi : Y \to Z$ are morphisms, then $\varphi \circ \psi : X \to Z$ is a morphism.*

We denote by $\mathbf{Var}_k$ the category of varieties over k. The category of affine varieties, denoted by $\mathbf{AffVar}_k$, is a full subcategory of $\mathbf{Var}_k$.

Example 2.5.

1. Consider the standard open chart of projective space $U_i = \mathbb{P}^n \setminus H_i$. The map $\varphi_i : U_i \to \mathbb{A}^n$ defined by dehomogenization:

$$[a_0 : \cdots : a_n] \longmapsto \left(\frac{a_0}{a_i}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i} \right)$$

is an isomorphism of varieties.

2. The hyperplane $H_i \subseteq \mathbb{P}^n$ is isomorphic to $\mathbb{P}^{n-1}$. This isomorphism is given by projecting onto the coordinates other than X_i.

3. Let $Y \subseteq \mathbb{A}^n$ be a quasi-affine variety. We have the natural inclusion $Y \hookrightarrow \mathbb{A}^n$. Composing this with the isomorphism $\mathbb{A}^n \cong U_i \subset \mathbb{P}^n$, we obtain an embedding:

$$Y \longrightarrow \mathbb{A}^n \xrightarrow{\sim} U_i \subset \mathbb{P}^n.$$

   Then Y is isomorphic to its image in $\mathbb{P}^n$. This shows that every quasi-affine variety is a quasi-projective variety.

4. A regular function $f : Y \to \mathbb{A}^1$ is a morphism.

More generally, we have the following criterion for morphisms into affine space.

Lemma 2.6. Let X be a variety and let $Y \subseteq \mathbb{A}^n$ be an affine (or quasi-affine) variety. Let $\varphi : X \to Y$ be a map. Then φ is a morphism if and only if for every coordinate function X_i $(1 \leq i \leq n)$, the composition $X_i \circ \varphi$ is a regular function on X.

Proof. **Necessity ($\implies$):** Suppose φ is a morphism. For each i, the coordinate function $X_i : Y \to k$ is a regular function (by the definition of regular functions on affine space). By the definition of a morphism, the composition of a regular function with a morphism is regular. Therefore, $X_i \circ \varphi$ is regular on X.

Sufficiency ($\impliedby$): Suppose that $X_i \circ \varphi$ is a regular function on X for all $i = 1, \ldots, n$.

First, we show that φ is continuous. Let $f \in k[X_1, \ldots, X_n]$ be any polynomial. The composition $f \circ \varphi$ is a polynomial expression in the functions $X_i \circ \varphi$. Since the sum and product of regular functions are regular, $f \circ \varphi$ is a regular function on X. A basic closed set in Y is of the form $Z(f) \cap Y$. The preimage is:

$$\varphi^{-1}(Z(f) \cap Y) = \{x \in X \mid f(\varphi(x)) = 0\} = Z(f \circ \varphi).$$

Since $f \circ \varphi$ is regular, its zero set $Z(f \circ \varphi)$ is a closed subset of X. Since the preimage of any closed set is closed, φ is continuous.

Next, we show the regularity condition. Let $U \subseteq Y$ be an open set and let $\psi : U \to k$ be a regular function. Locally on U, ψ can be written as a quotient of polynomials $\psi = g/h$ where $Z(h) \cap U = \emptyset$. The composition is $\psi \circ \varphi = (g \circ \varphi)/(h \circ \varphi)$. As shown above, $g \circ \varphi$ and $h \circ \varphi$ are regular functions on X. Since h does not vanish on U, $h \circ \varphi$ does not vanish on $\varphi^{-1}(U)$. Therefore, the quotient is a regular function on $\varphi^{-1}(U)$. Thus, φ is a morphism. $\square$

Definition 2.7. Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of global regular functions on Y.
For a point $P \in Y$, we define the **local ring of P on Y**, denoted by $\mathcal{O}_{Y,P}$ or simply $\mathcal{O}_P$, as the direct limit of the rings of regular functions on open neighborhoods of P:

$$\mathcal{O}_{Y,P} := \varinjlim_{P \in U \subseteq Y} \mathcal{O}(U).$$

The set of open subsets containing P, $\{U \subseteq Y \text{ open} \mid P \in U\}$, forms a directed set under inclusion (specifically, we say $U \geq V$ if $V \subseteq U$, which induces the restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$).
Explicitly, elements of the direct limit are equivalence classes of pairs (germs):

$$\mathcal{O}_{Y,P} = \{\langle U, f\rangle \mid P \in U, f \in \mathcal{O}(U)\}/\sim,$$

where the equivalence relation is defined by:

$$\langle U, f\rangle \sim \langle V, g\rangle \iff f|_{U\cap V} = g|_{U\cap V}.$$

(Note: Since we are dealing with functions, equality on the intersection is sufficient. For general presheaves, they must agree on some smaller open neighborhood $W \subseteq U \cap V$).
If Y is an irreducible variety, we define the **function field** of Y, denoted by $K(Y)$, as the direct limit of the rings of regular functions over all non-empty open sets:

$$K(Y) := \varinjlim_{U \neq \emptyset} \mathcal{O}(U).$$

Remark. 1. ***Local Ring Property:*** *The ring $\mathcal{O}_P$ is a local ring. Indeed, consider the set of germs vanishing at P:*

$$\mathfrak{m}_P = \{\langle U, f\rangle \in \mathcal{O}_P \mid f(P) = 0\}.$$

This set $\mathfrak{m}_P$ is the unique maximal ideal. To see this, suppose $f \notin \mathfrak{m}_P$. Then $f(P) \neq 0$. Since f is regular, it is continuous, so there exists an open neighborhood of P where f is non-zero. Thus, the function $1/f$ is well-defined and regular in that neighborhood. This implies that the inverse f^{-1} exists in $\mathcal{O}_P$. Since every element outside $\mathfrak{m}_P$ is a unit, $\mathfrak{m}_P$ is maximal. Furthermore, the residue field is isomorphic to the base field:

$$\mathcal{O}_P/\mathfrak{m}_P \cong k, \quad \textit{via the evaluation map } f \longmapsto f(P).$$

2. ***Field Property:*** *If Y is irreducible, $K(Y)$ is indeed a field. If a regular function f is not identically zero on a non-empty open set U, then (since Y is irreducible) the locus where $f \neq 0$ is a non-empty open subset. Thus $1/f$ is a regular function on that smaller open set, meaning f has an inverse in the direct limit.*

3. ***Invariance:*** *The rings $\mathcal{O}(Y)$, $\mathcal{O}_P$, and the field $K(Y)$ are invariants of the variety. That is, if $Y \cong Y'$, then $\mathcal{O}(Y) \cong \mathcal{O}(Y')$, $\mathcal{O}_{Y,P} \cong \mathcal{O}_{Y',\varphi(P)}$, etc.*

Proposition 2.8. Let X be any variety and let Y be an affine variety. Then there is a natural bijection:

$$\alpha : \mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{k\text{-alg}}(A(Y), \mathcal{O}(X)).$$

Proof. **Step 1: Define the map α (Forward direction).** We define the map α by the pullback of functions:

$$\alpha(\varphi) = \varphi^* : f \longmapsto f \circ \varphi$$

for any morphism $\varphi : X \to Y$ and any $f \in A(Y)$.

Check well-definedness: Since Y is affine, $A(Y) = k[x_1, \ldots, x_n]/I(Y)$. If $f \in I(Y)$, then f vanishes on Y. Since $\mathrm{Im}(\varphi) \subseteq Y$, we have $f \circ \varphi = 0$ on X. This ensures that the map is well-defined on the quotient ring $A(Y)$.

Step 2: Construction of the Inverse. Given a k-algebra homomorphism $\xi \in \mathrm{Hom}(A(Y), \mathcal{O}(X))$, we want to construct a morphism $\psi : X \to Y$. Assume $Y \subseteq \mathbb{A}^n$. Let $x_1, \ldots, x_n$ be the coordinate functions (images of the variables) in $A(Y)$. Define regular functions on X by applying ξ:

$$\xi_i := \xi(x_i) \in \mathcal{O}(X).$$

Now, define the map $\psi : X \to \mathbb{A}^n$ by:

$$P \longmapsto (\xi_1(P), \ldots, \xi_n(P)).$$

Step 3: Check 1 – The image lies in Y. We need to show $\mathrm{Im}(\psi) \subseteq Y$. Let $f \in I(Y)$. We view f as a polynomial. Consider the composition $f \circ \psi$. Since ξ is a homomorphism:

$$f \circ \psi = f(\xi_1, \ldots, \xi_n) = \xi(f(x_1, \ldots, x_n)) = \xi(f).$$

Since $f \in I(Y)$, it is zero in the coordinate ring $A(Y)$, so $\xi(f) = \xi(0) = 0$. Thus, f vanishes on the image of ψ. This implies $\mathrm{Im}(\psi) \subseteq V(I(Y)) = Y$.

Step 4: Check 2 – ψ is a morphism. Since $\mathrm{Im}(\psi) \subseteq Y$, we have a map $\psi : X \to Y$. To verify it is a morphism, it suffices to check that the composition with coordinate functions is regular.

$$x_i \circ \psi = \xi_i.$$

By our construction, $\xi_i \in \mathcal{O}(X)$ is a regular function. Therefore, ψ is a morphism.

It is straightforward to verify that these constructions are inverse to each other. $\square$

Theorem 2.9. Let $Y \subseteq \mathbb{A}^n$ be an aff. var. Then:

(1) $\mathcal{O}(Y) \cong A(Y)$

(2) $\forall p \in Y$, $\mathcal{O}_p \cong A(Y)_{m_p}$

(3) $K(Y) \cong \mathrm{Frac}(A(Y))$ when Y is irreducible.

Proof. The proof can be broken down into the following steps:

- **Setup from Previous Results:**
  We know from a previous lecture that for any variety X, $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(A(Y), \mathcal{O}(X))$. Let's consider the specific case where $X = Y$. The identity morphism $id : Y \to Y$ on the left-hand side corresponds to a homomorphism $\alpha : A(Y) \to \mathcal{O}(Y)$ on the right-hand side. Our goal is to show that α is an isomorphism.

- **Injectivity of α:**
  Suppose $f \in k[x_1, \ldots, x_n]$ such that its image in $A(Y)$ is mapped to 0 by α. This means $\alpha(f) = 0$ as a regular function on Y. By definition, this implies f vanishes at every point in Y, so $f \in I(Y)$. Therefore, the image of f is 0 in $A(Y) = k[x_1, \ldots, x_n]/I(Y)$. This shows α is injective.

- **Surjectivity of α (Local Argument):**
  To show α is surjective, it suffices to show that the induced map on the stalks, $\alpha_p : A(Y)_{m_p} \to \mathcal{O}(Y)_{m_p}$, is surjective for all $p \in Y$. This relies on Hilbert's Nullstellensatz, which gives a correspondence between points in Y and maximal ideals of $A(Y)$, where $p \leftrightarrow m_p$.

- **Factoring through the Localization:**
  Consider the composition of maps $A(Y) \xrightarrow{\alpha} \mathcal{O}(Y) \to \mathcal{O}_p$. For any element $s \in A(Y) \setminus m_p$, its image $\alpha(s)$ is a regular function that does not vanish at p. Thus, $\alpha(s)$ is invertible in the local ring $\mathcal{O}_p$. By the universal property of localization, this map must factor uniquely

through the localization $A(Y)_{m_p}$, giving the map α_p.

$$\begin{tikzcd}
A(Y) \arrow[r, "\alpha"] \arrow[dr] & \mathcal{O}(Y) \arrow[r] & \mathcal{O}_p \\
 & A(Y)_{m_p} \arrow[r, "\alpha_p"] & \mathcal{O}(Y)_{m_p} \arrow[u, "\beta_p"']
\end{tikzcd}$$

- **β_p is injective:**
Suppose $f \in \mathcal{O}(Y)$, $g \in A(Y) \setminus m_p$ such that the element $\frac{f}{g} \in \mathcal{O}(Y)_{m_p}$ is mapped to 0 in $\mathcal{O}_p$. By definition of the zero element in the stalk $\mathcal{O}_p$, this means there exists an open set $U \ni p$ such that the restriction of the function is zero, i.e., $\frac{f}{g}\Big|_U = 0$.

  Now, take a function $h \in A(Y)$ with the properties that its zero set $Z(h)$ contains the complement of U (i.e., $Z(h) \supseteq Y \setminus U$) and also that $h(p) \neq 0$. Such a function exists because $Y \setminus U$ is a closed set not containing p.

  Since $h(p) \neq 0$, h is an element of $A(Y) \setminus m_p$. On the open set U, we have $f|_U = 0$. On the closed set $Y \setminus U$, we have $h = 0$. Therefore, the product function $h \cdot f$ is zero everywhere on Y. In the ring $\mathcal{O}(Y)_{m_p}$, we can write:

$$\frac{f}{g} = \frac{h \cdot f}{h \cdot g}$$

  Since $h \cdot f = 0$ as a global function on Y, the numerator is 0. The denominator $h \cdot g$ is not in m_p because $h(p) \neq 0$ and $g(p) \neq 0$. Thus, $\frac{f}{g} = \frac{0}{hg} = 0$ in $\mathcal{O}(Y)_{m_p}$. This shows that any element mapping to 0 in $\mathcal{O}_p$ must have been 0 to begin with in $\mathcal{O}(Y)_{m_p}$. Hence, β_p is injective.

- **Final step of proving (1) and (2)**
By definition $\mathcal{O}_p = \varinjlim_{p \in U} \mathcal{O}(U) = \{\frac{f}{g} \mid f, g \in A(Y), g(p) \neq 0\} = A(Y)_{m_p}$. So $\beta_p \circ \alpha_p$ is isomorphism, which implies β_p is surjective. So β_p is isomorphism, so α_p is isomorphism. This proves (i) and (ii).

- **Proof of (3)**
Y is irreducible $\implies A(Y) \cong \mathcal{O}(Y)$ are integral domains, which can be regarded as a subring of $K(Y)$ (a field). Moreover, every $f \in K(Y) = \varinjlim_U \mathcal{O}(U)$, which implies $f \in \mathcal{O}_p$ for some p. Since $\mathcal{O}_p \cong A(Y)_{m_p}$, this implies $f \in \mathrm{Frac}(A(Y))$. $\implies \mathrm{Frac}(A(Y)) = K(Y)$.

$\square$

There is an equivalence of categories:

$$\begin{tikzcd}
\{\mathrm{AffVar}_k\} \arrow[r, leftrightarrow, "\sim"] & \{\mathrm{Alg}_k^{\text{f.g. red}}\}^{\mathrm{op}} \\
X \arrow[r, mapsto] & A(X) \\
& \circlearrowleft \\
\mathrm{Spm}\ A & A \arrow[l, mapsfrom]
\end{tikzcd}$$

Spm A = maximal spectrum of A.

2.1 $\mathcal{O}(V)\&K(V)$ for projective varieties

Definition 2.10. Let S be a graded ring, and let $\mathfrak{p} \subset S$ be a graded prime ideal. Define $S_{(\mathfrak{p})}$ = subring of deg=0 elements in $T^{-1}S$, where $T = \{f \in S \setminus \mathfrak{p} \mid f \text{ is homogeneous}\}$. For $f \in S$ homogeneous, we define $S_{(f)}$ = subring of deg=0 elements in S_f.

Theorem 2.11. Let $Y \subseteq \mathbb{P}^n$ be a projective variety. Then:

(a) $\mathcal{O}(Y) \cong k \times \cdots \times k$, where the number of factors is the number of connected components of Y.

(b) For any point $p \in Y$, let $m_p \subset S(Y)$ be the homogeneous ideal $I(p)$ corresponding to p. Then the local ring at p is given by $\mathcal{O}_p \cong S(Y)_{(m_p)}$.

(c) If Y is irreducible, then its function field is $K(Y) \cong S(Y)_{((0))}$.

proof of (b) and (c). The proof strategy is to reduce to the affine case, which we have already established.

- **Setup on an Affine Chart:** Recall that $\mathbb{P}^n$ is covered by open sets $U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$, each isomorphic to $\mathbb{A}^n$. Assume the point p lies in the chart U_i. Let $Y_i = Y \cap U_i$. This is an affine variety in $U_i \cong \mathbb{A}^n$. The coordinate ring of this affine space is $k[\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}]$.

- **Relating Coordinate Rings:** The homogeneous coordinate ring of Y, $S(Y)$, is related to the affine coordinate ring of Y_i, $A(Y_i)$, by the localization map. Specifically, the degree-zero part of the localization of $S(Y)$ at the element x_i is isomorphic to $A(Y_i)$:

$$S(Y)_{(x_i)} \cong A(Y_i)$$

- **Using the Affine Case Result:** As we have seen, the local ring of Y at p, $\mathcal{O}_p$, is determined by the local structure, so it is isomorphic to the local ring of the affine variety Y_i at p. This, in turn, is the localization of the affine coordinate ring $A(Y_i)$ at the maximal ideal $\mathfrak{m}'_p \subset A(Y_i)$ corresponding to the point p.

$$\mathcal{O}_p \cong A(Y_i)_{\mathfrak{m}'_p}$$

- **Connecting the Ideals and Localizations:** The maximal ideal $\mathfrak{m}'_p$ in $A(Y_i)$ corresponds to the degree-zero part of the localization of the homogeneous ideal $\mathfrak{m}_p$ at x_i. That is, $(\mathfrak{m}_p)_{(x_i)} \cong \mathfrak{m}'_p$. Since $p \in U_i$, we know that x_i is not in the ideal of p, i.e., $x_i \notin \mathfrak{m}_p$. A key property of localization of graded rings is that localizing at $\mathfrak{m}_p$ is the same as first localizing at x_i and then at the corresponding prime ideal $\mathfrak{m}'_p$.

- **Conclusion:** We can now chain these isomorphisms together to get the final result:

$$S(Y)_{(\mathfrak{m}_p)} \cong \left(S(Y)_{(x_i)}\right)_{\mathfrak{m}'_p} \cong A(Y_i)_{\mathfrak{m}'_p} \cong \mathcal{O}_p$$

  This proves part (b).

  Part (c) is proven similarly. We run the same argument, but replace the maximal ideal $\mathfrak{m}_p$ with the prime ideal (0).

$\square$

Proof of (a). The proof proceeds in two main parts. First, we reduce the problem to the case where the projective variety Y is irreducible. Second, we prove that for an irreducible projective variety, the only global regular functions are the constants.

- **Reduction to the Irreducible Case:** Let $Y = \bigcup_\alpha Y_\alpha$ be the decomposition of Y into its irreducible components. A function $f \in \mathcal{O}(Y)$ is regular on all of Y. Its restriction to any irreducible component, $f|_{Y_\alpha}$, is therefore a regular function on Y_α.

  If we can prove that for any irreducible projective variety Y_α, the ring of regular functions $\mathcal{O}(Y_\alpha)$ consists only of the constant functions (i.e., $\mathcal{O}(Y_\alpha) \cong k$), then it follows that f must be constant on each Y_α.

  If Y is connected, it cannot be partitioned into disjoint non-empty open sets. Since the values of f are in the field k, if f were not globally constant, we could construct a separation of Y, which is a contradiction. Therefore, for a connected variety, f must be a single constant. If Y has multiple connected components, f can take a different constant value on each, leading to $\mathcal{O}(Y) \cong k \times \cdots \times k$.

  Thus, it is sufficient to prove the statement for an irreducible variety Y.

- **Proof for an Irreducible Variety Y:** Let $f \in \mathcal{O}(Y)$ be a global regular function. Since Y is covered by the standard affine charts $U_i = Y \cap \{x_i \neq 0\}$, the function f can be represented on each chart as a fraction.

  $$f|_{U_i} = \frac{g_i}{x_i^{N_i}}$$

  where g_i is a homogeneous polynomial of degree N_i in the homogeneous coordinate ring $S(Y)$, and the representation is in the ring $S(Y)_{(x_i)} \cong A(U_i)$.

  Since f is a single function, we can view it as an element of the function field $K(Y)$. The condition above means that for each $i \in \{0, \dots, n\}$, we have $x_i^{N_i} f \in S(Y)$.

  Let $S(Y)_d$ denote the d-th graded component of $S(Y)$. For any sufficiently large integer N, the space $S(Y)_N$ is spanned by monomials $m = x_0^{\alpha_0} \dots x_n^{\alpha_n}$ of degree N. If we choose N large enough such that $\alpha_i \geq N_i$ for all i, then for any such monomial m, we have:

  $$m \cdot f = (x_0^{\alpha_0} \dots x_i^{\alpha_i - N_i} \dots x_n^{\alpha_n}) \cdot (x_i^{N_i} f) \in S(Y)$$

  This implies that multiplication by f maps the finite-dimensional vector space $S(Y)_N$ to itself: $f \cdot S(Y)_N \subseteq S(Y)_N$.

  This means that f is integral over the ring $S(Y)$. Therefore, f must satisfy a monic polynomial equation with coefficients in $S(Y)$:

  $$f^m + a_1 f^{m-1} + \dots + a_m = 0, \quad \text{where } a_j \in S(Y).$$

  Since f is a regular function on a projective variety, it must have degree 0. We can take the degree-zero component of the entire equation. Let $\bar{a_j}$ be the degree-zero part of a_j. This gives:

  $$f^m + \bar{a_1} f^{m-1} + \dots + \bar{a_m} = 0$$

  The degree-zero component of the homogeneous coordinate ring $S(Y)$ is just the base field k. So, each $\bar{a_j} \in k$.

  The function f therefore satisfies a polynomial equation with coefficients in k. Since the field k is algebraically closed, this implies that f must be an element of k.

  Thus, any global regular function on an irreducible projective variety is constant, i.e., $\mathcal{O}(Y) \cong k$.

$\square$

Corollary 2.12. Let Y be a connected projective variety and X be a quasi-affine variety. Then every morphism $\varphi : Y \to X$ is constant.

Proof. Since X is a quasi-affine variety, it is an open subset of an affine variety W. The map $\varphi : Y \to X$ is therefore a morphism from Y to W.

Now, we know that $\mathrm{Hom}(Y, W) \cong \mathrm{Hom}_k(A(W), \mathcal{O}(Y))$. Since Y is a connected projective variety, we have previously shown that its ring of global regular functions is just the base field, $\mathcal{O}(Y) \cong k$. Thus, any morphism from Y to W corresponds to a k-algebra homomorphism $\psi : A(W) \to k$.

Every such algebra homomorphism factors through a maximal ideal $\mathfrak{m}_p$ for some point $p \in W$. This implies the image of the entire variety Y under φ is the single point p.

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[dr, "\text{const.}"'] & W \\ & \{p\} \arrow[u, hook] \end{tikzcd}$$

Therefore, the morphism is constant. □

Lecture 3: Morphism II

2.2 Forgetful Functor

Consider the functor that forgets the geometric structure:

$$\begin{aligned} \mathrm{Forg} : \mathrm{Var}_k &\to \mathrm{Set} \\ X &\mapsto \{\text{points in } X\} \end{aligned}$$

Lemma 2.13. The forgetful functor is faithful (not full).

Proof. If $f, g \in \mathrm{Hom}_{\mathrm{Var}_k}(X, Y)$, then if f and g are equal as maps on the underlying sets of points, they are equal as morphisms. This shows faithfulness.

The set of points of X can be identified with $\mathrm{Hom}_{\mathrm{Var}_k}(\mathbb{A}^0, X)$, where $\mathbb{A}^0 = \mathrm{Spec}\, k$ is a single point. A map $f : \mathbb{A}^0 \to X$ is given by its image, $f(\mathbb{A}^0) = x \in X$. Every such map corresponding to a point $x \in X$ is a (unique) morphism. This is because for any open affine neighborhood $U \subseteq X$ containing x, we can define the corresponding ring homomorphism $A(U) \to A(U)/\mathfrak{m}_x \cong k$, which is the coordinate ring map for the morphism.

Thus, we can identify the set of points with the set of morphisms from a point:

$$\mathrm{Forg}(X) \cong \mathrm{Hom}_{\mathrm{Var}_k}(\mathbb{A}^0, X)$$

□

Definition 2.14. A morphism $\varphi : X \to Y$ is said to be **injective** (resp. **surjective**) if the underlying map of sets $\mathrm{Forg}(\varphi) : \mathrm{Forg}(X) \to \mathrm{Forg}(Y)$ is injective (resp. surjective).

Proposition 2.15. Let $\varphi : X \to Y$ be a morphism in the category Var.

1. φ is injective $\iff$ φ is a monomorphism in Var.
2. φ is surjective $\implies$ φ is an epimorphism in Var. (The converse $\impliedby$ is not true in general).

Proof.

- **Proof of (1): Injectivity $\iff$ Monomorphism**
  ($\Rightarrow$) Recall that in the category Set, a map is injective if and only if it is a monomorphism. We are given that $\mathrm{Forg}(\varphi)$ is injective, so for any set Z, the induced map

$$\mathrm{Hom}_{\mathrm{Set}}(\mathrm{Forg}(Z), \mathrm{Forg}(X)) \xrightarrow{\varphi\circ -} \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Forg}(Z), \mathrm{Forg}(Y))$$

  is injective. Since the functor $\mathrm{Forg} : \mathrm{Var} \to \mathrm{Set}$ is faithful, this implies that the map

$$\mathrm{Hom}_{\mathrm{Var}}(Z, X) \xrightarrow{\varphi\circ -} \mathrm{Hom}_{\mathrm{Var}}(Z, Y)$$

  is injective for any variety Z. This is the definition of a monomorphism.

  ($\Leftarrow$) Exercise.

- **Proof of (2): Surjectivity $\implies$ Epimorphism**
  ($\Rightarrow$) We are given that $\mathrm{Forg}(\varphi)$ is surjective. In Set, surjectivity implies epimorphism. This means that for any set Z, the induced map

$$\mathrm{Hom}_{\mathrm{Set}}(\mathrm{Forg}(Y), \mathrm{Forg}(Z)) \xrightarrow{-\circ\varphi} \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Forg}(X), \mathrm{Forg}(Z))$$

is injective. Again, by the faithfulness of the Forg functor, this implies that the map

$$\mathrm{Hom}_{\mathrm{Var}}(Y, Z) \xrightarrow{-\circ\varphi} \mathrm{Hom}_{\mathrm{Var}}(X, Z)$$

is injective for any variety Z. This is the definition of an epimorphism.

□

Remark: Suppose $U \subseteq Y$ is a dense open subset and Z is a separated variety (which all our varieties are). If two morphisms $f, g : Y \to Z$ agree on U (i.e., $f|_U = g|_U$), then they must be equal, $f = g$.

Proposition 2.16. If a morphism $\varphi : X \to Y$ is surjective, then $\dim X \geq \dim Y$.

Proof. WLOG (Without Loss of Generality), assume Y is irreducible. Since $\dim Y = \dim U$ for any open nonempty subset U of Y. We can replace Y by a non-empty open affine subset U (which we will still denote by Y), since $\dim U = \dim Y$. Let $\varphi : X \to Y$ be the morphism. We can cover X by a finite number of open affine irreducible sets, X_i.

$$X = \bigcup_{i=1}^{N} X_i$$

- **Claim:** There must exist at least one i such that the induced ring homomorphism $\varphi_i^* : A(Y) \to A(X_i)$ is injective.

- **Proof of Claim:** Suppose for the sake of contradiction that the kernel is non-zero for all i. Let $J_i = \ker(\varphi_i^*) \subset A(Y)$. By our assumption, $J_i \neq (0)$ for all i. This means that for any $f \in J_i$, the function f vanishes on the image of X_i, i.e., $\varphi(X_i) \subseteq Z(J_i)$. The image of the entire map is the union of the images of these pieces:

  $$Y = \mathrm{Im}(\varphi) = \bigcup_{i=1}^{N} \varphi(X_i) \subseteq \bigcup_{i=1}^{N} Z(J_i)$$

  Since each J_i is a non-zero prime ideal in the integral domain $A(Y)$, $Z(J_i)$ is a proper closed subset of Y , which contradicts the assumption that Y is irreducible. Thus, the claim must be true.

- **Conclusion:** Let's say for $i = 1$, the map $\varphi_1^* : A(Y) \to A(X_1)$ is injective. Since $A(Y)$ and $A(X_1)$ are integral domains, this injection extends to an injection of their fields of fractions: $K(Y) \hookrightarrow K(X_1)$. From field theory, we know that the transcendence degree of a subfield is less than or equal to that of the larger field.

$$\mathrm{trdeg}_k K(Y) \leq \mathrm{trdeg}_k K(X_1)$$

For irreducible affine varieties, the dimension is the transcendence degree of the function field. Therefore:

$$\dim Y \leq \dim X_1 \leq \dim X$$

This completes the proof. □

2.3 Finite Morphism in AffVar_k

Definition 2.17. We say a morphism $\varphi : X \to Y$ in AffVar_k is **finite** if the corresponding ring homomorphism $\varphi^* : A(Y) \to A(X)$ is finite. That is, $A(X)$ is a finitely generated $A(Y)$-module.

Remark: If morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$ are finite, then their composition $\psi \circ \varphi : X \to Z$ is also finite.

Lemma 2.18. If a morphism $\varphi : X \to Y$ is finite and the ring map $\varphi^* : A(Y) \to A(X)$ is injective, then φ is surjective.

Non-Example: The inclusion morphism $\varphi : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is not a finite morphism. The corresponding ring map is $\varphi^* : k[x] \to k[x, x^{-1}]$, which is not a finite extension (since x^{-1} is not integral over $k[x]$).

Example 2.19. The morphism $\varphi : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ given by $x \mapsto x + \frac{1}{x}$ is a finite morphism. The corresponding ring map is $\varphi^* : k[y] \to k[x, x^{-1}]$ where $y \mapsto x + \frac{1}{x}$. The ring $k[x, x^{-1}]$ is a finite module over $k[y]$ because x satisfies the monic equation $x^2 - yx + 1 = 0$. The morphism is a 2-to-1 map, except at the points $x = \pm 1$.

Example 2.20. A closed immersion $i : Y \hookrightarrow X$ is a finite morphism. The corresponding ring map $i^* : A(X) \to A(Y)$ is given by $A(X) \to A(X)/I(X) = A(Y)$, which is surjective. Therefore, $A(Y)$ is a finitely generated (in fact, cyclic) $A(X)$-module.

Proof of Lemma. Let $A = A(Y)$ and $B = A(X)$. We are given that $\varphi^* : A \to B$ is an injective and finite ring map. We want to show that the corresponding morphism of varieties $\varphi : X \to Y$ is surjective. This is equivalent to showing that for every maximal ideal $\mathfrak{m} \subset A$, there exists a maximal ideal $\mathfrak{n} \subset B$ such that $(\varphi^*)^{-1}(\mathfrak{n}) = \mathfrak{m}$. This is a consequence of "Going-Up" theorem for integral extensions. □

Proposition 2.21. If $\varphi : X \to Y$ is a finite morphism, then it is a closed map. That is, for every closed subset $V \subseteq X$, the image $\varphi(V)$ is closed in Y.

Proof. It suffices to prove this for irreducible closed subsets, so WLOG (Without Loss of Generality), let us assume that $V \subseteq X$ is irreducible.

Let $\mathfrak{p} = I(V) \subset A(X)$ be the prime ideal corresponding to V. The restriction of the morphism to V, denoted $\varphi|_V : V \to Y$, corresponds to a ring homomorphism from $A(Y)$ to $A(V) = A(X)/\mathfrak{p}$.

Let $\mathfrak{q} = (\varphi^*)^{-1}(\mathfrak{p})$. Since $\mathfrak{p}$ is a prime ideal, $\mathfrak{q}$ is a prime ideal in $A(Y)$. Let $W = Z(\mathfrak{q})$ be the corresponding irreducible closed subvariety of Y. The morphism $\varphi|_V$ factors through W.

- **Claim:** The induced map $\psi : V \to W$ is surjective.

- **Proof of Claim:** The morphism ψ corresponds to a ring homomorphism $\psi^* : A(W) \to A(V)$. The coordinate ring of W is $A(W) = A(Y)/\mathfrak{q}$, and the coordinate ring of V is $A(V) = A(X)/\mathfrak{p}$. The map ψ^* is injective by construction. Since the original map $\varphi^* : A(Y) \to A(X)$ is finite, the induced map $\psi^* : A(Y)/\mathfrak{q} \to A(X)/\mathfrak{p}$ is also finite.

  We now have a finite and injective map of coordinate rings. By the previous lemma (which relies on the Going-Up Theorem), the corresponding morphism of varieties $\psi : V \to W$ must be surjective.

Since ψ is surjective, the image of V is exactly W.

$$\varphi(V) = \psi(V) = W = Z(\mathfrak{q})$$

As W is a closed subvariety of Y, we have shown that $\varphi(V)$ is closed. □

Lemma 2.22. If $\varphi : X \to Y$ is a finite morphism between irreducible affine varieties and the ring map $\varphi^* : A(Y) \to A(X)$ is injective, then $\dim X = \dim Y$.

Proof. Because $A(Y)$ and $A(X)$ are integral domains, this injection extends to an embedding of their function fields, $\varphi^* : K(Y) \hookrightarrow K(X)$, which is a finite field extension. So:

$$\dim X = \mathrm{trdeg}_k K(X) = \mathrm{trdeg}_k K(Y) = \dim Y$$

This proves the lemma. □

Proposition 2.23. If $\varphi : X \to Y$ is a finite morphism, then $\dim X \leq \dim Y$.

Proof. Let $Z \subseteq X$ be an irreducible component of X. Since $\dim X = \max_i(\dim Z_i)$ over all components Z_i, it is enough to show that $\dim Z \leq \dim Y$.

Let $W = \varphi(Z)$. By Proposition 2.21, W is a closed subset of Y. As the continuous image of an irreducible set, W is also irreducible. The restricted morphism $\varphi|_Z : Z \to W$ surjective, so $\varphi|_Z^* : A(W) \to A(Z)$ is injective (A simple conclusion). Since φ is finite, the induced map $\varphi|_Z : Z \to W$ is also finite. By Lemma 2.22, we have $\dim Z = \dim W$.

Since W is a closed subvariety of Y, its dimension cannot exceed the dimension of Y.

$$\dim Z = \dim W \leq \dim Y$$

This holds for any irreducible component of X, so the inequality holds for X itself. $\square$

Corollary 2.24. If $\varphi : X \to Y$ is finite and surjective, then $\dim X = \dim Y$.

Theorem 2.25 (Noether's Normalization Theorem)**.** Let A be a finitely generated k-algebra and an integral domain. Let $d = \dim(\mathrm{Spm}(A))$. Then there exist elements $x_1, \ldots, x_d \in A$ such that:

1. $\{x_1, \ldots, x_d\}$ are algebraically independent over k.
2. The inclusion map $k[x_1, \ldots, x_d] \hookrightarrow A$ is a finite ring homomorphism.

Corollary 2.26. For every $X \in \mathrm{AffVar}_k$, there exists a surjective finite morphism $\varphi : X \to \mathbb{A}^{\dim X}$.

2.4 Product of Affine Varieties

Definition 2.27. Let $\mathcal{C}$ be a category. Let $X, Y \in \mathcal{C}$ be objects. The **product** $X \times Y$ (if it exists) is an object in $\mathcal{C}$ together with morphisms $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ such that there is a natural equivalence of functors:

$$\mathrm{Hom}_{\mathcal{C}}(Z, X \times Y) \cong \mathrm{Hom}_{\mathcal{C}}(Z, X) \times \mathrm{Hom}_{\mathcal{C}}(Z, Y)$$

This is represented by the following universal property: for any object Z and morphisms $f : Z \to X$ and $g : Z \to Y$, there exists a unique morphism $h : Z \to X \times Y$ making the diagram commute.

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

Lemma 2.28. Let $X, Y \in \mathrm{Var}$. Then the product $X \times Y$ (if it exists) satisfies:

$$\mathrm{Forg}(X \times Y) \cong \mathrm{Forg}(X) \times \mathrm{Forg}(Y)$$

Proof Sketch. This follows from the Yoneda lemma. The set of points of a variety Z can be identified with $\mathrm{Hom}_{\mathrm{Var}}(\mathbb{A}^0, Z)$. Applying this to the definition of the product gives the result.

$$\mathrm{Hom}(\mathbb{A}^0, X \times Y) \cong \mathrm{Hom}(\mathbb{A}^0, X) \times \mathrm{Hom}(\mathbb{A}^0, Y)$$

$$\mathrm{Forg}(X \times Y) \cong \mathrm{Forg}(X) \times \mathrm{Forg}(Y)$$

□

In the category of k-algebras, Alg_k, the coproduct of two algebras A and B is their tensor product, $A \otimes_k B$. This is defined by the universal property:

$$\mathrm{Hom}_{\mathrm{Alg}_k}(A \otimes_k B, C) \cong \mathrm{Hom}_{\mathrm{Alg}_k}(A, C) \times \mathrm{Hom}_{\mathrm{Alg}_k}(B, C)$$

for any other k-algebra C.

Theorem 2.29. The product of two affine varieties exists in the category AffVar_k. If $X = \mathrm{Spm}(A)$ and $Y = \mathrm{Spm}(B)$, their product is $X \times Y = \mathrm{Spm}(A \otimes_k B)$.

This theorem relies on the fact that the category of affine varieties is anti-equivalent (dual) to the category of finitely generated, reduced k-algebras, i.e., $\mathrm{AffVar}_k \cong (\mathrm{Alg}^{\text{f.g., red}})^{\mathrm{op}}$. The product in AffVar_k corresponds to the coproduct (the tensor product) in the category of algebras. For this to work, we need the tensor product of two such algebras to remain in the category.

Proposition 2.30. Let $A, B \in \mathrm{Alg}$ be finitely generated k-algebras.

(i) If A, B are reduced, then $A \otimes_k B$ is also reduced.

(ii) If A, B are integral domains, then $A \otimes_k B$ is also an integral domain.

Remark: This proposition holds even without the assumption that A is finitely generated.

Proof.

- **Proof of (i):** Suppose $\alpha \in A \otimes_k B$ is a nilpotent element, so $\alpha^n = 0$ for some n. We want to show $\alpha = 0$. Write $\alpha = \sum_{i=1}^m a_i \otimes b_i$. We can do this such that the set $\{b_i\}$ is linearly independent over k.

  For any maximal ideal $\mathfrak{m} \subset A$, we can consider the image of α in the quotient ring:

  $$\bar{\alpha} \in (A/\mathfrak{m}) \otimes_k B$$

  Since A is a finitely generated k-algebra, by Hilbert's Nullstellensatz, $A/\mathfrak{m} \cong k$. So, $(A/\mathfrak{m}) \otimes_k B \cong k \otimes_k B \cong B$. The image $\bar{\alpha} = \sum \bar{a}_i \otimes b_i$ is an element of B. Since α is nilpotent, $\bar{\alpha}$ is also nilpotent. But B is reduced, so it has no non-zero nilpotents. Thus, $\bar{\alpha} = 0$.

  $$\bar{\alpha} = \sum_{i=1}^{m} \bar{a}_i b_i = 0 \in B$$

  Here, $\bar{a}_i$ is the image of a_i in $A/\mathfrak{m}$, so $\bar{a}_i \in k$. Because the set $\{b_i\}$ is linearly independent over k, this equation implies that $\bar{a}_i = 0$ for all $i = 1, \ldots, m$.

  This means that each a_i is in every maximal ideal $\mathfrak{m}$ of A. The intersection of all maximal ideals is the nilradical of A. Since A is reduced, its nilradical is the zero ideal. So, $a_i = 0$ for all i. This implies $\alpha = \sum 0 \otimes b_i = 0$. Therefore, $A \otimes_k B$ is reduced.

- **Proof of (ii):** Suppose $\alpha, \beta \in A \otimes_k B$ with $\alpha\beta = 0$, $\alpha \neq 0, \beta \neq 0$. Write $\alpha = \sum a_i \otimes b_i$ and $\beta = \sum a'_j \otimes b'_j$, where $\{b_i\}$ and $\{b'_j\}$ are linearly independent sets.

  For any maximal ideal $\mathfrak{m} \subset A$, the images $\bar{\alpha}, \bar{\beta}$ in $(A/\mathfrak{m}) \otimes_k B \cong B$ satisfy $\bar{\alpha}\bar{\beta} = 0$. Since B is an integral domain, this means $\bar{\alpha} = 0$ or $\bar{\beta} = 0$. This implies that for any point $p \in \mathrm{Spm}(A)$, either all the functions $\{a_i\}$ vanish at p, or all the functions $\{a'_j\}$ vanish at p.

  Let $Z_a = Z(\{a_i\})$ and $Z_{a'} = Z(\{a'_j\})$ be the zero sets of these collections of functions in $\mathrm{Spm}(A)$. What we have shown is that $\mathrm{Spm}(A) = Z_a \cup Z_{a'}$. Since A is an integral domain, its spectrum $\mathrm{Spm}(A)$ is an irreducible topological space. An irreducible space cannot be the union of two proper closed subsets. Therefore, we must have $\mathrm{Spm}(A) = Z_a$ or $\mathrm{Spm}(A) = Z_{a'}$.

  If $\mathrm{Spm}(A) = Z_a$, then all functions a_i are zero everywhere, which means $a_i = 0$ for all i (since A is reduced). This implies $\alpha = 0$. If $\mathrm{Spm}(A) = Z_{a'}$, then similarly $a'_j = 0$ for all j, which implies $\beta = 0$. This contradicts our assumption that α and β are non-zero. Thus, $A \otimes_k B$ must be an integral domain.

□

Proposition 2.31. Let $X, Y \in \mathrm{AffVar}_k$. Then their product in the category of affine varieties, $X \times Y = \mathrm{Spm}(A(X) \otimes_k A(Y))$, is also the categorical product of X and Y in the larger category Var_k.

Proof. Let Z be any variety in Var_k. We need to show that the universal property for the product holds. We have the following chain of natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Var}}(Z, X \times Y) &\cong \mathrm{Hom}_{\mathrm{Alg}}(A(X \times Y), \mathcal{O}(Z)) \quad \text{(by duality)} \\
&= \mathrm{Hom}_{\mathrm{Alg}}(A(X) \otimes_k A(Y), \mathcal{O}(Z)) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}}(A(X), O(Z)) \times \mathrm{Hom}_{\mathrm{Alg}}(A(Y), O(Z)) \quad \text{(by univ. prop. of } \otimes) \\
&\cong \mathrm{Hom}_{\mathrm{Var}}(Z, X) \times \mathrm{Hom}_{\mathrm{Var}}(Z, Y) \quad \text{(by duality)}
\end{aligned}$$

This shows that $X \times Y$ satisfies the universal property of the product in Var_k. □

Example 2.32.

1. **Affine Spaces:** $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$. This is because the coordinate rings satisfy:

$$k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m] \cong k[x_1, \dots, x_n, y_1, \dots, y_m]$$

2. **Projective Space and Punctured Affine Space:** There is a canonical surjective morphism $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Let $U_i = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n \mid a_i \neq 0\}$ be the i-th standard affine chart of $\mathbb{P}^n$. The preimage of this chart under the projection is the set $\pi^{-1}(U_i) = \{(a_0, \dots, a_n) \in \mathbb{A}^{n+1} \mid a_i \neq 0\}$.

3. **Projective Varieties:** Let $Y \subseteq \mathbb{P}^n$ be a projective variety and let $C(Y) \subseteq \mathbb{A}^{n+1}$ be its affine cone. The intersection of the cone with the set $\pi^{-1}(U_i)$ gives the preimage of the affine part of Y.

$$C(Y) \cap (\mathbb{A}^{n+1} \setminus \{x_i = 0\}) \longrightarrow Y_i = Y \cap U_i$$

Lemma 2.33. Let $\pi^{-1}(U_i) = \mathbb{A}^{n+1} \setminus Z(x_i)$. There is an isomorphism of varieties:

$$\pi^{-1}(U_i) \cong U_i \times (\mathbb{A}^1 \setminus \{0\})$$

(More precisely, since $U_i \cong \mathbb{A}^n$, we have $\pi^{-1}(U_i) \cong \mathbb{A}^n \times (\mathbb{A}^1 \setminus \{0\})$).

Lemma 2.34. Let $Y \subseteq \mathbb{P}^n$ be a projective variety and let $Y_i = Y \cap U_i$ be its i-th affine chart. Let $C(Y)$ be the affine cone over Y. Then there is an isomorphism:

$$C(Y) \cap (\mathbb{A}^{n+1} \setminus Z(x_i)) \cong Y_i \times (\mathbb{A}^1 \setminus \{0\})$$

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Recall: Last time we defined/showed that $X \times Y$ exists in AffVar_k.

$$X \times Y = \mathrm{Spm}(A(X) \otimes_k A(Y))$$

Moreover, $X \times Y$ is the product of X and Y in Var_k.

Noetherian Normalization Theorem For all $X \in \mathrm{AffVar}$, there exists a finite surjective morphism $X \to \mathbb{A}^{\dim X}$.

Corollary 3.1.

$$\dim(X \times Y) = \dim X + \dim Y$$

Proof. Take finite surjective morphisms $\phi : X \to \mathbb{A}^{\dim X}$ and $\psi : Y \to \mathbb{A}^{\dim Y}$. Then the product morphism

$$(\phi, \psi) : X \times Y \to \mathbb{A}^{\dim X} \times \mathbb{A}^{\dim Y} \simeq \mathbb{A}^{\dim X + \dim Y}$$

is also finite and surjective. Therefore, by **Corollary 2.24**:

$$\dim(X \times Y) = \dim(\mathbb{A}^{\dim X + \dim Y}) = \dim X + \dim Y$$

□

Corollary 3.2. Let $Y \subseteq \mathbb{P}^n$ be a projective variety. Then the dimension of its affine cone $C(Y)$ is

$$\dim C(Y) = \dim Y + 1$$

Proof. Let $Y = \bigcup_{i=0}^n Y_i$, where $Y_i = Y \cap U_i$ and $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_i \neq 0\}$. Then

$$\dim Y = \max_i \dim Y_i$$

Similar for the affine cone $C(Y)$:

$$\dim C(Y) = \max_i \dim C(Y) \cap \{x_i \neq 0\}$$

The open subset of the cone where $x_i \neq 0$ is isomorphic to the product of the affine variety Y_i and the punctured line $\mathbb{A}^1 \setminus \{0\}$:

$$C(Y) \cap \{x_i \neq 0\} \simeq Y_i \times (\mathbb{A}^1 \setminus \{0\})$$

So we have:

$$\begin{aligned}
\dim C(Y) &= \max_i \dim C(Y) \cap \{x_i \neq 0\} \\
&= \max_i \dim(Y_i \times (\mathbb{A}^1 \setminus \{0\})) \\
&= \max_i \dim Y_i + \dim(\mathbb{A}^1 \setminus \{0\}) \\
&= \max_i \dim Y_i + 1 \\
&= \dim Y + 1
\end{aligned}$$

□

3.1 Product in Var_k

Theorem 3.3. For all $X, Y \in \mathrm{Var}_k$, the product $X \times Y$ exists.

We prove a Lemma first.

Lemma 3.4. Suppose $X, Y \in \mathrm{Var}_k$ such that their product $X \times Y$ exists in Var_k. Let $Z \subseteq X$ and $W \subseteq Y$ be subvarieties (either open or closed). Then the product $Z \times W$ exists in Var_k.

Proof. Consider the product $X \times Y$ with the projection morphisms $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$.

$$\begin{tikzcd}
& X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\
X & & Y
\end{tikzcd}$$

Without loss of generality, we can assume $W = Y$. (The argument can be applied twice: first to construct $Z \times Y$ as a subvariety of $X \times Y$, and then to construct $Z \times W$ as a subvariety of $Z \times Y$).

Let $Z \subseteq X$ be a subvariety. Define V as the preimage of Z under the projection p_X:

$$V = p_X^{-1}(Z) = \{(x, y) \in X \times Y \mid x \in Z\}$$

We claim that V is the product $Z \times Y$.

To verify this, we check the universal property of a product. Suppose T is any variety in Var_k with morphisms $\varphi : T \to Z$ and $\psi : T \to Y$. Let $i : Z \hookrightarrow X$ be the inclusion morphism.

By the universal property of the product $X \times Y$, there exists a unique morphism $f : T \to X \times Y$ that makes the following diagram commute:

$$\begin{tikzcd}
T \arrow[dr, dashed, "f"] \arrow[drr, "i\circ\varphi"] \arrow[ddr, "\psi"'] & & \\
& X \times Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \\
& Y &
\end{tikzcd}$$

For any point $t \in T$, we have $p_X(f(t)) = (i \circ \varphi)(t) = \varphi(t)$. Since the image of φ is contained in Z, it follows that $p_X(f(t)) \in Z$. By the definition of V, this implies that the image of the morphism f is contained in V.

$$\mathrm{Im}(f) \subseteq p_X^{-1}(Z) = V$$

Therefore, f can be considered as a morphism $f : T \to V$. The uniqueness of this morphism is inherited from its uniqueness as a morphism to $X \times Y$. The projection morphisms from V are the restrictions of p_X to V (which maps to Z) and p_Y to V. Thus, V satisfies the universal property for the product $Z \times Y$. $\square$

Example 3.5 (Segre Embedding)**.** Consider the map

$$\varphi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$
$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_m]) \mapsto [\cdots : a_i b_j : \ldots]_{0\le i\le n, 0\le j\le m}$$

Claim:

1. φ is set-theoretically injective.
2. $\mathrm{Im}(\varphi)$ is closed in $\mathbb{P}^{(n+1)(m+1)-1}$.

(This gives a projective variety structure to $\mathbb{P}^n \times \mathbb{P}^m$.)

Proof Sketch. (1) is easy to check.

(2) Let the coordinates in the target space be c_{ij}. Let $U_{ij} = \{[c] \mid c_{ij} \neq 0\}$ be a standard affine open set. Let $U_i = \{[a] \in \mathbb{P}^n \mid a_i \neq 0\}$ and $V_j = \{[b] \in \mathbb{P}^m \mid b_j \neq 0\}$. Then

$$\varphi^{-1}(U_{ij}) = U_i \times V_j$$

The restriction of φ to this open set, $\varphi|_{U_i \times V_j} : U_i \times V_j \to U_{ij}$, is given in affine coordinates by

$$((x_k)_{k \neq i}, (y_l)_{l \neq j}) \mapsto (x_k y_l)_{k \neq i, l \neq j}$$

where $x_k = a_k/a_i$ and $y_l = b_l/b_j$. Using this coordinate description, we see that $\mathrm{Im}(\varphi) \cap U_{ij}$ is an affine variety. Moreover, the map

$$\varphi|_{U_i \times V_j} : U_i \times V_j \to \mathrm{Im}(\varphi) \cap U_{ij}$$

is an isomorphism in AffVar_k. Since the image of φ is closed in each open set of a standard open cover of the target space, it is a closed subvariety. $\square$

Lemma 3.6. Let $X, Y, W \in \mathrm{Var}_k$ with morphisms $W \underset{f}{\to} X$ and $W \underset{g}{\to} Y$. Assume $X = \bigcup_\alpha X_\alpha$ and $Y = \bigcup_\beta Y_\beta$ are open covers such that for each pair of indices (α, β), the product $f^{-1}(X_\alpha) \cap g^{-1}(Y_\beta)$ exists and is identified with $X_\alpha \times Y_\beta$, where the restrictions of f and g correspond to the natural projections to X_α and Y_β. Then $W \simeq X \times Y$ with projections to X and Y given by f and g.

Proof. (Proof of the gluing principle / universal property)

Let $T \in \mathrm{Var}_k$ be a variety with morphisms $\varphi : T \to X$ and $\psi : T \to Y$. Let us define an open cover for T by setting $T_{\alpha\beta} = \varphi^{-1}(X_\alpha) \cap \psi^{-1}(Y_\beta)$.

By assumption, for each (α, β) there exists a product $W_{\alpha\beta} \simeq X_\alpha \times Y_\beta$. The restrictions of φ and ψ define morphisms $\varphi|_{T_{\alpha\beta}} : T_{\alpha\beta} \to X_\alpha$ and $\psi|_{T_{\alpha\beta}} : T_{\alpha\beta} \to Y_\beta$. By the universal property of the product $W_{\alpha\beta}$, there exists a unique morphism $h_{\alpha\beta} : T_{\alpha\beta} \to W_{\alpha\beta}$ such that the following diagram commutes:

$$\begin{tikzcd}
T_{\alpha\beta} \arrow[dr, dashed, "h_{\alpha\beta}"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\psi"'] & & \\
& W_{\alpha\beta} \arrow[r, "p_X"] \arrow[d, "p_Y"] & X_\alpha \\
& Y_\beta &
\end{tikzcd}$$

Now for any two pairs of indices, (α, β) and (α', β'), we must show that the morphisms $h_{\alpha\beta}$ and $h_{\alpha'\beta'}$ agree on the intersection of their domains, $T_{\alpha\beta} \cap T_{\alpha'\beta'}$. Their restrictions are both maps to $W_{\alpha\beta} \cap W_{\alpha'\beta'} \simeq (X_\alpha \cap X_{\alpha'}) \times (Y_\beta \cap Y_{\beta'})$. (Remark: the existence of $(X_\alpha \cap X_{\alpha'}) \times (Y_\beta \cap Y_{\beta'})$ comes from **Lemma 3.4**) By the uniqueness property of the product on this intersection, the two maps must be identical.

$$h_{\alpha\beta}|_{T_{\alpha\beta} \cap T_{\alpha'\beta'}} = h_{\alpha'\beta'}|_{T_{\alpha\beta} \cap T_{\alpha'\beta'}}$$

Since these local morphisms agree on all overlaps, they glue together to form a unique global morphism $h : T \to W$, where W is the variety constructed by gluing the pieces $W_{\alpha\beta}$. This h is the unique morphism satisfying the universal property for the product of X and Y.

$$\begin{tikzcd} & T \arrow[dl, "\varphi"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, "\psi"] & \\ X & W \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

□

Remark: This lemma implies **Theorem 3.3** that the product of any two varieties exists.

Definition 3.7 (Segre Embedding)**.** The morphism

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

is called the **Segre embedding**.

3.2 Affine Covering of a Variety

Proposition 3.8. On any variety X, there exists a basis for the topology consisting of open affine subsets.

Proof. Any variety X has an open cover $X = \bigcup X_i$, where each X_i is quasi-affine (e.g., if $X \subseteq \mathbb{P}^n$, $X_i = X \cap U_i$). We can therefore assume that X is quasi-affine. It is therefore sufficient to show that any open subset of a quasi-affine variety has a basis of open affine sets. Since a quasi-affine variety X is an open subset of a closed set $Y \subseteq \mathbb{A}^m$, it is enough to prove the proposition for the affine variety Y, i.e. any open subset of a affine variety has a basis of open affine sets.

Let $U \subseteq Y$ be an open set and $p \in U$. The complement $Y \setminus U$ is closed. Let $I = I(Y \setminus U)$ be the ideal of functions vanishing on $Y \setminus U$. Since $p \notin Y \setminus U$, there must exist some $f \in I$ such that $f(p) \neq 0$.

Take $V = Y \setminus Z(f)$, where $Z(f)$ is the zero set of f in Y. This is a principal open set in Y. Since f vanishes on $Y \setminus U$, we have $Y \setminus U \subseteq Z(f)$, which implies $V = Y \setminus Z(f) \subseteq U$.

The set V is an open neighborhood of p contained in U. Principal open subsets of affine varieties are themselves affine. Specifically, V is isomorphic to an affine variety. For instance, if $Y = \mathbb{A}^m$, then

$$V = \mathbb{A}^m \setminus Z(f) \simeq \mathrm{Spm}(k[x_1, \ldots, x_m, t]/\langle f \cdot t - 1 \rangle)$$

In particular, V is affine.

Since for any point p in any open set U, we found an open affine neighborhood V of p contained in U, the collection of such open affine sets forms a basis for the topology. $\square$

Lecture 4: Rational Maps

Idea: For $X \in \mathrm{Var}_k$, the ring of regular functions is

$$\mathcal{O}(X) = \mathrm{Hom}(X, \mathbb{A}^1)$$

Question: If X is an irreducible variety, how can we characterize its function field $K(X)$?

Definition 3.9 (Function Field)**.** The **function field** $K(X)$ of an irreducible variety X is the direct limit of the rings of regular functions on its non-empty open subsets:

$$K(X) = \varinjlim_{U \subseteq X, U \neq \emptyset} \mathcal{O}(U) = \varinjlim_{U} \mathrm{Hom}(U, \mathbb{A}^1)$$

An element of $K(X)$ is an equivalence class of pairs $\langle U, \varphi_U \rangle$, where $U \subseteq X$ is a non-empty open set and $\varphi_U : U \to \mathbb{A}^1$ is a morphism. The equivalence relation is given by

$$\langle U, \varphi_U \rangle \sim \langle V, \varphi_V \rangle \quad \text{if} \quad \varphi_U|_{U \cap V} = \varphi_V|_{U \cap V}$$

Definition 3.10 (Rational Map)**.** Let X, Y be varieties, and assume X is irreducible. A **rational map** $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs $\langle U, \varphi_U \rangle$, where $U \subseteq X$ is a non-empty open subset and $\varphi_U : U \to Y$ is a morphism. The equivalence relation is

$$\langle U, \varphi_U \rangle \sim \langle V, \varphi_V \rangle \quad \text{if} \quad \varphi_U|_{U \cap V} = \varphi_V|_{U \cap V}$$

Lemma 3.11. Suppose φ and ψ are two morphisms from a irreducible variety X to a variety Y.

$$X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$$

If $\varphi|_U = \psi|_U$ for some non-empty open subset $U \subseteq X$, then $\varphi = \psi$.

Proof. We can take Y to be a subvariety of some projective space, $Y \subseteq \mathbb{P}^n$. Then consider the morphism

$$(\varphi, \psi) : X \to \mathbb{P}^n \times \mathbb{P}^n$$

Consider the diagonal $\Delta_{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^n$. Using the Segre embedding, $\mathbb{P}^n \times \mathbb{P}^n$ is a projective variety, and the diagonal is defined by the equations $x_i y_j = x_j y_i$ for all $0 \le i, j \le n$, where (x_i) and (y_j) are the homogeneous coordinates. Thus, $\Delta_{\mathbb{P}^n}$ is a closed subset.

The set where φ and ψ agree is the preimage of the diagonal under the morphism $(\varphi, \psi) : X \to \mathbb{P}^n \times \mathbb{P}^n$.

$$\{x \in X \mid \varphi(x) = \psi(x)\} = (\varphi, \psi)^{-1}(\Delta_{\mathbb{P}^n})$$

Since the diagonal is closed, this set is a closed subset of X. By assumption, this closed set contains the non-empty open subset U. Because X is irreducible, any non-empty open subset is dense, so its closure is X. Therefore, the closed set must be all of X.

$$(\varphi, \psi)^{-1}(\Delta_{\mathbb{P}^n}) = X$$

This implies $\varphi = \psi$. □

Remark: Any morphism is a rational map. **Lemma 3.11** implies that the natural map

$$\mathrm{Hom}(X, Y) \to \mathrm{Rat}(X, Y)$$

is injective.

Warning: Rational maps do not compose in general!!!!

Example 3.12. Consider the rational maps $\varphi : \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$ given by the constant map $x \mapsto 0$, and $\psi : \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$ given by $y \mapsto 1/y$. The map φ is a regular morphism defined everywhere. The map ψ is defined on the open set $U = \mathbb{A}^1 \setminus \{0\}$.
The image of φ is the single point $\{0\}$. This image does not intersect the domain of definition of ψ. Therefore, the composition $\psi \circ \varphi$ is not defined.

To have a meaningful notion of composition, the image of the first map must lie in the domain of the second map in a substantial way. This leads to the following definition.

Definition 3.13 (Dominant Rational Map)**.** A rational map $\varphi : X \dashrightarrow Y$ is **dominant** if for some (and therefore any) representative $\varphi_U : U \to Y$, the image $\mathrm{Im}(\varphi_U)$ is a dense subset of Y.

Remark: If $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ are rational maps, and φ is dominant, then the composition $\psi \circ \varphi : X \dashrightarrow Z$ is well-defined.

Let $\langle U, \varphi_U \rangle$ be a representative for φ and $\langle V, \psi_V \rangle$ be a representative for ψ. Since φ is dominant, $\mathrm{Im}(\varphi_U)$ is dense in Y. Because V is a non-empty open subset of Y, the intersection $\mathrm{Im}(\varphi_U) \cap V$ is non-empty. The preimage $U' = \varphi_U^{-1}(V)$ is a non-empty open subset of U (and hence of X). We can then define the composition on U' as $\psi_V \circ \varphi_U|_{U'}$.

Proposition 3.14. Suppose $f : Z \to W$ is a morphism between irreducible varieties. The following are equivalent:

1. f is dominant (i.e., $\mathrm{Im}(f)$ is dense in W).
2. There exists a non-empty open set $S \subseteq Z$ such that $f(S)$ is dense in W.
3. For every non-empty open affine subset $V \subseteq W$ and every non-empty open affine subset $U \subseteq f^{-1}(V)$, the pullback map on coordinate rings $f^* : \mathcal{O}(V) \to \mathcal{O}(U)$ is injective.

Proof. (1) $\implies$ (2): Obvious, just take $S = Z$.

(2) $\implies$ (3): Let $V \subseteq W$ and $U \subseteq f^{-1}(V)$ be non-empty open affine subsets. Let $S \subseteq Z$ be a non-empty open set such that $\overline{f(S)} = W$. Then $f(S) \cap V$ is dense in V.

Consider:

$$S \cap f^{-1}(V) = \bigcup_{\alpha=1}^{n} U_\alpha$$

where U_α are open affine subsets. The repersentation of union of finitely many open affine subsets is because: $Z \setminus (S \cap f^{-1}(V))$ is closed in Z, which means it can be denoted by $\mathcal{Z}(J)$ for an ideal J in $k[x_1, \ldots, x_m]$. Then J can be written into the form $J =< f_1, \ldots, f_n >$. Denote $Z = \mathcal{Z}(I)$, Then:

$$\begin{aligned}
S \cap f^{-1}(V) &= \mathcal{Z}(I) \setminus \mathcal{Z}(J) \\
&= \mathcal{Z}(I) \cap \left(\bigcap_{\alpha=1}^{n} \mathcal{Z}(f_\alpha) \right)^c \\
&= \bigcup_{\alpha=1}^{n} (\mathcal{Z}(I) \cap \mathcal{D}(f_\alpha)) \\
&\sim \bigcup_{\alpha=1}^{n} \mathrm{Spm}\left(k[x_1, x_2, \ldots, x_m, t]/(I, f_\alpha t - 1)\right)
\end{aligned}$$

Then there exists α such that $f(U_\alpha)$ is dense in V. Then it is easy to verify:

$$A(V) \to A(U_\alpha) \text{ is injective.}$$

From the limit definition of Function Field and **Theorem 2.9(3)**

$$K(V) \to K(U_\alpha) = K(Z) = K(U) \text{ is injective.}$$

(The condition of Theorem 2.9 tells us why we should consider open affine subsets rather than merely open subsets.)

$$\implies \mathcal{O}(V) = A(V) \to \mathcal{O}(U) = A(U) \text{ is injective.}$$

(3) $\implies$ (1): The injectivity of $\mathcal{O}(V) \to \mathcal{O}(U)$ implies that $f(U)$ is dense in V. So:

$$W \supseteq \overline{f(Z)} \supseteq \overline{f(U)} = \overline{\overline{f(U)}} \supseteq \overline{V} = W$$

In addition, there is another condition equivalent to the mentioned 3 conditions:

2'. For all $S \subseteq Z$ open, $f(S)$ is dense in W.

(2') $\implies$ (2) is obvious. For (3) $\implies$ (2'), choose $V \subseteq W$ open affine subset, and there exists open affine subset $U \subseteq S \cap f^{-1}(V) \subseteq Z$. From (3) $\implies$ (1):

$$W \supseteq \overline{f(S)} \supseteq \overline{f(U)} = W$$

$\square$

Remark: Let $\varphi : X \dashrightarrow Y \in \mathrm{Rat}(X, Y)$. Then

$$\operatorname{dom}\varphi = \bigcup\{U \mid \varphi \text{ is defined on } U\}$$

is an open subset of X, which is called the **domain of definition** of φ. And there exists a unique maximal domain for φ.

Observation: Dominant rational maps can be composed.

$X \overset{\varphi}{\dashrightarrow} Y \overset{\psi}{\dashrightarrow} Z$

Then $\mathrm{Im}(\varphi|_{\operatorname{dom}\varphi}) \cap \operatorname{dom}\psi \neq \emptyset$. $\implies$ We can define $\psi \circ \varphi$ on the preimage $\varphi^{-1}(\operatorname{dom}\psi)$.

Definition 3.15. We define a category IrrVar_k with

- **Objects:** Irreducible varieties over k.

- **Morphisms:** Dominant rational maps.

An isomorphism in IrrVar_k will be called a **birational map**. In the case there exists a birational map $X \dashrightarrow Y$, we say X is **birationally equivalent** to Y.

Theorem 3.16. There is an equivalence of categories

$$\mathrm{IrrVar}_k \simeq (\mathrm{Fields}_k)^{\mathrm{op}}$$

where (Fields_k) is the category of finitely generated field extensions of k.

Proof. **Faithfulness:** Let $\varphi, \psi : X \dashrightarrow Y$ be two dominant rational maps.

$$\begin{tikzcd} X \arrow[r, dashed, shift left, "\varphi"] \arrow[r, dashed, shift right, "\psi"'] & Y \end{tikzcd}$$

Take $U = \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$. Suppose that $\varphi^* = \psi^*$, where $\varphi^*, \psi^* : K(Y) \to K(X)$ are the induced homomorphisms of function fields.

By replacing Y by an open affine subset, and then replacing U by an open affine subset, we can assume that both U and Y are affine varieties. Then the condition $\varphi^* = \psi^*$ on function fields implies that the pullbacks on the coordinate rings are the same when restricted to U:

$$\varphi_U^* = \psi_U^* \in \mathrm{Hom}_k(A(Y), A(U))$$

Since for affine varieties the morphisms are determined by the homomorphisms of their coordinate rings, we have

$$\varphi_U = \psi_U$$

This means φ and ψ agree on the non-empty open set U, so they represent the same rational map.

$$\varphi = \psi$$

Fullness: Let $\theta \in \mathrm{Hom}_k(K(Y), K(X))$ be a k-algebra homomorphism between the function fields. Since Y is covered by open affine varieties, we can assume without loss of generality that Y is affine. Let $\{y_1, \ldots, y_n\}$ be a set of generators for the coordinate ring $A(Y) = \mathcal{O}(Y)$ as a k-algebra. Then $\{\theta(y_1), \ldots, \theta(y_n)\}$ is a set of elements in the function field $K(X)$. Each $\theta(y_i)$ is a rational function on X, so it is defined on some non-empty open subset of X. The intersection of these open sets is also a non-empty open subset. Therefore, there exists a non-empty open affine subset $U \subseteq X$ such that

$$\theta(y_i) \in \mathcal{O}(U) = A(U) \quad \text{for all } i = 1, \dots, n$$

This induces a k-algebra homomorphism $\theta : \mathcal{O}(Y) \to \mathcal{O}(U)$. Therefore, since U and Y are affine, θ corresponds to a morphism $f : U \to Y$ such that $\theta = f^*$. This defines a rational map $X \dashrightarrow Y$. The induced map on function fields $f^* : K(Y) \to K(X)$ is precisely the original homomorphism θ.

Essential Surjectivity: Let K/k be a finitely generated field extension. Let $\{y_1, \dots, y_n\}$ be a set of generators for K as a field extension of k. Define the k-algebra $R = k[y_1, \dots, y_n] \subseteq K$. Then $K = \mathrm{Frac}(R)$. Since R is a finitely generated k-algebra with no zero divisors (as it is a subring of a field), $X = \mathrm{Spm}(R)$ is an irreducible affine variety. The function field of X is

$$K(X) = K(\mathrm{Spm}(R)) = \mathrm{Frac}(R) = K$$

Thus, for any finitely generated field extension K of k, there exists an irreducible variety X such that $K(X) \simeq K$. $\square$

Corollary 3.17. For any two irreducible varieties X and Y, the following are equivalent:

1. X and Y are birational.
2. There exist non-empty open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \simeq V$.
3. $K(X) \simeq K(Y)$ as k-algebras.

Proof. (1) $\implies$ (2): Let $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ be mutually inverse dominant rational maps. This means $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$ as rational maps. Let $U_0 = \mathrm{dom}(\varphi)$ and $V_0 = \mathrm{dom}(\psi)$. The composition $\psi \circ \varphi$ is defined on the open set $\varphi^{-1}(V_0) \subseteq U_0$. The composition $\varphi \circ \psi$ is defined on the open set $\psi^{-1}(U_0) \subseteq V_0$. Let $U = \varphi^{-1}(V_0)$ and $V = \psi^{-1}(U_0)$. The condition $\psi \circ \varphi = \mathrm{id}_X$ means that for any $x \in U$, $\psi(\varphi(x)) = x$. The condition $\varphi \circ \psi = \mathrm{id}_Y$ means that for any $y \in V$, $\varphi(\psi(y)) = y$. Let $U' = U \cap \psi(V)$ and $V' = V \cap \varphi(U)$. The morphism $\varphi|_{U'} : U' \to \varphi(U')$ is an isomorphism with inverse $\psi|_{\varphi(U')}$. The set $\varphi(U')$ is an open subset of V. Thus we have found isomorphic open subsets.

(2) $\implies$ (3): This is obvious. If $U \simeq V$, then their function fields are isomorphic. Since U is a non-empty open subset of X, $K(U) = K(X)$. Similarly $K(V) = K(Y)$. Therefore $K(X) \simeq K(Y)$.

(3) $\implies$ (1): This follows directly from the main theorem. An isomorphism $K(Y) \simeq K(X)$ corresponds to an isomorphism in the category $(\mathrm{Fields}_k)^{\mathrm{op}}$. By the equivalence of categories,

this corresponds to an isomorphism $X \to Y$ in the category IrrVar_k, which is by definition a birational map. $\square$

Proposition 3.18 (Example of a birational map)**.** Every irreducible variety X of dimension N is birational to a hypersurface in $\mathbb{P}^{N+1}$.

Proof. Since $\mathrm{trdeg}_k K(X) = \dim X = n$, there exists a set of elements $\{x_1, \dots, x_n\} \subseteq K(X)$ which is algebraically independent over k, and the field extension $K(X)/k(x_1, \dots, x_n)$ is a finite field extension.

Since k is algebraically closed, it is a perfect field. $\implies$ Any field extension of k is separable. $\implies$ The extension $K(X)/k(x_1, \dots, x_n)$ is separable and finite. $\implies$ By the Primitive Element Theorem, the extension is generated by a single element, say $y \in K(X)$, so that $K(X) = k(x_1, \dots, x_n, y)$.

The element y satisfies a polynomial equation with coefficients in $k(x_1, \dots, x_n)$. By removing denominators from the coefficients, we get a polynomial relation $f(x_1, \dots, x_n, y) = 0$, where f is an irreducible polynomial in $k[T_1, \dots, T_n, S]$. Therefore, the function field $K(X)$ is isomorphic to the function field of the affine hypersurface defined by the zero set of f.

$$K(X) \simeq K(\mathrm{Spm} k[x_1, \dots, x_n, y]/\langle f \rangle)$$

This shows that X is birational to an affine hypersurface $H \subseteq \mathbb{A}^{n+1}$. By regarding $\mathbb{A}^{n+1} \subseteq \mathbb{P}^{n+1}$, the projective closure $\overline{H}$ is a hypersurface in $\mathbb{P}^{n+1}$. Then X is birational to $\overline{H}$. $\square$

3.3 Blowing-up

Let $O = (0, \dots, 0) \in \mathbb{A}^n$. Let $(x_1, \dots, x_n)$ be the coordinates of $\mathbb{A}^n$. Let $[y_1 : \cdots : y_n]$ be the homogeneous coordinates of $\mathbb{P}^{n-1}$.

Definition 3.19. The **Blowing-up** of $\mathbb{A}^n$ at the origin O, denoted $\mathrm{Bl}_O(\mathbb{A}^n)$, is the closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ given by the equations

$$\{(x, [y]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i \text{ for all } 1 \le i, j \le n\}$$

There is a natural projection map $\psi : \mathrm{Bl}_O(\mathbb{A}^n) \to \mathbb{A}^n$.

$$\begin{tikzcd}
\mathrm{Bl}_O(\mathbb{A}^n) \arrow[r, hook] \arrow[dr, "\psi"'] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d, "\mathrm{proj}_1"] \\
& \mathbb{A}^n
\end{tikzcd}$$

Proposition 3.20. Let $\psi : \mathrm{Bl}_O(\mathbb{A}^n) \to \mathbb{A}^n$ be the blowing-up of $\mathbb{A}^n$ at the origin O.

1. The fiber over the origin is $\psi^{-1}(O) = \{O\} \times \mathbb{P}^{n-1}$.
2. ψ induces an isomorphism $\psi|_{\mathrm{Bl}_O(\mathbb{A}^n)\setminus\psi^{-1}(O)} : \mathrm{Bl}_O(\mathbb{A}^n) \setminus \psi^{-1}(O) \to \mathbb{A}^n \setminus \{O\}$.
3. There is a one-to-one correspondence:
$$\{x \in \{O\} \times \mathbb{P}^{n-1}\} \longleftrightarrow \{L : \text{lines through } O \text{ in } \mathbb{A}^n\}$$
4. $\mathrm{Bl}_O(\mathbb{A}^n)$ is an irreducible variety.

Fact: The blowing-up $\mathrm{Bl}_O(\mathbb{A}^n)$ can be seen in another way. Consider the map

$$q = (q, id) : (\mathbb{A}^n \setminus \{O\}) \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$$

where $q : (x_1, \dots, x_n) \mapsto [x_1 : \cdots : x_n]$. Also consider the map:

$$\iota = (\iota, id) : (\mathbb{A}^n \setminus \{O\}) \times \mathbb{P}^{n-1} \to \mathbb{A}^n \times \mathbb{P}^{n-1}$$

We have:

$$\mathrm{Bl}_O(\mathbb{A}^n) = \text{closure of } q^{-1}(\Delta_{\mathbb{P}^{n-1}}) \text{ in } \mathbb{A}^n \times \mathbb{P}^{n-1}$$

Proof. (1) is obvious from the defining equations. If $x = (0, \dots, 0)$, the equations $x_i y_j = x_j y_i$ become $0 = 0$, so any $[y] \in \mathbb{P}^{n-1}$ is a valid choice.

(2) Consider the morphism

$$\begin{aligned} \varphi : \mathbb{A}^n \setminus \{O\} &\to \mathbb{A}^n \times \mathbb{P}^{n-1} \\ (a_1, \dots, a_n) &\mapsto ((a_1, \dots, a_n), [a_1 : \cdots : a_n]) \end{aligned}$$

The image $\mathrm{Im}(\varphi)$ is contained in $\mathrm{Bl}_O(\mathbb{A}^n)$, specifically in the part that lies over $\mathbb{A}^n \setminus \{O\}$. The composition with the projection is the identity: $\psi \circ \varphi = \mathrm{id}_{\mathbb{A}^n\setminus\{O\}}$.

Moreover, any point $((x_1, \dots, x_n), [y_1 : \cdots : y_n])$ in $\mathrm{Bl}_O(\mathbb{A}^n)$ with some $x_i \neq 0$ must satisfy $y_j = y_i(x_j/x_i)$, which implies that $[y_1 : \cdots : y_n] = [x_1 : \cdots : x_n]$. This means that any point in $\mathrm{Bl}_O(\mathbb{A}^n) \setminus \psi^{-1}(O)$ is in the image of φ. Therefore, $\mathrm{Im}(\varphi) = \mathrm{Bl}_O(\mathbb{A}^n) \setminus \psi^{-1}(O)$. The map φ is the inverse of ψ on this set, so ψ is an isomorphism.

(3) Let L be a line through the origin, parameterized by $x_i = a_i t$ for $t \in k$, where

$(a_1, \dots, a_n) \in \mathbb{A}^n \setminus \{O\}$. The preimage of the punctured line $L \setminus \{O\}$ under ψ is

$$\psi^{-1}(L \setminus \{O\}) = \{((a_1 t, \dots, a_n t), [a_1 : \cdots : a_n]) \mid t \in k^*\}$$

The closure of this set in $\mathrm{Bl}_O(\mathbb{A}^n)$ is obtained by allowing $t = 0$, which gives the point $(O, [a_1 : \cdots : a_n])$. This point lies on the exceptional divisor $\psi^{-1}(O)$. This establishes a bijection where the line L defined by the direction vector $(a_1, \dots, a_n)$ corresponds to the point $[a_1 : \cdots : a_n]$ in the projective space fiber over the origin.

(4) If $\mathrm{Bl}_O(\mathbb{A}^n) = S_1 \cup S_2$ as union of two closed subsets. Consider:

$$\psi^{-1}(\mathbb{A}^n \setminus \{O\}) = (\psi^{-1}(\mathbb{A}^n \setminus \{O\}) \cap S_1) \cup (\psi^{-1}(\mathbb{A}^n \setminus \{O\}) \cap S_2)$$

$\psi^{-1}(\mathbb{A}^n \setminus \{O\}) \simeq \mathbb{A}^n \setminus \{O\}$ is irreducible, so WLOG $S_1 \supseteq \psi^{-1}(\mathbb{A}^n \setminus \{O\})$.

Claim: $\psi^{-1}(O) \subseteq \overline{\psi^{-1}(\mathbb{A}^n \setminus \{O\})}$. This is because for all $(O, [a_1, \dots, a_n]) \in \psi^{-1}(O)$,

$$(O, [a_1, \dots, a_n]) \in \overline{\psi^{-1}(L \setminus \{O\})} \subseteq \overline{\psi^{-1}(\mathbb{A}^n \setminus \{O\})}$$

So:

$$S_1 = \overline{S_1} \subseteq \overline{\psi^{-1}(\mathbb{A}^n \setminus \{O\})} \subseteq \psi^{-1}(\mathbb{A}^n \setminus \{O\}) \cup \psi^{-1}(O) = \mathrm{Bl}_O(\mathbb{A}^n)$$

which means $\mathrm{Bl}_O(\mathbb{A}^n)$ is irreducible. □

Definition 3.21. Let $X \subseteq \mathbb{A}^n$ be a closed subvariety with the origin $O \in X$. We define the **blowing-up of X at O**, denoted $\mathrm{Bl}_O(X)$, to be the closure of the preimage of $X \setminus \{O\}$ under the blow-up map $\psi : \mathrm{Bl}_O(\mathbb{A}^n) \to \mathbb{A}^n$.

$$\mathrm{Bl}_O(X) = \overline{\psi^{-1}(X \setminus \{O\})}$$

where the closure is taken in $\mathrm{Bl}_O(\mathbb{A}^n)$.

Lecture 5: Sheaves

omitted.

Lecture 6: Schemes

omitted.

4 2025.11.10

4.1 Projective Spectrum

Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring. Let $S_+ = \bigoplus_{d>0} S_d$.

Definition 4.1. The **Projective Spectrum** of S, denoted by Proj S, is the set of all homogeneous prime ideals $\mathfrak{p}$ of S such that $\mathfrak{p} \not\supset S_+$.

$$\text{Proj } S = \{\mathfrak{p} \text{ is a homogeneous prime ideal of } S \mid S_+ \not\subseteq \mathfrak{p}\}$$

For a homogeneous ideal $I \subseteq S$, we define

$$V(I) = \{\mathfrak{p} \in \text{Proj } S \mid \mathfrak{p} \supset I\}$$

These sets define the Zariski topology on Proj S.

Definition 4.2. For an open set $U \subseteq \text{Proj } S$, we define $\mathcal{O}_{\text{Proj } S}(U)$ to be the set of sections $\{(s_\mathfrak{p} \in S_{(\mathfrak{p})})_{\mathfrak{p} \in U} \mid \forall \mathfrak{p} \in U, \exists \text{ open } V \ni \mathfrak{p}, \exists f, g \in S_d \text{ s.t. } \forall \mathfrak{q} \in V, g \notin \mathfrak{q} \text{ and } s_\mathfrak{q} = \frac{f}{g} \in S_{(\mathfrak{q})}\}$. This forms a sheaf of rings on Proj S.

The pair $(\text{Proj } S, \mathcal{O}_{\text{Proj } S})$ is the projective spectrum of S.

Proposition 4.3.

1. For any $\mathfrak{p} \in \text{Proj } S$, the stalk is $\mathcal{O}_{\text{Proj } S, \mathfrak{p}} \cong S_{(\mathfrak{p})}$.

2. For any homogeneous element $f \in S_d$ with $d > 0$, define $D_+(f) = \{\mathfrak{p} \in \text{Proj } S \mid f \notin \mathfrak{p}\}$. Then the ringed space $(D_+(f), \mathcal{O}_{\text{Proj } S}|_{D_+(f)})$ is isomorphic to $(\text{Spec } S_{(f)}, O_{\text{Spec } S_{(f)}})$.

Moreover, Proj $S = \bigcup_{f \in S_+, \text{homogeneous}} D_+(f)$, thus Proj S is a scheme.

Proof Sketch of (1). Consider the natural map $\alpha_\mathfrak{p} : \mathcal{O}_{\text{Proj } S, \mathfrak{p}} \to S_{(\mathfrak{p})}$. Check that it is an isomorphism. □

Proof Sketch of (2). For any $\mathfrak{p} \in \text{Proj } S$, there exists a homogeneous element $f \in S_+$ such that $f \notin \mathfrak{p}$. This implies $\mathfrak{p} \in D_+(f)$, and thus Proj $S = \bigcup_{f \in S_+, \text{homogeneous}} D_+(f)$. Let us define a map $\psi : D_+(f) \to \text{Spec } S_{(f)}$, where $S_{(f)}$ is the degree 0 part of S_f.

$$\psi : \mathfrak{p} \mapsto \mathfrak{p} S_f \cap S_{(f)}$$

which is a prime ideal of $S_{(f)}$.

Check:

- ψ is bijective.
- ψ is a homeomorphism.

Note that the stalk at $\mathfrak{p}$ is

$$\mathcal{O}_{\text{Proj } S,\mathfrak{p}} \cong S_{(\mathfrak{p})} \cong (S_{(f)})_{\psi(\mathfrak{p})} \cong \mathcal{O}_{\text{Spec } S_{(f)},\psi(\mathfrak{p})}$$

Using the above isomorphism on the stalks gives an isomorphism of ringed spaces

$$(D_+(f), \mathcal{O}_{\text{Proj } S}|_{D_+(f)}) \cong (\text{Spec } S_{(f)}, \mathcal{O}_{\text{Spec } S_{(f)}})$$

□

Lecture 7: First Properties of Schemes

4.2 Properties of Schemes

Definition 4.4. We say a scheme X is

- **connected** if its underlying topological space, $|X|$, is connected.
- **irreducible** if its underlying topological space, $|X|$, is irreducible.
- **reduced** if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is reduced.
- **integral** if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Lemma 4.5. A scheme X is reduced if and only if for all $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is a reduced ring.

Proof. If X is an affine scheme, $X = \text{Spec } A$, then we know that A is reduced if and only if $A_\mathfrak{p}$ is reduced for all $\mathfrak{p} \in \text{Spec } A$. So we have the "only if" direction.

For the "if" direction, we only need to show: If $U = \bigcup_i U_i$, where $U_i = \text{Spec } A_i$ and each A_i is reduced, then $\mathcal{O}_X(U)$ is reduced. This follows from the fact that if $\mathcal{O}_X(U_i)$ is reduced for all i, then from the canonical exact sequence:

$$0 \to \mathcal{O}_X(U) \to \prod_i \mathcal{O}_X(U_i) \to \dots$$

we know $\mathcal{O}_X(U)$ is reduced. $\square$

Remark: Let $A = k \times k$, where k is a field. Then Spec $A = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, and the stalks are $A_{\mathfrak{p}_i} \cong k$, which are reduced. But A is not an integral domain.

Proposition 4.6. Let $X = \operatorname{Spec} A$. Then X is:

1. **Irreducible** if and only if the nilradical $\sqrt{(0)}$ is a prime ideal.
2. **Reduced** if and only if $\sqrt{(0)} = (0)$.
3. **Integral** if and only if A is an integral domain.

Proof of (1). ($\Rightarrow$) If $\sqrt{(0)}$ is not prime, then there exist $f, g \notin \sqrt{(0)}$ such that $fg \in \sqrt{(0)}$. Then $X = V(f) \cup V(g)$, but $X \neq V(f)$ and $X \neq V(g)$, which is a contradiction.

($\Leftarrow$) If $X = V(I) \cup V(J) = V(IJ)$, then $IJ \subseteq \sqrt{(0)}$. Since $\sqrt{(0)}$ is prime, we have $I \subseteq \sqrt{(0)}$ or $J \subseteq \sqrt{(0)}$. This implies $V(I) = X$ or $V(J) = X$. $\square$

Proof of (3). ($\Leftarrow$) If A is an integral domain, then for any $f \in A$, A_f is also an integral domain. For any open set $U \subseteq X$, we can write $U = \bigcup_i D(f_i)$.

Claim: $\mathcal{O}_X(U)$ is an integral domain.

Take $a, b \in \mathcal{O}_X(U)$ such that $a \cdot b = 0$. Then for each $D(f_i)$ in the covering, the restriction of a or b to $D(f_i)$ must be zero. This **implies** that $U = V_U(a) \cup V_U(b)$, where

$$\begin{aligned} V_U(s) &= \{\mathfrak{p} \in U \mid s(\mathfrak{p}) = 0\} \\ &= \{\mathfrak{p} \in U \mid s_{\mathfrak{p}} \in \mathfrak{m}_{\mathfrak{p}} \subseteq \mathcal{O}_{X,\mathfrak{p}}\} \end{aligned}$$

Reason: For any $\mathfrak{p} \in U$, there exists some $D(f_i)$ such that $\mathfrak{p} \in D(f_i)$. Since $a|_{D(f_i)}$ or $b|_{D(f_i)} = 0$ in $\mathcal{O}_X(D(f_i)) = A_{f_i}$, either $(a|_{D(f_i)})_{\mathfrak{p}} = 0$ or $(b|_{D(f_i)})_{\mathfrak{p}} = 0$. In particular, either $a_{\mathfrak{p}} \in \mathfrak{m}_{\mathfrak{p}}$ or $b_{\mathfrak{p}} \in \mathfrak{m}_{\mathfrak{p}}$. $V_U(a)$ and $V_U(b)$ are closed subsets of U from **Hartshorne Chapter II Exercise 2.16**.

Since U is an open subset of the irreducible space Spec A, U is irreducible. Therefore, $U = V_U(a)$ or $U = V_U(b)$. WLOG, $U = V_U(a)$, which means for all $D(f_i)$ in the covering, and for all $\mathfrak{p} \in D(f_i)$, we have:

$$a|_{D(f_i)} \in \mathfrak{p}A_{f_i}$$

In particular, $(0) \in D(f_i)$ for every i because A is integral domain. Thus, $a|_{D(f_i)} = 0$ in A_{f_i} for every i. This implies $a = 0$ in $\mathcal{O}_X(U)$. $\square$

Proposition 4.7. A scheme X is integral if and only if X is reduced and irreducible.

Proof. ($\Leftarrow$) This is similar to what we have shown above.

($\Rightarrow$) By definition, if X is integral, then X is reduced. Suppose $X = Y \cup Z$ with $Y, Z \subsetneq X$ being proper closed subsets. Then there exists open affine subsets $U_1 \subseteq X \setminus Y$ and $U_2 \subseteq X \setminus Z$ and obviously $U_1 \cap U_2 = \emptyset$. From canonical exact sequence:

$$0 \to \mathcal{O}_X(U_1 \cup U_2) \to \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \to 0$$

This contradicts the fact that $\mathcal{O}_X(U_1 \cup U_2)$ is an integral domain. So X is irreducible. $\square$

Definition 4.8. We say a scheme X is **locally Noetherian** if there exists an open affine covering $X = \bigcup_i U_i$, where $U_i = \text{Spec } A_i$, such that each ring A_i is Noetherian.
We say a scheme X is **Noetherian** if it is locally Noetherian and quasi-compact. (This is equivalent to saying X can be covered by a finite number of open affine subsets $U_i = \text{Spec } A_i$, where each A_i is a Noetherian ring).

Remark: If a scheme X is Noetherian, then its underlying topological space $|X|$ is a Noetherian topological space. The converse is not true.

Example 4.9. Let k be a field and consider the ring $A = k[t^a \mid a \in \mathbb{R}_{\geq 0}]$. Let $X = \text{Spec } A$.

Check: The topological space $|X|$ consists of two points: the prime ideal $\mathfrak{p}_0 = (0)$ and the maximal ideal $\mathfrak{m} = (t^a \mid a > 0)$. The point $\{\mathfrak{m}\}$ is a closed set. The closure of the point $\{\mathfrak{p}_0\}$ is the entire space X. The chain of closed sets must stabilize (it can only be $\{\mathfrak{m}\} \subseteq X$), so $|X|$ is a Noetherian topological space.

However, the ring A is not Noetherian, because the ideal $\mathfrak{m}$ is not finitely generated.

Proposition 4.10. A scheme X is locally Noetherian if and only if for every open affine covering $X = \bigcup_i U_i$ with $U_i = \text{Spec } A_i$, each ring A_i is Noetherian.

Proof. The "$\Leftarrow$" direction is obvious from the definition.

For "$\Rightarrow$": It suffices to show that for any open affine subset $U \subseteq X$, if we write $U = \text{Spec } A$, then A must be a Noetherian ring.

Let $\{X_i\}$ be an open affine covering of X such that $X_i = \text{Spec } B_i$ and each B_i is a Noetherian ring. Then $U = \bigcup_i (U \cap X_i)$. Now we can cover each open set $U \cap X_i$ by principal open sets $D_{\mathrm{Spec}B_i}(f)$ for $f \in B_i$. This implies that U can be covered by open affine sets $V_\alpha = \mathrm{Spec}\ B_\alpha$ where each B_α is a localization of some B_i and is therefore Noetherian.

Further refinement: Since U is an affine scheme, it is quasi-compact. We can therefore find a finite refinement of the cover, $U = \bigcup_{j=1}^n D_{\mathrm{Spec}A}(f_j)$, such that for each j, $D_{\mathrm{Spec}A}(f_j) \subseteq V_\alpha$ for some α. Then from **Hartshorne Chapter II Exercise 2.16**:

$$D_{\mathrm{Spec}A}(f_j) = U_{f_j} = U_{f_j} \cap V_\alpha = D_{\mathrm{Spec}B_\alpha}(f_j|_{V_\alpha})$$

Hence we have isomorphism of rings $A_{f_j} \cong (B_\alpha)_{f_j|_{V_\alpha}}$, which is Noetherian.

To show that A is Noetherian, it suffices to show that every ideal I of A is finitely generated. Since $(A)_{f_j}$ is Noetherian, the ideal $I_{f_j} \subseteq A_{f_j}$ is finitely generated. Let $\{\frac{a_1}{1}, \ldots, \frac{a_k}{1}\}$ be a set of generators of I_{f_j}, where $a_1, \ldots, a_k \in I$. Let $E_j = \{a_{j,1}, \ldots, a_{j,n_j}\} \subseteq I$ be a finite set of elements whose images generate the ideal I_{f_j} in A_{f_j}. Take $E = \bigcup_{j=1}^n E_j$. This is a finite set of elements of I. It is a standard result that E generates I. $\square$

4.3 Properties of Morphisms

Definition 4.11. A morphism of schemes $f : X \to Y$ is **affine** if there exists an open affine covering $Y = \bigcup_i Y_i$, with $Y_i = \mathrm{Spec}\ B_i$, such that for each i, the preimage $f^{-1}(Y_i)$ is an affine scheme.

Definition 4.12. A morphism of schemes $f : X \to Y$ is **quasi-compact** if for every quasi-compact open subset $V \subseteq Y$, its preimage $f^{-1}(V)$ is also quasi-compact.

Definition 4.13. A morphism of schemes $f : X \to Y$ is **locally of finite type** if there exists an open affine covering $Y = \bigcup_i Y_i$ with $Y_i = \mathrm{Spec}\ B_i$, each preimage $f^{-1}(Y_i)$ can be covered by open affine schemes $X_{ij} = \mathrm{Spec}\ A_{ij}$ such that each A_{ij} is a finitely generated B_i-algebra. A morphism of schemes $f : X \to Y$ is **finite type** if it is locally of finite type and quasi-compact.

Definition 4.14. A morphism of schemes $f : X \to Y$ is **finite** if for every open affine subset $V = \mathrm{Spec}\ B \subseteq Y$, its preimage $f^{-1}(V)$ is affine, say $f^{-1}(V) = \mathrm{Spec}\ A$, and the ring A is a finitely generated B-module.

Remark: We have the following implications for morphisms:

$$\text{finite} \implies \text{affine}$$
$$\text{finite} \implies \text{finite type} \implies \text{quasi-compact}$$

We will prove affine $\implies$ quasi-compact.

We first state a useful proposition which is actuallt **Hartshorne Chapter II Exercise 2.17(b)**:

Proposition 4.15. A scheme X is affine if and only if there exists a set of elements $\{f_i\}_{i\in I} \subseteq \mathcal{O}_X(X)$ such that the open sets $X_{f_i} = \{x \in X \mid (f_i)_x \notin \mathfrak{m}_x\}$ are affine and the set $\{f_i\}$ generates the unit ideal in $\mathcal{O}_X(X)$.

Proposition 4.16. Let $f : X \to Y$ be an affine morphism. If Y is an affine scheme, then X is affine.

Proof. Write $Y = \operatorname{Spec} B = \bigcup_{i=1}^n \operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i) \cong \operatorname{Spec} A_i$. Take $h \in B$ such that $D(h) \subseteq \operatorname{Spec} B_i$. Then $f^{-1}(D(h)) \cong D(\overline{h})$ in $\operatorname{Spec} A_i$, where $\overline{h}$ is the image of $h|_{\operatorname{Spec} B_i}$ of $B_i \to A_i$. $\tilde{h}$ is the image of h of $B \to \mathcal{O}_X(X)$. Then $\overline{h}$ is the image of $\tilde{h}$ of $\mathcal{O}_X(X) \to A_i$.

We **claim** that $X_{\tilde{h}} = f^{-1}(Y_h)$.

- $\subseteq$: For all $x \in X_{\tilde{h}}$, we have $(\tilde{h})_x \notin \mathfrak{m}_x$. From the local homomorphism $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$, if $f(x) \notin Y_h$, then $h_{f(x)} \in \mathfrak{m}_{f(x)}$, then $(\tilde{h})_x = f_x^\#(h_{f(x)}) \in \mathfrak{m}_x$, which is a contradiction! So $f(x) \in Y_h$, i.e., $x \in f^{-1}(Y_h)$.

- $\supseteq$: For all $x \in f^{-1}(Y_h)$, we have $f(x) \in Y_h$, then $h_{f(x)} \notin \mathfrak{m}_{f(x)}$. Then the same: $(\tilde{h})_x = f_x^\#(h_{f(x)}) \notin \mathfrak{m}_x$, which means $x \in X_{\tilde{h}}$.

So we can get an open affine covering of X by

$$X = \bigcup_{i,h} D_{\operatorname{Spec} A_i}(\overline{h}) = \bigcup_{i,h} f^{-1}(D_{\operatorname{Spec} B_i}(h|_{\operatorname{Spec} B_i})) = \bigcup_{i,h} f^{-1}(Y_h) = \bigcup_{i,h} X_{\tilde{h}}$$

where $X_{\tilde{h}} = D_{\operatorname{Spec} A_i}(\overline{h})$ is affine.

Moreover, $\operatorname{Spec} B$ is covered by $\{D(h_i)\}$. This implies that $\{h_i\}$ generates the unit ideal of B. This implies that $\{\tilde{h}_i\}$ generates the unit ideal of $\mathcal{O}_X(X)$. Therefore, X is affine by **Proposition 4.15**. $\square$

Lemma 4.17. If $f : X \to Y$ is an affine morphism, then f is quasi-compact.

Proof. Let $U \subseteq Y$ be a quasi-compact open subset. Since Y has a basis of open affine subsets, we can cover U by a finite number of open affine subsets, say $U = \bigcup_{i=1}^n V_i$. Then $f^{-1}(U) = \bigcup_{i=1}^n f^{-1}(V_i)$. Since f is an affine morphism, from **Proposition 4.16**: each $f^{-1}(V_i)$ is an affine scheme, and thus is quasi-compact. A finite union of quasi-compact spaces is quasi-compact, so $f^{-1}(U)$ is quasi-compact. Therefore, f is a quasi-compact morphism. $\square$

Lemma 4.18. If a morphism $f : X \to Y$ is locally of finite type and $Y \cong \text{Spec } B$. Then for every open affine subset $U = \text{Spec } A$ of X, the ring A is a finitely generated B-algebra.

Theorem 4.19 (**Chevalley's Criterion of Affineness**)**.** Let $f : X \to Y$ be a finite and surjective morphism. Assume that X is affine, then Y is affine.

4.4 Open & Closed Immersions

Definition 4.20. A morphism $f : X \to Y$ is an **open immersion** if f induces an isomorphism from X onto an open subscheme of Y.

Definition 4.21.

- A morphism $f : X \to Y$ is a **closed immersion** if f induces a homeomorphism from the topological space $|X|$ onto a closed subset of $|Y|$, and furthermore the induced map of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.
- A morphism $f : X \to Y$ is an **immersion** if it factorizes into $X \xrightarrow{h} Z \xrightarrow{g} Y$, where g is an open immersion and h is a closed immersion.

Definition 4.22. A **closed subscheme** of X is an equivalence class of closed immersions $i : Y \to X$. (If $i_0 : Y_0 \to X$ and $i_1 : Y_1 \to X$ are in the same class, then there is an isomorphism $\phi : Y_0 \to Y_1$ such that $i_0 = i_1 \circ \phi$).

Example: Let $X = \text{Spec } A$. Then for any ideal $I \subseteq A$, the natural morphism $\text{Spec } (A/I) \to \text{Spec } A$ is a closed immersion.

Proposition 4.23. Every closed immersion $Y \to \text{Spec } A$ is equivalent to a morphism of the form $\text{Spec } (A/I) \to \text{Spec } A$ for some ideal $I \subseteq A$.

Proof. We consider Y as a closed subset of Spec A with the induced topology. Let $Y = \bigcup_i Y_i$ with Y_i open affine of Y. Then for each i, Y_i is a locally closed subset in Spec A. So there exists

U_i open in Spec A such that $Y_i = Y \cap U_i$.

For all U_i can be written as an principal open covering $U_i = \bigcup_j U_{ij}$, then $Y = \bigcup_i \bigcup_j (Y \cap U_{ij})$. Also Spec $A \setminus Y$ is open, so we can write Spec $A \setminus Y = \bigcup_k V_k$ where each V_k is principal open. $\{U_{ij}, V_k\}$ forms an open covering of Spec A. From quasi-compactness of Spec A, we can take a finite subcovering

$$Y = \bigcup_{k=1}^{m} (Y \cap D(f_k))$$

And

- For all $k \in [1, m]$ either: there exists Y_i such that $Y \cap D(f_k) \subseteq Y_i$; or: $Y \cap D(f_k) = \emptyset$.
- $\{f_k\}_{k=1}^{m}$ generates the unit ideal of A.

Denote the closed immersion by $\iota : Y \to$ Spec A with $\iota^\# : \mathcal{O}_{\mathrm{Spec}A} \to \iota_* \mathcal{O}_Y$

$$\overline{f_k} := \iota^\# (\mathrm{Spec}A)(f_k) \in \mathcal{O}_Y(Y)$$

Notice that for those k such that $Y \cap D(f_k) \neq \emptyset$:

$$Y_{\overline{f_k}} = \iota^{-1}(X_{f_k}) = Y \cap D(f_k) \subseteq Y_i$$

So:

$$Y_{\overline{f_k}} = Y_{\overline{f_k}} \cap Y_i = D_{Y_i}(\overline{f_k}|_{Y_i})$$

So $Y_{\overline{f_k}}$ is affine. Together with $Y_{\overline{f_k}}$ covers Y, by **Proposition 4.15**, Y is affine!

Alternatively, let $Y =$ Spec B and let the morphism ι be induced by a ring homomorphism $\varphi : A \to B$. Let $I = \ker(\varphi)$. Then ι factors through $Y =$ Spec $B \xrightarrow{g}$ Spec $(A/I) \xrightarrow{i}$ Spec A. Since the map $A/I \to B$ is injective, the corresponding sheaf map $g^\# : \mathcal{O}_{\mathrm{Spec}(A/I)} \to g_* \mathcal{O}_Y$ is injective. By the definition of a closed immersion, the map $\iota^\# : \mathcal{O}_{\mathrm{Spec}\ A} \to \iota_* \mathcal{O}_Y$ is surjective. This implies that $g^\#$ is surjective. A surjective and injective map of sheaves implies $g^\#$ is an isomorphism. Since g is also a homeomorphism, g is an isomorphism. □

Corollary 4.24. A closed immersion is a finite morphism.

Remark: An open immersion is locally of finite type, but an open immersion is not necessarily quasi-compact.

Example 4.25. Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$. Let $\mathfrak{m} = (x_1, x_2, \dots)$ be the maximal ideal corresponding to the origin. Let $U = X \setminus \{\mathfrak{m}\}$. This is an open immersion. U is not quasi-compact.

Facts:

$$\begin{tikzcd}
\text{closed immersion} \arrow[r, Rightarrow] & \text{finite} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{affine} \arrow[d, Rightarrow] \\
& \text{finite type} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{quasi-compact} \\
\text{open immersion} \arrow[r, Rightarrow] & \text{locally finite type} &
\end{tikzcd}$$

Proposition 4.26 (Reduced Closed Subscheme). Let X be a scheme, and let $Y \subseteq X$ be a closed subset. Then there exists a unique reduced closed subscheme structure on Y.

Construction Sketch.

- If $X = \operatorname{Spec} A$, then we take $Y = V(I)$ for some ideal $I \subseteq A$. We give Y the scheme structure of $\operatorname{Spec}(A/\sqrt{I})$. This scheme is reduced.
- In general, take an open affine covering $X = \bigcup_i X_i$. For each i, define a structure on $Y_i = Y \cap X_i$ using the construction above.
- Then on the intersection $Y_i \cap Y_j$, these two subscheme structures are canonically isomorphic.
- Moreover, on the triple intersection $Y_i \cap Y_j \cap Y_k$, these canonical isomorphisms ϕ_{ij} satisfy the cocycle condition, i.e., $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$.
- This gluing construction gives a reduced closed subscheme structure on Y.

□

Lemma 4.27. Let $f : Y' \to X$ be a closed immersion with $\operatorname{Im}(f) = Y$. Then there exists a unique morphism $i : Y \to Y'$ such that the following diagram commutes. Here, Y has the reduced scheme structure.

$$\begin{tikzcd}
Y \arrow[rr, dashed, "\exists\,!\,i"] \arrow[dr, "\iota"'] & & Y' \arrow[dl, "f"] \\
& X &
\end{tikzcd}$$

Proof. Suppose $X \cong \operatorname{Spec} A$. Then a closed subscheme can be written as $Y' \cong \operatorname{Spec}(A/J)$. The associated reduced closed subscheme is $Y \cong \operatorname{Spec}(A/\sqrt{J})$. The uniqueness follows from the fact that if another ideal I defines the same closed set, $V(I) = V(J)$, then it must be that $\sqrt{I} = \sqrt{J}$. There is a morphism $Y \to Y'$ which is given by the canonical surjective ring homomorphism $A/J \to A/\sqrt{J}$. $\square$

Example: Apply the above to the case where the closed set is the entire space, $Y = X$. Then the resulting reduced closed subscheme is denoted by X_{red}.

Lemma 4.28 (**Universal Property of Reduction**). If $f : Z \to X$ is a morphism of schemes and the scheme Z is reduced, then there exists a unique morphism $f_{red} : Z \to X_{red}$ such that f factors through the canonical immersion $i : X_{red} \to X$. That is, the following diagram commutes:

$$\begin{tikzcd} Z \arrow[d, "f_{red}"] \arrow[dr, "f"] & \\ X_{red} \arrow[r, "i"] & X \end{tikzcd}$$

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Proposition 5.1. Let $Y \subseteq X$ be a closed subspace of a scheme X. Then there exists a unique closed reduced subscheme structure on Y. It is the smallest scheme structure on Y.
i.e. For any closed subscheme $i : Z \hookrightarrow X$ such that $|Z| \supseteq |Y|$, then there exists a unique morphism $Y_{red} \to Z$ over X.

Apply the construction to X itself, we get $X_{red} = (\mathrm{sp}X, \mathcal{O}_X/\mathrm{nil}(\mathcal{O}_X))$.

Lemma 5.2. Let $f : T \to X$ be a morphism of schemes. Suppose T is a reduced scheme ($T = T_{red}$), then f factors through X_{red}.

Lemma 5.3. Every irreducible closed subscheme Z of X contains a unique point η such that $\overline{\{\eta\}} = Z$.

Proof. Endow Z with the reduced closed subscheme structure. Then Z is an integral scheme. Take an open affine subset $U = \operatorname{Spec} A \subseteq Z$. Then A is an integral domain. Define η to be the point corresponding to the zero ideal $(0) \in \operatorname{Spec} A$.

We **claim** that for all subset S of U:

$$\mathrm{cl}_Z(S) \cap U = \mathrm{cl}_U(S)$$

All closed subset of U must be of the form $V \cap U$ where V is a closed subset of Z. For all those closed subsets $V \cap U$ of U containing S, we have $V = \mathrm{cl}_Z(V) \supseteq \mathrm{cl}_Z(S)$. So $V \cap U \supseteq \mathrm{cl}_Z(S) \cap U$. From the property of closure we have $\mathrm{cl}_U(S) \subseteq \mathrm{cl}_Z(S) \cap U$.

So $\mathrm{cl}_Z(\eta) \cap U = \mathrm{cl}_U(\eta) = U$. Because Z is closed subset of X, $\mathrm{cl}_Z(\eta) = \mathrm{cl}_X(\eta) = \overline{\{\eta\}}$. $U \subseteq \overline{\{\eta\}}$. So:

$$Z = \overline{\{\eta\}} \cup (Z \setminus U)$$

Because Z is irreducible, $\overline{\{\eta\}} = Z$ or $U = \emptyset$ (which is impossible).

Uniqueness: Supose there is another point $\eta' \in Z$ satisfies $\overline{\{\eta'\}} = Z$, choose an open affine neighborhood $U' = \operatorname{Spec} A'$ of η' in Z. From $\mathrm{cl}_{U'}(\eta') = \mathrm{cl}_Z(\eta') \cap U' = Z \cap U' = U'$ and A' is integral domain, we know that η' corresponds to $(0) \in \operatorname{Spec} A'$.

Because Z is irreducible, $U \cap U'$ is a nonempty open subset of Z. There is an affine open subset $U'' = \operatorname{Spec} A'' \subseteq U \cap U'$. We have proved that the point $\eta'' \in U''$ corresponding to $(0) \in \operatorname{Spec} A''$ satisfies $\overline{\{\eta''\}} = Z$. Because $\eta'' \in U'$, $\eta'' = \eta'$ both corresponds to $(0) \in \operatorname{Spec} A'$. Similarly, $\eta'' = \eta$ both corresponds to $(0) \in \operatorname{Spec} A$. So $\eta = \eta'$. $\square$

Definition 5.4. η is the **generic point** of Z.

5.1 Dimension of Schemes

Definition 5.5. The **dimension of a scheme** X, denoted $\dim X$, is the dimension of its underlying topological space sp$X$.
For an irreducible closed subset $Z \subseteq X$, we define the **codimension** of Z in X as

$$\operatorname{codim}(Z, X) = \sup\{n \mid Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X, Z_i \text{ irreducible closed}\}$$

Example 5.6. If $X = \operatorname{Spec} A$, then

$$\dim X = \sup\{n \mid \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n, \mathfrak{p}_i \text{ prime}\} = \text{Krull dim } A.$$

Remark: For an irreducible variety X and an irreducible closed subvariety $Z \subseteq X$, the following formula holds:

$$\dim X = \dim Z + \operatorname{codim}(Z, X)$$

This does not hold in general for schemes!

Example 5.7. Let R be a DVR (e.g., $R = k[x]_{(x)}$). Let $X = \operatorname{Spec} R[t]$. Let $\mathfrak{m} \subset R$ be the maximal ideal, $\mathfrak{m} = (t)$. Let $k = R/\mathfrak{m}$ and $K = \operatorname{Frac}(R)$.
Let $Z = V(1 + tx) \subset X$. Since $1 + tx$ is irreducible, Z is an irreducible closed subset.

- The scheme X has a closed fiber $\operatorname{Spec} k[t]$ and a generic fiber $\operatorname{Spec} K[t]$. We have $\dim X = \dim R[t] = 2$.
- The intersection of Z with the closed fiber is $Z \cap \operatorname{Spec} k[t] = V(1 + tx, t) = V(1) = \emptyset$. The intersection with the generic fiber is $Z \cap \operatorname{Spec} K[t] = V(1 + tx)$, which is a point. So Z is a closed subset of dimension 0.
- We can have the following feeling: The only irreducible subset of Spec $K[t]$ strictly containing a closed point is itself. Spec $K[t]$ is "generic fiber" of X. So we can guess that the only irreducible subset of X strictly containing Z is itself, i.e. $\operatorname{codim}(Z, X) = 1$.

Then $\dim Z + \operatorname{codim}(Z, X) = 0 + 1 = 1 \neq 2 = \dim X$.

Lemma 5.8.

$$\dim X = \sup_{p\in X} \dim \mathcal{O}_{X,p}$$

If Z is an irreducible closed subset with generic point η, then

$$\operatorname{codim}(Z, X) = \dim \mathcal{O}_{X,\eta}.$$

Proof. It is enough to prove the second statement. This is because:

$$\dim X = \sup_{Z\subseteq X \text{ irr. cl.}} \operatorname{codim}(Z, X)$$

Observe that for any open set $U \subseteq X$ with $U \cap Z \neq \emptyset$, we have

$$\operatorname{codim}(Z, X) = \operatorname{codim}(Z \cap U, U).$$

Take $U = \operatorname{Spec} A$. Then $Z \cap U$ corresponds to a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. We have $Z \cap U = \operatorname{cl}_U(\{[\mathfrak{p}]\})$. Similar argument in **Lemma 5.3**, $\operatorname{cl}_Z(\{[\mathfrak{p}]\}) = Z$. From the uniqueness of generic point, $[\mathfrak{p}] = \eta$.

A chain of irreducible closed subsets $Z_0 \cap U \subsetneq Z_1 \cap U \subsetneq \cdots \subsetneq Z_n \cap U$ containing $Z \cap U$ corresponds to a chain of prime ideals $\mathfrak{p} \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ in $\operatorname{Spec} A$. The length of the longest such chain is the height of $\mathfrak{p}$, which is $\dim A_\mathfrak{p} = \dim \mathcal{O}_{\operatorname{Spec} A,[\mathfrak{p}]} = \dim \mathcal{O}_{X,\eta}$. Therefore, $\operatorname{codim}(Z \cap U, U) = \dim \mathcal{O}_{X,\eta}$.

□

Lecture 8. More Properties of Morphisms

5.2 Fiber product of schemes

Recall: In a category $\mathcal{C}$, consider two morphisms $f : X \to Z$ and $g : Y \to Z$. The **fiber product** of X and Y over Z consists of an object W (often denoted $X \times_Z Y$) and two morphisms (called projections) $p_X : W \to X$ and $p_Y : W \to Y$ such that the square on the bottom right commutes, i.e., $f \circ p_X = g \circ p_Y$.

This data must satisfy the following universal property: for any other object T with morphisms $u : T \to X$ and $v : T \to Y$ that also satisfy $f \circ u = g \circ v$, there exists a **unique** morphism $h : T \to W$ such that $p_X \circ h = u$ and $p_Y \circ h = v$. In other words, the entire diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists!\ h"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& W \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

> **Theorem 5.9.** The fiber product exists in the category of schemes (Sch).

Remark: The fiber product is unique up to a canonical isomorphism. We denote it by $X \times_Z Y$ if it exists.

Proof. **Step 1: The affine case.** Suppose $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $Z = \operatorname{Spec} C$ are affine schemes, with morphisms corresponding to ring homomorphisms $\phi : C \to A$ and $\psi : C \to B$. Then the fiber product $X \times_Z Y$ exists and is given by

$$X \times_Z Y = \operatorname{Spec}(A \otimes_C B)$$

This is because for any scheme T, the set of morphisms from T to an affine scheme $\operatorname{Spec} R$ is in one-to-one correspondence with the set of ring homomorphisms from R to the ring of global sections $\Gamma(T, \mathcal{O}_T)$.

$$\operatorname{Map}(T, \operatorname{Spec} A) = \operatorname{Hom}_{\mathrm{Ring}}(A, \Gamma(T, \mathcal{O}_T))$$

The universal property of the fiber product of schemes then translates to the universal property of the tensor product (pushout) in the category of rings. The ring $A \otimes_C B$ is the pushout of A and B over C in the category of rings:

$$\begin{tikzcd}
C \arrow[r, "\phi"] \arrow[d, "\psi"'] & A \arrow[d] \\
B \arrow[r] & A \otimes_C B
\end{tikzcd}$$

Step 2: Gluing with an open subscheme. Suppose $X \times_Z Y$ exists. Let $U \subseteq X$ be an open subscheme. Then the fiber product $U \times_Z Y$ exists and is a canonical open subscheme of $X \times_Z Y$. Specifically, if $p_X : X \times_Z Y \to X$ is the projection, then $U \times_Z Y$ is isomorphic to the open subscheme $p_X^{-1}(U)$. This follows from the universal property, considering the pullback square for $U \to X$:

$$\begin{tikzcd} U \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ U \arrow[r, "f|_U"] & Z \end{tikzcd} \quad \text{is isomorphic to} \quad \begin{tikzcd} p_X^{-1}(U) \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ U \arrow[r, "f|_U"] & Z \end{tikzcd}$$

Step 3: General construction by gluing. Suppose $X = \bigcup_i U_i$ is an open covering such that for each i, the fiber product $U_i \times_Z Y$ exists. Then by the uniqueness of the fiber product, we have canonical isomorphisms on the intersections:

$$\phi_{ij} : (U_i \times_Z Y)|_{U_i \cap U_j} \xrightarrow{\sim} (U_i \cap U_j) \times_Z Y \xrightarrow{\sim} (U_j \times_Z Y)|_{U_i \cap U_j}$$

These isomorphisms ϕ_{ij} satisfy the cocycle condition ($\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on the triple intersection). Then the gluing construction gives a scheme W. It is easy to see that W is a fiber product of X and Y over Z.

Step 4: The case where Z is affine. Let us show that $X \times_Z Y$ exists under the assumption that Z is affine. Take affine open coverings $X = \bigcup_i U_i$ and $Y = \bigcup_\alpha Y_\alpha$. By Step 1, the fiber product $U_i \times_Z Y_\alpha$ exists for all i, α (since U_i, Y_α, Z are all affine). By Step 3 (gluing), since $Y = \bigcup_\alpha Y_\alpha$, the fiber product $U_i \times_Z Y$ exists for each i. Again by Step 3, since $X = \bigcup_i U_i$, the fiber product $X \times_Z Y$ exists.

Step 5: The general case. Take an open affine cover $Z = \bigcup_\alpha Z_\alpha$. Let $X_\alpha = f^{-1}(Z_\alpha)$ and $Y_\alpha = g^{-1}(Z_\alpha)$. By Step 4, the fiber product $X_\alpha \times_{Z_\alpha} Y_\alpha$ exists for each α. Notice that we have a canonical isomorphism $X_\alpha \times_{Z_\alpha} Y_\alpha \cong X_\alpha \times_Z Y_\alpha$. Applying the gluing argument again, we can construct $X \times_Z Y$. □

Definition 5.10. Let $f : X \to Y$ be a morphism of schemes and let $y \in Y$ be a point. Let $k(y)$ be the **residue field** of y, defined as $k(y) := \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$. There is a canonical morphism $\operatorname{Spec} k(y) \to Y$ whose image is the point y. We define the **fiber** of the morphism f at the point y to be the scheme

$$X_y := X \times_Y \operatorname{Spec} k(y)$$

The fiber fits into the following pullback diagram:

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r] & Y \end{tikzcd}$$

Example 5.11. Let $X = \operatorname{Spec} k[x,y,t]/(ty - x^2)$ and $Y = \operatorname{Spec} k[t]$. The morphism $f : X \to Y$ is induced by the inclusion $k[t] \hookrightarrow k[x,y,t]/(ty - x^2)$. This can be viewed as a family of curves over the affine line Y. Let's compute the fiber X_a over a point $a \in Y$ corresponding to the maximal ideal $(t-a) \subset k[t]$.

$$\begin{aligned} X_a &= \operatorname{Spec}\left(k[x,y,t]/(ty - x^2) \otimes_{k[t]} k[t]/(t-a)\right) \\ &\cong \operatorname{Spec} k[x,y]/(ay - x^2). \end{aligned}$$

If $a \neq 0$, the fiber X_a is the affine scheme corresponding to the parabola $y = x^2/a$. If $a = 0$, the fiber X_0 is $\operatorname{Spec} k[x,y]/(-x^2)$, which is the y-axis with a non-reduced structure. You can see the geometric intuition in Hartshorne's Algebraic Geometry.

Definition 5.12. If Y is an irreducible scheme and $\eta \in Y$ is its generic point, we say that the fiber X_η is the **generic fiber** of the morphism $f : X \to Y$.

5.3 Diagonal Morphism

Definition 5.13. Let $f : X \to Y$ be a morphism of schemes. Consider the fiber product $X \times_Y X$. The **diagonal morphism** is the unique morphism $\Delta_{X/Y} : X \to X \times_Y X$ induced by the identity morphisms $id : X \to X$ and $id : X \to X$ via the universal property of the fiber product.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!\ \Delta_{X/Y}"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Proposition 5.14. The diagonal morphism $\Delta_{X/Y}$ is an immersion.

Proof. **Step 1: The affine case.** Assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then $X \times_Y X = \operatorname{Spec}(A \otimes_B A)$. The diagonal morphism $\Delta_{X/Y}$ is induced by the multiplication map (a ring homomorphism)

$$\begin{aligned} m : A \otimes_B A &\to A \\ a_1 \otimes a_2 &\mapsto a_1 a_2 \end{aligned}$$

which is surjective. A surjective ring homomorphism induces a closed immersion of the corresponding affine schemes. Therefore, $\Delta_{X/Y}$ is a closed immersion in this case.

Step 2: The general case. Let $\{U_\alpha\}$ be an open affine cover of Y and let $\{V_{\alpha i}\}$ be an open affine cover of $f^{-1}(U_\alpha) = X_\alpha$. The collection $\{V_{\alpha i}\}$ is an open cover of X. Let's denote V_i for these open affine sets that cover X. The diagonal morphism $\Delta_{X/Y}$ maps V_i to $V_i \times_Y V_i$. The scheme $V_i \times_Y V_i$ is an open subscheme of $X \times_Y X$. Moreover, the restriction of the diagonal morphism $\Delta_{X/Y}|_{V_i} : V_i \to V_i \times_Y V_i$ is the diagonal morphism $\Delta_{V_i/Y}$. Since V_i is affine, let's say $V_i \to Y$ factors through an affine open $U \subset Y$, then $\Delta_{V_i/Y}$ is the same as $\Delta_{V_i/U}$. By Step 1, this is a closed immersion.

Conclusion:

$$X \xrightarrow{\text{closed immersion}} \bigcup_{\alpha,i} V_{\alpha,i} \times_{Y_\alpha} V_{\alpha,i} \xrightarrow{\text{open immersion}} X \times_Y X$$

So $\Delta_{X/Y}$ is an immersion. □

Proposition 5.15. A morphism $f : X \to Y$ is a monomorphism in the category of schemes (Sch) if and only if the diagonal morphism $\Delta_{X/Y}$ is an isomorphism.

Proof. ($\Rightarrow$) Suppose f is a monomorphism. Let $p_1, p_2 : X \times_Y X \to X$ be the two projections. By the definition of the fiber product, we have $f \circ p_1 = f \circ p_2$. Since f is a monomorphism, this implies $p_1 = p_2$.

Now, consider the diagram for $\Delta_{X/Y}$. We have $p_1 \circ \Delta_{X/Y} = \mathrm{id}_X$. Since $p_1 = p_2$, we also have $p_2 \circ \Delta_{X/Y} = \mathrm{id}_X$.

Let's construct an inverse. Consider the morphism $p_1 : X \times_Y X \to X$. The composition $\Delta_{X/Y} \circ p_1 : X \times_Y X \to X \times_Y X$ satisfies $p_1 \circ (\Delta_{X/Y} \circ p_1) = \mathrm{id}_X \circ p_1 = p_1$ and $p_2 \circ (\Delta_{X/Y} \circ p_1) = \mathrm{id}_X \circ p_1 = p_1 = p_2$. By the uniqueness part of the universal property, any morphism satisfying these conditions must be the identity on $X \times_Y X$. Thus, $\Delta_{X/Y} \circ p_1 = \mathrm{id}_{X \times_Y X}$.

Since we already have $p_1 \circ \Delta_{X/Y} = \mathrm{id}_X$, we conclude that $\Delta_{X/Y}$ is an isomorphism.

($\Leftarrow$) Suppose $\Delta_{X/Y}$ is an isomorphism. Let $g_1, g_2 : T \to X$ be two morphisms such that $f \circ g_1 = f \circ g_2$. By the universal property of the fiber product, this induces a unique morphism $h : T \to X \times_Y X$ such that $p_1 \circ h = g_1$ and $p_2 \circ h = g_2$.

Since $\Delta_{X/Y}$ is an isomorphism, $p_1 = p_2$ (because $p_1 = \mathrm{id}_X \circ p_1 = (p_2 \circ \Delta_{X/Y}) \circ p_1 = p_2 \circ (\Delta_{X/Y} \circ p_1) = p_2 \circ \mathrm{id}_{X \times_Y X} = p_2$).

Therefore, $g_1 = p_1 \circ h = p_2 \circ h = g_2$. This shows that f is a monomorphism. □

Definition 5.16. We say that a morphism $f : X \to Y$ has a certain property if its diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ has a corresponding property, as listed below:

Property of f		**Property of $\Delta_{X/Y}$**
unramified	$\iff$	open immersion
separated	$\iff$	closed immersion
semi-separated	$\iff$	affine
quasi-separated	$\iff$	quasi-compact
radicial	$\iff$	surjective

Fact: We have the following chain of implications for properties of a morphism:

$$\begin{tikzcd}
\text{monomorphism} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{unramified} & & \\
\text{radicial} \arrow[r, Rightarrow] & \text{separated} \arrow[r, Rightarrow] & \text{semi-separated} \arrow[r, Rightarrow] & \text{quasi-separated}
\end{tikzcd}$$

1. Immersions are monomorphisms.

2. Let $\pi : \mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k$ be the morphism given by $a \mapsto a^2$, where $\operatorname{char} k \neq 2$. Then π is unramified.

3. Let $X = \operatorname{Spec} \mathbb{F}_p[x]$, where p is a prime number. Let $Fr : X \to X$ be the absolute Frobenius morphism induced by the ring homomorphism $\mathbb{F}_p[x] \to \mathbb{F}_p[x]$ given by $f \mapsto f^p$. Then Fr is a radicial morphism.

   Remark: For any scheme X over $\operatorname{Spec} \mathbb{F}_p$, we can define the absolute Frobenius morphism $Fr : X \to X$ by gluing the local Frobenius morphisms on each open affine subscheme. This morphism Fr is always radicial.

4. As we have seen, affine morphisms are separated.

5. Let k be a field. Consider the scheme X constructed by taking two copies of the affine n-space, $U_1 = \mathbb{A}^n_k$ and $U_2 = \mathbb{A}^n_k$, and gluing them along the open subset $V = \mathbb{A}^n_k \setminus \{0\}$ via the identity map. There is a natural morphism $X \to \operatorname{Spec} k$.

   – Then for $n = 1$, X is semi-separated but not separated. (This is the line with two origins).

   – For $n > 1$, X is quasi-separated but not semi-separated.

   – For $n = \infty$, X is not quasi-separated.

We now state a lemma:

Lemma 5.17. Suppose L/K is a field extension. Then $\operatorname{Spec}(L \otimes_K K')$ is a single point for any K'/K field extension if and only if the field extension L/K is purely inseparable.

Theorem 5.18. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

1. f is universally injective.

   i.e., for every morphism $g : Y' \to Y$, the base change morphism $f' : X \times_Y Y' \to Y'$ is injective as a map between the underlying topological spaces.

2. For any field K, and any morphism $\operatorname{Spec} K \to Y$, the fiber product $X \times_Y \operatorname{Spec} K$ is either empty or a point.

3. f is radicial.

4. f is injective, and for every $x \in X$, the field extension $k(x)/k(f(x))$ is purely inseparable.

   i.e. char $k(x) = p$ and for all $a \in k(x)$, there exists some n such that $a^{p^n} \in k(f(x))$.

Proof. (1) $\implies$ (2): This is tautological.

(2) $\implies$ (1): For any $Y' \to Y$ and any $y \in Y'$, $(X \times_Y Y')_y = (X \times_Y Y') \times_{Y'} \operatorname{Spec} k(y) = X \times_Y (Y' \times_{Y'} \operatorname{Spec} k(y)) = X \times_Y \operatorname{Spec} k(y)$. So the fiber of $f' : X \times_Y Y' \to Y'$ at any point is either empty or a point. So f' is injective.

(1) $\implies$ (3): All discussed set-theoretically: From f universally injective, $p_2 : X \times_Y X \to X$ is injective. We know that $p_2 \circ \Delta_{X/Y} = \mathrm{id}_X$, so p_2 is surjective. So p_2 is bijective, then $\Delta_{X/Y}$ is bijective. In particular, it is surjective.

(3) $\implies$ (4): We need the following **fact:**

- **The Stacks Project Lemma 26.17.5.** Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Points z of $X \times_S Y$ are in bijective correspondence to quadruples
  $$(x, y, s, \mathfrak{p})$$
  where $x \in X$, $y \in Y$, $s \in S$ are points with $f(x) = s$, $g(y) = s$ and $\mathfrak{p}$ is a prime ideal of the ring $k(x) \otimes_{k(s)} k(y)$. The residue field of z corresponds to the residue field of the prime $\mathfrak{p}$.

If f is radical, i.e. $\Delta_{X/Y}$ is surjective. Similar argument in (1) $\Longrightarrow$ (3): $p_1 = p_2 : X \times_Y X \to X$ is bijective. By the fact, set-theoretically the bijection must be of the form:

$$(x, x, f(x), \mathfrak{p}_x) \in X \times_Y X \leftrightarrow x \in X$$

Suppose there exists some $y \in Y$ such that $f(x_1) = f(x_2)$, there must be some prime ideal $\mathfrak{p}$ of $k(x_1) \otimes_{k(y)} k(x_2)$. (**Remark**: We should check that $k(x_1) \otimes_{k(y)} k(x_2) \neq 0$.) Then by the fact again,

$$(x_1, x_2, y, \mathfrak{p}) \in X \times_Y X \leftrightarrow x_1 \in X$$

So $x_1 = x_2$. So f is injective.

$\mathfrak{p}_x$ must be the only prime ideal of $k(x) \otimes_{k(f(x))} k(x)$. We **claim** that this implies the field extension $k(x)/k(f(x))$ is purely inseparable. Suppose $k(x)/k(f(x))$ is not purely separable,

- If $k(x)/k(f(x))$ is transcendental, $|\mathrm{Spec}\ k(x) \otimes_{k(f(x))} k(x)| > 1$.
- If $k(x)/k(f(x))$ is algebraic, we can choose α algebraic separable over $k(f(x))$, then $k(f(x))(\alpha) \subseteq k(x)$ is a separable extension.

  α has a minimal polynomial $P(T)$ over $k(f(x))$, $k(f(x))(\alpha) \simeq k(f(x))[T]/(P(T))$. So:

  $$k(f(x))(\alpha) \otimes_{k(f(x))} k(x) \simeq (k(f(x))[T]/(P(T))) \otimes_{k(f(x))} k(x) \simeq k(x)[T]/(P(T))$$

  Because $P(T)$ is separable, it has at least 2 different irreducible factors over $k(x)$ and one of them has root α. So $k(x)[T]/(P(T))$ has at least two prime ideals.

  Consider $k(x) \otimes_{k(f(x))} k(x) \leftarrow k(f(x))(\alpha) \otimes_{k(f(x))} k(x)$, it is integral extension (Reason: tensor product keeps integrality. $k(x)$ is free $k(f(x))$-module hence flat, so tensor product preserves inclusions). so it is surjective on spectra. So $k(x) \otimes_{k(f(x))} k(x)$ has at least two prime ideals.

So the claim is proved, i.e. $k(x)/k(f(x))$ is purely inseparable.

(4) $\implies$ (2): Still use the **fact** above. For any field K and any morphism $\iota : \mathrm{Spec}\, K \to Y$, the points of $X \times_Y \mathrm{Spec}\, K$ correspond to triples

$$(x, \mathrm{Spec}\, K, y_0, \mathfrak{p})$$

where $x \in X$, $y_0 \in Y$ with $f(x) = \iota(\mathrm{Spec}\ K) = y_0$ and $\mathfrak{p}$ is a prime ideal of the ring $k(x) \otimes_{k(y_0)} K$. Because f is injective, there is at most one such x for each y_0. If there exists such x, since $k(x)/k(y_0)$ is purely inseparable, by the **Lemma 5.17**, $k(x) \otimes_{k(y_0)} K$ has only one prime ideal. So $X \times_Y \mathrm{Spec}\, K$ is either empty or a point. $\square$

5.4 Permanence of properties of morphisms

Definition 5.19. Let P be a property of morphisms. We say that P satisfies:

(LOC) **(Local on the source)** Given $f : X \to Y$ and an open covering $X = \bigcup_i U_i$, if $f|_{U_i}$ has property P for all i, then f has property P.

(LOCT) **(Local on the target)** Given $f : X \to Y$ and an open covering $Y = \bigcup_i V_i$, let $U_i = f^{-1}(V_i)$. If $f|_{U_i} : U_i \to V_i$ has property P for all i, then f has property P.

(BC) **(Stable under base change)** If $f : X \to Y$ has property P, then for any morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ has property P.

(COMP) **(Stable under composition)** If $f : X \to Y$ and $g : Y \to Z$ have property P, then the composition $g \circ f : X \to Z$ has property P.

(CANC) **(Cancellation)** If $g \circ f$ has property P and g has property P, then f has property P.

Proposition 5.20. All the properties in the following diagram satisfy (LOCT), (BC), (COMP).

$$
\begin{tikzcd}
 & \text{closed immersion} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \text{finite} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{affine} \arrow[d, Rightarrow] \\
 & & \text{finite type} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{quasi-compact} \\
\text{open immersion} \arrow[r, Rightarrow] & \text{immersion} \arrow[r, Rightarrow] & \text{locally finite type} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{monomorphism} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{unramified} & & \\
\text{radicial} \arrow[r, Rightarrow] & \text{separated} \arrow[r, Rightarrow] & \text{semi-separated} \arrow[r, Rightarrow] & \text{quasi-separated}
\end{tikzcd}
$$

Proposition 5.21. If a property of morphisms P satisfies stability under base change (BC) and composition (COMP), and if all immersions have property P, then P also satisfies the cancellation property (CANC).

Proof. Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that the composition $g \circ f$ has property P, and g also has property P. We want to show that f has P.

The morphism f can be factored through the fiber product $X \times_Z Y$ as follows:

$$X \xrightarrow{(\mathrm{id}, f)} X \times_Z Y \xrightarrow{p_Y} Y$$

We **claim** that $(\mathrm{id}, f) : X \to X \times_Z Y$ and $p_Y : X \times_Z Y \to Y$ both have property P. If we prove this claim, then since $f = p_Y \circ (\mathrm{id}, f)$ and P is stable under composition (COMP), it follows that f has property P.

Well-known that we have diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times_Z Y \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

Since $Y \to Y \times_Z Y$ is the diagonal morphism $\Delta_{Y/Z}$ hence an immersion, it has property P by hypothesis. $X \to X \times_Z Y$ as the base change of an immersion, so has property P.

From the diagram:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

The projection $p_Y : X \times_Z Y \to Y$ is the base change of $g \circ f : X \to Z$. Since $g \circ f$ has property P and P is stable under base change, it follows that p_Y has property P. $\square$

Corollary 5.22. All the properties in the following diagram satisfy (CANC).

$$\text{immersion} \Longrightarrow \text{locally finite type}$$

$$\begin{tikzcd} \text{monomorphism} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{unramified} & & \\ \text{radicial} \arrow[r, Rightarrow] & \text{separated} \arrow[r, Rightarrow] & \text{semi-separated} \arrow[r, Rightarrow] & \text{quasi-separated} \end{tikzcd}$$

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Lecture 9 Separated, Universally Closed, Proper Morphism

6.1 Separated Morphism

Definition 6.1. A morphism $f : X \to Y$ is called **separated** if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion. A scheme X is separated if $X \to \operatorname{Spec} \mathbb{Z}$ is separated.

Example 6.2.

(1) Affine schemes are separated.

(2) (Non-example) $\mathbb{A}^n_k \cup_{\mathbb{A}^n_k \setminus \{0\}} \mathbb{A}^n_k$ is not separated.

Remark: The map $X \to \operatorname{Spec} \mathbb{Z}$ (quick way) comes from the natural isomorphism

$$\operatorname{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} \mathbb{Z}) \cong \operatorname{Hom}_{\mathrm{Ring}}(\mathbb{Z}, \mathcal{O}_X(X)) \quad (1 \mapsto 1)$$

Definition 6.3. Let S be a scheme. A **scheme over** S is a pair (X, S) where X is a scheme equipped with a morphism $f : X \to S$. Notation: The category of schemes over S is denoted by Sch_S. Morphisms: $X \xrightarrow{g} Y$ over S are morphisms making the diagram commute:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

Similar to **Proposition 5.21**, we have the following lemma:

Lemma 6.4. Suppose a property P of morphisms has (BC), (COMP) and closed immersion has P. Then P has (CANC) if the second morphism is separated. i.e. $X \xrightarrow{f} Y \xrightarrow{g} Z$ s.t. $g \circ f$ has P, g separated $\implies$ f has P.

Recall that separatedness has (BC), (COMP). Then we will immediately get:

Lemma 6.5. Suppose S is a separated scheme, then a scheme X over S is separated if and only if the structure morphism $X \to S$ is separated.

Remark: Separated schemes are the analogue of Hausdorff spaces in topological spaces. Namely, a topological space T is Hausdorff if and only if the diagonal $\Delta \subseteq T \times T$ is closed under the product topology.

Proposition 6.6 (Criterion of separatedness). A scheme X is separated if and only if there exists an open affine covering $X = \bigcup X_i$ such that for all i, j, the intersection $X_i \cap X_j =: X_{ij}$ is affine and the image of $\mathcal{O}_X(X_i)$ and $\mathcal{O}_X(X_j)$ in $\mathcal{O}_X(X_{ij})$ generates $\mathcal{O}_X(X_{ij})$ as a ring.

Proof. ($\Leftarrow$) Notice that $X \times X := X \times_{\operatorname{Spec}\mathbb{Z}} X = \bigcup_{i,j} X_i \times X_j$. And $\Delta_X^{-1}(X_i \times X_j) = X_i \cap X_j = X_{ij}$. The restriction $\Delta|_{X_{ij}} : X_{ij} \to X_i \times X_j$ is a morphism of affine schemes. The corresponding ring homomorphism is

$$\mathcal{O}_{X\times X}(X_i \times X_j) = \mathcal{O}_X(X_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(X_j) \longrightarrow \mathcal{O}_X(X_{ij})$$

By assumption, the image generates $\mathcal{O}_X(X_{ij})$ as a ring, which means the map is surjective. This implies that $\Delta|_{X_{ij}}$ is a closed immersion. Since being a closed immersion is local on the target, Δ_X is a closed immersion.

($\Rightarrow$) If Δ_X is a closed immersion. Take *any* open affine covering $X = \bigcup X_i$. The restriction of Δ_X to $X_{ij} \longrightarrow X_i \times X_j$ is a closed immersion. This implies X_{ij} is affine (as a closed subscheme of the affine scheme $X_i \times X_j$) and the map $\mathcal{O}_X(X_i) \otimes \mathcal{O}_X(X_j) \to \mathcal{O}_X(X_{ij})$ is surjective. $\square$

Example 6.7. Let R be a ring. Consider $\mathbb{P}_R^n = \operatorname{Proj} R[x_0, \ldots, x_n]$. Then $\mathbb{P}_R^n = \bigcup_{i=0}^n U_i$, where $U_i = D_+(x_i) \cong \operatorname{Spec} R[y_1, \ldots, y_n]$. The intersection $U_i \cap U_j \cong \mathbb{A}_R^{n-1} \times_{\operatorname{Spec} R} (\mathbb{A}_R^1 \setminus \{0\})$ is affine.

$$U_i \cap U_j \cong \operatorname{Spec} R[y_1, \ldots, y_j, y_j^{-1}, \ldots, \hat{y}_i, \ldots, y_n].$$

The map $\mathcal{O}_{\mathbb{P}_R^n}(U_i) \otimes_R \mathcal{O}_{\mathbb{P}_R^n}(U_j) \longrightarrow \mathcal{O}_{\mathbb{P}_R^n}(U_i \cap U_j)$ is surjective. So $\mathbb{P}_R^n$ is separated.

6.2 Universally Closed Morphism

Definition 6.8. A morphism $f : X \to Y$ is **closed** if for any closed subset $Z \subseteq X$, the image $f(Z)$ is closed in Y. A morphism $f : X \to Y$ is **universally closed** if for any morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ is closed.

Example 6.9. Finite morphisms are universally closed.

Proposition 6.10. Universally closed morphisms have properties (BC), (LOCT), (COMP). They satisfy (CANC) if the second morphism is separated.

Proposition 6.11. The morphism $\pi : \mathbb{P}^n_R \to \operatorname{Spec} R$ is universally closed.

Proof. Without loss of generality, we show that π is closed. (Detail: Using "local on the target" and the fact: $\mathbb{P}^n_R \times_{\operatorname{Spec} R} \operatorname{Spec} R' \cong \mathbb{P}^n_{R'}$). Let $Z \subseteq \mathbb{P}^n_R$ be a closed subset ($Z = V_+(I)$). Our goal is to prove: $\operatorname{Spec} R \setminus \pi(Z)$ is open.

Let $p \in \operatorname{Spec} R \setminus \pi(Z)$, corresponding to the prime ideal $\mathfrak{p} \subseteq R$. $S := R[x_0, \ldots, x_n]$. Consider the fiber product over p:

$$\begin{tikzcd}
\mathbb{P}^n_R \times_{\operatorname{Spec} R} p \cong \mathbb{P}^n_{\kappa(p)} \arrow[r] \arrow[d] & p \cong \operatorname{Spec} \kappa(p) \arrow[d] \\
\mathbb{P}^n_R \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

Let $\bar{I}$ be the image of I in $\kappa(p)[x_0, \ldots, x_n] =: \bar{S}$. From $Z \cap \mathbb{P}^n_{\kappa(p)} = \emptyset$, we know $V_+(\bar{I}) = \emptyset$. It is well known that this implies:

$$\bar{I} \supseteq \bigoplus_{d \geq N} \bar{S}_d \quad \text{for some } N \in \mathbb{N}^*$$

In particular:

$$0 = \bar{S}_N/\bar{I}_N = (S_N/I_N) \otimes_R \kappa(p) = (S_N/I_N) \otimes_R R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \cong (S_N/I_N)_{\mathfrak{p}}/\mathfrak{p}(S_N/I_N)_{\mathfrak{p}}$$

S_N/I_N is obviously finitely generated R-module, so $(S_N/I_N)_{\mathfrak{p}}$ is finitely generated $R_{\mathfrak{p}}$-module. By Nakayama's Lemma:

$$(S_N/I_N)_{\mathfrak{p}} = 0$$

Because "finite generated", it is easily to find $g \in R \setminus \mathfrak{p}$ such that $(S_N/I_N)_g = 0$. So for any $q := \mathfrak{q} \in D(g)$, we have $(S_N/I_N)_\mathfrak{q} = 0$. Consider the fiber product over q:

$$\begin{tikzcd} \mathbb{P}^n_R \times_{\operatorname{Spec} R} q \cong \mathbb{P}^n_{\kappa(q)} \arrow[r, "\pi|_{\pi^{-1}(q)}"] \arrow[d, "\mathrm{pr}"] & q \cong \operatorname{Spec} \kappa(q) \arrow[d] \\ \mathbb{P}^n_R \arrow[r, "\pi"] & \operatorname{Spec} R \end{tikzcd}$$

$\mathrm{pr}^{-1}(\mathbb{P}^n_R) = \pi^{-1}(q)$. And $\mathrm{pr}^{-1}(Z) = V_+(\widetilde{I})$, where $\widetilde{I}$ is the image of I in $\kappa(q)[x_0, \ldots, x_n]$. $\widetilde{I}$ contains the N-degree part of $\kappa(q)[x_0, \ldots, x_n]$, so $\mathrm{pr}^{-1}(Z) = V_+(\widetilde{I}) = \emptyset$, which means $q \notin \pi(Z)$. So $\pi(Z) \cap D(g) = \emptyset$, where $D(g)$ is a neighborhood of p. □

Theorem 6.12. Universally closed morphisms are quasi-compact.

Proof. Quasi-compactness is (LOCT). So without loss of generality, we can assume $X \xrightarrow{f} Y \cong \operatorname{Spec} A$.

Suppose X is not quasi-compact. Then there exists $y \in Y$ such that for any open affine neighborhood U of y in Y, $f^{-1}(U)$ is not quasi-compact. (Using affine scheme is quasi-compact.)

Take an open affine covering $X = \bigcup_{i \in I} X_i$. Consider the base change $T \to Y$, where $T = \operatorname{Spec} A[t_i \mid i \in I]$ and $Y = \operatorname{Spec} A$. Consider the diagram:

$$\begin{tikzcd} X \times_Y T \arrow[r] \arrow[d, "f_T"] & X \arrow[d, "f"] \\ T \arrow[r] & Y \end{tikzcd}$$

Then f_T is closed.

Take $Z = X \times_Y T \setminus \bigcup_{i \in I}(X_i \times_Y T_i)$. Here $T_i = D(t_i)$. Then $f_T(Z)$ is closed in T.

Consider $T_y \cong \operatorname{Spec} \kappa(y)[t_i \mid i \in I]$. Take $\tau = \{t_i = 1, \forall i\} \in T_y$. Then $f_T^{-1}(\tau) \cong X_y$. Need details. Moreover:

$$(X_i \times_Y T_i) \cap f_T^{-1}(\tau) \cong X_i \cap X_y \quad \text{Need details.}$$

So $\bigcup_i (X_i \times_Y T_i)$ contains $f_T^{-1}(\tau)$, then $\tau \notin f_T(Z)$.

Since $f_T(Z)$ is closed, there exists $g \in A[t_i \mid i \in I]$ such that $\tau \in D(g)$ and $f_T(Z) \cap D(g) = \emptyset$. There exists a finite subset $J \subseteq I$ such that $g \in A[t_j \mid j \in J]$.

Consider a closed subscheme $S \subseteq T$ given by

$$S = \{t_j = 1, \forall j \in J; \quad t_i = 0, \forall i \notin J\}$$

Then we have $S \cong Y$ and $D(g) \cap S = D(\bar{g})$, where $\bar{g} = g|_{t_j = 1 \forall j \in J, t_i = 0 \forall i \notin J} = g|_{t_i = 1 \forall i \in I} \in A$,

$D(g) \subseteq T$, $D(\bar{g}) \subseteq S \simeq Y$, $y \in D(\bar{g})$ Need details.

So $f^{-1}(D(\bar{g}))$ can be covered by $\bigcup_{j\in J} X_j$. Need details, which contradicts to "$f^{-1}(U)$ is not quasi-compact." □

6.3 Valuation Criterion

Definition 6.13. Let A be a valuation ring. (A valuation ring A is an integral domain, $\operatorname{Frac} A = K$, s.t. $\forall x \in K^*$, either $x \in A$ or $x^{-1} \in A$). Equivalently, a **valuation ring** is a local integral domain $(A, \mathfrak{m})$ with fraction field $K = \operatorname{Frac} A$, such that $(A, \mathfrak{m})$ is the maximal element in the dominance partial order $\leq$ on the set of local subrings of K. The order is defined by $(R_1, \mathfrak{m}_1) \leq (R_2, \mathfrak{m}_2)$ if $R_1 \subseteq R_2$ and $\mathfrak{m}_2 \cap R_1 = \mathfrak{m}_1$.

Consider a morphism $f : X \to Y$. Suppose we have a commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "u"] \arrow[d, "j"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r, "v"'] \arrow[ur, dashed, "\tilde{v}"] & Y
\end{tikzcd}$$

where A is a valuation ring with field of fractions K, and j is the morphism induced by the inclusion $A \hookrightarrow K$.

Definition 6.14. A **lift** of v is a morphism $\tilde{v} : \operatorname{Spec} A \to X$ such that $u = \tilde{v} \circ j$ and $v = f \circ \tilde{v}$.

Definition 6.15. We say $f : X \to Y$ satisfies the **existence** (resp. **uniqueness**) **valuation criterion** if for every diagram as above, there exists at least one (resp. at most one) lift of v. We say $f : X \to Y$ satisfies the **valuation criterion** if there exists a unique lift for any diagram as above.

Theorem 6.16. Let $f : X \to Y$ be a morphism. Then:

1. f is separated $\iff$ f is quasi-separated and satisfies the uniqueness valuation criterion.

2. f is universally closed $\iff$ f is quasi-compact and satisfies the existence valuation criterion.

Theorem 6.17. Let Y be a locally Noetherian scheme and let $f : X \to Y$ be a morphism of finite type. Then

$$f \text{ is separated} \iff f \text{ satisfies the uniqueness Valuation Criterion for } \mathbf{DVR}!$$

Recall: DVR = Noetherian Valuation Rings.

Remark: We only prove $\Longrightarrow$ of **Theorem 6.16** and $\Longleftarrow$ of **Theorem 6.17**.

Proof of $\Rightarrow$ in Theorem 6.16(1). If $f : X \to Y$ is separated. Suppose we have the diagram:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "v"'] \arrow[ur, shift left, "\tilde{v}_1"] \arrow[ur, shift right, "\tilde{v}_2"'] & Y \end{tikzcd}$$

Then consider the map $(\tilde{v}_1, \tilde{v}_2) : \operatorname{Spec} A \longrightarrow X \times_Y X$. Let $\Delta(X)$ be the image of the diagonal morphism. Since X is separated, $\Delta(X)$ is closed in $X \times_Y X$. The preimage $(\tilde{v}_1, \tilde{v}_2)^{-1}(\Delta(X))$ is closed in $\operatorname{Spec} A$. It contains the generic point of $\operatorname{Spec} A$ (which corresponds to $\operatorname{Spec} K$ where the maps agree). Since the closure of the generic point in $\operatorname{Spec} A$ is the whole space, we have $(\tilde{v}_1, \tilde{v}_2)^{-1}(\Delta(X)) = \operatorname{Spec} A$. Hence $\tilde{v}_1 = \tilde{v}_2$. $\square$

Proof of $\Rightarrow$ in Theorem 6.16(2). We have already shown universally closed implies quasi-compact. Suppose we have the diagram:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "v"] & Y \end{tikzcd}$$

First take the base change $X_R = X \times_Y \operatorname{Spec} R$. We will have:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[dr, "s"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "j"'] & & \\ & X_R \arrow[r] \arrow[d, "f_R"] & X \arrow[d, "f"] \\ & \operatorname{Spec} R \arrow[r, "v"] & Y \end{tikzcd}$$

Then to find a lift $\tilde{v}$ is equivalent to finding a section $\tilde{s} : \operatorname{Spec} R \to X_R$ such that $f_R \circ \tilde{s} = id$.

Denote $x = s(\operatorname{Spec} K) \in X_R$. Then $\kappa(x) \cong K$. Let $Z = \overline{\{x\}}$ be a closed subset of X_R. Then $f_R(Z)$ is closed in $\operatorname{Spec} R$ and contains the generic point $\operatorname{Spec} K$. So $f_R(Z) = \operatorname{Spec} R$.

In particular, there exists $x' \in Z$ such that $f_R(x') = \operatorname{Spec} R/\mathfrak{m}$ (the closed point). Let $\tilde{R} := \mathcal{O}_{Z_{red},x'} \subseteq K$ is local with maximal ideal $\tilde{\mathfrak{m}}$.

Moreover, $\tilde{R} \supseteq R$ and $\tilde{\mathfrak{m}} \cap R = \mathfrak{m}$. Need details. Then $(\tilde{R}, \tilde{\mathfrak{m}}) \geq (R, \mathfrak{m})$. By the maximality of valuation rings, $\tilde{R} = R$. Then $\operatorname{Spec} R \to \operatorname{Spec} \tilde{R} \to Z \to X_R$ gives the desired section $\tilde{s}$. Need details. $\square$

Proof of $\Leftarrow$ in Theorem 6.17. We will need the following lemma:

Lemma 6.18. Let S be a Noetherian scheme. Let $x \rightsquigarrow y$ be a specialization (i.e. $y \in \overline{\{x\}}, y \neq x$). Then there exists a DVR A such that $\operatorname{Frac} A = k(x)$, together with a morphism

$$v : \operatorname{Spec} A \to S$$

such that $v(\operatorname{Spec} K) = x$ and $v(\operatorname{Spec} A/\mathfrak{m}) = y$.

Because of (LOCT), we may assume Y is Noetherian.

Since $\Delta_{X/Y}$ is an immersion, $\Delta(X)$ is a locally closed subset in $X \times_Y X$, in particular constructible. By **Hartshorne Chapter II Exercise 3.18(c)**, to prove $\Delta(X)$ is closed, it suffices to show that $\Delta(X)$ is stable under specialization, i.e., $\forall z \in \Delta(X)$, if $z \rightsquigarrow z'$, then $z' \in \Delta(X)$.

Using the lemma, we have a morphism $v : \operatorname{Spec} A \to X \times_Y X$ such that $v(\operatorname{Spec} K) = z$ and $v(\operatorname{Spec} A/\mathfrak{m}) = z'$.

Since f satisfies the uniqueness valuation criterion, we have $z' \in \Delta(X)$. $\square$

Proof of Lemma (sketch). Consider $\mathcal{O}_{\overline{\{x\}},y}$ which is a Noetherian local domain with $\operatorname{Frac}(\mathcal{O}_{\overline{\{x\}},y}) = \kappa(x)$. Take $X = \operatorname{Bl}_{\operatorname{Spec} \mathcal{O}_{\overline{\{x\}},y}}(y) = \operatorname{Proj} \bigoplus_{d \geq 0} \mathfrak{m}_y^d \xrightarrow{\pi} \operatorname{Spec} \mathcal{O}_{\overline{\{x\}},y}$.

Fact: $\pi^{-1}(y)$ is a divisor in X (i.e. codimension $= 1$).

Let y' be a generic point in $\pi^{-1}(y)$ and take $X' = \operatorname{Spec} \mathcal{O}_{X,y'}$. Then $\dim X' = 1$. $\mathcal{O}_{\overline{\{x\}},y} \subseteq \mathcal{O}_{X,y'}$ and $\pi(y') = y$. $\operatorname{Frac} \mathcal{O}_{X,y'} = \kappa(x)$.

Then take $A' =$ integral closure of $\mathcal{O}_{X,y'}$ in $k(x)$. By Krull-Akizuki Theorem, A' is Noetherian and $\dim A' = 1$, regular (since it is integrally closed of dimension 1). Now take a maximal ideal $\mathfrak{m}$ of A', and set $A = A'_{\mathfrak{m}}$. Then $\dim A = 1$. A is Noetherian and regular. So A is a DVR. This A has the expected properties ($v(\operatorname{Spec} K) = x$ and $v(\operatorname{Spec} A/\mathfrak{m}) = y$). $\square$

Theorem 6.19 (Krull-Akizuki). Let R be a Noetherian integral domain with $\dim R = 1$. Let $\overline{R}$ be the integral closure of R in $\mathrm{Frac}(R)$. Then $\overline{R}$ is Noetherian and $\dim \overline{R} = 1$. (See Wikipedia for proof).

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7.1 Proper Morphisms

Definition 7.1. A morphism $f : X \to Y$ is called **proper** if f is separated, of finite type, and universally closed.

Example 7.2. Finite morphisms are proper.

Remark: Finite $\iff$ proper + affine.

Proposition 7.3. Proper morphisms satisfy (LOCT), (BC), (COMP). They satisfy (CANC) if the second morphism is separated.

Definition 7.4. For a ring R, define $\mathbb{P}^n_R := \operatorname{Proj} R[x_0, \dots, x_n]$. More generally, given a scheme S, let $S = \bigcup_{i \in I} S_i$ be an open covering with $S_i \cong \operatorname{Spec} R_i$. We can glue $\mathbb{P}^n_{R_i}$ to form a scheme $\mathbb{P}^n_S$ over S.

Remark: $\mathbb{P}^n_S \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S$.

Proposition 7.5. The structure morphism $p : \mathbb{P}^n_S \to S$ is proper.

Proof. Last time we have proved p is separated and universally closed. p is obviously (by looking at $\mathbb{P}^n_S = \bigcup_{i=0}^n U_i$, where U_i is glued by Spec $R_j[x_0/x_i, \dots, x_n/x_i]$) of finite type. □

Theorem 7.6. A morphism f is proper $\iff$ f is quasi-separated, of finite type, and satisfies the Valuation Criterion.
Moreover, if f is a morphism of finite type between locally Noetherian schemes, then

$$f \text{ is proper} \iff f \text{ satisfies the Valuation Criterion for DVRs.}$$

***Proof*.** **Remark:** We only prove the second statement.

"$\Rightarrow$": Follows from last time.

"$\Leftarrow$": Because of (LOCT), we may assume Y is Noetherian. By **Theorem 6.17**, f is separated. It suffices to show f is universally closed.

Step 1. Claim: For any Noetherian scheme Y' together with $Y' \to Y$, the base change $X' := X \times_Y Y' \xrightarrow{f'} Y'$ is a closed morphism.

Proof of claim. Let $Z \subseteq X'$ be a closed subset. $f'(Z)$ is a constructible subset of Y'. To show $f'(Z)$ is closed, by **Hartshorne II Ex 3.18(c)**, we need to show that $f'(Z)$ is stable under specialization.

Take arbitrary $\eta \in f'(Z)$ and let $\eta \rightsquigarrow y$ be a specialization. From **Lemma 6.18**, let A be a DVR together with $\operatorname{Spec} A \xrightarrow{v} Y'$ such that $v(\operatorname{Spec} K) = \eta$ and $v(\operatorname{Spec} A/\mathfrak{m}) = y$.

Consider $Z_\eta := Z \times_{Y'} \eta$. It is a finite type scheme over $\operatorname{Spec} \kappa(\eta)$. Then we pick a closed point $\tilde{\eta} \in Z_\eta$. (**Remark**(Reason of existence): Let $W \to \operatorname{Spec} k$ be a finite type morphism. $W = \bigcup W_i$, $W_i \cong \operatorname{Spec} A_i$, $A_i \in \mathrm{Alg}_k^{\text{fin. gen.}}$. Then there is a bijection:

$$\{\text{closed points in } W\} \leftrightarrow \{\text{maximal ideals } \mathfrak{m} \text{ of } A_i \text{ for some } i\}$$

Moreover for any closed point $x \in W$, $\kappa(x)$ is a finite extension of k.)

So $\kappa(\tilde{\eta})$ is a finite extension of $\kappa(\eta) = K$. Take the normalization $\widetilde{A}$ of A in $\kappa(\tilde{\eta})$. By **Krull-Akizuki Theorem**, $\widetilde{A}$ is a regular Noetherian ring of dimension 1. Let $\widetilde{A}_\mathfrak{m}$ be the localization of $\widetilde{A}$ at a maximal ideal $\mathfrak{m}$. Then we have

$$\begin{tikzcd} \operatorname{Spec} \kappa(\tilde{\eta}) \arrow[r, "u"] \arrow[d] & Z \arrow[d] \\ \operatorname{Spec} \widetilde{A}_\mathfrak{m} \arrow[r, "\tilde{v}"'] \arrow[ur, dashed, "\exists!"] & Y' \end{tikzcd}$$

where the map $\operatorname{Spec} \widetilde{A}_\mathfrak{m} \to Y'$ factors through $\operatorname{Spec} A \xrightarrow{v} Y'$. We have $\tilde{v}(\operatorname{Spec} \operatorname{Frac} \widetilde{A}_\mathfrak{m}) = \eta$ and $\tilde{v}(\operatorname{Spec} \widetilde{A}_\mathfrak{m}/\mathfrak{m}) = y$.

We apply the valuation criterion to get the unique lift $\operatorname{Spec} \widetilde{A}_\mathfrak{m} \to Z \subseteq X'$. This shows $y \in f'(Z)$.

Step 2. We will use the following theorem. The proof of Chow's Lemma is omitted here, which is similar to the proof of **Hartshorne II Ex 4.10**.

Theorem 7.7 (Chow's Lemma)**.** Let $X \to S$ be a finite type separated morphism between Noetherian schemes. Then there exists a scheme $\tilde{X}$ together with a morphism $g : \tilde{X} \to X$ such that:

1. g is proper and surjective.
2. There exists an open dense subset $U \subseteq X$ such that $g|_{g^{-1}(U)} : g^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism.
3. There exists an immersion $i : \tilde{X} \to \mathbb{P}^n_S$ for some n such that the following diagram

commutes:

$$\begin{tikzcd} \tilde{X} \arrow[r, "i"] \arrow[d, "g"'] & \mathbb{P}^n_S \arrow[d, "p"] \\ X \arrow[r] & S \end{tikzcd}$$

Applying this to our situation: we find $\tilde{X} \xrightarrow{i,\text{imm.}} \mathbb{P}^n_Y$.

$$\begin{tikzcd} \tilde{X} \arrow[r, hook, "i"] \arrow[d, "g"'] & \mathbb{P}^n_Y \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where g is proper and surjective.

Step 3. By Step 1 and g is proper, $f \circ g$ is a closed morphism after base change to any Noetherian scheme. Because $\mathbb{P}^n_Y \to Y$ is proper, $\mathbb{P}^n_Y \to \mathbb{P}^n_Y \times_Y \mathbb{P}^n_Y$ is closed immersion, its base change $\tilde{X} \to \tilde{X} \times_Y \mathbb{P}^n_Y$ is also closed immersion. Because $\mathbb{P}^n_Y$ is Noetherian by Y is Noetherian, from Step 1, as a base change of $f \circ g$, $\tilde{X} \times_Y \mathbb{P}^n_Y \to \mathbb{P}^n_Y$ is closed morphism. So after composition, i is closed morphism. Since i is an immersion, this implies i is a closed immersion.

Closed immersions are proper. By (COMP), $f \circ g = p \circ i$ is proper. From g is proper surjective and surjectivity is preserved under base change, it is easy to see f is universally closed. $\square$

Lecture 10. Sheaves of Modules

Definition 7.8. Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of $\mathcal{O}_X$-modules** is a sheaf $\mathcal{F}$ on X, such that for every open subset $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for every inclusion $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.
A **morphism of $\mathcal{O}_X$-modules** is a sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ such that for every open $U \subseteq X$, $\mathcal{F}(U) \xrightarrow{\phi(U)} \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module map.

Notation: The category of $\mathcal{O}_X$-modules is denoted by $\mathcal{O}_X$-mod. $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ or $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$.

Remark: Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Then $\ker\varphi, \operatorname{coker}\varphi, \operatorname{im}\varphi$ are $\mathcal{O}_X$-modules. Moreover, $\bigoplus, \prod, \varinjlim, \varprojlim$ of $\mathcal{O}_X$-modules are $\mathcal{O}_X$-modules. So $\mathcal{O}_X$-mod is an abelian subcategory of $\mathrm{Ab}(X)$.

Remark: $\mathcal{O}_X$-mod has arbitrary coproducts, exact direct limits. And has a **generator**

(Grothendieck category). This means there is an object $\mathcal{G}$ such that the functor

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, -) : \mathcal{O}_X\text{-mod} \longrightarrow \mathrm{Set}$$

is fully faithful. In fact we can choose:

$$\mathcal{G} = \bigoplus_{U \subseteq X} (U \hookrightarrow X)_! \mathcal{O}_U$$

Fact: A Grothendieck category has enough injectives, so it has derived functors.

Definition 7.9. $\mathcal{F} \in \mathcal{O}_X$-mod is called **free** if $\mathcal{F}$ is a direct sum of copies of $\mathcal{O}_X$. It is called **locally free** if there exists an open covering $X = \bigcup U_i$ such that for all i, $\mathcal{F}|_{U_i}$ is free. We define $\mathrm{rk}(\mathcal{F}|_{U_i}) := \#$ of copies of $\mathcal{O}_{U_i}$ in the direct sum. When X is connected, the $\mathrm{rk}(\mathcal{F}|_{U_i})$ is constant, and we call it the **rank** of $\mathcal{F}$, denoted $\mathrm{rk}(\mathcal{F})$.
If $\mathcal{F}$ is locally free of rank 1, it is called an **invertible sheaf** or **line bundle**.

Definition 7.10. A **sheaf of ideals** is a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X$, denoted $\mathcal{I}$. i.e. for any open $U \subseteq X$, $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_X(U)$.

Definition 7.11. Let $\mathcal{F}, \mathcal{G} \in \mathcal{O}_X$-mod. Define

- $U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf, called $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. It is an $\mathcal{O}_X$-module.
- $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} := \big(U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)\big)^+$ (sheafification).

Definition 7.12. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

- If $\mathcal{F} \in \mathcal{O}_X$-mod, $f_*\mathcal{F}$ is naturally an $\mathcal{O}_Y$-module.

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \curvearrowleft \mathcal{O}_Y(V) \quad (\text{via } f^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))).$$

- If $\mathcal{G} \in \mathcal{O}_Y$-mod, then $f^{-1}\mathcal{G}$ is naturally an $f^{-1}\mathcal{O}_Y$-module. Define

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

Remark: We have the following adjunctions:

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$
$$f_*\underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

7.2 Sheaves of modules on affine schemes

Definition 7.13. Let A be a ring and $M \in A$-mod. Define a sheaf $\widetilde{M}$ on $X = \operatorname{Spec} A$ by:

$$\widetilde{M}(U) = \left\{ (s(\mathfrak{p}))_{\mathfrak{p}\in U} \in \prod_{\mathfrak{p}\in U} M_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \forall \mathfrak{p} \in U, \exists \text{ open neighborhood } \mathfrak{p} \in V \subseteq U, \exists m \in M, f \in A \\ \text{s.t. } \forall \mathfrak{q} \in V, s(\mathfrak{q}) = \dfrac{m}{f} \in M_{\mathfrak{q}} \end{array} \right\}$$

Proposition 7.14. Let $X = \operatorname{Spec} A$ and $M \in A$-mod.

(a) $\widetilde{M}$ is an $\mathcal{O}_X$-module.

(b) $\forall \mathfrak{p} \in X$, $\widetilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ as $A_{\mathfrak{p}}$-modules.

(c) $\forall f \in A$, $\widetilde{M}(D(f)) \cong M_f$ as A_f-modules.

(d) In particular $\widetilde{M}(X) \cong M$ as A-modules.

(Skip the proof.)

Proposition 7.15. Let $f : Y \to X$ be a morphism of affine schemes corresponding to a ring homomorphism $A \to B$, i.e., $X = \operatorname{Spec} A \xleftarrow{f} \operatorname{Spec} B = Y$. Then:

(a) The functor $M \longmapsto \widetilde{M}$ gives an exact fully faithful functor

$$A\text{-module} \longrightarrow \mathcal{O}_X\text{-module}$$

(b) If $M, N \in \mathrm{Mod}_A$, then $(\widetilde{M \otimes_A N}) \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.

(c) If $\{M_i\}_{i\in I}$ is a family of A-modules, then $(\widetilde{\bigoplus_{i\in I} M_i}) \cong \bigoplus_{i\in I} \widetilde{M_i}$.

(d) If $N \in \mathrm{Mod}_B$, then $f_*\widetilde{N} \cong (\widetilde{{}_A N})$. (Here ${}_A N$ denotes N viewed as an A-module via restriction of scalars).

(e) If $M \in \mathrm{Mod}_A$, then $f^*\widetilde{M} \cong (\widetilde{M \otimes_A B})$.

Remark: In general, $\widetilde{(\prod M_i)} \ncong \prod \widetilde{M_i}$.

Example 7.16. Let $A = \mathbb{Z}$, $M_i = \mathbb{Z}$, $I = \mathbb{N}$.

$$\widetilde{(\prod \mathbb{Z})}(D(f)) = (\prod \mathbb{Z})_f$$
$$(\prod \widetilde{\mathbb{Z}})(D(f)) = \prod (\mathbb{Z}_f)$$

Consider the element $(\frac{1}{f}, \frac{1}{f^2}, \frac{1}{f^3}, \dots)$. It belongs to $\prod(\mathbb{Z}_f)$ but not to $(\prod \mathbb{Z})_f$ (where the denominator power must be uniform).

Definition 7.17. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is called **quasi-coherent** if there exists an open affine covering $X = \bigcup U_i$, $U_i \cong \operatorname{Spec} A_i$, such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some $M_i \in A_i$-mod.
Moreover, we say:

- $\mathcal{F}$ is of **finite type** (finitely generated) if every M_i is a finitely generated A_i-module.
- $\mathcal{F}$ is of **finite presentation** if every M_i is a finitely presented A_i-module.
- If X is locally Noetherian, finite type $\iff$ finite presentation. And we call $\mathcal{F}$ **coherent** if $\mathcal{F}$ is of finite type.

Example 7.18.

1. $\mathcal{O}_X \in \operatorname{QCoh}(\mathcal{O}_X)$. In the case when X is locally Noetherian, $\mathcal{O}_X \in \operatorname{Coh}(\mathcal{O}_X)$.
2. **Non-example:** Let $U \subseteq X$ be an nonempty open subset, $U \neq X$. Let $j : U \hookrightarrow X$ be the inclusion. Then $j_! \mathcal{O}_U \notin \operatorname{QCoh}(\mathcal{O}_X)$.
3. Let $i : X \hookrightarrow Y$ be a closed immersion. Then $i_* \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Locally: $Y = \operatorname{Spec} B \leftarrow \operatorname{Spec} A = X$ corresponds to a surjective map $B \twoheadrightarrow A$. Then $i_* \mathcal{O}_X = \widetilde{({}_B A)}$.
4. Let X be an integral scheme. $\eta =$ generic point of X. Let $K = \kappa(\eta)$. Then the constant sheaf $K_X \in \operatorname{QCoh}(\mathcal{O}_X)$. Locally: $X = \operatorname{Spec} A$. $K_X \cong \widetilde{K}$.

   Remark: K_X is in general not coherent, even when X is locally Noetherian. For example: $X = \operatorname{Spec} \mathbb{Z}$. K_X corresponds to $\mathbb{Q}$. $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.

Proof of non-example. Say $X = \operatorname{Spec} A$. Then $j_! \mathcal{O}_U(X) = 0$. Suppose $j_! \mathcal{O}_U \cong \widetilde{M}$. Then

$M = 0 \implies j_! \mathcal{O}_U \cong 0$. Contradiction! (since $U \neq \emptyset$). $\square$

Proposition 7.19. Let $X = \operatorname{Spec} A$. Then the functor: $M \longmapsto \widetilde{M}$ induces an equivalence of categories

$$A\text{-mod} \longrightarrow \mathrm{QCoh}(\mathcal{O}_X)$$

To prove the proposition we'll need the following:

Lemma 7.20. Let $X = \operatorname{Spec} A$ and $D(f) \subseteq X$. Let $\mathcal{F} \in \mathrm{QCoh}(\mathcal{O}_X)$. Then:

(a) If $s \in \mathcal{F}(X)$ such that $s|_{D(f)} = 0$, then $f^n s = 0$ for some $n > 0$.

(b) If $t \in \mathcal{F}(D(f))$, then $f^n t$ extends to a global section on X for some $n \geq 0$.

Proof of Lemma. Take a covering $X = \bigcup D(g_i)$ with finitely many g_i. Since $\mathcal{F}$ is quasi-coherent, $\mathcal{F}|_{D(g_i)} \cong \widetilde{M_i}$ for some $M_i \in A_{g_i}$-mod.

(a) Suppose $s|_{D(f)} = 0$. Then $s|_{D(fg_i)} = 0$. Since $s|_{D(g_i)} = s_i \in M_i$, then $f^{n_i} s_i = 0$ in M_i (because restriction corresponds to localization). Take $n = \max\{n_i\}$, then $f^n s_i = 0$ for all i. This implies $f^n s = 0$.

(b) Suppose $t \in \mathcal{F}(D(f))$. Then $t|_{D(fg_i)} = \frac{m_i}{f^{n_i}}$ for some $m_i \in M_i$ and $n_i \geq 0$. Take $n = \max\{n_i\}$. Then $f^n t|_{D(fg_i)} = m_i'|_{D(fg_i)}$ where $m_i' \in \mathcal{F}(D(g_i))$ (specifically m_i' corresponds to $f^{n-n_i} m_i$). Consider $m_{ij}' := m_i'|_{D(g_i g_j)} - m_j'|_{D(g_i g_j)}$. This is in general can be $\neq 0$. However,

$$m_{ij}'|_{D(fg_ig_j)} = f^n t|_{D(fg_ig_j)} - f^n t|_{D(fg_ig_j)} = 0.$$

By part (a) (applied to the open set $D(g_ig_j)$), there exists $n_{ij} \geq 0$ such that $f^{n_{ij}} m_{ij}' = 0$. Let $n' = \max\{n_{ij}\}$. Then $f^{n'} m_{ij}' = 0$ for all i, j. This means the sections $f^{n'} m_i'$ glue to a global section on X. Then $f^{n+n'} t$ extends to X. $\square$

Proof of Proposition. Since the functor $M \longmapsto \widetilde{M}$ is fully faithful, it is enough to show that for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{O}_X)$, there exists $M \in A$-mod such that $\mathcal{F} \cong \widetilde{M}$.

Define $M = \mathcal{F}(X)$. Then there is a natural map

$$\alpha : \widetilde{M} \longrightarrow \mathcal{F} \quad \text{in } \mathrm{QCoh}(\mathcal{O}_X).$$

Cover $X = \bigcup_{g_i \in A} D(g_i)$ such that $\mathcal{F}|_{D(g_i)} \cong \widetilde{M_i}$. Apply the lemma to $f = g_i$, and we see that

$$M_{g_i} \cong M_i \quad \text{as } A_{g_i}\text{-modules.}$$

Then it follows that $\alpha|_{D(g_i)}$ is an isomorphism. Hence α is an isomorphism. $\square$

Remark: In the setting of the proposition:

- $\widetilde{M}$ is of finite type $\iff$ M is a finitely generated A-module.
- $\widetilde{M}$ is of finite presentation $\iff$ M is a finitely presented A-module.

Idea: Take $M = \varinjlim_\alpha M_\alpha$ where M_α are finitely generated submodules of M. Then it is easy to see $\widetilde{M} = \varinjlim_\alpha \widetilde{M_\alpha}$. Take $X = \bigcup_i D(g_i)$ an open affine covering such that $\widetilde{M}|_{D(g_i)}$ is a finitely generated A_{g_i}-module. Then $\widetilde{M}(D(g_i)) = M_{g_i}$. There exists α_i such that $M_{g_i} = (M_\alpha)_{g_i}$ for all $\alpha > \alpha_i$. Then for $\alpha > \max_i\{\alpha_i\}$, we have $M = M_\alpha$.

Proposition 7.21. Let X be a scheme. Let $\mathcal{F} \in \mathcal{O}_X$-mod. Then $\mathcal{F} \in \mathrm{QCoh}(\mathcal{O}_X)$ if and only if $\mathcal{F}|_U \cong \widetilde{M}$ for **every** affine open $U \subseteq X$.
Moreover, $\mathcal{F}$ is of finite type (resp. finite presentation) if and only if M in the above is finitely generated (resp. finitely presented) for any affine open U.

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Proposition 8.1 (Cohomology Vanishing). Let X be an affine scheme. Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume that $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Then the induced sequence of global sections

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow 0$$

is a short exact sequence.

Proof. The global section functor $\Gamma(X, -)$ is left exact (see Hartshorne, Exercise II.1.8), so we only need to show that the map $\Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H})$ is surjective.

Let $h \in \Gamma(X, \mathcal{H})$ be a global section. By the surjectivity of the sheaf morphism $\mathcal{G} \longrightarrow \mathcal{H}$, there exists an open covering of X. Since X is affine, we can choose a covering by distinguished open sets $X = \bigcup_{a \in A} D(a)$, such that for each $a \in A$, there exists a local section $g_a \in \Gamma(D(a), \mathcal{G})$ where g_a lifts the restriction $h|_{D(a)}$.

Claim: For every $a \in A$, there exists an integer $n \geq 0$ such that $a^n h$ lifts to a global section $\tilde{g}_a \in \Gamma(X, \mathcal{G})$.

In fact, consider the intersection of two open sets $D(a) \cap D(b) = D(ab)$. The difference of the local sections, $g_a|_{D(ab)} - g_b|_{D(ab)}$, is mapped to $0 \in \Gamma(D(ab), \mathcal{H})$ because both g_a and g_b lift h. Since the sequence is left exact, there exists a section $f_{ab} \in \Gamma(D(ab), \mathcal{F})$ such that f_{ab} is mapped to the difference $g_a|_{D(ab)} - g_b|_{D(ab)}$.

By the extension lemma for quasi-coherent sheaves, since $\mathcal{F}$ is quasi-coherent, there exists an integer $n_b \geq 0$ such that the section $a^{n_b} f_{ab} \in \Gamma(D(ab), \mathcal{F})$ extends to a section $f'_{ab} \in \Gamma(D(b), \mathcal{F})$.

Define a section on $D(b)$ by $g'_b := a^{n_b} g_b + f'_{ab} \in \Gamma(D(b), \mathcal{G})$. Restricting to the intersection $D(ab)$, we have:

$$g'_b|_{D(ab)} = a^{n_b} g_b|_{D(ab)} + a^{n_b}(g_a|_{D(ab)} - g_b|_{D(ab)}) = a^{n_b} g_a|_{D(ab)}.$$

Since X is quasi-compact (being affine), we can choose a finite subcover. Let $n = \max\{n_b\}$ over the finite set of indices. Then the section $a^n g_a$ (defined on $D(a)$) extends to a global section $\tilde{g}_a \in \Gamma(X, \mathcal{G})$. Furthermore, $\tilde{g}_a$ is mapped to $a^n h$ under the morphism $\mathcal{G} \to \mathcal{H}$. This is because locally on $D(a)$, we have $\tilde{g}_a|_{D(a)} = a^n g_a$, which maps to $a^n h|_{D(a)}$.

Since $X = \bigcup_{a \in A} D(a)$ is an open covering, the ideal generated by $\{a^n\}_{a \in A}$ is the unit ideal.

Thus, we can write

$$1 = \sum_{i=1}^{k} c_i a_i^n$$

for some $c_i \in \Gamma(X, \mathcal{O}_X)$ and indices a_i. Therefore, the global section h can be written as

$$h = 1 \cdot h = \sum_{i=1}^{k} c_i a_i^n h.$$

Since each $a_i^n h$ lifts to $\tilde{g}_{a_i}$, the sum lifts to $\sum_{i=1}^{k} c_i \tilde{g}_{a_i} \in \Gamma(X, \mathcal{G})$. This proves the surjectivity. $\square$

Proposition 8.2. Let X be a scheme. The category of quasi-coherent sheaves, denoted $\mathrm{Qcoh}(\mathcal{O}_X)$, is an abelian subcategory of the category of $\mathcal{O}_X$-modules $\mathcal{O}_X$-mod. That is, the kernel, cokernel, and image of any homomorphism between quasi-coherent sheaves are quasi-coherent.
Moreover, extensions of quasi-coherent sheaves are quasi-coherent.
The same holds for the category of coherent sheaves $\mathrm{Coh}(\mathcal{O}_X)$ when X is locally Noetherian.

Proof. It suffices to show this for the affine case $X = \mathrm{Spec}(A)$. Recall that the functor $M \longmapsto \widetilde{M}$ from the category of A-modules (A-mod) to $\mathcal{O}_X$-mod is a fully faithful and exact functor. This implies that $\mathrm{Qcoh}(\mathcal{O}_X)$ is abelian.

Regarding extensions: Suppose

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is a short exact sequence in $\mathcal{O}_X$-mod such that $\mathcal{F}$ and $\mathcal{H}$ are in $\mathrm{Qcoh}(\mathcal{O}_X)$. Then, according to the **Proposition** above, since X is affine, the sequence of global sections

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow 0$$

is exact in the category of A-modules (where $A = \Gamma(X, \mathcal{O}_X)$).

Applying the functor $\widetilde{(\cdot)}$, which is exact, we obtain a short exact sequence in $\mathrm{Qcoh}(\mathcal{O}_X)$. We can form the following commutative diagram in the category of $\mathcal{O}_X$-modules:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\Gamma(X,\mathcal{F})} \arrow[r] \arrow[d, "\cong"] & \widetilde{\Gamma(X,\mathcal{G})} \arrow[r] \arrow[d, dashed] & \widetilde{\Gamma(X,\mathcal{H})} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{F}$ and $\mathcal{H}$ are quasi-coherent, the canonical maps $\widetilde{\Gamma(X,\mathcal{F})} \to \mathcal{F}$ and $\widetilde{\Gamma(X,\mathcal{H})} \to \mathcal{H}$ are isomorphisms. By the Five Lemma, the middle vertical map $\widetilde{\Gamma(X,\mathcal{G})} \to \mathcal{G}$ must also be an isomorphism. Therefore, $\mathcal{G}$ is quasi-coherent. $\square$

Remark: For an arbitrary $\mathcal{O}_X$-module $\mathcal{E}$, one might consider the canonical morphism $\widetilde{\Gamma(X,\mathcal{E})} \longrightarrow \mathcal{E}$. Does this map behave well?

Consider the stalks. The map factors through the localization of the global sections:

$$\begin{tikzcd} \Gamma(X,\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E}_p \\ \Gamma(X,\mathcal{E})_p \arrow[ur, "canonical"'] & \end{tikzcd}$$

The essential issue is that an element in the stalk $\mathcal{E}_p$ is not necessarily the germ of a *global* section. Thus, for a general $\mathcal{O}_X$-module, this map is not an isomorphism.

8.1 Quasi-coherence under Pullback and Pushforward

Proposition 8.3. Let $f : X \to Y$ be a morphism of schemes.

(a) If $\mathcal{G} \in \mathrm{Qcoh}(\mathcal{O}_Y)$, then the pullback sheaf $f^*\mathcal{G}$ is in $\mathrm{Qcoh}(\mathcal{O}_X)$.

Furthermore, if X and Y are locally Noetherian and $\mathcal{G} \in \mathrm{Coh}(\mathcal{O}_Y)$, then $f^*\mathcal{G} \in \mathrm{Coh}(\mathcal{O}_X)$.

(b) If the morphism f is qcqs. (quasi-compact and quasi-separated), and $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{O}_X)$, then the pushforward sheaf $f_*\mathcal{F}$ is in $\mathrm{Qcoh}(\mathcal{O}_Y)$.

Proof. **(a)** It suffices to prove the statement locally. Let us assume the local case where $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, and f corresponds to a ring homomorphism $B \to A$. If $\mathcal{G} = \widetilde{M}$ for some B-module M, then the pullback sheaf is isomorphic to the sheaf associated to the tensor product:

$$f^*\mathcal{G} \cong (\widetilde{M \otimes_B A}).$$

Since $M \otimes_B A$ is an A-module, $f^*\mathcal{G}$ is clearly in $\mathrm{Qcoh}(\mathcal{O}_X)$.

Furthermore, if A and B are Noetherian and M is a finitely generated B-module (which implies $\mathcal{G}$ is coherent), then the tensor product $M \otimes_B A$ is a finitely generated A-module. Consequently, $f^*\mathcal{G}$ is a coherent sheaf on X. $\square$

Proof. **(b)** Since quasi-coherence is local on the target, it suffices to show the result when $Y = \mathrm{Spec}(B)$ is affine. Since f is a qcqs (quasi-compact and quasi-separated) morphism and Y is affine, X is a qcqs scheme. Being quasi-compact, X can be covered by a finite number of affine open sets:

$$X = \bigcup_{i=1}^{n} X_i.$$

Since X is quasi-separated, the intersection of any two affine open sets $X_{ij} := X_i \cap X_j$ is quasi-compact (for all $1 \leq i < j \leq n$). Therefore, each intersection X_{ij} can be covered by finitely many affine open sets:

$$X_{ij} = \bigcup_{k=1}^{m_{ij}} X_{ij}^{(k)},$$

where each $X_{ij}^{(k)}$ is an affine open subset of X_{ij}.

We have the following exact sequence (part of the Čech complex):

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_{i=1}^{n} f_*(\mathcal{F}|_{X_i}) \longrightarrow \bigoplus_{1 \leq i < j \leq n} f_*(\mathcal{F}|_{X_{ij}}).$$

Notice that for each pair (i, j), the restriction map induces an injection:

$$f_*(\mathcal{F}|_{X_{ij}}) \hookrightarrow \bigoplus_{k=1}^{m_{ij}} f_*(\mathcal{F}|_{X_{ij}^{(k)}}).$$

Combining these, we obtain an exact sequence of $\mathcal{O}_Y$-modules:

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_{i=1}^{n} f_*(\mathcal{F}|_{X_i}) \longrightarrow \bigoplus_{i,j,k} f_*(\mathcal{F}|_{X_{ij}^{(k)}}).$$

The terms in the middle and on the right are finite direct sums of pushforwards of quasi-coherent sheaves along morphisms between affine schemes (specifically, from X_i to Y and $X_{ij}^{(k)}$ to Y). Since the pushforward of a quasi-coherent sheaf along an affine morphism is quasi-coherent, both $\bigoplus f_*(\mathcal{F}|_{X_i})$ and $\bigoplus f_*(\mathcal{F}|_{X_{ij}^{(k)}})$ are in $\mathrm{Qcoh}(\mathcal{O}_Y)$.

Since $\mathrm{Qcoh}(\mathcal{O}_Y)$ is an abelian category (kernels of morphisms between quasi-coherent sheaves are quasi-coherent), it follows that $f_*\mathcal{F} \in \mathrm{Qcoh}(\mathcal{O}_Y)$. □

Remerk: If X and Y are Noetherian and $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_X)$, the pushforward $f_*\mathcal{F}$ is **not** coherent in general.

Example 8.4. Let $X = \mathbb{A}^1_k$ and $Y = \mathrm{Spec}(k)$. Let $\mathcal{F} = \mathcal{O}_X$. Then

$$f_*\mathcal{F}(Y) = \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) = k[x].$$

This is not finite-dimensional over k, so it is not a coherent sheaf on Y.

However, the statement is true if f is a **proper** morphism.

8.2 Ideal Sheaf

Definition 8.5. Let $i : Y \hookrightarrow X$ be a closed immersion. We define the **ideal sheaf** of Y, denoted by $\mathscr{I}_Y$, to be the kernel of the comorphism $i^\#$:

$$\mathscr{I}_Y = \ker(i^\# : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y).$$

Remark: The ideal sheaf $\mathscr{I}_Y$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

Proposition 8.6. Let X be a scheme. There is a one-to-one correspondence between the set of closed subschemes of X and the set of quasi-coherent sheaves of ideals on X:

$$\{\text{closed subschemes of } X\} \longleftrightarrow \{\text{quasi-coherent sheaves of ideals } \mathcal{I} \subseteq \mathcal{O}_X\}.$$

The correspondence is given by mapping a closed subscheme Y to its ideal sheaf $\mathscr{I}_Y$.

Proof. It suffices to prove the statement for the local case where $X \cong \mathrm{Spec}(A)$. In this case, the correspondence reduces to the known correspondence in commutative algebra:

$$\{\text{quasi-coherent submodules } \mathcal{I} \subseteq \mathcal{O}_X\} \longleftrightarrow \{\text{ideals } I \subseteq A\} \longleftrightarrow \{\text{closed subschemes of } \mathrm{Spec}(A)\}.$$

Specifically, a quasi-coherent sheaf of ideals $\mathcal{I}$ on $\mathrm{Spec}(A)$ corresponds to an ideal $I = \Gamma(X, \mathcal{I})$ of A, which in turn defines a closed subscheme $V(I)$. □

Remark: If X is locally Noetherian, then the ideal sheaf $\mathscr{I}_Y$ is coherent.

Proposition 8.7. Let $f : Y \to X$ be a quasi-compact morphism. Then the kernel of the morphism $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$, denoted by

$$\mathscr{I}_f := \ker(f^\# : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y),$$

is a quasi-coherent sheaf on X.

Proof. It suffices to show the result for the local case where $X = \mathrm{Spec}(A)$.

Since the morphism $f : Y \to X$ is quasi-compact and we are in the local case where X is affine, the scheme Y is quasi-compact. Thus, there exists a finite covering of Y by affine open subsets:

$$Y = \bigcup_{i=1}^{n} Y_i,$$

where each Y_i is an affine scheme. We can consider the sequence involving the restriction to these open sets:

$$0 \longrightarrow \mathscr{I}_f \longrightarrow \mathcal{O}_X \longrightarrow \bigoplus_{i=1}^{n} f_*(\mathcal{O}_{Y_i}).$$

Note that the map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is composed with the injection $f_*\mathcal{O}_Y \hookrightarrow \bigoplus f_*(\mathcal{O}_{Y_i})$. The term $\mathcal{O}_X$ is quasi-coherent. For each i, the restriction $f|_{Y_i} : Y_i \to X$ is a morphism between affine schemes, so the pushforward $f_*(\mathcal{O}_{Y_i})$ is a quasi-coherent $\mathcal{O}_X$-module. Consequently, the finite direct sum $\bigoplus_{i=1}^{n} f_*(\mathcal{O}_{Y_i})$ is quasi-coherent. Since $\mathrm{Qcoh}(\mathcal{O}_X)$ is an abelian category, the kernel of a morphism between quasi-coherent sheaves is quasi-coherent. Therefore, $\mathscr{I}_f$ is quasi-coherent. $\square$

Definition 8.8. Let $f : Y \to X$ be a quasi-compact morphism. We define the **scheme-theoretic image** of f, denoted by $\mathrm{Im}(f)$, to be the closed subscheme of X defined by the ideal sheaf $\mathscr{I}_f$.

Example 8.9. Let X be an integral scheme and let η be its generic point. Consider the inclusion morphism $f : \eta \to X$. The scheme-theoretic image $\mathrm{Im}(f)$ is the whole space X. Algebraically, if $X = \mathrm{Spec}(A)$, this corresponds to the fact that the kernel of the map $A \to \mathrm{Frac}(A)$ is zero (since A is a domain):

$$0 \longrightarrow 0 \longrightarrow A \longrightarrow \mathrm{Frac}(A).$$

Thus the ideal defining the image is the zero ideal, which corresponds to X itself.

8.3 Relative Spectrum

Definition 8.10. A **quasi-coherent $\mathcal{O}_X$-algebra** is an algebra object in the symmetric monoidal category $\mathrm{Qcoh}(\mathcal{O}_X)$. Explicitly, it is a quasi-coherent sheaf $\mathscr{A} \in \mathrm{Qcoh}(\mathcal{O}_X)$ equipped with two morphisms of $\mathcal{O}_X$-modules:

- A unit map $e : \mathcal{O}_X \longrightarrow \mathscr{A}$,
- A multiplication map $m : \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \longrightarrow \mathscr{A}$,

satisfying the following conditions:
1. **Associativity**: The following diagram commutes:

$$\begin{tikzcd} \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"'] & \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[d, "m"] \\ \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[r, "m"] & \mathscr{A} \end{tikzcd}$$

2. **Identity (Unitary property)**: The following diagram commutes (showing left and right identity):

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\simeq"] \arrow[rrr, bend right, "\mathrm{id}"'] & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[r, "e\otimes\mathrm{id}"] & \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[r, "m"] & \mathscr{A} \end{tikzcd}$$

Definition 8.11. We say a quasi-coherent $\mathcal{O}_X$-algebra $\mathscr{A}$ is **commutative** if the multiplication map $m : \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \to \mathscr{A}$ is symmetric. That is, the following diagram commutes:

$$\begin{tikzcd} \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[rr, "\sigma"'] \arrow[dr, "m"'] & & \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[dl, "m"] \\ & \mathscr{A} & \end{tikzcd}$$

where σ is the swap isomorphism $a \otimes b \mapsto b \otimes a$. We denote the category of commutative algebra objects in $\mathrm{Qcoh}(\mathcal{O}_X)$ by $\mathrm{CommAlg}(\mathrm{Qcoh}(\mathcal{O}_X))$.

Remark: Let $X = \mathrm{Spec}(B)$ be an affine scheme. Then the category of commutative quasi-coherent $\mathcal{O}_X$-algebras is equivalent to the category of B-algebras:

$$\mathrm{CommAlg}(\mathrm{Qcoh}(\mathcal{O}_X)) \cong \mathrm{CommAlg}(B\text{-mod}) \simeq B\text{-alg}.$$

Consequently, if $\mathscr{A}$ is such a sheaf, its global sections $\Gamma(X, \mathscr{A})$ form a B-algebra.

Definition 8.12 (**Relative Spectrum**)**.** Let $\mathscr{A} \in \mathrm{CommAlg}(\mathrm{Qcoh}(\mathcal{O}_X))$. We define the **relative spectrum** of $\mathscr{A}$, denoted by $\mathbf{Spec}_X\mathscr{A}$, to be the scheme constructed by gluing the affine schemes $\mathrm{Spec}(\Gamma(U, \mathscr{A}))$ for all affine open subsets $U \subseteq X$.
Explicitly, for each affine open $U \subseteq X$, let $A_U = \Gamma(U, \mathscr{A})$. Since $\mathscr{A}$ is an $\mathcal{O}_X$-algebra, A_U is a $\Gamma(U, \mathcal{O}_X)$-algebra. We define $Y_U = \mathrm{Spec}(A_U)$. For any two affine open sets U, V, the restrictions agree on the intersection, allowing us to glue Y_U and Y_V along the open subschemes corresponding to $U \cap V$:

$$\mathbf{Spec}(\Gamma(U, \mathscr{A})) \times_U (U \cap V) \simeq \mathbf{Spec}(\Gamma(V, \mathscr{A})) \times_V (U \cap V).$$

This construction yields a scheme $Y = \mathbf{Spec}_X\mathscr{A}$ equipped with a canonical morphism $\pi : \mathbf{Spec}_X\mathscr{A} \to X$.

Proposition 8.13. Let X be a scheme. There is a one-to-one correspondence between the set of affine morphisms to X and the set of quasi-coherent sheaves of commutative $\mathcal{O}_X$-algebras:

$$\{\text{affine morphisms } f : Y \to X\} \longleftrightarrow \{\mathscr{A} \in \mathrm{CommAlg}(\mathrm{Qcoh}(\mathcal{O}_X))\}.$$

The correspondence is given by:

- Given an affine morphism $f : Y \to X$, we associate the sheaf $\mathscr{A} = f_*\mathcal{O}_Y$.
- Given a quasi-coherent algebra sheaf $\mathscr{A}$, we associate the relative spectrum $Y = \mathbf{Spec}_X\mathscr{A}$ with the canonical projection to X.

Proof. It suffices to prove the statement locally. Let $X = \mathrm{Spec}(B)$. An affine morphism $f : Y \to X$ means Y is affine, say $Y = \mathrm{Spec}(A)$, and f corresponds to a ring homomorphism $B \to A$. Thus A is a B-algebra. Conversely, a quasi-coherent $\mathcal{O}_X$-algebra $\mathscr{A}$ corresponds to a B-algebra $A = \Gamma(X, \mathscr{A})$. The equivalence of categories $\mathrm{Aff}/X \simeq B\text{-alg}$ establishes the result. $\square$

Example 8.14 (Closed Subschemes)**.** The set of closed subschemes of X corresponds to the set of quasi-coherent commutative $\mathcal{O}_X$-algebras $\mathscr{A}$ such that the unit map $\mathcal{O}_X \to \mathscr{A}$ is surjective.

$$\{\text{closed subschemes of } X\} \longleftrightarrow \{\mathscr{A} \in \mathrm{CommAlg}(\mathrm{Qcoh}(\mathcal{O}_X)) \mid \mathcal{O}_X \twoheadrightarrow \mathscr{A}\}.$$

Specifically, if $\mathcal{O}_X \to \mathscr{A}$ is surjective, then $\mathscr{A} \cong \mathcal{O}_X/\mathscr{I}$ for some ideal sheaf $\mathscr{I}$, which defines a closed subscheme.

8.4 Bundles and Symmetric Algebras

Definition 8.15 (**Symmetric Power**). Let $\mathcal{E} \in \mathrm{Qcoh}(\mathcal{O}_X)$. We define the n-th **symmetric power** of $\mathcal{E}$, denoted by $\mathrm{Sym}^n\mathcal{E}$, as the quotient of the n-fold tensor product $\mathcal{E}^{\otimes n}$ by the action of the symmetric group S_n.
Explicitly, the group S_n acts on $\mathcal{E}^{\otimes n}$ by swapping factors. $\mathrm{Sym}^n\mathcal{E}$ is the sheaf of coinvariants $(\mathcal{E}^{\otimes n})_{S_n}$, defined by:

$$\mathrm{Sym}^n\mathcal{E} = \mathcal{E}^{\otimes n} \Big/ \sum_{g \in S_n} \mathrm{Im}(g - \mathrm{id}).$$

You can simply think that locally, this quotients out the relations generated by $v \otimes w - w \otimes v$.

Definition 8.16 (**Symmetric Algebra**). We define the **symmetric algebra** of $\mathcal{E}$, denoted by $\mathrm{Sym}^\bullet\mathcal{E}$, as the direct sum of all symmetric powers:

$$\mathrm{Sym}^\bullet\mathcal{E} = \bigoplus_{n \geq 0} \mathrm{Sym}^n\mathcal{E} = \mathcal{O}_X \oplus \mathcal{E} \oplus \mathrm{Sym}^2\mathcal{E} \oplus \cdots$$

It is equipped with the structure of a quasi-coherent $\mathcal{O}_X$-algebra, where the multiplication is induced by the tensor product maps:

$$\mathrm{Sym}^n\mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Sym}^m\mathcal{E} \longrightarrow \mathrm{Sym}^{n+m}\mathcal{E},$$

and the unit is the canonical inclusion $\mathcal{O}_X \cong \mathrm{Sym}^0\mathcal{E} \hookrightarrow \mathrm{Sym}^\bullet\mathcal{E}$. Thus, $\mathrm{Sym}^\bullet\mathcal{E} \in \mathrm{CommAlg}(\mathrm{Qcoh}(\mathcal{O}_X))$.

Definition 8.17 (**Bundle**). The **bundle** associated to $\mathcal{E}$, denoted by $\mathbb{V}_X(\mathcal{E})$, is defined as the relative spectrum of the symmetric algebra of $\mathcal{E}$:

$$\mathbb{V}_X(\mathcal{E}) = \mathbf{Spec}_X(\mathrm{Sym}^\bullet\mathcal{E}).$$

Remark: When $\mathcal{E}$ is a locally free sheaf of rank n, $\mathbb{V}_X(\mathcal{E})$ is a vector bundle of rank n over X in the geometric sense.

To see this, we compute locally. Let $X = \mathrm{Spec}(A)$ and assume $\mathcal{E}$ is free, i.e., $\mathcal{E} \cong \mathcal{O}_X^{\oplus n}$. The global sections of the symmetric algebra are given by the symmetric algebra of the A-module $A^{\oplus n}$:

$$\Gamma(X, \mathrm{Sym}^\bullet \mathcal{E}) \cong \mathrm{Sym}_A(A^{\oplus n}) \cong A[x_1, \ldots, x_n].$$

Therefore, the relative spectrum is the affine space over A:

$$\mathbb{V}_X(\mathcal{E}) = \mathrm{Spec}(A[x_1, \ldots, x_n]) = \mathbb{A}^n_A = \mathbb{A}^n_X.$$

8.5 Relative Projective Spectrum

Definition 8.18 (**Relative Projective Spectrum**)**.** Let $\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{S}_d$ be a graded commutative algebra object in $\mathrm{Qcoh}(\mathcal{O}_X)$. We define the **relative projective spectrum** of $\mathcal{S}$, denoted by $\mathbf{Proj}_X \mathcal{S}$, as the scheme constructed by gluing the projective spectra of the sections over affine open sets.
Explicitly, for each affine open subset $U \subseteq X$, let $S_U = \Gamma(U, \mathcal{S})$. This is a graded algebra over $\Gamma(U, \mathcal{O}_X)$. We form the scheme $\mathrm{Proj}(S_U)$. These schemes are glued together along intersections to form a global scheme $\mathbf{Proj}_X \mathcal{S}$ over X.

Example 8.19. The projective space over X, denoted by $\mathbb{P}^n_X$, is defined as the relative projective spectrum of the symmetric algebra of a free sheaf of rank $n+1$:

$$\mathbb{P}^n_X = \mathbf{Proj}_X \left(\mathrm{Sym}^\bullet(\mathcal{O}_X^{\oplus n+1}) \right).$$

8.6 Quasi-coherent Sheaves on Projective Spectrum

Let $S = \bigoplus_{d \geq 0} S_d$ be a graded algebra (for instance, S could be the ring of sections of a graded sheaf over an affine base). Let $X = \mathrm{Proj}(S)$. We have a functor from the category of graded S-modules to the category of quasi-coherent sheaves on X:

$$\begin{aligned} \mathrm{Gr}(S\text{-mod}) &\longrightarrow \mathrm{Qcoh}(\mathcal{O}_X) \\ M &\longmapsto \widetilde{M} \end{aligned}$$

Conversely, there is a functor $\mathcal{F} \mapsto \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$.

Remark: This functor $M \mapsto \widetilde{M}$ is **not** an equivalence of categories! However, it induces an equivalence between the category of quasi-coherent sheaves on $\mathrm{Proj}(S)$ and the quotient category of graded S-modules modulo the subcategory of ”saturated” modules.

$$(\mathrm{Gr}(S\text{-mod}))^{\mathrm{sat}} \simeq \mathrm{Qcoh}(\mathcal{O}_{\mathrm{Proj} S}).$$

Definition 8.20. Let $M \in \mathrm{Gr}(S\text{-mod})$. We define the sheaf $\widetilde{M}$ on $X = \mathrm{Proj}(S)$ as follows. For any open set $U \subseteq X$,

$$\widetilde{M}(U) := \left\{ (s_{\mathfrak{p}})_{\mathfrak{p}\in U} \prod_{\mathfrak{p}\in U} M_{(\mathfrak{p})} \;\middle|\; \text{gluing conditions} \right\}.$$

Here $M_{(\mathfrak{p})}$ denotes the homogeneous localization of M at the prime ideal $\mathfrak{p} \in \mathrm{Proj}(S)$ (which consists of elements of degree 0 in the localization $S_{\mathfrak{p}}^{-1}M$). The gluing condition requires that for every $\mathfrak{p} \in U$, there exists a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and homogeneous elements $m \in M, f \in S$ of the same degree such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s_{\mathfrak{q}} = m/f$ in $M_{(\mathfrak{q})}$.

Proposition 8.21. Let $X = \mathrm{Proj}(S)$ and M be a graded S-module.

(a) For any point $\mathfrak{p} \in X$, the stalk of $\widetilde{M}$ at $\mathfrak{p}$ is isomorphic to the homogeneous localization:

$$(\widetilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})}.$$

(b) For any homogeneous element $f \in S_+$, the restriction of $\widetilde{M}$ to the distinguished open set $D_+(f)$ is isomorphic to the sheaf associated to the degree-0 part of the localization M_f:

$$\widetilde{M}|_{D_+(f)} \cong \widetilde{(M_{(f)})},$$

where $M_{(f)}$ is the degree 0 part of M_f. Note that $D_+(f) \cong \mathrm{Spec}(S_{(f)})$.

(c) $\widetilde{M} \in \mathrm{Qcoh}(\mathcal{O}_X)$. Furthermore, if S is a Noetherian ring and M is a finitely generated graded S-module, then $\widetilde{M}$ is a coherent sheaf, i.e., $\widetilde{M} \in \mathrm{Coh}(\mathcal{O}_X)$.

Proof. The proof is similar to the affine case. □

8.7 Twisting Sheaves

Definition 8.22. Let S be a graded ring and let M be a graded S-module. We define a new graded S-module, denoted by $M(n)$ (the **shifted module**), by shifting the grading degrees:

$$M(n)_d = M_{n+d}.$$

Let $X = \mathrm{Proj}(S)$. We define the **twisting sheaf of Serre**, denoted by $\mathcal{O}_X(n)$, to be the sheaf associated to the shifted ring $S(n)$:

$$\mathcal{O}_X(n) := \widetilde{S(n)} \in \mathrm{Qcoh}(\mathcal{O}_X).$$

For any quasi-coherent sheaf $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{O}_X)$, we define the twisted sheaf $\mathcal{F}(n)$ by:

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

Assumption: From now on, we assume that S is generated as an S_0-algebra by S_1. This ensures that the distinguished open sets $D_+(f)$ for $f \in S_1$ cover X.

Proposition 8.23. Let $X = \mathrm{Proj}(S)$.

(a) The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf (i.e., a line bundle) on X.

(b) For any graded S-module M, there is a canonical isomorphism:

$$\widetilde{M(n)} \cong \widetilde{M}(n).$$

In particular, we have the isomorphism of tensor products:

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n).$$

Proof. Notice that for any homogeneous element $f \in S_1$ of degree 1, we can define a map between the homogeneous localizations. Recall that $M_{(f)} = \{\frac{m}{f^d} \mid m \in M_d\}$. Consider the map defined by multiplication by f^n:

$$\begin{aligned} M_{(f)} &\xrightarrow{\cdot f^n} M(n)_{(f)} \\ \frac{m}{f^d} &\longmapsto \frac{m}{f^{d-n}}. \end{aligned}$$

Note that if $m \in M_d$, then $m \in M(n)_{d-n}$, so the element $\frac{m}{f^{d-n}}$ is indeed a degree 0 element in the localization $M(n)_f$. Since f is invertible in the localization S_f, this map is an isomorphism with inverse given by multiplication by f^{-n}.

This local isomorphism implies the global properties:

(b) The isomorphism $M_{(f)} \cong M(n)_{(f)}$ implies that locally on $D_+(f)$, $\widetilde{M} \cong \widetilde{M(n)}$ (up to the twist by f^n which corresponds to tensoring with $\mathcal{O}(n)$). More precisely, $\widetilde{M(n)} \cong \widetilde{M} \otimes \mathcal{O}_X(n)$.

(a) Applying the above to $M = S$, we have $S(n)_{(f)} \cong S_{(f)}$ as $S_{(f)}$-modules. Since $D_+(f) = \mathrm{Spec}(S_{(f)})$, this means that restricted to $D_+(f)$, $\mathcal{O}_X(n)$ is isomorphic to the structure sheaf $\mathcal{O}_X$. Since X is covered by such $D_+(f)$ (as S is generated by S_1), $\mathcal{O}_X(n)$ is locally free of rank 1, hence an invertible sheaf.

□

8.8 Graded Module Associated to a Sheaf

Definition 8.24. Let S be a graded ring and let $X = \mathrm{Proj}(S)$. Let $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{O}_X)$ be a quasi-coherent sheaf on X. We define the **graded module associated to** $\mathcal{F}$, denoted by $\Gamma_*(X, \mathcal{F})$, as the direct sum of the global sections of the twisted sheaves:

$$\Gamma_*(X, \mathcal{F}) := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n)).$$

This object has the structure of a graded S-module. The scalar multiplication is defined as follows: for any homogeneous element $s \in S_d$ and any section $t \in \Gamma(X, \mathcal{F}(n))$, the product $s \cdot t$ is the image of $t \otimes s$ under the canonical map:

$$\Gamma(X, \mathcal{F}(n)) \otimes_{\Gamma(X, \mathcal{O}_X)} S_d \longrightarrow \Gamma(X, \mathcal{F}(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)) \cong \Gamma(X, \mathcal{F}(n+d)).$$

Thus, $s \cdot t \in \Gamma(X, \mathcal{F}(n+d))$.

Theorem 8.25. Let $X = \mathrm{Proj}(S)$. For any quasi-coherent sheaf $\mathcal{F}$ on X, there is a natural isomorphism between the sheaf associated to the graded module $\Gamma_*(X, \mathcal{F})$ and the sheaf $\mathcal{F}$ itself:

$$\widetilde{\Gamma_*(X, \mathcal{F})} \xrightarrow{\sim} \mathcal{F}.$$

Remark: The converse statement does not hold in general. That is, for a graded S-module M, the graded module associated to the sheaf $\widetilde{M}$ is not necessarily isomorphic to M:

$$\Gamma_*(X, \widetilde{M}) \ncong M.$$

(For instance, this can happen even when $M = S$; the map $S \to \Gamma_*(X, \mathcal{O}_X)$ is an isomorphism only if S satisfies certain saturation properties).

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Lecture 11. Quasi-coherent Sheaves on Projective Schemes

We begin by reviewing some key properties of quasi-coherent sheaves established in previous lectures:

1. **Cohomology on Affine Schemes:** If X is an affine scheme, then the higher cohomology of quasi-coherent sheaves vanishes. That is, for any $\mathcal{F} \in \mathrm{QCoh}(X)$, we have $H^1(X, \mathcal{F}) = 0$ (and indeed $H^i(X, \mathcal{F}) = 0$ for all $i > 0$).

2. **Pullback:** The inverse image functor preserves quasi-coherence. If $f : X \to Y$ is a morphism of schemes, then $f^*\mathcal{G} \in \mathrm{QCoh}(X)$ for any $\mathcal{G} \in \mathrm{QCoh}(Y)$.

3. **Pushforward:** The direct image functor preserves quasi-coherence under mild conditions. Specifically, if $f : X \to Y$ is a quasi-compact and quasi-separated (qcqs) morphism, then $f_*\mathcal{F} \in \mathrm{QCoh}(Y)$ for any $\mathcal{F} \in \mathrm{QCoh}(X)$.

4. **Ideal Sheaves and Closed Subschemes:** There is a one-to-one correspondence between quasi-coherent ideal sheaves and closed subschemes:

$$\{\text{Quasi-coherent ideal sheaves } \mathcal{I} \subset \mathcal{O}_X\} \xleftrightarrow{1:1} \{\text{Closed subschemes } Z \hookrightarrow X\}.$$

5. **Relative Constructions:**

   - Given a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$, we can form the relative spectrum $\mathbf{Spec}_X(\mathcal{A})$. A key example is the geometric vector bundle associated to a locally free sheaf $\mathcal{E}$, denoted by $\mathbb{V}_X(\mathcal{E}) = \mathbf{Spec}_X(\mathrm{Sym}^\bullet \mathcal{E})$.
   - Similarly, given a graded quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{S}$, we can form the relative projective spectrum $\mathbf{Proj}_X(\mathcal{S})$. An example is the projective bundle $\mathbb{P}_X(\mathcal{E}) = \mathbf{Proj}_X(\mathrm{Sym}^\bullet \mathcal{E})$.

9.1 The Graded Module-Sheaf Correspondence

Let S be a graded ring and let $X = \mathrm{Proj}(S)$. We have a relationship between the category of graded S-modules and the category of quasi-coherent sheaves on X.

There is a pair of adjoint functors:

$$\text{gr-}S\text{-mod} \underset{\Gamma_*(\mathcal{F}) \leftarrow \mathcal{F}}{\overset{M \mapsto \widetilde{M}}{\rightleftarrows}} \mathrm{QCoh}(\mathcal{O}_X)$$

where the functor from sheaves to modules is defined by the graded global sections:

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

Remark (Exactness of the Tilde Functor)**.** *The functor $M \mapsto \widetilde{M}$ is exact. This means that if we have a short exact sequence of graded S-modules:*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

applying the tilde functor yields a short exact sequence of quasi-coherent sheaves on X:

$$0 \longrightarrow \widetilde{M'} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M''} \longrightarrow 0.$$

Furthermore, the stalk of the associated sheaf at a point $P \in \mathrm{Proj}(S)$ corresponds to the homogeneous localization of the module. Specifically, for any homogeneous prime ideal $\mathfrak{p} \in \mathrm{Proj}(S)$, we have:

$$(\widetilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})},$$

where $M_{(\mathfrak{p})}$ denotes the elements of degree 0 in the localization $M_{\mathfrak{p}}$.

Example 9.1. Let R be an integral domain and let $S = R[X_0, \dots, X_n]$ be the polynomial ring with the standard grading. Let $X = \mathbb{P}^n_R = \mathrm{Proj}(S)$.
Consider a homogeneous polynomial $f \in S_d$ of degree d. This polynomial defines a hypersurface $V_+(f) \subseteq X$.
We can describe the structure sheaf of this hypersurface using an exact sequence of graded modules. Multiplication by f gives an injective map from the shifted ring $S(-d)$ to S (since R is a domain):

$$0 \longrightarrow S(-d) \xrightarrow{\cdot f} S \longrightarrow S/(f) \longrightarrow 0.$$

Applying the exact functor $M \mapsto \widetilde{M}$, we obtain the corresponding exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow \widetilde{S/(f)} \longrightarrow 0.$$

Note that the sheaf $\widetilde{S/(f)}$ is isomorphic to $i_*\mathcal{O}_{V_+(f)}$, where $i : V_+(f) \hookrightarrow X$ is the inclusion of the closed subscheme defined by the ideal (f). Thus, we have the ideal sheaf sequence:

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_{V_+(f)} \longrightarrow 0.$$

Proposition 9.2. Let R be a ring and let $S = R[X_0, \dots, X_n]$ be the polynomial ring with the standard grading. Then there is an isomorphism of graded rings:

$$\Gamma_*(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}) \cong S.$$

Proof. We construct a natural map of graded rings $\alpha = (\alpha_d)_{d \in \mathbb{Z}}$, where

$$\alpha_d : S_d \longrightarrow \Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)).$$

This map is defined by sending a homogeneous polynomial $f \in S_d$ to the global section defined locally by the fraction $f/1$. Specifically, for any point $\mathfrak{p} \in \mathbb{P}^n_R$, the germ of the section is given by:

$$f \longmapsto \left(\frac{f}{1} \in (\mathcal{O}_{\mathbb{P}^n_R}(d))_{\mathfrak{p}} \cong S(d)_{(\mathfrak{p})} \right)_{\mathfrak{p} \in \mathbb{P}^n_R} \in \Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)).$$

Step 1: α is injective. Suppose $\alpha_d(f) = 0$ for some $f \in S_d$. This implies that the section defined by f is zero everywhere. In particular, restricting to the standard affine open cover $D_+(X_i) \cong \operatorname{Spec}(S_{(X_i)})$, the element f must be zero in the coordinate ring. We have the isomorphism $\Gamma(D_+(X_i), \mathcal{O}_{\mathbb{P}^n_R}(d)) \cong S(d)_{(X_i)}$. The image of f in this localization is zero, which means there exists some integer N such that:

$$X_i^N \cdot f = 0 \quad \text{in } S.$$

Since X_i are not zero-divisors in the polynomial ring S, this implies that $f = 0$. Thus, α_d is injective for all d.

Step 2: α is surjective. Suppose we have a global section $t \in \Gamma_*(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R})$. We can represent t by its restrictions to the standard open cover. Let $t = (t_i)_{i=0}^n$, where

$$t_i \in \bigoplus_{d \in \mathbb{Z}} \Gamma(D_+(X_i), \mathcal{O}_{\mathbb{P}^n_R}(d)) \cong \bigoplus_{d \in \mathbb{Z}} S(d)_{(X_i)} = S_{X_i}.$$

For t to glue to a global section, these local sections must satisfy the compatibility condition on the intersections:

$$t_i|_{D_+(X_i) \cap D_+(X_j)} = t_j|_{D_+(X_i) \cap D_+(X_j)} \quad \text{in } S_{X_i X_j}.$$

Notice that we have the following short exact sequence relating the ring S to its localizations (related to the Cech complex):

$$0 \longrightarrow S \longrightarrow \bigoplus_{i=0}^{n} S_{X_i} \longrightarrow \bigoplus_{0\le i<j\le n} S_{X_iX_j}.$$

The element $(t_i)_{i=0}^n$ lies in the kernel of the map to $\bigoplus S_{X_iX_j}$ because the sections agree on overlaps. By the exactness of the sequence at the middle term, there exists a unique element $T \in S$ that maps to $(t_i)_{i=0}^n$. Thus, $\alpha(T) = t$, proving surjectivity. $\square$

Definition 9.3. Let X be a scheme and let $\mathcal{L}$ be a line bundle (invertible sheaf) on X. Let $f \in \Gamma(X, \mathcal{L})$ be a global section. We define the open set X_f as the locus where the section does not vanish:

$$X_f := \{x \in X \mid \text{the image of } f \text{ in } \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \text{ is nonzero}\}.$$

Here, $\kappa(x)$ is the residue field at x, and the map to the fiber is induced by the natural map $i_x : \operatorname{Spec} \kappa(x) \to X$.

Example 9.4. If $X = \operatorname{Spec} A$ is an affine scheme and $\mathcal{L} \cong \mathcal{O}_X$ is the trivial line bundle, then a global section corresponds to an element $f \in A$. In this case, the set X_f is exactly the distinguished open set $D(f)$.

In the affine case, let $X = \operatorname{Spec} A$. The inclusion of a point $i_x : \operatorname{Spec} \kappa(x) \to X$ corresponds to the canonical map to the residue field:

$$A \longrightarrow A_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong \kappa(\mathfrak{p}),$$

where $\mathfrak{p}$ is the prime ideal corresponding to x. Under this map, the condition that the image $\bar{f} \neq 0$ in $\kappa(\mathfrak{p})$ is equivalent to $f \notin \mathfrak{p}$.

Globally, we can show that X_f is an open set. Let $X = \bigcup X_i$ be an open cover where each X_i is affine and the line bundle trivializes locally, i.e., $\mathcal{L}|_{X_i} \cong \mathcal{O}_{X_i}$. Under this trivialization, the restriction of f corresponds to a function on X_i. Then we have:

$$X_f = \bigcup (X_i \cap X_f) = \bigcup D(f|_{X_i}).$$

Since the distinguished open set $D(f|_{X_i})$ is open in X_i (and thus in X), their union X_f is an open set of X.

The following lemma generalizes standard properties of localizations in commutative algebra to the setting of schemes and line bundles.

Lemma 9.5. Let X be a scheme and let $\mathcal{L}$ be a line bundle on X (i.e., $\mathcal{L} \in \mathrm{Pic}(X)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $f \in \Gamma(X, \mathcal{L})$.

(a) Suppose X is quasi-compact. Let $s \in \Gamma(X, \mathcal{F})$ be a global section such that its restriction to the non-vanishing locus is zero, i.e., $s|_{X_f} = 0$. Then there exists an integer $n > 0$ such that

$$f^n s = 0 \quad \text{in } \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}).$$

(b) Suppose X is quasi-compact and quasi-separated (qcqs). Let $t \in \Gamma(X_f, \mathcal{F})$ be a section defined on the open set X_f. Then there exists an integer $m > 0$ such that the section $f^m t$ extends to a global section in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$.

Proof of (a). Since X is quasi-compact, we can choose a finite open cover $X = \bigcup_{i=1}^N X_i$, where each X_i is affine and trivializes the line bundle, $\mathcal{L}|_{X_i} \cong \mathcal{O}_{X_i}$.

Identify the restriction $f|_{X_i}$ with an element $f_i \in \Gamma(X_i, \mathcal{O}_{X_i})$. The condition $s|_{X_f} = 0$ implies that s vanishes on $X_f \cap X_i = D(f_i)$. Using the affine version of this result (properties of localization of modules), for each i, there exists an integer n_i such that

$$f_i^{n_i} \cdot (s|_{X_i}) = 0 \quad \text{in } \Gamma(X_i, \mathcal{F} \otimes \mathcal{L}^{\otimes n_i}).$$

Since there are finitely many X_i, we can choose $n = \max\{n_1, \ldots, n_N\}$. Then

$$f^n s|_{X_i} = 0$$

for all i. Since the sections vanish on an open cover, the global section $f^n s$ is zero in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. $\square$

Proof of (b). We retain the notation from part (a). Let $X = \bigcup X_i$ be a finite affine open cover such that $\mathcal{L}|_{X_i} \cong \mathcal{O}_{X_i}$. Let $t \in \Gamma(X_f, \mathcal{F})$. On each affine piece X_i, the intersection $X_i \cap X_f$ is a distinguished open set $D(f|_{X_i})$ inside X_i. By the standard properties of affine schemes, there exists an integer $m_i' > 0$ such that the section $f^{m_i'} t|_{X_i \cap X_f}$ extends to a section $t_i \in \Gamma(X_i, \mathcal{F} \otimes \mathcal{L}^{\otimes m_i'})$. Since there are finitely many X_i, we can choose a single large integer $m' > 0$ working for all i. Thus, for each i, we have a section $t_i \in \Gamma(X_i, \mathcal{F} \otimes \mathcal{L}^{\otimes m'})$ extending $f^{m'} t$.

Now we consider the overlaps. Let $X_{ij} = X_i \cap X_j$. Since X is quasi-separated, the intersection of affine opens X_{ij} is quasi-compact. The two extensions t_i and t_j may not agree on X_{ij}, but they do agree on the open subset $X_{ij} \cap X_f$, where both are equal to $f^{m'}t$. Therefore, the section $(t_i - t_j)|_{X_{ij}}$ vanishes on $X_{ij} \cap X_f$. We can apply part (a) of the Lemma to the quasi-compact scheme X_{ij}. There exists an integer $m'' > 0$ such that:

$$f^{m''}(t_i|_{X_{ij}} - t_j|_{X_{ij}}) = 0 \quad \text{in } \Gamma(X_{ij}, \mathcal{F} \otimes \mathcal{L}^{\otimes(m'+m'')}).$$

By taking m'' large enough to work for all pairs i, j, and setting $m = m' + m''$, the local sections $f^{m''}t_i$ glue to form a global section in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$ which extends $f^m t$. □

9.2 Graded S-modules vs. Quasi-coherent Sheaves on $\operatorname{Proj} S$

Let S be a graded ring and $X = \operatorname{Proj} S$. We investigate the relationship between graded S-modules and quasi-coherent sheaves on X.

We define the natural maps relating the functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$.

Definition 9.6 (The Unit Map α)**.** For any graded S-module M, we define a morphism of graded S-modules:

$$\alpha : M \longrightarrow \Gamma_*(\widetilde{M}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \widetilde{M}(n)).$$

For a homogeneous element $x \in M_n$, we define $\alpha(x)$ to be the global section of $\widetilde{M}(n)$ given locally by the fraction $x/1$. Specifically, at a point $\mathfrak{p} \in \operatorname{Proj} S$, the germ is:

$$x \longmapsto \left(\frac{x}{1} \in (\widetilde{M}(n))_{\mathfrak{p}} \cong M(n)_{(\mathfrak{p})} \right)_{\mathfrak{p} \in X}.$$

Definition 9.7 (The Counit Map β)**.** For any quasi-coherent sheaf $\mathcal{F}$ on X, we define a morphism of sheaves:

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}.$$

We define this map locally on the distinguished open sets $D_+(f)$ for homogeneous $f \in S_d$. Recall that the sections of $\widetilde{\Gamma_*(\mathcal{F})}$ on $D_+(f)$ are given by degree 0 fractions of the form x/f^n, where $x \in \Gamma_*(\mathcal{F})_{nd} = \Gamma(X, \mathcal{F}(nd))$. We define the map $\beta|_{D_+(f)}$ by:

$$\frac{x}{f^n} \longmapsto \frac{x|_{D_+(f)}}{\alpha_{nd}(f^n)} \in \Gamma(D_+(f), \mathcal{F}),$$

where we interpret the division by f^n via the isomorphism $\mathcal{F}(nd)|_{D_+(f)} \cong \mathcal{F}|_{D_+(f)}$ induced by the trivialization of $\mathcal{O}_X(nd)$ on $D_+(f)$.
One verifies that these local maps are compatible on intersections. Specifically,

$$\beta|_{D_+(f)}|_{D_+(fg)} = \beta|_{D_+(g)}|_{D_+(fg)}.$$

Thus, the local morphisms glue to a global morphism of $\mathcal{O}_X$-modules $\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$.

From now on, we assume that S is generated as an S_0-algebra by finitely many elements in S_1.

Remark. *Under this assumption, there is a surjective homomorphism of graded S_0-algebras from a polynomial ring to S:*

$$S_0[T_0, \dots, T_n] \longrightarrow S, \quad T_i \longmapsto f_i,$$

where $f_0, \dots, f_n$ are the generators of S_1. Geometrically, this surjection induces a closed immersion of schemes:

$$\operatorname{Proj} S \hookrightarrow \mathbb{P}^n_{S_0}.$$

We now prove that the counit map β constructed in the previous section is an isomorphism. This result is crucial for establishing the equivalence of categories.

Lemma 9.8. Let S be a graded ring generated by S_1 over S_0, and let $X = \operatorname{Proj} S$. For any quasi-coherent sheaf $\mathcal{F}$ on X, the morphism

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$$

is an isomorphism of $\mathcal{O}_X$-modules.

Proof. It suffices to show that the restriction of β to the basic open sets $D_+(f)$ is an isomorphism for every $f \in S_d$ $(d > 0)$. Let $X_f = D_+(f)$ and let $\mathcal{L} = \mathcal{O}_X(1)$.

Recall that the sections of $\widetilde{\Gamma_*(\mathcal{F})}$ over $D_+(f)$ are given by the localization $(\Gamma_*(\mathcal{F}))_{(f)}$. An element in this localization is of the form x/f^n, where $x \in \Gamma(X, \mathcal{F}(nd))$. The map β sends x/f^n to the restriction $x|_{X_f}$ (viewed as a section of $\mathcal{F}$ via the isomorphism $\mathcal{F}(nd)|_{X_f} \cong \mathcal{F}|_{X_f}$).

Injectivity: Suppose $\beta(x/f^n) = 0$. This means that the restriction of the global section x to the open set X_f is zero. By the "Annihilation Lemma" (part (a) of the previous Lemma), there exists an integer $N \gg 0$ such that

$$f^N \cdot x = 0 \quad \text{in } \Gamma(X, \mathcal{F}(nd + Nd)).$$

In the localized module $(\Gamma_*(\mathcal{F}))_{(f)}$, the element x/f^n is equal to $(f^N x)/f^{n+N}$. Since the numerator is zero, the fraction is zero. Thus, the kernel of $\beta|_{D_+(f)}$ is trivial.

Surjectivity: Let $t \in \Gamma(X_f, \mathcal{F})$. By the "Extension Lemma" (part (b) of the previous Lemma), there exists an integer $k > 0$ such that the section $f^k t$ extends to a global section $\tilde{t} \in \Gamma(X, \mathcal{F}(kd))$. Consider the element $\tilde{t}/f^k$ in the localized module $(\Gamma_*(\mathcal{F}))_{(f)}$. The map β sends this element to:

$$\beta\left(\frac{\tilde{t}}{f^k}\right) = \frac{\tilde{t}|_{X_f}}{f^k|_{X_f}} = \frac{f^k t}{f^k} = t.$$

Thus, $\beta|_{D_+(f)}$ is surjective. □

While every quasi-coherent sheaf comes from a graded module (namely $\Gamma_*(\mathcal{F})$), not every graded module corresponds perfectly to its associated sheaf via the α map. We isolate the "good" modules.

Definition 9.9. A graded S-module M is called **saturated** if the canonical map

$$\alpha : M \longrightarrow \Gamma_*(\widetilde{M})$$

is an isomorphism.

Let gr- S -mod$^{\text{sat}}$ denote the full subcategory of gr- S -mod consisting of saturated modules.

Theorem 9.10. The functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$ induce an equivalence of categories:

$$\text{gr-}\,S\text{-mod}^{\text{sat}} \overset{\sim}{\longleftrightarrow} \mathrm{QCoh}(\mathcal{O}_X).$$

Proof. To prove an equivalence of categories, we must show that the unit and counit of the adjunction are isomorphisms.

1. We have already proven in the Lemma above that the counit map $\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{O}_X)$.

2. Now, let $M \in \text{gr-}\,S\text{-mod}^{\text{sat}}$. By definition, the unit map $\alpha : M \to \Gamma_*(\widetilde{M})$ is an isomorphism.

It remains to check consistency. Let $M = \Gamma_*(\mathcal{F})$ for some sheaf $\mathcal{F}$. We check that this module is saturated. Consider the composition of natural maps:

$$\Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathcal{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F}).$$

This composition is the identity map on $\Gamma_*(\mathcal{F})$. Since β is an isomorphism (by the Lemma), the induced map on global sections $\Gamma_*(\beta)$ is also an isomorphism. Therefore, α must be an isomorphism. This implies that for any quasi-coherent sheaf $\mathcal{F}$, the module of sections $\Gamma_*(\mathcal{F})$ is saturated. □

9.3 Closed Subschemes of $\operatorname{Proj} S$

We now apply the correspondence between graded modules and sheaves to classify closed subschemes of projective spectra.

Lemma 9.11. Let S be a graded ring and let $X = \operatorname{Proj} S$. Let M be a graded S-module and let $\mathcal{F}$ be a quasi-coherent subsheaf of $\widetilde{M}$ (i.e., $\mathcal{F} \subseteq \widetilde{M}$). Then there exists a graded submodule $N \subseteq M$ such that $\widetilde{N} \cong \mathcal{F}$. Moreover, if M is a saturated module, then we can choose N to be saturated.

Proof. Consider the canonical map $\alpha : M \to \Gamma_*(\widetilde{M})$. We are given a subsheaf $\mathcal{F} \subseteq \widetilde{M}$. Taking global sections yields a submodule $\Gamma_*(\mathcal{F}) \subseteq \Gamma_*(\widetilde{M})$. We define N to be the preimage of this submodule under α:

$$N := \alpha^{-1}(\Gamma_*(\mathcal{F})) \subseteq M.$$

Applying the exact functor $M \mapsto \widetilde{M}$ to the inclusion $N \hookrightarrow M$, we obtain a commutative diagram of sheaves:

$$\begin{tikzcd} \widetilde{N} \arrow[r, hook] \arrow[d, two heads] & \widetilde{M} \arrow[d, equal] \\ \mathcal{F} \arrow[r, hook] & \widetilde{M} \end{tikzcd}$$

The surjective vertical map on the left arises from $\widetilde{N} \to \widetilde{\Gamma_*(\mathcal{F})} \simeq \mathcal{F} \to 0$ exact. Consequently, $\widetilde{N} \cong \mathcal{F}$.

Now, suppose that M is saturated. By definition, the map $\alpha : M \to \Gamma_*(\widetilde{M})$ is an isomorphism. In this case, the definition of N simplifies to $N \cong \Gamma_*(\mathcal{F})$. We know from the previous theorem that for any quasi-coherent sheaf $\mathcal{F}$, the module of sections $\Gamma_*(\mathcal{F})$ is saturated. Therefore, N is saturated. □

Proposition 9.12. Let S be a graded ring and let $Z \subseteq \operatorname{Proj} S$ be a closed subscheme. Then there exists a homogeneous ideal $I_Z \subseteq S$ with S_+ not contained in I_Z such that

$$Z \cong \operatorname{Proj}(S/I_Z).$$

Furthermore, if S is saturated as a module over itself, then there exists a unique saturated ideal I_Z such that $Z \cong \operatorname{Proj}(S/I_Z)$.

Proof. A closed subscheme Z is determined by a quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, where $X = \operatorname{Proj} S$. We apply the previous Lemma with $M = S$ and the subsheaf $\mathcal{F} = \mathcal{I} \subseteq \widetilde{S} = \mathcal{O}_X$. By the Lemma, there exists a homogeneous ideal $I_Z \subseteq S$ such that $\widetilde{I_Z} \cong \mathcal{I}$.

The structure sheaf of the closed subscheme is given by $\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{I}$. Using the exactness of the tilde functor, we have:

$$\mathcal{O}_Z \cong \widetilde{S}/\widetilde{I_Z} \cong \widetilde{S/I_Z}.$$

Therefore, as schemes, $Z \cong \operatorname{Proj}(S/I_Z)$.

If S is saturated, the Lemma guarantees that we can choose I_Z to be a saturated submodule of S (which is a saturated ideal). Uniqueness follows from the equivalence of categories between saturated modules and quasi-coherent sheaves: the ideal sheaf $\mathcal{I}$ corresponds uniquely to the saturated ideal $\Gamma_*(\mathcal{I})$. □

Remark (Explicit Construction of the Ideal)**.** *Suppose explicitly that S is a saturated S-module and is generated as an S_0-algebra by a finite set of elements $\{f_0, \dots, f_n\} \subseteq S_1$. Let $Z \subseteq \operatorname{Proj} S$ be a closed subscheme defined by the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\operatorname{Proj} S}$. Let $\mathcal{I}_i$ denote the restriction of the ideal sheaf to the distinguished open set $D_+(f_i)$, i.e., $\mathcal{I}_i = \mathcal{I}|_{D_+(f_i)}$.*

The homogeneous components $I_{Z,d}$ of the saturated ideal I_Z corresponding to Z can be described explicitly. An element $x \in S_d$ belongs to $I_{Z,d}$ if and only if its localizations fall into the restricted ideal sheaf:

$$I_{Z,d} := \left\{ x \in S_d \;\middle|\; \frac{x}{f_i^d} \in \Gamma(D_+(f_i), \mathcal{I}_i) \subseteq S_{(f_i)} \text{ for all } i \right\}.$$

Example 9.13 (Saturation in Polynomial Rings)**.** Consider the case where $S = R[X_0, \dots, X_n]$ is the polynomial ring with the standard grading. We know that S is saturated as a module over itself. Suppose a closed subscheme Z is given by $\operatorname{Proj}(S/J)$ for some homogeneous ideal $J \subseteq S$. Then the unique *saturated* ideal I defining Z (such that $Z \cong \operatorname{Proj}(S/I)$ and I corresponds to $\Gamma_*(\mathcal{I}_Z)$) is given by the saturation of J with respect to the irrelevant ideal. Explicitly:

$$J^{\text{sat}} := \left\{ a \in S \;\middle|\; \exists N > 0 \text{ such that } \forall i \in \{0, \dots, n\}, X_i^N a \in J \right\}.$$

Example 9.14 (Hypersurfaces in $\mathbb{P}^n_k$)**.** Consider the case of projective space $X = \mathbb{P}^n_k$ where k is a field. Let $Z \subseteq \mathbb{P}^n_k$ be a closed subscheme defined by the ideal sheaf $\mathcal{I}$. The degree d component of the associated saturated ideal is given by global sections:

$$I_d = \Gamma(\mathbb{P}^n_k, \mathcal{I}(d)).$$

This vector space consists of homogeneous polynomials of degree d that vanish on Z. We have a one-to-one correspondence between the set of hypersurfaces of degree d containing Z and the projectivization of this vector space:

$$\left\{\begin{matrix}\text{Hypersurfaces of degree } d \\ \text{that contain } Z\end{matrix}\right\} \xleftrightarrow{1:1} \frac{\Gamma(\mathbb{P}^n_k, \mathcal{I}(d)) \setminus \{0\}}{k^*}.$$

9.4 Generation by Global Sections

Proposition 9.15 (Serre's Theorem A)**.** Let S be a graded ring generated as an S_0-algebra by finitely many elements in S_1. Let $X = \operatorname{Proj} S$. Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on X. Then there exists an integer $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$, the twisted sheaf $\mathcal{F}(n)$ is generated by global sections. In other words, there exists a surjective morphism

$$\mathcal{O}_X^{\oplus k} \longrightarrow \mathcal{F}(n)$$

for some integer $k > 0$.

Corollary 9.16. Any finitely generated quasi-coherent sheaf $\mathcal{F}$ on $\operatorname{Proj} S$ is a quotient of a direct sum of twisted structure sheaves. That is, $\mathcal{F}$ is a quotient of a sheaf of the form $\mathcal{E} = \bigoplus_{i=1}^N \mathcal{O}_X(q_i)$.

***Proof of Proposition*.** Since S is finitely generated as an S_0-algebra by elements $\{f_1, \dots, f_m\} \in S_1$, the scheme X is covered by the affine open sets $X_{f_i} = D_+(f_i)$:

$$X = \bigcup_{i=1}^m X_{f_i}.$$

Since $\mathcal{F}$ is finitely generated, for each i, the restriction $\mathcal{F}|_{X_{f_i}}$ is generated by a finite number of sections. Let $\{t_{ij}\}_j$ be a finite set of sections in $\Gamma(X_{f_i}, \mathcal{F})$ that generate $\mathcal{F}|_{X_{f_i}}$.

We apply the "Extension Lemma" (part (b) from the previous lemma). There exists an integer $N_0 \in \mathbb{Z}$ such that for all $n \geq N_0$, the sections $f_i^n \cdot t_{ij}$ extend to global sections of the twisted sheaf. Let

$$\tilde{t}_{ij} \in \Gamma(X, \mathcal{F}(n))$$

be these extensions. The collection of global sections $\{\tilde{t}_{ij}\}_{i,j}$ generates the sheaf $\mathcal{F}(n)$ at every point, because their restrictions generate the sheaf on the open cover. Each global section defines a morphism $\mathcal{O}_X \to \mathcal{F}(n)$. Taking the direct sum over all such sections, we obtain a surjection:

$$\mathcal{O}_X^{\oplus k} \longrightarrow \mathcal{F}(n).$$

□

9.5 Projective Pushforward and Coherent Sheaves

Theorem 9.17. Let A be a Noetherian ring. Let X be a closed subscheme of $\mathbb{P}^N_A$ and let $f : X \to Y = \operatorname{Spec} A$ be the structure morphism (which is projective). Then for any coherent sheaf $\mathcal{F}$ on X (i.e., $\mathcal{F} \in \operatorname{Coh}(X)$), the direct image sheaf is coherent on Y:

$$f_*\mathcal{F} \in \operatorname{Coh}(\mathcal{O}_Y).$$

(This means $f_*\mathcal{F}$ corresponds to a finitely generated A-module).

Remark. *In today's lecture, we will prove this for the case where A is a finitely generated k-algebra, where k is a field.*

***Proof*.** It suffices to prove the theorem for the case $X = \mathbb{P}^n_A$. This is because the morphism f factors as a closed immersion $i : X \hookrightarrow \mathbb{P}^n_A$ followed by the projection $p : \mathbb{P}^n_A \to \operatorname{Spec} A$. We have $f_* = p_* \circ i_*$. Since i is a closed immersion (a finite morphism), $i_*\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_A$ if $\mathcal{F}$ is coherent on X. Thus, we can replace $\mathcal{F}$ with $i_*\mathcal{F}$ and work on the projective space.

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_A$. We know that $\mathcal{F} \cong \widetilde{M}$ for some graded module M.

Claim: We can choose the graded module M to be finitely generated over the polynomial ring $S = A[x_0, \ldots, x_n]$.

In general, any graded module M is the direct limit of its finitely generated graded submodules:

$$M = \varinjlim M_i,$$

where M_i are finitely generated graded submodules of M.

Since the sheafification functor commutes with direct limits, we have:

$$\mathcal{F} = \varinjlim \widetilde{M_i}.$$

Since $\mathcal{F}$ is a coherent sheaf on a Noetherian scheme, it is a Noetherian object in the category of sheaves. Therefore, the ascending chain of subsheaves stabilizes. There exists an index j such that $\widetilde{M_i} = \mathcal{F}$ for all $i \geq j$. Thus, we can assume $\mathcal{F} = \widetilde{M}$ where M is a finitely generated graded S-module.

Consider a short exact sequence of graded S-modules:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Applying the tilde functor and then the global sections functor $\Gamma(X, -)$, we obtain a left-exact sequence of A-modules:

$$0 \longrightarrow \Gamma(X, \widetilde{A}) \longrightarrow \Gamma(X, \widetilde{B}) \longrightarrow \Gamma(X, \widetilde{C}).$$

If $\Gamma(X, \widetilde{A})$ and $\Gamma(X, \widetilde{C})$ are finitely generated A-modules, then $\Gamma(X, \widetilde{B})$ is also a finitely generated A-module (since A is Noetherian).

Since M is a finitely generated graded module over the Noetherian ring $S = A[x_0, \ldots, x_n]$, there exists a filtration of graded submodules:

$$0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^m = M,$$

such that each quotient M^{i+1}/M^i is a cyclic graded S-module, isomorphic to a shifted quotient of the ring, $\widetilde{(S/I)}(d)$. By the induction argument on the filtration above, it is enough to show that $\Gamma(X, \widetilde{M^{i+1}/M^i})$ is a finitely generated A-module. Thus, we reduce the problem to proving the statement for sheaves of the form $\widetilde{(S/I)}(n)$.

Let $I = \bigcap_{j=1}^k \mathfrak{q}_j$ be a primary decomposition of the ideal I. Applying a similar filtration argument based on the primary decomposition, the problem boils down to the case where the module is of the form $S/\mathfrak{p}(n)$ for a homogeneous prime ideal $\mathfrak{p}$. This reduction technique is known as **Grothendieck's Dévissage**.

We redefine $S := S/\mathfrak{p}$. Now S is a graded integral domain which is a finitely generated A-algebra. We would like to show that $\Gamma(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S}(m))$ is a finitely generated S_0-module (where S_0 is a quotient of A).

Since S is an integral domain, multiplication by a non-zero element $x \in S_1$ induces an injective homomorphism of sheaves $\mathcal{O}_X(m) \to \mathcal{O}_X(m+1)$, and consequently an injective map on global sections:

$$\Gamma(X, \mathcal{O}_X(m)) \longrightarrow \Gamma(X, \mathcal{O}_X(m+1)).$$

It suffices to show that $\Gamma(X, \mathcal{O}_X(n))$ is finitely generated for n sufficiently large.

Definition 9.18. Let $S' = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$.

Note that S' is a graded S-algebra. For each i, the restriction of a global section to the affine open set $D_+(X_i)$ lies in $S_{(X_i)}$. Therefore, S' is contained in the intersection of these localizations inside the fraction field of S:

$$S' \subseteq \bigcap_{i=0}^{n} S_{X_i} \subseteq \mathrm{Frac}(S).$$

Claim: S' is integral over S.

Proof of Claim. Let $s' \in S'_d$ be a homogeneous element of degree d. Since $s' \in S_{X_i}$, there exists an integer n_i such that $X_i^{n_i} s' \in S$. Since there are finitely many generators X_i, we can find a uniform integer N_0 such that for all $n \geq N_0$ and for all i, we have $X_i^n s' \in S$. This implies that for all $n \geq N_0$, and for any monomial $y \in S_n$ of degree n, the product $y \cdot s'$ lies in S. In particular, take $y = X_0^n$. Then $s' \in X_0^{-n} S$. This shows that s' is contained in a finitely generated S-submodule of $\mathrm{Frac}(S)$ (namely $X_0^{-n} S$). A standard result in commutative algebra states that if an element is contained in a finitely generated submodule of the fraction field, it is integral over the ring. Therefore, s' is integral over S. Since s' was arbitrary, S' is integral over S. □

Since S is a finitely generated A-algebra and A is a finitely generated k-algebra (where k is a field), it follows that S is a finitely generated k-algebra. We invoke the following finiteness theorem from commutative algebra (related to the Noether Normalization Lemma and finiteness of integral closure):

Theorem 9.19 (Finiteness of Integral Closure)**.** Let S be a finitely generated algebra over a field k which is an integral domain. Let L be a finite field extension of $\mathrm{Frac}(S)$. If R is an S-subalgebra of L such that R is integral over S, then R is a finitely generated S-module.

Applying this theorem to the inclusion $S \subseteq S' \subseteq \mathrm{Frac}(S)$ (with $L = \mathrm{Frac}(S)$), we conclude that S' is a finitely generated S-algebra. Moreover, since it is integral and finitely generated as an algebra, it is finitely generated as an S-module. Since S is Noetherian, the graded components S'_n are finitely generated S_0-modules for all n. This proves that $\Gamma(X, \mathcal{O}_X(n))$ is a finitely generated A-module for all n.

□

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Lecture 12. Divisors

10.1 Degree of rational functions

In this section:

- A: Noetherian ring of dimension 1.
- R: multiplicative subset of **regular** elements in A. (Regular = not zero-divisor).
- $K = R^{-1}A$. $K^* \subseteq K$ invertible elements in K. This procedure generalizes fraction field (when A is a domain).

Proposition 10.1. There exists a map $e_A : A\text{-mod}_{\text{f. g.}} \times K^* \longrightarrow \mathbb{Z}$.

$$(M, f) \longmapsto e_A(M, f).$$

uniquely determined by

(1) $e_A(M, fg) = e_A(M, f) + e_A(M, g)$.

(2) For all short exact sequences $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, have:

$$e_A(M, f) = e_A(M', f) + e_A(M'', f).$$

(3) If $l_A(M) < +\infty$, then $e_A(M, f) = 0$ for all $f \in K^*$. (where $l_A(M)$ is the length of M as A-module).

(0) $e_A(A, f) = l_A(A/fA)$ for all $f \in R$.

Remark: $e_A(M, f)$ is a special case of Herbrand quotient.

Sketch of proof. For $f \in R$, we define

$$e_A(M, f) := l_A(M/fM) - l_A(\mathrm{Ann}(M, f)).$$

Consider the exact sequence:

$$0 \longrightarrow \mathrm{Ann}(M, f) \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0.$$

We verify that $l_A(\mathrm{Ann}(M, f)) < \infty$ and $l_A(M/fM) < \infty$ (using the **fact** that $\dim A = 1$).

(1) **Notation:** Let ${}_fM := \operatorname{Ann}(M, f)$ and $M_f := M/fM$. Consider the following long exact sequence:

$$0 \longrightarrow {}_fM \longrightarrow {}_{gf}M \longrightarrow {}_gM \xrightarrow{\delta} M_f \longrightarrow M_{gf} \longrightarrow M_g \longrightarrow 0.$$

From the additivity of length, we deduce the required equality.

(2) Consider a short exact sequence $0 \to M' \to M \to M'' \to 0$. We have the following commutative diagram with exact rows and columns (Snake Lemma):

$$\begin{tikzcd}
0 \arrow[r] & {}_fM' \arrow[r] \arrow[d] & {}_fM \arrow[r] \arrow[d] & {}_fM'' \arrow[d] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
& M'_f \arrow[r] & M_f \arrow[r] & M''_f \arrow[r] & 0
\end{tikzcd}$$

The sequence of kernels and cokernels (highlighted in red) is exact. The additivity of $e_A(M, f)$ follows from the additivity of length in this long exact sequence.

(3) If $l_A(M) < \infty$. There exists a composition series $0 = M^0 \subsetneq M^1 \subsetneq \cdots \subsetneq M^n = M$ such that each quotient M^{i+1}/M^i is a simple A-module. This implies $e_A(M^{i+1}/M^i, f) = 0$. By property (2), we have:

$$e_A(M, f) = \sum_{i=0}^{n-1} e_A(M^{i+1}/M^i, f) = 0.$$

Remark: We haven't proved uniqueness yet. □

Definition 10.2. Let A be a local Noetherian ring of dimension 1. For $f \in K^*$, we define the **degree** of f to be

$$\deg_A(f) := e_A(A, f).$$

When $f \in R$ (the set of regular elements), we have $\deg_A(f) = l_A(A/fA)$. In the general case, if $f = \frac{g}{h}$ with $g, h \in R$, then $\deg_A(f) = \deg_A(g) - \deg_A(h)$.

Remark: If A is a DVR, then $\deg_A(f) = v_A(f)$, where $v_A : K^* \to \mathbb{Z}$ is the valuation of A.

Proposition 10.3 (**Projection formula**)**.** Let A be a local Noetherian domain of dimension 1. Let $A \to B$ be an integral map such that $\mathrm{Frac}(B) = L$ is a finite extension of $\mathrm{Frac}(A) = K$. Then (by the **Krull-Akizuki Theorem**), B is a semi-local Noetherian domain of dimension 1 (i.e., it has finitely many maximal ideals).
Let $\mathrm{Nm}_{L/K} : L^* \to K^*$ be the norm map defined by

$$g \in L \longmapsto \det_K(g\cdot : L \to L) \in K$$

(the determinant of the multiplication map on the K-vector space L).
Then for any $g \in L^*$, we have:

$$\deg_A(\mathrm{Nm}_{L/K}(g)) = \sum_{\substack{\mathfrak{p} \subset B \\ \text{max ideals}}} \deg_{B_\mathfrak{p}}(g) \cdot [B/\mathfrak{p} : A/\mathfrak{m}],$$

where $\mathfrak{m}$ is the maximal ideal of A.

Skip the proof for now.

10.2 Weil divisors

Assume: X is an integral Noetherian scheme.

Definition 10.4. A **prime divisor** is a codimension 1 integral subscheme Y of X. This is equivalent to saying that $\dim \mathcal{O}_{X,\eta} = 1$, where η is the generic point of Y.
A **Weil divisor** is an element of the free abelian group generated by prime divisors. It is a formal sum:

$$D = \sum n_i Y_i \quad (n_i \in \mathbb{Z}),$$

where the sum runs over the set of prime divisors. **Notation:** $\mathrm{Div}(X) := \mathbb{Z}^{\{\text{prime divisors}\}}$.
A divisor D is called **effective** if $n_i \geq 0$ for all i.

Definition 10.5. Let $f \in K(X)^* = \mathcal{O}_{X,\eta}^*$, where η is the generic point of X. The **degree** of f at a prime divisor Y (with generic point y) is defined as:

$$\deg_Y(f) := \deg_{\mathcal{O}_{X,\eta}}(f).$$

Lemma 10.6. For any $f \in K(X)^*$, there are only finitely many prime divisors Y such that $\deg_Y(f) \neq 0$.

Proof. **Claim:** There exists an open subset $U \subseteq X$ such that $f \in \mathcal{O}_X(U)^*$. In fact, take an affine open subset $V \subseteq X$, with $V \cong \operatorname{Spec} A$. Then f can be written as $f = \frac{g}{h}$ with $g, h \in A$. Then take $U = D(hg)$.

Then for any $y \in U$ with $\dim \mathcal{O}_{X,y} = 1$, we have $\deg_{\mathcal{O}_{X,y}}(f) = 0$ because the multiplication map $\mathcal{O}_{X,y} \xrightarrow{f} \mathcal{O}_{X,y}$ is an isomorphism (since f is a unit in $\mathcal{O}_{X,y}$).

Notice that $X \setminus U$ is Noetherian. In particular, it has finitely many irreducible components. Since any prime divisor Y with $\deg_Y(f) \neq 0$ must be an irreducible component of $X \setminus U$, there are only finitely many such divisors. □

Definition 10.7. For any $f \in K(X)^*$, we define the **principal divisor** associated to f as:

$$(f) := \sum_Y -\deg_Y(f) \cdot Y.$$

This sum is finite by the previous Lemma.

Remark: Since $(f/g) = (f) - (g)$, the map $f \mapsto (f)$ defines a group homomorphism:

$$K(X)^* \longrightarrow \operatorname{Div}(X).$$

Definition 10.8. Two divisors $D, D' \in \operatorname{Div}(X)$ are called **linearly equivalent**, denoted by $D \sim D'$, if $D - D'$ lies in the image of the map $K(X)^* \to \operatorname{Div}(X)$. In other words, $D - D' = (f)$ for some $f \in K(X)^*$.
We define the **Weil divisor class group** of X to be the quotient group:

$$\operatorname{Cl}(X) := \operatorname{Div}(X)/\sim.$$

Proposition 10.9. Let A be a Noetherian domain. Then A is a UFD if and only if $X = \operatorname{Spec} A$ is normal and $\operatorname{Cl}(X) = 0$.

Proof. It is well known in [AM] that if A is a UFD, then A is integrally closed. We also know that A is a UFD if and only if A is integrally closed and every prime ideal of height 1 is principal [AM].

We now show that if A is integrally closed, then every prime ideal of height 1 being principal is equivalent to $\mathrm{Cl}(X) = 0$.

($\Rightarrow$): Trivial. Every prime divisor Y of X corresponds to a prime ideal $\mathfrak{p}$ of height 1. If $\mathfrak{p} = (f)$ for some $f \in A$, then $Y = V(f) = (f)$ (as a principal divisor), i.e., $Y \sim 0$.

($\Leftarrow$): Let $\mathfrak{p} \in X$ with $\mathrm{ht}(\mathfrak{p}) = 1$. Then $V(\mathfrak{p}) \sim 0$, so $1 \cdot V(\mathfrak{p}) = (f)$ for some $f \in K^*$. Then for all $\mathfrak{q} \in X$ with $\mathrm{ht}(\mathfrak{q}) = 1$, we have $\deg_{\mathfrak{q}}(f) \geq 0$ (since the coefficient of $\mathfrak{q}$ in the divisor (f) is 1 if $\mathfrak{q} = \mathfrak{p}$ and 0 otherwise). Therefore, $f \in A_{\mathfrak{q}}$.

Proposition 10.10. If A is a Noetherian integrally closed domain, then

$$A = \bigcap_{\mathrm{ht}(\mathfrak{q})=1} A_{\mathfrak{q}}.$$

Therefore, $f \in A$. $\square$

Example 10.11. $\mathrm{Cl}(\mathbb{A}^n_k) = 0$, since $k[x_1, \ldots, x_n]$ is a UFD.

Question: How to compute $\mathrm{Cl}(\mathbb{P}^n_k) =$?

Proposition 10.12. Let $X = \mathbb{P}^n_k$. Then the map

$$\deg : D = \sum n_i Y_i \longmapsto \sum n_i \deg Y_i$$

(where $\deg Y_i$ is the degree of Y_i as a hypersurface) induces an isomorphism:

$$\mathrm{Cl}(X) \cong \mathbb{Z}.$$

More precisely:

(1) If $\deg D = d$, then $D \sim dH$, where $H = V_+(X_0)$.

(2) If $f \in K(X)^*$, then $\deg((f)) = 0$.

Proof of (2). Every $f \in K(X)^*$ can be written as $f = \frac{g}{h}$, where $g, h \in k[X_0, X_1, \ldots, X_n]$ are homogeneous polynomials of the same degree. Let $g = \prod g_i^{n_i}$ be the irreducible decomposition of g. Then $V_+(g_i)$ is a prime divisor. Similarly, let $h = \prod h_j^{m_j}$, where each $V_+(h_j)$ is a prime divisor. Then the principal divisor is given by (using the definition from the previous section):

$$(f) = -\sum n_i V_+(g_i) + \sum m_j V_+(h_j).$$

This implies:

$$\begin{aligned}\deg((f)) &= -\sum n_i \deg(g_i) + \sum m_j \deg(h_j) \\ &= -\deg(g) + \deg(h) \\ &= 0.\end{aligned}$$

□

Proof of (1). If $\deg D = d$, then let us write $D = D_1 - D_2$, where D_1, D_2 are effective divisors. Say $D_1 = \sum \alpha_i Z_i$ and $D_2 = \sum \beta_j W_j$. We can write them as $D_1 = \sum \alpha_i V_+(z_i)$ and $D_2 = \sum \beta_j V_+(w_j)$, where z_i, w_j are irreducible homogeneous polynomials. We have $d = \deg D_1 - \deg D_2$. Then

$$\begin{aligned}D - dH &= \sum \alpha_i V_+(z_i) - \Big(\sum \beta_j V_+(w_j) + d \sum V_+(X_0)\Big) \\ &= \left(\frac{X_0^d \prod w_j^{\beta_j}}{\prod z_i^{\alpha_i}}\right).\end{aligned}$$

The term inside the parenthesis is in $K(X)^*$. Therefore, $D \sim dH$. □

10.3 Excision Sequence

Let X be a Noetherian integral scheme. Let $Z \subseteq X$ be a closed subset, and let $U = X \setminus Z$.

Proposition 10.13.

(a) There exists a surjective homomorphism $\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U)$, defined by

$$D = \sum n_i Y_i \longmapsto \sum n_i (Y_i \cap U),$$

where we ignore the prime divisors Y_i such that $Y_i \cap U = \emptyset$.

(b) If $\mathrm{codim}(Z, X) \geq 2$, then $\mathrm{Cl}(X) \cong \mathrm{Cl}(U)$.

(c) If Z is a prime divisor of X, then we have a short exact sequence:

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where the first map is defined by $1 \longmapsto [Z]$.

Proof.

(a) This is obvious.

(b) Since $\mathrm{Div}(X)$ and $\mathrm{Cl}(X)$ are defined using only codimension 1 subschemes, and $U = X \setminus Z$ where $\mathrm{codim}(Z, X) \geq 2$, the removal of Z does not affect the set of codimension 1 subschemes. Thus $\mathrm{Div}(X) \cong \mathrm{Div}(U)$, which implies $\mathrm{Cl}(X) \cong \mathrm{Cl}(U)$.

(c) Suppose $D \in \mathrm{Cl}(X)$ is in the kernel, i.e., $D|_U = (f)|_U$ for some $f \in K(U)^* = K(X)^*$. Then the divisor $D - (f)$ in $\mathrm{Cl}(X)$ restricts to 0 on U. This means its support is contained in $X \setminus U = Z$. Since Z is a prime divisor, we must have $D - (f) = m \cdot Z$ for some integer m. Therefore, the kernel of the map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is generated by the class $[Z]$.

□

Example 10.14. Let $Y \subseteq \mathbb{P}^n_k$ be a hypersurface of degree d. Then

$$\mathrm{Cl}(\mathbb{P}^n_k \setminus Y) \cong \mathbb{Z}/d\mathbb{Z}.$$

Proof. We have the following exact sequence:

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & \mathrm{Cl}(\mathbb{P}^n_k) & \longrightarrow & \mathrm{Cl}(\mathbb{P}^n_k \setminus Y) & \longrightarrow & 0 \\ \\ 1 & \longmapsto & [Y] = d[H] & & & & \end{array}$$

We know that $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$ is generated by the class of a hyperplane H, and since Y has degree d, we have $[Y] = d[H]$. Thus the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by d. The cokernel is $\mathbb{Z}/d\mathbb{Z}$. □

Example 10.15. Let $A = k[x, y, z]/(xy - z^2)$ and let $X = \operatorname{Spec} A$. Then we claim that

$$\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}.$$

X looks like the nilpotent cone of $\mathfrak{sl}_2$ (a quadratic cone).

Proof. First, let $Y = V(y, z) \subseteq X$. We have

$$Y \cong \operatorname{Spec} k[x] \cong \mathbb{A}^1_k.$$

Notice that Y is a prime divisor of X. We apply the excision sequence:

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X \setminus Y) \longrightarrow 0.$$

Set-theoretically, $Y = V(y)$ (since $y = 0 \implies z^2 = 0 \implies z = 0$ in A). Therefore, the open complement is

$$\begin{aligned} X \setminus Y &\cong \operatorname{Spec} k[x, y, y^{-1}, z]/(xy - z^2) \\ &\cong \operatorname{Spec} k[z, y, y^{-1}] \quad (\text{via } x = y^{-1}z^2). \end{aligned}$$

The ring $k[z, y, y^{-1}]$ is a UFD. This implies $\mathrm{Cl}(X \setminus Y) = 0$.

Notice that as divisors, $2 \cdot Y = (y)$. (Reason: In the local ring $\mathcal{O}_{X,Y}$, the element x becomes a unit. The relation $y = x^{-1}z^2$ implies that the maximal ideal is generated by the uniformizer z. Consequently, y differs from z^2 by a unit, meaning y corresponds to a quadratic term in the local parameter, i.e., $\mathrm{val}_Y(y) = 2$.)

Consequently, the map $\mathbb{Z} \to \mathrm{Cl}(X)$ sends $1 \mapsto [Y]$, and $2 \mapsto 2[Y] = [(y)] = 0$. Thus, we have a surjection:

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Cl}(X).$$

To prove that $\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$, we need to show that $[Y] \neq 0$. Suppose $[Y] = 0$, i.e., $Y = (f)$ for some $f \in K^*$. Notice that A is integrally closed. If $\mathrm{Cl}(X) = 0$, then A is a UFD. This implies that the prime ideal corresponding to Y, which is $\mathfrak{p} = (y, z)$, must be principal. So $(y, z) = (f)$ for some $f \in A$. Consider the maximal ideal $\mathfrak{m} = (x, y, z)$. If $(y, z) = (f)$, then $\mathfrak{m} = (x, f)$. This implies that the dimension of the cotangent space is

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq 2.$$

However, $\mathfrak{m}/\mathfrak{m}^2 = \mathrm{Span}_k\{\bar{x}, \bar{y}, \bar{z}\}$ has dimension 3 (since the relation $xy - z^2$ is in $\mathfrak{m}^2$). This is a contradiction! Therefore, $\mathrm{Cl}(X) \neq 0$, and we conclude $\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$. □

10.4 Thom Isomorphism

Let X be an integral Noetherian scheme and $\mathcal{E}$ be a locally free sheaf on X of rank n. Let $Y = \mathbb{V}_X(\mathcal{E}) \xrightarrow{\pi} X$ be the associated vector bundle.

Proposition 10.16. The map $\pi^* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ defined by

$$D = \sum n_i D_i \mapsto \sum n_i \pi^{-1}(D_i)$$

gives an isomorphism.

Proof. **Well-definedness:** Need to show:

- $Y = \mathbb{V}_X(\mathcal{E})$ is integral Noetherian scheme.

- If D is a prime divisor of X, then $\pi^{-1}(D)$ is a prime divisor of Y.

This is omitted in the lecture.

Injectivity: Suppose $D = \sum n_i D_i$ is such that $\sum n_i \pi^{-1}(D_i) = (f)$ for some $f \in K(Y)^*$. Let $Y_\eta = Y \times_X \{\eta\}$, where η is the generic point of X. Then for every prime ideal $\mathfrak{p}$ of $\mathcal{O}_{Y_\eta}(Y_\eta) \cong K(X)[X_1, \dots, X_n]$ of height 1, we have $\deg_{\mathfrak{p}}(f) = 0$. (Need details) It follows that f is invertible in $K(X)[X_1, \dots, X_n]$, and thus $f \in K(X)^*$. Therefore, $\sum n_i D_i = (f)$ in $\mathrm{Cl}(X)$.

Surjectivity: Suppose $W \subseteq Y$ is a prime divisor. There are two possibilities:

(a) $W_\eta := W \times_X \{\eta\}$ is empty, i.e., $\pi(W) \not\ni \eta$. Since π is a morphism of finite type of Noetherian schemes(Need details), by **Hartshorne Chapter II Exercise 3.19**, $\pi(W)$ is constructible. Then there exists an open subset $U \subseteq X$ such that $\pi(W) \subseteq X \setminus U$. W must be of the form $\pi^{-1}(Z)$ for some irreducible $Z \subseteq X \setminus U$ of codimension 1. (Note: π is flat with geometrically irreducible fibers). (Need details)

(b) (Need details) $W_\eta \neq \emptyset$. In this case, since $\mathrm{Cl}(Y_\eta) = 0$ (as $Y_\eta \cong \mathbb{A}^n_{K(X)}$), we have $W_\eta = (g)$ for some $g \in K(Y)^*$. Therefore, $W - (g) = \sum m_i W_i$ with each W_i of type (a).

This proves surjectivity. $\square$

Definition 10.17. Let $\bar{Y} := \mathbb{P}_X(\mathcal{E}) \xrightarrow{\bar{\pi}} X$ be the associated projective bundle.

Proposition 10.18. We have a short exact sequence:

$$0 \longrightarrow \mathrm{Cl}(X) \xrightarrow{\bar{\pi}^*} \mathrm{Cl}(\bar{Y}) \xrightarrow{\deg_\eta} \mathbb{Z} \longrightarrow 0.$$

Here, the first map is $D = \sum n_i D_i \longmapsto \sum n_i \bar{\pi}^{-1}(D_i)$, and the second map is $W \longmapsto \deg W_\eta$. Note that $\bar{Y}_\eta = \bar{Y} \times_X \eta \cong \mathbb{P}^{n-1}_{K(X)}$. $\deg W_\eta$ is the degree of the divisor W_η in $\mathbb{P}^{n-1}_{K(X)}$ which was defined in **Proposition 10.12**.

Sketch of proof. (Need a lot of details) The proof is similar to the previous one.

- **Injectivity of $\bar{\pi}^*$:** This is the same as the injectivity of π^*.
- **Surjectivity of $\deg_\eta$:** Take $H_\eta = V_+(X_0)$. Then the closure of H_η is a prime divisor of $\bar{Y}$ with degree 1.
- **Exactness:** Suppose $W \in \mathrm{Cl}(\bar{Y})$ such that $\deg W_\eta = 0$. Then $W_\eta = (h)$ for some $h \in K(\bar{Y})^*$ (since $\mathrm{Cl}(\mathbb{P}^{n-1}_{K(X)}) \cong \mathbb{Z}$ via degree). Using the same argument as above, $W - (h)$ in $\mathrm{Cl}(\bar{Y})$ has the form $\sum m_i W_i$ with $W_i \cap \bar{Y}_\eta = \emptyset$ for all i. This implies $W_i = \bar{\pi}^{-1}(D_i)$ for some prime divisor $D_i \subseteq X$.

□

Example 10.19. Let $Q = V_+(xy - zw) \subseteq \mathbb{P}^3_k$. Then $Q \cong \mathbb{P}^1_k \times_k \mathbb{P}^1_k$. We have $\mathrm{Cl}(Q) \cong \mathbb{Z}^2$.

Proof. Consider the vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ on $\mathbb{P}^1$. Then $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \cong \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p} \mathbb{P}^1$. (Need details) Using the exact sequence for the class group of a projective bundle:

$$0 \longrightarrow \mathrm{Cl}(\mathbb{P}^1) \xrightarrow{p^*} \mathrm{Cl}(Q) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Since $\mathrm{Cl}(\mathbb{P}^1) \cong \mathbb{Z}$, the sequence becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}(Q) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Since $\mathbb{Z}$ is a free module, the sequence splits, and $\mathrm{Cl}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

10.5 Push-forward along finite morphism

Let $\varphi : X \to Y$ be a finite morphism of Noetherian integral schemes.

Definition 10.20. We define the **push-forward homomorphism** $\varphi_* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ as follows: For an irreducible divisor $D \in \mathrm{Div}(X)$, let η be the generic point of D.

$$\varphi_*(D) := [\kappa(\eta) : \kappa(\varphi(\eta))] \cdot \varphi(D).$$

(Note: Since φ is finite, $\varphi(D)$ is a prime divisor in Y). (Need details)

Proposition 10.21. φ_* descends to a homomorphism:

$$\varphi_* : \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y).$$

Sketch of proof. (Need details) For any $f \in K(X)^*$, we have

$$\varphi_*((f)) = (\mathrm{Nm}_{K(X)/K(Y)}(f)).$$

This follows from the projection formula. □

Definition 10.22. Suppose X is a <u>normal</u> scheme. We define the **pull-back homomor-**

phism $\varphi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ as follows: For an irreducible divisor $Z \in \mathrm{Div}(Y)$,

$$\varphi^*(Z) := \sum n_i D_i,$$

where D_i are the irreducible components of $X \times_Y Z$ (the preimage of Z), and the coefficient n_i is given by

$$n_i = \deg_{\mathcal{O}_{X,\eta_i}}(x),$$

where x is a generator of the maximal ideal of $\mathcal{O}_{Y,\eta}$ (η is the generic point of Z), and η_i is the generic point of D_i. (Since X is normal, $\mathcal{O}_{X,\eta_i}$ is a DVR, so this degree is well-defined as the valuation of x).

Proposition 10.23. φ^* descends to a homomorphism:

$$\varphi^* : \mathrm{Cl}(Y) \longrightarrow \mathrm{Cl}(X).$$

In fact, for all $f \in K(Y)^*$,

$$\varphi^*((f)) = (\varphi^\flat(f))$$

Moreover, for all $D \in \mathrm{Div}(Y)$, we have:

$$\varphi_* \varphi^*(D) = [K(X) : K(Y)]D$$

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Lecture 13. Cartier Divisors and Line Bundles

11.1 Cartier Divisors

Let X be a scheme, and let $U \subseteq X$ be an open subscheme.

Definition 11.1. For any open set U, we define $R_X(U)$ to be the set of regular sections that are not zero divisors. Specifically:

$$R_X(U) := \left\{ a \in \mathcal{O}_X(U) \mid \text{the map } \mathcal{O}_X|_U \xrightarrow{\cdot a} \mathcal{O}_X|_U \text{ is injective} \right\}.$$

Fact: The assignment $U \mapsto R_X(U)$ forms a sheaf of multiplicative monoids, which we denote by R_X.

Remark: If U is an affine open set, say $U \cong \operatorname{Spec} A$, then $R_X(U)$ corresponds precisely to the set of non-zero divisors in A:

$$R_X(U) = \{a \in A \mid a \text{ is not a zero-divisor}\}.$$

Definition 11.2. Let $\mathcal{K}_X$ be the sheaf associated to the presheaf defined by the localization:

$$U \longmapsto R_X(U)^{-1}\mathcal{O}_X(U).$$

The sheaf $\mathcal{K}_X$ is called the **sheaf of total quotient rings** of X.

Note that $\mathcal{O}_X$ is naturally a subsheaf of $\mathcal{K}_X$. In fact, $\mathcal{O}_X$ is a sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$.

Proposition 11.3. If X is an integral scheme, then $\mathcal{K}_X$ is the constant sheaf associated to the function field $K(X)$.

Remark: Let η be the generic point of an integral scheme X, and let $i_\eta : \{\eta\} \hookrightarrow X$ be the inclusion map. Then we have an isomorphism:

$$\mathcal{K}_X \simeq (i_\eta)_* K(X) \simeq (i_\eta)_* \mathcal{O}_{X,\eta}.$$

In particular, $\mathcal{K}_X$ is a quasi-coherent sheaf in this case. However, $\mathcal{K}_X$ is **not** quasi-coherent in general (for arbitrary schemes).

Example 11.4. Consider the ring $A = k[x,y]_{(x,y)} \oplus k$. This is the trivial extension of the local ring $R = k[x,y]_{(x,y)}$ by its residue field k. The multiplication operation is defined by:

$$(a,b) \cdot (a',b') = (aa', \bar{a}b' + \bar{a}'b),$$

where $\bar{a}$ denotes the image of a in the residue field $k \cong k[x,y]_{(x,y)}/(x,y)$.
Then $\mathcal{K}_X$ is **not** quasi-coherent.

Proof. Let $X := \mathrm{Spec} A$. We analyze the sheaf of total quotient rings $\mathcal{K}_X$.

First, consider the global sections. The set of non-zero divisors on X is:

$$R_X(X) = \{a \in A \mid a \text{ is invertible}\}.$$

This is because any element in the maximal ideal of A is a zero divisor (it annihilates elements in the ideal $(0) \oplus k$). Consequently, the total quotient ring of global sections is simply the ring itself:

$$R_X(X)^{-1}\mathcal{O}_X(X) = A.$$

However, consider the open set $U = D(x)$ (where x becomes invertible). One can show that:

$$R_X(D(x))^{-1}\mathcal{O}_X(D(x)) \cong k(x,y) = \mathrm{Frac}(k[x,y]).$$

More generally, it can be shown that for an open set $U \subseteq X$:

$$R_X(U)^{-1}\mathcal{O}_X(U) = \begin{cases} A & \text{if } 0 \in U \text{ (the closed point)}, \\ k(x,y) & \text{if } 0 \notin U. \end{cases}$$

Therefore, the sheaf $\mathcal{K}_X$ satisfies $\mathcal{K}_X(X) = A$, but $\mathcal{K}_X(U) = k(x,y)$ for any U not containing the closed point.

This implies that $\mathcal{K}_X$ is **not** quasi-coherent. If it were quasi-coherent, then for the affine scheme $X = \mathrm{Spec} A$, the sections on a basic open set $D(f)$ should be the localization A_f. However, here the restriction maps do not align with the localization property of the global sections. □

Lemma 11.5. If X is a locally Noetherian reduced scheme, then there is an isomorphism:

$$\mathcal{K}_X \simeq \prod_{\eta \in \text{gen. pts of } X} (i_\eta)_* \mathcal{O}_{X,\eta},$$

where the product runs over the generic points of the irreducible components of X, and

$i_\eta : \{\eta\} \hookrightarrow X$ is the inclusion map.
In particular, in this case, $\mathcal{K}_X \in \mathrm{Qcoh}(\mathcal{O}_X)$.

Proof. Let Y be the scheme defined by the disjoint union of the spectra of the local rings at the generic points:

$$Y = \bigsqcup_{\eta \in \text{gen. pts of } X} \mathrm{Spec}\mathcal{O}_{X,\eta}.$$

Let $i : Y \to X$ be the natural morphism. We have the natural adjunction map (the unit of the adjunction):

$$\varphi : \mathcal{O}_X \longrightarrow i_* i^* \mathcal{O}_X = \prod_\eta (i_\eta)_* \mathcal{O}_{X,\eta}.$$

We claim that the map $\mathcal{O}_X \xrightarrow{\varphi} i_* \mathcal{O}_Y$ induces a unique isomorphism $\mathcal{K}_X \simeq i_* \mathcal{O}_Y$ such that its restriction to $\mathcal{O}_X$ is φ.

It suffices to prove the claim locally. Thus, without loss of generality, we may assume that X is a Noetherian scheme. Consequently, the set Y of generic points consists of finitely many points.

Let $U = \mathrm{Spec}A$ be an affine open subset of X. The set of non-zero divisors in A is given by

$$R_X(U) = A \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i,$$

where $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \mathrm{Ass}(A)$ is the set of associated primes of the zero ideal (0).

Since X is a reduced scheme, the ring A is reduced. Therefore, the nilradical is zero:

$$\sqrt{(0)} = \bigcap_{i=1}^{n} \mathfrak{p}_i = (0).$$

Since A is reduced, every associated prime $\mathfrak{p}_i$ is a minimal prime (refer to Atiyah-MacDonald). Geometrically, the set $\{\mathfrak{p}_i\}$ corresponds exactly to the set of generic points of U.

We compute the section of the total quotient sheaf:

$$R_X(U)^{-1}\mathcal{O}_X(U) = \left(A \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i \right)^{-1} A.$$

By the Chinese Remainder Theorem (and the property that localization commutes with finite products in this context), we have an isomorphism:

$$\left(A \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i \right)^{-1} A \xrightarrow{\sim} \prod_{i=1}^{n} A_{\mathfrak{p}_i} = i_* \mathcal{O}_Y(U).$$

By the universal property of localization, the isomorphism above is the unique one that extends the natural map $A \longrightarrow \prod_{i=1}^{n} A_{\mathfrak{p}_i}$. This completes the proof. $\square$

Remark: Let $X = \operatorname{Spec} A$ be an affine Noetherian scheme, where A is not necessarily reduced. Then the global sections of the sheaf of total quotient rings are given by:

$$\mathcal{K}_X(X) = \left(A \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i \right)^{-1} A,$$

where $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \operatorname{Ass}(A)$ are the associated primes of A. And the relation with the nilradical still holds:

$$\sqrt{(0)} = \bigcap_{i=1}^{n} \mathfrak{p}_i$$

Definition 11.6 (**Cartier Divisor**)**.** Let $\mathcal{K}_X^*$ be the subsheaf of invertible elements in the sheaf of total quotient rings $\mathcal{K}_X$. Similarly, let $\mathcal{O}_X^*$ denote the subsheaf of invertible elements in the structure sheaf $\mathcal{O}_X$ (i.e., the sheaf of units).
A **Cartier divisor** on X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$.
Concretely, a Cartier divisor can be described by a collection $\{(U_i, f_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open cover of X, and for each i, $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ is a section such that for all i, j:

$$\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*).$$

Definition 11.7 (**Principal Divisor and Linear Equivalence**)**.** A Cartier divisor is called **principal** if it is in the image of the natural map:

$$\Gamma(X, \mathcal{K}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Two Cartier divisors D_1 and D_2 are said to be **linearly equivalent**, denoted by $D_1 \sim D_2$, if their difference $D_1 - D_2$ is a principal divisor.

Question: How do we relate Weil divisors to Cartier divisors?

Construction: From Cartier to Weil Divisors

Let X be an integral Noetherian scheme. We define a group homomorphism from the group of Cartier divisors to the group of Weil divisors:

$$\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow \operatorname{Div}(X).$$

This induces a homomorphism on the class groups:

$$\mathrm{CaCl}(X) \longrightarrow \mathrm{Cl}(X).$$

Given a Cartier divisor represented by the system $D = \{(U_i, f_i)\}$, we associate a Weil divisor to it as follows:

For every prime divisor $Y \subset X$, let the coefficient of Y be the degree of the local defining function. Specifically, we define the coefficient to be $\deg_Y(f_i)$ for any index i such that $Y \cap U_i \neq \emptyset$.

Note that since f_i/f_j is a unit on $U_i \cap U_j$, $\deg_Y(f_i) = \deg_Y(f_j)$, so this coefficient is well-defined.

Following the construction, we obtain a well-defined Weil divisor:

$$D = \sum_Y \deg_Y(f_i)Y.$$

If $f \in \Gamma(X, \mathcal{K}_X^*)$ is a global section (a rational function), then the associated Weil divisor is the principal divisor (f). Since principal Cartier divisors map to principal Weil divisors, we obtain a well-defined group homomorphism between the divisor class groups:

$$\mathrm{CaCl}(X) \longrightarrow \mathrm{Cl}(X).$$

Proposition 11.8. Let X be a normal Noetherian scheme. Then the natural homomorphism

$$\mathrm{CaCl}(X) \longrightarrow \mathrm{Cl}(X)$$

is injective. Moreover, under the above assumption, this map is an isomorphism if and only if X is locally factorial (i.e., for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a Unique Factorization Domain).

***Proof of injectivity*.** Suppose $D = \{(U_i, f_i)\}$ is a Cartier divisor whose associated Weil divisor is zero. By definition, this means that for every prime divisor Y, the coefficient is zero:

$$\deg_Y(f_i) = 0 \quad \text{for all prime divisors } Y.$$

We need to show that D is the zero Cartier divisor, which means showing that $f_i \in \mathcal{O}_X(U_i)^*$.

Since X is a normal Noetherian scheme, for any open set U_i, the ring of sections $\mathcal{O}_X(U_i)$

satisfies the algebraic property:

$$\mathcal{O}_X(U_i) = \bigcap_{\substack{p\in U_i \\ \dim \mathcal{O}_{X,p}=1}} \mathcal{O}_{X,p}.$$

The local rings $\mathcal{O}_{X,p}$ of dimension 1 are actually DVRs. The condition $\deg_Y(f_i) = 0$ implies that f_i has valuation 0 in each of these DVRs. Consequently, f_i is a unit in each $\mathcal{O}_{X,p}$.

Therefore, f_i is a unit in the intersection, i.e., $f_i \in \mathcal{O}_X(U_i)^*$.

This implies that the collection $\{(U_i, f_i)\}$ represents the zero element in the group $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Thus, the map from Cartier divisors to Weil divisors is injective.

Consider the following commutative diagram with exact rows. The top row corresponds to the sequence defining Cartier divisors, and the bottom row corresponds to Weil divisors:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathcal{O}_X^*) \arrow[r] \arrow[d, equal] & \Gamma(X,\mathcal{K}_X^*) \arrow[r] \arrow[d, equal] & \Gamma(X,\mathcal{K}_X^*/\mathcal{O}_X^*) \arrow[r] \arrow[d, hook] & \mathrm{CaCl}(X) \arrow[r] \arrow[d, hook, dashed] & 0 \\
0 \arrow[r] & \Gamma(X,\mathcal{O}_X^*) \arrow[r] & \Gamma(X,\mathcal{K}_X^*) \arrow[r] & \mathrm{Div}(X) \arrow[r] & \mathrm{Cl}(X) &
\end{tikzcd}$$

By the Five Lemma, establishing an injective homomorphism from $\mathrm{CaCl}(X)$ to $\mathrm{Cl}(X)$. □

Proof of equivalence. $\Longleftarrow$: Assume that X is locally factorial. Let $D \in \mathrm{Div}(X)$ be a Weil divisor. We want to show that D comes from a Cartier divisor.

Take any point $x \in X$. The divisor D induces a Weil divisor D_x on the local scheme $\mathrm{Spec}(\mathcal{O}_{X,x})$. If we write $D = \sum n_Y Y$, then locally this corresponds to:

$$D_x = \sum_{Y \ni x} n_Y \cdot Y_x,$$

where $Y_x = Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$.

Since $\mathcal{O}_{X,x}$ is a UFD, we know that the divisor class group of its spectrum is trivial:

$$\mathrm{Cl}(\mathrm{Spec}\, \mathcal{O}_{X,x}) = 0.$$

This implies that every Weil divisor on $\mathrm{Spec}(\mathcal{O}_{X,x})$ is principal. Therefore, there exists a rational function $f_x \in \mathcal{K}_X^* = \Gamma(X, \mathcal{K}_X^*)$ (viewed as an element of the fraction field of $\mathcal{O}_{X,x}$) such that:

$$D_x = \mathrm{div}(f_x) \quad \text{on } \mathrm{Spec}(\mathcal{O}_{X,x}).$$

Consider the difference $D - \operatorname{div}(f_x)$. This is a Weil divisor on X whose support does not contain x.

Consequently, there exists an open neighborhood $U_x \ni x$ such that the restriction of the divisors agrees:

$$D|_{U_x} = \operatorname{div}(f_x)|_{U_x}.$$

This gives us a collection $\{(U_x, f_x)\}_{x\in X}$. Since D is globally defined, on overlaps $U_x \cap U_y$, the functions f_x and f_y define the same divisor D, so f_x/f_y has divisor zero, implying $f_x/f_y \in \mathcal{O}_X(U_x \cap U_y)^*$.

Thus, D is represented by the Cartier divisor $\{(U_x, f_x)\}$. This proves that the map $\operatorname{CaCl}(X) \to \operatorname{Cl}(X)$ is surjective. Combined with the injectivity proved earlier, we have $\operatorname{CaCl}(X) \cong \operatorname{Cl}(X)$.

$\Longrightarrow$: If $\operatorname{CaCl}(X) \cong \operatorname{Cl}(X)$, then every prime divisor corresponds to a Cartier divisor. Locally, this implies that for every $x \in X$ and every prime ideal $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X,x}$ with height $\operatorname{ht}(\mathfrak{p}) = 1$, the ideal $\mathfrak{p}$ is principal. That is, $\mathfrak{p} = (f)$ for some $f \in \mathcal{O}_{X,x}$. This characterizes the local ring as a UFD. □

Fact: Every regular local ring is a UFD.

Consequently, if X is a **regular scheme** (meaning that for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a regular local ring), then X is locally factorial. Therefore, we have the isomorphisms:

$$\operatorname{CaCl}(X) \cong \operatorname{Cl}(X) \cong \operatorname{Pic}(X).$$

Example 11.9 (The Quadric Cone)**.** Let $A = k[x, y, z]/(xy - z^2)$ and let $X = \operatorname{Spec} A$. This is a singular surface (a cone) with a singularity at the origin.

Consider the closed subscheme $Y = V(y, z)$, which corresponds to a line passing through the origin on the cone. The class of Y defines a Weil divisor. However, Y is **not** in the image of the map $\operatorname{CaCl}(X) \longrightarrow \operatorname{Cl}(X)$.

This is because the ideal (y, z) is not principal in the local ring $A_{(x,y,z)}$. Geometrically, the line Y cannot be defined by a single equation in any neighborhood of the cone point. Thus, $\operatorname{CaCl}(X) \subsetneq \operatorname{Cl}(X)$.

Example 11.10 (Affine Elliptic Curve)**.** This example illustrates a locally factorial Noetherian ring that is **not** a UFD itself.

Let k be an algebraically closed field ($k = \bar{k}$). Consider the ring:

$$A = k[x, y]/(y^2 - x(x-1)(x-\lambda)), \quad \text{where } \lambda \in k \setminus \{0, 1\}.$$

Let $X = \operatorname{Spec} A$. This is an affine elliptic curve. Since the curve is smooth (non-singular), the ring A is regular, and hence locally factorial.
However, A is not a UFD. The class group is non-trivial. In fact, we have an isomorphism:

$$\operatorname{CaCl}(X) \cong \operatorname{Pic}(X) = X(k),$$

where $X(k)$ denotes the set of k-rational points on the curve, which carries a group structure (the group law on the elliptic curve). The non-triviality of the group structure implies that there are divisors that are not principal, even though the ring is locally factorial everywhere.

11.2 Line Bundles (Invertible Sheaves)

Definition 11.11. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is called a **line bundle** (or an **invertible sheaf**) if it is locally free of rank 1. That is, there exists an open covering $\{U_i\}$ of X such that for each i, there is an isomorphism of $\mathcal{O}_{U_i}$-modules:

$$\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}.$$

Lemma 11.12. Let X be a scheme and let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf. Then $\mathcal{F}$ is a line bundle if and only if there exists a finitely generated quasi-coherent sheaf $\mathcal{G}$ such that:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X.$$

Proof. **"Only if" part:** Suppose $\mathcal{F}$ is a line bundle. This direction is trivial. We can take $\mathcal{G} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, the dual sheaf of $\mathcal{F}$. The natural evaluation map

$$\operatorname{ev} : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{O}_X, \quad (\alpha \otimes f) \longmapsto f(\alpha)$$

is an isomorphism. (This can be verified locally: since $\mathcal{F}$ is locally isomorphic to $\mathcal{O}_X$, the map locally corresponds to $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\vee} \cong \mathcal{O}_X$).

"If" part: We prove this by analyzing the stalks. We start with a trivial case in linear algebra and then lift it to local rings.

Step 1: Vector Spaces. Let V and W be finite-dimensional vector spaces over a field k such that $V \otimes_k W \cong k$. Taking dimensions, we have:

$$\dim_k(V) \cdot \dim_k(W) = \dim_k(V \otimes W) = \dim_k(k) = 1.$$

Since dimensions are non-negative integers, this implies $\dim_k V = 1$ and $\dim_k W = 1$.

Step 2: Local Rings. Let $x \in X$ be a point. Let $A = \mathcal{O}_{X,x}$ be the local ring at x, and let $\mathfrak{m}$ be its maximal ideal. Let $M = \mathcal{F}_x$ and $N = \mathcal{G}_x$ be the stalks, which are finitely generated A-modules. The condition $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$ implies $M \otimes_A N \cong A$.

We tensor with the residue field $k = A/\mathfrak{m}$:

$$(M \otimes_A N) \otimes_A k \cong A \otimes_A k \cong k.$$

Using the property $(M \otimes_A N) \otimes_A k \cong (M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N)$, we have:

$$(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) \cong k.$$

By Step 1, we conclude that $\dim_k(M/\mathfrak{m}M) = 1$.

Since M is a finitely generated A-module, by **Nakayama's Lemma**, the minimum number of generators of M is equal to $\dim_k(M/\mathfrak{m}M)$. Thus, M is generated by exactly one element.

This implies there exists a surjective homomorphism $\varphi : A \to M$. Similarly, there exists a surjection $\psi : A \to N$. Let their kernels be denoted by $\ker \varphi$ and $\ker \psi$. We have isomorphisms $M \cong A/\ker \varphi$ and $N \cong A/\ker \psi$.

Substituting these back into the tensor product:

$$M \otimes_A N \cong (A/\ker \varphi) \otimes_A (A/\ker \psi) \cong A/(\ker \varphi + \ker \psi).$$

We are given that $M \otimes_A N \cong A$. Therefore:

$$A/(\ker \varphi + \ker \psi) \cong A.$$

This isomorphism implies that the ideal $\ker \varphi + \ker \psi$ must be the zero ideal. Since $\ker \varphi$ and $\ker \psi$ are ideals of A, $\ker \varphi + \ker \psi = 0$ implies $\ker \varphi = 0$ and $\ker \psi = 0$.

Consequently, $M \cong A$ (and $N \cong A$).

Step 3: Globalization. From the previous step, we know that for every point $x \in X$, the stalks $\mathcal{F}_x$ and $\mathcal{G}_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. Thus, we have isomorphisms of stalks:

$$\mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{F}_x \quad \text{and} \quad \mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{G}_x.$$

We can lift the isomorphism at the stalk level to a homomorphism on an open affine neighborhood. That is, there exists an open neighborhood U of x and morphisms:

$$f : \mathcal{O}_U \longrightarrow \mathcal{F}|_U \quad \text{and} \quad g : \mathcal{O}_U \longrightarrow \mathcal{G}|_U.$$

By the condition that the stalk maps are isomorphisms and $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, we can easily prove that f, g are both injective sheaf morphisms.

Since $\mathcal{F}$ and $\mathcal{G}$ are finitely generated, we can easily prove that there exists a smaller open neighborhood (which we still denote by U) on which f is surjective sheaf morphism. Thus, we have isomorphism, which means $\mathcal{F}$ is a line bundle on X. □

Remark: If $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles (invertible sheaves), then their tensor product $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ is also a line bundle.

Definition 11.13 (**Picard Group**)**.** For a scheme X, we define the **Picard group** of X, denoted by $\mathrm{Pic}(X)$, to be the group of isomorphism classes of line bundles on X, under the operation of tensor product.

$$\mathrm{Pic}(X) := \{\text{line bundles on } X\}/\text{isomorphism}, \quad \text{with group operation } \otimes.$$

The identity element is the structure sheaf $\mathcal{O}_X$, and the inverse of a line bundle $\mathcal{L}$ is its dual $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$.

Definition 11.14 (**Sheaf Associated to a Cartier Divisor**)**.** Let $D = \{(U_i, f_i)\}$ be a Cartier divisor on X, where $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$. We define an invertible sheaf associated to D, denoted by $\mathcal{L}(D)$, to be the $\mathcal{O}_X$-submodule of the sheaf of total quotient rings $\mathcal{K}_X$, generated locally on U_i by f_i^{-1}.
Explicitly:

$$\mathcal{L}(D)|_{U_i} = \mathcal{O}_{U_i} \cdot f_i^{-1} \subset \mathcal{K}_X|_{U_i}.$$

This is well-defined because on the intersection $U_i \cap U_j$, the ratio f_i/f_j is a unit in $\mathcal{O}_X(U_i \cap U_j)$. Therefore, f_i^{-1} and f_j^{-1} differ by a unit, and thus they generate the same $\mathcal{O}_X$-submodule inside $\mathcal{K}_X$.

Proposition 11.15. Let X be a scheme.

(a) For every Cartier divisor $D \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, the associated sheaf $\mathcal{L}(D)$ is a line bundle (invertible sheaf). Moreover, the map $D \mapsto \mathcal{L}(D)$ gives a one-to-one correspondence between Cartier divisors and invertible $\mathcal{O}_X$-submodules of $\mathcal{K}_X$.

(b) The map is a homomorphism. Specifically, for any two Cartier divisors D_1 and D_2, we have an isomorphism:

$$\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes_{\mathcal{O}_X} \mathcal{L}(D_2)^{-1}.$$

(Note: $\mathcal{L}(D)^{-1}$ denotes the dual line bundle).

(c) Two Cartier divisors are linearly equivalent, $D_1 \sim D_2$, if and only if their associated line bundles are isomorphic, i.e., $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$.

Proof. **(a)** Let D be represented by the collection $\{(U_i, f_i)\}$. By definition, $\mathcal{L}(D)$ is locally generated by f_i^{-1}. Consider the map:

$$\mathcal{O}_{U_i} \longrightarrow \mathcal{O}_{U_i} \cdot f_i^{-1} = \mathcal{L}(D)|_{U_i}, \quad 1 \longmapsto f_i^{-1}.$$

Since f_i is a unit in the total quotient ring $\mathcal{K}_X(U_i)$, multiplication by f_i^{-1} is injective (and surjective onto the submodule generated by it). Thus, $\mathcal{L}(D)|_{U_i} \cong \mathcal{O}_{U_i}$, which shows that $\mathcal{L}(D)$ is a line bundle.

Conversely, suppose $\mathcal{L} \subseteq \mathcal{K}_X$ is an invertible $\mathcal{O}_X$-submodule. Since it is locally free of rank 1, we can locally choose a generator $g_i \in \mathcal{K}_X(U_i)$ such that $\mathcal{L}|_{U_i} = \mathcal{O}_{U_i} \cdot g_i$. Then $\mathcal{L}$ corresponds to the Cartier divisor defined by the collection $\{(U_i, g_i^{-1})\}$. This establishes the one-to-one correspondence.

(b) Let $D_1 = \{(U_i, f_i)\}$ and $D_2 = \{(U_i, g_i)\}$. The difference $D_1 - D_2$ is represented by the collection $\{(U_i, f_i/g_i)\}$.

The sheaf $\mathcal{L}(D_1 - D_2)$ is locally generated by $(f_i/g_i)^{-1} = g_i/f_i$.

On the other hand, $\mathcal{L}(D_1)$ is generated by f_i^{-1}, and $\mathcal{L}(D_2)$ is generated by g_i^{-1}. The inverse sheaf $\mathcal{L}(D_2)^{-1}$ (which is the dual) corresponds to the divisor $-D_2$, so it is locally generated by $(g_i^{-1})^{-1} = g_i$.

The tensor product $\mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$ is locally generated by the product of the generators:

$$f_i^{-1} \cdot g_i = \frac{g_i}{f_i}.$$

Since the local generators match, we have the isomorphism:

$$\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}.$$

(c) By part (b), $D_1 \sim D_2$ is equivalent to $D_1 - D_2 \sim 0$. Let $D = D_1 - D_2$. We need to show $D \sim 0 \iff \mathcal{L}(D) \cong \mathcal{O}_X$.

Suppose $\mathcal{L}(D) \cong \mathcal{O}_X$. Since $\mathcal{L}(D)$ is a submodule of $\mathcal{K}_X$, this isomorphism identifies the global section $1 \in \Gamma(X, \mathcal{O}_X)$ with a global section $f \in \Gamma(X, \mathcal{L}(D)) \subset \Gamma(X, \mathcal{K}_X)$.

Since the isomorphism maps a generator to a generator, f must be invertible in $\mathcal{K}_X$, i.e., $f \in \Gamma(X, \mathcal{K}_X^*)$.

Locally, $\mathcal{L}(D)$ is generated by f_i^{-1}. The fact that it is globally generated by f means that $f = u_i f_i^{-1}$ for some unit $u_i \in \mathcal{O}_X(U_i)^*$. Thus, $f_i = u_i f^{-1}$. This implies that the divisor D is the principal divisor associated to f^{-1}. Hence $D \sim 0$. □

Corollary 11.16. The map $D \mapsto \mathcal{L}(D)$ induces an injective group homomorphism:

$$\mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X).$$

In terms of sheaf cohomology, this corresponds to the natural map:

$$H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow H^1(X, \mathcal{O}_X^*).$$

Remark: This map is not necessarily surjective. There exist schemes where not every line bundle comes from a Cartier divisor. (See references in Hartshorne for counterexamples, typically involving non-separated schemes or schemes with embedded points in a specific configuration).

Proposition 11.17. Let X be a scheme. Assume that one of the following conditions holds:

(1) X is integral.

(2) X is locally Noetherian and contains an open affine subset U that is **scheme-theoretically dense** in X. (This means that the restriction map $\mathcal{O}_X \to i_*\mathcal{O}_U$ is injective, where $i : U \hookrightarrow X$ is the inclusion).

Then the map is an isomorphism:

$$\mathrm{CaCl}(X) \cong \mathrm{Pic}(X).$$

Sketch of Proof. **Goal:** We want to show that every line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ admits an embedding $\mathcal{L} \hookrightarrow \mathcal{K}_X$ into the sheaf of total quotient rings. If such an embedding exists, then $\mathcal{L}$ is isomorphic to a subsheaf of $\mathcal{K}_X$, which by the previous proposition corresponds to a Cartier divisor.

Step 1: Extension from a dense open set. Let $U \subseteq X$ be an open affine subset that is scheme-theoretically dense.

- In case (1) where X is integral, every non-empty open affine subset satisfies this condition.
- In case (2), such a U exists by assumption.

Actually, we can just assume we have a trivialization on U, or more generally, embed $\mathcal{L}|_U$ into $\mathcal{K}_U$. Since $\mathcal{K}_X$ is the sheaf of total quotient rings, on the dense open set U, we have $\mathcal{K}_U = \mathcal{K}_X|_U$.

Assume we have an embedding $\mathcal{L}|_U \hookrightarrow \mathcal{K}_U$. We claim that this embedding uniquely extends to an embedding $\mathcal{L} \hookrightarrow \mathcal{K}_X$.

Indeed, the condition that U is scheme-theoretically dense means that the natural map $\mathcal{O}_X \to i_*\mathcal{O}_U$ is injective. Since $\mathcal{L}$ is locally free of rank 1 (locally isomorphic to $\mathcal{O}_X$), tensoring with $\mathcal{L}$ preserves injectivity (flatness of line bundles). Thus, the map:

$$\mathcal{L} \longrightarrow i_*(\mathcal{L}|_U)$$

is injective.

On the other hand, we have an isomorphism $\mathcal{K}_X \simeq i_*\mathcal{K}_U$.

- In case (1): If X is integral, then $\mathcal{K}_X$ is the constant sheaf associated to the function field $K(X)$, and similarly $\mathcal{K}_U$ is associated to $K(U) = K(X)$. Thus, the isomorphism is trivial.
- In case (2): Assume $X = \operatorname{Spec} A$ is affine (or cover by affines). The condition that $\mathcal{O}_X \to i_*\mathcal{O}_U$ is injective implies that every associated prime of A is contained in U. Consequently, for any open set $V \subseteq X$, the sections of the total quotient ring are determined by their restriction to $V \cap U$:
$$\Gamma(V, \mathcal{K}_X) = \Gamma(V \cap U, \mathcal{K}_X).$$

Therefore, $\mathcal{K}_X \simeq i_*\mathcal{K}_U$.

To conclude Step 1, we have the following composition of injections:

$$\mathcal{L} \longrightarrow i_*(\mathcal{L}|_U) \longrightarrow i_*\mathcal{K}_U \cong \mathcal{K}_X.$$

This gives a global embedding of $\mathcal{L}$ into $\mathcal{K}_X$, provided we can embed $\mathcal{L}|_U$ into $\mathcal{K}_U$.

Step 2: The Affine Case. We may assume that X is affine, say $X = \operatorname{Spec} A$. Then the line bundle $\mathcal{L}$ corresponds to a rank 1 projective A-module M, i.e., $\mathcal{L} \cong \widetilde{M}$.

Let $S = A \backslash \bigcup_i \mathfrak{p}_i$, where $\{\mathfrak{p}_i\} = \operatorname{Ass}(A)$ is the set of associated primes of A. The localization $S^{-1}A$ is the total quotient ring of A.

We have the natural map $M \to S^{-1}M$. Since M is projective (hence flat) and locally free, the map is injective.

Moreover, the ring $S^{-1}A$ is a **semi-local ring** (it has finitely many maximal ideals, corresponding to the maximal elements among the associated primes).

We use the algebraic fact that **every locally free module of finite rank over a semi-local ring is free**.

In our situation, this implies that $S^{-1}M$ is a free $S^{-1}A$-module of rank 1:

$$S^{-1}M \cong S^{-1}A \cong \Gamma(X, \mathcal{K}_X).$$

This isomorphism provides the desired embedding:

$$\mathcal{L} = \widetilde{M} \hookrightarrow \widetilde{S^{-1}M} \cong \mathcal{K}_X.$$

This completes the proof that every line bundle is isomorphic to a subsheaf of $\mathcal{K}_X$, and thus comes from a Cartier divisor.

□

Remark: The conditions of the proposition hold when X is a subscheme of a projective space $\mathbb{P}^N_A$ over a Noetherian ring A, via the immersion $X \hookrightarrow \mathbb{P}^N_A$.

Example 11.18. Let $X = \mathbb{P}^n_k$ where k is a field. Then we have the isomorphisms:

$$\mathrm{CaCl}(X) \cong \mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1).$$

This is also isomorphic to the divisor class group:

$$\mathrm{Cl}(X) \cong \mathbb{Z} = \mathbb{Z} \cdot [H],$$

where $[H]$ is the class of a hyperplane.
In particular, every line bundle on $\mathbb{P}^n_k$ is isomorphic to $\mathcal{O}_X(m)$ for some integer $m \in \mathbb{Z}$.

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A Some completion of details

If you wanna use citations of the context, use in the following form:

- hypertarget{label-1}{xxxx}
- xxxx
- hyperlink{label-1}{xxxx}

B Completion of details for Lecture 9

Details for Theorem 6.12

Here we provide the details for the proof that universally closed morphisms are quasi-compact.

1. The fiber isomorphism $f_T^{-1}(\tau) \cong X_y$

- The point $\tau \in T$ is defined by $t_i = 1$ for all i. This corresponds to the kernel of the evaluation map $A[t_i] \to \kappa(y)$ given by $t_i \mapsto 1$.
- The fiber of the morphism $T \to Y$ over y is Spec $(\kappa(y) \otimes_A A[t_i]) \cong$ Spec $\kappa(y)[t_i]$. The point τ can be viewed as a $\kappa(y)$-rational point in this fiber. Thus, the residue field is $\kappa(\tau) \cong \kappa(y)$.
- By the property of fiber products, the fiber of $f_T : X \times_Y T \to T$ over τ is:

$$\begin{aligned}(X \times_Y T) \times_T \operatorname{Spec} \kappa(\tau) &\cong X \times_Y (T \times_T \operatorname{Spec} \kappa(\tau)) \\ &\cong X \times_Y \operatorname{Spec} \kappa(\tau) \\ &\cong X \times_Y \operatorname{Spec} \kappa(y) \\ &= X_y\end{aligned}$$

2. The intersection $(X_i \times_Y T_i) \cap f_T^{-1}(\tau) \cong X_i \cap X_y$

- Recall that $T_i = D(t_i) \subset T$. Since the coordinate of τ is $t_i = 1 \neq 0$, we have $\tau \in T_i$.
- The set $X_i \times_Y T_i$ is an open subscheme of $X \times_Y T$. Intersecting this open set with the fiber over τ is equivalent to restricting the fiber X_y to the open set X_i:

$$\begin{aligned}(X \times_Y T)_\tau \cap (X_i \times_Y T_i) &\cong X_y \cap (X_i \times_{\operatorname{Spec} A} \operatorname{Spec} \kappa(y)) \\ &\cong X_y \cap X_i\end{aligned}$$

3. The contradiction via $D(g)$

- We constructed a closed subscheme $S \subseteq T$ defined by $t_j = 1$ for $j \in J$ and $t_k = 0$ for $k \notin J$. The projection $S \to Y$ is an isomorphism, so $S \cong Y = \operatorname{Spec} A$.
- We found $g \in A[t_j \mid j \in J]$ such that $\tau \in D(g)$ and $f_T(Z) \cap D(g) = \emptyset$.
- Let $\bar{g}$ be the restriction of g to S (substituting $t_j = 1$). Since $\tau \in S$ and $g(\tau) \neq 0$, we have $y \in D(\bar{g}) \subseteq S \cong Y$.
- The condition $f_T(Z) \cap D(g) = \emptyset$ implies that over the open set $D(g)$, the preimage is contained in the union of open sets indexed by J.
- Pulling this back to $S \cong Y$, we find that $f^{-1}(D(\bar{g}))$ is covered by $\bigcup_{j \in J} X_j$.
- Since J is a finite set and each X_j is affine (hence quasi-compact), the finite union is quasi-compact. This implies $f^{-1}(D(\bar{g}))$ is quasi-compact, which contradicts the assumption that y has no such neighborhood.

Details for Theorem 6.17 (Page 61)

Here we provide the details for the direction $\Leftarrow$ of Theorem 6.17 (Proof of Theorem 6.16(2)), specifically regarding the valuation ring argument.

1. The relation between $\tilde{R}$ and R

- Let $x \in X_R$ be the image of the generic point $\operatorname{Spec} K$. Let $Z = \overline{\{x\}}$ be the closure of x in X_R with the reduced scheme structure.
- The morphism $f_R : Z \to \operatorname{Spec} R$ is closed (since f is universally closed). Since the image contains the generic point, $f_R(Z) = \operatorname{Spec} R$.
- Therefore, there exists a point $x' \in Z$ such that $f_R(x') = \mathfrak{m}$ (the closed point of $\operatorname{Spec} R$).
- Let $\tilde{R} = \mathcal{O}_{Z,x'}$ be the local ring of Z at x'. Let $\tilde{\mathfrak{m}}$ be its maximal ideal. Since Z is integral with function field K, we have $\tilde{R} \subset K$.
- The morphism $Z \to \operatorname{Spec} R$ induces a local homomorphism of local rings on the stalks:

  $$(f_R)^{\#}_{x'} : \mathcal{O}_{\operatorname{Spec} R, \mathfrak{m}} \longrightarrow \mathcal{O}_{Z,x'}$$

  Since R is a valuation ring, $\mathcal{O}_{\operatorname{Spec} R, \mathfrak{m}} = R$. Thus we have a homomorphism $R \to \tilde{R}$.
- This homomorphism is compatible with the identity on the function field K, so it is an inclusion $R \subseteq \tilde{R}$.

- Since the map is local, the preimage of $\tilde{\mathfrak{m}}$ is $\mathfrak{m}$. Thus $\tilde{\mathfrak{m}} \cap R = \mathfrak{m}$.
- In the language of valuation rings, this means $(\tilde{R}, \tilde{\mathfrak{m}})$ dominates $(R, \mathfrak{m})$, denoted as $(\tilde{R}, \tilde{\mathfrak{m}}) \geq (R, \mathfrak{m})$.

2. Maximality and the Section $\tilde{s}$

- **Fact:** A valuation ring R of a field K is a maximal element in the set of all local subrings of K ordered by the dominance relation.
- Since R is a valuation ring and $(\tilde{R}, \tilde{\mathfrak{m}}) \geq (R, \mathfrak{m})$, we must have $\tilde{R} = R$.
- Consequently, we have an isomorphism $i : \text{Spec } R \xrightarrow{\sim} \text{Spec } \tilde{R}$.
- We can now construct the section $\tilde{s} : \text{Spec } R \to X_R$. It is the composition:

$$\tilde{s} : \text{Spec } R \xrightarrow{\cong} \text{Spec } \tilde{R} = \text{Spec } \mathcal{O}_{Z,x'} \longrightarrow Z \hookrightarrow X_R$$

- This morphism $\tilde{s}$ satisfies $f_R \circ \tilde{s} = \text{id}$, providing the existence of the lift required for the valuation criterion.

C Completion of details for Lecture 12

I think we will use the following lemma.

Lemma C.1. A is Noetherian and M is finitely generated, then M admits a prime filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_{i+1}/M_i \cong A/\mathfrak{p}_i$ for some prime ideals $\mathfrak{p}_i \subset A$.

To be written for 10.16.
To be written for 10.18.
To be written for 10.19.
To be written for 10.20.
To be written for 10.21.