[Aside]

14.2.K. EXERCISE. Suppose $\phi: \mathscr{E} \to \mathscr{F}$ is a morphism of finite rank locally free sheaves, and $p \in X$. Show that the natural map $(\ker \phi)|_p \to \ker(\phi|_p)$ is surjective if and only if ϕ is a map of vector bundles in some neighborhood of p.

pf. If ϕ is a map of vb, then $\operatorname{im}\phi$ is a vb and we may assume

$\operatorname{im}\phi|_U = \mathcal{O}_U^{\oplus a} \hookrightarrow \mathcal{O}_U^{\oplus(a+c)} = \mathcal{F}|_U$ (near p).

Then $(\operatorname{im}\phi|_p = k(p)^{\oplus a} \hookrightarrow k(p)^{\oplus(a+c)} = \mathcal{F}|_p)$ is inj.

We have

$$\begin{tikzcd} 0 \arrow[r] & \ker\phi \arrow[r] & \mathcal{E} \arrow[r] \arrow[dr, "\phi"'] & \operatorname{im}\phi \arrow[r] \arrow[d, hook] & 0 \\ & & & \mathcal{F} & \end{tikzcd}$$

in $QCoh(X)$

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & (\ker\phi)_p \arrow[r] & \mathcal{E}_p \arrow[r] \arrow[dr, "\phi_p"'] & (\operatorname{im}\phi)_p \arrow[r] \arrow[d, hook] & 0 \\ & & & \mathcal{F}_p & \end{tikzcd}$$

in $\mathcal{O}_{X,p}$-mod

$$\Rightarrow \begin{tikzcd} (\ker\phi)|_p \arrow[r] & \mathcal{E}|_p \arrow[r] \arrow[dr, "\phi|_p"'] & (\operatorname{im}\phi)|_p \arrow[r] \arrow[d, hook] & 0 \\ & & \mathcal{F}|_p & \end{tikzcd}$$

in $k(p)$-v.s.

$\Rightarrow (\ker\phi)|_p \to \ker(\phi|_p)$ is surj.

Conversely, if $(\ker\phi)|_p \to \ker(\phi|_p)$ is surj. ($\operatorname{im}\phi \in QCoh(X)$).

Then $(\operatorname{im}\phi)|_p \to \mathcal{F}|_p$ is inj $\Rightarrow (\operatorname{im}\phi)_p \to \mathcal{F}_p$ is inj by Nakayama

$\Rightarrow \operatorname{im}\phi|_U \to \mathcal{F}|_U$ is inj for some nbd of p, U, s.t. $\mathcal{F}|_U$ is free

$\Rightarrow \operatorname{im}\phi|_U$ is free. $\Rightarrow \phi$ is a map of v.b.s.

Geometric Nakayama

X scheme $\mathcal{F} \in QCoh.$ of f.t. $U \overset{\circ}{\hookrightarrow} X$.

If $a_1, \dots, a_n \in \mathcal{F}(U)$ s.t. $\mathcal{F}|_p$ is generated by $a_i|_p$'s, then $\exists\ V \overset{\circ}{\hookrightarrow} U$, $p \in V$ s.t. $a_i|_V$ generate $\mathcal{F}(V)$

pf. WMA $\mathcal{F}|_U = \widetilde{M}$, $M \in \mathrm{Mod}_{\mathcal{O}_X(U)}$. Let $N = \sum \mathcal{O}_X(U) a_i$. Then $(M/N) \otimes k(p) = 0$

By Nakayama, $(M/N)_p = 0$. Since M/N is finitely generated, $\exists\ f \in \mathcal{O}_X(U)$ s.t. $(M/N)_f = 0$

Take $V = D(f) \subseteq U$.

§14.4 Pushforwards

Recall: ① $Mod_A \xrightarrow{\simeq} QCoh(\operatorname{Spec} A)$ $\quad M \longmapsto \widetilde{M}$ $\quad$ (exact functor)

② $\mathcal{F}_i \in QCoh(X)$ $\quad \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact.

$\Leftrightarrow$ $\forall$ affine open $U \hookrightarrow X$, $\to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U) \to 0$ exact

$\Leftrightarrow$ $\exists$ affine open cover U_i's of X s.t. $\to \mathcal{F}_1(U_i) \to \mathcal{F}_2(U_i) \to \mathcal{F}_3(U_i) \to 0$ exact

14.4.A $\quad \pi : \operatorname{Spec} A \to \operatorname{Spec} B$. $\quad M \in Mod_A$.

Then $\pi_* \widetilde{M} = \widetilde{M_B}$.

p.f. $\pi_* \widetilde{M}(D(g)) = M_{\pi^\#(g)} = \widetilde{M_B}(D(g))$ where $\pi^\# : B \to A$ corresponds to π

14.4.B. If $\pi : X \to Y$ is affine, then $\pi_* : QCoh(X) \to QCoh(Y)$ is exact.

p.f. Obvious.

14.4.C. For QCQS $\pi : X \to Y$, $\pi_* : QCoh(X) \to QCoh(Y)$.

p.f. Let $Y = \bigcup_i Y_i$ be an affine open cover of Y. then it suffices to show

$\pi_* \mathcal{F}|_{Y_i} = (\pi_i)_*(\mathcal{F}|_{\pi^{-1}(Y_i)})$ is quasi-coherent where $\pi_i : \pi^{-1}(Y_i) \to Y_i$

WMA Y is affine. Then X is qcqs. Easy: $\pi_* \mathcal{F}|_V = (\pi|_{\pi^{-1}(V)})_*(\mathcal{F}|_{\pi^{-1}(V)})$

Let $M = \pi_* \mathcal{F}(Y) = \mathcal{F}(X)$, then by qcqs lemma,

$\forall g \in \mathcal{O}_Y(Y)$, $M_g = M_{\pi^\#(g)} = \mathcal{F}(X)_{\pi^\#(g)} = \mathcal{F}(X_{\pi^\#(g)}) = \mathcal{F}(\pi^{-1}(D(g))) = \pi_* \mathcal{F}(D(g))$

$\Rightarrow \pi_* \mathcal{F} = \widetilde{M} \in QCoh(Y)$

14.4.D Suppose $\pi : X \to Y$ is finite and $\mathcal{F} \in QCoh(X)$ is of f.t., then $\pi^* \mathcal{F}$ is of f.t.

$\mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0 \Rightarrow \pi_*(\mathcal{O}_X^{\oplus n}) \longrightarrow \pi_* \mathcal{F} \to 0$ exact by 14.4.B

$\pi_*(\mathcal{O}_X^{\oplus n}) = (\pi_* \mathcal{O}_X)^{\oplus n}$. Since π finite, $\pi_* \mathcal{O}_X$ is of f.t. $\Rightarrow$ So does $\pi_* \mathcal{F}$.

§14.5. Pullbacks

Let $\pi: X \to Y$ be a morphism of schemes, we want to get a functor $\pi^*: QCoh(Y) \to QCoh(X)$, (Let $\mathcal{G} \in QCoh(Y)$), such that

(a) (Affine locally) If $\operatorname{Spec} B \hookrightarrow Y$, $\operatorname{Spec} A \hookrightarrow \pi^{-1}(\operatorname{Spec} B) \hookrightarrow X$, $\mathcal{G}|_{\operatorname{Spec} B} = \widetilde{N}$, $N \in Mod_B$, then $\pi'^*(\mathcal{G}|_{\operatorname{Spec} B}) \cong \widetilde{N \otimes_B A}$ where $\pi' := (\operatorname{Spec} A \xrightarrow{\pi} \operatorname{Spec} B)$.

(b) ((Universal property, adjointness)) $\mathcal{F} \in \mathcal{O}_X\text{-mod}$ $\mathcal{G} \in \mathcal{O}_Y\text{-mod}$.

$\operatorname{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \xleftrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F})$ (We could extend $\pi^*: \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$)

(c) (Inherit from ringed space level) For $X \xrightarrow{\pi} Y$ ringed spaces, $\mathcal{F} \in \mathcal{O}_X\text{-mod}$, $\mathcal{G} \in \mathcal{O}_Y\text{-mod}$,

$\pi^*\mathcal{G} := (\pi^{-1}\mathcal{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X)^{\dagger}$ $\quad \pi^*_{(\text{schemes})} = \pi^*_{(\text{ringed spaces})}|_{(\text{schemes})}$

We could use (a) to give a construction by glueing.

If $Y = \operatorname{Spec} B$ is affine, then $\mathcal{G} = \widetilde{N}$, $N \in Mod_B$.

$\forall \operatorname{Spec} A \hookrightarrow X$, Define $\Gamma(\operatorname{Spec} A, \pi^*\mathcal{G}) = N \otimes_B A$

Then $\Gamma(\operatorname{Spec} A_f, \pi^*\mathcal{G}) = N \otimes_B A_f = \Gamma(\operatorname{Spec} A, \pi^*\mathcal{G})_f$

By Ex6.2.D, this implies that we can obtain $\pi^*\mathcal{G} \in QCoh(X)$ satisfying the condition.

14.5.A When $Y = \operatorname{Spec} B$, $\operatorname{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \xleftrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F})$ holds. $\mathcal{F} \in \mathcal{O}_X\text{-mod}$.

pf: Write $\mathcal{G} = \widetilde{N}$, $N \in Mod_B$. $Y = \bigcup_j V_j$. $V_j = \operatorname{Spec}(B_{g_j})$.

Write $\pi^{-1}(V_\ell) = \bigcup_i U_i$, $U_i = \operatorname{Spec} A_i$, $U_i \cap U_j = \bigcup_k U_{ijk}$. $U_{ijk} = \operatorname{Spec} A_{ijk}$

1° $\forall \varphi \in \operatorname{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F})$. Let $\varphi \mapsto \psi$.

$\psi(V_\ell): N_{g_\ell} = \mathcal{G}(V_\ell) \to (\pi_*\mathcal{F})(V_\ell) = \mathcal{F}(\pi^{-1}(V_\ell))$

We give the definition of

$\mathcal{F}(\pi^{-1}(V_\ell)) = eq\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)\right) = eq\left(\prod_i \mathcal{F}(U_i) \to \prod_{ijk} \mathcal{F}(U_{ijk})\right)$

$U = \operatorname{Spec} A \hookrightarrow X$, $\widetilde{\varphi}_U: (\pi^*\mathcal{G})(U) \to \mathcal{F}(U)$ induced by φ

The data of
$$\left(\begin{tikzcd} \tilde{\varphi}_{U_i} : \pi^*\mathcal{G}(U_i) \arrow[r] \arrow[d, equal] & \mathcal{F}(U_i) \\ N \arrow[r] & N \otimes_B A_i \arrow[u] \end{tikzcd}, \quad \begin{tikzcd} \tilde{\varphi}_{U_{ijk}} : \pi^*\mathcal{G}(U_{ijk}) \arrow[r] \arrow[d, equal] & \mathcal{F}(U_{ijk}) \\ N \arrow[r] & N \otimes_B A_{ijk} \arrow[u] \end{tikzcd}\right)$$

gives a morphism $N \to \mathcal{F}(\pi^{-1}(V_\ell))$

2° $\forall \psi \in \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F}) \overset{1:1}{\rightsquigarrow} \forall \ell.\ N_{g_\ell} \to \mathcal{F}(\pi^{-1}(V_\ell)) \rightsquigarrow N_{g_\ell} \otimes_{B_{g_\ell}} A \to \mathcal{F}(X) \to \mathcal{F}(\operatorname{Spec} A) = M$, $A \in \{A_i, A_{ijk}\}$

$\rightsquigarrow \pi^*\mathcal{G}|_{U_i} \to \mathcal{F}|_{U_i} \rightsquigarrow \pi^*\mathcal{G} \xrightarrow{\varphi} \mathcal{F}$

3° Check the two maps are inverse to each other.

Use $\mathrm{Hom}_A(N \otimes_B A, M) = \mathrm{Hom}_B(N, M_B)$.

When we admit the universal property of π^* i.e. π^* is left adjoint to π_*, we could easily know $\pi^*\mathcal{G}$ is unique up to a unique isom. by abstract nonsense. And it is true for affine Y.

14.5.B. $i: U \hookrightarrow X$, $\mathcal{F} \in \mathcal{O}_X\text{-mod}$, then $i^*\mathcal{F} \cong \mathcal{F}|_U$

pf. It suffices to show $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{E}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_*\mathcal{E})$ $\forall \mathcal{E} \in \mathcal{O}_U\text{-mod}$. and functorial in $\mathcal{E}$

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{E}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_*\mathcal{E})$$

"$\to$": $\varphi: \mathcal{F}|_U \to \mathcal{E}$. We define the corresponding $\psi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_*\mathcal{E})$

$\forall V \hookrightarrow X$, $\psi(V): \mathcal{F}(V) \to \mathcal{E}(V \cap U)$ is given by $(\mathcal{F}(V) \to \mathcal{F}(V \cap U) = \mathcal{F}|_U(V \cap U) \xrightarrow{\varphi} \mathcal{E}(V \cap U))$

"$\leftarrow$": $\psi: \mathcal{F} \to i_*\mathcal{E}$. We define the corresponding $\varphi \in \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{E})$

$\forall V \hookrightarrow U$, $\varphi(V): \mathcal{F}|_U(V) \to \mathcal{E}(V)$ is given by $(\mathcal{F}|_U(V) = \mathcal{F}(V) \xrightarrow{\psi} i_*\mathcal{E}(V) = \mathcal{E}(V))$

It's easy to show they are inverse to each other.

Prop. $V \overset{j}{\hookrightarrow} Y$, $U \overset{\text{open}}{\subseteq} \pi^{-1}(V)$.
$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, "\pi|_U"'] \arrow[dr, phantom, "2"] & X \arrow[d, "\pi"] \\ V \arrow[r, hook, "j"] & Y \end{tikzcd}$$
, then $\pi|_U^*(\mathcal{G}|_V) = (\pi^*\mathcal{G})|_U$

p.f. We'll show $(\pi^*\mathcal{G})|_U$ satisfies the universal property of $\pi|_U^*(\mathcal{G}|_V)$.

Let $\mathcal{F}' \in \mathcal{O}_{U}$-mod.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_U}((\pi^*\mathcal{G})|_U, \mathcal{F}') &\xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_U}(i^*(\pi^*\mathcal{G}), \mathcal{F}') \\
&\xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, i_*\mathcal{F}') \\
&\xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \pi_* i_*\mathcal{F}') \\
&\xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_*(\pi|_U)_*\mathcal{F}') \\
&\xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_V}(j^*\mathcal{G}, (\pi|_U)_*\mathcal{F}') \\
&\xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G}|_V, (\pi|_U)_*\mathcal{F}')
\end{aligned}$$

which is functorial in $\mathcal{F}'$.

Thus, the previous construction of π^* satisfies the universal property when Y is an open subscheme of an affine scheme.

For general Y, we glue! (Routine arguement). We omit the details.

Then we turn to study the ringed spaces

$\pi: X \longrightarrow Y$ $\mathcal{F} \in \mathcal{O}_X$-mod. $\mathcal{G} \in \mathcal{O}_Y$-mod

Recall that we defined that $\pi^*\mathcal{G} := (\pi^{-1}\mathcal{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X)^{\dagger}$: $\mathcal{O}_Y$-mod $\to$ $\mathcal{O}_X$-mod.

Ex 7.2.D / 14.5.C. told us that $\mathrm{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F})$

Actually, it is easy.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) &\xleftrightarrow[\text{univ of } (\,)^{\dagger}]{1:1} \mathrm{Hom}_{\mathcal{O}_X}(\pi^{-1}\mathcal{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathcal{F}) \\
&\xleftrightarrow[\text{tensor adjoint}]{1:1} \mathrm{Hom}_{\pi^{-1}\mathcal{O}_Y}(\pi^{-1}\mathcal{G}, \mathcal{F}) \\
&\xleftrightarrow[\pi_*,\ \pi^{-1} \text{ adjoint}]{1:1} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \pi_*\mathcal{F})
\end{aligned}$$

which is functorial in $\mathcal{F}$

$\Rightarrow$ universal property holds

$$\left(\begin{tikzcd} (X, \mathcal{O}_X) \arrow[r, "i"] \arrow[rr, bend right, "\pi"'] & (X, \pi^{-1}\mathcal{O}_Y) \arrow[r, "\pi'"] & (Y, \mathcal{O}_Y) \end{tikzcd} \right)$$

Thus, it remains to show $\pi^*: QCoh(Y) \to QCoh(X)$ in scheme case

Before proving this, we'll rewrite some properties we have shown by $\underset{\text{univ. prop.}}{14.5.C}$

14.5.B' $i: U \hookrightarrow X$, $\mathcal{F} \in \mathcal{O}_X\text{-mod}$, then $i^*\mathcal{F} \cong \mathcal{F}|_U$

(Still check universal property)

Prop' $V \overset{j}{\hookrightarrow} Y$, $U \overset{\text{open}}{\subseteq} \pi^{-1}(V)$. $\begin{array}{ccc} U & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle \pi|_U}\downarrow & & \downarrow{\scriptstyle \pi} \\ V & \overset{j}{\hookrightarrow} & Y \end{array}$, then $\pi|_U^*(\mathcal{G}|_V) = (\pi^*\mathcal{G})|_U$

(still check universal property).

Finally, 14.5.D. When $X, Y \in Sch$, $\mathcal{G} \in QCoh(Y)$, $\pi^*\mathcal{G} \in QCoh(X)$

pf. It suffices to show: $\exists \{U_i\}$, an open covering for X s.t. $(\pi^*\mathcal{G})|_{U_i} \in QCoh(U_i)$.

By prop' we may assume $Y = \operatorname{Spec} B$ is affine! Write $\mathcal{G} = \tilde{N}$, $N \in Mod_B$.

$\forall$ affine open $U \hookrightarrow X$, $(\pi^*\mathcal{G})|_U = (\pi|_U)^*\mathcal{G}$. Write $U = \operatorname{Spec} A$

$\Rightarrow (\pi|_U)^*\mathcal{G} = (\tilde{N} \otimes \mathcal{O}_U)^+ = (\widetilde{N \otimes_B A})^+ = \widetilde{N \otimes_B A}$.

$\Rightarrow \pi^*\mathcal{G} \in QCoh(X)$.

Thus, we have another rigorous construction for π^*.

14.5.7. Prop $\pi: X \to Y$. qcqs. then $\pi_*: QCoh(X) \to QCoh(Y)$, $\pi^*: QCoh(Y) \to QCoh(X)$ are adjoint to each other

$Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \overset{1:1}{\longleftrightarrow} Hom_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F})$ functorial in both $\mathcal{G}, \mathcal{F}$.

14.5.E Pullback is right-exact.

pf. If $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is exact in $QCoh(Y)$.

Then we'll show $\pi^*\mathcal{G}' \to \pi^*\mathcal{G} \to \pi^*\mathcal{G}'' \to 0$ is exact in $QCoh(X)$.

Write $X = \bigcup_i U_i$. $Y = \bigcup_j V_j$ $\quad U_i \xrightarrow{\pi|_{U_i}} V_j$ for some j, $\forall i$.

It suffices to show $\pi^*\mathcal{G}' \to \pi^*\mathcal{G} \to \pi^*\mathcal{G}'' \to 0$ is exact on every U_i.

By 14.5.B $\quad (\pi^*\mathcal{G}')|_{U_i} \to (\pi^*\mathcal{G})|_{U_i} \to (\pi^*\mathcal{G}'')|_{U_i} \to 0$ exact

$\iff (\pi|_{U_i})^*(\mathcal{G}'|_{V_j}) \to (\pi|_{U_i})^*(\mathcal{G}|_{V_j}) \to (\pi|_{U_i})^*(\mathcal{G}''|_{V_j}) \to 0$ exact.

$\iff (\pi|_{U_i})^* : QCoh(V_j) \to QCoh(U_i)$ is exact

$\iff - \otimes_{B_j} A_i : B_j\text{-mod} \to A_i\text{-mod}$ is exact $\quad$ ✓.

14.5.F Pullback preserves f.t. and f.p.

p.f.
$$\begin{tikzcd} X \arrow[r, "\pi"] & Y \\ \operatorname{Spec} A \arrow[u, hook] \arrow[r] \arrow[d, equal] & \operatorname{Spec} B \arrow[u, hook] \arrow[d, equal] \\ U & V \end{tikzcd} \qquad \pi^*\mathcal{O}_V = \mathcal{O}_U$$

$\mathcal{G} \in QCoh(Y)$ $\exists$ cover V's s.t. $\mathcal{O}_V^{\oplus n} \to \mathcal{G}|_V \to 0$ exact
of f.t.

$\Rightarrow \quad \mathcal{O}_U^{\oplus n} \to (\pi^*\mathcal{G})|_U \to 0$ exact.

of f.p. $\quad \mathcal{O}_V^{\oplus m} \to \mathcal{O}_V^{\oplus n} \to \mathcal{G}|_V \to 0$ exact

$\Rightarrow \quad \mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to (\pi^*\mathcal{G})|_U \to 0$ exact.

14.5.G $\xi : W \to X$ $\quad \pi : X \to Y$. Prove $(\pi \circ \xi)^* = \xi^* \pi^*$

p.f. Check the universal property. It holds for $\pi_* \xi_* = (\pi \circ \xi)_*$.

We do this in the $\mathcal{O}_X, \mathcal{O}_Y, \mathcal{O}_W$-mod level.

14.5.H. $\pi : X \to Y$. $\pi(p) = q$ $\quad \mathcal{G} \in \overset{\mathcal{O}_Y\text{-mod}}{QCoh(Y)}$.

(a) $(\pi^*\mathcal{G})_p \xrightarrow{\sim} \mathcal{G}_q \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{X,p}$

p.f. $(\pi^*\mathcal{G})_p = (\pi^{-1}\mathcal{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X)_p = \varinjlim_{U \ni p} (\pi^{-1}\mathcal{G})(U) \otimes_{\pi^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U) = (\pi^{-1}\mathcal{G})_p \otimes_{(\pi^{-1}\mathcal{O}_Y)_p} \mathcal{O}_{X,p} = \mathcal{G}_q \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{X,p}$

(b) $(\pi^*\mathcal{G})|_p \xrightarrow{\sim} \mathcal{G}|_q \otimes_{k(q)} k(p)$

pf. By (c), it suffices to show: $(\mathcal{G}_q \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{X,p}) \otimes_{\mathcal{O}_{X,p}} k(p) = \mathcal{G}|_q \otimes_{k(q)} k(p)$

LHS $= \mathcal{G}_q \otimes_{\mathcal{O}_{Y,q}} k(p) = \mathcal{G}_q \otimes_{\mathcal{O}_{Y,q}} (k(q) \otimes_{k(q)} k(p)) = \mathcal{G}_q \otimes_{k(q)} k(p)$

14.5.I. $\mathcal{G}, \mathcal{G}' \in QCoh(Y)$. $\pi^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{G}') = \pi^*\mathcal{G} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}'$

pf. Take an affine covering of Y, write $Y = \bigcup_j V_j$, $V_j = \operatorname{Spec} B_j$. $\mathcal{G}|_{V_j} = \widetilde{N_j}$, $\mathcal{G}'|_{V_j} = \widetilde{N_j'}$.

Write $\pi^{-1}(V_j) = \bigcup_i U_{ji}$, $U_{ji} = \operatorname{Spec}(A_{ji})$

Then $\pi^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{G}')|_{U_{ji}} = (\pi|_{U_{ji}})^*((\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{G}')|_{V_j}) = (\pi|_{U_{ji}})^*(\widetilde{N_j \otimes_{B_j} N_j'}) = \widetilde{N_j \otimes_{B_j} N_j' \otimes_{B_j} A_{ji}}$

$(\pi^*\mathcal{G})|_{U_{ji}} = \widetilde{N_j \otimes_{B_j} A_{ji}}$, $(\pi^*\mathcal{G}')|_{U_{ji}} = \widetilde{N_j' \otimes_{B_j} A_{ji}}$

$(\pi^*\mathcal{G} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}')|_{U_{ji}} = (\widetilde{N_j \otimes_{B_j} A_{ji} \otimes_{A_{ji}} A_{ji} \otimes_{B_j} N_j'}) = \widetilde{N_j \otimes_{B_j} N_j' \otimes_{B_j} A_{ji}}$

Thus, the assertion follows by checking locally.

14.5.K.
$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

$\mathcal{F} \in \mathcal{O}_X$-mod. $\exists$ natural $\psi^*\pi_*\mathcal{F} \longrightarrow \pi'_*\varphi^*\mathcal{F}$

$$\begin{aligned}
\operatorname{Hom}(\psi^*\pi_*\mathcal{F}, \pi'_*\varphi^*\mathcal{F}) &\xleftrightarrow{1:1} \operatorname{Hom}(\pi'^*\psi^*\pi_*\mathcal{F}, \varphi^*\mathcal{F}) \\
&\xleftrightarrow{1:1} \operatorname{Hom}(\varphi^*\pi^*\pi_*\mathcal{F}, \varphi^*\mathcal{F}) \\
&\longleftarrow \operatorname{Hom}(\pi^*\pi_*\mathcal{F}, \mathcal{F}) \\
&\xleftrightarrow{1:1} \operatorname{Hom}(\pi_*\mathcal{F}, \pi_*\mathcal{F}) \\
&\qquad\qquad \ni \mathrm{id}
\end{aligned}$$

$\mathcal{F} \in QCoh(X)$. need π, π' qcqs. same arguement.

14.5.L $\pi: X \to Y$ qcqs, $\mathcal{F} \in QCoh(X)$ $\mathcal{G} \in QCoh(Y)$

Take an affine covering of Y, write $Y = \bigcup_j V_j$, $V_j = \operatorname{Spec} B_j$. $\mathcal{G}|_{V_j} = \widetilde{N_j}$

Write $U_j = \pi^{-1}(V_j) = \bigcup_i U_{ji}$ $U_{ji} = \operatorname{Spec}(A_{ji})$ $\mathcal{F}|_{U_{ji}} = \widetilde{M_{ji}}$

$$((\pi_* \mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G})|_{V_j} = (\pi_* \mathcal{F})|_{V_j} \otimes_{\mathcal{O}_{V_j}} \widetilde{N_j} = (\pi|_{U_j})_* (\mathcal{F}|_{U_j}) \otimes_{\mathcal{O}_{V_j}} \widetilde{N_j}$$

$$\pi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^* \mathcal{G})|_{V_j} = (\pi|_{U_j})_* (\mathcal{F}|_{U_j} \otimes_{\mathcal{O}_{U_j}} (\pi^* \mathcal{G})|_{U_j}) = (\pi|_{U_j})_* (\mathcal{F}|_{U_j} \otimes_{\mathcal{O}_{U_j}} (\pi|_{U_j})^* (\widetilde{N_j}))$$

$D(g) \subseteq V_j$. $((\pi_* \mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G})|_{V_j} (D(g)) = \mathcal{F}((U_j)_{\pi^\#(g)}) \otimes_{(B_j)_g} (N_j)_g$

$$\pi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^* \mathcal{G})|_{V_j} (D(g)) = eq\left(\prod_i ((M_{ji})_{\pi^\#(g)} \otimes_{(A_{ji})_{\pi^\#(g)}} ((A_{ji})_{\pi^\#(g)} \otimes_{(B_j)_g} (N_j)_g) \longrightarrow *** \right)$$

$$= eq\left(\prod_i (M_{ji})_{\pi^\#(g)} \otimes_{(B_j)_g} (N_j)_g \longrightarrow *** \right)$$

$\Rightarrow ((\pi_* \mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G})|_{V_j} (D(g)) \longrightarrow \pi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^* \mathcal{G})|_{V_j} (D(g))$ ....

(a). $\exists$ natural $((\pi_* \mathcal{F}) \otimes \mathcal{G}) \to \pi_* (\mathcal{F} \otimes \pi^* \mathcal{G})$

(b) If $\mathcal{G}$ is locally free, we may assume $N_j = B_j^{\oplus m}$

Then by direct calculating, $\pi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^* \mathcal{G})|_{V_j} (D(g)) \xrightarrow{\sim} \pi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^* \mathcal{G})|_{V_j} (D(g))$

key: $M \otimes_B B^m \cong M^m$

(c) If π is affine, then U_j is affine! $\mathcal{F}|_{U_j} = \widetilde{M_j}$

$$(\pi|_{U_j})_* (\mathcal{F}|_{U_j} \otimes_{\mathcal{O}_{U_j}} (\pi|_{U_j})^* (\widetilde{N_j})) = (\pi|_{U_j})_* (\widetilde{M_j} \otimes_{A_j} \widetilde{A_j \otimes_{B_j} N_j}) = (\pi|_{U_j})_* (\widetilde{M_j \otimes_{B_j} N_j}) = M_j \otimes_{B_j} N_j$$

$$(\pi|_{U_j})_* (\mathcal{F}|_{U_j}) \otimes_{\mathcal{O}_{V_j}} \widetilde{N_j} = \widetilde{M_j} \otimes_{B_j} \widetilde{N_j} = M_j \otimes_{B_j} N_j$$

Rmk. $\mathcal{F}, \mathcal{F}' \in \mathcal{O}_X$-mod. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}' :=$ the sheafification of $(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}'(U))$

When $X = \operatorname{Spec} A$, $\mathcal{F} = \widetilde{M}$, $\mathcal{F}' = \widetilde{M'}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}' = \widetilde{M \otimes_A M'}$

Pulling back ideal sheaves

$i: X \hookrightarrow Y$ closed emb $\mu: Y' \to Y$ morphism. $X' := X \times_Y Y'$ $i': X' \to Y'$ closed emb.

$$\begin{array}{ccc} X' & \to & X \\ {\scriptstyle i'}\downarrow & \square & \downarrow{\scriptstyle i} \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

Then $\mu^* \mathcal{I}_{X/Y} = \mathcal{I}_{X'/Y'}$ may not be true.

$0 \to I \to A \to A/I \to 0$

$\_\ I \otimes B \to B \to B/IB \to 0$

§14.6.

A is a ring, $S_\bullet$ is a graded ring with $S_0 = A$, $S_+ = \bigoplus_{n\geq 1} S_n$ is a f.g. ideal

$M_\bullet$ is a $\mathbb{Z}$-graded module.

We'll construct $\widetilde{M_\bullet}$, a sheaf on $\operatorname{Proj} S_\bullet$

$\forall$ homogenous $f \in S_+$, define $\widetilde{M_\bullet}|_{D_+(f)} = \widetilde{(M_f)_0}$. $D_+(f) = \operatorname{Spec}((S_\bullet)_f)_0$.

$\deg f = n$, $\deg g = m$. $D_+(fg) = \begin{cases} D(\frac{f^m}{g^n}) \subseteq D_+(g) = \operatorname{Spec}(((S_\bullet)_f)_0) \\ D(\frac{g^n}{f^m}) \subseteq D_+(f) = \operatorname{Spec}(((S_\bullet)_g)_0) \end{cases}$

$$(\widetilde{M_\bullet}|_{D_+(f)})|_{D_+(fg)} = \widetilde{((M_f)_0)_{\frac{g^n}{f^m}}} = \widetilde{(M_{fg})_0} = \cdots = \widetilde{M_\bullet}|_{D_+(g)}|_{D_+(fg)}$$

$\Rightarrow \widetilde{M_\bullet}$ is a sheaf glued by $\widetilde{(M_f)_0}$'s and hence a quasi-coherent sheaf.

14.6.A. (Stalk) $(\widetilde{M_\bullet})_{\mathfrak{p}} = ((M_\bullet)_{\mathfrak{p}})_0$

pf. WMA $[\mathfrak{p}] \in D_+(f)$. Then $[\mathfrak{p}] = (\mathfrak{p}(S_\bullet)_f)_0 \triangleq \mathfrak{q}$. Write $n = \deg f$.

$(\widetilde{M_\bullet})_{\mathfrak{p}} = [((M_\bullet)_f)_0]_{\mathfrak{q}} \underset{\Delta}{=} ((M_\bullet)_{\mathfrak{p}})_0$

Δ: $\forall \frac{m}{s} \in RHS$, $s \in M_\bullet \setminus \mathfrak{p}$, $\deg s = \deg m = k$ $\frac{m}{s} = (\frac{s^{n-1}m}{f^k}) / \frac{s^n}{f^k}$, $\frac{s^n}{f^k} \in ((M_\bullet)_f)_0 \setminus \mathfrak{q} \Rightarrow \frac{m}{s} \in LHS$

$\forall (\frac{m}{f^t})/(\frac{w}{f^\ell}) \in LHS$, $\frac{w}{f^\ell} \in \mathfrak{q} \Rightarrow w \notin \mathfrak{p} \Rightarrow (\frac{m}{f^t})/(\frac{w}{f^\ell}) = \frac{f^\ell m}{f^t w} \in RHS$

14.6.B. $\sim$ is an exact functor from the category of graded $S_\bullet$-module to the category of quasicoherent sheaves on $\operatorname{Proj}(S_\bullet)$

p.f. It is a functor since for any $\phi: M \to N$, we have the following diagram:

$$\begin{tikzcd} ((M_\bullet)_f)_0 \arrow[r, "\phi_f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & ((N_\bullet)_f)_0 \arrow[d] \\ ((M_\bullet)_{fg})_0 \arrow[r, "\phi_{fg}"] & ((N_\bullet)_{fg})_0 \end{tikzcd}$$

"$\sim$" is exact since "localization" and "take degree 0 part" are exact.

Notation: $(M_\bullet)_{\geq n} := \bigoplus_{k\geq n} M_k$

14.6.C. If $(M_\bullet)_{\geq n} = (M'_\bullet)_{\geq n}$ for some $n \in \mathbb{N}_+$, then $\widetilde{M_\bullet} \cong \widetilde{M'_\bullet}$.

p.f. It suffices to show: for any $f \in (S_\bullet)_+$ homogenous,

$((M_\bullet)_f)_0 = ((M'_\bullet)_f)_0$.

By symmetry, it suffices to show $((M_\bullet)_f)_0 \subseteq ((M'_\bullet)_f)_0$

$\forall \frac{m}{f^t} \in ((M_\bullet)_f)_0$, $\frac{m}{f^t} = \frac{f^n m}{f^{t+n}} \in ((M'_\bullet)_f)_0$ ✓

14.6.D. $\exists$ map: $M_\bullet \longrightarrow \Gamma(\operatorname{Proj} S_\bullet, \widetilde{M}_\bullet)$

p.f. Take an open covering $\operatorname{Proj} S_\bullet$, write $\operatorname{Proj}(S_\bullet) = \bigcup_i D_+(f_i)$

Then $\Gamma(\operatorname{Proj} S, \widetilde{M}_\bullet) = \operatorname{eq}\left(\prod_i ((M_\bullet)_{f_i})_0 \rightrightarrows \prod_{i,j} ((M_\bullet)_{f_if_j})_0\right)$

M_0

14.6.E. $0 \longrightarrow I_\bullet \longrightarrow S_\bullet \longrightarrow S_\bullet/I_\bullet \longrightarrow 0$

Apply $\sim$ $\Rightarrow$ $0 \longrightarrow \widetilde{I_\bullet} \longrightarrow \widetilde{S_\bullet} \longrightarrow \widetilde{S_\bullet/I_\bullet} \longrightarrow 0$

corresponds to $\operatorname{Proj}(S_\bullet/I_\bullet) \hookrightarrow \operatorname{Proj}(S_\bullet)$.

15.1.4 Let $M_\bullet$ be a graded $S_\bullet$-mod, define $M(m)_\bullet$ by $(M(m)_\bullet)_d := M_{m+d}$.

Thus, $\Gamma(D_+(f), \widetilde{M(m)_\bullet}) = ((M_\bullet)_f)_m$

15.1.5. If $S_\bullet$ is a graded ring generated in degree 1, we define $\mathcal{O}_{\operatorname{Proj} S_\bullet}(m)$ by $\widetilde{S(m)_\bullet}$.

15.1.G. $\mathcal{O}(m)$ is an invertible sheaf on $\operatorname{Proj}(S_\bullet)$

p.f. Assume $f_1, \dots, f_n \in S_1$ generate $S_\bullet$.

Then $\Gamma(D_+(f_j), \mathcal{O}(m)) = ((S_\bullet)_{f_j})_m \longrightarrow ((S_\bullet)_{f_j})_0$

$\frac{x}{f_j^t} \longmapsto \frac{x}{f_j^{m+t}}$ (inj)

Inverse to each other

$\frac{f_j^m x}{f_j^\ell} \longleftarrow \frac{x}{f_j^\ell}$ (inj)

$\Rightarrow \mathcal{O}(m)|_{D_+(f_j)} \cong \mathcal{O}|_{D_+(f_j)}$. $\Rightarrow \mathcal{O}(m)$ is an invertible sheaf.

If $S_\bullet$ is generated in degree 1, and $\mathcal{F} \in QCoh(\operatorname{Proj} S_\bullet)$ Define $\mathcal{F}(m) := \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(m)$
This is called twisting $\mathcal{F}$ by m.

15.1.H. If $S_\bullet$ is generated in degree 1, then $\widetilde{M_\bullet}(m) \cong \widetilde{M(m)_\bullet}$.

p.f. Assume $f_1, \ldots, f_n \in S_1$ generate $S_\bullet$.

$$\Gamma(D_+(f_j), \widetilde{M_\bullet}(m)) = ((M_\bullet)_{f_j})_0 \otimes_{((S_\bullet)_{f_j})_0} ((S_\bullet)_{f_j})_m \xrightarrow{\sim} ((M_\bullet)_{f_j})_m = \Gamma(D_+(f_j), \widetilde{M(m)_\bullet})$$

Note $((S_\bullet)_{f_j})_m = ((S_\bullet)_{f_j})_0 \cdot f_j^m$

15.1.I. If $S_\bullet$ is generated in degree 1, then $\mathcal{O}(m_1+m_2) \cong \mathcal{O}(m_1) \otimes \mathcal{O}(m_2)$

$$\mathcal{O}(m_1) \otimes \mathcal{O}(m_2) = \mathcal{O}(m_1)(m_2) \cong (\widetilde{S(m_1)(m_2)_\bullet}) = \widetilde{S(m_1+m_2)_\bullet} = \mathcal{O}(m_1+m_2)$$