16.4. Grassmanian.

$S = \mathrm{Spec} A$ affine Noetherian, V/S vector bundle.

Recall functor

$\underline{\mathrm{Grass}_S}(V,r) : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$

$(T,f) \longmapsto \{\underset{\substack{\| \\ f^*V}}{V_T} \twoheadrightarrow Q \mid Q$ locally free of rank $r\}/\sim$ as quotient

$\underline{\mathrm{Grass}}^I_S(\mathcal{O}_S^n, r) := (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$

$T \longmapsto \{[\mathcal{O}_T^n \twoheadrightarrow Q] \in \underline{\mathrm{Grass}_S}(\mathcal{O}_S^n, r)(T) \mid \mathcal{O}_T^I \xrightarrow{e^I} \mathcal{O}_T^n \twoheadrightarrow Q$ isomorphism$\}$

Fact we have proved:

① $\{\underline{\mathrm{Grass}}^I_S(\mathcal{O}_S^n, r)\}$ open cover of $\underline{\mathrm{Grass}_S}(V,r)$

② $\underline{\mathrm{Grass}}^I_S(\mathcal{O}_S^n, r)$ is represented by $\mathbb{A}_S^{r\times(n-r)}$

then $\underline{\mathrm{Grass}_S}(\mathcal{O}_S^n, r)$ is representable by $\mathrm{Grass}_S(\mathcal{O}_S^n, r) =: \mathrm{Gr}(r,n)$

*Geometric side of Grassmanian:

Now we assume $S = \mathrm{Spec} k$, k field

A k point of $\mathrm{Gr}(r,n)$ is $\mathrm{Spec} k \xrightarrow{f} \mathrm{Gr}(r,n)$

by definition we have $\mathrm{Hom}(\mathrm{Spec} k, \mathrm{Gr}(r,n)) = \{V \twoheadrightarrow Q \mid Q$ v.b of rank $r\}/\sim$

$= \{R \hookrightarrow V$ sub v.b. of rank $n-r\}$

[Slogan: S-point of $\mathrm{Gr}(r,n)$ is subbundle of V.]

Want to show $\mathrm{Grass}_S(\mathcal{O}_S^n, r)$ smooth irreducible projective

① Smooth is done (cover by $\mathbb{A}_S^{r(n-r)}$)

② Projectivity [Plücker Embedding]

consider $\mathbb{P}(\hat{\wedge}\mathcal{O}_S^n) = \underline{\mathrm{Grass}_S}(\wedge^r\mathcal{O}_S^n, 1) : (Sch/S)^{op} \longrightarrow Set.$

$T \longmapsto \{\hat{\wedge}\mathcal{O}_T^n \twoheadrightarrow L \mid L \text{ is line bundle}\}/\!\sim$

free of rank $\binom{n}{r}$

Claim. $\mathbb{P}(\underset{\smile}{\hat{\wedge}\mathcal{O}_S^n})$ represented by $\mathrm{Proj}(\mathrm{Sym}(\wedge^r\mathcal{O}_S^n)^*) = \mathbb{P}^{\binom{n}{r}-1}$

"proj space is 1-dim subspace of a vector space"

Preparation: $\boxed{17.2.C}$ Relative Proj

Given $\mathcal{I}_\cdot = \bigoplus_{n\geq 0}\mathcal{I}_n \in \mathrm{Qcoh}(X)$ graded $\mathcal{O}_X$-algebra

$\forall\, U = \mathrm{Spec}A \hookrightarrow X \qquad \underbrace{\mathcal{I}(U) = \bigoplus_{n\geq 0}\mathcal{I}_n(U)}$ is graded A-alg.

then $\mathrm{Proj}\,\mathcal{I}(U) \longrightarrow A \qquad (M \leftrightarrow \tilde{M})$

By glueing we can define $\mathrm{Proj}_X(\mathcal{I}_\cdot)/X$

$\boxed{17.2.4}$ (Projectivization of v.b.)

How to do "proj space is 1-dim subspace of a vector space"

$\Longleftrightarrow$ 1-dim Quotient space of its dual.

Simple case: $V/\mathrm{Spec}\,k$ of rank n, then $\mathrm{Proj}(\mathrm{Sym}\,V^*) = \mathrm{Proj}\,k[x_1, \dots, x_n] = \mathbb{P}_k^{n-1}$

dual basis

General case: V/S vb of rank n, then $\mathbb{P}V := \mathrm{Proj}(\mathrm{Sym}\,V^*)$

Back to Claim, it suffices to prove $\mathbb{P}V$ represents

$\underline{\mathrm{Grass}}_S(V^*,1): (Sch/S)^{op} \to Set.$

$T \mapsto \{V_T^* \twoheadrightarrow L \mid L \text{ is line bundle}\}/\sim$

Now if $V^* \cong \mathcal{O}_S^n$, this is proved. (In general, $\mathcal{O}_S^n \twoheadrightarrow V^*$)

Why Dual: Given morphism of v.b. $W \hookrightarrow V$, we have $W^* \twoheadrightarrow V^*$

$\underline{\mathrm{Grass}}_S(V^*,1) \hookrightarrow \underline{\mathrm{Grass}}_S(W^*,1)$ which is $\mathrm{Proj}\,\mathrm{Sym}\,V^* \to \mathrm{Proj}\,\mathrm{Sym}\,W^*$ [check]

Now $\mathcal{P}(\hat{\wedge}\mathcal{O}_S^n)$ represented by $\mathrm{Proj}(\mathrm{Sym}(\wedge^r\mathcal{O}_S^n)^*) = \mathbb{P}^{\binom{n}{r}-1}$

$\Phi: \mathrm{Grass}_S(\mathcal{O}_S^n, r) \longrightarrow \mathcal{P}(\hat{\wedge}\mathcal{O}_S^n)$

$[\mathcal{O}_T^n \twoheadrightarrow Q] \longmapsto [\hat{\wedge}\mathcal{O}_T^n \twoheadrightarrow \hat{\wedge}Q]$

Claim: Φ is closed immersion.

$\mathcal{P}^I(\hat{\wedge}\mathcal{O}_S^n): (Sch/S)^{op} \longrightarrow Set$

$T \longmapsto \{[\hat{\wedge}\mathcal{O}_T^n \twoheadrightarrow L] \mid \hat{\wedge}\mathcal{O}_T^I \xrightarrow{\hat{\wedge}e^I} \hat{\wedge}\mathcal{O}_T^n \twoheadrightarrow L \text{ isomorphism}\}$

$\mathcal{P}^I(\hat{\wedge}\mathcal{O}_S^n) \hookrightarrow \mathcal{P}(\hat{\wedge}\mathcal{O}_S^n)$ open covering.

Now
$$\begin{tikzcd}
\mathrm{Grass}_S^I(\mathcal{O}_S^n, r) \arrow[r] \arrow[d, "\phi"'] & \mathcal{P}^I(\hat{\wedge}\mathcal{O}_S^n) \arrow[d, "\phi"] \\
\mathrm{Grass}_S(\mathcal{O}_S^n, r) \arrow[r] & \mathcal{P}(\hat{\wedge}\mathcal{O}_S^n)
\end{tikzcd}$$
is it fibre product?

$\forall T$ $(\mathcal{P}^I \times_{\mathcal{P}} Gr)(T) = \{(\hat{\wedge}\mathcal{O}_S^n \twoheadrightarrow \hat{\wedge}Q,\ \mathcal{O}_S^n \twoheadrightarrow Q) \mid \hat{\wedge}\mathcal{O}_T^I \xrightarrow{\hat{\wedge}e^I} \hat{\wedge}\mathcal{O}_T^n \twoheadrightarrow \hat{\wedge}Q \text{ isomorphism}\}$

$= \{\mathcal{O}_S^n \twoheadrightarrow Q \mid \mathcal{O}_T^I \longrightarrow \mathcal{O}_T^n \twoheadrightarrow Q \text{ isomorphism}\}$

(on stalk if $\hat{\wedge}\mathcal{O}_T^I \xrightarrow{\hat{\wedge}f} \hat{\wedge}Q$ $\quad \mathcal{O}_{T,x} \xrightarrow{(\hat{\wedge}f)_x} \mathcal{O}_{T,x}$ $\quad (\hat{\wedge}f)_x \in \mathcal{O}_{T,x}^\times$

then $\det(f_x) \in \mathcal{O}_{T,x}^\times \Rightarrow f_x$ isomorphism) $\quad \overset{\|}{\hat{\wedge}f_x}$ (check affine locally)

$$\begin{array}{ccc} \mathbb{A}^{r(n-r)} & & \mathbb{A}^{\binom{n}{r}-1} \\ \| & & \| \end{array}$$

then since $Grass_S^{I}(\mathcal{O}_S^n, r) \longrightarrow \mathbb{P}^{I}(\overset{r}{\wedge}\mathcal{O}_S^n)$ morphism of affine scheme

$$f \longmapsto \overset{r}{\wedge} f$$

$$A = \left(I_r \;\middle|\; * \right) \overset{\overset{r}{\wedge}}{\longmapsto} \left[\left(\det\left(A\begin{pmatrix} 1 & \cdots & r \\ i_1 & \cdots & i_r \end{pmatrix}\right) e_{i_1} \wedge \cdots \wedge e_{i_r} \right)_{\binom{n}{r}} \right]$$

$$\Big\updownarrow \; \left|A\begin{pmatrix} 1 & \cdots & r \\ 1 & \cdots & r \end{pmatrix}\right| = 1$$

$$\left(\left|A\begin{pmatrix} 1 & \cdots & r \\ i_1 & \cdots & i_r \end{pmatrix}\right| e_{i_1} \wedge \cdots \wedge e_{i_r} \right)_{(i_1 \cdots i_r) \neq (1 \cdots r)}$$

on ring homomorphism $k[x_I]_{\binom{n}{r}-1} \longrightarrow k[y_{ij}]_{\substack{1 \le i \le r \\ r+1 \le j \le n}}$.

$$x_{(i_1 \cdots i_r)} \longmapsto \left| (y_{ij}) \begin{pmatrix} 1 & \cdots & r \\ i_1 & \cdots & i_r \end{pmatrix} \right|$$

which is surjective since $x_{(1, \cdots i-1, i+1 \cdots r, j)} \longmapsto \pm y_{ij}$

since closed immersion is local on the target,

$Grass_S(\mathcal{O}_S^n, r) \longrightarrow \mathbb{P}(\overset{r}{\wedge}\mathcal{O}_S^n)$ closed embedding

then $Grass_S(\mathcal{O}_S^n, r)$ projective.

* Now for general S, take affine covering $S = \bigcup S_i$

$S_i \hookrightarrow S$, then

$$\begin{tikzcd} \underline{Grass_{S_i}(\mathcal{O}_{S_i}^n, r)} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & S_i \arrow[d] \\ \underline{Grass_S(\mathcal{O}_S^n, r)} \arrow[r] & S \end{tikzcd}$$

$Grass_{S_i}(\mathcal{O}_{S_i}^n, r)$ open subfunctor of $Grass_S(\mathcal{O}_S^n, r)$

then $Grass_S(\mathcal{O}_S^n, r)$ representable

$$\begin{tikzcd} \mathbb{P}_{S_i}^N \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S_i \arrow[d] \\ \mathbb{P}_S^N \arrow[r] & S \end{tikzcd}$$

$$\begin{tikzcd} Grass_{S_i}(\mathcal{O}_{S_i}^n, r) \arrow[r, hook] \arrow[d, hook] & \mathbb{P}_{S_i}^N \arrow[d] \\ Grass_S(\mathcal{O}_S^n, r) \arrow[r] & \mathbb{P}_S^N \end{tikzcd}$$

closed immersion is local on the target.

then $Grass_S(\mathcal{O}_S^n, r)$ is projective.

$\square$

13.4. Bertini's Theorem

13.4.1. Define Dual Proj space of V/S being $\mathbb{P}V^*$
i.e. S-point of $\mathbb{P}V^*$ is rank $n-1$ subbundle

We focus on case $/k$. where $V = k^{n+1}$ vector space
then a k-point of $\mathbb{P}V^*$ is a n subspace of V (1-quotspace of V)
which is a hyperplane of $\mathbb{P}V$ ($W \longrightarrow V$ $V^* \twoheadrightarrow W^*$ $\mathbb{P}W \hookrightarrow \mathbb{P}V$)
Δ In general, $\forall p \in \mathbb{P}V^*$ p is $k(p) \longrightarrow \mathbb{P}V^*$, where $k(p) \xrightarrow{f} k$
$k(p) \longrightarrow \mathbb{P}V^*$ induces $f^*V \twoheadrightarrow L$ $\rightsquigarrow$ n subspace of f^*V
which induces $H \hookrightarrow \mathbb{P}(f^*V) = \mathbb{P}^n_{k(p)} = \mathbb{P}^n_k \times k(p)$

Theorem (Bertini)
X smooth closed subvariety of finite type, reduced, separated. of $\mathbb{P}^n_k$ $(k=\bar{k}$ $\mathrm{char}\,k=0)$ of pure dim d
then $\exists U \hookrightarrow \mathbb{P}^{n\vee}_k$ s.t. $\forall p \in U$ assume $p \leftrightarrow H \hookrightarrow \mathbb{P}^n_{k(p)}$
we have H doesn't contain any component of $X_{k(p)}$ (set theoretically)
and $H \cap X_{k(p)}$ smooth over $k(p)$ of pure dim $d-1$.

proof. X smooth $\Rightarrow X$ regular $\Rightarrow X$ locally integral $\xrightarrow{\text{locally integral + connect = integral}}$
irre component of X is connected component of X which is clopen
then WMA X is integral of dim d
consider $\mathbb{P}^n \times \mathbb{P}^{n\vee} = \mathrm{Proj} \bigoplus_{n \geq 0} (\mathrm{Sym}^r V^* \otimes \mathrm{Sym}^{n-r} V)$
then $I = (v^* \otimes v)_{v^* \in \mathrm{Sym}^1 V^*,\, v \in \mathrm{Sym}^1 V}$ then $Y := \mathrm{Proj} \bigoplus_{n \geq 0} (\mathrm{Sym}^r V^* \otimes \mathrm{Sym}^{n-r} V)/I$
$Y(k) = \{(p, H) \mid p \in H\}$ k-points of Y.

X smooth. Suppose X is cut out by $f_1 \cdots f_r$

then $\left(\frac{\partial f_i}{\partial x_j}\right)$ is of constant rank $n-d$

We want $\left(\frac{\partial f_i}{\partial x_j} \middle| \begin{matrix} a_0 \\ \vdots \\ a_n \end{matrix}\right)$ of rank $\leq n-d$. $(=n-d)$ (sufficient condition)

which is $J = ($ $n-d+1$ 子式 of $\left(\frac{\partial f_i}{\partial x_j} \middle| \begin{matrix} y_1 \\ \vdots \\ y_n \end{matrix}\right)$ $)$

$Y' := \mathrm{Proj} \bigoplus_{r\geq 0} (\mathrm{Sym}^r V^* \otimes \mathrm{Sym}^{n-r} V)/J$ $\quad Z := X \cap Y \cap Y'$ closed subscheme of $\mathbb{P}^n \times \mathbb{P}^{n\vee}$

If $\dim Z \leq n-1$, by 5.4.10 $\mathbb{P}^n_k \times \mathrm{Spec}\, k[\frac{y_0}{y_i} \cdots \frac{y_n}{y_i}] \to \mathrm{Spec}\, k[\frac{y_0}{y_i} \cdots \frac{y_n}{y_i}]$

is closed, then $\mathbb{P}^n_k \times \mathbb{P}^{n\vee}_k \xrightarrow{\pi} \mathbb{P}^{n\vee}_k$ closed

$\dim \pi(Z) \leq n-1 \Rightarrow \mathbb{P}^{n\vee}_k \setminus Z \neq \emptyset \Rightarrow$ dense open

now $\forall p \in X$ closed, $W_p :=$ fiber of $Z \to X$ at p.

$$\begin{tikzcd} W_p \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}^{n\vee} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & k(p) \arrow[d] \\ Z \arrow[r, hook] & X \times \mathbb{P}^{n\vee} \arrow[r] & X \end{tikzcd}$$

$W_p(k) = \{(p,H) \mid \overset{\mathrm{Ker}\, v^*}{\overset{\parallel}{T_pH}} \supseteq T_pX\}$, which is define by $d+1$ linear equation $\begin{cases} p \in H \\ T_pH \supseteq T_pX \end{cases}$

Krull $\Rightarrow \dim W_p = n-d-1$ then by 12.4.A $\dim Z \leq n-1$

$\square$

Valuative Criterion

Theorem. 16.10.13. (Valuative Criterion)

$f: X \to Y$ morphism of schemes. TFAE

① f separated. (universally closed, proper)

② f quasi-separated. (quasi-compact, quasi-separated and of finite type)

and $\forall$

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "v"] \arrow[ur, "\tilde{v}"] & Y \end{tikzcd} \qquad (A \hookrightarrow K)$$

where A valuation ring, $K := \operatorname{Frac} A$

$\exists$ at most 1 (at least 1, unique) lifting $\tilde{v}$

s.t. $f\tilde{v} = v$ $\tilde{v}j = u$

proof ① $\Rightarrow$ ②

(a) If f separated

since $\tilde{v}, \tilde{v}' \in \operatorname{Mor}(Sch/Y)$ $\operatorname{Ker}(\tilde{v}, \tilde{v}') \hookrightarrow \operatorname{Spec} A$

since $\tilde{v}j = \tilde{v}'j$ $\{\eta\} \in \operatorname{Ker}(\tilde{v}, \tilde{v}')$

thus $\operatorname{Ker}(\tilde{v}, \tilde{v}') = \operatorname{Spec} A$ since A reduced.

hence $\tilde{v} = \tilde{v}'$

(b) If f universally closed.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[dr, "\varphi"] \arrow[rr, bend left, "u"] \arrow[ddr, bend right, "j"'] & & X \arrow[dd, "f"] \\ & X \times_Y A \arrow[r, "g"] \arrow[d, "p", shift left] & X \\ & \operatorname{Spec} A \arrow[u, dashed, "r", shift left] \arrow[r, "v"] \arrow[uur, dashed, "\tilde{v}"] & Y \end{tikzcd}$$

if $\exists r: \operatorname{Spec} A \to X \times_Y A$ s.t. $pr = id$. and $rj = \varphi$

then gr satisfy $fgr = vpr = v$, $grj = g\varphi = u$

which is the desired lifting.

Let x be image of φ $X' := \overline{\{x\}}$

X' can be seem as a reduced closed subscheme

f universally closed $\Rightarrow$ p closed

$\eta \in p(X') \Rightarrow p(X') = \operatorname{Spec} A$ $\exists\, x' \in X'$ map to m

then $A = \mathcal{O}_{\operatorname{Spec} A, m} \longrightarrow \mathcal{O}_{X', x'} \underset{\text{generalize}}{\longrightarrow} \mathcal{O}_{X', x} \longrightarrow \mathcal{O}_{\operatorname{Spec} K, 0} = K$

since X' integral $\mathcal{O}_{X', x}$ field $\mathcal{O}_{X', x'} \hookrightarrow \mathcal{O}_{X', x} \hookrightarrow K$

since the composition is exactly $A \hookrightarrow K$

$A \hookrightarrow \mathcal{O}_{X', x'}$ thus $\mathcal{O}_{X', x} = K$ and

$\mathcal{O}_{X', x'}$ local ring dominating A $(m_{x'} \supseteq m)$

which implies $A = \mathcal{O}_{X', x'}$ $\operatorname{Spec} A \xrightarrow{\sim} \operatorname{Spec} \mathcal{O}_{X', x'} \longrightarrow X' \hookrightarrow X \times_Y A$

is the desired section

(C) If f proper, f is separated and universally closed.

② $\Longrightarrow$ ① **Lemma.** $f: X \to Y$ quasi-compact, f is closed iff $\forall y \in f(X)$, $y' \in \overline{\{y\}}$, $y' \in f(X)$.

Lemma. $x \in \overline{\{x'\}}$ iff $x' \in \mathrm{Im}(\operatorname{Spec} \mathcal{O}_{X,x} \to X)$

(a) If f quasi-compact, $S \to Y$, $X' := S \times_Y X$

then $\forall$

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u"] \arrow[d] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r, "v"] \arrow[ur, dashed, "\tilde{v}"] \arrow[urr, dashed, "v_0"'] & S \arrow[r] & Y \end{tikzcd}$$

$\exists$ lifting $v_1 : \operatorname{Spec} A \to X$ which induces $\tilde{v}$

then assume $s \in \overline{\{s'\}} \subseteq S$, $s' \in f'(X')$ and $x' \in f'^{-1}(s') \subseteq X'$

$K := \kappa(x')$ $\quad K | \kappa(s')$ extension of field

we have

$$\begin{tikzcd} \operatorname{Spec} \kappa(s) \arrow[r] \arrow[dr] & \operatorname{Spec} \mathcal{O}_{S,s} \arrow[r, "\alpha"] & S \\ & \operatorname{Spec} \mathcal{O}_{S,s'} \arrow[u] & \end{tikzcd}$$

which induces

$\mathcal{O}_{S,s} \xrightarrow{\beta} \kappa(s') \hookrightarrow K$ by $\boxed{\text{Chevalley}}$

$\exists A \subseteq K$ valuation ring of K s.t.

$\beta(\mathcal{O}_{S,s}) \subseteq A$ and $m_A \cap \beta(\mathcal{O}_{S,s}) = \beta(m_s)$

$\mathcal{O}_{S,s} \xrightarrow{\beta} A$ induces $\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} \mathcal{O}_{S,s} \xrightarrow{\alpha} S$

$m_A \mapsto m_s \mapsto s$

thus we obtain

$$\begin{tikzcd} \operatorname{Spec} K \arrow[rr, "0 \mapsto x'"] \arrow[d] & & X' \arrow[d, "f'"] \\ \operatorname{Spec} A \arrow[r] \arrow[urr, "\tilde{v}"] & \operatorname{Spec} \mathcal{O}_{S,s} \arrow[r] & S \end{tikzcd}$$

$(\kappa(X'))$, $\tilde{v}$ lifting of $\operatorname{Spec} A \to S$

$\kappa(\eta)$ $\quad 0 \mapsto \ker \beta \mapsto s'$ $\quad \kappa(s')$

$x := \tilde{v}(m_A)$ then $f'(x) = s$ and $x \in \overline{\{x'\}}$ $\quad f'(x) = s$

$s \in f(X)$ thus by Lemma f is closed

S is arbitrary thus f universally closed.

(b) if f quasi-separated. $\Delta_{X/Y}$ quasi-compact.

it sufficient to show $\forall$
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r, "v"] & X \times_Y X \end{tikzcd}$$
$\exists$ lifting of v

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u"] \arrow[d, "j"] & X \arrow[d] & \\ \operatorname{Spec} A \arrow[r, "v"] \arrow[rr, bend left, "b"] \arrow[dr, bend right, "a"] & X \times_Y X \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

then $fa = fb : \operatorname{Spec} A \to Y$

$aj = pvj = pbu = u$

$bj = qvj = qbu = u$

by
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u"] \arrow[d, "j"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[ur, shift left, "a"] \arrow[ur, shift right, "b"'] \arrow[r, "fa=fb"] & Y \end{tikzcd}$$
a, b both lifting of $fa=fb$

thus $a=b$

$p\Delta a = a \quad q\Delta a = a = b$

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "u"] \arrow[d, "j"] & X \arrow[d, "\Delta"] \\ \operatorname{Spec} A \arrow[ur, "a=b"] \arrow[r, "v"] & X \times_Y X \end{tikzcd}$$
by universal property $\Delta a = \Delta b = v$

thus Δa is a lifting

by (a) $\Delta_{X/Y}$ is (universally) closed

thus $f: X \to Y$ is separated.

(c) f of finite type $\Longrightarrow$ f quasi-compact

thus by (a) (b) f is proper. $\square$