Chapter 12 Dimension.

12.1 Dimension and codimension

12.1.1 $\dim X = \sup$ (length of irr. subset sequence $X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_n$)

$\dim A = \sup$ (length of prime ideal sequence)

$= \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht} \mathfrak{p}$

12.1.A $\dim A = \dim \operatorname{Spec} A$.

12.1.2. $\dim \mathbb{A}^1_k = \dim \operatorname{Spec} \mathbb{Z} = 1$. $\dim \operatorname{Spec} k = 0$

$\dim \underbrace{\operatorname{Spec} k[X]/(X^2)}_{\Downarrow} = 0$.

The only prime ideal of $k[X]$ containing X^2 is (X)

12.1.3. pure dimensional:

if every irr. component has same dim.

pure dim 1 is called curve

2 surface

n n-fold.

12.1.B. A scheme has dim n iff admits an open affine covering of dim at most n, where equality is achieved for some affine open set.

Pf. Use the Hint:

$\forall$ topo. sp. X $U \subset X$. There is bij. (with including)

$\{$irr. closed subset of $U\} \longleftrightarrow \{$irr. closed subset of X meeting $U\}$.

If the Hint is correct.

"$\Longrightarrow$" If $\dim X = n$

$X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \dots \subsetneq X_n$ irr. cl. subset sequence

~~Choose~~ $X = \bigcup_{i\in I} U_i$. $\exists U_0 \in \{U_i \mid i\in I\}$

s.t. $U \cap X_0 \neq \emptyset$.

Then $\forall j \in [0,n]$ $U \cap X_j \neq \emptyset$.

From Hint:

$X_0\cap U \subsetneq X_1 \cap U \subsetneq \dots \subsetneq X_n \cap U$ irr. cl. subset sequence of U. $\therefore n \leq \dim U$.

$\therefore n \leq \max\{\dim U_i \mid i\in I\}$

If $\exists U' \in \{U_i \mid i \in I\}$ $\dim U_i \geq n+1$.

From the Hint: Exists $X_0 \subsetneq \dots \subsetneq X_{n+1}$

s.t. $X_0 \cap U' \subsetneq \dots \subsetneq X_{n+1}\cap U'$ irr. cl. subset chain of U'

contradicts with $\dim X = n$.

$\therefore n \geq \max\{\dim U_i \mid i \in I\}$.

"$\Longleftarrow$". As "$\Longrightarrow$" we proved.

$\dim X = \sup\{\dim U_i \mid i\in I\}$ $\left(\bigcup_{i\in I} U_i = X \text{ open (aff) covering.}\right)$

pf of Hint:

$$\begin{array}{ccc}
\{\text{irr. cl. subset of } X \text{ meeting } U\} & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & \{\text{irr. cl. subset of } U\} \\
Z & \longmapsto & Z\cap U. \\
cl_X(S). & \longleftarrow\!\shortmid & S
\end{array}$$

•

~~$S \subseteq cl_X(S)\cap U$. ($\because S \subseteq cl_X(S)$. $S\subseteq U$).~~

~~β reasonable~~

~~cl_X~~ Claim: $cl_X(S)\cap U=S$.

reason: $S\subseteq cl_X(S)$, $S\subseteq U \Rightarrow S\subseteq cl_X(S)\cap U$.

$S=V\cap U$. V closed in X. $S\subseteq V$.

$cl_X(S)\cap U\subseteq cl_X(V)\cap U=V\cap U=S$.

- β reasonable:

  If $cl_X(S)=V_1\cup V_2$ V_1, V_2 closed in X.

  $(V_1\cup V_2)\cap U=(V_1\cap U)\cup(V_2\cap U)=cl_X(S)\cap U=S$.

  $\therefore V_1\cap U=S$ or $V_2\cap U=S$.

  $\therefore S\subseteq V_1$ or $S\subseteq V_2$ $\therefore cl_X(S)\subseteq V_1$ or V_2.

- $\alpha\circ\beta=\mathrm{id}$.

- ~~$\beta\circ\alpha=\mathrm{id}$ i.e. $cl_X(Z\cap U)=Z$.~~

- α reasonable:

  If $Z\cap U=S_1\cup S_2$. S_1, S_2 closed in U.

  $S_1=V_1\cap U$, $S_2=V_2\cap U$.

  V_1, V_2 closed in X.

$$Z=(Z\setminus U)\cup(Z\cap U)$$
$$\subseteq(Z\setminus U)\cup\big((V_1\cup V_2)\cap Z\big)$$
$$=(Z\setminus U)\cup(V_1\cap Z)\cup(V_2\cap Z)\subseteq Z.$$

Z irr. $\therefore \underset{\text{contradiction}}{Z\cap U=\emptyset}$ or $V_1\supseteq Z$ or $V_2\supseteq Z$.

$\therefore Z\cap U=S_1$ or S_2.

- $\beta\circ\alpha=\mathrm{id}$.

i.e. $Z=cl_X(Z\cap U)$ for irr. $\overset{cl.}{Z}$ in X.

$cl_X(Z\cap U)\subseteq cl_X(Z)=Z$.

$Z \subseteq (Z \setminus U) \cup (Z \cap U) \subseteq (Z \setminus U) \cup cl_X(Z \cap U) = Z$.

$Z \setminus U \neq Z$, $\therefore cl_X(Z \cap U) = Z$.

From α, β easily know the bij. keeps inclusion. $\blacksquare$

<u>12.1.C</u> Noetherian scheme of dim 0 has a finite number of points.

Pf. X Noetherian scheme of dim 0.

$X = \bigcup_{i \in I} U_i = \bigcup_{i \in I} \operatorname{Spec} A_i$. $\dim A_i = 0$. $|I| < +\infty$.

i.e. A_i Noetherian, every prime ideal ~~is~~ are maximal.

$\Rightarrow |\operatorname{Spec} A_i| < +\infty$. (Noetherian has fin. many minimal prime ideal in the case: $\forall$ prime ideal are minimal)

<u>12.1.D</u> Surjection of integral domains of the same dim must be an isomorphism. $\blacksquare$

Pf. $\varphi: A \twoheadrightarrow B$ surjection of integral domains.

If $\ker\varphi \neq 0$. Then $\underset{(0)}{p_0} \subsetneq \cdots \subsetneq p_n$ prime ideal chain of B.

$\rightsquigarrow 0 \subsetneq \ker\varphi \subsetneq \varphi^{-1}(p_0) \subsetneq \cdots \subsetneq \varphi^{-1}(p_n)$ prime ideal chain of A.

$\Rightarrow \dim A \geq \dim B + 1$. contradiction! $\blacksquare$

<u>12.1.E</u> $\pi: X \to Y$ integral morphism.

Show: every (nonempty) fiber of π has $\dim = 0$.

Pf. integral implies affine.

$\therefore$ reduced to the case $X = \operatorname{Spec} B$. $Y = \operatorname{Spec} A$.

~~with B, A integral.~~

with $A \to B$ integral!

$$\begin{tikzcd} B\otimes_A A_p/pA_p & A_p/pA_p \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

$p \in \operatorname{Spec} A$. fiber of $p \simeq \operatorname{Spec}(B\otimes_A A_p/pA_p)$

$\forall$ prime q of $B\otimes_A A_p/pA_p$.

$(B\otimes_A A_p/pA_p)/q$ is integral over $\underbrace{A_p/pA_p}_{\text{field}}$

$\therefore (B\otimes_A A_p/pA_p)/q$ is field. $\therefore \dim B\otimes_A A_p/pA_p = 0$. $\boxed{/\!/}$

$\star$ 12.1.F If $\pi: \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral extension of rings, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

Pf. For prime ideal chain of A $\quad \pi^{\#}(\operatorname{Spec} B) = \varphi: B \hookrightarrow A$.

$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$.

$\underset{q_0}{\underset{\parallel}{\varphi^{-1}(p_0)}} \subseteq \underset{q_1}{\underset{\parallel}{\varphi^{-1}(p_1)}} \subseteq \cdots \subseteq \underset{q_n}{\underset{\parallel}{\varphi^{-1}(p_n)}}$ prime ideal of B.

~~Claim: all "$\subseteq$" are "$\subsetneq$"~~

$$\cancel{\begin{tikzcd} A\otimes_B B_q/qB_q & B_q/qB_q \arrow[l] \\ A \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd}}$$

~~$0 \to p \to A \to A/p \to 0$ $\quad \otimes$ right-exact.~~

~~$\Rightarrow A\otimes_B B_q/qB_q \simeq A/p \otimes_B B_q/qB_q$, $p\otimes_B B_q/qB_q$, integral field.~~

Claim: all "$\subseteq$" are "$\subsetneq$". $\quad p_0 \subsetneq p_1$, $q_0 \subseteq q_1$

$S = \varphi(B \setminus q_1)$.

$B_{q_1} \xrightarrow{\varphi} S^{-1}A$ is integral extension.

~~$\forall q' \subsetneq q$ $\quad q' \in \operatorname{Spec} B$~~ corresponding to $P \subseteq A$ s.t. $\varphi^{-1}(P) \subseteq q_1$.

Consider $S^{-1}p_1$.

$\tilde{\varphi}^{-1}(S^{-1}p_1) \supseteq q_1 B_{q_1}$ maximal ideal of B_{q_1}.

$\therefore$ $S^{-1}p_1$ is maximal ideal of $S^{-1}A$.

$p_0 \subsetneq p_1$. $\therefore$ $S^{-1}p_0$ is not maximal ideal of $S^{-1}A$.

$\therefore$ $\tilde{\varphi}^{-1}(S^{-1}p_0)$ is not maximal ideal of B_{q_1}

$\therefore$ $q_0 \subsetneq q_1$.

$\therefore$ $\dim \operatorname{Spec} B \geq \dim \operatorname{Spec} A$.

For prime ideal chain of B $q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_n$

Going up: $\exists$ $p_0 \subsetneq p_1 \cdots\cdots \subsetneq p_n$ of A.

$$\begin{matrix} p_0 & & p_n \\ \wr & & \wr \\ q_0 & & q_n \end{matrix}$$

$\therefore$ $\dim A \geq \dim B$.

$\therefore$ $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$. $\blacksquare$

<u>12.1.G</u>. Show: If $\nu: \tilde{X} \to X$ normalization of a scheme then $\dim \tilde{X} = \dim X$.

pf. $\dim X = \sup(\dim U_i)$ $\quad U_i = \operatorname{Spec} A_i$. $\bigcup_{i \in I} U_i = X$.

$= \sup(\dim \operatorname{Spec} A_i)$

$= \sup(\dim \operatorname{Spec}(A_i)_{red})$

$= \sup(\dim \operatorname{Spec}(\text{integral closure of } (A_i)_{red}))$ by <u>12.1.F</u>

$= \dim \tilde{X}$. $\blacksquare$

扫描全能王 创建

復旦大學
Fudan University

<u>12.1.H</u>

X: aff. k-scheme. K/k alg. field extn.

(a). Suppose X has pure dim n.

Show that: $X_K := X\times_k K$ pure dim n. ~~reduced~~

pf. ~~$X=\mathrm{Spec}A$. A: k-alg. $X\simeq X_{red}$ ∴ WLOG A integral~~

~~only need to prove~~

~~only need to prove the case of A integral~~

(~~If this case is done. irr component of~~

$X=\mathrm{Spec}A$. A: k-alg. $X_K\simeq \mathrm{Spec}(A\otimes_k K)$.

$$\begin{tikzcd} X_K \arrow[r] \arrow[d] & \mathrm{Spec}K \arrow[d] \\ X \arrow[r] & \mathrm{Spec}k \end{tikzcd}$$

For affine $\mathrm{Spec}B$. $\{\text{irr. components}\} \overset{1:1}{\longleftrightarrow} \{\text{minimal prime}\}$.

$\forall q \in \{\text{minimal prime ideal of } A\otimes_k K\}$.

$\varphi: A \longrightarrow A\otimes_k K$. $\varphi^{-1}(q) =: p$

The irr. component of X_K is $\simeq \mathrm{Spec}(A\otimes_k K/q)$

$A/p \longrightarrow A\otimes_k K/q$ is integral extension.

($k\longrightarrow K$ int extn. $\Rightarrow A\otimes_k k \longrightarrow A\otimes_k K$ int.)

From <u>12.1.F</u>. $\dim \mathrm{Spec}\, A\otimes_k K/q = \dim A/p \overset{?}{=} n$

only need to show p is minimal prime ideal of A.

~~If not. $p'\subsetneq p$~~

Recall:

6.6.2} Any element of any minimal prime $\mathfrak{p}'$ is a zerodivisor

~~Suppose~~ $\mathfrak{p}'$ is the minimal prime ideal $\mathfrak{p}' \subseteq \mathfrak{p}$ of A.

Claim The natural $A/\mathfrak{p}' \longrightarrow A\otimes_k K/\mathfrak{q}$ ~~has~~ inj.

reason: $\bar{a} \in$ kernel. $a \in A$

$\therefore a\otimes 1 \in \mathfrak{q}$. From recall $a\otimes 1$ is a zerodivisor of $A\otimes_k K$. In particular. ~~$\bar{a}\otimes 1$ is a zero divisor of~~

If $a \notin \mathfrak{p}'$. Then $\bar{a}\otimes 1$ is a zero divisor of

$A/\mathfrak{p}' \otimes_k K$ $A/\mathfrak{p}' \otimes_k K$ is free $A/\mathfrak{p}'$-module with finite rank.

integral domain

multiply by $\bar{a} \in A/\mathfrak{p}' \doteq$ multiply $\bar{a}\otimes 1 \in A/\mathfrak{p}'\otimes K$ is injective of $A/\mathfrak{p}'\otimes_k K \longrightarrow A/\mathfrak{p}'\otimes_k K$.

$\therefore a \in \mathfrak{p}$.

$\mathfrak{p}' = \text{kernel} = \mathfrak{p}$. $\therefore \mathfrak{p}$ is minimal prime ideal of A.

(b). Prove the converse to (a).

$\forall\, \mathfrak{p} \in \operatorname{Spec} A$. minimal

going up $\Rightarrow \exists\, \mathfrak{q} \in \operatorname{Spec} A\otimes_k K$ s.t. $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$

$\varphi: A \longrightarrow A\otimes_k K$ natural

In 12.1.F: we proved: If $\mathfrak{q} \subsetneq \mathfrak{q}^*$. Then $\varphi^{-1}(\mathfrak{q}') \subsetneq \varphi^{-1}(\mathfrak{q})$.

Because $\mathfrak{p}$ minimal $\therefore \mathfrak{q}$ minimal.

From $\dim A/\mathfrak{p} = \dim A\otimes_k K/\mathfrak{q} = n$.

X pure dim n. $\blacksquare$

复旦大学 Fudan University

12.1.4 Codim.

X ~~Aut~~ $\Rightarrow$ "$\dim X - \dim Y$" (= $\operatorname{codim} Y$) is not a good defn.

codim of an irr. subset $Y \subseteq X$ $\left(\operatorname{codim}_X Y \right)$

$\sup\{$ length of ... starting by $Y\}$

"linear sight" $\operatorname{codim}_X Y \simeq \dim(X/Y)$.

12.1.J. If Y irr. closed subset of scheme X. η generic pt. then $\operatorname{codim}_Y X = \dim \mathcal{O}_{X,\eta}$

pf. • $\operatorname{codim}_Y X \leq \dim \mathcal{O}_{X,\eta}$.

Choose an open affine $\eta \in U \simeq \operatorname{Spec} A$. $\eta \leftrightarrow \mathfrak{p}_\eta$.

$\forall$ chain $\overline{Y} = Y = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n$.
irr. closed subset

consider $Z_0 \cap U \subseteq Z_1 \cap U \subseteq \cdots \subseteq Z_n \cap U$.

From 12.1.B. This is a strict "$\subsetneq$" chain of irr. closed subset of U.

The chain corresponds to descending prime ideal chain of A.

by $A \supseteq I(Z_0 \cap U) \supsetneq I(Z_1 \cap U) \supsetneq \cdots \supsetneq I(Z_n \cap U)$.
all contained in $\mathfrak{p}_\eta$.

$\therefore$ The chain corresponds to descending prime ideal chain of $A_{\mathfrak{p}_\eta} \simeq \mathcal{O}_{X,\eta}$.

$\therefore$ $\operatorname{codim}_Y X \leq \dim \mathcal{O}_{X,\eta}$.

- $\dim \mathcal{O}_{X,\eta} \le \operatorname{codim}_X Y$.

$\Leftrightarrow$ prime ideal chain of $A_{\mathfrak{p}_\eta}$

$\Leftrightarrow$ prime ideal chain of A contained in $\mathfrak{p}_\eta$: $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \dots \supsetneq \mathfrak{p}_n$

$\Leftrightarrow$ irr. closed chain of $\operatorname{Spec} A = U$

$$\underset{\subseteq V(\mathfrak{p}_\eta)}{V(\mathfrak{p}_0)} \subsetneq V(\mathfrak{p}_1) \subsetneq \dots \subsetneq V(\mathfrak{p}_n)$$

$\Leftrightarrow$ irr. closed chain of ~~irr~~ X by

$$\underset{\subseteq cl_X(\{\mathfrak{p}_\eta\}) = Y}{cl_X(V(\mathfrak{p}_0))} \subsetneq cl_X(V(\mathfrak{p}_1)) \subsetneq \dots \subsetneq cl_X(V(\mathfrak{p}_n))$$

(proved in 12.1.B) $\blacksquare$

~~12.1.5 irr. component~~

12.1.K If Y is an irr. closed subset of a scheme X, show that

$$\operatorname{codim}_X Y + \dim Y \le \dim X$$

Pf. LHS $= \sup\{$length of irr. closed subset ~~containing~~ ~~one~~ $V_0 \subsetneq V_1 \cdots \subsetneq V_n$ s.t. $\exists j \in [0,n]$ s.t. $V_j = Y\} \le \dim X$ $\blacksquare$

FACT: equality holds when X and Y are irreducible varieties.

12.1.6 UFO. codim $= 1$ prime ideals are principal
i.e. ht $= 1$

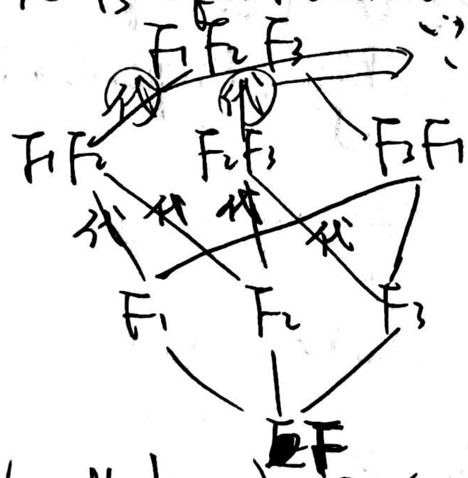
12.1.7 Lem Pf. (When A Noetherian integral domain. the converse ✓.
i.e. ~~ht=1~~ principal ~~non~~ prime has ht $= 1$)
prove later?

pf. If $\mathfrak{p}$ codim=1 prime. $f \neq 0 \in \mathfrak{p}$.
$f = f_1 f_2 \dots f_n$ f_i are prime/irr. elt.
$\exists i \in [1,n]$ s.t $f_i \in \mathfrak{p}$. called g
$(0) \subseteq (g) \subseteq \mathfrak{p}$ $\therefore (g) = \mathfrak{p}$. ▨

☆ **2.1.9 prop** (without proof: all Comm. Alg.).
$(A, \mathfrak{m})$. Noe. loc. ring. TFAE:
(1) $\dim A = 0$.
(2) $\mathfrak{m} = \mathfrak{R}(A)$.
(3) for some $n > 1$. $\mathfrak{m}^n = 0$.
(4) A fin. length.
(5) every descending sequence of ~~p~~ ideals stable (Artin)

12.2 Dimension, trdeg and Noetherian normalization

12.2.A $F' \sim F'' \Leftrightarrow F'F''$ alg. over F', F''
(a). $\sim$ is equ. relation



$\because F_1F_2F_3 \subseteq \overline{F_2}$.

(b). (well-known). E/F two tr. bases have same cardinality.

12.2.1 Thm (dim = trdeg).

A: fin. gen. domain over field k. Then $\dim \operatorname{Spec} A = \operatorname{trdeg} K(A)/k$

12.2.B (well-known). (But this is as a consequence of **12.2.1 Thm**)

$A = k[X_1 \dots X_n]/I$. Show that the residue field of any maximal ideal of A is finite extn. of k.

証. pf. $M \in \operatorname{Spec} A$. $\underbrace{\dim \operatorname{Spec} A/M}_{=0} = \operatorname{trdeg} (A/M)/k$. $\blacksquare$

12.2.C $\pi: X \to Y$ dominant morphism of irr. k-varieties.
Then $\dim X \geq \dim Y$.

pf. dominant morphism of irr.: (gen) $\mapsto$ (gen).

(gen) of X: η. (gen) of Y: ξ.

$\rightsquigarrow$ $\mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,\eta}$ local morphism.

$\rightsquigarrow$ $K(\xi) \to K(\eta)$ field homomorphism

$\therefore$ $\underbrace{\operatorname{trdeg} K(\xi)/k}_{\dim Y} \leq \underbrace{\operatorname{trdeg} K(\eta)/k}_{\dim X}$ $\longrightarrow$ 12.2.1 $\blacksquare$

12.2.D $f_1(x_1, x_2, \dots, x_n)$ ~ $f_m(x_1, x_2, \dots, x_n) \in k[x_1 \dots x_n]$ are m polynomials in n variables over field k

$m > n$ Show that $f_1 \dots f_m$ are alg. dependent.

($m = n$ false when char $k \neq 0$).

pf. Suppose not dependent.

$n = \operatorname{trdeg} k(x_1 \dots x_n) \geq \operatorname{trdeg} k(f_1 \dots f_m) = m$.

Contradiction! $\blacksquare$

Fudan University

12.2.E. Show that $wz-xy=0$, $wy-x^2=0$, $xz-y^2=0$ cut out an integral surface S in $\mathbb{A}_k^4$.

pf. $I=(wz-xy,\ wy-x^2,\ xz-y^2)$ $\quad A= k[x,y,z,w]/I$.

- First to prove I prime ~~(I think Vakil didn't realise to prove this first)~~.

~~$\cdot$ Claim: $\forall f\in k[y,z,w]$ homog. If $f(y,z,w)\in I$. Then $f=0$.~~ (3.6.F)

Consider $k[a^3, a^2b, ab^2, b^3] \xleftarrow{\varphi} k[x,y,z,w]/I$.

$k[a,b]$

$$\begin{aligned} a^3 &\longleftarrow \bar{z}\\ a^2b &\longleftarrow \bar{y}\\ ab^2 &\longleftarrow \bar{x}\\ b^3 &\longleftarrow \bar{w}. \end{aligned}$$

φ is well defined

i.e. show that $f(x,y,z,w)\in I$ Then $f(a^3, a^2b, ab^2, b^3)=0$.

obviously.

φ is surj. obviously. φ is inj ?

i.e. For all $f\in k[z,y,x,w]$ s.t. $f(a^3, a^2b, ab^2, b^3)=0$.

Then $f\in I$.

Reason: ~~$a=0$ $b=t$ $\rightsquigarrow$ $f(0,0,0,w)=0$.~~

~~$a=t$ $b=0$ $\rightsquigarrow$ $f(z,0,0,0)=0$~~

I is homog. ideal. $\therefore$ We can only do the case f is homogeneous. $f=\sum_{\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} k_\alpha z^{\alpha_1}y^{\alpha_2}x^{\alpha_3}w^{\alpha_4}$.

$$\Rightarrow \sum_\alpha^{\alpha_1+\alpha_2+\alpha_3+\alpha_4} k_\alpha\, a^{3\alpha_1+2\alpha_2+\alpha_3}\, b^{\alpha_2+2\alpha_3+3\alpha_4}=0.$$

$$\begin{cases} 3\alpha_1+2\alpha_2+\alpha_3 = 3\alpha_1'+2\alpha_2'+\alpha_3' \\ \alpha_2+2\alpha_3+3\alpha_4 = \alpha_2'+2\alpha_3'+3\alpha_4' \end{cases}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}}_{\varphi} \begin{pmatrix} \alpha_1-\alpha_1' \\ \alpha_2-\alpha_2' \\ \alpha_3-\alpha_3' \\ \alpha_4-\alpha_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution: $\mathbb{Z}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

This means. $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = \alpha_1'+\alpha_2'+\alpha_3'+\alpha_4'$

$$f(z^{\alpha_1}y^{\alpha_2}x^{\alpha_3}w^{\alpha_4}) = f(z^{\alpha_1'}y^{\alpha_2'}x^{\alpha_3'}w^{\alpha_4'})$$

$$\Leftrightarrow z^{\alpha_1}y^{\alpha_2}x^{\alpha_3}w^{\alpha_4}\cdot(z^{-1}y^{2}x^{-1})^m\cdot(z^{-2}y^{3}w^{-1})^n = z^{\alpha_1'}y^{\alpha_2'}x^{\alpha_3'}w^{\alpha_4'}$$

For these $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ $(\alpha_1',\alpha_2',\alpha_3',\alpha_4')$. $m,n\in\mathbb{Z}$.

$$z^{\alpha_1}y^{\alpha_2}x^{\alpha_3}w^{\alpha_4} - z^{\alpha_1'}y^{\alpha_2'}x^{\alpha_3'}w^{\alpha_4'}$$
$$= z^{\alpha_1}y^{\alpha_2}x^{\alpha_3}w^{\alpha_4}\left(1-\left(\frac{y^2}{xz}\right)^m\cdot\left(\frac{xy}{wz}\right)^{n}\right).$$
$$\in I.$$

Consider the equivalence class of $\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\}$

$\Rightarrow$ The coefficient of $\underbrace{\quad}_{z^{\alpha_1}y^{\alpha_2}x^{\alpha_3}w^{\alpha_4}}$ $\bar{f} \in k[z,y,x,w]/I$ is 0.

$\Rightarrow \bar{f} = 0 \in k[x,y,z,w]/I$

$\therefore \varphi$ isomorphism. to an integral domain

$\therefore I$ prime

~~$\dim A$~~ $= A = k[X, y, z, w]/I$

$\bar{x}, \bar{w}$ alg. independent in $k[\bar{x}, \bar{y}, \bar{z}, \bar{w}]$.

$\dim A = \operatorname{trdeg} k(A)$.

notice that $\bar{y}$ is alg. over $k(\bar{x}, \bar{w})$

$\Rightarrow \bar{z}$ is alg. over $k(\bar{x}, \bar{w})$.

(好的做法：$w \longleftrightarrow b^3$, $x \longleftrightarrow ab^2$ 令 a
$\therefore$ x 必不在 $k[\bar{w}]$ 上代数)

$\therefore \operatorname{trdeg} k(A) \le 2$

Consider $\operatorname{Spec}\left(k[X, y, z, w]/I\right)_{\bar{w}}$ principal affine open of $\operatorname{Spec} A$.

$\dim A \geq \dim \operatorname{Spec}\left(k[X, y, z, w]/I\right)_{\bar{w}}$

$\cong k[\bar{x}, \bar{w}]_{\bar{w}}$ $\left(\because \bar{y} = \frac{1}{\bar{w}} \cdot (\bar{x})^2,\ \bar{z} = \frac{1}{\bar{w}} \bar{x}\bar{y} = \frac{1}{\bar{w}^2}(\bar{x})^3\right)$

$=$ ~~dim~~ $\operatorname{trdeg} k(\bar{x}, \bar{w}) = 2$.

12.2.2 $\lambda: X \dashrightarrow Y$ dominant rational map of irr. k-varieties of the same dim.

$k(\xi) \hookrightarrow k(\eta)$ degree of this extn $= \deg \lambda$.

not dominant $\Longrightarrow \dim \lambda = 0$.

$\deg(\rho \circ \lambda) = \deg(\rho) \cdot \deg(\lambda)$.

12.2.4 Noether Normalization Lem.

A integral domain. fin. gen. over k. $\operatorname{trdeg} K(A)/k = n$.

Then $\exists x_1, \ldots, x_n \in A$. alg. independent. over k

s.t. A finite. extn. of $k[x_1 \ldots x_n]$.

geom. sight: integral affine fin. type X.
$\exists$ surj. finite morphism $X \longrightarrow \mathbb{A}^n_k$.

<u>12.2.5</u> pf: $A = k[y_1 \dots y_m]/p$ By induction.

<u>$m=n$</u>. $p=0$. obviously.

If true for $\leq m-1$. for m.
$p \neq 0$ (or contradicts with $\mathrm{trdeg} = n < m$).
$\exists f \in k[y_1 \text{—} y_m]$ s.t. $f(\bar{y}_1, \bar{y}_2 \dots \bar{y}_m) = 0$.
consider $z_1 = y_1 - y_m^{r_1}$ —— $z_{m-1} = y_{m-1} - y_m^{r_{m-1}}$
(some coefficients is going to be defined)
$\Rightarrow f(\overline{z_1 + y_m^{r_1}} \dots \overline{z_{m-1} + y_m^{r_{m-1}}}, \bar{y}_m) = 0$.
Choose r_1 — — r_{m-1} great enough.
s.t. The ~~first~~ greatest of $f(z_1 + y_m^{r_1} \dots z_{m-1} + y_m^{r_{m-1}}, y_m)$
is y_m^N

$\Rightarrow \bar{y}_m$ is integral over $k[\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{m-1}]$.
Then use induction $\longleftrightarrow$ finite over some $k[a_1 \text{—} a_n]$ ▨

<u>12.2.F</u> If X, Y irr. k-var. then $\dim X \times_k Y = \dim X + \dim Y$.

pf. X irr. k-var.
$\therefore X$ has pure dim m.
<u>12.1.H</u> $\Rightarrow X_{\bar{k}} := X \times_k \bar{k}$ pure dim m.
similarly $Y_{\bar{k}} := Y \times_k \bar{k}$ pure dim n.
$\forall$ irr. component X' of $X_{\bar{k}}$. Y' of $Y_{\bar{k}}$.
<u>11.2.N</u> $\Rightarrow X' \times_{\bar{k}} Y'$ irr.

复旦大学 Fudan University

$U \subseteq X'$ affine open. $V \subseteq Y'$ affine open.

$\Rightarrow U \times_{\bar{k}} V \subseteq X' \times_{\bar{k}} Y'$ affine open

$\Rightarrow \dim X' \times_{\bar{k}} Y' = \dim U \times_{\bar{k}} V$. $\dim X' = \dim U$ $\dim Y' = \dim V$.

need to show $\dim U \times_{\bar{k}} V = \dim U + \dim V$.

Noetherien normalization theorem:

$U \longrightarrow \mathbb{A}^m_{\bar{k}}$ $V \longrightarrow \mathbb{A}^n_{\bar{k}}$ surjective, finite.

finite, surj. ~~is~~ morphism is preserved by base change

($\underline{10.4.D}$). $\mathbb{A}^{m+n}_{\bar{k}} \simeq \mathbb{A}^m_{\bar{k}} \times_{\bar{k}} \mathbb{A}^n_{\bar{k}}$.

$\therefore U \times_{\bar{k}} V \longrightarrow \mathbb{A}^{m+n}_{\bar{k}}$ surjective finite.

$\therefore \underline{12.1.F}$ $\dim U \times_{\bar{k}} V = \dim \mathbb{A}^{m+n}_{\bar{k}} = m+n$.

(~~need~~ I think it needs to show:

$k[X_1 - X_m] \hookrightarrow A$ $\underline{\text{finite exten.}}$

$k[y_1 - y_n] \hookrightarrow B$

$\Rightarrow k[X_1 - X_m y_1 - y_n] \hookrightarrow A \otimes_k B$ finite extension)

inj. ($\because$ free k-module is flat).

~~$X' \times_{\bar{k}} Y'$ irr. subset of $X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}}$. irreducible~~

~~$X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}} \supseteq X' \times_{\bar{k}} Y' \supseteq U \times_{\bar{k}} V$~~

~~$X \times_{\bar{k}} Y$ is closed subset of X~~

as a "component". $X' \subseteq X_{\bar{k}}$ $Y' \subseteq Y_{\bar{k}}$ closed.

$\therefore \underline{X' \times_{\bar{k}} Y'} \subseteq X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}}$ closed ($pr_1^{-1}(X') \cap pr_2^{-1}(Y')$).

irr. topo. space.

$\therefore \dim X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}} \geq m+n$.

$X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}}$ is covered by $\{X' \times_{\bar{k}} Y'\}$

∈ Noetherian topo. space — no one contained in another.

~~$X'' \cap X' = \emptyset$. $Y'' \cap Y' = \emptyset$ (property of component)~~ ~~irreducible~~ — finite union because

For Noe. top. space. closed subset can be written as $Z = U_1 \cup \dots \cup U_n$ (prop 3.6.15)

~~$\therefore X' \times_{\bar{k}} Y' \cap X'' \times_{\bar{k}} Y'' = \emptyset$ if $X' \neq X''$ or $Y' \neq Y''$~~ — irr. closed — $\{X'\}\{Y'\}$ finite cover because Noetherian of $X_{\bar{k}}$

~~So~~ $\{X' \times_{\bar{k}} Y'\}$ form irreducible components of $X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}}$

(U irr. $U = \bigcup (X' \times Y') \cap U \Rightarrow U \subseteq X' \times Y'$)

$\therefore X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}}$ has pure dim $m+n$.

$X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}} \simeq (X \times_k Y) \times_k \bar{k}$.

~~$(X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}} \simeq (X_{\bar{k}} \times_{\bar{k}} Y \times_k \bar{k})$~~

$(\because RHS \simeq (X \times_k Y_{\bar{k}}) \simeq X \times_k (\bar{k} \times_{\bar{k}} Y_{\bar{k}}) \simeq X_{\bar{k}} \times_{\bar{k}} Y_{\bar{k}})$.

by 12.1.H $X \times_k Y$ has pure dim $m+n$. $\blacksquare$

12.2.G 跳

12.2.7 pf of 12.2.1 Thm

X integral affine k-sch of fin. type

~~$\dim X =$~~ Prove by induction

~~$n=0$ trivial~~.

Noether Norm. Lem $X \to \mathbb{A}_k^n$ surj. finite

12.1.F $\Rightarrow \dim X = \dim \mathbb{A}_k^n$ ~~trdeg $k(X) = n$~~

~~need to show~~ need to show $\dim \mathbb{A}_k^n = n$. $\forall n \in \mathbb{N}$.

$n=0$ trivial

$\leq n+1$ For n. $(0) \subsetneq (x_1) \subsetneq \dots \subsetneq (x_1, x_2, \dots, x_n)$

$\Rightarrow \dim \mathbb{A}_k^n \geq n$.

~~$\mathfrak{p} = (X_1)$~~

Suppose have a chain of prim. ideals length $\geq n+1$.

$0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_*$.

$k[X_1 \dots X_n]$ UFD. $\exists\, g \in \mathfrak{p}_1$ nonzero. irr./prime elt.

$k[X_1 \dots X_n]/(g)$ has chain $\mathfrak{p}_1/(g) \subsetneq \mathfrak{p}_2/(g) \subsetneq \dots \subsetneq \mathfrak{p}_*/(g)$.

contradiction.

$\operatorname{Trdeg} \cancel{k} \operatorname{Frac}\left(k[X_1 \dots X_n]/(g)\right) = n-1$. By induction. [illegible]

先读12.2.9

<u>12.2.11</u> A: <u>catenary</u>: if $\forall$ pair $\mathfrak{p} \subseteq \mathfrak{q} \subseteq A$.

Every strictly increasing chain of primes ~~from p to q~~ can be extended to the maximal length. $\mathfrak{p} \subsetneq \dots\dots\dots \subsetneq \mathfrak{q}$

~~A~~ Show that: If A is a localization of fin. gen. ring over k

Then A is catenary.

pf. If ~~[illegible]~~ R catenary. <u>Claim</u>: $S^{-1}R$ is catenary

<u>reason</u>: $\{\text{prime of } S^{-1}R\} \longleftrightarrow \{\text{prim of } R \text{ which doesn't meet } S\}$.

(or $\subseteq (R \setminus S)$.)

(Remark for myself: ~~[illegible]~~ catenary 的定义并非向两边延长 只要在内部扩充就行).

If R catenary ~~Cl~~ obviously R/I is catenary.

$\therefore$ ~~[illegible]~~ only need to do the case of

$A = k[X_1, X_2, \dots\dots X_n]$

待续……

~~Claim: For $\forall$ pair $p \subseteq q \in \operatorname{Spec} R$ with ht $q/p \geq 2$.~~

~~Then we can put~~

12.2.9 Thm X pure dim k-scheme locally of fin. type (e.g. irr. k-variety). Y: irr. closed subset. $\overline{\{\eta\}} = Y$.

Then $\dim Y + \dim \mathcal{O}_{X,\eta} = \dim X$

$\dim Y + \operatorname{codim}_X Y$. (12.1.J).

12.2.11 pf. of Thm

12.2.I $\boxed{\text{can be Reduced to:}}$ X irr. affine k-variety, Z closed irr. subset maximal among those smaller than X (the only larger one is X) then $\dim Z = \dim X - 1$.

pf of why can reduce (Goal: If have 12.2.I. Then 12.2.9 ✓).

In 12.1.K, $\dim Y + \operatorname{codim}_X Y \leq \dim X$. only "$\geq$"

For X, Y in 12.2.9 $Y \subseteq X'$ an irr. component of X

$\dim X' = \dim X$. Consider the topo. of X' Y irr. closed

Choose affine open nbhd of η in X'. denote by U.

$U \cap Y$ irr. closed in U. $U \cap Y$ open in Y

$\dim U \cap Y = \dim Y$.

$\dim U = \dim X$

$\left\{\begin{array}{l}\text{irr. closed chain of } X' \\ \text{containing } Y\end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{l}\text{irr. closed chain of } U \\ \text{containing } U \cap Y\end{array}\right\}$

$\therefore \operatorname{codim}_X Y = \operatorname{codim}_{X'} Y = \operatorname{codim}_U U \cap Y$ U aff irr. k-variety

$\because$ For the any ~~length~~ sequence of ~~prime~~ ~~of~~ irr. closed of U which can't extend any more.

$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m$

Z_i also irr. k-var. $\therefore Z_{i-1}$ must be maximal one of Z_i.

~~$\therefore \dim Z_{i-1} = \dim Z - 1$ From the statement of 12.2.I~~

$\therefore \underset{\overset{\|}{0}}{\dim Z_0} = \dim Z_1 - 1 = \cdots = \dim Z_m - m = \dim X - m.$

($\because Z_0$ closed pt.)

$\Longrightarrow \dim U \cap Y + \operatorname{codim}_U U \cap Y = \dim U.$ ▨

The process proved 12.2.H ▨

For pf of the statement of 12.2.I

$d = \dim X$. Noether Norm. Lem. $X \to \mathbb{A}^d_k$ surj. fin.

8.3.L Integral morphism are closed.

irr closed $\longmapsto$ closed. Then image is irr.

$\therefore \pi(Z)$ is irr. closed in $\mathbb{A}^d_k$

12.2.J Show it suffices to show $\pi(Z)$ is hypersurface

pf. If prove $\pi(Z) = H$ is hypersurface

H is defined by (f).

Then $\dim H = \operatorname{trdeg} \operatorname{Frac} k[x_1 \ldots x_d]/(f) = d-1.$

$\pi|_{\pi^{-1}(\pi(Z))} : \underbrace{\pi^{-1}(\pi(Z))}_{\in \text{ also affine}} \longrightarrow \pi(Z)$ surj. fin.

12.1.F $\Rightarrow \dim \underbrace{\pi^{-1}(\pi(Z))}_{\in \text{ closed. irr.}} = \dim \pi(Z).$

$\because Z$ maximal $\therefore \pi^{-1}(\pi(Z)) = Z$ or X, $\in \dim = d$ contradiction!

$\therefore \dim Z = \dim \pi^{-1}(\pi(Z)) = d-1$ ▨

pf of why $\pi(Z)$ must be hypersurface

~~If not. $\pi(Z) \subsetneq H$ for one hypersurface~~ ~~irr. closed.~~

needs Going-down Lem.

12.2.12 Thm $B \hookrightarrow A$ fin. extn. of ring B integral closed A integral domain. given $q \subseteq q'$ of B p' of A s.t. $p' \cap B = q'$. $\exists\ p \in \operatorname{Spec} A$ s.t. $q = p \cap B$. $p \subseteq p'$.

(pf of Going-Down 略).

If $\pi(Z)$ not a hypersurface $\exists H$ hypersurface

$\pi(Z) \subsetneq H$ corresponds to $q \subsetneq q'$ of $k[x_1 \ldots x_d]$

$X \to \mathbb{A}^d_k$ corresponds to $k[x_1 \ldots x_d] \hookrightarrow A$.

~~proved~~ have proved $\pi^{-1}(\pi(Z)) = Z$. by maximal of Z.

corresponding to p. $\exists\ p' \in \operatorname{Spec} A$. $p' \cap \text{—} = q'$.

$Z \subsetneq V(p') \subsetneq X$. contradicts to the maximality of Z.

$\therefore \pi(Z)$ hypersurface $\blacksquare$

12.2.13, 12.2.14, 12.2.15 略

12.2.K X Noetherian scheme. Define $\dim_X(\cdot): X \to \mathbb{Z}^{\geq 0}$

$\dim_X p =$ dim of largest irr. component of X containing p

the dim of X at p

Show: $\dim_X(\cdot)$ upper semicontinuous function ~~if $\forall x \in \mathbb{R}$.~~

(i.e. $\forall x \in \mathbb{R}$ $f^{-1}((-\infty, x))$ open).

lower semi $\Leftrightarrow$ $f^{-1}((x, +\infty))$ open.

pf. ~~X Noetherian scheme ⇒ has finitely irr. component.~~ $\because$ Noetherian topo. space, $\{Z_i\}_{i=1}^{m}$.

$F_n := \{p \in X \mid \dim_X p \geq n\}$

$p \in F_n \Leftrightarrow \exists$ irreducible component Z s.t. $p \in Z$, $\dim Z \geq n$.

$\therefore$ If $p \in F_n$. Then all irr. component Z containing p. for $Z \subseteq F_n$.

$\therefore \forall i \in [1, m]$. $Z_i \subseteq F_n$ or $Z_i \cap F_n = \emptyset$.

$\therefore F_n$ closed subset of X. $\forall n \in \mathbb{Z}_{\geq 0}$. $\blacksquare$

12.2.16 listed definitions which satisfies semicontinuousness.

12.2.L. p closed pt. of a loc. fin. type k-scheme X

Show that: TFAE of the integer:

(1). $\dim_X p$ (2). $\dim \mathcal{O}_{X,p}$. (3). $\operatorname{codim}_X p$.

pf. $\dim_X p =$ sup (length of irr. cl. chain of X ~~cont~~ starting at $\{p\}$.)

$= \operatorname{codim}_X \{p\} \underset{12.1.J}{=} \dim \mathcal{O}_{X,p}$. $\blacksquare$

12.2.M (The dim. of a loc. fin. type k-sch. is preserved by any field extn.) cf. 12.1.H(A)

X: loc. finite type k-scheme pure $\dim = n$.

K/k field extn. (not necessarily alg. !!!)

Show that X_K has pure dim n.

~~FACT: X pure dim $n \Leftrightarrow \forall$ open U, $\dim U = n$.~~

~~pf of FACT: $\forall$ irr. component X'. $\exists$ open $U \subseteq X$ s.t $U \cap X' \neq \phi$ and $\dim X' = \dim U\cap X' = \dim U$~~ ~~open in X'~~ ~~irr.~~

Fact: X pure dim $n \Leftrightarrow \exists$ open affine covering of X $\{U_i\}_{i\in I}$, U_i has pure dim $= n$.

pf of FACT: "$\Leftarrow$" $\forall$ irr. component X'. $\exists$ open affine $U \subseteq X$ s.t $U \cap X' \neq \phi$.

$U\cap X'$ open in X' $\therefore \dim X' = \dim U \cap X'$.

$U \cap X'$ irr. closed of U.

Recall: $\{$irr. closed of $U\} \longleftrightarrow \{$irr. closed of X meeting $U\}$

with intersection relation

$\therefore$ $U\cap X'$ must be irr. component of U.

(otherwise $\exists X'' \supsetneq X'$ ~~s.t. $X'' \cap U$~~ irr. contradiction!)

$\therefore \dim X' = n$.

"$\Rightarrow$" $U \cap X'$ irr. comp of $U \Leftrightarrow X'$ irr. comp of X

$\dim U \cap X' = \dim X' = n$. $\boxed{}$

If prove the case of X affine.

general X. U_K is open affine covering of X_K

$\therefore$ From the FACT above. X pure $n \Rightarrow X_K$ pure dim n.

$$\begin{array}{ccc} X \text{ pure } n & \Rightarrow & X_K \text{ pure dim } n \\ \Updownarrow & & \Updownarrow \\ \exists \text{ open affine } \{U\},\ \{U\} \text{ pure } n & \Longrightarrow & \{U_K\} \text{ pure } n \end{array}$$

Goal: affine case!

$X = \operatorname{Spec} A$.

$$\begin{tikzcd} \operatorname{Spec} A\otimes_k K \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

From 2.1.H only prove the case K/k purely transcendental

Fudan University 復旦大學

~~irreducible component~~ $\overset{\mathrm{Spec}A}{\Longleftrightarrow}$ ~~minimal prime ideal of A~~

Now that Spec A is Noetherian scheme

$\therefore$ $X = \bigcup_{i=1}^{m} X_i$ fin. union of irr. components. $\forall i \neq j$ $X_i \neq X_j$

$X_K = \bigcup_{i=1}^{m} (X_i)_K$ union of irr. ~~components~~ subsets needs

X_K is Noetherian scheme.

~~3.6.11 Every closed subset of X_K can be written.~~

~~$\forall i \neq j$ $(X_i)_K \neq (X_j)_K$~~

Every irr. cl. subset of X_K. $Z = \bigcup_{i=1}^{m} ((X_i)_K \cap Z)$

$\therefore \exists i \in [1.m]$ $Z \subseteq (X_i)_K$.

$\therefore$ $\{(X_i)_K\}$ are irr. components.

After moving out containing

$\therefore$ If we show ~~$(X_i)_K$ has dim~~

$\dim (X_i)_K = \dim X_i$ $\forall i \in [1.m]$.

X_K has pure dim n.

Goal: X ~~integral~~ $= \mathrm{Spec}A$ A integral case

Claim: $A \otimes_k K$ is integral domain.

Claim: $\dim A \otimes_k K = \dim A$.

Claim ①: K/k transcendental basis $\{e_i\}_{i\in I}$.

$$A\otimes_k K \simeq \cancel{A[\{e_i\}]}\ S^{-1}(A[\{e_i\}])$$

$$S = \{k[\{e_i\}] \setminus \{0\}\}$$

$A[\{e_i\}]$ integral domain. localization is also integral domain.

Claim ② Because irreducibility

$$\dim A = \operatorname{trdeg} K(A)/k \cancel{\ \}}$$

$$\dim A\otimes_k K = \operatorname{trdeg} K(A\otimes_k K)/K \quad \blacksquare$$

~~12.3 Krull's Theo~~

~~先讲12.4 Dimensions of fibers of morph. of varieties~~

~~(用不到新东西).~~

12.3. Krull's Theorems

12.3.3 (Principal Ideal Thm Alg version)

A Noether ring. $f\in A$. $\mathfrak{p}$ minimal among (f).

Then $\operatorname{codim}\mathfrak{p} \le 1$.

If f is not a zerodivisor. $\operatorname{codim}\mathfrak{p} = 1$.

12.3.2 (Geom. version).

X loc. Noetherian scheme f function.

The irreducible components of $V(f)$ are codim 0 or 1

12.3.A. Show that an irr. homog. polynomial in $n+1$ variables over a field k describes an integral scheme of dim $n-1$ in $\mathbb{P}^n_k$ (略).

(can be proved without Krull Principal Ideal Theorem).

~~$f\ (x_1 \ldots)$~~ ~~$\frac{x_1}{x_0}, \frac{x_2}{x_0}$~~

~~Pf. $\dim \operatorname{Spec}\left(k[x_0 \ldots x_n]/(f)\right)$~~ ~~$\operatorname{Spec}\left(k[x_0 \ldots x_n]_f\right)_{\deg = 0}$~~ ~~integral domain~~

~~$\dim = \operatorname{trdeg} \operatorname{Frac}\left((k[x_0 \ldots x_n]_f)_{\deg=0}\right)/k$~~

12.3.4 跳. 12.3.B 跳

☆ 12.3.C (有用 Applications).

k field. $\{p_1 \ldots p_m\} \in \mathbb{P}^n_k$ points. not necessarily closed. none of which are gen. pt. of $\mathbb{P}^n_k$

Show that for $N \gg 0$. $\exists$ nonempty hypersurface deg N meeting none of $\{p_i\}$.

Pf. By induction.

$m=1$ trivial

m ✓. For $m+1$.

Choose hypersurface not containing $p_1 \ldots p_m$.

If doesn't vanish on p_{m+1} ✓. If vanishes, call f_{m+1}.

$p_2, p_3, p_4 \ldots p_m \rightsquigarrow f_1$

$p_1, p_3, p_4 \ldots p_m \rightsquigarrow f_2$

$\vdots$

$\rightsquigarrow f_m$.

Consider

$\sum_i f_1 \ldots \hat{f_i} \ldots f_{m+1}$

~~For p_j~~ For p_j

$f_j(p_j) = 0$.

$\left(\sum_i f_1 \ldots \hat{f_i} \ldots f_{m+1}\right)(p_j)$

$= (f_1 \ldots \hat{f_j} \ldots f_{m+1})(p_j) \neq 0$. ▨

12.3.D (a)

$H = V_+(f)$.

$Z_1 = V_+(p)$

p prime homog. ideal of $A = k[x_0 \_\_ x_n]$.

pf. $\dim X \geq 1$.

$\Rightarrow \exists$ irr closed chain $Z_0 \subsetneq Z_1 \subseteq X$.

Claim: $Z_1 \cap H \neq \emptyset$.

needs to show: $(\operatorname{cone}(Z) \cap \operatorname{cone}(H))^{\circ} \neq \emptyset$ in $\mathbb{A}^{n+1}_k$.

$\operatorname{cone}(Z) \simeq \operatorname{Spec} A/p$.

$(\because \dim X \geq 1)$

If $f \in p$. $V_+(f) \supseteq V_+(p)$. $H \cap X = X \neq \emptyset$.

If $f \notin p$ $\because A/p$ integral. $\therefore \bar{f} \in A/p$ not a zero-divisor

Krull Principal Ideal Thm $\Rightarrow$ $\operatorname{codim}_{\operatorname{cone}(Z)} \operatorname{cone}(Z) \cap \operatorname{cone}(H) = 1$.

i.e. $\forall$ minimal among $\bar{f}$ of A/p. q/p has codim $= 1$

$\bar{f} \in q/p \Rightarrow \operatorname{cone}(Z) \cap \operatorname{cone}(H) \supseteq (V(q/p)$ of $\operatorname{Spec} A/p)$

~~not empty~~

$\operatorname{Spec} A/p$ irr. k-~~scheme~~ variety

~~because~~

$\therefore \dim V(q/p) + \operatorname{codim}(q/p)$

$= \underline{\underline{\dim A/p}} \geq 2$ $(\because \{0\} \subsetneq \operatorname{cone}(Z_0) \subsetneq \operatorname{cone}(Z_1))$

$\therefore \dim V(q/p) \geq 1$.

$\therefore (\operatorname{cone}(Z) \cap \operatorname{cone}(H))^{\circ} \neq \emptyset$. $\boxtimes$

(b) Suppose $X \hookrightarrow \mathbb{P}^n_k$ closed subset of dim r.
Show that any codim r linear space meets X.

pf. After linear transformation
The linear space can be assumed.
$V_+(x_0, x_1, \ldots\ldots, x_{r-1})$

$\dim X = r$.

$\therefore \exists\ Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r$ irr. cl. subset chain.

From the process of (a).

$\exists$ irr. $\tilde{Z}_r$ s.t. $\tilde{Z}_r \subseteq Z_r \cap V_+(x_0)$
with $\dim \tilde{Z}_r \geq \dim(Z_r) - 1 = r+1-1 = r$.

$\tilde{Z}_r \longleftrightarrow$ a prime ideal containing x_0, $I(Z_r)$.

consider $S/I(\tilde{Z}_r)$ $(\bar{x}_0, \bar{x}_1) = (\bar{x}_1)$

From the process of (a)

$\exists$ irr. $\tilde{Z}_{r-1}$, s.t. $\tilde{Z}_{r-1} \subseteq \tilde{Z}_r \cap V_+(x_1)$
with $\dim \tilde{Z}_{r-1} \geq r-1$.

consider $S/I(\tilde{Z}_{r-1})$ $(\bar{x}_0, \bar{x}_1, \bar{x}_2) = (\bar{x}_2)$

$\vdots$

$\Rightarrow \exists$ irr. $\tilde{Z}_0$ s.t. $\tilde{Z}_{1} \subseteq \tilde{Z}_{2} \cap V_+(x_{r-1})$
with $\dim \tilde{Z}_1 \geq 1$.

$\therefore$ not empty $\blacksquare$

(A better notation: ~~codim~~ $\dim \emptyset = -\infty$).

(c). ① Show further ~~that~~ there is an intersection of $r+1$ nonempty hypersurfaces missing X.

~~[illegible]~~ ② k infin. then "hypersurfaces" $\longrightarrow$ "hyperplanes" ✓

~~[illegible]~~

(d). If k infinite $\exists$ intersection of r hyperplanes meeting X in a fin. number of points.

pf of (c)

① $\dim X = 0$ 时. X is finite set of points of $\mathbb{P}^n_k$.

123.C. $\exists$ hypersurface $H = V_+(f)$. $H \cap X = \emptyset$.

If correct for $\leq r-1$. For r. ($\mathbb{P}$ is "good")

X has finitely many irr. components ~~in this case~~

$$X = \underbrace{X_1}_{\eta_1} \cup \underbrace{X_2}_{\eta_2} \cup \cdots \cup \underbrace{X_s}_{\eta_s}.$$

$\exists H_1$ hypersurface $\{\eta_1 - \eta_s\} \cap H_1 = \emptyset$. $\therefore X_i \not\subseteq H_1\ \forall i$

Consider $X' = X \cap H_1$.

$$X' = \bigcup_{i=1}^{s} \underline{(X_i \cap H_1)}$$

$$\dim X_i \cap H_1 + \underline{\operatorname{codim}_{X_i} X_i \cap H_1} = \dim X_i.$$

$= 1$ From Krull's Principal Theorem (otherwise $X_i \cap H_1 = X_i$ contradiction!).

$\therefore$ ~~$\dim X_i \cap H_1$~~ $\dim X' \leq \max \dim X_i \cap H \leq r-1$.

($\because$ irr. $U \subseteq X'$ must contain in one of $X_i \cap H_1$)

Induction Hypothese $\Rightarrow$

$\exists\ H_2, \dots, H_{r+1}$ hypersurface

s.t. $X' \cap (H_2 \cap H_3 \cap \dots \cap H_{r+1}) = \emptyset$

$\therefore \underbrace{(X \cap H_1) \cap (H_2 \cap \dots \cap H_{r+1})}_{\text{what we find}} = \emptyset$. $\boxed{\equiv}$ ①.

② same process

Essential: In 12.3.C k infinite case.

$\{p_1 - p_m\}$. $\exists$ hyperplane H.

$p_1 \dots p_m \notin H$. $\boxed{\equiv}$ ②

pf of (d)

$\dim X = 0$. fin. many pts. trivial.

Correct for $\dim X \le r-1$. For $\dim X = r$.

$X = X_1 \cup \dots \cup X_s$.

$\eta_1 \qquad \eta_s$.

$\exists$ hyperplane H_1. s.t. $\{\eta_1 - \eta_s\} \cap H_1 = \emptyset$. $\therefore X_i \not\subseteq H$.

$X' = X \cap H_1$ ~~dim~~

$\dim X' \le \dim X - 1$.

$\exists\ H_2 \dots H_r$ hyperplane

s.t. $\#(X' \cap H_2 \cap \dots \cap H_r) < +\infty$.

$\therefore \#(X \cap H_1 \cap \dots \cap H_r) < +\infty$. $\boxed{\equiv}$

12.3.E 跳

12.3.F $k=\bar{k}$. X, Y irr. cl. subvarieties of codim m, n respectively in $\mathbb{A}^d_k$. Show that every irr. component of $X\cap Y$ has codim $\le m+n$ in $\mathbb{A}^d_k$.

Pf. $\cancel{\mathbb{A}^d_k \cap \Delta \subseteq \mathbb{A}^d_k \times_k \mathbb{A}^d_k}$

$$\begin{array}{ccc}
\begin{array}{ccc} X\times_k Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & k \end{array}
& \subseteq &
\begin{array}{ccc} \mathbb{A}^d_k\times_k \mathbb{A}^d_k & \longrightarrow & \mathbb{A}^d_k \\ \downarrow & & \downarrow \\ \mathbb{A}^d_k & \longrightarrow & k \end{array}
\end{array}$$

k alg. closed. $\therefore$ X irr. Y irr. $\Rightarrow$ $X\times_k Y$ irr.
and $\dim X\times_k Y = \dim X + \dim Y$. From 12.2.F
$= (d-m)+(d-n) = 2d-m-n$.

$\Delta: \mathbb{A}^d_k \hookrightarrow \mathbb{A}^d_k\times\mathbb{A}^d_k$ diagonal

we can see. $(X\times_k Y)\cap\Delta(\mathbb{A}^d_k) = \Delta(X\cap Y)$.

$\Delta(\mathbb{A}^d_k)\subseteq \mathbb{A}^d_k\times\mathbb{A}^d_k$ is defined by $x_1-y_1, x_2-y_2 \ldots x_d-y_d$.

$X\times_k Y \rightsquigarrow p \in \operatorname{Spec} \underbrace{k[x_1 \ldots x_d, y_1 \ldots y_d]}_{= A}$.

~~之 irr. component of~~ $X\cap Y$
~~is irr. in~~ $\cancel{X\times_k Y \rightsquigarrow P}$
~~consider~~ $\cancel{A_P / pA_P}$ ~~integral~~
~~and~~ $\cancel{(x_1-y_1, x_2-y_2, \ldots, x_d-y_d)}$

~~Similar proof in 12.3.1(b)~~

~~$\Rightarrow$ codim~~ $\cancel{\dim A_p/pA_p \le \dim A/p = 2d-m-n}$.

$\cancel{(\overline{x_1-y_1}, \dots, \overline{x_d-y_d}) \subseteq q \subseteq pA_p}$

$\cancel{q \in \operatorname{Spec} A_p/pA_p}$

~~corresponding to~~ $\cancel{(X \times_k Y) \cap \Delta(A) = X \cap Y}$

$\cancel{\operatorname{codim}_{A_p/pA_p} q \le d}$.

component of $X \cap Y \leftrightarrow$ min. prime ideals of A containing $(x_1-y_1, \dots, x_d-y_d)$ and p

~~It suffices to show~~ $\cancel{\dim A/(x_1-y_1, \dots, x_d-y_d, p) \le m+n}$.

~~codim~~

$Z \longleftrightarrow Q$

$\operatorname{codim}_{A_k^d} Z = \sup(\text{length of primes of } A \text{ containing } (x_1-y_1, \dots, x_d-y_d)\text{, end at } Q)$.

$= d + \dim A/Q$

$\dim A/Q = \dim (A/p)/(Q/p) = \dim A/p - \operatorname{codim}_{A/p} Q/p$.

$\bar{0} \subseteq (x_1-y_1, p)/p \subseteq (x_1-y_1, x_2-y_2, p)/p \subseteq \dots \subseteq (x_1-y_1, \dots, x_d-y_d, p)/p$

$\subseteq Q/p$

$\exists\, Q'$ among $x_1-y_1, \dots, x_d-y_d$ ~~codim~~ $\cancel{(x_1-y_1, \dots, x_d-y_d, p)/p \le d}$

s.t. $\operatorname{codim} Q'/p \le d$. Similar in 12.3.1(b)

$\because Q$ minimal $\therefore \operatorname{codim} Q/p \le d$.

$\therefore \dim A/Q \ge 2d-m-n-d = d-m-n$

$\therefore \operatorname{codim}_{A_k^d} Z \le d-(d-m-n) = m+n$ ∎

12.3.G. X, Y pure dim d. subvarieties of $\mathbb{P}^n$ of codim d, e respectively $d+e \le n$.

Show that $X \cap Y \neq \emptyset$.

Pf. It suffices to show $\dim \operatorname{cone}(X) \cap \operatorname{cone}(Y) \geq 1$.

$\operatorname{codim} \operatorname{cone}(X) = d$. $\operatorname{codim} \operatorname{cone}(Y) = e$

~~$\operatorname{codim} \operatorname{cone}(X) \cap \operatorname{cone}(Y)$~~

~~$\dim(\operatorname{cone}(X) \cap \operatorname{cone}(Y)) = \dim(\operatorname{cone}(X)_{\bar{k}} \cap \operatorname{cone}(Y)_{\bar{k}})$~~

~~codim~~ ~~codim~~

~~$\operatorname{codim}(\operatorname{cone}(X)_{\bar{k}} \cap \operatorname{cone}(Y)_{\bar{k}})$~~

WLOG, X, Y irr. $\Rightarrow$ $\operatorname{cone}(X), \operatorname{cone}(Y)$ irr.

"$\mathbb{P}^n$ is good"

$X \cap Y = Z_1 \cup \_\_\_ \cup Z_s$ union of irr. components.

$\Rightarrow (X \cap Y)_{\bar{k}} = Z_{1\bar{k}} \cup \_\_\_ \cup Z_{s\bar{k}}$

keep irreducible component

~~$\dim Z_i$~~ $\dim Z_1 = \dim Z_{1\bar{k}}$

$= \dim \mathbb{A}_{\bar{k}}^{n+1} - \operatorname{codim}_{\mathbb{A}_{\bar{k}}^n} Z_{1\bar{k}}$

$\geq n+1-d-e = 1$ (From 12.3.F).

$\boxed{}$. 后面交换代数内容先跳过
可以参考笔记(抄书).

12.4 ~~Krull~~ Dimensions of fibers of morphisms of var.

12.3.11 Def

(A, m) Noetherian local ring of $\dim = d$.

$f_1 \dots f_d$ <u>system of parameters</u> for (A, m) if

$$m = \sqrt{(f_1, f_2 \dots f_d)}$$

12.4.A $\pi: X \to Y$ morphism of locally Noetherian schemes.

$p \in X$. $q \in Y$. $\pi(p) = q$

Show: $\operatorname{codim}_X p \le \operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1} q} p$.

Pf. It suffices to show

$$\dim \mathcal{O}_{X,p} \le \dim \mathcal{O}_{Y,q} + \dim \mathcal{O}_{\pi^{-1}(q), p}.$$

$$\begin{array}{ccc} \mathcal{O}_{Y,q} & \longrightarrow & \mathcal{O}_{X,p} \\ m_q & \longrightarrow & m_p \end{array}$$

locally:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \\ B_q & \longrightarrow & A_p \end{array}$$

$A \otimes_B k(q) = A \otimes_B B_q / qB_q$

$\cong A_q / qA_q$

pA_q / qA_q is ~~maximal~~ prime ideal of A_q / qA_q

~~$\varphi(B \setminus q) \supseteq A \setminus p$~~

~~$\varphi^{-1}(p) = q \qquad p \supseteq \varphi(q)$~~

$p \cap \varphi(B \setminus q) = \emptyset$. $\therefore A \setminus p \supseteq \varphi(B \setminus q)$.

$\therefore (A_q / qA_q)_{(pA_q / qA_q)} \simeq$ <u>$A_p / \varphi(q) A_p$</u>.

i.e. prove $\dim A_p \le \dim B_q + \dim A_p / \varphi(q) A_p$.

From 12.3.K(b) (existence of system of parameters).

$\exists\, g_1 \ldots\ldots g_d$ system of parameters of B_q ~~dim B_q~~

$g_1 \ldots g_d \in q$.

$h_1 \ldots h_e$ the system of parameters of $A_p / \varphi(q)A_p$

$h_1 \ldots h_e \in p$.

$d = \dim B_q \qquad e = \dim A_p/\varphi(q)A_p$.

Hope: $\sqrt{(\varphi(g_1), \varphi(g_2), \ldots, \varphi(g_d), h_1 \ldots h_e)} = m_p \subseteq A_p$.

$\forall\, x \in m_p.\ \exists\, n \in \mathbb{N}$.

$$\bar{x}^n = \sum_{i=1}^{e} \bar{y_i}\,\bar{h_i} \in A_p/\varphi(q)A_p \qquad y_i \in A_p.$$

$\Rightarrow x^n - y_i h_i \in \underline{\varphi(q)A_p}$

$\Rightarrow (x^n - y_i h_i)^m = \sum_{j=1}^{d} z_j \varphi(g_j)$ (小跳步). $z_j \in A_p$. ✓

Use 12.3.K(a) $\quad \dim A_p \leq d+e$ ∎

12.4.1 Thm $\pi: X \longrightarrow Y$ fin. type dominant morphism of integral schemes s.t. trdeg $K(X)/K(Y) = r$.

Then $\exists\, U \subseteq Y$ open nonempty s.t. $\forall\, q \in U$.

$\pi^{-1}(q)$ nonempty and has pure dim r.

12.4.B Coro $\pi: X \to Y$ fin. type morphism of irr. k-var. $\dim X = m$, $\dim Y = n$. Then $\exists$ open subset $U \subseteq Y$ s.t. $\forall q \in U$, fiber over q has pure dimension $m-n$, or is empty.

$m-n$时　　　　$m<n$时

pf. omitted

pf of Thm 先跳

12.4.D $\pi: X \to Y$ proper morphism to irr. var.
all the fibers are nonempty, irr. of the same dim
Show that X is irr.

pf. X var. ($\because$ fin. type over k) $\therefore$ Noetherian scheme
$\therefore X = Z_1 \cup Z_2 \cup \dots \cup Z_s$ union of irr. components.
π proper $\therefore$ closed $\therefore \pi(Z_i)$ closed
$\therefore \pi(Z_i)$ irr. (otherwise Z_i reducible contradict!)
~~$\therefore \pi(Z_i)$~~
$\because$ all the fibers are nonempty $\therefore \pi$ surjective.　　$X_y := \pi^{-1}(y)$

~~$Y = \pi(X) = \bigcup_{i=1}^{s} \pi(Z_i)$ irr.~~
~~$\therefore \exists\, i \in [1, s]$ s.t. $\pi(Z_i) = Y$ denote Z_1~~
~~Consider $\pi|_{Z_1}: Z_1 \to Y$, surj.~~
~~$\pi^{-1}(y) =: X_y$　$X_y = \bigcup_{i=1}^{s} X_y \cap Z_i$~~
~~$\because X_y$ irr. $\therefore \exists\, i$ s.t. $X_y \subset Z_i$~~
~~which means: If $x \in Z_i$, $x \notin Z_j$ $\forall j \neq i$~~
~~Then $Z_i \supseteq X_{\pi(x)}$ ($\because \forall j \neq i$, never $Z_j \supseteq X_{\pi(x)} \ni x$)~~

復旦大學

Fudan University

~~$\forall y \in Y$~~

~~$\pi_i : Z_1, \dots, Z_i \to Y$~~

Claim: $\forall i \in [1,s]$, $\pi(Z_i) = Y$.

Suppose $\pi(Z_1) \subsetneq Y$.

$\pi|_{Z_1} : Z_1 \longrightarrow \pi(Z_1)$

12.4.2: $\exists$ dense open $V \subseteq \pi(Z_1)$ s.t. $\forall y \in V$,

$\dim X_y \cap Z_1 = \dim Z_1 - \dim \pi(Z_1)$.

also $\pi|_{Z_1} : Z_1 \longrightarrow Y$.

12.4.2: $\exists$ dense open $U \subseteq Y$ s.t. $\forall y \in U$,

$\dim X_y \cap Z_1 = \dim Z_1 - \dim Y$

$\dim \pi(Z_1) < \dim Y$ ($\because \pi(Z_1)$ irr.)

$\therefore U \cap$

- **Claim**: $\dim Z_1 = \dim Z_2 = \dots = \dim Z_s$.

If Z_j satisfies: $\forall y \in Y$ $X_y \cap Z_j \subsetneq X_y$

$X_y = \bigcup_{i=1}^{s} (X_y \cap Z_i)$ irr.

$\therefore \exists j' \in [1,s]$ s.t. $X_y \subseteq Z_{j'}$

$\therefore \forall x \in Z_j$ $\exists j' \in [1,s]$ s.t. $x \in X_{\pi(x)} \subseteq Z_{j'}$.

$\therefore$ can remove Z_j out of $\{Z_i\}_{i=1}^{s}$,

$\therefore \forall i \in [1,s]$ $\exists y \in Y$ s.t. $X_y \subseteq Z_i$.

$\therefore \pi_i = \pi|_{Z_i} : Z_i \longrightarrow Y$. From 2.4.2

$\exists$ (dense) open $U_i \subset Y$. s.t $\forall y \in U_i$.

pure $\dim X_y \cap Z_i = -\dim Y + \dim Z_i$.

$U = \bigcap_{i=1} U_i$ (dense) open of Y.

$\forall y \in U$. $\exists j(y) \in [1, s]$. $X_y \subseteq Z_{j(y)}$

$\therefore \dim Z_{j(y)} = \underbrace{\dim Y + \dim X_y}$ constant.

Hope: $j : U \longrightarrow [1, s]$ is surjective.

If not. $\exists j_0 \in [1, s]$. s.t. $\forall y \in U$.

$X_y \cap Z_{j_0} \subsetneq X_y$

$\therefore \underbrace{Z_{j_0} \cap \pi^{-1}(U)}_{\text{irr. cl. of } \pi^{-1}(U)} \subseteq \bigcup_{i=1}^{s} (\pi^{-1}(U) \cap Z_i)$

$\therefore Z_{j_0} \cap \pi^{-1}(U) \subseteq Z_{i_0} \cap \pi^{-1}(U)$.

$\left\{ \begin{matrix} \text{irr. of } X \\ \text{meeting } \pi^{-1}(U) \end{matrix} \right\} \longleftrightarrow \left\{ \text{irr. of } \pi^{-1}(U) \right\}$

$Z_{j_0} \subseteq Z_{i_0}$. Contradiction!

$\therefore$ ~~dim~~ X is of pure dim.

$\therefore$ ~~$\forall y \in Y$~~ $\forall y \in U$. $\forall i \in [1, s]$.

pure $\dim X_y \cap Z_i$.

$= -\dim Y + \dim Z_i = -\dim Y + \dim Z_{j(y)}$

$= \dim X_y$

~~4.7 Results on homology~~

~~$H \leq G$~~

~~Thm (Shapiro's Lem)~~ ~~#~~

$\therefore \forall y \in U.\ X_y \subseteq Z_i.\ \therefore \pi^{-1}(U) \subseteq Z_i.$
$\forall i \in [1.s]$

π surj. $\therefore U \subseteq \pi(Z_i)$. π closed

$\therefore Y = \overline{U} \subseteq \pi(Z_i)$

$\therefore \underline{\forall i \in [1.s]\ \pi(Z_i) = Y}.$

$\forall$ $\underline{\text{Coro of 12.4.A}}$

$\forall y \in Y$. pure dim $X_y \cap Z_i \geq \dim Z_i - \dim Y$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{\dim X_y}.$

$\therefore \forall y \in Y.\ X_y \subseteq Z_i.$

$\therefore X \subseteq Z_i$. $\therefore X$ irreducible $s = 1$ ▢

Think: $\underline{\text{12.4.A}} \Longrightarrow \underline{\text{inequality version of}}$
$\underline{\text{12.4.1}}$

大致上取遍12.4.A中的p.q.
利用不可约$\Leftrightarrow \{\overline{p}\}$. (待验证).

12.4.3 Thm $\pi: X \to Y$ fin. type k-schemes.

(a). $\dim$(largest component of $\pi^{-1}(\pi(p))$ containing p) is upper-semicontinuous

$p \mapsto$

(b). $q \longmapsto \dim$(largest component of $\pi^{-1}(q)$) upper-semicontinuous

12.4.E. Show it suffices to prove the case of X, Y integral, π dominant.
To prove (a)

pf Y obviously can be considered irreducible

Goal: $F_n := \{p \in X \mid \dim(\text{——}) \geq n\}$ closed in X.

$X = X_1 \cup X_2 \cup \cdots \cup X_s$ union of fin. irr. components.

Claim: $F_n = F_{1,n} \cup \cdots \cup F_{s,n}$.

$\forall$ component B of $\pi^{-1}(\pi(p))$.

$B = \bigcup_{i=1}^{s} (B \cap F_{i,n})$

B irr. $\therefore B \subseteq$ some $F_{i,n}$. $F_n \subseteq$ —— ✓

$\supseteq$ trivial.

$\therefore X$ can be considered irreducible

after reduced $\Rightarrow$ integral $\longrightarrow$ integral.

X irr. $\Rightarrow \overline{\pi(X)}$ irr. Y 被 $\overline{\pi(X)}$ 替代. dominant. ∎

pf of (a) ~~induction~~. $r = \dim X - \dim Y$

$n \leq r$ $F_n = X$ by 12.4.1

$n > r$. If proved F_{n-1} closed.

$\exists U$ ~~closed~~ dense open $\subseteq Y$. s.t. $\forall y \in U$.

pure dim of fiber $= r$.

pure $\dim Y \setminus U \leq \dim Y - 1$. $\Rightarrow F_{r+1}$. Induction. ∎

~~12.4.E~~. $\pi(F_n) = G_n := \{ y \in Y \mid \dim(\pi^{-1}(y)) \geq n \}$. closed by π closed ▨

Counter-example of (b) without closedness:

$\pi : \mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k \quad (x, y) \longmapsto (x, xy)$.

$G_1 = \{(0,0)\}$. $G_0 = (\mathbb{A}^2_k \setminus V(u)) \cup \{0,0\}$ not closed!

$\therefore$ not upper-semicontinuous!

$Z = V(xy-1) \rightsquigarrow \{(u, v) \mid u \neq 0, v = 1\}$ not closed.

12.4.4 prop $\pi : X \longrightarrow Y$ generically finite morphism of irr. k-var. of dim n. Then $\exists$ dense open $V \hookrightarrow Y$ s.t. $\pi|_{\pi^{-1}(V)}$ is finite.

Pf. (use 10.3.G)

~~12.4.G Show that π is closed.~~

Pf. Y affine case. (Fact: only need to prove this case).

π: dominant.

$X = \bigcup_{i=1}^{n} U_i$. affine open covering

~~Recall~~: 10.3.G $\Rightarrow$ $\exists$ dense open V_i of Y

~~s.t. $\pi|_{U_i}$ finite : $U_i \longrightarrow V_i$.~~

s.t. $\pi|_{U_i \cap \pi^{-1}(V_i)}$ is finite.

$V \subseteq \bigcap_{i=1}^{n} V_i$. affine open. then $\pi|_{\pi^{-1}(V)}$ is finite.

~~Goal: $\pi|_{\pi^{-1}(V)}$ finite~~

12.4.G $\pi|_{\pi^{-1}(V)}$ is closed

pf. $\pi^{-1}(V) = \bigcup_{i=1}^{n} (U_i \cap \pi^{-1}(V))$

$S \subseteq \pi^{-1}(V)$ closed

$\pi(S) = \pi\left(\bigcup_{i=1}^{n} (U_i \cap \pi^{-1}(V)) \cap S\right) = \bigcup_{i=1}^{n} \pi(\underbrace{U_i \cap \pi^{-1}(V) \cap S}_{\text{closed of } U_i \cap \pi^{-1}(V)})$

$\underbrace{}_{\text{closed of } V}$ (finite $\Rightarrow$ closed).

closed in V. ▨

$\therefore$ $\pi^{-1}(V) \setminus (U_i \cap \pi^{-1}(V))$ closed in $\pi^{-1}(V)$.

$\therefore$ $\pi(\pi^{-1}(V) \setminus (U_i \cap \pi^{-1}(V)))$ closed in V.

12.4.H $\pi(\pi^{-1}(V) \setminus (U_i \cap \pi^{-1}(V))) \neq V$.

pf. ~~$U_i \cap \pi^{-1}(V)$ contains generic pt.~~

~~η generic pt. of V.~~

~~$\pi^{-1}(\eta)$ ... generic pt of $\pi^{-1}(V)$.~~

Claim: $\eta \notin \pi(\pi^{-1}(V) \setminus (U_i \cap \pi^{-1}(V)))$

~~otherwise consider~~

~~$\pi: \pi^{-1}(V) \to V$.~~

~~$\pi: (\pi^{-1}(V) \setminus (U_i \cap \pi^{-1}(V))$ an irreducible component of $\pi^{-1}(V) \setminus (U_i \cap \pi^{-1}(V))) \to V$.~~

we have assumed π is dominant.

$\therefore$ $\pi^{-1}(\eta) = \xi$ generic pt of $\pi^{-1}(V)$.

$\xi \notin \pi^{-1}(V) \setminus (U_i \cap \pi^{-1}(V))$. ▨ (感觉有问题)

$\tilde{V} := V \setminus \pi(\pi^{-1}(V) \setminus (U_i \cap \pi^{-1}(V)))$ ✓ ▨

扫描全能王 创建