$P \subseteq \mathrm{Mor}(\text{scheme})$, P is a "Reasonable" class of morphisms

(i) preserved by compositions:

$\pi: X \to Y,\ \rho: Y \to Z \in P \Rightarrow \rho\circ\pi: X \to Z \in P.$

(ii) preserved by base change:

$\pi: X \to Y \in P \Rightarrow \forall Y' \to Y,\ X \times_Y Y' \to Y' \in P.$

(iii) Local on the target:

(a) If $\pi: X \to Y \in P$, then $\forall$ open $V \subseteq Y$.

$\pi^{-1}(V) \to V \in P.$

(b) $\forall \pi: X \to Y$, if $\exists$ open cover $\{V_i\}$ of Y s.t

$\forall i,\ \pi^{-1}(V_i) \to V_i \in P.$ then $\pi \in P.$

(affine-local on the target)

8.1A: (i),(ii) $\Rightarrow$ (iv): preserved by product.

S-scheme morphisms $X \to Y,\ X' \to Y' \in P$, then $X \times_S X' \to Y \times_S Y'$ $\in P$.

pf: $X \times_S X' \xrightarrow[\in P]{} X \times_S Y' \xrightarrow[\in P]{} Y \times_S Y'$

$X' \xrightarrow{cp} Y'$ $\qquad$ $X \xrightarrow{cp} Y$ 12

Rem: (i), (ii) $\Rightarrow$ (v) Cancellation Thm.

8.1 B: "isomorphisms" is a reasonable class

pf: ✓. 12

8.1.C, D: "open embeddings" reasonable.

pf: (i)(ii)(iii) ✓. 12

8.1 E: Open embeddings of sch. are monomorphisms of Sch.

pf: Directly verify. 12.

8.1 F: $\pi: X \to Y$ open embedding,

(a) Y locally Noe $\Rightarrow$ X locally Noe

(b) Y Noe $\Rightarrow$ X Noe

(c) Y qc $\not\Rightarrow$ X qc

pf: (a) By def: ✓.

(b) Suffices to show X qc.

$Y = \bigcup_i \operatorname{Spec} A_i$, A_i Noe ring.

Suffices to show $X \cap A_i$ qc, $\forall i$.

Wlog assume $Y = \operatorname{Spec} A$, A Noe ring.

Then $Y - X = V(I) \overset{\text{Noe}}{=} V(f_1, \cdots, f_n) = \bigcap_{k=1}^{n} V(f_k)$

$\Rightarrow X = \bigcup_{k=1}^{n} D(f_k)$ qc

(c) Counter example: $A = k[x_1, x_2, \cdots]$,

$Y = \operatorname{Spec} A$, $X = Y - V(m)$, $m = (x_1, x_2, \cdots)$.

then $X = Y - \bigcap_{n=1}^{\infty} V(x_n) = \bigcup_{n=1}^{\infty} D(x_n)$.

Assume $X = \bigcup_{n=1}^{\infty} D(x_n)$ has fin subopencover $\bigcup_{k=1}^{N} D(x_{n_k})$.

$\bigcup_{k=1}^{N} D(x_{n_k}) = Y - V((x_{n_1}, \cdots, x_{n_N})) \subsetneq Y - V(m) = X$. X.

□

8.1.3 Def

Property P is local on the source if

(a) If $\pi: X \to Y \in P$, then $\forall i: U \hookrightarrow X$, $\pi \circ i: U \to Y \in P$

(b) If $\exists$ open cover $X = \bigcup_i U_i$, s.t $\forall i$,
$$\begin{tikzcd} U_i \arrow[r] \arrow[d, "\varphi"'] & Y \in P \\ X \arrow[ur, "\pi"'] & \end{tikzcd}$$

then $\pi \in P$.

($\Longleftrightarrow$ affine local).

8.1.G "open embeddings" is not local on the source

pf: $X \amalg X \xrightarrow{id \amalg id} X$. $\square$

8.2 Another alg. interlude: Lying over and Nakayama
插曲

Def: Ring hom $\phi: B \to A$ is called integral if $\forall a \in A$, a is integral over $\phi(B)$.

8.2.A $\phi: B \to A$ ring hom, $(b_1, \cdots, b_n) = 1$ in B, and $\phi_{b_i}: B_{b_i} \to A_{\phi(b_i)}$ integral, $\forall i$. then ϕ integral pf: $\forall a \in A$, $\frac{a}{1} \in A_{\phi(b_i)}$ integral over B_{b_i}

$\Rightarrow \exists m_i \geq 1, t_i \geq 0$, s.t $b_i^{t_i} a^{m_i} \in Ba^{m_i-1} + \cdots + B$ (1)

Let $m = m_1 \cdots m_n$, then (1) $\Rightarrow$

$b_i^{s_i} a^m \in Ba^{m-1} + \cdots + B$, $s_i = t_i \prod_{j \neq i} m_j$.

let $s \geq s_1 + \cdots + s_n$, then $b_i^s a^m \in Ba^{m-1} + \cdots + B$.

$(b_1^s, \cdots, b_n^s) \supseteq (b_1, \cdots, b_n)^{ns} = (1)$ in B

$\Rightarrow (b_1^s, \cdots, b_n^s) = (1)$ in B

$\Rightarrow a^m \in Ba^{m-1} + \cdots + B$ ✓ $\square$

8.2.B

(a) $\phi: B \to A$ integral, then

$T^{-1}B \to \phi(T)^{-1}A$, $B/J \to A/\phi(J)A$, $B \to A/I$ integral.

but $B \to S^{-1}A$ not necessary integral.

(b) $\phi: B \to A$ integral extension, then

$T^{-1}B \to \phi(T)^{-1}A$ integral extension.

but $\underline{B/J \to A/\phi(J)A}$, $B \to A/I$, $B \to S^{-1}A$ not necessary integral extension)

(c) e.g.

8.2.1 Lemma: $\phi: B \to A$ ring hom,
$\forall a \in A$, a is integral over B
$\Longleftrightarrow$ $\exists$ subalg $C \subseteq A$ s.t. C is a f.g B-mod and $a \in C$.
$\Longleftrightarrow$ $B[a]$ is a f.g B-mod.

pf: (Atiyah Chapter 5) $\square$

8.2.2. Cor: finite $\Rightarrow$ integral.

8.2.C: $C \to B$, $B \to A$ are integral ring hom
$\Rightarrow$ $C \to A$ integral.

pf: ✓ $\square$

8.2.D: $\phi: B \to A$, then $\{a \in A \mid a \text{ integral over } B\}$ is a subalg of A.

a subset of A.

Pf: ✓.

8.2.5: The Lying Over Thm:

$\phi: B \to A$ integral extension. then $\forall$ prime $q \trianglelefteq B$,

$\exists$ prime ideal $p \trianglelefteq A$ s.t $p \cap B = q$.

The Going-up Thm:

$\phi: B \to A$ integral. $q_1 \subset \cdots \subset q_n$ prime ideals of B.

$p_1 \subset \cdots \subset p_m$ prime ideals of A s.t

$\phi^{-1}(p_i) = q_i$. $(1 \leq m < n)$,

then $p_1 \subset \cdots \subset p_m$ can be extended to

$p_1 \subset \cdots \subset p_n$ s.t $\phi^{-1}(p_i) = q_i$.

Nakayama!

8.2.H (A, m) local ring, M f.g A-mod, $f_1, \cdots, f_n \in M$

s.t $\bar{f}_1, \cdots, \bar{f}_n$ generating M/mM, then $f_1, \cdots, f_n$

generates M.

pf: Let $N = Af_1 + \cdots + Af_n \subseteq M$, then

$N \to M \to M/N \to 0$ exact

$\xrightarrow{-\otimes_A A/m} N/mN \to M/mM \to (M/N)/m(M/N) \to 0$

$N/mN \to M/mM$ surj $\Rightarrow (M/N)/m(M/N) = 0 \Rightarrow M/N = 0$

$\Rightarrow N = M$. ✓ .

8.3 A gazillion finiteness conditions on morphisms

Def: $\pi: X \to Y$,

(1) π is qc $\Leftrightarrow \pi^{-1}(qc) = qc \Leftrightarrow \pi^{-1}(aff) = qc$.

(2) π is qs $\Leftrightarrow \pi^{-1}(qs) = qs \underset{(*)}{\Leftrightarrow} \pi^{-1}(aff) = qs$.

qs sch: $qc \cap qc = qc = \bigcup_{\text{finite}} aff$. (qc, qc, aff are all open!)

(*): Fact.

Rem: X is qc (or qs) $\Leftrightarrow$ $X \to \operatorname{Spec}\mathbb{Z}$ is qc (or qs)

8.3.A qc (or qs) is preserved by composition

8.3.B (a) Any morphism from a Noe sch is qc.

(b) Any morphism from a qs sch is qs

(Cor: Any mor from a locally Noe sch is qs)

pf: (a) Let X Noe,

Let $X = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$, A_i Noe ring, $\forall$ open $U \hookrightarrow X$.

$$U = \bigcup_{i=1}^{n} (U \cap \operatorname{Spec} A_i) = \bigcup_{i=1}^{n} \bigcup_{\substack{j=1,\dots,n_i \\ f_{ij} \in A_i \\ D(f_{ij}) \subseteq U \cap \operatorname{Spec} A_i}} D(f_{ij}) \text{ is qc. } \checkmark$$

(b). Let X qs. $\forall U \hookrightarrow X$, Let qc $C, D \hookrightarrow U$,

then $C, D \hookrightarrow X \Rightarrow C \cap D$ qc. so U qs $\checkmark$.

8.3.2 Rem: $\pi: X \to Y$,

X qc $\not\Rightarrow$ π qc.

X qc, Y qs $\Rightarrow \pi$ qc

8.3.C (a) qc is affine-local on the target

(b) qs -

Pf: (a) $\pi: X \to Y$, Suffices to show

$\exists$ affine open cover $Y = \bigcup_i U_i$ s.t $\pi^{-1}(U_i) \to U_i$ qc,

then π qc.

Affine communication lemma 5.3.2

$P \subseteq$ {affine open subsets of a sch X}

If (i) $\operatorname{Spec} A \in P \Rightarrow \forall f \in A$, $\operatorname{Spec} A_f \in P$.

(ii) $(f_1, \cdots, f_n) = A$ and $\operatorname{Spec} A_{f_i} \in P$, $\forall i$

$\Rightarrow \operatorname{Spec} A \in P$.

(iii) $X = \bigcup_i \operatorname{Spec} A_i$, $\forall i$, $\operatorname{Spec} A_i \in P$

Then $P =$ {affine open subsets of X}

$P = \{ \operatorname{Spec} A \hookrightarrow Y \mid \pi^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A \text{ qc} \}$

(i) $A_f \hookleftarrow A$: ✓. (or $\pi^{-1}(\operatorname{Spec} A)$ qc?)

(ii) $\pi^{-1}(\operatorname{Spec} A_{f_i}) \to \operatorname{Spec} A_{f_i}$ qc

$\forall$ qc $C \hookrightarrow \operatorname{Spec} A$, $C \cap \operatorname{Spec} A_{f_i}$ qc

$\Rightarrow \pi^{-1}(C) = \bigcup_{i=1}^{n} \pi^{-1}(C \cap \operatorname{Spec} A_{f_i})$ qc.

(iii) ✓.

(b) $P = \{\operatorname{Spec} A \overset{o}{\hookrightarrow} Y \mid \pi^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A \text{ qs}\}$.

(i) ✓.

(ii) $\forall$ qc $C \overset{o}{\hookrightarrow} \operatorname{Spec} A$, ???

(iii) ✓.

8.3.3 Def: $\pi: X \to Y$ is aff if $\pi^{-1}(\text{aff}) = \text{aff}$

8.3D: aff mor is qc and qs. ✓.

8.3.4 Prop ("aff" is aff.-local on the target)

pf: Use 5.3.2. $\pi: X \to Y$

(i) $\operatorname{Spec} B \overset{o}{\hookrightarrow} Y$, $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$;

$\forall s \in B$, $\pi^{-1}(\operatorname{Spec} B_s) = \operatorname{Spec} A_{\pi^\# s}$.

(ii) $\pi: X \to \operatorname{Spec} B$, $(s_1, \cdots, s_n) = B$.

$X_{\pi^\# s_i} = \pi^{-1}(\operatorname{Spec} B_{s_i}) =: \operatorname{Spec} A_i$ aff.

Show X is aff:

Let $A = \Gamma(X, \mathcal{O}_X)$, then

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\alpha"'] & & \operatorname{Spec} B \\ & \operatorname{Spec} A \arrow[ur, "\beta"'] & \end{tikzcd}$$

induced by $id: A \to A$

induced by $\pi^\#: B \to \Gamma(X, \mathcal{O}_X) = A$

By 8.3.C, X is qcqs.

$\xRightarrow{\text{qcqs lemma 6.2.9}}$ $A_{\beta^\# s_i} = \Gamma(X, \mathcal{O}_X)_{\beta^\# s_i} \cong \Gamma(X_{\beta^\# s_i}, \mathcal{O}_X) = A_i$

$\Rightarrow \alpha: \operatorname{Spec} A_i \xrightarrow{\sim} \operatorname{Spec} A_{\beta^\# s_i}$.

$\{\operatorname{Spec} A_{\beta^\# s_i}\}$ covers A, $\{\operatorname{Spec} A_i\}$ covers X

$\Rightarrow \alpha: X \xrightarrow{\sim} \operatorname{Spec} A \Rightarrow X$ aff. 12.

8.3.E: Suppose Z is a closed subset of aff sch $\operatorname{Spec} A$, locally cut out by one equation.

(i.e $\exists$ open cover $\operatorname{Spec} A = \bigcup_i U_i$, and $\forall i$, $\exists f_i \in \Gamma(U_i, \mathcal{O}_{\operatorname{Spec} A})$

s.t $Z \cap U_i = U_i - (U_i)_{f_i}$)

Then Y $\mathrm{Spec}A - Z$ is aff.

Pf: Wlog let U_i aff.

Consider $\pi: Y \hookrightarrow \mathrm{Spec}A$

For $\mathrm{Spec}A = \bigcup_i U_i$, $\pi^{-1}(U_i) = (U_i)_{f_i}$ is aff

$\Rightarrow Y$ is aff.

$\square$