Def. 9.1.1

A morphism $\pi: X \to Y$ is a closed embedding (or closed immersion) if it is an affine morphism, and for every affine open subset $\operatorname{Spec} B \hookrightarrow Y$, with $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$, the map $B \to A$ is surjective. (i.e. of the form $B \to B/I$)

If $X \subseteq Y$, we say that X is a closed subscheme of Y.

Prop.

① closed embedding is affine-local on the target 9.1.D

② $X \hookrightarrow Y$, $\mathcal{I}_{X/Y} = \ker(\mathcal{O}_Y \to \pi_* \mathcal{O}_X)$ 9.1.E

$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_Y \to 0$

③ Closed subscheme $\xleftrightarrow{1:1}$ QCoh of ideals 9.1.F

④ Closed embeddings & open embeddings are monomorphisms

Def. (Locally closed embedding / immersion) 9.2.0

We say a morphism $\pi: X \to Y$ is a locally closed embedding if π can be factored in to $X \overset{\rho}{\hookrightarrow} Z \overset{\tau}{\hookrightarrow} Y$

If $X \subseteq Y$, we say X is a locally closed subscheme of Y. 9.2.1

Prop.

locally closed embeddings are locally of finite type. 9.2.A

Pf. This is followed from open embeddings & closed embeddings are locally of finite type

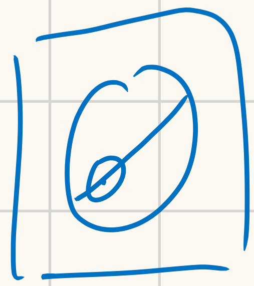
Prop. 9.2.B

Suppose $\pi: X \to Y$ is a locally closed embedding whose image is a closed subset of Y, then π is a closed embedding.

Pf. By closed embedding is local on target,

$\forall\ x \in X,\ \exists\ U_x \subseteq Y$ s.t. $\pi(x) \in U_x$,

$\pi|_{\pi^{-1}(U_x)}$ is a closed embedding.



$Y = \left(\bigcup_{x\in X} U_x\right) \cup (Y \setminus \mathrm{im}\,\pi)$,

$\pi|_{\pi^{-1}(U_x)}$ and $\pi|_{\pi^{-1}(\mathrm{im}\,\pi)}$ are all closed embedding.

Def. (Intersection) (May be) 9.1.I

Let $i_1: X_1 \to Y$, $i_2: X_2 \to Y$ be closed embedding or open embedding,

$X_1 \cap X_2 := X_1 \times_Y X_2$.

Prop. 9.2.C

$V \to X$ is a morphism

(i) V is the intersection of an open subscheme of X and a closed subscheme of X.

(ii) V is an open subscheme of a closed subscheme of X.

(iii) V is a closed subscheme of a open subscheme of X.

(i.e π is a locally closed embedding)

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

Pf. (i) ⇒ (ii), (iii):

$$\begin{tikzcd} V \arrow[r, hookrightarrow] \arrow[d, hookrightarrow] \arrow[dr, phantom, "\square"] & U \arrow[d, hookrightarrow] \\ Z \arrow[r, hookrightarrow] & X \end{tikzcd} \Rightarrow \quad \iota : V \hookrightarrow U \overset{\circ}{\hookrightarrow} X$$

$$\text{or } \iota : V \overset{\circ}{\hookrightarrow} Z \hookrightarrow X.$$

(ii) ⇒ (i): $\iota : V \overset{\circ}{\hookrightarrow} Z \hookrightarrow X$

$\forall v \in V$, $\exists\, U_v \overset{\circ}{\hookrightarrow} X$ affine open, s.t. $U_v \cap Z \subseteq V$.

(Locally, $Z = \operatorname{Spec} A/I \to X = \operatorname{Spec} A$,

every principle open subset of $\operatorname{Spec} A/I$ is induced by a principle open subset of $\operatorname{Spec} A$).

Let $U = \bigcup_{v \in V} U_v$, then $U \overset{\circ}{\hookrightarrow} X$, $V = U \cap Z$.

Cor. 9.2.D

The composition of two locally closed embedding is a locally closed embedding.

Pf.

$$\begin{tikzcd} X \arrow[r, hookrightarrow] \arrow[rr, bend right, "\iota_1"'] & U \arrow[r, "\circ", hookrightarrow] & Y \arrow[r, hookrightarrow] \arrow[rr, bend right, "\iota_2"'] & V \arrow[r, "\circ", hookrightarrow] & Z \end{tikzcd}$$

$$\begin{tikzcd} U \arrow[r, "\circ", hookrightarrow] \arrow[dr, hookrightarrow] & Y \arrow[r, hookrightarrow] & V \\ & Y' \arrow[ur, "\circ"', hookrightarrow] & \end{tikzcd}$$ by last prop.

⇒ $\iota_2 \circ \iota_1 : X \hookrightarrow U \hookrightarrow Y' \overset{\circ}{\hookrightarrow} V \overset{\circ}{\hookrightarrow} Z$ is a locally closed embedding.

Def. (Scheme-theoretic image) 9.4.2

Suppose $i : Z \hookrightarrow Y$ closed subscheme,

$\rightsquigarrow 0 \to \mathcal{I}_{Z/Y} \to \mathcal{O}_Y \to i_* \mathcal{O}_Z \to 0$ exact

We say that the image of $\pi : X \to Y$ lies in Z if the composition $\mathcal{I}_{Z/Y} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ is zero.

If the image of π lies in Z_j, $j \in J$,
then it's clearly the image of π lies in their intersection.
(Check on affine open subset)
We then define the scheme-theoretic image of π,
a closed subscheme of Y, as the intersection of all closed subschemes containing the image.

Example.

① $\pi: \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to \operatorname{Spec} k[x]$ given by $x \mapsto \varepsilon$,
$\operatorname{im} \pi = \operatorname{Spec} k[x]/(x^2) \hookrightarrow \operatorname{Spec} k[x]$.

② $\pi: \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to \operatorname{Spec} k[x]$ given by $x \mapsto 0$
$\operatorname{im} \pi = \operatorname{Spec} k[x]/(x) \hookrightarrow \operatorname{Spec} k[x]$.

③ $\pi: \operatorname{Spec} k[t, t^{-1}] = \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 = \operatorname{Spec} k[u]$ given by $u \mapsto t$,
$f(t, t^{-1}) \mapsto 0 \Leftrightarrow f(t, t^{-1}) = 0$.
$\operatorname{im} \pi = \mathbb{A}^1$.

Thm. 9.4.4

Suppose $\pi: X \to Y$ is a morphism of schemes, If X is reduced or π is quasi-compact, then the scheme-theoretic image of π may be computed affine locally:
on $\operatorname{Spec} A \hookrightarrow Y$, it cut out by the functions (elt. of A) that pull back to the function 0 (on $\pi^{-1}(\operatorname{Spec} A)$)
i.e. ker of $A = \mathcal{O}_Y(\operatorname{Spec} A) \to \pi_* \mathcal{O}_X(\operatorname{Spec} A)$

Pf. $\operatorname{Spec} B \hookrightarrow Y$, $I(B) := \ker(B \to \Gamma(\operatorname{Spec} B, \pi_* \mathcal{O}_X))$

We need to check that $\phi: I(B)_g \xrightarrow{\sim} I(B_g)$, $\forall$ such B, $g \in B$.

then we will have defined the scheme-theoretic image as a closed subscheme.

ϕ is inj. : $I(B)_g \to I(B_g) \to B_g$ inj. by $I(B) \hookrightarrow B$.

ϕ is surj : $\forall \frac{r}{g^n} \in I(B_g)$, we need to check that

$g^m r \in I(B)$ for some m.

Let $\pi^{-1}(\operatorname{Spec} B) = \bigcup_i \operatorname{Spec} A_i$.

$\pi_i : \pi|_{\operatorname{Spec} A_i} : \operatorname{Spec} A_i \to \operatorname{Spec} B$,

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B_g & \longrightarrow & A_{\pi_i^\#(g)} \end{array}$$

So $\pi^\#(\frac{r}{g^n}) = 0 \iff \exists m_i,\ \pi_i^\#(g)^{m_i} \cdot \pi_i^\#(r) = 0$.

(i) If $\pi^{-1}(\operatorname{Spec} B)$ is reduced,

$\pi_i^\#(g)^{m_i} \pi_i^\#(r) = 0 \Rightarrow \pi_i^\#(g) \pi_i^\#(r) = 0$, $\forall i$

$\Rightarrow \pi^\#(gr) = 0$.

(Notice that a function on a reduced scheme is zero if it has value zero at every point.)

(ii) if $\pi^{-1}(\operatorname{Spec} B)$ is quasi-compact,

$\pi^{-1}(\operatorname{Spec} B) = \bigcup_{i=1}^n \operatorname{Spec} A_i$. Let $m = \max_{1 \le i \le n} \{m_i\}$,

then $\pi^\#(g^m r) = 0$.

Cor. 9.4.5

Under the hypotheses of Thm 9.4.4,

the closure of the set-theoretic image of π

is the underlying set of the scheme-theoretic image.

Pf. Check affine-locally.

Prop.

If X is reduced, the scheme-theoretic image of $\pi: X \to Y$ is reduced.

Pf. By the last thm, we can check locally:

$Y = \bigcup_i \operatorname{Spec} B_i$, $I(B_i) = \ker(B_i \to \Gamma(\operatorname{Spec} B_i, \pi_* \mathcal{O}_X))$ is a radical ideal. So $B_i / I(B_i)$ is reduced.

Def. (Scheme-theoretic closure of a locally closed subschemes)

We define the scheme-theoretic closure of a locally closed embedding $\pi: X \hookrightarrow Y$ as the scheme-theoretic image of π.

Prop.

If a locally closed embedding $V \hookrightarrow X$ is quasicompact, or if V is reduced, then (iii) $\Rightarrow$ (i), (ii) in 9.2C.

Pf. $\pi: V \overset{\text{cl.}}{\hookrightarrow} U \overset{\text{open}}{\hookrightarrow} X$.

Let Z be the scheme-theoretic image of π.

Then $Z = \overline{V}$ in X by 9.4.5

$$\begin{tikzcd} V \arrow[r] \arrow[dr, hook] & \overline{V} \cap U \arrow[d, hook] \\ & U \end{tikzcd}$$

$\Rightarrow V \hookrightarrow \overline{V} \cap U \Rightarrow V = \overline{V} \cap U$.

Def. (The reduced subscheme structure on a closed subset.) 9.4.9

Suppose X^{set} is a closed subset of a scheme Y.

Then we can define a canonical scheme structure X on X^{red} that is reduced.

① $\operatorname{Spec} B \hookrightarrow Y$, $\mathcal{I}(\operatorname{Spec} B) := I(X^{set} \cap \operatorname{Spec} B)$,

X is defined by $\mathcal{I}$.

② $W := \coprod_{x \in X^{set}} \operatorname{Spec} k(x)$ (reduced), $\rho: W \to Y$ induced by the canonical map $\operatorname{Spec} k(x) \to Y$.

X is the scheme-theoretical image of ρ

③ X is the smallest closed subscheme whose underlying set contains X^{set}.

These definitions are the same.

Def. 9.4.10

Let $X^{set} = Y$, we obtain a reduced closed subscheme $Y^{red} \hookrightarrow Y$. (Correspond to Liu Qing 2.4.2)

Def. (Locally principle closed subschemes, effective Cartier divisors) 9.5.1

A closed subscheme is locally principle if it is locally cut out by a single equation.

i.e. $\pi: X \hookrightarrow Y$, s.t. $Y = \bigcup_i U_i$, $U_i = \operatorname{Spec} A_i \hookrightarrow Y$,

$\pi^{-1}(U_i) \xrightarrow{\sim} V(s_i) \hookrightarrow \operatorname{Spec} A_i$ for some $s_i \in \Gamma(U_i, \mathcal{O}_Y)$.

If the ideal sheaf is locally generated near every point by a function that is not a zero divisor, we call the closed subscheme an effective Cartier divisor.

When we say "slicing by an effective Cartier divisor", we mean "restrict to an effective divisor.

Prop. 9.5.A

Suppose $t \in A$ is a non-zerodivisor.

Then t is a non-zerodivisor in $A_\mathfrak{p}$, $\forall \mathfrak{p} \in \operatorname{Spec} A$.

Pf. $\forall \frac{a}{u} \in A_\mathfrak{p}$, $t \cdot \frac{a}{u} = 0 \Leftrightarrow \exists u' \in A \setminus \mathfrak{p}$, $u'ta = 0$

$\Leftrightarrow u'a = 0 \Leftrightarrow \frac{a}{u} = 0$.

Rmk. 9.5.2

If D is an effective Cartier divisor on an affine scheme $\operatorname{Spec} A$,

it is not necessarily true that $D = V(t)$ for some $t \in A$.

i.e. Being an effective Cartier divisor is not an affine-local condition.

Prop. 9.5.B

A locally principle closed subscheme of X is an effective Cartier divisor if it doesn't contain any associated points of X.

Rmk. Liu Qing 7.1.8

For an l.c.i scheme over an regular locally Noetherian, associated points are just generic points.

Prop. 9.5.C

Suppose $V(t) = V(t') \hookrightarrow \operatorname{Spec} A$ is an effective Cartier divisor, with t and t' non-zerodivisor in A. Then $t = st'$ for some invertible $s \in A$.

Pf. $V(t)=V(t') \Rightarrow tA=t'A$ as ideals.

So $\exists\, s, s' \in A$, s.t. $ts=t'$, $t's'=t$

Then $t=tss'$, $t(1-ss')=0$, $ss'=1$.

Def. (regular sequence) 9.5.4

If M is an A-mod, a sequence $x_1, \cdots, x_r \in A$ is called an M-regular sequence (or a regular sequence for M) if

(i) For each i, x_i is not a zerodivisor for $M/(x_1, \cdots, x_{i-1})M$

(ii) $(x_1, \cdots, x_i) \subsetneq M$

We say r is the length of the regular sequence $x_1, \cdots, x_r \in A$.

An A-regular sequence is just called a regular sequence.

Prop. 9.5.D

The condition (i) is preserved under flat base change.

Pf. $\forall\, i$, $0 \to M/(x_1, \cdots, x_{i-1}) \xrightarrow{\times x_i} M/(x_1, \cdots, x_{i-1})$ exact

$\Rightarrow \quad 0 \to M/(x_1, \cdots, x_{i-1}) \otimes_A N \xrightarrow{\times(x_i \otimes 1)} M/(x_1, \cdots, x_{i-1}) \otimes_A N$ exact,

where N flat over A.

9.5.E

Prop.

If x, y is an M-regular sequence, x^N, y is also an M-regular sequence.

Pf. x is a non-zerodivisor $\Rightarrow x^N$ is a non-zerodivisor.

if $y \cdot (m + x^N M) = x^N M$, $y \cdot m \in x^N M \subseteq xM$

By y is a non-zerodivisor of M/xM, $m \in xM$. Let $m = xm'$

So $xy \cdot m' \in x^N m$, by x is a non-zerodivisor, $ym' \in x^{N-1} m$.

Using induction, we can finally get $y \cdot m \in x^N M$.

Cor.

If $x_1, \cdots, x_n$ is a M-regular sequence, so is $x_1^{a_1}, \cdots, x_n^{a_n}$, $a_n \in \mathbb{N}^*$.

Thm. (Liu Qing Ex 6.3.1) 9.5.6

Suppose (A, m) is a Noetherian local ring, and M is a finitely generated A-module. Then any M-regular sequence $(x_1, \cdots, x_r)$ in m remains a regular sequence upon any reordering.

Pf. It suffice to consider the case $r=2$.

If x, y is a M-regular sequence,

Let $N := \{m \in M \mid ym = 0\}$,

① Claim: $N = xN$, hence by Nakayama's Lemma, $N = 0$.

$\forall n \in N$, $yn = 0 \in xM \Rightarrow n \in xM$. Let $n = xn'$

$yn = yxn' = 0 \Rightarrow yn' = 0$, $n' \in N$. so $n = xn' \in xN$.

Hence y is a non-zerodivisor of M.

② $\forall\ xm \in yM$, $\exists m' \in M$ s.t. $xm = ym'$.

Then $ym' \in xM \Rightarrow m' \in xM$, $\exists m'' \in M$ s.t. $m' = xm''$.

Now $xm = ym' = xym'' \Rightarrow m = ym'' \in yM$.

Hence x is a non-zerodivisor of M/yM.

Def. (Regular embeddings) 9.5.7

Suppose $\pi: X \hookrightarrow Y$. We say that π is a regular embedding (of codimension r) at a point $p \in X$ if in the local ring $\mathcal{O}_{Y,p}$, the ideal of X is generated by a regular sequence (of length r). We say that π is a regular embedding (of codimension r) if it is a regular embedding (of codimension r) at all $p \in X$.

Prop. 9.5.G

If a locally closed embedding $\pi: X \hookrightarrow Y$ of locally Noetherian schemes is a regular embedding at p, then it is a regular embedding in some open nbd. of p in X.

Pf. By the prop. is local, W.M.A π is a closed embedding, $X = \operatorname{Spec} B/I \to Y = \operatorname{Spec} B$.

By the definition of regular embedding, $I_p = (f_1, \cdots, f_r)$ where $f_1, \cdots, f_r \in B$ is a regular sequence in B_p.

Let $I_i = \operatorname{Ann}(f_i + (f_1, \cdots, f_{i-1})) \subseteq B/(f_1, \cdots, f_{i-1})$, then I_i is a fin. gen. B-mod due to B is Notherian. $I_{i,p} = 0$.

$\Rightarrow \exists\, p \in D(g) \subseteq \operatorname{Spec} B$, $I_{i,g} = 0$, $\forall\, 1 \le i \le n$.

Let $N_i = I/(f_1, \cdots, f_r)$, then N_i is also fin. gen. B-mod. $N_{i,p} = 0$

$\Rightarrow \exists\, p \in D(h) \subseteq \operatorname{Spec} B$, $N_{i,h} = 0$, $\forall\, 1 \le i \le n$.

Let $U = D(hg)$. Then $p \in U$ satisfies the condition.

Cor.

If X is locally of finite type over a field, then to check that a closed embedding π is regular, it suffices to consider closed pt.s

Prop. 9.5.4

Let $\iota: X \hookrightarrow Y$ be a closed imbedding of locally Noetherian scheme, Then ι is a regular embedding of codim 1 iff X is an effective Cartier divisor on Y.

Pf. "$\Rightarrow$": $\forall p \in X$, $\exists\, U \ni p$, s.t. $\mathcal{I}_p = (f_p)$ for some $f \in \mathcal{O}_Y(U)$.
As the proof of last prop, $\exists\, V \hookrightarrow U$, s.t. $\mathcal{I}(V) = (f|_V)$.
f is not a zero divisor on a nbd. of p by f_p is not a zero divisor.
So X is cut off by a non-zerodivisor locally.
i.e. X is an effective Cartier divisor on Y.
"$\Leftarrow$" trivial by definition.

Def. 9.5.8

A codim r complete intersection in a scheme Y is a closed subscheme X that can be written as the scheme-theoretic intersection of r effective Cartier divisors $D_1, \cdots, D_r$ s.t. $\forall p \in X$, the equations corresponding to $D_1, \cdots, D_r$ form a regular sequence.

Def. (local complete intersection, l.c.i.) Liu Qing 6.3.17

Let Y be a locally Noetherian scheme, $f: X \to Y$ be a morphism of finite type. We say that f is a l.c.i. at x if $\exists\, U \ni x$, $U \hookrightarrow X$, s.t.

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ U \arrow[ur, hook, "i"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes

where i is a regular embedding and g is a smooth morphism. We say that f is a l.c.i. if it is a l.c.i. at all of its points.

Lemma. Liu Qing 6.3.6

Let A be a ring. $I=(a_1,\cdots,a_r)$ is an ideal generated by a regular sequence. then the image of a_i in I/I^2 forms a basis of I/I^2 over A/I. In particular, I/I^2 is a free A/I-mod of rank n.

Pf. Claim 1 $\sum_{i=1}^{n} a_i x_i = 0$, $x_i \in A \Rightarrow x_i \in I$, $\forall i$.

We use induction. This is obvious for $n=1$. ($a_1 x_1 = 0 \Rightarrow x_1 = 0$)

Let us suppose that this is true for $n-1$.

As a_n is not a zerodivisor in $A/(a_1,\cdots,a_{n-1})$, we have $x_n \in (a_1,\cdots,a_{n-1})$. Write $x_n = \sum_{i=1}^{n-1} a_i y_i$, $y_i \in A$.

It follows that $\sum_{i=1}^{n-1} a_i(x_i + a_n y_i) = 0$. By the induction hypothesis, $x_i + a_i y_i \in I \Rightarrow x_i \in I$, $i \le n-1$. So $x_i \in I$, $\forall i$.

Now let $x_1,\cdots,x_n \in A$ be such that $\sum_{i=1}^{n} a_i x_i \in I^2 = \sum_{i=1}^{n} a_i I$.

We have $\sum_{i=1}^{n} a_i(x_i - z_i) = 0$ for some $z_i \in I$.

From above, $x_i - z_i \in I \Rightarrow x_i \in I$.

This shows that the image of a_i in I/I^2 form a free family over A/I. It's therefore a basis since the a_i generate I.

Def. Liu Qing 6.3.7

Let $f: X \hookrightarrow Y$ be an (locally closed) immersion.
Let $V \hookrightarrow Y$ s.t. $f: X \overset{i}{\hookrightarrow} V \hookrightarrow Y$
Let $\mathcal{J}$ be the sheaf of ideal defining i.
$\mathcal{C}_{X/Y} := i^*(\mathcal{J}/\mathcal{J}^2)$ on X is called the conormal sheaf of X in Y.
($\mathcal{C}_{X/Y}$ does not depend on the choice of V)
As $\mathcal{J}/\mathcal{J}^2$ is killed by $\mathcal{J}$, we have $i_*\mathcal{C}_{X/Y} = \mathcal{J}/\mathcal{J}^2$.
$\mathcal{N}_{X/Y} := \mathcal{C}_{X/Y}^\vee$ is called the normal sheaf of X in Y.

Cor. Liu Qing 6.3.8

Let $f: X \to Y$ be a regular immersion.
Then $\mathcal{C}_{X/Y}$ is a locally free sheaf on X,
of rank n if f is a regular immersion of codim n.

Pf. This is easy from the lemma.

Prop.

Let X, Y, Y', Z be locally Noetherian schemes.
(a) Let $f: X \to Y$, $g: Y \to Z$ be regular immersions (resp. of codim n, m)
Then $g \circ f$ is a regular immersion (resp. of codim $n+m$)
Moreover, we have a canonical exact sequence
$$0 \to f^*\mathcal{C}_{Y/Z} \to \mathcal{C}_{X/Z} \to \mathcal{C}_{X/Y} \to 0$$
(b) Let $i: X \hookrightarrow Y$ is a regular immersion of codim n,
Then for any irreducible component Y' of Y,
we have $\operatorname{codim}(X \cap Y', Y') = n$ if $X \cap Y' \neq \emptyset$.
Moreover, $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} - n$ for all $x \in X$.

(c) Let $f: X \hookrightarrow Y$ be a regular immersion, $Y' \to Y$ be a morphism. $X' := X \times_Y Y' \xrightarrow{p} Y'$,

Then we have a canonical surjection $p^* \mathcal{C}_{X/Y} \twoheadrightarrow \mathcal{C}_{X'/Y'}$.

(d) Keeping the hypotheses of (c), let us support that $Y' \to Y$ is flat. Then $X' \to Y'$ is a regular immersion, of codim n if f is of codim n. Moreover, $p^* \mathcal{C}_{X/Y} \xrightarrow{\sim} \mathcal{C}_{X'/Y'}$.

Pf. (a) By construction, $f^* \mathcal{C}_{Y/Z} \to \mathcal{C}_{X/Z} \xrightarrow{\alpha} \mathcal{C}_{X/Y} \to 0$ exact.

The sheaves in the sequence are locally free and coherent.

It follows that $\operatorname{Ker} \alpha$ is flat and coherent, and hence locally free. The homo. $f^* \mathcal{C}_{Y/Z} \to \operatorname{Ker} \alpha$ is surj. and the two sheaves have stalks of same rank.

So it is an isom.

(b) As the property is local, W.M.A $Y = \operatorname{Spec} A$, where A is a Noetherian local ring, with closed pt y.

Then $\mathcal{I}_y$ is gen. by a regular sequence $a_1, \cdots, a_n$.

By induction on n, it suffice to show the property when $n=1$. (hence $X = V(a_1)$)

Let p be the minimal prime ideals correspond to Y', let b be the image of a_1 in A/p, then $b \neq 0$. It follows that $\operatorname{codim}(X \cap Y', Y') = 1$, and $\dim X \cap Y' = \dim Y' - 1$. The last equality implies that $\dim X = \dim Y - 1$, by varying Y.

(c) is easy to check.

(d) p flat by 9.S.D

$p^* \mathcal{C}_{X/Y} \to \mathcal{C}_{X'/Y'}$ is isom. because it is surj. and the two sheaves are, locally on X, free of the same rank.

Prop. Liu Qing 6.3.13

Let S be a locally Noetherian scheme, X, Y be smooth schemes over S. Then any immersion $f: X \to Y$ of S-schemes is a reguler immersion, and we have a canonical exact sequence on X:

$$0 \to \mathcal{C}_{X/Y} \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0.$$