Review

① X is a scheme, $x \in X \rightsquigarrow T^\vee_{X,x}$ and $T_{X,x}$

② X is a k-scheme of locally finite type $\exists\ JC : X \to \mathbb{Z}_{\geq 0}$

③ X is a scheme. X is regular at $x \Leftrightarrow \mathcal{O}_{X,x}$ is regular (otherwise X is called singular)

$\Leftrightarrow \dim T_{X,x} = \dim \mathcal{O}_{X,x}$

X is regular $\Leftrightarrow X$ is regular at all $x \in X$

④ X is a k-scheme of locally finite type of pure dimension d

$x \in X(k)$ is regular $\Leftrightarrow JC(x) = d$

④' X is a k-scheme of finite type of pure dimension d

$x \in X(\bar{k})$ ($\Leftrightarrow x$ is closed) is regular $\Leftrightarrow JC(x) = d$.

13.2.E

⑤ X is a k-scheme of locally finite type of pure dimension d

X is smooth (over k) $\overset{\text{def}}{\Leftrightarrow} \exists$ affine open cover $\{U_i\}_{i \in I}$ s.t. $\forall x \in U_i$, $JC_{U_i}(x) = d$

⑤' (By 13.1.J) $\Leftrightarrow JC(x) = d \quad \forall x \in X$

If we define that X is smooth at x if $JC(x) = d$, then X is smooth $\Leftrightarrow X$ is smooth at all $x \in X$.

⑥ (13.2.H) X is a k-scheme of locally finite type of pure dimension d

X is smooth over $k \Leftrightarrow X_\ell$ is smooth over $\ell \quad \forall \ell/k$.

⑦ X is smooth at x if $\forall y \in p^{-1}(x)$ is regular in $X_{\bar{k}}$.

$$\begin{array}{c} X_{\bar{k}} \\ \downarrow p \\ X \end{array}$$

(Liu Qing's def)

Then there is no difference between Liu Qing and Vakil if $k = \bar{k}$ and X is pure dim and compact.

When $k = \bar{k}$. Liu. x is smooth $\Leftrightarrow x$ is regular

Vakil. x is smooth $\Leftrightarrow JC(x) = d$ ($= \dim X$)

By 12.3.E, the two def coincides when x is closed.

⑧ X is a k-scheme of locally finite type of pure dim d

Then by 13.1.M, $JC(x) \geq \dim_X x = d$

(lemma) 1. $\forall x \in X$, $\exists U \ni x$ s.t. $\forall y \in U$, $JC(y) \leq JC(x)$.

2. $\forall x \in X$, $\forall y \in \overline{\{x\}}$, $JC(y) \geq JC(x)$

3. $\forall x \in X$, $\exists y \in \overline{\{x\}}$, $JC(y) = JC(x)$.

$\Rightarrow$ x is smooth $\Rightarrow \exists$ open $U \ni x$ s.t. U is smooth

⑨ (Characterize smoothness on closed points) X over k, lft. puredim $= d$.

$x \in X$ $\quad$ x is smooth $\Rightarrow \exists$ closed pt $y \in \overline{\{x\}}$ s.t. y is smooth

$\exists y \in \overline{\{x\}}$ smooth $\Rightarrow \exists U \ni y$ s.t. U is smooth $\Rightarrow x$ is smooth.

Thus, if $S (\subseteq X)$ is the smooth locus of X, then S is an open subscheme

let S° be the smooth closed pts of X, then $S^\circ \subseteq S$ and we could get S by find all the points specialized by the points in S°.

⑩ (Fact) X is a k-scheme of finite type of pure dimension d (TFAE)

(i) X is smooth at x (Liu Qing's def)

(ii) $\Omega^1_{X,x}$ is free of rank $\dim_x X = d$ ($\because$ pure dimensional)

(iii) $JC(x) = d$. (ie. X is smooth at x defined by Vakil)

(iv) X is smooth in a nbd of x, U_x (smooth at every $y \in U_x$ Liu)

(v) X is smooth in a nbd of x, U_x (smooth at every $y \in U_x$ Vakil)

pf: ref Prop 6.2.2 in Liu Qing ((i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv))

Since $JC(x) = \dim_{k(x)}(\Omega^1_{X,x} \otimes_{\mathcal{O}_{X,x}} k(x))$, then (ii) $\Rightarrow$ (iii). By ⑨, (iii) $\Rightarrow$ (v)

It remains to show (v) $\Rightarrow$ (i).

U is smooth $\Rightarrow U_{\bar{k}}$ is smooth by 13.2.H $\Rightarrow U_{\bar{k}}$ is regular $\Rightarrow X$ is smooth (Liu).

13.3 Examples

13.3.A $\mathbb{A}^1_k$ is regular since $\dim(k[X]_{\mathfrak{p}}) = \begin{cases} 1 & \mathfrak{p}=(p(x)) \\ 0 & \mathfrak{p}=(0) \end{cases}$ ......

$\mathbb{A}^2_{\bar{k}}$ is regular since $\bar{k}[X,Y]_{(X-\alpha,Y-\beta)}$ has dim 2 and $\dim_{\bar{k}}((X-\alpha,Y-\beta)/(X-\alpha,Y-\beta)^2)=2$

and $\operatorname{Max} \mathbb{A}^2_{\bar{k}} = \{(X-\alpha, Y-\beta) \mid (\alpha,\beta)\in \bar{k}^2\}$.

(If you want to check the primes of $\bar{k}[X,Y]$ of dim 1, note that $\bar{k}[X,Y]$ is a UFD, and every such prime ideal comes from an irreducible element.)

$\mathbb{P}^1_k$ and $\mathbb{P}^2_{\bar{k}}$ are thus regular since it's a local property

13.3.B $\mathbb{P}^n_k$, $f\in k[X_0,\dots,X_n]$ homogenous

(a) Show that $\{\text{non-smooth pts in } V_+(f)\} = V_+(f)\cap \bigcap_{j=0}^{n} V_+(\frac{\partial f}{\partial X_j}) \triangleq L$

We just check the closed pts

Let $x=[\alpha_0,\dots,\alpha_n]\in V_+(f)$ closed $(\alpha_j\in \bar{k}) \rightsquigarrow \ker(k[\frac{X_0}{X_i},\dots,\frac{X_n}{X_i}]\to \bar{k},\ \frac{X_j}{X_i}\mapsto \frac{\alpha_j}{\alpha_i})$

$x\in LHS \Leftrightarrow JC_f(x)\neq n-1 \Leftrightarrow \operatorname{rank}(\frac{\partial \tilde{f}}{\partial X_0} \cdots \widehat{\frac{\partial \tilde{f}}{\partial X_i}} \cdots \frac{\partial \tilde{f}}{\partial X_n})=0$, $\tilde{f}(X_0,\dots,\hat{X_i},\dots,X_n)=f(X_0,\dots,1,\dots,X_n)$

$\Leftrightarrow \frac{\partial \tilde{f}}{\partial X_0}(x)=\cdots=\frac{\partial \tilde{f}}{\partial X_n}(x)=0 \Leftrightarrow \frac{\partial f}{\partial X_j}(\frac{\alpha_0}{\alpha_i},\dots,1,\dots,\frac{\alpha_n}{\alpha_i})=0,\ \forall j\neq i \Leftrightarrow \frac{\partial f}{\partial X_j}(\alpha_0,\dots,\alpha_n)=0,\ \forall j\neq i$

Since $\deg f \cdot f = X_1\frac{\partial f}{\partial X_1}+\cdots+X_n\frac{\partial f}{\partial X_n}$ and $f(\alpha_1,\dots,\alpha_n)=0$, then $\overset{①}{\Rightarrow}$ $\frac{\partial f}{\partial X_j}(\alpha_1,\dots,\alpha_n)=0,\ \forall j$

For a non closed pt x, if x is smooth, then $\exists$ closed $y\in\overline{\{x\}}$ s.t. y is smooth

$\Rightarrow y\in L^c$ (open) $\Rightarrow x\in L^c$ If $x\in L^c$, then x is smooth, since

$\forall$ closed pts in L^c is smooth $\Rightarrow L^c$ is smooth.

13.3.C. $f(X,Y,Z)=Y^2Z-X^3+XZ^2$ $\frac{\partial f}{\partial X}=-3X^2+Z^2$, $\frac{\partial f}{\partial Y}=2YZ$, $\frac{\partial f}{\partial Z}=Y^2+2XZ$

$$\begin{cases} Y^2Z-X^3+XZ^2=0 & ① \\ Z^2=3X^2 & ② \\ YZ=0 & ③ \\ Y^2+2XZ=0 & ④ \end{cases}$$

$\because YZ=0 \Rightarrow Y=0$ or $Z=0$

1° $Y=0 \overset{④}{\Rightarrow} XZ=0 \overset{①}{\Rightarrow} X=0 \overset{②}{\Rightarrow} Z=0$ no solution

2° $Z=0 \overset{④}{\Rightarrow} Y=0 \overset{1°}{\Rightarrow}$ no solution

irr: $Y^2Z-X^3+XZ^2\in(k[Y,Z])[X]$ is irr by Eisenstein's criterion.

13.3.E $X^d+Y^d+Z^d=0$ is a regular projetive plane curve over $k=\bar{k}$ chark=0.

$$\begin{cases} X^d+Y^d+Z^d=0 \\ dX^{d-1}=0 \\ dY^{d-1}=0 \\ dZ^{d-1}=0 \end{cases} \Rightarrow X=Y=Z=0 \quad \text{has no solution.}$$

13.3.F $k=\bar{k}$. find the singular closed pts.

(a) $Y^2=X^2+X^3$ $\quad \begin{cases} 2Y=0 \\ 2X+3X^2=0 \end{cases} \Rightarrow (X,Y)=(0,0)$ node

(b) $Y^2=X^3$ $\quad (X,Y)=(0,0)$ cusp

(c) $Y^2=X^4$ $\quad (X,Y)=(0,0)$ tacnode

For details about node, cusp, tacnode, see 13.3.2. remark.

13.3.G The twisted cubic $\mathrm{Proj}(k[w,x,y,z]/(wz-xy, wy-x^2, xz-y^2))$ is smooth. One way to show it is to use the fact that it is isom. to $\mathbb{P}^1_k$. Another solid solution is to calculate.

13.1.N. EXERCISE (COMPUTING THE JACOBIAN CORANK FUNCTION FOR PROJECTIVE k-SCHEMES). Suppose $X \subset \mathbb{P}^n_k$ is cut out by homogeneous polynomials $f_i \in k[x_0, \dots, x_n]$ $(1 \le i \le r)$. Show that the Jacobian corank function of X is one less than the corank of the "projective Jacobian matrix":

$$\text{(13.1.9.1)} \qquad JC(p) = \operatorname{corank} \begin{pmatrix} \frac{\partial f_1}{\partial x_0}(p) & \cdots & \frac{\partial f_r}{\partial x_0}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix} - 1.$$

You will have to make sense of the right side, as the projective Jacobian matrix has entries which are not functions; what does the corank even mean?

$$4-\operatorname{rank}\begin{pmatrix} z & y & 0 \\ -y & -2x & z \\ -x & w & -2y \\ w & 0 & x \end{pmatrix} \neq 1 \iff \operatorname{rank}\begin{pmatrix} z & y & 0 \\ -y & -2x & z \\ -x & w & -2y \\ w & 0 & x \end{pmatrix} \neq 3$$

$\Rightarrow \dots\dots$

Tangent planes and tangent lines

Suppose $X = V(f_1, \cdots, f_r) \subseteq \mathbb{A}^n$ and $a = (a_1, \cdots a_n) \in X$ is a k-point.
If X is smooth at a of dim d (i.e. $JC(a) = \dim_X a = d$), then
the tangent d-plane to X at a is given by the r equations:

$$\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}(a)\right)(x_j - a_j) = 0 \quad i = 1, \cdots, r$$

, and it is denoted by $T_a X$

$$\dim(T_a X) \underset{\uparrow}{=} \dim_{T_a X} a \underset{\uparrow}{=} \dim_X a = d.$$

Since $T_a X$ is cut out by linear system

Since they enjoy the same J-matrix at a

13.3.H. $T_a X$ is independent of the choice of $f_1, \cdots, f_r$ by the same arguement in 13.1.J.

13.3.I. Caculate $T_a X$ where $a = (1,1)$, $X = \{Y^2 = X^3\}$.
$T_a X = \{-3(X-1) + 2(Y-1) = 0\} = \{-3X + 2Y + 1 = 0\}$.

13.3.K. Define the tangent d-plane to X at a where $X = V_+(f_1, \cdots, f_r)$
$\subseteq \mathbb{P}_k^n$ and a is smooth $\quad a = [a_0, \cdots, a_n]$

$$\left(\frac{\partial \tilde{f}_i}{\partial x_1}(a)\right)(a_0 x_1 - a_1 x_0) + \cdots + \left(\frac{\partial \tilde{f}_i}{\partial x_n}(a)\right)(a_0 x_n - a_n x_0) \quad (a_0 \neq 0) \qquad i = 1, \cdots, r$$

13.3.5 Arithmetic examples

13.3.M. Spec $\mathbb{Z}$ is regular

$\mathbb{Z}_{(p)}$ is of dim 1, $(p)/(p)^2 \cong \mathbb{Z}/p\mathbb{Z}$ DVR $\quad \forall p$ prime, $\mathbb{Z}_{(0)} = \mathbb{Q}$ is a field.

13.3.N Spec $\mathbb{Z}[i]$ is regular.

similar to 13.3.M. $\mathbb{Z}[i]$ is PID

13.3.0. $[(5,5i)]$ is the unique singular point of $\operatorname{Spec}(\mathbb{Z}[5i])$

Since $\mathbb{Z}[5i]_p = \mathbb{Z}[i]_p$ for $p \not\ni 5$ $\because \mathbb{Z}[5i]_5 = \mathbb{Z}[i]_5$

Since $\mathbb{Z}[5i] \xhookrightarrow{\text{finite}} \mathbb{Z}[i] \Rightarrow \dim \mathbb{Z}[5i] = \dim \mathbb{Z}[i] = 1$

And $(1+2i) \cap \mathbb{Z}[5i] = (1-2i) \cap \mathbb{Z}[5i] \supseteq (5, 5i)$ and $\mathbb{Z}[5i]/(5,5i)$

$= \mathbb{Z}[X]/(5,X) = \mathbb{Z}/5\mathbb{Z}$

$\Rightarrow$ If $p \ni 5 \Rightarrow p = (5, 5i)$

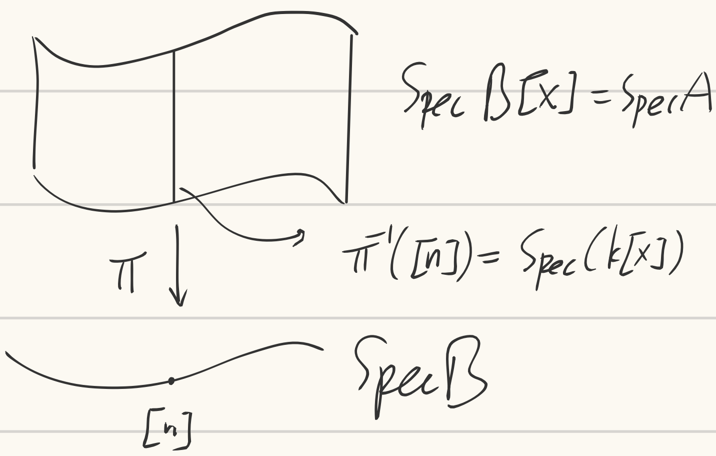
$\dim_{\mathbb{F}_5}(p/p^2) = 2$ $\quad (a5 + b(5i) = 5a + 5bi \in p^2 \Leftrightarrow 5 | a,\ 5 | b)$.

$\Rightarrow p$ is not regular.

13.3.6. $\mathbb{A}_k^n$ is regular

13.3.7. Proposition. — *Suppose* $(B, \mathfrak{n}, k)$ *is a regular local ring of dimension* d. *Let* $\phi\colon B \to B[x]$. *Suppose* $\mathfrak{p}$ *is a prime ideal of* $A := B[x]$ *such that* $\mathfrak{n}B[x] \subset \mathfrak{p}$. *Then* $A_\mathfrak{p}$ *is a regular local ring.*

$\mathfrak{n} = (f_1, \cdots, f_d)$ $\qquad 0 = q_0 \subset q_1 \subset \cdots \subset q_d = \mathfrak{n}$



$[p] \in \pi^{-1}([n])$

① $[p]$ is the generic point. Then $p = \mathfrak{n}B[x]$

(Since $B[x]/\mathfrak{n}B[x] \cong k[x]$)

$0 = q_0[x] \subset q_1[x] \cdots \subset q_d[x] = p \Rightarrow \operatorname{ht} p \geq d$

$p = (f_1, \cdots, f_d) \Rightarrow \operatorname{ht} p \leq d$

$\Downarrow$

$\dim p/p^2 \leq d \Rightarrow$ regular

② $[p]$ is a closed pt in $\pi^{-1}([n])$.

Then $0 = q_0[x] \subset q_1[x] \cdots \subset q_d[x] \subset p$ $\quad \Rightarrow \operatorname{ht} p = d+1$

$p = (f_1, \cdots, f_d, g)$ for some g $\because k[x]$ is a PID

$\Downarrow$

$\dim p/p^2 \leq d+1 \Rightarrow$ regular

13.3.P. If X regular, then $X \times \mathbb{A}^1$ is regular. In particular $\mathbb{A}_k^n$ is regular.

p.f. Check locally. $\operatorname{Spec} A$ regular $\Rightarrow$ $\operatorname{Spec} A[x]$ regular by 13.3.7.

13.4 Bertini's thm

Def. $\mathbb{P}_k^n$ with cordinates $x_0, \dots, x_n$ $(\mathbb{P}_k^n)^\vee$ with coordinates $a_0, a_1, \dots, a_n$.

$\underline{\text{dual basis}}$

$\mathbb{P}_k^n = \operatorname{Proj}(\operatorname{Sym}_\cdot V)$ $\quad \mathbb{P}_k^{n\vee} = \operatorname{Proj}(\operatorname{Sym}_\cdot V^*)$

$[v]$ $\quad [v^*]$

Then $\forall\ [v^*] \in \mathbb{P}_k^n(k)$ corresponds to a hyperplane $v^*(x) = 0$ in $\mathbb{P}_k^n$.

13.4.2 Bertini's thm

$X \hookrightarrow \mathbb{P}_k^n$ of puredim d. Then $\exists$ nonempty open $U \subseteq \mathbb{P}_k^{n\vee}$ s.t. $\forall$ k-pt $p = [H] \in U$

H does not contain any component of X, and $H \cap X$ is smooth over k of (pure) dim $d-1$.

13.5 Discrete valuation rings, and Algebraic Hartog's Lemma

13.5.10. Let X be a one dim variety. Since the normalization morphism is birational, by 7.5.5, $\exists$ dense open $U \subseteq X$, $V \subseteq X^{\text{norm}}$ s.t. $U \cong V \Rightarrow U$ is normal. $\Rightarrow \forall x \in U$, $\mathcal{O}_{X,x}$ is a DVR and hence regular $\Rightarrow x$ is regular. $\Rightarrow \{\text{non-regular pts}\} \subseteq X \setminus U$

Since $X \setminus U$ is Noe. of dim $0 \Rightarrow (X \setminus U)_{\text{set}}$ is finite $\Rightarrow X \setminus U$ consists with finite many closed pts. $\Rightarrow X$ is regular at almost all pts except for finite many closed pts.

13.5.13. *Definition: Regular in codimension one* (R_1). We say that a Noetherian ring A is **regular in codimension** 1 (or R_1 for short) if its localization at all codimension one primes are regular local rings. We say a locally Noetherian scheme A is **regular in codimension** 1 (or R_1) if it is regular at its codimension one points. By Theorem 13.5.8, Noetherian normal schemes are regular in codimension 1.

13.5.14. *Definition: Valuation at an* R_1 *point.* Suppose X is a locally Noetherian scheme. Then for any regular codimension 1 point p (i.e., any point p where $\mathscr{O}_{X,p}$ is a regular local ring of dimension 1), we have a discrete valuation val_p. If f is any nonzero element of the fraction field of $\mathscr{O}_{X,p}$ (e.g., if X is integral, and f is a nonzero element of the function field of X), then if $val_p(f) > 0$, we say that f has a **zero of order** $val_p(f)$ **at** p, and if $val_p(f) < 0$, we say that f has a **pole of order** $-val_p(f)$ **at** p.

13.5.G X integral Noetherian. $\forall f \in K(X)$ has finite many zeros and poles.

pf. Since X is quasi-compact, we may assume $X = \operatorname{Spec} A$ is affine.

$\forall f \in K(X)^* = (\operatorname{Frac} A)^*$. write $f = \frac{f_1}{f_2}$. $f_1, f_2 \in A^*$

Then it suffices to show $\forall f \in A$, if A is Noe. then $\{ f \in \mathfrak{p} \mid \operatorname{ht}\mathfrak{p} = 1 \}$ is finite.

Since $(f) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ by thm 7.13 in Atiyah, then $\forall\, \mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) = 1$, $\mathfrak{p} \supseteq r(\mathfrak{q}_j)$

for some j by thm 4.6 in Atiyah. $\Rightarrow \mathfrak{p} \in \{ r(\mathfrak{q}_j) \mid 1 \le j \le n \} \Rightarrow \{ f \in \mathfrak{p} \mid \operatorname{ht}\mathfrak{p} = 1 \}$ is finite.

13.5.15. Any normal Noe. scheme is regular at its codim 1 points.

But it could be non-regular at other points.

ex. $\{ x^2 + y^2 = z^2 \} \subseteq \mathbb{A}^3_{k}$ (char $k \neq 2$) is normal Noe, but it is singular at $(0,0,0)$.

13.5.16 For Noe. rings UFD $\Rightarrow$ integrally closed $\Rightarrow$ regular in codim 1

Noe. schemes factorial $\Rightarrow$ normal $\Rightarrow$ regular in codim 1.

examples agaist "$\Leftarrow$"

13.5.H Let $A = k[x^3, x^2, xy, y] = \{ \sum_{(i,j) \text{ finite}} a_{ij} x^i y^j \mid a_{10} = 0 \} \subseteq k[x,y]$

① x is integral over A. $\because (x)^2 - (x^2) = 0$

② $k[x,y]$ is integrally closed (easy)

$\Rightarrow$ A is not integrally closed. and $A \xrightarrow{\text{finite}} k[x,y]$

③ $\forall\, \mathfrak{p} \in \operatorname{Spec} A$, $\operatorname{ht}\mathfrak{p} = 1$.

1° $x^2 \notin \mathfrak{p} \Rightarrow \mathfrak{p} \rightsquigarrow \mathfrak{p}' \in A_{x^2} = k[x,y]_{x^2} \rightsquigarrow \tilde{\mathfrak{p}} \in \mathbb{A}^2_k$ $\operatorname{ht}\tilde{\mathfrak{p}} = 1$ $\because A_{\mathfrak{p}} = (A_{x^2})_{\mathfrak{p}'} = k[x,y]_{\mathfrak{p}'}$

Since $k[x,y]$ is regular $\Rightarrow \mathfrak{p}$ is regular

2° $Y \notin \mathfrak{p} \Rightarrow$ similar $\Rightarrow \mathfrak{p}$ is regular

$\because \mathrm{ht}\,\mathfrak{p}=1$

3° $X^2, Y \in \mathfrak{p} \Rightarrow (X^3)^2 \in \mathfrak{p} \Rightarrow X^3 \in \mathfrak{p},\ (XY)^2 \in \mathfrak{p} \Rightarrow XY \in \mathfrak{p} \Rightarrow \mathfrak{p} = (1)$, a contradiction!

13.5.I. 1. $k[W,X,Y,Z]/(WZ-XY)$ integrally closed but not UFD

2. K is a number field with class number bigger than 1, then $\mathcal{O}_K$ is not a UFD since UFD $\Leftrightarrow$ PID for Dedekind domains (taught by cmf)

13.5.19 Algebraic Hartogs's Lemma

A Noe. normal domain. Then $A = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} A_{\mathfrak{p}}$

pf. ref LiuQing Lemma 4.1.13.

13.5.20. It is worth *geometrically* interpreting Algebraic Hartogs's Lemma as saying that *a rational function on a Noetherian normal scheme with no poles is in fact regular* (an element of A). Informally: *"Noetherian normal schemes have the Hartogs property."*

Another generalization. $A \subseteq K$, then the integral closure of A in K is the intersection of all valuation rings of K containing A.

Thm 13.5.25. A Noe. integral domain (R_1). Then

(a) A is normal $\Leftrightarrow$ (b) $A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$

13.6. Smooth (and étale) morphism

13.6.2. *Definition.* A morphism $\pi: X \to Y$ is **smooth of relative dimension** n if there exist open covers $\{U_i\}$ of X and $\{V_i\}$ of Y, with $\pi(U_i) \subset V_i$, such that for every i we have a commutative diagram

$$\begin{tikzcd} U_i \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\pi|_{U_i}"'] & W \arrow[r, hook, "\text{open}"] \arrow[d, "\rho|_W"] & \operatorname{Spec} B[x_1, \ldots, x_{n+r}]/(f_1, \ldots, f_r) \arrow[dl, "\rho"] \\ V_i \arrow[r, leftrightarrow, "\sim"] & \operatorname{Spec} B & \end{tikzcd}$$

where ρ is induced by the obvious map of rings in the opposite direction, and W is an open subscheme of Spec $B[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)$, such that the determinant of the Jacobian matrix of the f_i's with respect to the *first* r x_i's

$$\det\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j\le r} \tag{13.6.2.1}$$

is an invertible (= nowhere zero) function on W.

Étale means *smooth of relative dimension* 0.

Def. X is smooth (resp Étale) at x iff X is smooth (resp Étale) at a nbd of x.
$\Rightarrow$ the smooth (resp. Étale) locus is open

Open imm. is étale: $\operatorname{Spec} A_f = \operatorname{Spec} A[x]/(fx-1) \to \operatorname{Spec} A$

$\det\left(\frac{\partial (fx-1)}{\partial x}\right) = f$ is invertible on $\operatorname{Spec} A_f$

$X \overset{\circ}{\hookrightarrow} Y$, take an affine covering for $Y = \bigcup_i V_i'$. Take a distinguished open cover for $X \cap V_i' = \bigcup_{j \in I_i} U_j$. $J \doteq \bigsqcup_i I_i$. Then $X = \bigcup_{j\in J} U_j$, $Y = \bigcup_{j \in J} V_j$. $V_j = V_i'$ $(j \in I_i)$

$\mathbb{A}^n \times Y \to Y$ is smooth of rdim n. $\{U_i\}_i$ covers $Y \Rightarrow \{\mathbb{A}^n \times U_i\}_i$ covers $\mathbb{A}^n \times Y$

13.6.B "smooth of rdim n" is local on source and on target.

pf. "target":

(a). If $\pi: X \to Y$ is smooth, then $\pi^{-1}(V) \to V$ $(V \overset{\circ}{\hookrightarrow} Y)$ is smooth:

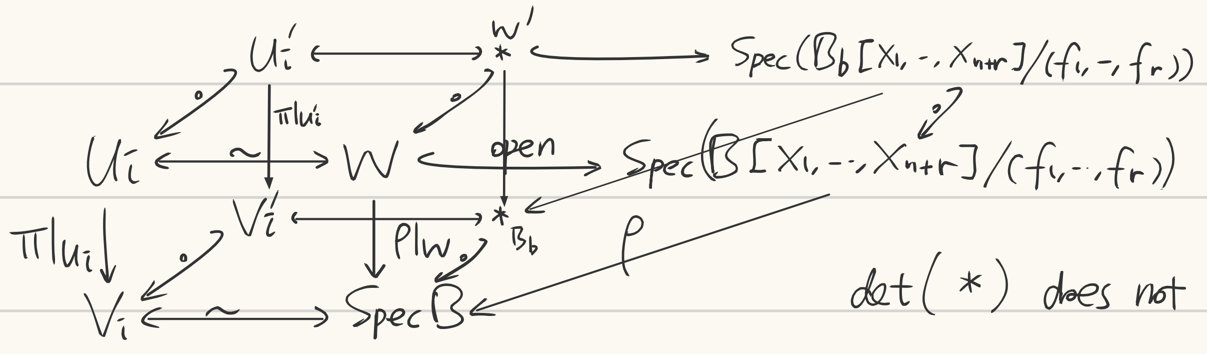
By definition, $Y = \bigcup_i V_i$, $X = \bigcup_i U_i$. Let $V_i' = V_i \cap V$, $U_i' = U_i \cap \pi^{-1}(V)$

By modifying slightly as above, we may assume W is special affine.

Write $W = \operatorname{Spec}(B[x_0, x_1, \dots, x_{n+r}]/(f_1, \dots, f_r, gx_0 - 1))$ $g \in B[x_1, \dots, x_{n+r}]$

Do it again, we may assume $V_i' \overset{\sim}{\longleftrightarrow} \operatorname{Spec}(B_b) = \operatorname{Spec}(B[y]/(by-1))$

$U_i' \overset{\sim}{\longleftrightarrow} \operatorname{Spec}(B_b[x_0, \dots, x_{n+r}]/(f_1, \dots, f_r, gx_0 - 1)) = W'$



$\det(*)$ does not change and hence still invertible in $W'(\subseteq W)$.

(b) Glueing is clear.

"Source":

(a) $\forall U \hookrightarrow X \xrightarrow{\pi} Y \quad U_i' = U \cap U_i$ Similar to above

$$\begin{tikzcd}
 & U_i' \arrow[r, leftrightarrow] \arrow[dl, hook, "\circ"'] \arrow[ddl, "\pi|_{U_i'}"] & W' \arrow[r, hook, "\circ"] \arrow[dl, hook, "\circ"'] \arrow[dr, hook, "\circ"] \arrow[dd] & \mathrm{Spec}(B_b[x_1,\dots,x_{n+r}]/(f_1,\dots,f_r)) \arrow[d, hook, "\circ"'] \\
U_i \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\pi|_{U_i}"'] & W \arrow[rr, hook, "\text{open}"] \arrow[d] & & \mathrm{Spec}(B[x_1,\dots,x_{n+r}]/(f_1,\dots,f_r)) \arrow[dll, "\rho"] \\
V_i \arrow[r, leftrightarrow, "\sim"] & \mathrm{Spec}\, B & &
\end{tikzcd}$$

$W' = \mathrm{Spec}(B[y, x_0, \dots, x_{n+r}]/(by-1, gx_0-1, f_1, \dots, f_r))$

$$\text{matrix} = \begin{pmatrix} b & & \\ & g & \\ 0 & * & \boxed{\text{origional matrix}} \\ & \vdots & \\ & * & \end{pmatrix} \qquad \begin{array}{c} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x_0} \\ \vdots \\ \frac{\partial}{\partial x_r} \end{array}$$

$\Rightarrow \det(\cdot) = b \cdot g \cdot \underbrace{\det(\text{origional})}_{\text{still invertible}}$ $\quad b, g$ are natural invertible on W'

(b) Glueing is clear.

We can thus make sense of the phrase "$\pi: X \to Y$ is smooth of relative dimension n at $p \in X$": it means that there is an open neighborhood U of p such that $\pi|_U$ is smooth of relative dimension n.

The phrase **smooth morphism** (without reference to relative dimension n) often informally means "smooth morphism of some relative dimension n", but sometimes can mean "smooth of some relative dimension in the neighborhood of every point".

13.6.C smoothness of rdim n is preserved by base change

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d] \\
X \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

π smooth of rdim n

$X = \bigcup_i U_i \quad Y = \bigcup_i V_i$

We may assume $V_i \cong \mathrm{Spec}\, B_i$, $\to \mathrm{Spec}\, B_i[x_1, \dots, x_{n+r_i}]/(f_1^{(i)}, \dots, f_{r_i}^{(i)})$

$U_i = W_i$ is a distinguished open subset.

WMA $Y' = \bigcup_i V_i'$, V_i' affine open $\longrightarrow V_i$

Thus, we may assume $X = \mathrm{Spec}(B[x_0, x_1, \dots, x_{n+r}]/(gx_0 - 1, f_1, \dots, f_r))$

$$\begin{tikzcd}
X' = \mathrm{Spec}(B'[\dots]/(\dots)) \arrow[r, hook, "\circ"] \arrow[d, "p"'] \arrow[dd, dashed, bend left] & \mathrm{Spec}(B'[\dots]/(\dots)) \arrow[d] \arrow[ddl, dashed] \\
X \arrow[r, hook, "\circ"] \arrow[ddd] & \mathrm{Spec}(B[\dots]/(f_1, \dots, f_r)) \arrow[ddd] \\
Y' = \mathrm{Spec}\, B' \arrow[dd, dashed] & \\
 & \\
Y = \mathrm{Spec}\, B &
\end{tikzcd}$$

$\det(-)$ inv on $X \Rightarrow p^{\#}(\det(-))$ inv on X'

13.6.D $X \xrightarrow{\pi} Y \xrightarrow{\rho} Z$, the $\rho\circ\pi$ is smooth of rdim $n+m$.

π: smooth of rdim m $\qquad$ ρ: smooth of rdim n

$X=\bigcup X_i \to Y=\bigcup_i Y_i \to Z=\bigcup Z_i$ (distinguished; affine)

$X_i=\pi^{-1}(Y_i)$

$X_i \to Y_i$ is smooth $\Rightarrow$ $X'_j \to Y'_j$ simultaneously distinguished open in Y_i (distinguished open)

$X_i = \bigcup_j X'_j$, $Y_i = \bigcup_j Y'_j$

$(Z'_j = Z_i)$ $\Rightarrow$ $X=\bigcup_j X'_j$, $Y=\bigcup_j Y'_j$, $Z=\bigcup_j Z'_j$

It suffices to consider:

$$\begin{array}{c} W \\ \| \end{array}$$

$$X = \operatorname{Spec} A_g[Y_0,Y_1,\dots,Y_{m+r}]/(hY_0-1, f'_1,\dots,f'_r) \hookrightarrow \operatorname{Spec} A_g[Y_1,\dots,Y_{m+r}]/(f'_1,\dots,f'_r) \hookrightarrow \operatorname{Spec} A[Y_0,Y_1,\dots,Y_{m+r}]/(gY_0-1,f'_1,\dots,f'_r)$$

$f'_j \in A[Y_1,\dots,Y_{m+r}]$

$$Y = \operatorname{Spec} \underbrace{B[X_0,X_1,\dots,X_{n+t}]/(gX_0-1,f_1,\dots,f_t)}_{\cong A_g} \hookrightarrow \operatorname{Spec} \underbrace{B[X_1,\dots,X_{n+t}]/(f_1,\dots,f_t)}_{\cong A}$$

$$Z = \operatorname{Spec} B$$

$$A[Y_1,\dots,Y_{m+r}]/(f'_1,\dots,f'_r) = B[X_1,\dots,X_t,Y_1,\dots,Y_r,\dots]/(f_1,\dots,f_t,f'_1,\dots,f'_r)$$

$$\begin{array}{c|c} & f_1 - f_t,\ gY_0-1,\ f'_1 - f'_r \\ \hline \begin{array}{c}\frac{\partial}{\partial X_1}\\ \vdots \\ \frac{\partial}{\partial X_t} \\ \frac{\partial}{\partial Y_0} \\ \vdots \\ \frac{\partial}{\partial Y_r}\end{array} & \operatorname{matrix}(X,Z) = \begin{pmatrix} \boxed{\begin{matrix} \frac{\partial f_1}{\partial X_1} & \text{matrix } YZ & \\ & \ddots & \\ & & \frac{\partial f_t}{\partial X_t}\end{matrix}} & * & * \\ 0 & g & 0 \\ 0 & * & \boxed{\begin{matrix} \frac{\partial f'_1}{\partial Y_1} & \text{matrix } XY & \\ & \ddots & \\ & & \frac{\partial f'_r}{\partial Y_r}\end{matrix}} \end{pmatrix} \end{array}$$

$\Rightarrow \det(\cdot) = \underbrace{g}_{\text{inv on } Y \text{ thus on } X} \cdot \underbrace{\det(YZ)}_{\text{inv on } Y \text{ thus on } X} \cdot \underbrace{\det(XY)}_{\text{inv on } X}$ $\Rightarrow$ inv on X ✓

13.6.4. *Observation.* Suppose $\pi: X \to Y$ is smooth of relative dimension n. Then locally on X, π can be described as an étale cover of $\mathbb{A}^n_Y$. More precisely, for every $p \in X$, there is an open neighborhood U of p, such that $\pi|_U$ can be factored into

$$U \xrightarrow{\alpha} \mathbb{A}^n_Y \xrightarrow{\beta} Y$$

where α is étale and β is the obvious projection.

$$\begin{tikzcd}
 & \operatorname{Spec} B[X_{r+1},\dots,X_{n+r}][X_1,\dots,X_r]/(f_1,\dots,f_r) \arrow[d, phantom, "\parallel"] \\
U = W \arrow[r, hook, "\circ"] \arrow[dr, "\text{étale } \alpha"] \arrow[d, "\text{smooth of rdim } n"'] & \operatorname{Spec} B[X_1,\dots,X_{n+r}]/(f_1,\dots,f_r) \arrow[dl] \arrow[d] \\
\operatorname{Spec} B & \mathbb{A}^n_Y = \operatorname{Spec} B[X_{r+1},\dots,X_{r+n}] \arrow[l, "\beta"'] \\
Y \arrow[u, phantom, "\parallel"] &
\end{tikzcd}$$

13.6.E. EXERCISE. Show that $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ is not an étale morphism.

13.6.E Since $\operatorname{Spec} k$, $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ are both points

If $\operatorname{Spec} k \longrightarrow \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$ is étale, then

$$x \longmapsto x$$

$$\begin{tikzcd}
X = \operatorname{Spec} k \arrow[r, hook, "\circ"] \arrow[d] & \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)[X_1,\dots,X_r]/(f_1,\dots,f_r)) = Z \arrow[dl] \\
Y = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) &
\end{tikzcd}$$

$x \mapsto y$ $\circlearrowleft$

$\det\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j} \neq 0$ on $\operatorname{Spec} k$

Consider $T_{X,x}$ and $T_{Y,y}$

$T_{X,x} = T_{Z,x}$ since $X \overset{\circ}{\hookrightarrow} Z \Rightarrow \underset{m\sim x}{m/m^2} = T^\vee_{X,x} = 0 \Rightarrow m = m^2$

Write $B = k[\varepsilon]/(\varepsilon^2)$ $\underset{n = m\cap B}{B} \xrightarrow{\varphi} B[X_1,\dots,X_r]/(f_1,\dots,f_r)$ $m \rightsquigarrow x$

Then $n = (\varepsilon)$ and $n = n/n^2 \xrightarrow{\varphi} m/m^2 = 0 \rightsquigarrow \varphi(\varepsilon) = 0 \Rightarrow \varepsilon \in (f_1,\dots,f_r)$

$\Rightarrow B[X_1,\dots,X_r]/(f_1,\dots,f_r) = k[\varepsilon, X_1,\dots,X_r]/(\varepsilon^2, f_1,\dots,f_r) = k[\varepsilon, X_1,\dots,X_r]/(f_1,\dots,f_r)$ $\because \varepsilon \in (f_1,\dots,f_r)$

$\exists g$ s.t. $\operatorname{Spec} k = D(g) \subseteq \operatorname{Spec}(B[X_1,\dots,X_r]/(f_1,\dots,f_r)) \Rightarrow k = B[Y, X_1,\dots,X_r]/(gY-1, f_1,\dots,f_r)$

$\Rightarrow k = k[\varepsilon, Y, X_1,\dots,X_r]/(gY-1, f_1,\dots,f_r)$

$\Rightarrow 0 = \dim(k[\varepsilon, Y, X_1,\dots,X_r]/(gY-1, f_1,\dots,f_r)) \geq 2 + r - (r+1) = 1$, a contradiction

13.6.F. EXERCISE. Suppose k is a field of characteristic not 2. Let $Y = \operatorname{Spec} k[t]$, and $X = \operatorname{Spec} k[u, 1/u]$. Show that the morphism $\pi\colon X \to Y$ induced by $t \mapsto u^2$ is étale. (Sketch π!) Show that there is no nonempty open subset U of X on which π is an isomorphism.

13.6.F $X = \operatorname{Spec} k[u, 1/u] \longrightarrow Y = \operatorname{Spec} k[t]$. $t \mapsto u^2$

$$\begin{tikzcd} X=\operatorname{Spec} k[u,1/u] \arrow[r, hook, "\circ"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{Spec}(k[t][x]/(x^2-t)) \arrow[dl] \\ Y=\operatorname{Spec} k[t] & \end{tikzcd}$$

$u^2 \mapsfrom t$

$\operatorname{Spec}(k[t][x]/(x^2-t)) = \operatorname{Spec} k[x]$, $t=x^2$

$D(f(t)) \simeq D(\varphi(f(t)))$

No local isom: If $k[t,\frac{1}{f(t)}] \cong k[u,\frac{1}{f(u^2)},\frac{1}{u}]$ i.e. $k[u^2,\frac{1}{f(u^2)}] = k[u,\frac{1}{f(u^2)},\frac{1}{u}]$

$\Rightarrow \frac{1}{u} \in k[u^2,\frac{1}{f(u^2)}] \subseteq k(u^2) \Rightarrow k(u) = k(u^2)$. But $[k(u) : k(u^2)] = 2$! a contradiction

13.6.6. Theorem. — *Suppose X is a k-scheme. Then the following are equivalent.*

(i) *X is smooth of relative dimension n over $\operatorname{Spec} k$ (Definition 13.6.2).*
(ii) *X has pure dimension n, and is smooth over k (in the sense of Definition 13.2.4).*

pf. (i) $\Rightarrow$ (ii) Check affine locally, we just need to show:

If X is an open subscheme of $Z := \operatorname{Spec}(k[x_1,x_2,\dots,x_{n+r}]/(f_1,\dots,f_r))$ s.t. $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j\le r}$ is invertible on X, then X is smooth and has puredim n.

Since "JC" is an inherent function defined on a scheme

$\forall x \in X$, $JC(x) = n+r - \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{1\le i\le r\\ 1\le j\le n+r}} = n+r-r = n \Rightarrow$ It remains to show X has puredim n.

By 12.1.H, it suffices to show $X_{\bar{k}}$ has puredim n, i.e. WMA $k=\bar{k}$.

We know that $X \overset{\circ}{\hookrightarrow} V(f_1,\dots,f_r) \hookrightarrow \mathbb{A}^{n+r}_k$

$V(f_1,\dots,f_r) = \bigcup_{j=1}^{m} V_j$ Irr components

$\{\text{Irr components of } X\} = \{X\cap V_j \mid 1\le j\le m,\ X\cap V_j \ne \emptyset\} \triangleq \{X\cap V_j \mid j\in J\}$

It suffices to show "$\dim(X\cap V_j) \ge n$, $\forall j\in J$" and "$\dim X \le n$"

Step 1. $\dim(X\cap V_j) \ge n$, $\forall j \in J$.

It suffices to show $\dim(V_j) \ge n$.

By 12.3.10. $\operatorname{codim}_{\mathbb{A}^{n+r}_k} V_j \le r \Rightarrow \dim V_j = \dim \mathbb{A}^{n+r}_k - \operatorname{codim}_{\mathbb{A}^{n+r}_k} V_j \ge n+r-r = n$.

Step 2. $\dim X \le n$

By 13.1.M, $\forall p\in X$, $\dim_X p \le JC(p) = n \Rightarrow \dim X \le n$

(ii)$\Rightarrow$(i): We just need to check locally.

WMA $X = \mathrm{Spec}(k[x_1, \dots, x_{n+r}]/(f_1, \dots, f_N))$

We only need to solve the following problem:

"Assumption: $\forall\, p \in X$, $\exists$ open nbhd U s.t. U is of puredim n and $\forall x \in U$, $\mathrm{rank}\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j} = r$

Want to show: $X \to \mathrm{Spec}\, k$ is smooth of relative dim n at p."

By permuting f_i and shinking U, WMA $\mathrm{rank}\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{1 \le i \le r \\ 1 \le j \le n+r}} = r$, $\forall x \in U$

Then $Y := \mathrm{Spec}(k[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r))$ is smooth of rdim n at p over $\mathrm{Spec}\, k$

$\Rightarrow \exists\, V$ open nbhd of p in Y, V is smooth of rdim n over $\mathrm{Spec}\, k$.

$$\begin{tikzcd} X \arrow[r, hook] & Y \\ U \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\square"] & V \arrow[u, hook] \\ p \arrow[u, phantom, "\in"] & p \arrow[u, phantom, "\in"] \end{tikzcd}$$
(shinking U again)

By the arguement (i)$\Rightarrow$(ii), V is of puredim n

Since $U \hookrightarrow V$ and U has puredim n

$\Rightarrow \forall$ irr comp V' of V, $V' \cap U = V'$ or $\emptyset$.

$\Rightarrow U \cong V \Rightarrow U$ is smooth of relative dim n over $\mathrm{Spec}\, k$

$\Rightarrow X$ is smooth of relative dim n at p.

Valuation criterion for separatedness and properness

13.7.1. Theorem (valuative criterion for separatedness, DVR version; see Figure 13.6). — *Suppose $\pi\colon X \to Y$ is a morphism of finite type of locally Noetherian schemes. Then π is separated if and only if the following condition holds: for any discrete valuation ring A, and any diagram of the form*

(13.7.1.1)
$$\begin{tikzcd} \mathrm{Spec}\, K(A) \arrow[r] \arrow[d, hook, "\text{open emb.}"'] & X \arrow[d, "\pi"] \\ \mathrm{Spec}\, A \arrow[r] & Y \end{tikzcd}$$

(where the vertical morphism on the left corresponds to the inclusion $A \hookrightarrow K(A)$), there is at most one morphism $\mathrm{Spec}\, A \to X$ such that the diagram

(13.7.1.2)
$$\begin{tikzcd} \mathrm{Spec}\, K(A) \arrow[r] \arrow[d, hook, "\text{open emb.}"'] & X \arrow[d, "\pi"] \\ \mathrm{Spec}\, A \arrow[r] \arrow[ur, dotted, "\le 1"] & Y \end{tikzcd}$$

13.7.A. If π is separated, the the criterion holds

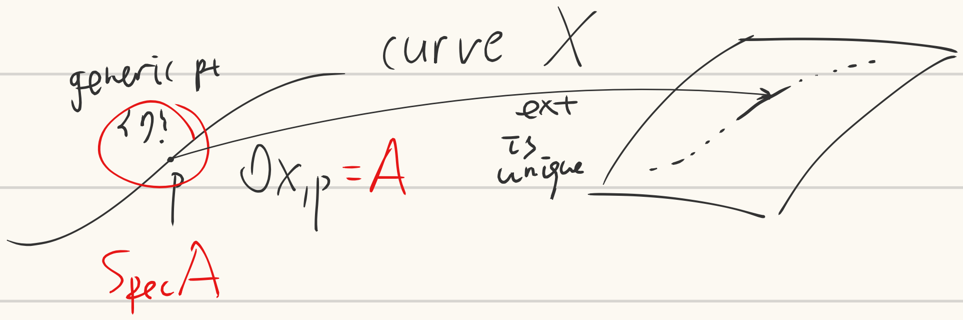
If $\operatorname{Spec} A \underset{f_2}{\overset{f_1}{\rightrightarrows}} X$, then $f_1|_{\{\eta\}} = f_2|_{\{\eta\}} \Rightarrow f_1$ and f_2 coincide on a dense open subset

$\{\eta, s\}$ $\Rightarrow$ QED.

13.7.B. EXERCISE. Suppose X is an integral Noetherian separated curve. If $p \in X$ is a regular closed point, then $\mathcal{O}_{X,p}$ is a discrete valuation ring, so each regular point yields a discrete valuation on $K(X)$. Use the previous exercise to show that distinct points yield distinct discrete valuations.

p.f. obvious.

Intuition: A DVR



13.7.C

See why the line with two origions is not separable

A good choice to understand the intuition of separatedness, see the note written by WZH

13.7.D' If Z is a locally closed subset contained in X, a locally Noe. scheme and Z is not closed. Then $\exists\, q \in Z$ and $p \in X \setminus Z$ s.t. $p \in \overline{q}$ and if $r \in X$ s.t. $p \in \overline{r}$, $r \in \overline{q}$, then $r = p$ or q.

By 8.4.C, $\exists\, q' \in Z$ s.t. $\overline{q'} \not\subseteq Z$ the rising sea-247 Let $p' \in \overline{q'} \setminus Z$

Then take $U = \operatorname{Spec} A$ open and $U \ni q'$ Assume $U \cap Z = V$

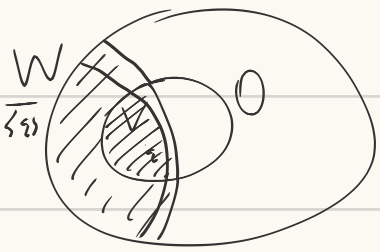
Then q', p' correspond to prime ideals $\mathfrak{q}', \mathfrak{p}'$ s.t. $\mathfrak{q}' \subsetneq \mathfrak{p}'$ $\mathfrak{q}' \in V$, $\mathfrak{p}' \notin V$

Let$_{\mathfrak{q}' \in}$ $S_1 = \{\mathfrak{q} \text{ primes} \mid \mathfrak{q}' \subseteq \mathfrak{q} \subsetneq \mathfrak{p}', \mathfrak{q} \in V\} \neq \emptyset$ $\exists$ maximal $\mathfrak{q} \in S$ since A is Noe.

Let$_{\mathfrak{p}' \in}$ $S_2 = \{\mathfrak{p} \text{ primes} \mid \mathfrak{p} \supsetneq \mathfrak{q} \text{ and } \mathfrak{p} \notin V\} \neq \emptyset$

$V = W\cap O$, $O \hookrightarrow U$. $W \hookrightarrow U$. $q\in V \Rightarrow \bar q \subseteq W \Rightarrow \bar q \not\subseteq O$ ∴ $\bar q\not\subseteq V$

U $\emptyset \neq \overline{\{q\}}\setminus O$ is closed. Take a irr component of $\overline{\{q\}}\setminus O$.



Let $p \sim \mathfrak{p}$ be the corresponding generic point

Then $\mathfrak{p}$ is a minimal elt in S_2

Then $q\in Z$, $p\in \bar q\setminus Z$

If $r\in X$ s.t. $p\in\bar r$, $r\in \bar q \Rightarrow r\in U$ Write $r \rightsquigarrow \mathfrak{p}'' \in \operatorname{Spec} A$

$\Rightarrow \mathfrak{q}\subseteq \mathfrak{p}''\subseteq \mathfrak{p}$. 1° $\mathfrak{p}''=\mathfrak{p}' \Rightarrow \mathfrak{p}''=\mathfrak{p}=\mathfrak{p}'' \Rightarrow r=p$

2° $\mathfrak{p}''\neq\mathfrak{p}'$ and $\mathfrak{p}''\notin V \Rightarrow \mathfrak{q}\subsetneq \mathfrak{p}''\subseteq\mathfrak{p} \Rightarrow \mathfrak{p}''\in S_2\Rightarrow \mathfrak{p}''=\mathfrak{p}$.

3° $\mathfrak{p}''\in V\Rightarrow \mathfrak{q}'\subseteq \mathfrak{q}\subseteq \mathfrak{p}''\subsetneq \mathfrak{p}\subseteq\mathfrak{p}' \Rightarrow \mathfrak{p}''\in S_1\Rightarrow \mathfrak{p}''=\mathfrak{q}$

13.7.D. EXERCISE. Show that you can find points p not in the diagonal Δ of $X \times_Y X$ and q in Δ such that $p \in \overline{q}$, and there are no points "between p and q" (no points r distinct from p and q with $p \in \overline{r}$ and $r \in \overline{q}$). (Exercise 8.4.C may shed some light.)

Let Q be the scheme obtained by giving the induced reduced subscheme structure to $\overline{q}$. Let $B = \mathscr{O}_{Q,p}$ be the local ring of Q at p.

13.7.E. EXERCISE. Show that B is a Noetherian local integral domain of dimension 1.

Integral: Since Q is reduced and irr ($\bar q$ is irr) $\Rightarrow$ Q is integral $\Rightarrow$ B is int

dim = 1 by 13.7.D

Then Let $\bar B$ be the normalization of B. $\exists\ \bar{\mathfrak{p}}\in\operatorname{Spec}\bar B$ s.t. $\bar{\mathfrak{p}}\cap B=\mathfrak{p}$ by going up Then $\bar B_{\bar{\mathfrak{p}}}$ is a DVR.

$\operatorname{Spec}\bar B_{\bar{\mathfrak{p}}} = \{s,\eta\}$ $\eta\longmapsto q\in\Delta$ induced by $B\hookrightarrow \bar B_{\mathfrak{p}}$, and

$s\longmapsto p\notin\Delta$ $\operatorname{Spec} B\longrightarrow Q\hookrightarrow X\times_Y X$

$$\begin{tikzcd}
 & \{\eta\} \arrow[d, phantom, "\subseteq" rotate=-90] & \\
 & \operatorname{Spec}\bar B_{\bar{\mathfrak{p}}} \arrow[d, "\psi"] & \\
 & X\times_Y X \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\
X \arrow[r, "="] \arrow[dr] & X \arrow[u, "\Delta"] \arrow[d, "\pi"] & X \arrow[l, "="'] \arrow[dl] \\
 & Y &
\end{tikzcd}$$

$\varphi_1\circ\psi\neq\varphi_2\circ\psi$

$\varphi_1\circ\psi|_\eta=\varphi_2\circ\psi|_\eta$

a contradiction.

Then we conclude the thm.