Effective Cartier divisor = invertible ideal sheaf

Recall. Def. (Effective Cartier divisor)

A closed subscheme Y of X, s.t.
it is "generated" by non-zero divisors,
i.e.,

$$\exists \{U_i \cong \operatorname{Spec} A_i\}_i \twoheadrightarrow X, \text{ s.t.}$$
$$Y|_{U_i} = V(t_i), \quad t_i \in A_i \text{ non-zero divisor.}$$

Weil divisor : linear combination of irr. closed subsets.

Given an eff. Cartier div. $Y \overset{i}{\hookrightarrow} X$

$$\Updownarrow$$

$$Y = \bigcup_i V(t_i) \quad \text{(scheme theoretically)}$$

t_i non-zero divisor

ideal sheaf corresponding to Y

$$\mathcal{I}(Y) = \bigcup_i (t_i)_{\operatorname{Spec} A_i}$$

$$A_i \xrightarrow[\sim]{\times t_i} (t_i) \qquad \text{because } t_i \nmid 0.$$

$\Rightarrow \mathcal{I}(Y)$ is a line bundle.

Def. D eff. Cartier div.

The line bundle corresponding to D is defined as

$$\mathcal{O}(D) := \mathcal{I}(D)^{\vee} \qquad \text{"}\vee\text{": dual}$$

Recall: $\mathcal{I}(D)^{\vee} = \mathcal{H}om(\mathcal{I}(D), \mathcal{O}_X) = \mathcal{I}(D)^{\otimes(-1)}$

$$\mathcal{O}_X(D)_{\text{Weil}}(U) = \{ t \mid t|_U \geq -D|_U \}$$

Prop. 15.6.B. If X is Noetherian and normal, D is an eff. Cartier div., then

$$\mathcal{O}(D) \cong \mathcal{O}(D_W)$$

Pf. Noetherian + normal $\Rightarrow X = \coprod_{j=1}^{n} X_j$ integral normal Noetherian

$D = \bigcup_i V(f_i) \qquad f_i \in A_i \qquad U_i \cong \operatorname{Spec} A_i$

$X = \bigcup_i U_i$

$D_W := \sum_Y v_Y(f_i)\, Y$

$D \mapsto D_W$

inj.

Recall: $Y \rightsquigarrow \mathcal{O}_{X,\eta_Y} \quad v_Y$

independency of U_i:

$V(f_i)|_{U_i \cap U_j} = V(f_j)|_{U_i \cap U_j}$

$\Downarrow$

$\dfrac{f_i|_{U_i\cap U_j}}{f_j|_{U_i\cap U_j}} \in \mathcal{O}_X(U_i \cap U_j)^*$

$$\Gamma(\mathcal{O}(D), U_i) \overset{def}{=} \mathcal{H}om(\mathcal{I}(D), \mathcal{O}_X)(U_i)$$

$$\overset{def}{=} Hom(\mathcal{I}(D)|_{U_i}, \mathcal{O}_X|_{U_i})$$

$$= Hom_{A_i\text{-mod}}(\underbrace{(f_i)}_{\text{ideal}}, A_i)$$

$$\cong (f_i^{-1}) \subseteq Frac(A_i)$$

$$s \longmapsto \frac{s(f_i)}{f_i}$$

$$\Gamma(\mathcal{O}(D_W), U_i) \overset{def}{=} \{s \in (Frac A_i)^* \mid v_Y(s|_{U_i}) + v_Y(D_W|_{U_i}) \geq 0 \ \forall Y \cap U_i \neq \emptyset\} \cup \{0\}$$

$$= \{s \mid \underline{v_Y(sf_i) \geq 0}, \forall Y \cap U_i \neq \emptyset\} \cup \{0\}$$

codim 1

$$v_Y(D|_{U_i}) = v_Y(f_i)$$

$$\underline{\underline{\text{Alg. Hartogs' Thm.}}} \ \{s \mid sf_i \in A_i\} = (f_i^{-1})$$

Def. $\mathcal{O}(nD) := \mathcal{O}(D)^{\otimes n}$

$\mathcal{O}(-D) = \mathcal{I}(D)$.

Def. (Canonical section of $\mathcal{O}(D)$).

$\mathcal{I}(D) \hookrightarrow \mathcal{O}_X$

$\sigma_D : \mathcal{O}_X \longrightarrow \mathcal{I}(D)^\vee = \mathcal{O}(D)$

$\sigma_D(X)(1) \triangleq s_D$ the canonical section of $\mathcal{O}(D)$.

<u>Prop. 15.6.C.</u> D is cut out by s_D.

Recall: "cut out"

$$\psi_i : \mathcal{O}(D)|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$$

$$D = V(s_D) := \bigcup_i V(\psi_i(s_D|_{U_i}))$$

<u>Pf.</u> $\underbrace{\sigma_D(X)(1)|_{U_i}} = 1 \in (f_i^{-1}) \cong \mathcal{O}(D)(U_i)$

$$\sigma_D(U_i) : \mathcal{O}_X(U_i) \longrightarrow \mathcal{O}(D)(U_i) = \mathcal{I}(D)^\vee(U_i)$$

$$\mathcal{O}_X \cong \mathcal{I}(D) \otimes \mathcal{I}(D)^\vee$$

$$\mathcal{I}(D)^\vee \cong \mathcal{O}_X \otimes \mathcal{I}(D)^\vee$$

$$\mathcal{I}(D)(U_i) \longrightarrow \mathcal{O}_X(U_i)$$
$$t \longmapsto t$$

$$\begin{tikzcd}
\mathcal{I}(D)(U_i) \otimes \mathcal{I}(D)^\vee(U_i) \arrow[r] \arrow[d, "s"] & \mathcal{O}_X(U_i) \otimes \mathcal{I}(D)^\vee(U_i) \arrow[d, "s"] \\
\mathcal{O}_X(U_i) & \mathcal{I}(D)^\vee(U_i)
\end{tikzcd}$$

$$t \otimes s \longmapsto t \otimes s$$

$$s(t) \quad - \overline{\sigma_D(U_i)} \Rightarrow {}^{t}s : x \mapsto s(tx)$$

$t_1 = f_i \; s_1$ such that $s_1(U_i) : x \mapsto \dfrac{x}{f_i}$

$$t_1 \otimes s_1 \quad \longleftrightarrow \quad 1 \in \mathcal{O}_X(U_i)$$

$$\sigma_D(U_i)(1) = {}^{t_1}s_1 \; : \; x \mapsto \frac{t_1 x}{f_i} = x$$

$$\mathcal{J}(D)^{\vee}(U_i) \xrightarrow{\sim} (f_i^{-1}) \cong \mathcal{O}(D)(U_i)$$

$$(x \mapsto x) \longmapsto 1$$

$$\psi_i : \mathcal{O}(D)(U_i) = (f_i^{-1}) \xrightarrow{\sim} \mathcal{O}_X(U_i) = A_i$$

$$x \longmapsto f_i x$$

$$\Rightarrow V(\psi_i(s_D|_{U_i})) = V(\psi_i(1)) = V(f_i) = D|_{U_i} .$$

$\square$

$D \mapsto s_D \qquad s_D|_{U_i}$ non-zero div.

<u>Prop. 15.6.D.</u> $\mathcal{L}$: line bundle $\quad s \in \mathcal{L}(X)$, s.t.

$s|_{U_i}$ non-zero div.

Then

$$D \triangleq V(s)$$

is eff. Cartier div., and $\mathcal{O}(D) \cong \mathcal{L}$

$\square$

$$\boxed{\begin{array}{ccc} \text{eff. Cartier div.} & \longrightarrow & \text{invertible ideal sheaf} \\ D & \longmapsto & \mathcal{O}(D) \\ V(s) & \longleftarrow & \mathcal{L},\ s \in \mathcal{L}(X) \end{array}}$$

Def. $\mathcal{I}$, $\mathcal{J}$: ideal sheaf

$$(\mathcal{I}\mathcal{J})(\operatorname{Spec} A) := \mathcal{I}(\operatorname{Spec} A) \cdot \mathcal{J}(\operatorname{Spec} A)$$

Def. For eff. Cartier divisors D, D',

$$\mathcal{I}_D^\vee = \mathcal{O}(D) \qquad \mathcal{I}_{D'}^\vee = \mathcal{O}(D')$$

define $D + D'$, s.t.

$$\mathcal{O}(D+D') \cong \mathcal{I}_D^\vee \mathcal{I}_{D'}^\vee$$

Prop. 15.6.E. $\mathcal{O}(D) \otimes \mathcal{O}(D') \cong \mathcal{O}(D+D')$. $\square$

Def. (Normal line bundles to eff. Cartier div.)

$$\mathcal{N}_{D/X} := \mathcal{O}(D)|_D$$