Thm 16.1.1: $\forall$ fin type Qcoh $\mathcal{F}$ on Proj S. can be presented as the form: $\bigoplus_{fin} \mathcal{O}(-m) \to \mathcal{F} \to 0$.

Def: X sch, $\mathcal{F}$ $\mathcal{O}_X$-mod $\mathcal{F}$ globally generated

if $\exists$ surj $\mathcal{O}_X^{\oplus I} \twoheadrightarrow \mathcal{F}$ (glo. g)

$\mathcal{F}$ globally f.g if $\exists$ surj $\mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{F}$. (glo f.g)

$\mathcal{F}$ glo. g at p if $\exists$ $\mathcal{O}_X^{\oplus I} \to \mathcal{F} \rightsquigarrow \mathcal{O}_{X,p}^{\oplus I} \twoheadrightarrow \mathcal{F}_p$

Rem: $\mathcal{F}$ glo g $\Leftrightarrow$ $\mathcal{F}$ glo g at all pts.

"$\Leftarrow$": $\bigoplus \mathcal{O}_{X,p}^{\oplus I_p} \twoheadrightarrow \mathcal{F}_p \rightsquigarrow \bigoplus_{p \in X} \mathcal{O}_X^{\oplus I_p} \twoheadrightarrow \mathcal{F}$.

16.1.A. $\mathcal{L}$ invertible. then $\mathcal{L}$ glo.g $\Leftrightarrow$ $\forall p \in X$, $\exists$ global section of $\mathcal{L}$ not vanishing at p. i.e $\mathcal{L}$ base-pt-free (Def 15.2.1)

pf: Obviously $\forall p \in X$, $s \in \mathcal{L}(X)$, we have s not vanishing at $p \Leftrightarrow \mathcal{L}_p = \mathcal{O}_{X,p} \cdot s_p \Leftrightarrow s_p \notin \mathfrak{m}_{\mathcal{O}_{X,p}} \cdot \mathcal{L}_p$

"$\Rightarrow$" $\mathcal{L}$ glo.g $\Rightarrow \exists\, \mathcal{O}_X^{\oplus I} \twoheadrightarrow \mathcal{L}$

Assume all global section of $\mathcal{L}$ vanishing at $p \in X$,

$$\begin{tikzcd} \mathcal{O}_X^{\oplus I}(X) \arrow[r] \arrow[d] & \mathcal{L}(X) \arrow[d] \\ \mathcal{O}_{X,p}^{\oplus I} \arrow[r] & \mathcal{L}_p \end{tikzcd} \qquad \begin{tikzcd} e_i \arrow[r, mapsto] \arrow[d, mapsto] & {} \arrow[d, mapsto] \\ e_i \arrow[r, mapsto] & \mathfrak{m}_{\mathcal{O}_{X,p}}\mathcal{L}_p. \end{tikzcd} \quad \times.$$

"$\Leftarrow$" $\forall p \in X$, $\exists\, s(p) \in \mathcal{L}(X)$ s.t $s(p)$ not vanishing at p.

Consider $\bigoplus_{p\in X} \mathcal{O}_X \longrightarrow \mathcal{L}$, $e_p \mapsto s(p)$. ✓. $\square$.

16.1.B ① Qcoh sheaf on aff sch glo g.

② fin type Qcoh sheaf on aff sch glo. f.g.

pf: Every mod is a quotient of a free mod. $\square$

16.1.C. $\mathcal{F}, \mathcal{G}$ glo. g $\Rightarrow$ $\mathcal{F}\otimes\mathcal{G}$. $\mathcal{F}\oplus\mathcal{G}$ glo. g

pf: $\mathcal{O}_X^{\oplus I} \twoheadrightarrow \mathcal{F}$, $\mathcal{O}_X^{\oplus J} \twoheadrightarrow \mathcal{G} \Rightarrow (\mathcal{O}_X^{\oplus I}) \otimes (\mathcal{O}_X^{\oplus J}) \twoheadrightarrow \mathcal{F}\otimes\mathcal{G}$. $\square$

16.1.D $\mathcal{F}$ fin type Qcoh sheaf on X.

(a) $\mathcal{F}$ glo.g at $p\in X \Leftrightarrow \mathcal{F}(X) \twoheadrightarrow \mathcal{F}_p/\mathfrak{m}_{\mathcal{O}_{X,p}}\mathcal{F}_p$.

(b) $\mathcal{F}$ glo.g at $p\in X \Rightarrow \exists$ open nbd U of p. $\forall q\in U$. $\mathcal{F}$ glo.g at q

(c) Suppose X qc,

$\mathcal{F}$ glo.g at all cl pts $\Rightarrow$ $\mathcal{F}$ glo.g.

pf: (a) "$\Rightarrow$" ✓.

"$\Leftarrow$" Let $\mathcal{F}_p'$ generated by the image of $\mathcal{F}(X)$ on $\mathcal{F}_p$

$\mathcal{F}(X) \twoheadrightarrow \mathcal{F}_p/\mathfrak{m}_{\mathcal{O}_{X,p}}\mathcal{F}_p \Rightarrow \mathcal{F}_p' \twoheadrightarrow \mathcal{F}_p/\mathfrak{m}_{\mathcal{O}_{X,p}}\mathcal{F}_p$

$\Rightarrow \mathcal{F}_p = \mathcal{F}_p' + \mathfrak{m}_{\mathcal{O}_{X,p}}\mathcal{F}_p \xRightarrow{\text{Nakayama}} \mathcal{F}_p = \mathcal{F}_p'$.

(b) Let $V = \operatorname{Spec} A \hookrightarrow X$, $\mathcal{F}|_V = \widetilde{M}$.

The image of $\mathcal{F}(X)$ on $\mathcal{F}(V)$ generates $M' \subseteq M$.

$\mathcal{F}$ glo.g at $q\in V \Leftrightarrow \left(\widetilde{M/M'}\right)_q = 0$

M f.g A-mod $\Rightarrow M/M' = \sum_{i=1}^{n} A\cdot m_i$.

$\{q\in V \mid (m/m')_q = 0\} = \{q\in V \mid \forall i=1,\dots,n,\ (m_i)_q = 0\}$.

$= \bigcap_{i=1}^{n} \underbrace{\{q\in V \mid q\notin \mathrm{nil}(m_i)\}}_{\text{open}}$

(c) $Y = \{q\in X \mid \mathcal{F}$ not glo.g at $q\}$ closed by (b).

Assume $Y\neq\emptyset$,

Given Y a cl subsch structure, Y qc $\Rightarrow Y$ has a cl pt

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16.1 E. $\mathcal{F}$ fin type Qcoh sheaf on X, X qc.

Then $\mathcal{F}$ glo.g $\Leftrightarrow \mathcal{F}$ generated by a fin number of global

sections, i.e. $\exists s_1,\dots,s_n\in\mathcal{F}(X)$. s.t $\forall p\in X$, $\mathcal{F}_p = \sum_{i=1}^{n}\mathcal{O}_{X,p} s_i$.

pf: "$\Leftarrow$" ✓

"$\Rightarrow$" $\forall$ aff open $U \hookrightarrow X$, $\mathcal{F}|_U = \widetilde{M}$.

$\mathcal{F}$ glo.g $\Rightarrow$ let $M'\subseteq M$ generated by $\mathcal{F}(X)$, then

$M' = M$.

$M' = M$ f.g $\Rightarrow \exists s_1, \dots, s_n \in \mathcal{F}(X)$. $\forall p \in U$, $\mathcal{F}_p = \sum_{i=1}^{n} \mathcal{O}_{X,p} s_i$.

X qc ✓. $\square$

In this section. $S_\bullet$ generated in deg 1. f.g over $A = S_0$

Thm 16.1.5 (Serre thm A).

$\mathcal{F}$ fin type Qcoh on $\operatorname{Proj} S_\bullet$. Then $\exists m_0 \geq 1$

s.t $\forall m \geq m_0$. $\mathcal{F}(m)$ glo. f.g.

Thm 16.1.1

$\mathcal{F}$ fin type Qcoh on $\operatorname{Proj} S_\bullet$, then $\mathcal{F}$ can be presented in the form $\bigoplus_{\text{fin}} \mathcal{O}(-m) \twoheadrightarrow \mathcal{F} \to 0$.

pf of thm 16.1.1 assuming thm 16.1.5:

$\exists m \geq 1$ s.t $\mathcal{O}_{\operatorname{Proj} S_\bullet}^{\oplus n} \twoheadrightarrow \mathcal{F}(m)$

$\Rightarrow \mathcal{O}(-m)^{\oplus n} \twoheadrightarrow \mathcal{F} \to 0$ $\square$

16.2. Ample, very ample line bundles.

Def: $\pi: X \to \operatorname{Spec} A$, $\mathcal{L}$ invertible on X,

$\mathcal{L}$ very ample over A if $X \cong \operatorname{Proj} S_\bullet$, $S_\bullet$ f.g graded ring over A, generated in deg 1, and $\mathcal{L} \cong \mathcal{O}_{\operatorname{Proj} S_\bullet}(1)$.

(i.e
$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}_A^n \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}, \quad \mathcal{L} \cong i^*\mathcal{O}(1) \;)$$

(In Hartshorne:
$$\begin{tikzcd} X \arrow[r, hook, "\text{immersion}", "i"'] \arrow[dr] & \mathbb{P}_A^n \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}, \quad \mathcal{L} \cong i^*\mathcal{O}(1) \;)$$

16.2.B. $\mathcal{L}$ very ample on X over A, then $\mathcal{L}$ base-pt-free

pf: $\mathcal{L} \cong \mathcal{O}_{\operatorname{Proj} S}(1)$. $S_\bullet$ generated by deg 1 $\square$.

16.2.C. (very ample $\otimes$ base-pt-free = very ample)

$\mathcal{L}, \mathcal{M}$ invertible on proper A-sch X. $\mathcal{L}$ very ample over A, $\mathcal{M}$ base-pt-free, then $\mathcal{L}\otimes\mathcal{M}$ very ample

pf: $\mathcal{L} \rightsquigarrow X \hookrightarrow \mathbb{P}^m_A$, $\mathcal{M} \rightsquigarrow X \to \mathbb{P}^n_A$

(base-pt-free $\overset{16.1A}{\Longrightarrow}$ $\mathcal{M}$ gb.g)

$\rightsquigarrow X \to \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ cl emb

$$\left(\begin{tikzcd} X \arrow[r] \arrow[dr, hook] & \mathbb{P}^m_A \times_A \mathbb{P}^n_A \arrow[d, "sep"] \\ & \mathbb{P}^m_A \end{tikzcd}\right)$$

$\rightsquigarrow X \hookrightarrow \mathbb{P}^m_A \times_A \mathbb{P}^n_A \overset{\text{(Segre emb)}}{\hookrightarrow} \mathbb{P}^{mn+m+n}_A$

$\mathcal{L}\otimes\mathcal{M}$ $\quad \mathcal{O}(1)\otimes\mathcal{O}(1)$ $\quad \mathcal{O}(1)$ (15.2.10). $\square$

16.2.12. (very ample $\boxtimes$ very ample = very ample).

X, Y proper A-sch, $\mathcal{L}$ (resp $\mathcal{M}$) very ample invertible on X (resp Y).

$$\begin{tikzcd} X\times_A Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ Y \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

$\rightsquigarrow \pi_X^*\mathcal{L}\otimes\pi_Y^*\mathcal{M}$ very ample on $X\times_A Y$.

Pf: $\mathcal{L}, \mathcal{M} \rightsquigarrow X \hookrightarrow \mathbb{P}^m_A, \ Y \hookrightarrow \mathbb{P}^n_A$.

$\rightsquigarrow X \times_A Y \hookrightarrow \mathbb{P}^m_A \times_A \mathbb{P}^n_A \hookrightarrow \mathbb{P}^{mn+m+n}_A$

$\pi_X^*\mathcal{L} \otimes \pi_Y^*\mathcal{M} \quad \mathcal{O}(1) \otimes \mathcal{O}(1) \qquad \mathcal{O}(1)$. $\quad \square$

Def: $\mathcal{L}$ invertible on proper A-sch X is ample over A if one of following equivalent conditions holds:

Projective Geo: (a) $\exists N > 0$, $\mathcal{L}^{\otimes N}$ very ample over A

(a') $\exists N > 0$, $\forall n \geq N$, $\mathcal{L}^{\otimes n}$ very ample over A.

Glo generation: (b) $\forall$ fin type Qcoh $\mathcal{F}$, $\exists n_0$, s.t. $\forall n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ glo.g.

Top: (c) $X_f = \{p \in X \mid f(p) \neq 0\}$ ($\forall f \in \mathcal{L}^{\otimes n}$, $\forall n > 0$) form a top base of X

(c') affine X_f form a top base of X.

(c'') affine X_f cover X.

Thm 16.2.2. 以上等价.

Application:

Thm 16.1.5 (Serre thm A): $\mathcal{O}(1)$ very ample

"(a') ⇒ (b)" ⇒ $\forall m \gg 0$, $\mathcal{F}(m)$ glo.g. □.

16.2.E. (Imp) $\mathcal{L}$, $\mathcal{M}$ invertible sheaves on a proper A-sch X,

$\mathcal{L}$ ample ⇒ $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ very ample for $n \gg 0$.

pf: (a') ⇒ $\exists n_0$, $\forall n \geq n_0$ $\mathcal{L}^{\otimes n}$ very ample

(b) ⇒ $\exists m_0$, $\forall m \geq m_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ glo.g.

16.1.A ⇒ $\forall m \geq m_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ base-pt-free.

16.2.C ⇒ $\forall n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n_0} \otimes \mathcal{L}^{\otimes m} = \mathcal{F} \otimes \mathcal{L}^{\otimes (n_0+m)}$ very ample

16.2.F: (Imp). $\mathcal{L}$ line bundle on X, X projective

$\exists$ very ample invertible sheaves $\mathcal{M}$, $\mathcal{N}$ s.t $\mathcal{L} \cong \mathcal{M} \otimes \mathcal{N}^\vee$.

pf: X projective. $\mathcal{O}(1)$ very ample.

16.2.E $\Rightarrow \exists m \geq 1$, s.t $\mathcal{L} \otimes \mathcal{O}(m)$ very ample

$\Rightarrow \mathcal{L} \cong (\mathcal{L} \otimes \mathcal{O}(m)) \otimes \mathcal{O}(m)^\vee$. $\square$

16.2.G (Imp) $\pi: X \to Y$ fin. X, Y proper A-sch, $\mathcal{L}$ ample line bundle on Y. then $\pi^*\mathcal{L}$ ample on X.

pf: $\forall$ fin type Qcoh $\mathcal{F}$ on X, $\pi_*\mathcal{F}$ fin type Qcoh on Y.

(14.5.L Projection formula)

$\forall n \gg 0$, $\pi_*(\mathcal{F} \otimes \pi^*\mathcal{L}^{\otimes n}) \cong (\pi_*\mathcal{F}) \otimes \mathcal{L}^{\otimes n}$ glo.g.

$\Rightarrow \forall$ aff open $U \hookrightarrow Y$. $\pi_*(\mathcal{F} \otimes \pi^*\mathcal{L}^{\otimes n})(U)$ is generated by $\pi_*(\mathcal{F} \otimes \pi^*\mathcal{L}^{\otimes n})(Y)$

$\Rightarrow \mathcal{F} \otimes \pi^*\mathcal{L}^{\otimes n}(\underbrace{\pi^{-1}(U)}_{\text{aff}})$ generated by $\mathcal{F} \otimes \pi^*\mathcal{L}^{\otimes n}(X)$

$\Rightarrow \mathcal{F} \otimes \pi^*\mathcal{L}^{\otimes n}$ glo.g on $\pi^{-1}(U)$

$\Rightarrow \mathcal{F} \otimes \pi^*\mathcal{L}^{\otimes n}$ glo.g By (b) ✓. $\square$

16.2.H. $\mathcal{L}, \mathcal{M}$ invertible on proper A-sch X. $\mathcal{L}$ ample. $\mathcal{M}$ base-pt-free. then $\mathcal{L} \otimes \mathcal{M}$ ample

Pf: By (a), $\forall n \gg 0$ s.t $\mathcal{L}^n$ very ample.

$\overset{16.2.C}{\Longrightarrow}$ $\mathcal{M}^n \otimes \mathcal{L}^n = (\mathcal{M} \otimes \mathcal{L})^n$ very ample

$\overset{(a)}{\Longrightarrow}$ $\mathcal{M} \otimes \mathcal{L}$ ample. $\square$

16.2.I: (less imp).

16.2.J: X projective k-sch. $\mathcal{L}$ ample line bundle on X.

$\mathcal{M}$ a line bundle on X. $q_1, \dots, q_m \in X$.

Then $\forall N \gg 0$, $\exists$ a section of $\mathcal{L}^{\otimes N} \otimes \mathcal{M}$ not vanish on any of the q_i.

pf: ???

Pf of thm 16.2.2 (In case X Noe).

$$\begin{tikzcd}
 & (a) \arrow[r, shift left] \arrow[dl] & (a') \arrow[l, shift left] \\
(c) \arrow[r, shift right] & (c') \arrow[l, shift right] \arrow[r] & (c'') \arrow[ul] \arrow[dl] \\
 & (b) \arrow[ul] &
\end{tikzcd}$$

• clear

• harder

$(c)\Rightarrow(c')$ $p\in U \overset{o}{\hookrightarrow} X$, want: $\exists$ aff X_f, $p\in X_f\subseteq U$.

Wlog let U aff, then $\exists\, X_f$ s.t $p\in X_f\subseteq U$.

U aff $\Rightarrow X_f$ aff ✓.

$(a)\Rightarrow(c)$ $\exists N$ s.t $\mathcal{L}^{\otimes N}$ very ample.

$\mathcal{L}^{\otimes N} \rightsquigarrow X \cong \operatorname{Proj} S$.

Given cl subset $Z\subseteq X$, $p\in X\setminus Z$, want: $\exists$ a section of $\mathcal{L}^{\otimes n}$ vanishes on Z, not on p.

$(\exists n\geq 0)$

$Z = V(I(Z))$. $I(Z)$ homogeneous ideal of S.

$\exists$ homogeneous $s \in I(Z)$ s.t s not vanish on p.

$\Rightarrow s \in \mathcal{L}^{\otimes \deg s}(X)$. vanishes on Z, not on p.

(b) $\Rightarrow$ (c) $\forall p \in U \hookrightarrow X$, want: $\exists$ a section of $\mathcal{L}^{\otimes N}$ vanishes on $X \setminus U$, not at p.

$\mathcal{F}$ is the ideal sheaf of reduced cl subsch $X \setminus U$.

X Noe $\Rightarrow \mathcal{F}$ fin type. By (b), $\mathcal{F} \otimes \mathcal{L}^{\otimes N}$ glo.g for some N.

$\overset{?}{\Longrightarrow} \exists$ global section not vanish on p.

(c'') $\Rightarrow$ (b):

① (c'') $\Rightarrow \exists N > 0$ s.t $\mathcal{L}^{\otimes N}$ glo.g:

X qc $\overset{(c'')}{\Longrightarrow} X = \bigcup_{i=1}^{n} X_{f_i}$. $f_i \in \Gamma(X, \mathcal{L}^{\otimes N_i})$.

$\mathcal{L}|_{X_{f_i}}$ free on X_{f_i}. X_{f_i} aff.

Replace f_i by $f_i^{\ldots/N_i}$, we can assume $\forall i, N_i = N$,

Then $\mathcal{L}^{\otimes N}$ generated by global sections.

② Assume $\forall$ fin type Qcoh $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^{\otimes mN}$ glo.g for $m \gg 0$:

Consider $\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \cdots, \mathcal{F} \otimes \mathcal{L}^{(N-1)}$

$\Rightarrow \forall m \gg 0$, $\mathcal{F} \otimes \mathcal{L}^{m}$ glo.g.

So wlog let $\mathcal{L} = \mathcal{L}^{\otimes N}$ glo.g.

③ $\forall$ cl pt p, $\exists$ aff open nbd X_f of p. (by c'')

Then $\mathcal{F}|_{X_f}$ generated by fin glo sections. (16.1.B)

15.4.N $\Rightarrow \Gamma(X_f, \mathcal{F}) \cong \left(\left(\bigoplus_{n \geq 0} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \right)_f \right)_0$

$\exists M(p) \geq 0$ s.t $\mathcal{F} \otimes \mathcal{L}^{\otimes M(p)}$ glo.g at p

16.1 D(b) $\Rightarrow \exists$ open nbd $U_p \ni p$ s.t $\mathcal{F} \otimes \mathcal{L}^{\otimes M(p)}$ glo.g on U_p

$\mathcal{L}$ glo.g, X qc $\Rightarrow \forall M \gg 0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ glo.g

(c'') $\Rightarrow$ (a) $X = \bigcup_{i=1}^{n} X_{a_i}$, X_{a_i} affine open. $a_i \in \mathcal{L}^{\otimes N}$.

$X_{a_i} = \operatorname{Spec} A_i$. $A_i = A[a_{i1}, \cdots, a_{ij_i}]/I_i$ (π fin type)

15.4.N $\Rightarrow a_{ij} = \dfrac{s_{ij}}{a_i^{m_{ij}}}$. $s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes N m_{ij}})$.

let $m = \max\limits_{i,j} m_{ij}$. $\forall i,j$, $a_{ij} = (s_{ij} a_i^{m-m_{ij}})/a_i^m$.

Let $b_i = a_i^m$, $b_{ij} = s_{ij} a_i^{m-m_{ij}} \in \mathcal{L}^{\otimes mN}$.

$X = \bigcup X_{a_i} = \bigcup D(b_i)$

15.2.A $\rightsquigarrow f: X \longrightarrow \mathbb{P}^Q$. $Q = \# b_i + \# b_{ij} - 1$

$f^* x_i = b_i$, $f^* x_{ij} = b_{ij}$.

$\Rightarrow X_{a_i} \hookrightarrow D(x_i)$ ($a_{ij} \hookleftarrow \dfrac{x_{ij}}{x_i} \Rightarrow$ surj ring hom).

$\Rightarrow f$ locally cl emb.

X proper $\Rightarrow \mathrm{Im} f$ cl $\Rightarrow f$ cl emb. $\mathcal{L}^{\otimes mN}$ very ample.

(a),(b) $\Rightarrow$ (a'): $\underset{(a)}{\mathcal{L}^{\otimes N} \text{ very ample}}$. $\underset{(b)}{\exists n_0, \forall n \geq n_0, \mathcal{L}^{\otimes n}}$ base-pt-free.

$\Rightarrow \forall m \geq n_0 + N$, $\mathcal{L}^{\otimes m}$ very ample. $\square$

Def: $\mathcal{L}$ invertible on qc X, we say $\mathcal{L}$ ample if

$\{ X_f = \{p \in X \mid f(p) \neq 0\} \mid f \in \mathcal{L}^{\otimes n},\ n > 0 \}$ is a top base of X.

e.g. (i) X aff. then every line bundle ample

(ii) X projective, A-sch. $\mathcal{O}(1)$ ample.

16.2. M (a) Fix n, $\mathcal{L}$ ample $\Longleftrightarrow \mathcal{L}^{\otimes n}$ ample.

(b) $Z \overset{\varphi}{\hookrightarrow} X$ cl emb. $\mathcal{L}$ ample. then $\mathcal{L}|_Z$ ample on Z

pf: (a) $X_f = X_{f^{\otimes n}}$.

(b) $X_f \cap Z = X_{\varphi^\# f}$.

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