Chapter 13. Regularity and smoothness

§13.1 The Zariski tangent space

Lemma.

For a ring A and $m \in \mathrm{Spec}_{max} A$,
$m/m^2 \simeq mA_m / m^2 A_m$.

Pf. $m/m^2 = m \otimes_A A/m \simeq m \otimes_A A_m/mA_m = mA_m / m^2 A_m$.
$\left(A/m = (A/m)_m = A_m / mA_m \right)$.

Def. 13.1.1

(a) The Zariski cotangent space of a local ring (A, m) is m/m^2. It's a vector space over A/m.
The dual vector space $(m/m^2)^\vee$ is called the Zariski tangent space.

(b) If X is a scheme,
the Zariski cotangent space $T_{X,p}^\vee := m_p/m_p^2$,
the Zariski tangent space $T_{X,p} := (T_{X,p}^\vee)^\vee$. $\forall p \in X$.

(c) Elts. of the Zariski cotangent space are called cotangent vector or differentials, dx, dy
Elts. of the Zariski tangent space are called tangent vectors. v, w

Rmk.

(a) the cotangent space is more naturally determined in terms of functions on a space.

(b) If X is locally Noetherian, $\dim_{k(p)} T_{X,p} \geq \dim \mathcal{O}_{X,x}$.

Lemma. Liu Qing 6.2.1*

Let A be a fin. gen. k-alg, $x \in \operatorname{Spec} A$ be a closed pt correspond to a maximal ideal m s.t. $k(x)/k$ is a finite seperated ext. Then the canonical homom.

$$\begin{aligned} \delta : m/m^2 &\longrightarrow \Omega^1_{A/k} \otimes_A k(x) \\ a &\longmapsto da \otimes 1 \end{aligned}$$

is an isomorphism.

Pf. Recall that for $C = B/I$, we have an exact seq.

$$I/I^2 \xrightarrow{\delta} \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to 0$$

$$\Rightarrow m/m^2 \xrightarrow{\delta} \Omega^1_{A/k} \otimes_k k(x) \to \Omega^1_{k(x)/k} \to 0 \quad \text{exact}$$

Where $\Omega^1_{k(x)/k} = 0$, it suffices to show δ inj.

Write $A = B/I$, where $B = k[T_1, \cdots, T_n]$,

Let n be the inverse image of m in B.

$$\left(\begin{aligned} \operatorname{Spec} A &\hookrightarrow \operatorname{Spec} B \\ m &\longmapsto n \end{aligned} \right)$$

We have a commutative diagram of exact rows:

$$\begin{tikzcd} I \arrow[r] \arrow[d, equal] & n/n^2 \arrow[r] \arrow[d, "\delta'"] & m/m^2 \arrow[r] \arrow[d, "\delta"] & 0 \arrow[d, equal] \\ I \arrow[r, "\gamma"] & \Omega^1_{B/k} \otimes k(x) \arrow[r] & \Omega^1_{A/k} \otimes k(x) \arrow[r] & 0 \end{tikzcd}$$

Where the first row is given by

$$I \to n \to m \to 0 \quad \text{as } B\text{-mod}$$

$$\rightsquigarrow \begin{tikzcd} I \otimes_B B/n \arrow[r] & n \otimes_B B/n \arrow[r] & m \otimes_B B/n \arrow[r] & 0 \\ I \arrow[u] \arrow[ur, dashed] & & & \end{tikzcd}$$

The second row is given by

$$I/I^2 \to \Omega^1_{B/k} \otimes_B A \to \Omega^1_{A/k} \to 0$$

$$\xrightarrow{\otimes_A k(x)} \quad I/I^2 \otimes_A k(x) \xrightarrow{\delta'} \Omega^1_{B/k} \otimes_B k(x) \to \Omega^1_{A/k} \otimes_A k(x) \to 0$$

$$\begin{tikzcd} I/I^2 \otimes_A k(x) \arrow[r] & \Omega^1_{B/k} \otimes_B k(x) \\ I \arrow[u] \arrow[ur, dashed] & \end{tikzcd}$$

It remains to show δ' is inj.

But $B = k[T_1, \dots, T_n]$, $\underline{n = (T_i - X_i)_i}$,

$n/n^2 \simeq \bigoplus_{i=1}^n k \, \overline{T_i - X_i}$,

$\Omega^1_{B/k} \otimes k(x) \simeq \left(\bigoplus_{i=1}^n B \, dT_i \right) \otimes k(x)$

$\simeq \bigoplus_{i=1}^n k(x) \, dT_i$

(or use δ' surj & they are both k-vector space of dim n).

Rmk. 13.1.1

Recall the universal property of $\Omega^1_{B/A}$:

For $M \in \mathrm{Mod}_B$, $d' : B \to M \in \mathrm{Der}_A(B, M)$,

$$\begin{tikzcd} B \arrow[r, "d'"] \arrow[d, "d"'] \arrow[dr, phantom, "\circlearrowleft"] & M \\ \Omega^1_{B/A} \arrow[ur, dashed, "\exists ! \phi"'] & \end{tikzcd}$$

That is, we have a canonical isom. as A-mod:

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M)$$

$$\Rightarrow \mathrm{Hom}_A(\Omega^1_{A/k}, k(x)) \simeq \mathrm{Der}_k(A, k(x))$$

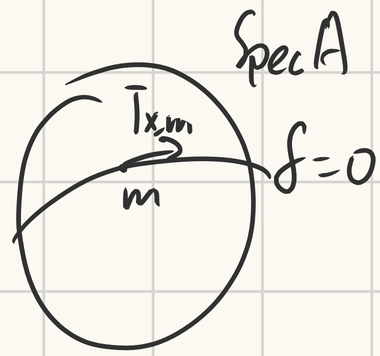
$$\simeq \mathrm{Hom}_{k(x)}(\Omega^1_{A/k} \otimes k(x), k(x)) \simeq \mathrm{Hom}_{k(x)}(m/m^2, k(x)).$$

If $k(x) = k$, this is to say

$$(m/m^2)^\vee \simeq \mathrm{Der}_k(A, k).$$

Prop. 13.1.B*

A ring, $\mathfrak{m} \in \operatorname{Spec}_{max} A$, if an ideal $I \subseteq \mathfrak{m}$,
the Zariski tangent space of A/I is cut out in
the Zariski tangent space of A by $I \pmod{\mathfrak{m}^2}$.
(i.e. $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I \hookrightarrow X$, $\mathfrak{m} \rightsquigarrow$ pt. $p \in X$,
$T_{Y,p} = \{ v \in T_{X,p} \mid v(I) = 0 \}$.)



Pf. We have an exact seq. of $k(x)$-vector space:

$$0 \to I/I \cap \mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \frac{\mathfrak{m}/(f)}{(\mathfrak{m}/(f))^2} \to 0$$

$$\mathfrak{m}/\mathfrak{m}^2 = T_{X,p}^{\vee}, \qquad \frac{\mathfrak{m}/(f)}{(\mathfrak{m}/(f))^2} = T_{Y,p}^{\vee}$$

By taking dual, we get

$$0 \to T_{Y,p} \to T_{X,p} \to (I/I \cap \mathfrak{m}^2)^{\vee} \to 0.$$

This is what we want.

Cor. 13.1.B

Let $I = (f)$, $Y = \operatorname{Spec} A/(f)$,
Then the Zariski tangent space of $A/(f)$ is cut out in
the Zariski tangent space of A by $f \pmod{\mathfrak{m}^2}$.

Prop. 13.1.C

Let Y, Z be both closed subscheme of X, $p \in Y \cap Z$. Then
(a) $T_{Y,p}$ is naturally a sub-$k(p)$-vector space of $T_{X,p}$
(b) $T_{Y \cap Z, p} = T_{Y,p} \cap T_{Z,p}$
(c) $T_{Y,p} + T_{X,p} \subseteq T_{Y \cup Z, p}$ ($Y \cup Z$ is scheme-theoretic union)
(d) $T_{Y \cup Z, p}$ can be strict larger than $T_{Y,p} + T_{X,p}$.

Pf. As the prop.s are local, W.M.A $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I$, $Z = \operatorname{Spec} A/J$.

Then $Y \cap Z = \operatorname{Spec} A/_{I+J}$, $Y \cup Z = \operatorname{Spec} A/_{I \cap J}$.

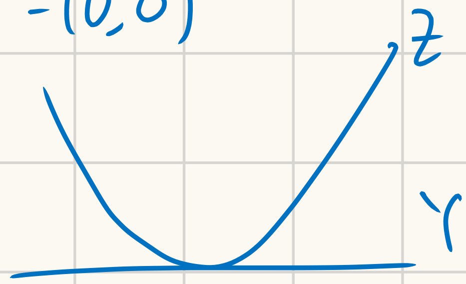
(a), (b), (c) follows from the last prop.

(d): Let $X = \operatorname{Spec} k[X,Y]$, $I = (Y)$, $J = (Y - X^2)$, $P = (0,0)$

Then $I \cap J = (Y(Y-X^2))$, $Y(Y-X^2) \in \mathfrak{m}^2$.

$\Rightarrow \forall\, v \in T_{X,P}$, $v(I \cap J) = 0$

$T_{Y \cup Z, P} = T_{X,P}$, $T_{Y,P} = T_{Z,P} = \{ v \in T_{X,P} \mid v(Y) = 0 \}$.



Example. 13.1.3, 13.1.4

$A = k[X,Y,Z]/(XY - Z^2)$, (X, Z) is not a principle ideal of A.

Moreover, by 12.1.12, this shows that A is not a UFD.

Pf. $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/(X,Z)$.

$k[X,Y,Z]$ is an integral domain $\Rightarrow \dim A = 2$.

$A/(X,Z) \simeq k[Y]$, $\dim A/(X,Z) = 1$ $\Rightarrow \operatorname{codim}_X Y = 1$. (Thm 12.2.9)

$XY - Z^2 = \left(\frac{X+Y}{2}\right)^2 - \left(\frac{X-Y}{2}\right)^2 - Z^2 \Rightarrow X = \operatorname{Spec} A \hookrightarrow \mathbb{A}_k^3$ is a cone.

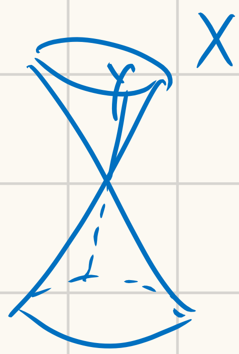
$Y \hookrightarrow \mathbb{A}_k^3$ is a line on the cone.

Let $P = (0,0,0)$ correspond to the maximal ideal $\mathfrak{m} = (X,Y,Z)$,

$(XY - Z^2) \subseteq \mathfrak{m}^2 \Rightarrow \dim T_{X,P} = 3$.

$\dim T_{Y,P} = 1$ by direct caculation.

But if $(X,Y) = (f)$, by 13.1.B, $\dim T_{Y,P} \geq \dim T_{X,P} - 1$. Contradiction!



Let $\pi: X \to Y$ be a morphism, $\pi(P) = q$, 13.1.6

then $\pi_P^\#: \mathcal{O}_{Y,q} \to \mathcal{O}_{X,P}$

$\rightsquigarrow T_{Y,q}^\vee = \mathfrak{m}_q/\mathfrak{m}_q^2 \longrightarrow T_{X,P}^\vee = \mathfrak{m}_P/\mathfrak{m}_P^2$

$\rightsquigarrow T_{\pi,X}: T_{X,P} \longrightarrow T_{Y,q} \otimes_{k(q)} k(P)$ as $k(P)$-linear space.

Exercise 13.1.G

For a k-scheme X, there is a bijection

$$\mathrm{Mor}_k(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2), X) \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{data of a } k\text{-rational pt } p \\ \text{and a tangent vector at } p \end{array} \right\}$$

Pf. Denote the only pt. of $\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)$ by $*$.

$\mathrm{Mor}_k(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2), X)$

$\leftrightarrow \{$data of a k-rational pt p

and a k-homom. $\mathcal{O}_{X,p} \to k[\varepsilon]/(\varepsilon^2)\}$.

Notice that $m_* = (\varepsilon)/(\varepsilon^2)$, $m_*^2 = (0)$.

So $m_p \to m_*$, $m_p^2 \to m_*^2 = (0)$.

$\mathcal{O}_{X,p} \to k[\varepsilon]/(\varepsilon^2) \longleftrightarrow \mathcal{O}_{X,p}/m_p^2 \to k[\varepsilon]/(\varepsilon^2)$.

$$\begin{tikzcd} 0 \arrow[r] & m_p/m_p^2 \arrow[r] \arrow[d] & \mathcal{O}_{X,p}/m_p^2 \arrow[r] \arrow[d, dashed] & \mathcal{O}_{X,p}/m_p \arrow[r] \arrow[d, equal, "\text{(both } k\text{)}"] & 0 \\ 0 \arrow[r] & \underset{=k}{(\varepsilon)/(\varepsilon^2)} \arrow[r] & k[\varepsilon]/(\varepsilon^2) \arrow[r] & k[\varepsilon]/(\varepsilon) \arrow[r] & 0 \end{tikzcd}$$

So $\mathcal{O}_{X,p} \to k[\varepsilon]/(\varepsilon^2) \longleftrightarrow (m_p/m_p^2 \to k) \in T_{X,p}$.

Exercise 13.1.H

$X = \mathrm{Spec}\, \mathbb{Z}[2i]$, $p = [(2, 2i)] \in X$, $\dim T_{X,p} = 2$.

$X = \mathrm{Spec}\, \mathbb{Z}[x]/(x^2+4)$, $p = [(2, x)]$,

$m_p^2 = (2, x)^2 = (4, 2x) = 2(2, x)$.

$m_p/m_p^2 = (2,x)/2(2,x) = \mathbb{Z}[x]/(2, x^2+4) = \mathbb{Z}_2[x]/(x^2)$.

Prop.

Suppose X is a finite type k-scheme. Then locally it is of the from $\operatorname{Spec} k[x_1,\dots,x_n]/(f_1,\dots,f_r)$. For a k-rational pt. $P\in X$, the Zariski cotangent space at P is given by the coker of the Jacobian map $k^r\to k^n$ given by the matrix $J=\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{ij}$.

Pf. Let P correspond to a maximal ideal $m\in \operatorname{Spec} k[x_1,\dots,x_n]/(f_1,\dots,f_r)$,

Consider $X=\operatorname{Spec} k[x_1,\dots,x_n]/(f_1,\dots,f_r) \hookrightarrow Y=\operatorname{Spec} k[x_1,\dots,x_n]$,

Denote n be the inverse image of m, view $P\in \operatorname{Spec} k[x_1,\dots,x_n]$,

$I:=(f_1,\dots,f_r)\subseteq n$, $m=(x_i-c_i)_i$, Then same as 13.1.B*,

$$0\to I/I\cap n^2 \to \underbrace{n/n^2}_{=T_{Y,P}^\vee} \to \underbrace{m/m^2}_{=T_{X,P}^\vee} \to 0 \text{ exact.}$$

Where $T_{Y,P}^\vee \simeq \bigoplus_{j=1}^n k\,\overline{x_j-c_j} \simeq k^n$,

$I/I\cap n^2 \simeq \operatorname{span}\{\overline{f_i}\}_i$

Let $J: k^r\to k^n$ given by

$$\begin{array}{ccccccc} k^r & \longrightarrow & I/I\cap n^2 & \longrightarrow & n/n^2 & \xrightarrow{\sim} & k^n \\ (\lambda_i)_i & \longmapsto & \sum_{i=1}^r \lambda_i \overline{f_i} & \longmapsto & \sum_{i=1}^r \lambda_i \overline{f_i}. & & \end{array}$$

Notice that $\overline{f_i}=\sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(P)\,\overline{x_j-c_j}$,

So J exactly correspond to the matrix $\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{ij}$.

Rmk.

(a) $\dim_k T_{X,P}=\dim_k(\operatorname{coker} J(P))$ in this situation.

(b) This result can be extended to closed pts of X whose residue field is separable over k.

(c) For arbitrary $P\in X$, $J=\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{ij}\in M_{n\times r}(k(P))$.

Prop.

Under the same notation as above, the Jacobian corank at $P \in X$ is defined by $\dim(\operatorname{coker} J)$. ($P$ is not necessarly closed pt)

J is determined by choosing $P \in U = \operatorname{Spec} A \hookrightarrow X$, $A \simeq k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$

The corank is independent with the choice of $x_1, \cdots, x_n$; $f_1, \cdots, f_r$. and U.

Pf. (a) $I := (f_1, \cdots, f_r)$, if $g \in I$, write $g = \sum_{i=1}^{r} h_i f_i$, $h_i \in k[x_1, \cdots, x_n]$.

Then $\left(\frac{\partial g}{\partial x_j}\right)_j = \left(\sum_{i=1}^{r} \frac{\partial h_i f_i}{\partial x_j}\right)_j = \left(\sum_{i=1}^{r} f_i \frac{\partial h_i}{\partial x_i} + \frac{\partial f_i}{\partial x_i} h_i\right)$

By $I \subseteq P$, $\left(\frac{\partial g}{\partial x_j}\right)_j = \left(\sum_{i=1}^{r} h_i(P) \frac{\partial f_i}{\partial x_j}(P)\right)_j = \sum_{i=1}^{r} h_i(P) \left(\frac{\partial f_i}{\partial x_j}(P)\right)_j$.

That is, if $J' = \left(\begin{array}{c|c} J & \begin{matrix} \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{matrix} \end{array}\right)$, Then $\operatorname{Im} J = \operatorname{Im} J'$.

So $\operatorname{rank} J = \operatorname{rank} J'$.

Now if $I = (f_1, \cdots, f_r) = (g_1, \cdots, g_{r'})$,

Then
$$\operatorname{Im}\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix} = \operatorname{Im}\left(\begin{array}{ccc|ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_1} & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_{r'}}{\partial x_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} & \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_{r'}}{\partial x_n} \end{array}\right)$$
$$= \operatorname{Im}\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_r}{\partial x_n} \end{pmatrix}$$

i.e. $\dim(\operatorname{coker} J)$ is independent of the choice of $f_1, \cdots, f_r$.

(b) A is generated by $x_1, \cdots, x_n$ as an k-algebra

If $h = Y - q(x_1, \cdots, x_n) \in k[x_1, \cdots, x_n, Y]$, $q \in k[x_1, \cdots, x_n]$,

Then $\frac{\partial h}{\partial Y} = 1$, $\frac{\partial h}{\partial x_j} = -\frac{\partial q}{\partial x_j}$.

$$J' := \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_r}{\partial X_1} & \frac{\partial h}{\partial X_1} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_1}{\partial X_n} & \cdots & \frac{\partial f_r}{\partial X_n} & \frac{\partial h}{\partial X_n} \\ \frac{\partial f}{\partial Y} & \cdots & \frac{\partial f_r}{\partial Y} & \frac{\partial h}{\partial Y} \end{pmatrix} = \left(\begin{array}{c|c} J & \begin{matrix} \frac{\partial h}{\partial X_1} \\ \vdots \\ \frac{\partial h}{\partial X_n} \end{matrix} \\ \hline 0 & 1 \end{array}\right)$$

So $\operatorname{rank} J' = \operatorname{rank} J + 1$,

$\dim \operatorname{coker} J' = \dim \operatorname{coker} J$.

Now if A is also generated by $Y_1, \cdots, Y_{n'}$ as k-alg,

Let $Y_j = q_j(X_1, \cdots, X_n)$, $q_j \in k[X_1, \cdots, X_n]$.

$h_j := Y_j - q_j(X_1, \cdots, X_n) \in k[X_1, \cdots, X_n, Y_j]$, $1 \le j \le n'$

$X_j = P_j(Y_1, \cdots, Y_{n'})$, $P_j \in k[Y_1, \cdots, Y_{n'}]$

$l_j := X_j - P_j(Y_1, \cdots, Y_{n'}) \in k[Y_1, \cdots, Y_{n'}, X_j]$, $1 \le j \le n$.

$A \simeq k[X_1, \cdots, X_n]/(f_1, \cdots, f_r) \simeq k[Y_1, \cdots, Y_{n'}]/(g_1, \cdots, g_{r'})$

Then notice that $A \simeq k[X_1, \cdots, X_n, Y_1, \cdots, Y_{n'}]/(f_1, \cdots, f_r, h_1, \cdots, h_{n'})$

$\simeq k[X_1, \cdots, X_n, Y_1, \cdots, Y_{n'}]/(g_1, \cdots, g_{r'}, l_1, \cdots, l_n)$

So by (a) and above observation, $\dim(\operatorname{coker} J)$ is independent of the choice of $X_1, \cdots, X_n$.

(c) Assume $A \simeq k[X_1, \cdots, X_n]/(f_1, \cdots, f_r)$,

For $q(X_1, \cdots, X_n) \in k[X_1, \cdots, X_n]$, $\overline{q(X_1, \cdots, X_n)} \notin P$,

Let $h = Y q(X_1, \cdots, X_n) - 1 \in k[X_1, \cdots, X_n, Y]$.

$\left(k[X_1, \cdots, X_n]/(f_1, \cdots, f_r)\right)_{\bar{q}} \simeq k[X_1, \cdots, X_n, Y]/(f_1, \cdots, f_r, h)$.

$$J' := \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_r}{\partial X_1} & \frac{\partial h}{\partial X_1} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_1}{\partial X_n} & \cdots & \frac{\partial f_r}{\partial X_n} & \frac{\partial h}{\partial X_n} \\ \frac{\partial f}{\partial Y} & \cdots & \frac{\partial f_r}{\partial Y} & \frac{\partial h}{\partial Y} \end{pmatrix} = \left(\begin{array}{c|c} J & \begin{matrix} \frac{\partial h}{\partial X_1} \\ \vdots \\ \frac{\partial h}{\partial X_1} \end{matrix} \\ \hline 0 & q \end{array}\right)$$

where $q(P) \neq 0$ by $\overline{q(X_1, \cdots, X_n)} \notin m$.

So the same as above, $\dim \operatorname{coker} J = \dim \operatorname{coker} J'$.
Now for $P \in U \cap U'$, choose an open subset V of $U \cap U'$ contains P, s.t. V is a principle open subset of both U and U'.
By (a)(b) and above observation, $\dim(\operatorname{coker} J)$ is independent of the choice of U.

Thus every finite type k-scheme X comes with an intrinsic "Jacobian corank function" from X to $\mathbb{Z}_{\geq 0}$, whose value at a k-rational pt $p \in X$ is $\dim_k T_{X,p}$.
We temporarily call this function $JC : X \to \mathbb{Z}_{\geq 0}$.

Prop. 21.2.E

$\forall p \in X$, $JC(x) = \dim_{k(x)} (\Omega^1_{X/k})_x \otimes_{\mathcal{O}_{X,x}} k(x)$.

Pf. W.M.A $X = \operatorname{Spec} A$, $A = k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$. Let $p \rightsquigarrow \mathfrak{p} \subseteq A$.
$B := k[x_1, \cdots, x_n]$,
$\Omega^1_{A/k} \simeq \left(\bigoplus_{j=1}^n B\,dx_j\right) / \left(\sum_{i=1}^r B\,df_i\right)$.
So $B^r \xrightarrow{J} B^n \to \Omega^1_{A/k} \to 0$ exact.
$(\lambda_i)_i \mapsto \sum_{i=1}^r \lambda_i df_i$
where J is given by the matrix $J = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j} \in M_{n \times r}(B)$.
Tensor by $\otimes_B k(x)$, we get
$$k(x)^r \xrightarrow{J \otimes \mathrm{id}_{k(x)}} k(x)^n \to (\Omega_{X/k})_x \otimes_{\mathcal{O}_{X,x}} k(x) \to 0 \quad \text{exact}.$$
This is what we want.

Prop. 13.1.K

JC is upper semicontinuous.

Pf. $\forall p \in X = \operatorname{Spec} A$, if $JC(p) = t$, $\operatorname{rank}_{k(p)} \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{ij} = n-t$.

That is, there exist a submatrix J' of $J = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij} \in M_{n \times r}(A)$,

s.t. $J'(p) \in M_{(n-t)\times(n-t)}(k(p))$, $\operatorname{rank}_{k(p)} J'(p) = n-t$.

So $\det(J'(p)) \neq 0 \in k(p)$, $\det J' \notin \mathfrak{p}$. $\det J' \in A$.

$p \in D(\det J') \subseteq \operatorname{Spec} A$.

Now $\forall q \in D(\det J')$, $\det(J'(q)) \neq 0 \in k(q)$.

Hence $\operatorname{rank}_{k(q)} J'(q) = n-t$, $\operatorname{rank}_{k(q)} J(q) \geq \operatorname{rank}_{k(q)} J'(q) = n-t$.

$JC(q) \leq n-(n-t) = t$. JC is upper semicontinuous.

Prop. 13.1.L

JC is perserved by field ext. of k.

If X is of finite type over k, l/k is a field ext,

$X \times_k l \to X$, $p \mapsto q$, Then $JC(p) = JC(q)$.

Pf. W.M.A $X = \operatorname{Spec} A$, $A = k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$.

for any $k \times k$ submatrix J' of $J = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij} \in M_{n \times r}(A)$,

Notice that $X \times_k l = \operatorname{Spec} A \otimes_k l$, $A \otimes_k l = l[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$.

So the Jacobi matrix $\tilde{J}$ of $X \times_k l$ is $\tilde{J} = \left(\frac{\partial f_i}{\partial x_j} \otimes 1\right) \in M_{n \times r}(A \otimes_k l)$.

Let $\tilde{J}'$ be the $k \times k$ submatrix of $\tilde{J}$ corresponding to the place of J,

then $\det(\tilde{J}'(q)) = \det(J'(p)) \in k(q)$. $(k(p) \hookrightarrow k(q))$

So it follows that $JC(p) = JC(q)$.

Prop.

$\forall P \in X$, $JC(P) \geq \dim_X P$.

Pf. Case 1. P is a closed pt, $k = \bar{k}$.

$JC(P) = \dim_{k(P)} T_{X,P} \geq \dim \mathcal{O}_{X,P} = \dim_X P$.

Case 2. P is a closed pt, k arbitrary

Consider

$$\begin{tikzcd} \operatorname{Spec} k(P) \otimes_k \bar{k} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_k \bar{k} \arrow[d] \\ \operatorname{Spec} k(P) \arrow[r, hook] & X \end{tikzcd},$$

By 12.2.M, the dimension of a locally finite type k-scheme is preserved by any field ext., $\dim \operatorname{Spec} k(P) = \dim k(P) \otimes_k \bar{k} = 0$.

So $\forall q \in \operatorname{Spec} k(P) \otimes_k \bar{k}$, q is a closed pt.

By 13.1.L, $JC(P) = JC(q)$.

$\dim_{X \times_k \bar{k}} q = \dim X \times_k \bar{k} = \dim X$. ($X \times_k \bar{k}$ is of pure dimension).

So by case 1, $JC(P) \geq \dim_X P$.

Case 3. P, k arbitrary.

W.M.A $X = \operatorname{Spec} A$, $P \rightsquigarrow \mathfrak{p} \subseteq A$,

By JC is upper semicontinuous, $\exists U \ni P$, $U \hookrightarrow X$,

s.t. $\forall q \in U$, $JC(u) \leq JC(P)$.

Let $p' \in U \cap \overline{\{p\}}$ be a closed pt.

(a closed pt of U is a closed pt. of X)

Then $JC(P') \leq JC(P)$.

Notice that for any sub matrix J' of $J = \left(\frac{\partial f_i}{\partial x_j}\right)$,

$\det J'(P) = 0 \Leftrightarrow \det J' \in \mathfrak{p} \Rightarrow \det J' \in \mathfrak{m} \Leftrightarrow \det J'(P') = 0$.

That is, $JC(P') \geq JC(P)$. $JC(P') = JC(P)$.

So $JC(P) = JC(P') \geq \dim_X P' \geq \dim_X P$

§ 13.2 Regularity, and smooth over a field.

Def. 13.2.1

(a) We say a Noetherian local ring (A, m, k) $(k = A/m)$ satisfies $\dim_k m/m^2 = \dim A$, we say that A is a regular local ring.

(b)* If a Noetherian ring is regular at all of its prime ideals, then A is said to be a regular ring.

(c) A locally Noetherian scheme X is regular at a pt P if the local ring $\mathcal{O}_{X,P}$ is regular (or nonsingular). It is singular (or nonregular) at P otherwise, and we say that the point is a singularity.

(d) A scheme is regular (or nonsingular) if it is regular at all points. It is singular otherwise.

Example. 13.2.A

A Noetherian regular local ring of dim 0 is a field.

Prop. 13.2.B

Suppose (A, m) is a regular local ring of dim $n > 0$, $f \in A$. Then $A/(f)$ is a regular local ring of dim $n-1$ iff $f \in m/m^2$.

Pf. If $f \in A \setminus m$, then $(f) = (1)$, $A/(f) = 0$.

If $f \in m \setminus m^2$, then $\bar{f} \neq 0 \in m/m^2$.

Let $\overline{f}, \overline{g_1}, \cdots, \overline{g_{n-1}}$ be a k-basis of m/m^2, then $m = (f, g_1, \cdots, g_{n-1})$ by Nakayama's Lemma.

By Krull's Principal Ideal Thm, $\dim A/(f) = n-1$.

By 13.1.B, $n := m/(f)$, we have an exact seq. of k-vector space

$$0 \to (f)/(f)\cap m^2 \to m/m^2 \to n/n^2 \to 0$$

Where $\dim_k (f)/(f)\cap m^2 = 1$.

Hence $\dim_k n/n^2 = \dim_k m/m^2 - 1 = \dim A - 1 = \dim A/(f)$.

$A/(f)$ is a regular local ring in this situation.

If $f \in m^2$, similarly $\dim A/(f) = n-1$,

But $\dim m/m^2 = \dim n/n^2$, $A/(f)$ is not regular.

Prop. 13.2.C

Suppose X is a Noetherian scheme, D is an effective Cartier divisor on X, $P \in D$. Then if P is a regular pt of D, P is also a regular pt of X.

Pf. W.M.A $X = \operatorname{Spec} A$, $D = \operatorname{Spec} A/(f)$ where f is not a zerodivisor.

By Krull's Principle ideal thm, $\dim D = \dim A - 1$.

By same exact seq., $\dim m/m^2 \le \dim n/n^2 + 1$.

But $\dim m/m^2 \ge \dim A = \dim D + 1 = \dim n/n^2 + 1$,

We must have $\dim m/m^2 = \dim A$. i.e. P is a regular pt of X.

Prop. (The Jacobian criterion for regularity for k-rational pt) 13.2.E

Suppose $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_n)$ has pure dimension d,

Then a k-rational pt $P \in X$ is regular iff $JC(P) = d$.

Pf. By 13.1.7, $JC(P) = \dim_{k(P)} T_{X,P}$.

By 12.2.L, $\dim \mathcal{O}_{X,P} = \dim_X P = d$ (X has pure dimension d).

So P is regular $\iff \dim \mathcal{O}_{X,P} = \dim_{k(P)} T_{X,P} = d$

$\iff JC(P) = d$.

Thm*. (The Jacobian criterion) GTM256 Thm 13.10

Let $X = \operatorname{Spec} k[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$, $P \in X$, Then

(a) $JC(P) \geq \dim_X P$.

(b) If equality holds in (a), then P is a regular point.

(c) If $k(P)$ is a (not necessarily finite) seperable field ext. of k, then the converse of (b) holds.

In particular, this condition is satisfied when k is a perfect field.

Def. 13.2.4.

(a) A k-scheme is smooth of dimension d over k, if it is of pure dimension d, and there exists a cover by affine open sets $\operatorname{Spec} k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ where the Jacobian matrix has corank d at all points (not just closed pts). (In particular, smooth schemes are implicitly locally of finite type.)

(b) A k-scheme is smooth over k if it is smooth of some dimension

By 13.1.7, for any pure dimension d variety, this can be checked on any affine open cover.

Lemma.

Suppose A is a fin. gen. k-alg, l/k is an arbitrary field ext,

(a) The going down property holds for $A \to A \otimes_k l$

(b) $\operatorname{Spec} A \otimes_k l \to \operatorname{Spec} A$ has finite fiber of dim 0.

Pf. (a) $A \otimes_k l$ is free over A, use GTM256 Lemma 7.16.

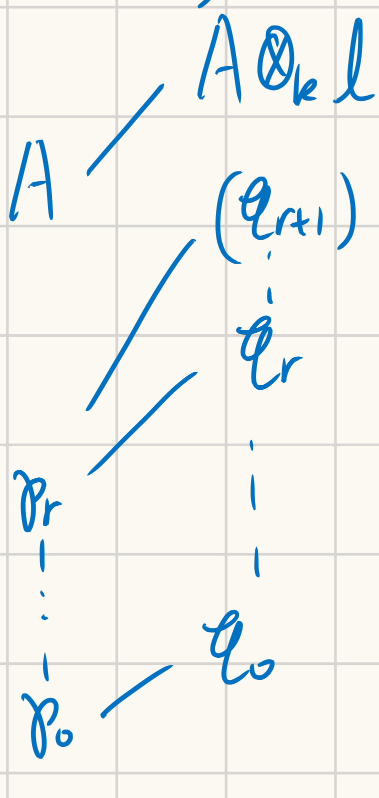
(b) $\forall\, x \in \operatorname{Spec} A$, $\operatorname{Spec} A \otimes_k l \times_{\operatorname{Spec} A} \operatorname{Spec} k(x) = \operatorname{Spec} k(x) \otimes_k l$.

It is finite over $\operatorname{Spec} l$.

Prop. 13.2.H

Suppose X is a locally finite type k-scheme, l/k is a field ext.
Then X is smooth over k iff $X \times_k l$ is smooth over l.

Pf. "$\Rightarrow$" if X is smooth over k of pure dimension d,
By 12.2.M $X \times_k l$ is of pure dimension d.
$\forall\, p \in X \times_k l$, $X \times_k l \to X$, $p \mapsto q$,
Then By 13.1.L $JC(p) = JC(q) = d$.
"$\Leftarrow$" Clearly $X \times_k l \to X$ is surjective.
It's suffices to show that X is of pure dimension d.
W.M.A $X = \operatorname{Spec} A$, A is fin. gen. over k.
Then $X \times_k l = \operatorname{Spec} A \otimes_k l$.
For any minimal prime $\mathfrak{p}_0 \in \operatorname{Spec} A$,
Suppose $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ is a maximal chain of prime ideal,
By $X \times_k l \to X$ surj, $\exists\, \mathfrak{q}_r \in \operatorname{Spec} A \otimes_k l$ lying over $\mathfrak{p}_r$.
Then by going down property, $\exists\, \mathfrak{q}_0, \cdots, \mathfrak{q}_{r-1} \in \operatorname{Spec} A \otimes_k l$,
s.t. $\mathfrak{q}_i$ is lying over $\mathfrak{p}_i$ $\forall i$.
Here $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_r$,
By $\operatorname{Spec} A \otimes_k l$ is of pure dimension d, $r \le d$.
Moreover, if $r < d$, $\exists\, \mathfrak{q}_r \subsetneq \mathfrak{q}_{r+1} \in \operatorname{Spec} A \otimes_k l$.
By $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ is maximal,
$\mathfrak{q}_{r+1}$ is also lying over $\mathfrak{p}_r$
Contradiction to part (b) of the Lemma!
So $r = d$, X is of pure dimension d.



Prop. 13.2.I

If the Jacobian matrix for $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ has corank d at all closed pts, then it has corank d at all pts.

Pf. This has been proved in 13.1.M.

13.2.J

Prop.

Suppose X is a locally finite type scheme of pure dimension d over an algebraically closed field $k = \bar{k}$.

Then X is regular at its closed pts iff it is smooth.

Pf. For any closed pt $p \in X$, by 13.1.7,

$\dim_k T_{X,p} = JC(p)$

So X is regular at its closed pts

$\Leftrightarrow d = \dim_k T_{X,p}$

$\Leftrightarrow d = JC(p), \ \forall$ closed pt $p \in X$

$\Leftrightarrow d = JC(p), \ \forall p \in X$, 13.2.I

$\Leftrightarrow X$ is smooth.

Thm. (Smoothness-Regularity Comparison Thm) 13.2.7

(a) If k is perfect, every regular finite type k-scheme is smooth over k.

(b) Every smooth k-scheme is regular.

Rmk.

Let X be a locally finite type k-scheme of pure dimension d, then X is smooth iff $X_{\bar{k}}$ is regular.

(Compare with the definition of smooth in Liu Qing.)

Pf. By 13.2.H, X is smooth $\Leftrightarrow X_{\bar{k}}$ is smooth.

By 13.2.7, $X_{\bar{k}}$ is smooth $\Leftrightarrow X_{\bar{k}}$ is regular.

Prop. 13.2.M

(a) Suppose (A, m, k) is a regular local ring of dim n,

$I \subseteq A$ is an ideal s.t. A/I is a regular local ring of dim d.

Then $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ is a regular embedding.

(i.e. $r := n-d$, I is gen. by an regular seq. $f_1, \cdots, f_r$.)

(b) Suppose $\pi: X \hookrightarrow Y$ is a closed embedding of regular scheme.

Then π is a regular embedding.

Pf.(a) Recall that we have an exact seq. of k-vector space

$$0 \to I/I\cap m^2 \to m/m^2 \to n/n^2 \to 0.$$

where $n = m/(I) \in \operatorname{Spec} A/I$, $\dim_k m/m^2 = n$, $\dim_k n/n^2 = d$.

So $\dim_k I/I\cap m^2 = r$, Let $I/I\cap m^2$ is generated by $\bar{f_1}, \cdots, \bar{f_r} \in m/m^2$,

We use induction to show that $f_1, \cdots, f_r$ is an regular seq.

and $A/(f_1, \cdots, f_r)$ is regular of dim $n-r = d$.

It's clear that f_1 is not a zerodivisor in A, $\dim A/(f_1) = n-1$

By A is an integral domain, Krull's principal ideal Thm, 13.1.B.

If k has been proved, notice that

$f_{k+1} \bmod (f_1, \cdots, f_r) \in (m/(f_1, \cdots, f_r)) \setminus (m/(f_1, \cdots, f_r))^2$

By $\bar{f_1}, \cdots, \bar{f_r} \bmod m^2$ is a basis of m/m^2.

So by the same conclusion, we get $A/(f_1 \cdots f_{k+1})$ is regular of dim $n-k+1$.

Now $A/(f_1, \cdots, f_r)$ and A/I are both regular local ring of dim d.

By 12.1.D, $A/(f_1, \cdots, f_r) = A/I$ That is, $I = (f_1, \cdots, f_r)$

is generated by a regular seq.

(b) $\forall x \in X$, $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a surjection of regular local ring.
By (a), this is an regular embedding.

Lemma. 13.2.13

Smooth k-scheme is reduced.

Pf. Suppose X is a k-scheme, $X_{\bar{k}} = X \times_k \bar{k}$.

13.2.H $\Rightarrow$ $X_{\bar{k}}$ is smooth.

13.2.J $\Rightarrow$ $X_{\bar{k}}$ is regular at its closed pts.

13.2.11 $\Rightarrow$ The local ring of $X_{\bar{k}}$ at closed pts are integral domain, hence reduced.

9.4.10 $\Rightarrow$ $X_{\bar{k}}$ is reduced.

$\Rightarrow$ X is reduced.

($\mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \otimes_k \bar{k}$ is inj. By flatness over k)

Proof of 13.2.7

Let X be a k-scheme of finite type. $X_{\bar{k}} := X \times_k \bar{k} \xrightarrow{\pi} X$.

Assume X is of pure dimension d.

(a) (21.2.Y) k perfect, regular $\Rightarrow$ smooth

$\forall$ closed pt $p \in X$, let $p = \pi(q)$, $q \in X_{\bar{k}}$, q is also a closed pt.

By Liu Qing 6.2.1*, $T_{X,p}^\vee \simeq (\Omega^1_{X/k})_p \otimes_{\mathcal{O}_{X,p}} k(p)$.

$T_{X_{\bar{k}},q}^\vee \simeq (\Omega^1_{X_{\bar{k}}/\bar{k}})_q \otimes_{\mathcal{O}_{X_{\bar{k}},q}} k(q)$

Recall that $(\Omega^1_{X_{\bar{k}}/\bar{k}})_q = (\pi^* \Omega^1_{X/k})_q = (\Omega^1_{X/k})_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X_{\bar{k}},q}$

So $T_{X_{\bar{k}},q}^\vee = T_{X,p}^\vee \otimes_{k(p)} k(q)$. $\dim_{k(p)} T_{X,p}^\vee = \dim_{k(q)} T_{X_{\bar{k}},q}^\vee$

X regular $\Rightarrow$ $\dim_{k(q)} T_{X_{\bar{k}},q}^\vee = \dim_{k(p)} T_{X,p}^\vee = d = \dim X_{\bar{k}}$.

By 13.2.7, $X_{\bar{k}}$ is smooth $\Rightarrow$ X is smooth.

(b) smooth $\Rightarrow$ regular

W.M.A $X = \operatorname{Spec} A$, $A = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. $I := (f_1, \dots, f_r)$

Under same notation with the pf of 13.1.7, $\forall P \in X$,

we have an exact seq. of $k(P)$-linear space

$$0 \to I/I \cap n^2 \to n/n^2 \to m/m^2 \to 0 \quad \text{exact.}$$

Notice that $I \bmod n^2 \supseteq \operatorname{span}\left\{\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \overline{x_i - c_i}\right\}$,

$\dim_{k(P)} m/m^2 = n - \dim_{k(P)}(I/I \cap n^2) \leq n - \operatorname{rank}(J(P)) = JC(P)$

By X smooth, $JC(P) = d$. So $\dim_{k(P)} m/m^2 = d$,

x is a regular pt. in X.

By Fact 13.8.2, X is regular.