Applied Mathematics & Optimization (2022) 85:5
https://doi.org/10.1007/s00245-022-09863-2

Continuous Data Assimilation Algorithm for the Two Dimensional Cahn–Hilliard–Navier–Stokes System

Bo You[1] · Qing Xia[1]

Accepted: 12 February 2022 / Published online: 13 April 2022

Abstract
Based on the fact that dissipative dynamical systems possess finite degrees of freedom, a new continuous data assimilation algorithm for the two dimensional Cahn–Hilliard–Navier–Stokes system is introduced. In this paper, we provide some suitable conditions on the nudging parameters and the size of the spatial coarse mesh observables, which are sufficient to show that the solution of the proposed algorithm converges at an exponential rate, asymptotically in time, to the unique exact unknown reference solution of the original system under the assumption that the observed data are free of error. Thus, we can make the future predictions of the exact solution by the approximation solution of the continuous data assimilation algorithm if the initial data is missing, which usually appears in the fields of geophysical and biological sciences.

Keywords Cahn–Hilliard–Navier–Stokes system · Continuous data assimilation · Signal synchronization · Nudging · Downscaling

Mathematics Subject Classification 35Q30 · 93C20 · 37C50 · 76B75 · 34D06

1 Introduction

Accurate numerical simulations of nonlinear systems require high precision in the initial data. In general, initial data is ideally defined on the whole physical domain for most applications, but it can be measured only discretely with inadequate resolution. The goal of continuous data assimilation is to use low spatial resolution observational data, obtained continuously in time, along with dynamical principles pertaining to

✉ Bo You
youb2013@xjtu.edu.cn

Qing Xia
xiaqing151701@stu.xjtu.edu.cn

[1] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China

the underlying mathematical model, to recover the corresponding exact solution from which future predictions can be made. It was first proposed in [1] for atmospheric predictions, such as weather forecasting. In general, as a result of the low spatial and/or temporal resolution of the given observational measurements, it is very difficult to produce accurate information of the true state of the atmosphere or a fluid at a given time. More precisely, we assume that the discrete spatial measurements, continuously in time, of an exact solution to a given model is available, while the exact solution itself is unknown. For instance, the datas for weather prediction is collected at weather stations spread over a discrete spatial grid, nearly continuously in time.

Inspired by the ideas from control theory [2, 3], a new continuous data assimilations algorithm was developed in [4] for the two-dimensional Navier–Stokes equations, which is different from the classical method of continuous data assimilation in [1], but it is actually applicable to a large class of dissipative evolution equations, which possess a finite number of determining parameters, such as determining modes, nodes and local volume averages (see [5–15]). In this algorithm, a feedback control term is introduced into the original evolution equation of the system, which forces the coarse spatial scales of the solution of the new model, i.e., the approximating solution, toward the coarse spatial scales of the solution of the original system, i.e., the reference solution (see [16, 17]).The motivation of this approach is the fact that instabilities in dissipative evolution equations occur at the large spatial scales, hence these coarse scales in downscaling algorithms need to be controlled, stabilized, or nudged. This type of technique was previously called Newtonian nudging or dynamic relaxation method (see [18]) and usually considered in much simpler scenarios. The advantage of this new algorithm is that no derivatives of the observational measurements are required, but it is required in [1]. Thus, the method in [4] works for a general class of interpolant operators without modification.

The main idea of such algorithm for a general evolution equation can be formally described as follows: suppose that $u(t)$ is a solution of some dissipative dynamical system generated by the following abstract evolution equation:

$$\frac{du}{dt} = F(u), \tag{1.1}$$

where the initial data $u(0) = u_0$ is missing. Let $I_h(u(t))$ be an interpolant operator based on the observational measurements of this system at a coarse spatial resolution of size h for $t \in [0, T]$. The algorithm proposed in [4] is to construct an approximate solution $v(t)$ that satisfies the following equations

$$\begin{cases} \frac{dv}{dt} = F(v) - \mu(I_h(v) - I_h(u)), \\ v(0) = v_0, \end{cases} \tag{1.2}$$

where $\mu > 0$ is a relaxation (nudging) parameter and v_0 is arbitrarily given initial data. If problem (1.2) is globally well-posed and $I_h(v)$ converges to $I_h(u)$ in time, then one can recover the reference solution $u(t)$ from the approximate solution $v(t)$. The goal is to provide some conditions on $\mu > 0$ and $h > 0$ in terms of physical parameters of the evolution Eq. (1.1), such that the approximate solution $v(t)$ approaches the reference solution $u(t)$ with increasing accuracy as more continuous data in time is supplied. After some large enough time $T > 0$, the solution $v(T)$ can be used as an initial condition of problem (1.1) to make future predictions of the reference solution $u(t)$ for $t \geq T$, or one can continue to use the solution of problem (1.2) itself as long as more observable measurements are provided. Recently, the approach of [4] is extended to the case that the observations are contaminated with random errors in [19]. Moreover, the ideas from [2] have also been applied to many dissipative dynamical systems, such as the 3D Bénard convection in porous medium (see [20]), 2D Bénard convection system (see [21, 22]), 2D Navier–Stokes equations (see [17, 23–29]), simplified Bardina model (see [30]) as well as to the 3D Brinkman–Forchheimer-extended Darcy model of porous media (see [31]), 3D Navier–Stokes-α model (see [32]), quasi-geostrophic equation (see [33]). Moreover, there are some works concerning the numerical algorithm (see [34–37]). In particular, the authors in [36] have proposed a numerical approximation method for the Cahn–Hilliard equations that incorporates continuous data assimilation to achieve long time accuracy by using of a $\mathcal{C}^0$ interior penalty spatial discretization of the fourth order Cahn–Hilliard equations and a backward Euler temporal discretization. They proved the method was long time stable and long time accurate for arbitrarily inaccurate initial conditions under the condition that enough data measurements are incorporated into the simulation. Additionally, there is another method of data assimilation named variational data assimilation used in many applications (see [38–41]), which is based on the minimization of a cost function to estimate parameters of a partial differential equation from the observation of one (or possibly several) solution of this equation. This cost function measures the difference between the predicted and actual observations. Comparing to the variational data assimilation, the continuous data assimilation do not need to solving the corresponding optimalization problem and the convergence rate is faster than the one of variational data assimilation. Therefore, we will consider the continuous data assimilation of the two dimensional Cahn–Hilliard–Navier–Stokes system in this paper.

We mainly investigate the two dimensional Cahn–Hilliard–Navier–Stokes system given by:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p + \lambda\phi\nabla\mu = g(x), \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ \nabla\cdot u = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ \frac{\partial\phi}{\partial t} + u\cdot\nabla\phi - \gamma\Delta\mu = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ \mu = -\Delta\phi + f(\phi), \quad (x,t)\in\Omega\times\mathbb{R}^+. \end{cases} \tag{1.3}$$

Equation (1.3) is subject to the following homogeneous Neumann conditions

$$u = 0, \quad \frac{\partial\phi}{\partial\mathbf{n}} = \frac{\partial\mu}{\partial\mathbf{n}} = 0, \quad (x,t)\in\Gamma\times\mathbb{R}^+ \tag{1.4}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0(x), \quad x\in\Omega, \\ \phi(x,0) = \phi_0(x), \quad x\in\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ and $\mathbb{R}^+ = [0, +\infty)$, $\nu > 0$ is the viscosity, $\lambda > 0$ is a surface tension parameter, $\gamma > 0$ is the elastic relaxation time (see [42]), $g(x) = (g_1(x), g_2(x))$ is the external force, $u(x,t) = (u_1(x,t), u_2(x,t))$ denotes the average velocity and ϕ is the difference of the two fluid concentrations, p is the fluid pressure, $\mathbf{n}$ is the unit external normal vector on Γ.

Diffuse-interface methods in fluid mechanics are widely used by many researchers to describe the behavior of complex fluids (see [43]). A diffuse interface variant of Cahn–Hilliard–Navier–Stokes system has been proposed to model the motion of an isothermal mixture of two immiscible and incompressible fluids subject to phase separation (see [44, 45]). The coupled system consists of a convective Cahn–Hilliard equation for the order parameter, i.e., the difference of the relative concentrations of the two phases, coupled with the Navier–Stokes equations for the average fluid velocity. The Cahn–Hilliard–Navier–Stokes system has been investigated from the numerical (see [46–48]) and analytical (see, e.g., [49–52]) viewpoint in several papers. The long-time behavior of solutions and well-posedness for the two dimensional Cahn-Hilliard-Navier-Stokes system were proved in [50]. In [51], the authors have considered the instability of two-phase flows and provided a lower bound on the dimension of the global attractor of the Cahn–Hilliard–Navier–Stokes system.

In this paper, we propose a new continuous data assimilation algorithm for the construction of $v(t)$ and $\psi(t)$ that approximates the velocity $u(t)$ and the order parameter $\phi(t)$, respectively, from the observational measurement $I_h(u(t))$ of the velocity along with $I_h(\phi(t))$ of the order parameter of $t \in [0, T]$, which satisfies the following problem:

$$\begin{cases} \frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla q + \lambda\psi\nabla\eta = g(x) - \beta\left(I_h(v) - I_h(u)\right), \ (x,t) \in \Omega\times\mathbb{R}^+, \\ \nabla\cdot v = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ \frac{\partial \psi}{\partial t} + v\cdot\nabla\psi - \gamma\Delta\eta = -\chi\left(I_h(\psi) - I_h(\phi)\right), \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ \eta = -\Delta\psi + f(\psi), \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ v = 0, \ \frac{\partial\psi}{\partial\mathbf{n}} = \frac{\partial\eta}{\partial\mathbf{n}} = 0, \quad (x,t)\in\Gamma\times\mathbb{R}^+, \\ v(x,0) = v_0(x), \quad x\in\Omega, \\ \psi(x,0) = \psi_0(x), \quad x\in\Omega, \end{cases} \tag{1.6}$$

where β and χ are two positive relaxation (nudging) parameters, which relaxes the coarse spatial scales of (v, ψ) toward the observed data, q is the approximate pressure, and the initial data (v_0, ψ_0) is arbitrarily chosen.

To study problem (1.3)–(1.5) and problem (1.6), we assume the following conditions:
(H_1) the function $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies that there exists a positive constant C_1 such that

$$|f'(r) - f'(s)| \le C_1|r-s|(|r|^{p-3} + |s|^{p-3} + 1) \tag{1.7}$$

for any $r, s \in \mathbb{R}$ and

$$c_1|r|^p - k_1 \le f(r)r \le c_2|r|^p + k_1, \tag{1.8}$$

where $c_i > 0$ $(i = 1, 2)$, $p \ge 3$, $k_1 > 0$.

(H_2) the function $g \in C(\mathbb{R}, \mathbb{R})$ satisfies that there exists a positive constant C_2 such that

$$|g(r) - g(s)| \leq C_2|r - s|(|r|^{q-2} + |s|^{q-2} + 1) \tag{1.9}$$

for any $r, r \in \mathbb{R}$ and

$$c_3|r|^q - k_2 \leq g(r)r \leq c_4|r|^q + k_2, \tag{1.10}$$

where $c_i > 0$ $(i = 3, 4)$, $q > 2$, $k_2 > 0$.

The observational measurement operator I_h is given by a linear interpolant operator $I_h : H^1(\Omega) \to L^2(\Omega)$ or $I_h : V \to H$ satisfying the approximation property

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)} \leq \gamma_0 h \|\varphi\|_{H^1(\Omega)} \tag{1.11}$$

for every $\varphi \in H^1(\Omega)$ or $\varphi \in V$, where $\gamma_0 > 0$ is a dimensionless constant. One example of an interpolant observable that satisfies (1.11) is the orthogonal projection onto the low Fourier modes with wave numbers k such that $|k| \leq \frac{1}{h}$.

In this paper, we will establish some convergence results about the data assimilation algorithm to the two dimensional Cahn–Hilliard–Navier–Stokes system in the absence of measurement errors. The paper is organized as follows: first, we recall the functional setting of the two dimensional Cahn–Hilliard–Navier–Stokes system and give a lemma needed to study our continuous data assimilation. Subsequently, in Sect. 3, we establish some regularity results about problem (1.3). Finally, in Sect. 4, we give some suitable conditions on β, χ and h under which the approximate solutions obtained by this algorithm will converge to the reference solution of the two dimensional Cahn–Hilliard–Navier–Stokes system.

2 Preliminaries

For the sake of completeness, this section presents some preliminary material and notations commonly used in the mathematical study of fluids, in particular in the study of the Navier–Stokes equations and the Euler equations. Define

$$\mathcal{V} = \{u \in (\mathcal{C}_c^\infty(\Omega))^2 : \nabla \cdot u = 0\}.$$

Denote by H and V the closure of $\mathcal{V}$ with respect to the norms in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Denote by X^* the dual space of X and let P be the Leray orthogonal projector from $L^2(\Omega)$ onto H, and define the Stokes operator $A_1 : V \to V^*$ by $A_1 u = -P\Delta u$ with domain $\mathcal{D}(A_1) = V \cap H^2(\Omega)$. The linear operator A_1 is self-adjoint and positive definite with compact inverse $A_1^{-1} : H \to H$. Thus, there exists a complete orthonormal set of eigenfunctions ω_i in H such that $A_1\omega_i = \lambda_i\omega_i$ for any $i \in \mathbb{N}$, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \lambda_{i+1} \leq \cdots \to +\infty$. Let A_2 be the Laplacian operator with homogeneous Neumann boundary conditions.

In order to define the variational setting for the two dimensional Cahn–Hilliard–Navier–Stokes system, we also need to introduce the bilinear operators B_1, B_2 (and their related trilinear forms b_1 and b_2) as well as the coupling mapping R which are defined, from $V \times V$ into V^*, $V \times H^1(\Omega)$ into $(H^1(\Omega))^*$ and $L^2(\Omega) \times H^2(\Omega)$ into V^*, respectively. More precisely, we set

$$
\begin{aligned}
\langle B_1(u,v),w\rangle &= \int_\Omega [(u(x)\cdot\nabla)v(x)]\cdot w(x)\,dx = b_1(u,v,w), \quad \forall\ u,\ v,\ w\in V,\\
\langle B_2(u,\phi),\psi\rangle &= \int_\Omega (u(x)\cdot\nabla\phi(x))\psi(x)\,dx\\
&= b_2(u,\phi,\psi), \quad \forall\ u\in V,\ \phi,\ \psi\in H^1(\Omega),\\
\langle R(\mu,\phi),u\rangle &= \int_\Omega (u(x)\cdot\nabla\phi(x))\mu(x)\,dx\\
&= b_2(u,\phi,\mu), \quad \forall\ u\in V,\ \phi\in H^2(\Omega),\ \mu\in L^2(\Omega).
\end{aligned}
$$

Based on the above notations, we can reformulate problem (1.3)–(1.5) and problem (1.6) into the functional form

$$
\begin{cases}
\frac{\partial u}{\partial t} + \nu A_1 u + B_1(u,u) + \lambda R(\mu,\phi) = k(x),\\
\frac{\partial \phi}{\partial t} + B_2(u,\phi) + \gamma A_2\mu = 0,\\
\mu = A_2\phi + f(\phi),\\
u(0) = u_0,\\
\phi(0) = \phi_0
\end{cases}
\tag{2.1}
$$

and

$$
\begin{cases}
\frac{\partial v}{\partial t} + \nu A_1 v + B_1(v,v) + \lambda R(\eta,\psi) = k(x) - \beta P\left(I_h(v) - I_h(u)\right),\\
\frac{\partial \psi}{\partial t} + B_2(v,\psi) + \gamma A_2\eta = -\chi\left(I_h(\psi) - I_h(\phi)\right),\\
\eta = A_2\psi + f(\psi),\\
v(0) = v_0,\\
\psi(0) = \psi_0,
\end{cases}
\tag{2.2}
$$

where $k = Pg$. Without loss of generality, we assume that $g\in H$ such that $k = Pg = g$. Furthermore, inequalities (1.11) implies that

$$
\|w - I_h(w)\|_{L^2(\Omega)} \le c_0 h\|w\|_{H^1(\Omega)} \tag{2.3}
$$

for every $w\in H^1(\Omega)$ or $w\in V$, where $c_0 = \gamma_0$.

Remark 2.1 We will use the same notation indiscriminately for both scalar and vector Lebesgue and Sobolev spaces, which should not be a source of confusion.

In what follows, we recall a lemma used in the sequel.

Lemma 2.2 [31] *Assume that A, B, D and β are positive constants. Let $\alpha(t)$ be a measurable nonnegative function satisfying*

$$
\int_0^t \alpha(s)\,ds \le At + B
$$

for all $t\ge 0$. Suppose that $Y(t)$ is a nonnegative and absolutely continuous function that satisfies the following inequality

$$
\begin{cases}
\frac{dY(t)}{dt} \le (D-\beta)Y(t) + \alpha(t)Y(t),\\
Y(0) = Y_0.
\end{cases}
$$

Let β be such that

$$\beta > \max\left\{4A, \frac{8}{3}De^B\right\}.$$

Then, we have

(1) $Y(t) \leq 2e^B Y_0$ *for all* $t \geq 0$.
(2) $Y(t) \to 0$ *at an exponential rate as* $t \to +\infty$.

In what follows, we give the definition of solutions to problem (2.1).

Definition 2.3 Assume that $g \in H$ and (H_1)-(H_2) hold. For any $(u_0, \phi_0) \in H \times H^1(\Omega)$ and any fixed $T > 0$, a function (u, ϕ) is called a weak solution of problem (2.1) on $(0, T)$, if

$$\mu \in L^2(0, T; H^1(\Omega)) \text{ is given by the third equation of problem (2.1)}$$

and

$$\begin{aligned}
&\phi \in \mathcal{C}([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),\\
&u \in \mathcal{C}([0, T]; H) \cap L^2(0, T; V),\\
&(u_t, \phi_t) \in L^2(0, T; V^* \times (H^1(\Omega))^*),
\end{aligned}$$

which satisfies the following equalities

$$\begin{aligned}
&\int_\Omega (u_t \cdot v + \nu \nabla u \cdot \nabla v)\, dx + b_1(u, u, v) + \lambda b_2(v, \mu, \phi) = \int_\Omega g \cdot v\, dx,\\
&\int_\Omega \phi_t \psi\, dx + b_2(u, \phi, \psi) + \gamma \int_\Omega \nabla\mu \cdot \nabla\psi\, dx = 0,\\
&\int_\Omega (\nabla\phi \cdot \nabla\theta + f(\phi)\theta)\, dx = \int_\Omega \mu\theta\, dx
\end{aligned}$$

for all test functions $v \in V$ and $\psi, \theta \in W = \{w \in H^1(\Omega) : \frac{\partial w}{\partial \mathbf{n}} = 0\}$.

We recall the following well-posedness and the existence of a finite dimensional global attractor for problem (2.1) in [50].

Theorem 2.4 (Well-posedness) *Assume that $g \in H$ and (H_1)-(H_2) hold. Then for any $u_0 \in H$ and $\phi_0 \in H^1(\Omega)$, there exists a unique weak solution $(u(t), \phi(t))$ for problem (2.1) such that $m\phi(t) = m\phi_0$, which depends continuously on the initial data (u_0, ϕ_0) with respect to the norm in $H \times H^1(\Omega)$.*

Theorem 2.5 (Existence of global attractor) *Assume that $g \in H$ and (H_1)-(H_2) are in force. Then problem (2.1) possesses a finite dimensional global attractor $\mathcal{A}$ in $H \times V_I$, where $V_I = \{\phi \in H^1(\Omega) : \bar{\phi} = \frac{1}{|\Omega|}\int_\Omega \phi(x)\, dx = I\}$ for every fixed $I \in \mathbb{R}$.*

Remark 2.6 In this work, we will assume that the reference solution of problem (2.1), that we are trying to approximate, has evolved enough in time to satisfy the estimate provided in the following Theorem. That is, we will assume that the solution satisfies this estimate at $t = 0$.

3 Regularity Results

Theorem 3.1 *Assume that $g \in H$ and (H_1)–(H_2) hold. Then for any $u_0 \in H$ and $\phi_0 \in H^1(\Omega)$, the solution of problem (2.1) satisfies*

$$\|u(t)\|^2_{L^2(\Omega)} + \|\phi(t)\|^2_{H^1(\Omega)} \leq K^2$$

for any $t \geq 0$, where K is given in (3.7).

Proof Multiplying the first equation, the second equation and the third equation of Eq. (2.1) by u, $\lambda\mu$ and ϕ, respectively, and integrating by parts and adding the resulting equalities, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\phi(t)\|^2_{H^1(\Omega)} + \lambda\int_\Omega F(\phi(t))\,dx\right)\\
&\quad + \lambda\gamma\|\nabla\mu(t)\|^2_{L^2(\Omega)} + \nu\|\nabla u(t)\|^2_{L^2(\Omega)}\\
&\quad + \|\phi(t)\|^2_{H^1(\Omega)} + \int_\Omega f(\phi(t))\phi(t)\,dx\\
&= \int_\Omega \mu(x,t)\phi(x,t)\,dx + \int_\Omega g(x)\cdot u(x,t)\,dx\\
&\leq \|\mu(t) - \bar{\mu}(t)\|_{L^2(\Omega)}\|\phi(t)\|_{L^2(\Omega)} + \bar{\mu}\overline{\phi_0}|\Omega| + \|g\|_{L^2(\Omega)}\|u(t)\|_{L^2(\Omega)}\\
&\leq C\|\nabla\mu(t)\|_{L^2(\Omega)}\|\phi(t)\|_{L^2(\Omega)} + |\bar{\mu}||\overline{\phi_0}||\Omega| + \|g\|_{L^2(\Omega)}\|u(t)\|_{L^2(\Omega)},
\end{aligned} \tag{3.1}$$

where $F(s) = \int_0^s f(r)\,dr$ is the primitive function of f.

Taking the $L^2(\Omega)$ inner product of the third equation of Eq. (2.1) with the constant 1, we find

$$\left|\int_\Omega \mu(x,t)\,dx\right| \leq \|f(\phi(t))\|_{L^1(\Omega)}. \tag{3.2}$$

Combining assumptions (1.7)–(1.10) and inequalities (3.1)–(3.2) with Hölder's inequality and Young's inequality, we find that there exist two positive constants δ and ϱ such that

$$\frac{d}{dt}J(u(t),\phi(t)) + \lambda\gamma\|\nabla\mu(t)\|^2_{L^2(\Omega)} + \nu\|\nabla u(t)\|^2_{L^2(\Omega)} + \delta J(u(t),\phi(t)) \leq \varrho, \tag{3.3}$$

where

$$J(u,\phi) = \|u\|^2_{L^2(\Omega)} + \lambda\|\phi\|^2_{H^1(\Omega)} + 2\lambda\int_\Omega F(\phi)\,dx.$$

From the classical Gronwall inequality, we infer

$$\begin{aligned}
J(u(t),\phi(t)) &\leq e^{-\delta t}J(u(0),\phi(0)) + \frac{\varrho}{\delta}\\
&\leq e^{-\delta t}J(u_0,\phi_0) + \frac{\varrho}{\delta}.
\end{aligned} \tag{3.4}$$

From assumptions (1.7)–(1.10), we deduce that there exist four positive constants δ_1, δ_2, l_1 and l_2 such that

$$\begin{aligned}\delta_1\left(\|u\|^2_{L^2(\Omega)}+\lambda\|\phi\|^2_{H^1(\Omega)}+\|\phi\|^p_{L^p(\Omega)}\right)-l_1&\le J(u,\phi)\\ &\le\delta_2\left(\|u\|^2_{L^2(\Omega)}+\lambda\|\phi\|^2_{H^1(\Omega)}+\|\phi\|^p_{L^p(\Omega)}\right)+l_2.\end{aligned}\tag{3.5}$$

By virtue of inequalities (3.4)–(3.5), we obtain

$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega)}+\lambda\|\phi(t)\|^2_{H^1(\Omega)}+\|\phi(t)\|^p_{L^p(\Omega)}\\ &\quad\le\frac{1}{\delta_1}e^{-\delta t}J(u_0,\phi_0)+\frac{\varrho}{\delta_1\delta}+\frac{l_1}{\delta_1}.\end{aligned}\tag{3.6}$$

Thus, inequality (3.6) shows that

$$\|u(t)\|^2_{L^2(\Omega)}+\lambda\|\phi(t)\|^2_{H^1(\Omega)}+\|\phi(t)\|^p_{L^p(\Omega)}\le K^2$$

for any $t\ge 0$ with

$$K^2=\frac{1}{\delta_1}J(u_0,\phi_0)+\frac{\varrho}{\delta_1\delta}+\frac{l_1}{\delta_1}.\tag{3.7}$$

Integrating inequality (3.3) from 0 to t, we obtain

$$\begin{aligned}&\lambda\gamma\int_0^t\|\nabla\mu(s)\|^2_{L^2(\Omega)}\,ds+\nu\int_0^t\|\nabla u(s)\|^2_{L^2(\Omega)}\,ds\\ &\quad\le(\varrho+l_1\delta)t+J(u_0,\phi_0)+l_1.\end{aligned}\tag{3.8}$$

We infer from inequalities (1.7)–(1.10), (3.2), (3.6), Sobolev embedding Theorem and the regularity theory of second order elliptic equation that

$$\begin{aligned}&\|\mu(t)\|_{L^2(\Omega)}+\|\phi(t)\|_{H^2(\Omega)}\\ &\quad\le C\left(\|\mu(t)-f(\phi(t))+\phi(t)\|_{L^2(\Omega)}+\|\mu(t)\|_{L^2(\Omega)}\right)\\ &\quad\le C(\|\phi(t)\|_{L^2(\Omega)}+\|f(\phi(t))\|_{L^2(\Omega)}+\|\mu(t)\|_{L^2(\Omega)})\\ &\quad\le C(\|\phi(t)\|_{L^2(\Omega)}+\|f(\phi(t))\|_{L^2(\Omega)}+\|\nabla\mu(t)\|_{L^2(\Omega)})\\ &\quad\le C(1+\|\phi(t)\|^{p-1}_{H^1(\Omega)}+\|\nabla\mu(t)\|_{L^2(\Omega)}).\end{aligned}\tag{3.9}$$

It follows from inequalities (3.8)–(3.9) that

$$\begin{aligned}&\int_0^t(\|\mu(r)\|^2_{L^2(\Omega)}+\|\phi(r)\|^2_{H^2(\Omega)})\,dr\\ &\quad\le C\left(1+K^{2(p-1)}+\varrho+l_1\delta\right)t+C(J(u_0,\phi_0)+l_1).\end{aligned}\tag{3.10}$$

Inequality (3.8) and inequality (3.10) show that there exists a dimensional positive constant K_1, such that for any $t \geq 0$,

$$\begin{aligned}\lambda\gamma \int_0^t \|\mu(s)\|^2_{H^1(\Omega)}\,ds + \nu \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \|\phi(r)\|^2_{H^2(\Omega)}\,dr \\ \leq K_1 \left(1 + K^{2(p-1)} + \varrho + l_1\delta\right) t + K_1(J(u_0,\phi_0) + l_1). \qquad (3.11)\end{aligned}$$

□

4 Convergence Result

In this section, we will derive some conditions on β, χ and h under which the approximate solution (v, ψ) of the data assimilation system (2.2) converges to the reference solution (u, ϕ) of problem (2.1) as $t \to +\infty$ when the observables operator satisfy (2.3).

Theorem 4.1 *Assume that $g \in H$ and (H_1)–(H_2) hold. If I_h satisfies the approximation property (2.3) and $(u(x,t), \phi(x,t))$ is a weak solution in the global attractor $\mathcal{A}$ of problem (2.1), $\beta > 0$ and $\chi > 0$ are arbitrary and let $h \ll 1$ be chosen such that $\beta c_0^2 h^2 \leq \nu$ and $c_0^4 h^4 \leq \min\left\{\frac{\gamma}{54\chi}, \frac{1}{16}\right\}$. Then for any $(v_0, \psi_0) \in H \times H^1(\Omega)$, problem (2.2) has a unique weak solution (v, ψ) in the sense of Definition 2.3, which depends continuously on the initial data in the $H \times H^1(\Omega)$-norm.*

Let $\alpha > 0$ be arbitrary, but fixed, such that $\alpha > K$, where the positive constant K is given in Theorem 3.1. If (u_0, ϕ_0), $(v_0, \psi_0) \in H \times H^1(\Omega)$ such that

$$\begin{aligned}\|u_0\|^2_{L^2(\Omega)} + \lambda\|\phi_0\|^2_{H^1(\Omega)} &\leq \alpha^2,\\ \|v_0\|^2_{L^2(\Omega)} + \lambda\|\psi_0\|^2_{H^1(\Omega)} &\leq \alpha^2\end{aligned}$$

and we choose β, $\chi > 0$ large enough such that

$$\min\{\beta,\chi\} > \max\left\{4K_2A, \frac{8}{3}K_2(1 + K^4 + K^{4(p-2)} + e^{4K_2B} + e^{4(p-2)K_2B})e^{K_2B}\right\}.$$

Then

$$\|u(t) - v(t)\|^2_{L^2(\Omega)} + \lambda\|\phi(t) - \psi(t)\|^2_{H^1(\Omega)} \to 0$$

at an exponential rate as $t \to +\infty$.

Proof The existence of the weak solution (v, ψ) of problem (2.2) is obtained by using the Galerkin method and the Aubin compactness theorem (see [53]). Here, we omit its proof.

Define $w = u - v$, $\theta = \mu - \eta$ and $\varphi = \phi - \psi$. From assumption, we know that (u, ϕ) is a weak solution in the global attractor $\mathcal{A}$ of problem (2.1), which satisfies the global estimates in Theorem 3.1, then showing the global existence, in time, of the solution $(w(t), \varphi(t))$ is equivalent to showing the global existence, in time, of the solution $(v(t), \psi(t))$ of system (2.2). To be concise here, we will show the global existence of the solution $(w(t), \varphi(t))$ and that $\|w(t)\|^2_{L^2(\Omega)} + \|\varphi(t)\|^2_{H^1(\Omega)}$ decays exponentially, in time, which will imply that the convergence of the approximate solution $(v(t), \psi(t))$ to the exact solution $(u(t), \phi(t))$, exponentially in time.

It is clear that (w, φ) satisfies the following problem

$$\begin{cases} \frac{\partial w}{\partial t} + \nu A_1 w + B_1(w, u) + B_1(v, w) = -\lambda R(\theta, \phi) - \lambda R(\eta, \varphi) - \beta P I_h(w), \\ \frac{\partial \varphi}{\partial t} + B_2(w, \phi) + B_2(v, \varphi) + \gamma A_2 \theta = -\beta I_h(\varphi), \\ \theta = \mu - \eta = A_2 \varphi + f(\phi) - f(\psi). \end{cases} \tag{4.1}$$

Equation (4.1) is subject to the following initial conditions

$$\begin{cases} w(0) = u_0 - v_0, \\ \varphi(0) = \phi_0 - \psi_0. \end{cases} \tag{4.2}$$

Next, we will prove some formal a priori estimates that are essential in proving the global existence of solutions for problem (4.1)–(4.2). These estimates can be justified rigorously by using the Galerkin method and the Aubin compactness theorem.

Multiplying the first equation of Eq. (4.1) by w and integrating by parts, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \nu\|\nabla w(t)\|^2_{L^2(\Omega)} \\ &= -b_1(w, u, w) - \lambda b_2(w, \phi, \theta) - \lambda b_2(w, \varphi, \eta) - \beta \int_\Omega I_h(w(x, t)) \cdot w(x, t)\, dx \\ &\leq -b_1(w, u, w) + \lambda b_2(w, \theta, \phi) - \lambda b_2(w, \theta, \varphi) + \lambda b_2(w, \mu, \varphi) \\ &\quad + \beta\|w(t) - I_h(w(t))\|_{L^2(\Omega)}\|w(t)\|_{L^2(\Omega)} - \beta\|w(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.3}$$

Taking the $L^2(\Omega)$ inner product of the second equation of Eq. (4.1) with $-\lambda\Delta\varphi + \lambda\varphi$, we obtain

$$\begin{aligned} &\lambda\frac{d}{dt}\|\varphi(t)\|^2_{H^1(\Omega)} + \lambda\gamma\|\nabla\Delta\varphi(t)\|^2_{L^2(\Omega)} + 2\lambda\gamma\|\Delta\varphi(t)\|^2_{L^2(\Omega)} + \lambda\gamma\|\nabla\theta(t)\|^2_{L^2(\Omega)} \\ &= \lambda\gamma\int_\Omega \nabla(f(\phi) - f(\psi)) \cdot (\nabla\theta + \nabla\Delta\varphi)\, dx - 2\lambda b_2(w, \phi, \varphi) + 2\lambda b_2(w, \phi, \Delta\varphi) \\ &\quad + 2\lambda b_2(u, \varphi, \Delta\varphi) - 2\lambda b_2(w, \varphi, \Delta\varphi) + 2\lambda\gamma\int_\Omega (f(\phi) - f(\psi))\Delta\varphi\, dx \\ &\quad - 2\chi\int_\Omega I_h(\varphi(x, t))(-\lambda\Delta\varphi + \lambda\varphi)\, dx. \end{aligned} \tag{4.4}$$

In what follows, we will estimate each term of the right hand side of inequalities (4.3)–(4.4) by Hölder's inequality.

$$\begin{aligned} |b_1(w, u, w)| &\leq \|w\|^2_{L^4(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\ &\leq C\|w\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} |\lambda b_2(w, \theta, \phi)| &\leq \|w\|_{L^4(\Omega)}\|\nabla\theta\|_{L^2(\Omega)}\|\phi\|_{L^4(\Omega)} \\ &\leq C\|w\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla w\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla\theta\|_{L^2(\Omega)}\|\phi\|_{H^1(\Omega)}, \end{aligned} \tag{4.6}$$

$$|\lambda b_2(w, \theta, \varphi)| \leq \|w\|_{L^4(\Omega)}\|\nabla\theta\|_{L^2(\Omega)}\|\varphi\|_{L^4(\Omega)}$$

$$\leq C\|w\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla w\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\theta\|_{L^2(\Omega)}\|\varphi\|_{H^1(\Omega)}, \tag{4.7}$$

$$\begin{aligned}|\lambda b_2(w,\mu,\varphi)| &\leq \|w\|_{L^4(\Omega)}\|\nabla\mu\|_{L^2(\Omega)}\|\varphi\|_{L^4(\Omega)}\\ &\leq C\|w\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla w\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\mu\|_{L^2(\Omega)}\|\varphi\|_{H^1(\Omega)}.\end{aligned} \tag{4.8}$$

With the help of Hölder's inequality and the Sobolev embedding inequality $\|u\|_{L^p(\Omega)} \leq C\|u\|_{H^1(\Omega)}$ for any $1 \leq p < +\infty$ and any $u \in H^1(\Omega)$ as well as the following Gagliardo–Nirenberg inequality:

$$\|\nabla\varphi\|_{L^4(\Omega)} \leq C\|\Delta\varphi\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\varphi\|_{L^2(\Omega)}^{\frac{1}{2}} + C\|\nabla\varphi\|_{L^2(\Omega)},$$

we obtain

$$\begin{aligned}
&\lambda\gamma\left|\int_\Omega \nabla(f(\phi)-f(\psi))\cdot(\nabla\theta+\nabla\Delta\varphi)\,dx\right|\\
&\leq \lambda\gamma\left|\int_\Omega (f'(\phi)-f'(\psi))\nabla\phi\cdot(\nabla\theta+\nabla\Delta\varphi)\,dx\right|\\
&\quad+\lambda\gamma\left|\int_\Omega f'(\psi)\nabla\varphi\cdot(\nabla\theta+\nabla\Delta\varphi)\,dx\right|\\
&\leq C\|\nabla\phi\|_{L^4(\Omega)}(1+\|\phi\|_{L^{8(p-3)}(\Omega)}^{p-3}+\|\psi\|_{L^{8(p-3)}(\Omega)}^{p-3})\|\varphi\|_{L^8(\Omega)}(\|\nabla\theta\|_{L^2(\Omega)}+\|\nabla\Delta\varphi\|_{L^2(\Omega)})\\
&\quad+C(1+\|\psi\|_{L^{4(p-2)}(\Omega)}^{p-2})\|\nabla\varphi\|_{L^4(\Omega)}(\|\nabla\theta\|_{L^2(\Omega)}+\|\nabla\Delta\varphi\|_{L^2(\Omega)})\\
&\leq C\|\nabla\phi\|_{L^4(\Omega)}(1+\|\phi\|_{H^1(\Omega)}^{p-3}+\|\psi\|_{H^1(\Omega)}^{p-3})\|\varphi\|_{H^1(\Omega)}(\|\nabla\theta\|_{L^2(\Omega)}+\|\nabla\Delta\varphi\|_{L^2(\Omega)})\\
&\quad+C(1+\|\psi\|_{H^1(\Omega)}^{p-2})\|\nabla\varphi\|_{L^4(\Omega)}(\|\nabla\theta\|_{L^2(\Omega)}+\|\nabla\Delta\varphi\|_{L^2(\Omega)})\\
&\leq C\|\nabla\phi\|_{L^4(\Omega)}(1+\|\phi\|_{H^1(\Omega)}^{p-3}+\|\psi\|_{H^1(\Omega)}^{p-3})\|\varphi\|_{H^1(\Omega)}(\|\nabla\theta\|_{L^2(\Omega)}+\|\nabla\Delta\varphi\|_{L^2(\Omega)})\\
&\quad+C(1+\|\psi\|_{H^1(\Omega)}^{p-2})(\|\nabla\varphi\|_{L^2(\Omega)}+\|\nabla\varphi\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\Delta\varphi\|_{L^2(\Omega)}^{\frac{1}{2}})(\|\nabla\theta\|_{L^2(\Omega)}+\|\nabla\Delta\varphi\|_{L^2(\Omega)})\\
&\leq C\|\nabla\phi\|_{L^4(\Omega)}(1+\|\phi\|_{H^1(\Omega)}^{p-3}+\|\psi\|_{H^1(\Omega)}^{p-3})\|\varphi\|_{H^1(\Omega)}(\|\nabla\theta\|_{L^2(\Omega)}+\|\nabla\Delta\varphi\|_{L^2(\Omega)})\\
&\quad+C(1+\|\psi\|_{H^1(\Omega)}^{p-2})(\|\varphi\|_{H^1(\Omega)}+\|\varphi\|_{H^1(\Omega)}^{\frac{1}{2}}\|\nabla\Delta\varphi\|_{L^2(\Omega)}^{\frac{1}{2}})(\|\nabla\theta\|_{L^2(\Omega)}+\|\nabla\Delta\varphi\|_{L^2(\Omega)}),
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&2\lambda\gamma\left|\int_\Omega (f(\phi)-f(\psi))\Delta\varphi\,dx\right|\\
&\leq 2\lambda\gamma\|f(\phi)-f(\psi)\|_{L^2(\Omega)}\|\Delta\varphi\|_{L^2(\Omega)}\\
&\leq C(1+\|\phi\|_{L^{4(p-2)}(\Omega)}^{p-2}+\|\psi\|_{L^{4(p-2)}(\Omega)}^{p-2})\|\varphi\|_{L^4(\Omega)}\|\Delta\varphi\|_{L^2(\Omega)}\\
&\leq C(1+\|\phi\|_{H^1(\Omega)}^{p-2}+\|\psi\|_{H^1(\Omega)}^{p-2})\|\varphi\|_{H^1(\Omega)}\|\Delta\varphi\|_{L^2(\Omega)},
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&2\lambda|b_2(w,\phi,\Delta\varphi)| \leq 2\lambda\|w\|_{L^4(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)}\|\phi\|_{L^4(\Omega)}\\
&\leq C\|w\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla w\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\Delta\varphi\|_{L^2(\Omega)}\|\phi\|_{H^1(\Omega)},
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&2\lambda|b_2(u,\varphi,\Delta\varphi)| \leq 2\lambda\|u\|_{L^4(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)}\|\varphi\|_{L^4(\Omega)}\\
&\leq C\|\nabla u\|_{L^2(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)}\|\varphi\|_{H^1(\Omega)},
\end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&2\lambda|b_2(w,\varphi,\Delta\varphi)| \le 2\lambda\|w\|_{L^4(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)}\|\varphi\|_{L^4(\Omega)}\\
&\le C\|w\|^{\frac12}_{L^2(\Omega)}\|\nabla w\|^{\frac12}_{L^2(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)}\|\varphi\|_{H^1(\Omega)}, \qquad (4.13)\\
&2\lambda|b_2(w,\phi,\varphi)| \le 2\lambda\|w\|_{L^4(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\phi\|_{L^4(\Omega)}\\
&\le C\|w\|^{\frac12}_{L^2(\Omega)}\|\nabla w\|^{\frac12}_{L^2(\Omega)}\|\Delta\varphi\|^{\frac12}_{L^2(\Omega)}\|\varphi\|^{\frac12}_{H^1(\Omega)}\|\phi\|_{H^1(\Omega)}, \qquad (4.14)\\
&-2\chi\lambda\int_\Omega I_h(\varphi(x,t))(-\Delta\varphi+\varphi)\,dx\\
&=2\chi\lambda\int_\Omega(\varphi(x,t)-I_h(\varphi(x,t)))(-\Delta\varphi+\varphi)\,dx-2\chi\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\\
&\le 2\chi\lambda\|\varphi(t)-I_h(\varphi(t))\|_{L^2(\Omega)}\|-\Delta\varphi+\varphi\|_{L^2(\Omega)}-2\chi\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\\
&\le 2\chi\lambda c_0 h\|\nabla\varphi(t)\|_{L^2(\Omega)}(\|\nabla\Delta\varphi\|^{\frac12}_{L^2(\Omega)}\|\nabla\varphi\|^{\frac12}_{L^2(\Omega)}+\|\varphi\|_{L^2(\Omega)})-2\chi\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\\
&\le \frac{\lambda\gamma}{4}\|\nabla\Delta\varphi(t)\|^2_{L^2(\Omega)}+\frac34(2\chi\lambda c_0h)^{\frac43}(\lambda\gamma)^{-\frac13}\|\nabla\varphi(t)\|^2_{L^2(\Omega)}-\frac32\chi\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\\
&\le \frac{\lambda\gamma}{4}\|\nabla\Delta\varphi(t)\|^2_{L^2(\Omega)}+\frac34(2\chi\lambda c_0h)^{\frac43}(\lambda\gamma)^{-\frac13}\|\nabla\varphi(t)\|^2_{L^2(\Omega)}-\frac32\chi\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\\
&\le \frac{\lambda\gamma}{4}\|\nabla\Delta\varphi(t)\|^2_{L^2(\Omega)}-\chi\lambda\|\varphi(t)\|^2_{H^1(\Omega)}. \qquad (4.15)
\end{aligned}
$$

We infer from inequalities (4.3)–(4.15) and Young's inequality that there exists a positive dimensionless constant K_2, such that on the time interval $[0,\tilde T)$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\|w(t)\|^2_{L^2(\Omega)}+\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\right)+\frac{\lambda\gamma}{2}\|\nabla\Delta\varphi(t)\|^2_{L^2(\Omega)}+\frac{\lambda\gamma}{2}\|\nabla\theta(t)\|^2_{L^2(\Omega)}\\
&\quad+\lambda\gamma\|\Delta\varphi(t)\|^2_{L^2(\Omega)}+\frac{\nu}{2}\|\nabla w(t)\|^2_{L^2(\Omega)}+\beta\|w(t)\|^2_{L^2(\Omega)}+\chi\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\\
&\le K_2(\nu\|\nabla u\|^2_{L^2(\Omega)}+\|\phi\|^4_{H^1(\Omega)}+\|\psi\|^4_{H^1(\Omega)}+\lambda\gamma\|\nabla\mu\|^2_{L^2(\Omega)})\|w\|^2_{L^2(\Omega)}\\
&\quad+K_2\lambda(1+\lambda\gamma\|\nabla\mu\|^2_{L^2(\Omega)}+\nu\|\nabla u\|^2_{L^2(\Omega)}+\|\psi\|^{4(p-2)}_{H^1(\Omega)}+\|\phi\|^{4(p-2)}_{H^1(\Omega)})\|\varphi\|^2_{H^1(\Omega)}\\
&\quad+K_2\lambda\|\phi\|^2_{H^2(\Omega)}\|\varphi\|^2_{H^1(\Omega)}. \qquad (4.16)
\end{aligned}
$$

Denote by

$$
\begin{aligned}
\alpha(t)&=\min\{\beta,\chi\}-K_2(1+\|\phi\|^4_{H^1(\Omega)}+\|\psi\|^4_{H^1(\Omega)}+\|\psi\|^{4(p-2)}_{H^1(\Omega)}+\|\phi\|^{4(p-2)}_{H^1(\Omega)}),\\
\gamma(t)&=K_2\left(\lambda\gamma\|\nabla\mu\|^2_{L^2(\Omega)}+\nu\|\nabla u\|^2_{L^2(\Omega)}+\|\phi\|^2_{H^2(\Omega)}\right),
\end{aligned}
$$

it follows from inequality (4.16) that on the time interval $[0,\tilde T)$,

$$\begin{aligned}
&\frac{d}{dt}\left(\|w(t)\|^2_{L^2(\Omega)}+\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\right)+\frac{\lambda\gamma}{2}\|\nabla\Delta\varphi(t)\|^2_{L^2(\Omega)}+\frac{\lambda\gamma}{2}\|\nabla\theta(t)\|^2_{L^2(\Omega)}\\
&\quad+\lambda\gamma\|\Delta\varphi(t)\|^2_{L^2(\Omega)}+\frac{\nu}{2}\|\nabla w(t)\|^2_{L^2(\Omega)}\\
&\quad\le(\gamma(t)-\alpha(t))\left(\|w(t)\|^2_{L^2(\Omega)}+\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\right).
\end{aligned}\tag{4.17}$$

Denote by

$$\begin{aligned}
A=&K_1\left(1+K^{2(p-1)}+\varrho+l_1\delta\right),\\
B=&K_1(J(u_0,\phi_0)+l_1),
\end{aligned}$$

it follows from inequality (3.11) and Theorem 3.1 that for any $t\ge 0$,

$$\|u(t)\|^2_{L^2(\Omega)}+\lambda\|\phi(t)\|^2_{H^1(\Omega)}\le\alpha^2\tag{4.18}$$

and

$$\lambda\gamma\int_0^t\|\mu(r)\|^2_{H^1(\Omega)}\,dr+\nu\int_0^t\|\nabla u(r)\|^2_{L^2(\Omega)}\,dr+\int_0^t\|\phi(r)\|^2_{H^2(\Omega)}\,dr\le At+B.\tag{4.19}$$

Since $\|v_0\|^2_{L^2(\Omega)}+\lambda\|\psi_0\|^2_{H^1(\Omega)}\le\alpha^2$, it follows from the continuity of $\|v(t)\|^2_{L^2(\Omega)}+\lambda\|\psi(t)\|^2_{H^1(\Omega)}$ that there exists a short time interval $[0,\tilde T)$ such that

$$\|v(t)\|^2_{L^2(\Omega)}+\lambda\|\psi(t)\|^2_{H^1(\Omega)}\le 12e^{K_2B}\alpha^2\tag{4.20}$$

for any $t\in[0,\tilde T)$.

To begin with, we assume that $[0,\tilde T)$ is the maximal interval such that inequality (4.20) holds, then we will show that $\tilde T=+\infty$ by contradiction. Thus, we assume that $\tilde T<+\infty$, then it is clear that

$$\limsup_{t\to\tilde T}(\|v(t)\|^2_{L^2(\Omega)}+\lambda\|\psi(t)\|^2_{H^1(\Omega)})=12e^{K_2B}\alpha^2.$$

Thus, if

$$\min\{\beta,\chi\}>\max\left\{4K_2A,\frac{8}{3}K_2(1+K^4+K^{4(p-2)}+e^{4K_2B}+e^{4(p-2)K_2B})e^{K_2B}\right\},\tag{4.21}$$

we infer from Lemma 2.2 that

$$\|w(t)\|^2_{L^2(\Omega)}+\lambda\|\varphi(t)\|^2_{H^1(\Omega)}\le 2e^{K_2B}\left(\|w_0\|^2_{L^2(\Omega)}+\lambda\|\varphi_0\|^2_{H^1(\Omega)}\right)$$

for any $t\in[0,\tilde T]$. Therefore, we obtain

$$\|v(t)\|^2_{L^2(\Omega)}+\lambda\|\psi(t)\|^2_{H^1(\Omega)}$$

$$
\begin{aligned}
&\leq 2\left(\|w(t)\|^2_{L^2(\Omega)} + \lambda\|\varphi(t)\|^2_{H^1(\Omega)}\right) + 2\left(\|u(t)\|^2_{L^2(\Omega)} + \lambda\|\phi(t)\|^2_{H^1(\Omega)}\right)\\
&\leq 4e^{K_2 B}\|\zeta_0\|^2 + 2\alpha^2\\
&\leq 8e^{K_2 B}\alpha^2 + 2\alpha^2\\
&\leq 10e^{K_2 B}\alpha^2
\end{aligned}
$$

for any $t \in [0, \tilde{T}]$, which contradicts with the definition of $\tilde{T}$. Thus, we deduce that $\tilde{T} = +\infty$. By virtue of Lemma 2.2, we obtain

$$
\|u(t) - v(t)\|^2_{L^2(\Omega)} + \|\phi(t) - \psi(t)\|^2_{H^1(\Omega)} \to 0
$$

at an exponential rate as $t \to +\infty$. □

5 Numerical Test

In this section, we consider the shape relaxation with the binary phases flow as shown in Fig. 1. The computational domain is set as $\Omega = [-1, 1] \times [-1, 1]$ with the 256×256 mesh grid. We have used 16,384 points for the nudging. We choose the initial condition as

$$
\begin{cases}
\phi(x, y, 0) = \tanh\left(-\max\left((|x| - 0.5, |y| - 0.5)\right) / \left(\sqrt{2\gamma}\right)\right),\\
u_1(x, y, 0) = 0, u_2(x, y, 0) = 0, p(x, y, 0) = 0
\end{cases}
\tag{5.1}
$$

where $\sqrt{\gamma} = 4h/(2\sqrt{2}\text{atanh}(0.9))$ is the interfacial thickness. For the discretization system, we use the projection method for the incompressible NS equation and use the operator splitting method for the nonlinear CH equation [54]. Meanwhile, we have used the Crank–Nicolson scheme to obtain the second-order temporal accuracy. It should be pointed that a standard essentially non-oscillatory scheme has been used for the second-order spatial accuracy. In this paper, we choose $g(x) = 0$ and $f(\phi) = \phi^3 - \phi$ as the double-well potential. We choose the two positive relaxation parameter for the nudging as $\beta = 1$ and $\chi = 1$. The other parameter are $\Delta t = 0.01h$ and $\lambda = 1$. As can be seen from the results, the isolated irregular interface relaxes to a circle due to the isotropy of the mobility and the influence of surface tension.

By using the results obtained by the original NSCH system, we recalculate based on the CDA of NSCH system with the following initial conditions:

$$
\begin{cases}
\psi = \text{rand}(x, y, 0), q(x, y, 0) = \text{rand}(x, y, 0),\\
v_1 = \text{rand}(x, y, 0), v_2 = \text{rand}(x, y, 0).
\end{cases}
\tag{5.2}
$$

The results have been demonstrated in Fig. 2a with the same parameters. It can be seen from the results that although different initial values are used here, the chaotic phase field gradually forms the same circle as Fig. 1 under the influence of the previous results. Figure 2b is the results obtained by the original NSCH system with the same initial condition. By comparing the results, we can find that the proposed system can indeed work well for capturing the original phase separation. In Fig. 3, we perform the results by the two models in the same frame and plot the 0-contour line at the indicated times $t = 0, 1, 5$ and 10, respectively. The blue lines and red lines are the results obtained by the CDA of NSCH model and the observational measurement

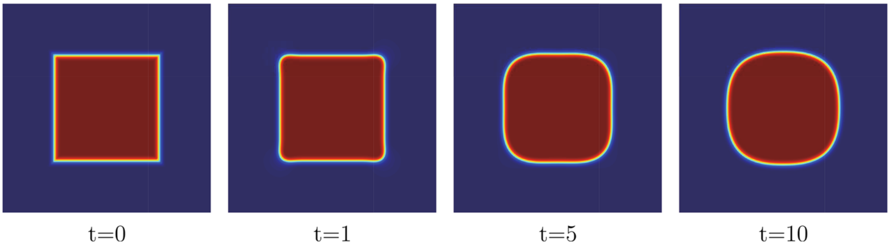

Fig. 1 Snapshots of the relaxation of a square shape by the original NSCH system. From left to right, $t = 0$, 1, 5, 10, respectively

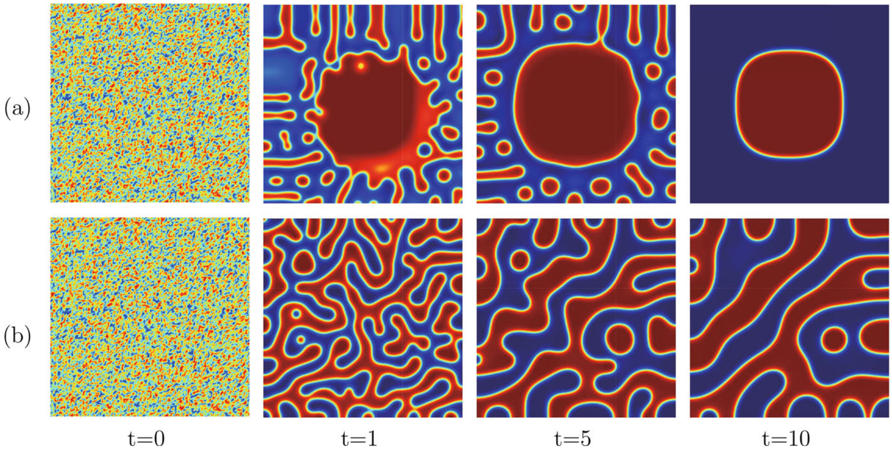

Fig. 2 Temporal evolution of the results obtained by the CDA of NSCH system (**a**) and the original NSCH system (**b**). From left to right, the indicated times are t = 0, 1, 5 and 10, respectively

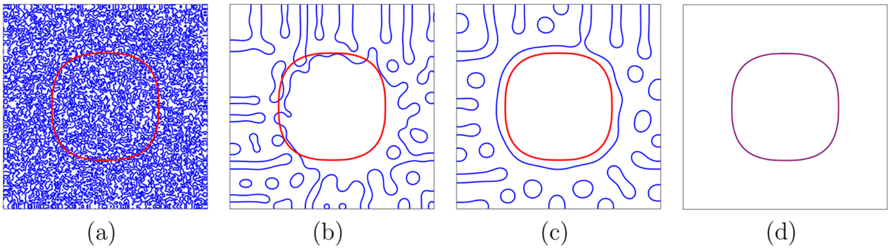

Fig. 3 Comparison of the phase-field interface between the CDA of NSCH model and the original NSCH model. The blue lines and red lines are the results obtained by the CDA of NSCH model and the observational measurement of this system, respectively. From (**a**–**d**), the indicated times are $t = 0, 1, 5$ and 10, respectively (Color figure online)

of this system, respectively. As we expected, the blue line gradually coincides with the red line under the influence of the penalty term.

Acknowledgements This work was supported by the National Science Foundation of China Grant (11401459,11801427, 11871389), the Natural Science Foundation of Shaanxi Province (2018JQ1009, 2018JM1012) and the Fundamental Research Funds for the Central Universities (xjj2018088).

References

1. Daley, R.: Atmos. Data Anal. Cambridge University Press, New York (1991)
2. Azouani, A., Titi, E.S.: Feedback control of nonlinear dissipative systems by finite determining parameters-a reaction-diffusion paradigm. Evol. Equ. Control Theory **3**(4), 579–594 (2014)
3. Lunasin, E., Titi, E.S.: Finite determining parameters feedback control for distributed nonlinear dissipative systems: A computational study. Evol. Equ. Control Theory **6**(4), 535–557 (2017)
4. Azouani, A., Olson, E., Titi, E.S.: Continuous data assimilation using general interpolant observables. J. Nonlinear Sci. **24**(2), 277–304 (2014)
5. Cockburn, B., Jones, D.A., Titi, E.S.: Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems. Math. Comput. **66**, 1073–1087 (1997)
6. Foias, C., Manley, O., Rosa, R., Temam, R.: Navier–Stokes Equations and Turbulence. Cambridge University Press, New York (2001)
7. Foias, C., Prodi, G.: Sur le comportement global des solutions non-stationnaires des equations de Navier–Stokes en dimension 2. Rend. Semin. Mat. Univ. Padova **39**, 1–34 (1967)
8. Foias, C., Temam, R.: Sur la détermination dun écoulement fluide par des observations discrétes. C. R. l'Acad. Sci. I **295**(3), 239–241 (1982)
9. Foias, C., Temam, R.: Determination of the solutions of the Navier–Stokes equations by a set of nodal values. Math. Comput. **43**, 117–133 (1984)
10. Foias, C., Titi, E.S.: Determining nodes, finite difference schemes and inertial manifolds. Nonlinearity **4**(1), 135–153 (1991)
11. Holst, M.J., Titi, E.S.: Determining projections and functionals for weak solutions of the Navier–Stokes equations. Contemp. Math. **204**, 125–138 (1997)
12. Jones, D.A., Titi, E.S.: Determining finite volume elements for the 2D Navier–Stokes equations. Physica D **60**, 165–174 (1992)
13. Jones, D.A., Titi, E.S.: Upper bounds on the number of determining modes, nodes and volume elements for the Navier–Stokes equations. Indiana Univ. Math. J. **42**(3), 875–887 (1993)
14. Olson, E., Titi, E.S.: Determining modes for continuous data assimilation in 2D turbulence. J. Stat. Phys. **113**, 799–840 (2003)
15. Olson, E., Titi, E.S.: Determining modes and Grashof number in 2D turbulence. Theor. Comput. Fluid Dyn. **22**(5), 327–339 (2008)
16. Foias, C., Jolly, M.S., Kravchenko, R., Titi, E.S.: A unified approach to determining forms for the 2D Navier–Stokes equations-the general interpolants case. Russ. Math. Surv. **69**(2), 177–200 (2014)
17. Foias, C., Mondaini, C.F., Titi, E.S.: A discrete data assimilation scheme for the solutions of the two-dimensional Navier–Stokes equations and their statistics. SIAM J. Appl. Dyn. Syst. **15**(4), 2109–2142 (2016)
18. Hoke, J., Anthes, R.: The initialization of numerical models by a dynamic relaxation technique. Monthly Weather Rev. **104**, 1551–1556 (1976)
19. Bessaih, H., Olson, E., Titi, E.S.: Continuous data assimilation with stochastically noisy data. Nonlinearity **28**, 729–753 (2015)
20. Farhat, A., Lunasin, E., Titi, E.S.: Data assimilation algorithm for 3D Bénard convection in porous media employing only temperature measurements. J. Math. Anal. Appl. **438**, 492–506 (2016)
21. Farhat, A., Jolly, M.S., Titi, E.S.: Continuous data assimilation for the 2D Bénard convection through velocity measurements alone. Physica D **303**, 59–66 (2015)
22. Farhat, A., Lunasin, E., Titi, E.S.: Continuous data assimilation for a 2D Bénard convection system through horizontal velocity measurements alone. J. Nonlinear Sci. **27**, 1065–1087 (2017)
23. Biswas, A., Foias, C., Mondaini, C.F., Titi, E.S.: Downscaling data assimilation algorithm with applications to statistical solutions of the Navier–Stokes equations. Ann. l'Inst. Henri Poincaré C **36**, 295–326 (2019)
24. Carlson, E., Hudson, J., Larios, A.: Parameter recovery for the 2 dimensional Navier–Stokes equations via continuous data assimilation. SIAM J. Sci. Comput. **42**(1), A250–A270 (2020)
25. Carlson, E., Larios, A.: Sensitivity analysis for the 2D Navier–Stokes equations with applications to continuous data assimilation. J. Nonlinear Sci. **31**(5), 84 (2021)
26. Farhat, A., Lunasin, E., Titi, E.S.: Abridged continuous data assimilation for the 2D Navier–Stokes equations utilizing measurements of only one component of the velocity field. J. Math. Fluid Mech. **18**, 1–23 (2016)

27. Garcia-Archilla, B., Novo, J., Titi, E.S.: Uniform in time error estimates for a finite element method applied to a downscaling data assimilation algorithm for the Navier–Stokes equations. SIAM J. Numer. Anal. **58**(1), 410–429 (2020)
28. Gardner, M., Larios, A., Rebholz, L.G., Vargun, D., Zerfas, C.: Continuous data assimilation applied to a velocity-vorticity formulation of the 2D Navier–Stokes equations. Electron. Res. Arch. **29**(3), 2223–2247 (2021)
29. Mondaini, C.F., Titi, E.S.: Uniform-in-time error estimates for the postprocessing Galerkin method applied to a data assimilation algorithm. SIAM J. Numer. Anal. **56**(1), 78–110 (2018)
30. Albanez, D.A.F., Benvenutti, M.J.: Continuous data assimilation algorithm for simplified Bardina model. Evol. Equ. Control Theory **7**(1), 33–52 (2018)
31. Markowich, P.A., Titi, E.S., Trabelsi, S.: Data assimilation for the three-dimensional Brinkman–Forchheimer-extended Darcy model. Nonlinearity **29**, 1292–1328 (2016)
32. Albanez, D.A.F., Lopes, H.J.N., Titi, E.S.: Continuous data assimilation for the three-dimensional Navier–Stokes-α model. Asymp. Anal. **97**, 139–164 (2016)
33. Jolly, M.S., Martinez, V.R., Olson, E.J., Titi, E.S.: Continuous data assimilation with blurred-in-time measurements of the surface quasi-geostrophic equation. Chin. Ann. Math. B **40**(5), 721–764 (2019)
34. Chen, N., Li, Y.C., Lunasin, E.: An efficient continuous data assimilation algorithm for the sabra shell model of turbulence. Chao **31**(10), 103123 (2021)
35. Chow, Y. T., Leung, W. T., Pakzad, A.: Continuous data assimilation for two-phase flow: analysis and simulations. http://arxiv.org/abs/2110.13491v1 (2021)
36. Diegel, A. E., Rebholz, L. G.: Continuous data assimilation and long-time accuracy in a $\mathcal{C}^0$ interior penalty method for the Cahn–Hilliard equation. http://arxiv.org/abs/2106.14744v1 (2021)
37. Rebholz, L.G., Zerfas, C.: Simple and efficient continuous data assimilation of evolution equations via algebraic nudging. Numer. Methods Partial Diff. Equ. **37**(3), 2588–2612 (2021)
38. Dimet, F.X.L., Talagrand, O.: Variational algorithms for analysis and assimilation of meteorological observations: Theoretical aspects. Tellus A **38**, 97–110 (1986)
39. Fehrenbach, J., Masmoudi, M., Souchon, R., Trompette, P.: Detection of small inclusions using elastography. Inverse Problem **22**, 1055–1069 (2006)
40. Fehrenbach, J., Oudry, J., Sandrin, L.: Variational data assimilation to estimate the velocity in the wave equation. Inverse Problem **26**, 115005 (2010)
41. Korn, P.: Strong solvability of a variational data assimilation problem for the primitive equations of large-scale atmosphere and ocean dynamics. J. Nonlinear Sci. **31**(3), 56 (2021)
42. Cherfils, L., Petcu, M.: On the viscous Cahn–Hilliard–Naveri–Stokes equations with dynamic boundary conditions. Commun. Pure Appl. Anal. **15**(4), 1419–1449 (2016)
43. Anderson, D.M., McFadden, G.B., Wheeler, A.A.: Diffuse-interface methods in fluid mechanics. Annu. Rev. Fluid Mech. **30**, 139–165 (1998)
44. Hohenberg, P.C., Halperin, B.I.: Theory of dynamical critical phenomena. Rev. Mod. Phys. **49**, 435–479 (1977)
45. Jasnow, D., Vinals, J.: Coarse-grained description of thermo-capillary flow. Phys. Fluids **8**, 660–669 (1996)
46. Feng, X.B.: Fully discrete element approximations of the Navier–Stokes–Cahn–Hilliard diffuse interface model for two-phase fluid flows. SIAM J. Numer. Anal. **44**(3), 1049–1072 (2006)
47. Kay, D., Styles, V., Welford, R.: Finite element approximation of a Cahn–Hilliard–Navier–Stokes system. Interfaces Free Bound **10**, 15–43 (2008)
48. Kay, D., Welford, R.: Efficient numerical solution of Cahn–Hilliard–Navier–Stokes fluids in 2D. SIAM J. Sci. Comput. **29**, 2241–2257 (2007)
49. Cao, C.S., Gal, C.G.: Global solutions for the 2D Navier–Stokes–Cahn–Hilliard model for a two-phase flow of viscous, incompressible fluids with mixed partial viscosity and mobility. Nonlinearity **25**, 3211–3234 (2012)
50. Gal, C.G., Grasselli, M.: Asymptotic behavior of a Cahn–Hilliard–Navier–Stokes system in 2D. Ann. l'Inst. Henri Poincaré C **27**, 401–436 (2010)
51. Gal, C.G., Grasselli, M.: Instability of two-phase flows: a lower bound on the dimension of the global attractor of the Cahn–Hilliard–Navier–Stokes system. Physica D **240**, 629–635 (2011)
52. Starovoitov, V.N.: The dynamics of a two-component fluid in the presence of capillary forces. Math. Notes **62**, 244–254 (1997)
53. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York (1997)

54. Li, Y., Choi, J., Kim, J.: Multi-component Cahn–Hilliard system with different boundary conditions in complex domains. J. Comput. Phys. **323**, 1–16 (2016)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.