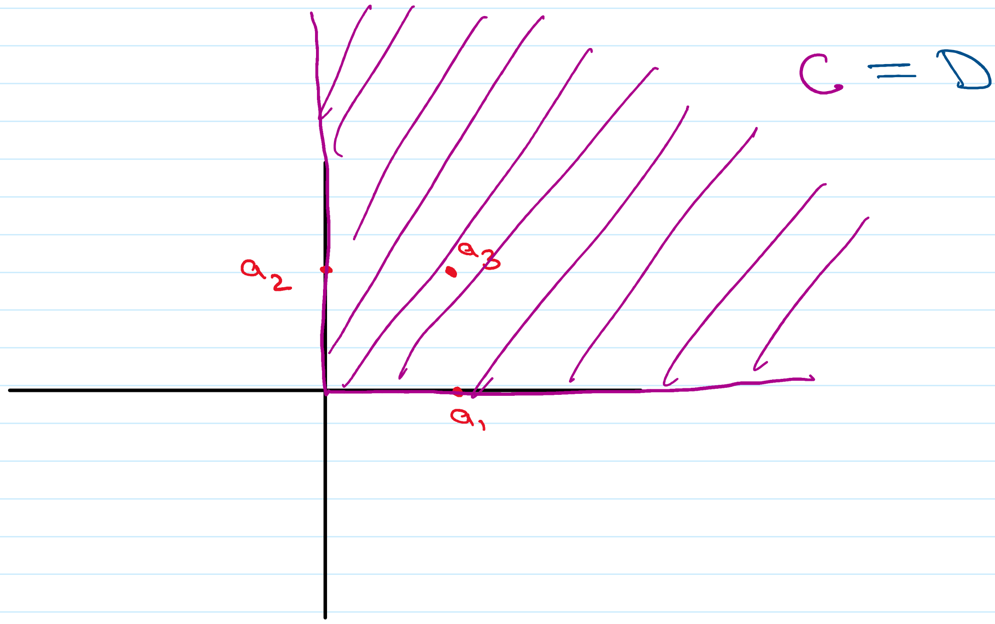
Problem 1

Solution:



Both $\{a_1, a_2\}$ and $\{a_1, a_2, a_3\}$ generate the same conic hull.

Problem 2

Solution. We consider two points $(\bar{x}, \bar{y}_1 + \bar{y}_2)$, $(\tilde{x}, \tilde{y}_1 + \tilde{y}_2) \in S$, *i.e.*, with

$$(\bar{x}, \bar{y}_1) \in S_1, \qquad (\bar{x}, \bar{y}_2) \in S_2, \qquad (\tilde{x}, \tilde{y}_1) \in S_1, \qquad (\tilde{x}, \tilde{y}_2) \in S_2.$$

For $0 \le \theta \le 1$,

$$\theta(\bar{x}, \bar{y}_1 + \bar{y}_2) + (1-\theta)(\tilde{x}, \tilde{y}_1 + \tilde{y}_2) = (\theta\bar{x} + (1-\theta)\tilde{x}, (\theta\bar{y}_1 + (1-\theta)\tilde{y}_1) + (\theta\bar{y}_2 + (1-\theta)\tilde{y}_2))$$

is in S because, by convexity of S_1 and S_2,

$$(\theta\bar{x} + (1-\theta)\tilde{x}, \theta\bar{y}_1 + (1-\theta)\tilde{y}_1) \in S_1, \qquad (\theta\bar{x} + (1-\theta)\tilde{x}, \theta\bar{y}_2 + (1-\theta)\tilde{y}_2) \in S_2.$$

Problem 3

Solution.

(a) The polar is the intersection of hyperplanes $\{y \mid y^T x \le 1\}$, parametrized by $x \in C$, so it is convex.

Problem 4

Solution: The hypograph of g is:

$$\text{hypo } g = \{(x,t) : t \le g(x)\}$$

Since f is increasing, so if $t \le g(x)$

then $f(t) \le f(g(x))$

$$\Rightarrow \text{hypo } g = \{(x,t) : f(t) \le f(g(x))\}$$

But $f(g(x)) = x$ since $g = f^{-1}$

$$\Rightarrow \text{hypo } g = \{(x,t) : f(t) \le x\}$$

$$= \{(x,t) : (t,x) \in \text{epi } f\}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{epi } f$$

Since hypo g is a linear transformation of epi f and since epi f is convex, because f is convex

$\Rightarrow$ hypo g is convex $\Rightarrow$ g is concave. $\blacksquare$

$\boxed{\text{Problem 5}}$

Solution.

(a) Define $g(t) = f(Z + tV)$, where $Z \succ 0$ and $V \in \mathbf{S}^n$.

$$\begin{aligned} g(t) &= \mathbf{tr}((Z+tV)^{-1}) \\ &= \mathbf{tr}\left(Z^{-1}(I + tZ^{-1/2}VZ^{-1/2})^{-1}\right) \\ &= \mathbf{tr}\left(Z^{-1}Q(I+t\Lambda)^{-1}Q^T\right) \\ &= \mathbf{tr}\left(Q^TZ^{-1}Q(I+t\Lambda)^{-1}\right) \\ &= \sum_{i=1}^{n}(Q^TZ^{-1}Q)_{ii}(1+t\lambda_i)^{-1}, \end{aligned}$$

where we used the eigenvalue decomposition $Z^{-1/2}VZ^{-1/2} = Q\Lambda Q^T$. In the last equality we express g as a positive weighted sum of convex functions $1/(1+t\lambda_i)$, hence it is convex.

$\boxed{\text{Problem 6}}$

(a) $f(x) = \max_{i=1,\dots,k} \|A^{(i)}x - b^{(i)}\|$, where $A^{(i)} \in \mathbf{R}^{m\times n}$, $b^{(i)} \in \mathbf{R}^m$ and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.

Solution. f is the pointwise maximum of k functions $\|A^{(i)}x - b^{(i)}\|$. Each of those functions is convex because it is the composition of an affine transformation and a norm.

$\boxed{\text{Problem 7}}$

(a) $f(x) = -\log(-\log(\sum_{i=1}^m e^{a_i^Tx+b_i}))$ on $\mathbf{dom}\, f = \{x \mid \sum_{i=1}^m e^{a_i^Tx+b_i} < 1\}$. You can use the fact that $\log(\sum_{i=1}^n e^{y_i})$ is convex.

Solution. $g(x) = \log(\sum_{i=1}^m e^{a_i^Tx+b_i})$ is convex (composition of the log-sum-exp function and an affine mapping), so $-g$ is concave. The function $h(y) = -\log y$ is convex and decreasing. Therefore $f(x) = h(-g(x))$ is convex.

$\boxed{\text{Problem 8}}$

Solution: Let x_1 and $x_2 \in I$.

we need to prove that

$$f(\theta x_1 + (1-\theta)x_2)\, g(\theta x_1 + (1-\theta)x_2)$$

$$\le \theta f(x_1) g(x_1) + (1-\theta) f(x_2) g(x_2)$$

$$\leq \theta f(x_1) g(x_1) + (1-\theta) f(x_2) g(x_2)$$

$$f(\theta x_1 + (1-\theta) x_2)\, g(\theta x_1 + (1-\theta) x_2)$$

$$\leq \left(\theta f(x_1) + (1-\theta) f(x_2)\right)\left(\theta g(x_1) + (1-\theta) g(x_2)\right)$$

$$= \theta^2 f(x_1) g(x_1) + \theta(1-\theta) f(x_1) g(x_2)$$

$$(*) \qquad + \theta(1-\theta) f(x_2) g(x_1) + (1-\theta)^2 f(x_2) g(x_2)$$

Recall : $a = \theta a + (1-\theta) b$

So: $\theta \left[f(x_1) g(x_1) \right] = \theta\left[\theta f(x_1) g(x_1) + (1-\theta) f(x_1) g(x_1)\right]$

$$= \theta^2 f(x_1) g(x_1) + \theta(1-\theta) f(x_1) g(x_1)$$

Similarly : $(1-\theta) f(x_2) g(x_2)$

$$= (1-\theta)\left[\theta f(x_2) g(x_2) + (1-\theta) f(x_2) g(x_2)\right]$$

$$= \theta(1-\theta) f(x_2) g(x_2) + (1-\theta)^2 f(x_2) g(x_2)$$

Thus: $(*)$ means

$$f(\theta x_1 + (1-\theta) x_2)\, g(\theta x_1 + (1-\theta) x_2)$$

$$\leq \theta f(x_1) g(x_1) - \theta(1-\theta) f(x_1) g(x_1)$$

$$+ \theta(1-\theta) f(x_1) g(x_2)$$

$$+ (1-\theta) f(x_2) g(x_2) - \theta(1-\theta) f(x_2) g(x_2)$$

$$+ \theta(1-\theta) f(x_2) g(x_1)$$

$$= \theta f(x_1) g(x_1) + (1-\theta) f(x_2) g(x_2)$$

$$+ \theta(1-\theta) f(x_1)\left(g(x_2) - g(x_1)\right)$$

$$- \theta(1-\theta) f(x_2)\left(g(x_2) - g(x_1)\right)$$

$$+\theta(1-\theta) f(x_1)\big(g(x_2)-g(x_1)\big)$$

$$-\theta(1-\theta)\, f(x_2)\big(g(x_2)-g(x_1)\big)$$

$$= \theta f(x_1) g(x_1) + (1-\theta) f(x_2) g(x_2)$$

$$+ \theta(1-\theta)\big(f(x_1)-f(x_2)\big)\big(g(x_2)-g(x_1)\big)$$

when f and g are both increasing or both decreasing, and positive:

$$\big(f(x_1)-f(x_2)\big)\big(g(x_2)-g(x_1)\big) \leq 0$$

$$\Rightarrow \quad \theta f(x_1) g(x_1) + (1-\theta) f(x_2) g(x_2)$$

$$+ \theta(1-\theta)\big(f(x_1)-f(x_2)\big)\big(g(x_2)-g(x_1)\big)$$

$$\leq \theta f(x_1) g(x_1) + (1-\theta) f(x_2) g(x_2)$$