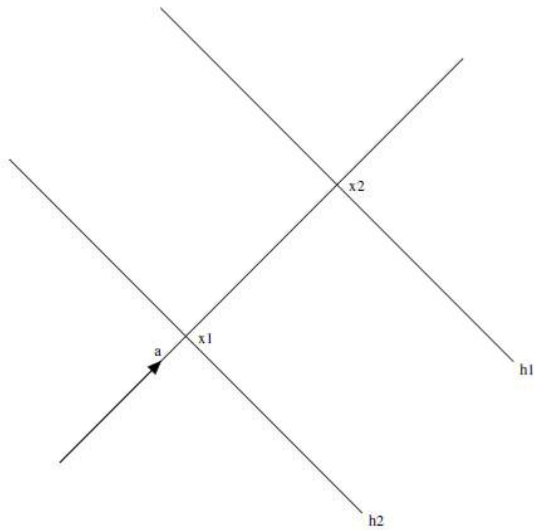
Problem 1

Solution. The distance between the two hyperplanes is $|b_1 - b_2|/\|a\|_2$. To see this, consider the construction in the figure below.



The distance between the two hyperplanes is also the distance between the two points x_1 and x_2 where the hyperplane intersects the line through the origin and parallel to the normal vector a. These points are given by

$$x_1 = (b_1/\|a\|_2^2)a, \qquad x_2 = (b_2/\|a\|_2^2)a,$$

and the distance is

$$\|x_1 - x_2\|_2 = |b_1 - b_2|/\|a\|_2.$$

Problem 2

(b) S is a polyhedron, defined by linear inequalities $x_k \geq 0$ and three equality constraints.

(c) S is not a polyhedron. It is the intersection of the unit ball $\{x \mid \|x\|_2 \leq 1\}$ and the nonnegative orthant $\mathbf{R}_+^n$. This follows from the following fact, which follows from the Cauchy-Schwarz inequality:

$$x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1 \iff \|x\|_2 \leq 1.$$

Although in this example we define S as an intersection of halfspaces, it is not a polyhedron, because the definition requires infinitely many halfspaces.

Problem 3

Solution.

(a) A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

(b) As in part (a), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.

Problem 4

Solution. We first note that the constraints $p_i \geq 0$, $i = 1, \ldots, n$, define halfspaces, and $\sum_{i=1}^{n} p_i = 1$ defines a hyperplane, so P is a polyhedron.
The first five constraints are, in fact, linear inequalities in the probabilities p_i.

(a) $\mathbf{E}\, f(x) = \sum_{i=1}^{n} p_i f(a_i)$, so the constraint is equivalent to two linear inequalities

$$\alpha \leq \sum_{i=1}^{n} p_i f(a_i) \leq \beta.$$

(b) $\mathbf{prob}(x \geq \alpha) = \sum_{i:\, a_i \geq \alpha} p_i$, so the constraint is equivalent to a linear inequality

$$\sum_{i:\, a_i \geq \alpha} p_i \leq \beta.$$

Problem 5

Solution

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

The Hessian of f is
$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is neither positive semidefinite, nor negative semidefinite.
Therefore, f is neither convex, nor concave.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

Solution:

The Hessian of f is

$$\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} 2/x_1^2 & 1/x_1 x_2 \\ 1/x_1 x_2 & 2/x_2^2 \end{bmatrix} \succcurlyeq 0$$

Therefore f is convex.

$\boxed{\text{Problem 6}}$

Solution: we will prove it using the definition of a convex set. Let $S_\alpha = \{x : x \in \operatorname{dom} f, f(x) \le \alpha\}$ be a sublevel set of a convex function f for any arbitrary $\alpha \in \mathbb{R}$.

Consider $x_1 \in S_\alpha$ and $x_2 \in S_\alpha$.

we need to show that $\theta x_1 + (1-\theta) x_2 \in S_\alpha \quad \forall \theta \in [0,1]$.

(1) Since x_1 and $x_2 \in S_\alpha \Rightarrow x_1$ and $x_2 \in \operatorname{dom} f$

$$\Rightarrow \theta x_1 + (1-\theta) x_2 \in \operatorname{dom} f$$

(2)
$$\begin{aligned} f(\theta x_1 + (1-\theta) x_2) &\le \theta f(x_1) + (1-\theta) f(x_2) \\ &\le \theta \alpha + (1-\theta)\alpha \\ &= \alpha \end{aligned}$$

$$\Rightarrow f(\theta x_1 + (1-\theta) x_2) \le \alpha$$

Since $\theta x_1 + (1-\theta) x_2 \in \operatorname{dom} f$ and

$$f(\theta x_1 + (1-\theta) x_2) \le \alpha$$

$$\Rightarrow \theta x_1 + (1-\theta) x_2 \in S_\alpha$$

Hence, S_α is a convex set.

$\boxed{\text{Problem 7}}$

Solution: In order to prove that $f(Ax+b)$ is convex, we need to show that dom $(f(Ax+b))$ is convex and for x_1 and $x_2 \in$ dom$(f(Ax+b))$, and

$$\tilde{x} = \theta x_1 + (1-\theta)x_2 \quad \forall \ \theta \in [0,1],$$

$$f(A\tilde{x}+b) \leq \theta f(Ax_1+b) + (1-\theta) f(Ax_2+b)$$

① since $f(\cdot)$ is convex $\Rightarrow$ dom(f) is convex.

dom$(f(Ax+b))$ is simply an affine transformation of the domain of f; any affine mapping of a convex set is convex. Thus, $f(Ax+b)$ has a convex domain.

② $$f(A\tilde{x}+b) = f(A(\theta x_1 + (1-\theta)x_2)+b)$$
$$= f(\theta Ax_1 + (1-\theta)Ax_2 + b)$$

But $b = \theta b + (1-\theta)b$

$$\Rightarrow f(A\tilde{x}+b) = f(\theta(Ax_1+b) + (1-\theta)(Ax_2+b))$$
$$\leq \theta f(Ax_1+b) + (1-\theta) f(Ax_2+b).$$

$\blacksquare$

Problem 8

Solution. The feasible set is shown in the figure.

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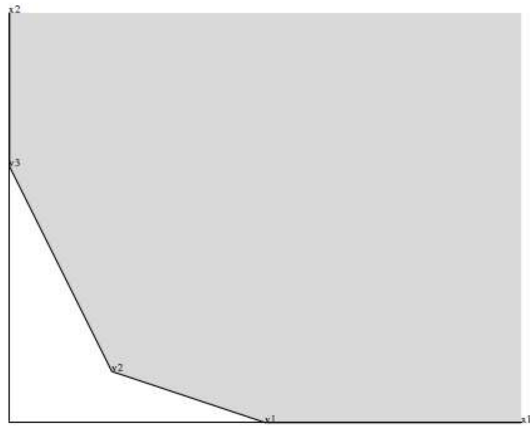

Problem 9

Solution. We verify that $x^\star$ satisfies the optimality condition (**??**). The gradient of the objective function at $x^\star$ is

$$\nabla f_0(x^\star) = (-1, 0, 2).$$

Therefore the optimality condition is that

$$\nabla f_0(x^\star)^T (y - x) = -1(y_1 - 1) + 2(y_2 + 1) \geq 0$$

for all y satisfying $-1 \leq y_i \leq 1$, which is clearly true.