Gershgorin circle Theorem

Let λ be an eigenvalue for the $n \times n$ matrix $A = \{a_{ij}\}_{\substack{i=1\\j=1}}^{n}$ with corresponding eigenvector $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq 0$.

① Let i_0 be an index so that

$$|x_{i_0}| = \max_{1 \le j \le n} |x_j| > 0$$

The i_0 entry of the equation $Ax = \lambda x$ reads

$$\lambda x_{i_0} = \sum_{j=1}^{n} a_{i_0 j} x_j ,$$

or

$$(\lambda - a_{i_0 i_0}) x_{i_0} = \sum_{j \neq i_0} a_{i_0 j} x_j$$

② Define $r_{i_0} := \sum_{j \neq i_0} |a_{i_0 j}|$

Then we get

$$|\lambda - a_{i_0 i_0}| = \left|\sum_{j \neq i_0} a_{i_0 j}\, x_j\right| \Big/ |x_{i_0}|$$

$$\leq \sum_{j \neq i_0} |a_{i_0 j}| \frac{|x_j|}{|x_{i_0}|} \leq 1$$

$$\leq r_{i_0}$$

and so $\lambda \in B_{r_{i_0}}(a_{i_0 i_0})$ circle of radius r_{i_0} centered at $a_{i_0 i_0}$

for this particular i_0.

We conclude that any eigenvalue of A lies in the union of these circles

$$\boxed{\lambda \in \bigcup_{i=1}^{m} B_{r_i}(a_{ii})}$$

circles in complex plane.