Linear Algebra and Applications
Homework #06

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(a)

If A & B are similar

$A = C^{-1} B C$, where c is an invertible matrix.

Now,

eigenvalues of B = roots of characteristic polynomial equation $\det(B - \lambda I) = 0$

eigenvalues of A = roots of characteristic polynomial equation $\det(A - \lambda I) = 0$

Now,

$$\det(A - \lambda I) = \det(C^{-1} B C - \lambda I)$$

$$= \det(C^{-1} B C - \lambda C^{-1} C) \qquad \Big| \; C^{-1}C = CC^{-1} = I$$

$$= \det[C^{-1}(B - \lambda I) C]$$

$$= \det(C^{-1}) . \det(B - \lambda I) . \det(C) \qquad \Big| \; \det(AB) = \det(A) . \det(B)$$

$$= \det(C^{-1} C) . \det(B - \lambda I)$$

$$= \det(I) . \det(B - \lambda I) \qquad \Big| \; \det(I) = 1$$

$$\boxed{\det(A - \lambda I) = \det(B - \lambda I)}$$

So, characteristic polynomials are the same for A & B.

So, $\boxed{\text{eigenvalues are same}}$.

(b)

A & B are similar: $A = C^{-1}BC$, C invertible matrix — with a C invertible matrix

A is diagonalizable: $\boxed{A = PDP^{-1}}$, P invertible matrix, D, Diagonal matrix

So,

$$PDP^{-1} = C^{-1}BC$$

$$\Rightarrow CPDP^{-1} = CC^{-1}BC$$

$$\Rightarrow CPDP^{-1} = BC$$

$$\Rightarrow CPDP^{-1}C^{-1} = BCC^{-1} = B \cdot I = B$$

$$\Rightarrow CP \cdot D \cdot (CP)^{-1} = B \qquad [(XY)^{-1} = Y^{-1}X^{-1}]$$

So $\boxed{B = QDQ^{-1}}$, with $Q = CP$

Now, $Q = CP$

C is invertible
P is invertible $\Big\} \to$ so $CP = Q$ is invertible

So, B is diagonalizable with the same diagonal matrix (D) of A and an invertible matrix $Q = CP$.

(c) let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda = 1, 1, 2}$$

let, $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

eigenvalues:

$$\det(B - \lambda I) = 0$$

$$\Rightarrow \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda)(1-\lambda) - 1(1-\lambda) = 0$$

$\Rightarrow (1-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$

$\Rightarrow (1-\lambda)\ (\lambda^2 - 4\lambda + 3) = 0$

$\Rightarrow \quad \lambda = 1$

$\lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 = 0$

$\Rightarrow \lambda(\lambda - 3) - 1(\lambda - 3) = 0$

$\Rightarrow (\lambda - 3)(\lambda - 1) = 0$

$\Rightarrow \lambda = 1, 3$

So, $\boxed{\lambda = 1, 1, 3}$

So, eigenvalues of A & B are not ~~similar~~ same.

So, those matrices are not similar.

(d)

$$A_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

eigenvalues: $\det(A_1 - \lambda I) = 0$

$$\Rightarrow \det \begin{bmatrix} 1-\lambda & 3 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda = 1, 1, 3}$$

$$A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{from 'c',} \quad \boxed{\lambda = 1, 1, 3}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

eigenvalues: $\det(A_3 - \lambda I) = 0$

$$\Rightarrow \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (1-\lambda)(1-\lambda)(3-\lambda) = 0$

$\Rightarrow \boxed{\lambda = 1, 1, 3}$

All three have same eigenvalues, so we now have to check diagonalizability.

<u>Now,</u> for A_1 algebric multicliplicity $= 2$ for $\lambda = 1$

<u>eigenvector for $\lambda = 1$ in A_1:</u>

$(A_1 - \lambda I)\underline{v} = 0$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} 3v_2 = 0 \\ 2v_3 = 0 \end{cases} \Rightarrow \begin{matrix} v_2 = 0 \\ v_3 = 0 \end{matrix}$$

<u>So,</u> $v = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

<u>So,</u> dimension of eigenspace = geometric multicipicity
$= \underline{1}$

So, for $\lambda = 1$ of A_1,

algebric multiplicity $\neq$ geometric multiplicity

so, A_1 is $\boxed{\text{not diagonalizable}}$.

Now,

$$A_2 \longrightarrow \text{real \& symmetric matrix}$$
$$\downarrow$$
$$\text{Always Diagonalizable}$$

A_3:

Algebric multiplicity = 2, for $\lambda=1$

Geometric Mul:

$$(A_3 - \lambda I)\underline{v} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} 3v_3 = 0 \\ 2v_3 = 0 \end{cases} \longrightarrow v_3 = 0$$

So, $v = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

So, dimension of eigen space = Geometric Multiplicity = 2

So $\boxed{A_3 \text{ is diagonalizable}}$

Conclusion: * A_1 is not similar to A_2 or A_3 since A_1 is not diagonalizable

* A_2 & A_3 are similar, as they both are diagonalizable with same eigen values and thus have the same diagonal jordan form.