# Class #25

Friday, April 25, 2025 10:22 AM

Reminders:

$g(\lambda, \nu) \leq f_0(x) \quad \forall x \in dom(P_0)$

① $\underline{g(\lambda,\nu)} \leq p^*$, for the case $\lambda \succeq 0$

② Dual problem

$$\max_{\lambda,\nu} \; g(\lambda,\nu) \qquad = d^*$$
$$s.t \quad \lambda \succeq 0$$

③ $d^* \leq p^* \Rightarrow$ weak duality

$p^* - d^* \Rightarrow$ called duality gap

④ we are interested in

$p^* = d^* \Rightarrow$ strong duality

⑤ Slater's Condition for strong duality of convex.

$$D = dom(f_0) \cap \bigcap_{i=1}^{m} dom\, f_i \cap \bigcap_{i=1}^{p} dom\, h_i$$

$$C = \{x : \boxed{f_i(x) \leq 0}_{i=1,\dots,m},\; h_i(x) = 0_{\; i=1,\dots,p}\}$$

↳ Any $x \in C$ is called 'feasible' (primal feasible)

**Corollary:** Any quadratic program that is feasible has strong duality. This means we can solve the has strong duality. This means we can solve the dual problem to get $(\lambda^*, \nu^*)$ and then recover $x^*$ from there.

$$\min_x \quad x^T P x$$

Standard form

$$\text{s.t.} \quad \boxed{Ax = b} \implies Ax - b = 0$$

Quadratic $\implies P \succeq 0$ (if $P \not\succeq 0$ then $x^T P x$ is unbounded below)

Feasible $\implies b \in \mathcal{R}(A)$

$$L(x, \nu) = x^T P x + \nu^T (Ax - b)$$

Not analytical form

$$g(\nu) = \boxed{\inf_x} \left( \underline{x^T P x + \nu^T A x - \nu^T b} \right)$$

$$\nabla_x L(x,\nu) = \nabla_x \left( x^T P x + \nu^T A x - \nu^T b \right)$$

$$= 2Px + A^T \nu = 0$$

$$\implies 2Px = -A^T \nu$$

$$Px = -\frac{A^T \nu}{2}$$

Let's assume $P \succ 0$

$$\implies x = -P^{-1} \frac{A^T \nu}{2}$$

$$g(\nu) = \frac{1}{4} \nu^T \underbrace{A P^{-T} \cdot P \cdot P^{-1} A^T \nu} + \underbrace{\nu^T A \left( -\frac{P^{-1} A^T \nu}{2} \right)} - \nu^T b$$

concave

concave

$$= \nu^T B \nu - \nu^T b$$

$B \preceq 0$

Find $\nu^*$ by Solving

$$\nu^* = \arg\max_{\nu} g(\nu)$$

$$= \arg\min_{\nu} -g(\nu)$$

Use GD, SD, or Newton's

Now obtain $L(x, \nu^*) = x^T P x + \nu^* (Ax - b)$

Quadratic in $x$

Solve for $x^*$ by $\arg\min_{x} L(x, \nu^*)$

Summary of Steps

① If possible, obtain an analytical form of the dual function.

② If the primal problem had no inequality constraints, then the dual problem will be an unconstrained problem

$\Rightarrow$ Solve it using any unconstrained optimization solver. This will give us $\nu^*$.

③ Plug $\nu^*$ into the Lagrangian to get

$L(x, \nu^*)$

$\Rightarrow$ Solve $\min_x L(x, \nu^*)$ to get $x^*$

Find part: $f_0(x^*) = p^*$ if strong duality was there.

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Given: A constrained optimization problem with primal variable $x$ and primal objective function $f_0(x)$ and dual variables $(\lambda, \nu)$ and dual objective function $g(\lambda, \nu)$.

when $(\lambda, \nu)$ is a feasible pair (i.e., $\lambda \succeq 0$)

$$g(\lambda,\nu) \le p^* \le f_0(x) \quad \forall$$

feasible $x$.

If $(\lambda, \nu)$ and $x$ are dual feasible and primal feasible, respectively, then:

$$p^* \in [g(\lambda,\nu), f_0(x)]$$

$f_0(x) - g(\lambda, \nu)$ is called duality gap between $(\lambda, \nu)$ and $x$.

$(\lambda, \nu)$ and $x$.

If $g(\lambda^*, \nu^*) = f_0(x^*)$ for any $(\lambda^*, \nu^*)$ and $x^*$

$\Rightarrow$ $(\lambda^*, \nu^*)$ is dual optimal

$x^*$ is primal optimal $\Rightarrow$ There can be multiple $x^*$

<u>Similarly</u>: $\forall\, x$, and $(\lambda, \nu)$ primal and dual feasible:

$$g(\lambda, \nu) \leq f_0(x)$$

$$\Rightarrow d^* = \underset{\lambda \geq 0}{\operatorname{argmax}}\, g(\lambda, \nu) \leq f_0(x)$$

$$d^* \geq g(\lambda, \nu) \quad \text{(by definition)}$$

for $(\lambda, \nu)$ feasible.

$$\Rightarrow d^* \in [g(\lambda, \nu), f_0(x)]$$

Many primal-dual optimization methods use this as a stopping criterion (assuming strong duality holds).

<u>Primal-dual methods</u>

They produce a sequence of

feasible primal variable $x^{(k)}$

feasible dual variable $(\lambda^{(k)}, \nu^{(k)})$

feasible dual variable $(\lambda^{(k)}, \nu^{(k)})$

Let $\epsilon > 0$. we stop when:

$$f_0(x^{(k)}) - g(\lambda^{(k)}, \nu^{(k)}) \leq \epsilon$$

<u>Lemma:</u> Let $(\lambda^*, \nu^*)$ be the solution of the dual problem. Then

$L(x, \lambda^*, \nu^*)$, considered only as a function of $x$, is minimized by $x^*$, which is the solution of the primal problem.

<u>Proof:</u> Let $x^*$ be the primal solution:

$$\boxed{f_0(x^*)} = g(\lambda^*, \nu^*)$$

Plugging the definition of $g(\lambda, \nu)$ $\leftarrow$

$$\boxed{=} \inf_x \left( f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \right)$$

$$=$$

$$\boxed{\leq} f_0(x^*) + \underbrace{\sum_{i=1}^m \lambda_i^* f_i(x^*)}_{\leq 0} + \underbrace{\sum_{i=1}^p \nu_i^* h_i(x^*)}_{=0}$$

$$=$$

$$\boxed{\leq} f_0(x^*)$$

$$a = b \leq c \leq d \leq a$$

$$\rightarrow L(x^*, \lambda^*, \nu^*) = \inf \left( L(x, \lambda^*, \nu^*) \right)$$

$$\rightarrow L(x^*, \lambda^*, \nu^*) = \inf_x \left( L(x, \lambda^*, \nu^*) \right)$$

**Note:** $L(x^*, \lambda, \nu) \neq \inf_x \left( L(x, \lambda, \nu) \right)$

unless $(\lambda, \nu)$ are dual solutions

## Complementary Slackness

Notice in the proof of the lemma that

$$\cancel{f_0(x^*)} + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) = \cancel{f_0(x^*)}$$

$$\Leftrightarrow \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \underbrace{\sum_{i=1}^{p} \nu_i^* h_i(x^*)}_{=0} = 0$$

$$\Leftrightarrow \boxed{\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0}$$

↳ Complementary slackness

**Remember:** $f_i(x^*) \leq 0$

$\lambda_i^* \geq 0$

The only way $\sum^{m} \lambda_i^* f_i(x^*) = 0$

The only way $\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$

$$\Leftrightarrow \lambda_i^* f_i(x^*) = 0 \ \forall \ i=1,\dots,m$$

$\Rightarrow$ If $\lambda_i^* > 0 \Rightarrow f_i(x^*) = 0$

If $f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0$

$\Downarrow$

They lead to Karush-Kuhn-Tucker Conditions (KKT conditions) for strong duality.

KKT conditions are necessary conditions for strong duality to hold for any constrained optimization problem.

In the case of convex optimization, if the KKT conditions hold then they are sufficient for strong duality.

Convex optimization approach

$\hookrightarrow$ verify Slater's condition and then use KKT conditions to solve problems.

(or) we verify KKT conditions and also use them to solve problems.