Solving systems of ODEs

Consider

$$\frac{d\underline{u}}{dt} = A\underline{u} \qquad \underline{u}(0) = \underline{u}_0$$

where A is $n \times n$, real and symmetric

($\underline{u}$ is a vector in $\mathbb{R}^n$ that depends on t, $\underline{u}(t)$ and A is independent of t)

Since A is real and symmetric

$$AQ = Q \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

where Q is an orthagonal matrix (whose columns are the eigenvectors of A)

We introduce

$$\underline{v}(t) = Q^T \underline{u}(t) \qquad \left(= Q^{-1} \underline{u}(t) \right)$$

Then

why?

$$\frac{d}{dt}\underline{v}(t) = Q^T \frac{d}{dt}\underline{u}(t)$$

$$= Q^T A\, \underline{u}(t)$$

$$= Q^T A\, Q\, \underline{v}(t)$$

why?

$$= \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \underline{v}(t)$$

and so

$$v_i(t) = c_i\, e^{\lambda_i t}$$

or

$$\underline{v}(t) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{\underline{c}}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \underline{v}(0)$$

why?

$$= \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} Q^T \underline{u}(0)$$

It follows that

$$\underline{u}(t) = Q\,\underline{v}(t)$$

$$= Q \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} Q^T \underline{u}(0)$$

$$\left(= \quad e^{At}\,\underline{u}(0) \right)$$

Now what if A is not symmetric and thus possibly non-diagonalizable?

Ex:

$$\frac{d}{dt}\underline{u}(t) = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} \underline{u}(t), \quad \underline{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This matrix has one eigenvalue (4) of <u>algebraic multiplicity</u> 2 but of <u>geometric multiplicity</u> 1

(i.e. only <u>one</u> linearly ind eigenvector)

eigenvalue	eigenvector
4	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The matrix cannot be diagonalized.

Instead

$$\underset{A}{\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}} \underset{Q}{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}} = \underset{Q}{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}} \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}$$

with $\underline{v}(t) = Q^T \underline{u}(t) \quad (= Q^{-1} \underline{u}(t))$

we thus get

$$(*) \quad \left[\begin{aligned} \frac{d}{dt}\underline{v} &= Q^T \frac{d}{dt}\underline{u}(t) = Q^T A \underline{u}(t) \\ &= Q^T A Q Q^T \underline{u}(t) = \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix} \underline{v}(t) \\ \underline{v}(0) &= Q^T \underline{u}(0) = Q^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned} \right.$$

solving for v_2 in $(*)$ we get

$$v_2(t) = -\frac{1}{\sqrt{2}} e^{4t}$$

and $v_1(t)$ must now satisfy

$$\boxed{\begin{aligned} \frac{d}{dt} v_1 &= 4v_1 + 2v_2 = 4v_1 - \sqrt{2}\, e^{4t} \\ v_1(0) &= 1/\sqrt{2} \end{aligned}} \quad (**)$$

(**) is equivalent to

$$e^{-4t}\left(\frac{d}{dt}v_1 - 4v_1\right) = -\sqrt{2} \qquad \text{or,}$$

$$\frac{d}{dt}\left(v_1 e^{-4t}\right) = -\sqrt{2} \qquad \text{, i.e.,}$$

$$v_1 = -\sqrt{2}\, t\, e^{+4t} + v_1(0)\, e^{+4t}$$

$$= -\sqrt{2}\, t\, e^{+4t} + 1/\sqrt{2}\; e^{+4t}$$

so $\underline{u}(t)$ has the form

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = Q\underline{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\sqrt{2}te^{4t} + \frac{1}{\sqrt{2}}e^{4t} \\ -\frac{1}{\sqrt{2}}e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} -te^{4t} + e^{4t} \\ -te^{4t} \end{pmatrix}$$

Please check that this is a solution!

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