Convex Optimization
Homework #07

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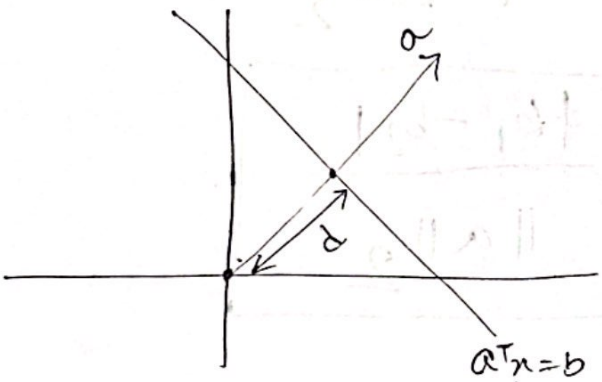
Problem 1:

Exercise 2.5:

A hyperplane is defined by:

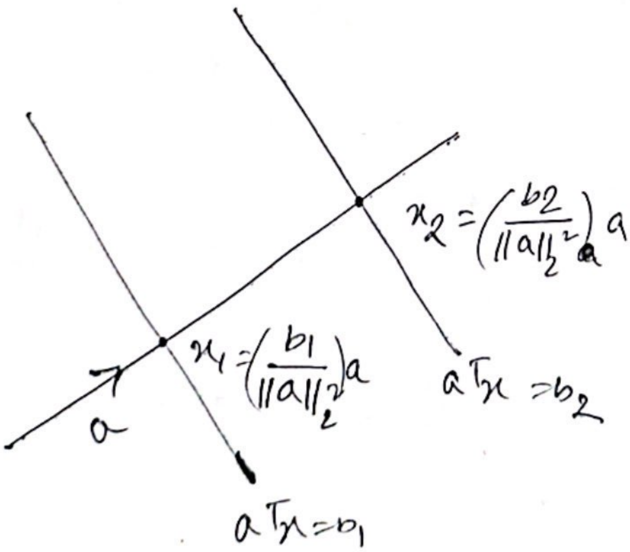
$$H = \{x : a^T x = b\} \text{ for fixed } a \in \mathbb{R}^n, \; a \neq 0 \; \& \; b \in \mathbb{R}$$



d = distance of a hyperplane from origin in parallel to normal vector a

$$= \frac{|b|}{\|a\|_2}$$

Now, for $a^T x = b_1$ & $a^T x = b_2$:



Here, x_1 & x_2 are the points where the hyperplane intersects the line through the origin parallel to normal vector 'a' for the points:

$$x_1 = \frac{b_1}{\|a\|_2^2} a, \qquad x_2 = \frac{b_2}{\|a\|_2^2} a$$

So, the distance between the hyperplanes is the distance between the points (x_1 & x_2)

So, distance $= \|x_1 - x_2\|_2$

$$= \boxed{\frac{|b_1 - b_2|}{\|a\|_2}}$$

Problem 2:

Exercise 2.8 (b):-

$$S = \left\{ x \in \mathbb{R}^n \mid x \succeq 0,\ \mathbf{1}^T x = 1,\ \sum_{i=1}^{n} x_i a_i = b_1,\ \sum_{i=1}^{n} x_i a_i^2 = b_2 \right\}, \quad a_1 \ldots a_n \in \mathbb{R},\ b_1, b_2 \in \mathbb{R}$$

Here,

the constraints are:

(i) $x \succeq 0 \Rightarrow x_k \geq 0$ (represent 'n' linear inequalities)

(ii) $\mathbf{1}^T x = 1$

$\Rightarrow \sum_{k=1}^{n} x_k = 1$ (represent linear equality)

(iii) $\sum_{i=1}^{n} x_i a_i = b_1$ (represent linear equality)

(iv) $\sum_{i=1}^{n} x_i a_i^2 = b_2$ (linear equality)

So, S is defined by finite linear equalities & inequalities $\longrightarrow$ $\boxed{\text{polyhedron}}$

2.8 (c):

$$S = \{x \in \mathbb{R}^n \mid x \succcurlyeq 0,\ x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$$

Here, $x^T y \leq 1$ with all $y : \|y\|_2 = 1$

means maximum of $x^T y$ over all unit-norm y must be ≤ 1

So, $\sup\limits_{\|y\|_2 = 1} x^T y \leq 1$ reduces to:

$$\|x\|_2 \leq 1$$

So, $S = \{x \in \mathbb{R}^n \mid x \succcurlyeq 0,\ \|x\|_2 \leq 1\}$

Here, $x \succcurlyeq 0 \rightarrow$ defined by n linear inequalities $\rightarrow$ polyhedron

But, $\|x\|_2 \leq 1 \rightarrow$ is not a polyhedron, since it is defined by infinitely many linear inequalities $x^T y \leq 1$ for all $\|y\|_2 = 1$

So, S being an intersection of polyhedron & a non-polyhedron, is $\boxed{\text{not a polyhedron}}$

Problem 3 :-

Exercise 2.12 (a):

A slab is an intersection of two half spaces ($a^T x \geq \alpha$ & $a^T x \leq \beta$), which are both convex.
The intersection of convex set is convex
So, A slab is $\boxed{\text{convex}}$

2.12 (b)

Rectangle of this form is the intersection of 2n halfspaces ($x_i \geq \alpha_i$ & $x_i \leq \beta_i$ for each i), which are ~~both~~ each convex.

So, the rectangle (intersection) is $\boxed{\text{convex}}$

2.12 (c):

A wedge is the intersection of two halfspaces, which are both convex. $\left(a_1^T x \le b_1,\ a_2^T x \le b_2\right)$

So, it is a $\boxed{\text{convex}}$ set.

Problem 4:

Exercise 2.15

Here,

$$P = \{ p \mid 1^T p = 1,\ p \succeq 0 \}$$

From constraints,

$p_i \geq 0,\ i = 1, \dots n \longrightarrow$ define 'n' halfspaces

$1^T p = 1$

$\Rightarrow \sum_{i=1}^{n} p_i = 1 \longrightarrow$ defines a hyperplane

So, P is a polyhedron here which is convex ~~bein~~ for being intersection of convex sets.

Now, **(a)** $E f(x) = \sum_{i=1}^{n} p_i f(a_i)$

So,

$$\alpha \le E f(x) \le \beta$$

$$\Rightarrow \alpha \le p_i f(a_i) \le \beta$$

So, this represent two linear inequalities in the probabilities p_i. So, the inequalities define intersection of two half-spaces (which are each convex), hence this condition is $\boxed{\text{convex}}$

(b) $\text{prob}(x \ge \alpha) = \sum_{i:\, a_i \ge \alpha} (p_i)$

So, $\text{Prob}(x \ge \alpha) \le \beta$

$$\Rightarrow \sum_{i:\, a_i \ge \alpha} p_i \le \beta$$

So, this is a linear inequality in terms of p.

And since 'P' is convex and linear inequalities preserve Convexity, the set of p satisfying this condition is $\boxed{\text{convex}}$.

Problem 5:

Exercise 3.16:

(b) Given, $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So, the hessian neither positive semi definite nor negative semi definite.

<u>eigenvalues:</u>

$$\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

So, the function is $\boxed{\text{nether Convex OR concave}}$

(c) $f(x_1, x_2) = \dfrac{1}{x_1 x_2}$ where $x_1, x_2 > 0$

$$\nabla f(x) = \begin{bmatrix} -\dfrac{1}{x_1^2 x_2} \\ -\dfrac{1}{x_1 x_2^2} \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} \dfrac{2}{x_1^3 x_2} & \dfrac{1}{x_1^2 x_2^2} \\ \dfrac{1}{x_1^2 x_2^2} & \dfrac{2}{x_1 x_2^3} \end{bmatrix}$$

Now, for any $v = (v_1, v_2) \in \mathbb{R}^2$

$$v^T \nabla^2 f(x) v = \frac{2v_1^2}{x_1^3 x_2} + \frac{2v_2^2}{x_1 x_2^3} + \frac{2v_1 v_2}{x_1^2 x_2^2}$$

this is non negative (dominated by quadratic even when v_1 & v_2 are of opposite signs)

So, $\nabla^2 f(x) \succeq 0$

So, f is $\boxed{\text{Convex}}$

Problem 6:

A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex

if $\forall\, x, y \in \text{dom} f$ & $\theta \in [0,1]$

$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$$

Sublevel set:

for ~~any~~ given $\alpha \in \mathbb{R}$, sublevel set:

$$S_\alpha = \{ x \in \text{dom}(f) \mid f(x) \leq \alpha \}$$

Now, let $x, y \in S_\alpha$

so, $f(x) \leq \alpha$

$f(y) \leq \alpha$

Now, let's consider a convex combination

of x & y

$z = \theta x + (1-\theta) y$, $\theta \in [0,1]$

since f is convex

$f(z) = f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$

$f(z) \leq \theta \alpha + (1-\theta) \alpha$

$f(z) \leq \alpha$

So,

$z \in S_\alpha$

So, the sublevel set of a convex function is convex

Problem 7:

Let, $h(x) = f(Ax + b)$ | $A \in \mathbb{R}^{m \times n}$

for $f: \mathbb{R}^m \to \mathbb{R}$ convex | $b \in \mathbb{R}^m$

| $x \in \mathbb{R}^n$

Let, $x, y \in \mathbb{R}^n$ & $\theta \in [0, 1]$

Now, $g(x) = f(Ax + b)$

So, $g(\theta x + (1-\theta) y) = f(A(\theta x + (1-\theta) y) + b)$

$$= f(\theta A x + (1-\theta) A y + b)$$

$$= f(\theta (Ax + b) + (1-\theta)(Ay + b))$$

$$\leq \theta f(Ax + b) + (1-\theta) f(Ay + b)$$

(As f is convex)

So,

$$\boxed{g(\theta x+(1-\theta)y) \leq \theta g(x) + (1-\theta)\, g(y)}$$

So, $g(x) = f(Ax+b)$ is also convex.

problem 8:-

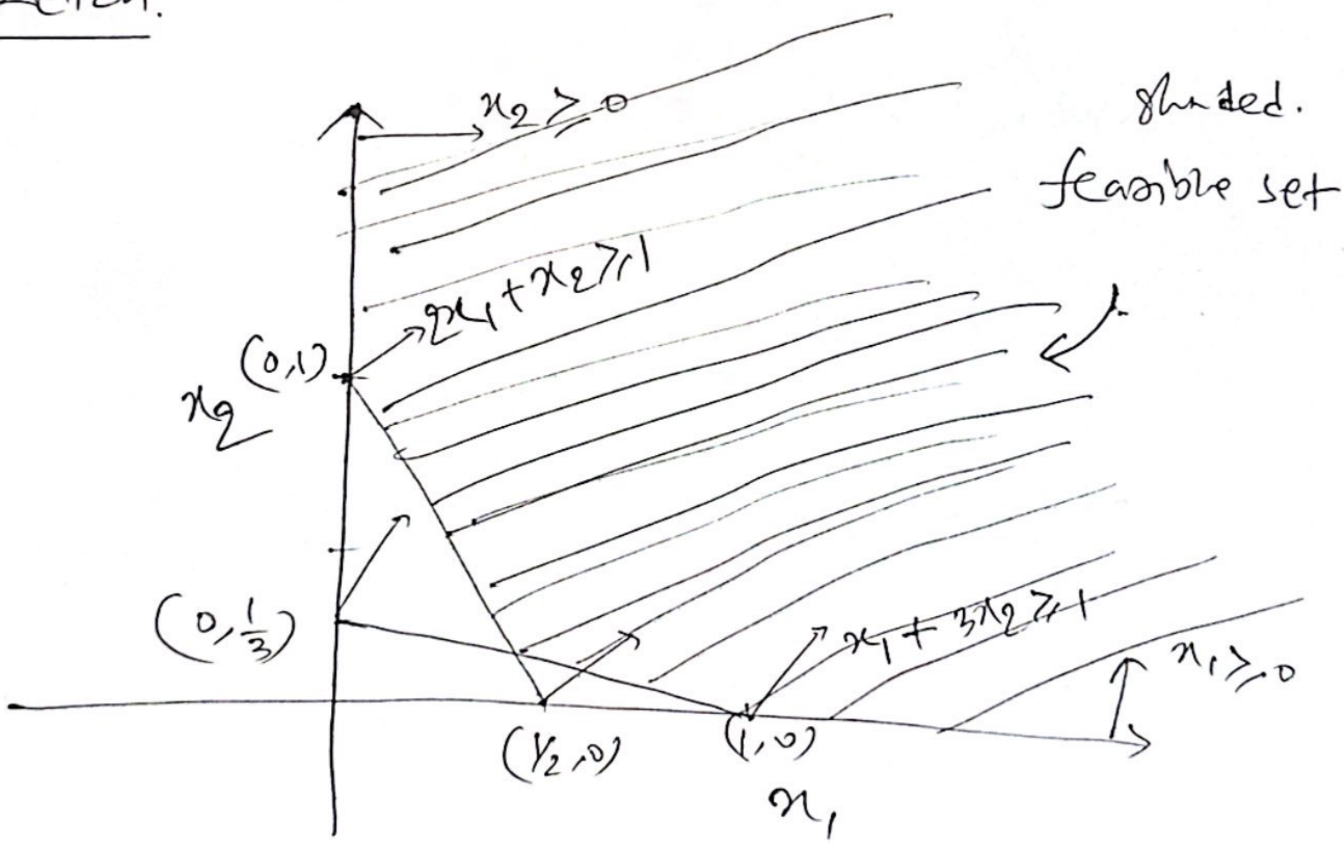
minimize $f_0(x_1, x_2)$

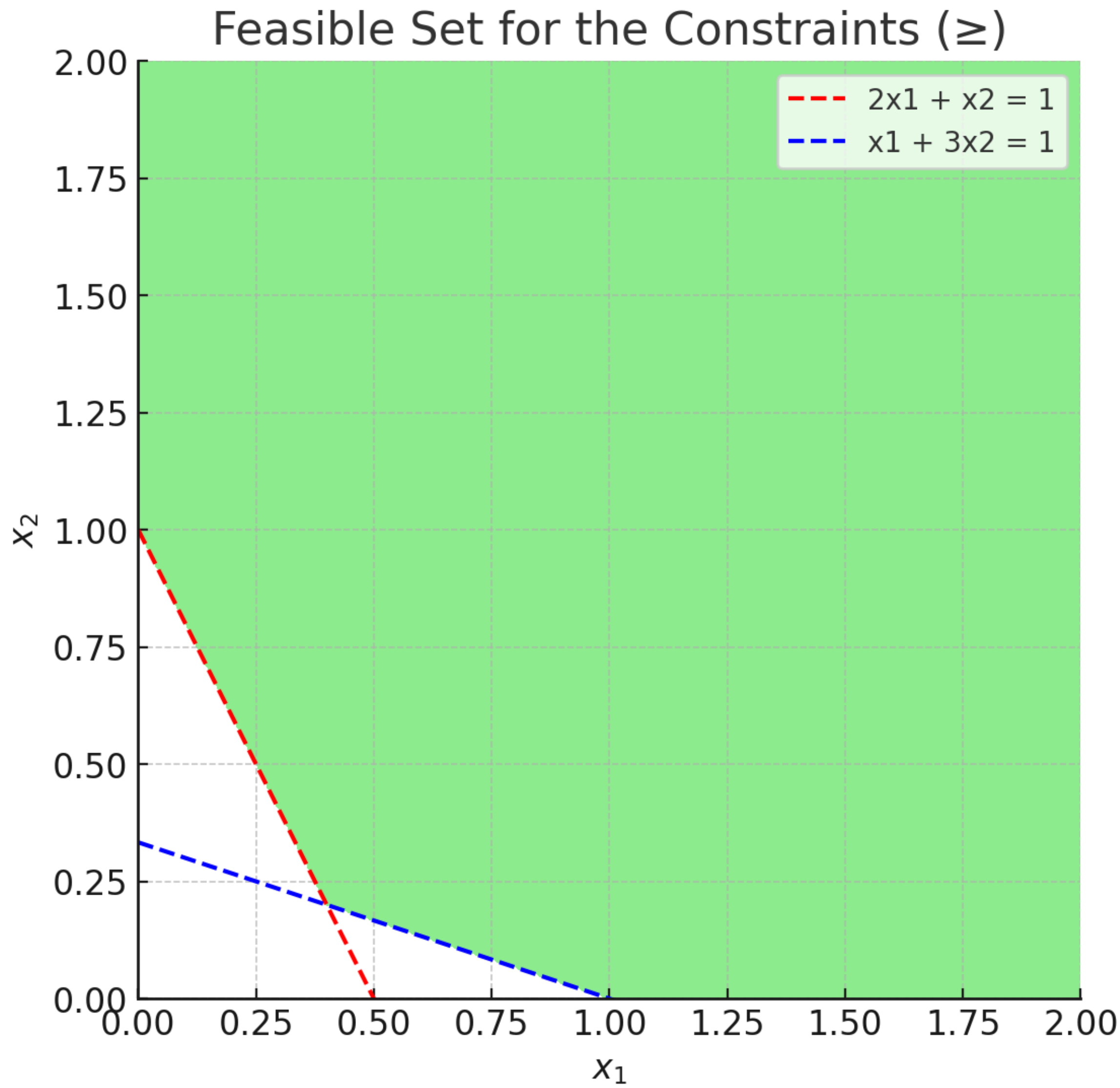
subject to $2x_1 + x_2 \geq 1$

$x_1 + 3x_2 \geq 1$

$x_1 \geq 0, \quad x_2 \geq 0$

Sketch:





Feasible Set for the Constraints (≥)
2x1 + x2 = 1
x1 + 3x2 = 1
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x_1
x_2

Problem 09:-

$$\min \ \frac{1}{2}x^T P x + q^T x + r$$

such that: $\quad -1 \leq x_i \leq 1, \ i = 1,2,3$

Here

$$\nabla f(x) = Px + q$$

$$\nabla f(x^*) = Px^* + q$$

$$= \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

For primal feasibility of x^*:

$$\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}}_{x^*} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, all components of x^* is within the given contraint bound

Now, the KKT conditions for this box-constrained convex optimization are:

$$\nabla f(x^*) + \lambda - \mu = 0$$

where,

$\lambda_i \geq 0$ for active lower bounds $x_i = -1$

$\mu_i \geq 0$ for active upper bounds $x_i = +1$

complementary slackness:

$$\lambda_i (x_i + 1) = 0, \quad \mu_i (x_i - 1) = 0$$

Now, For Lagrange multipliers λ_i, μ_i:

- $x_1 = 1$: upper bound active: $\mu_1 > 0$, $\lambda_1 = 0$
- $x_2 = 0.5$: inside bounds : $\mu_2 = \lambda_2 = 0$
- $x_3 = -1$: lower bound active: $\lambda_3 > 0$, $\mu_3 = 0$

From KKT stationary:

$$\nabla f(x^*) + \lambda - \mu = 0$$

$$\Rightarrow \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \lambda_3 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} \mu_1 = 1 \\ \lambda_3 = 2 \end{cases}$$

So, All multipliers ≥ 0 & satisfy Complementary slackness.

So, $\boxed{x^* \text{ is optimal}}$.

Problem 10:

4.1 Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$.

(b) $f_0(x_1, x_2) = -x_1 - x_2$.

(c) $f_0(x_1, x_2) = x_1$.

(d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$.

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

Matching with the given framework with **CVXPY**, the five objective functions were defined as such:

```
# Define all five objective functions
objectives = {
   "(a) minimize x1 + x2": cvx.Minimize(x1 + x2),
   "(b) minimize -x1 - x2": cvx.Minimize(-x1 - x2),
   "(c) minimize x1": cvx.Minimize(x1),
   "(d) minimize max{x1, x2}": cvx.Minimize(cvx.maximum(x1,
x2)),
   "(e) minimize x1^2 + 9*x2^2": cvx.Minimize(cvx.square(x1) + 9
* cvx.square(x2))
}
```

Full code :

```
import cvxpy as cvx
```

```
# Define variables
x1 = cvx.Variable()
x2 = cvx.Variable()

# Define constraints (inequalities)
constraints = [
   2 * x1 + x2 >= 1,
   x1 + 3 * x2 >= 1,
   x1 >= 0,
   x2 >= 0
]

# Define all five objective functions
objectives = {
   "(a) minimize x1 + x2": cvx.Minimize(x1 + x2),
   "(b) minimize -x1 - x2": cvx.Minimize(-x1 - x2),
   "(c) minimize x1": cvx.Minimize(x1),
   "(d) minimize max{x1, x2}": cvx.Minimize(cvx.maximum(x1,
x2)),
   "(e) minimize x1^2 + 9*x2^2": cvx.Minimize(cvx.square(x1) + 9
* cvx.square(x2))
}

# Solve each problem and print results
for label, obj in objectives.items():
   prob = cvx.Problem(obj, constraints)
   prob.solve()

   print(f"{label}")
   print(f"  Optimal value: {prob.value}")

   # Check for solution
```

```
if x1.value is not None and x2.value is not None:
    print(f"  Optimal x1: {x1.value:.4f}")
    print(f"  Optimal x2: {x2.value:.4f}")
else:
    print("  Problem is unbounded or infeasible.")

print("-" * 40)
```

Output :

```
(a) minimize x1 + x2
  Optimal value: 0.6000000001640435
  Optimal x1: 0.4000
  Optimal x2: 0.2000
----------------------------------------
(b) minimize -x1 - x2
  Optimal value: -inf
  Problem is unbounded or infeasible.
----------------------------------------
(c) minimize x1
  Optimal value: -1.957293364650049e-11
  Optimal x1: -0.0000
  Optimal x2: 1.6916
----------------------------------------
(d) minimize max{x1, x2}
  Optimal value: 0.3333333337083394
  Optimal x1: 0.3333
  Optimal x2: 0.3333
----------------------------------------
```

(e) minimize x1^2 + 9*x2^2
Optimal value: 0.5000000000000002
Optimal x1: 0.5000
Optimal x2: 0.1667

Here, for problem b :

minimize $-x_1 - x_2$
subject to the constraints:

$2x_1 + x_2 \geq 1$
$x_1 + 3x_2 \geq 1$
$x_1 \geq 0$
$x_2 \geq 0$

is equivalent to maximizing $x_1 + x_2$ over the same feasible region, due to the negative sign in the objective. This region is unbounded, so we can make $x_1 + x_2$ arbitrarily large, which means $\mathbf{-x_1 - x_2 \to -\infty}$. So, there is no optimal value of x1 and x2 that satisfy this condition.