Practice Problems (Week 7)

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Problem: Let R be an $n \times n$ matrix, not necessarily symmetric, and define the quadratic function $f(x) = x^\top Rx$. Show that $f(x)$ can always be written as $f(x) = x^\top Px$, where P is a symmetric matrix. Conclude that every quadratic function of this form corresponds to a symmetric matrix.

Solution: Since R is not necessarily symmetric, decompose it into its symmetric and skew-symmetric components, $R = P + S$, where $P = \frac{1}{2}(R + R^\top), \quad S = \frac{1}{2}(R - R^\top)$. Here, P is symmetric since $P^\top = \left(\frac{1}{2}(R + R^\top)\right)^\top = \frac{1}{2}(R^\top + R) = P$, while S is skew-symmetric, satisfying $S^\top = \left(\frac{1}{2}(R - R^\top)\right)^\top = \frac{1}{2}(R^\top - R) = -S$.

Substituting this decomposition into the quadratic form, $f(x) = x^\top(P + S)x = x^\top Px + x^\top Sx$. The term $x^\top Sx$ vanishes for all x, as follows from the skew-symmetry of S: $x^\top Sx = -x^\top Sx$. Since this expression equals its own negative, it must be zero, leaving $f(x) = x^\top Px$.

Since every quadratic function of the form $f(x) = x^\top Rx$ can be rewritten as $f(x) = x^\top Px$ with a symmetric matrix P, it follows that every quadratic function corresponds to a symmetric matrix.

8.1 *Minimizing a quadratic function.* Consider the problem of minimizing a quadratic function:

$$\text{minimize} \quad f(x) = (1/2)x^T Px + q^T x + r,$$

where $P \in \mathbf{S}^n$ (but we do not assume $P \succeq 0$).

(a) Show that if $P \not\succeq 0$, *i.e.*, the objective function f is not convex, then the problem is unbounded below.

(b) Now suppose that $P \succeq 0$ (so the objective function is convex), but the optimality condition $Px^\star = -q$ does not have a solution. Show that the problem is unbounded below.

Solution.

(a) If $P \not\succeq 0$, we can find v such that $v^T Pv < 0$. With $x = tv$ we have

$$f(x) = t^2(v^T Pv/2) + t(q^T v) + r,$$

which converges to $-\infty$ as t becomes large.

(b) This means $q \notin \mathcal{R}(P)$. Express q as $q = \tilde{q} + v$, where $\tilde{q}$ is the Euclidean projection of q onto $\mathcal{R}(P)$, and take $v = q - \tilde{q}$. This vector is nonzero and orthogonal to $\mathcal{R}(P)$, *i.e.*, $v^T Pv = 0$. It follows that for $x = tv$, we have

$$f(x) = tq^T v + r = t(\tilde{q} + v)^T v + r = t(v^T v) + r,$$

which is unbounded below.

8.3 *Initial point and sublevel set condition.* Consider the function $f(x) = x_1^2 + x_2^2$ with domain $\mathbf{dom}\, f = \{(x_1, x_2) \mid x_1 > 1\}$.

(a) What is $p^\star$?

(b) Draw the sublevel set $S = \{x \mid f(x) \leq f(x^{(0)})\}$ for $x^{(0)} = (2, 2)$. Is the sublevel set S closed? Is f strongly convex on S?

(c) What happens if we apply the gradient method with backtracking line search, starting at $x^{(0)}$? Does $f(x^{(k)})$ converge to $p^\star$?

Solution.

(a) $p^\star = \lim_{x \to (1,0)} f(x_1.x_2) = 1$.

(b) No, the sublevel set is not closed. The points $(1 + 1/k, 1)$ are in the sublevel set for $k = 1, 2, \ldots$, but the limit, $(1, 1)$, is not.

(c) The algorithm gets stuck at $(1, 1)$.

Problem: Consider the following subsets of $\mathbb{R}^2$:

1. $S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$
2. $S_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$
3. $S_3 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1\}$
4. $S_4 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1\}$
5. $S_5 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

For each set, determine whether it is open, closed, both, or neither in $\mathbb{R}^2$. Justify your answers.

Solution: Recall that a set is open if every point in the set has a small open ball around it that is fully contained in the set. A set is closed if its complement is open.

The set S_1 is open because every point strictly inside the disk has a small open ball that remains inside. It is not closed because its complement, which includes the boundary, is not open.

The set S_2 is closed because its complement, the set $(x, y) \mid x^2 + y^2 > 1$, is open. It is not open because points on the boundary do not have open balls around them that stay entirely inside S_2.

The set S_3 is open because every point in S_3 has a small open ball that remains in S_3. It is not closed because its complement, which includes the boundary, is not open.

The set S_4 is closed because its complement, the set $(x, y) \mid x^2 + y^2 < 1$, is open. It is not open because points on the boundary do not have open balls around them that stay entirely in S_4.

The set S_5 is not open but is closed. It is not open because any small open ball around a point on the boundary contains points both inside and outside the unit disk, meaning it does not fully belong to S_5. However, S_5 is closed because its complement, which consists of both the interior and exterior of the disk, is a union of two open sets and is therefore open.

Problem: Why is it important for initialization sublevel sets to be closed in descent-based optimization methods? What could go wrong if the set is not closed?

Solution: In descent-based optimization, the initialization sublevel set consists of all points where an optimization algorithm may start and remain during iterations. If this set is closed, it contains all its limit points, ensuring that any convergent sequence of iterates has its limit within the feasible region.

If the initialization sublevel set is not closed, an optimization sequence could approach a boundary point that is not part of the set. This can lead to instability, failure to find a minimizer, or premature termination of the algorithm. For example, if the iterates approach a point outside the set that the algorithm cannot access, the optimization process may not converge properly. Ensuring that initialization sublevel sets are closed guarantees that the method remains well-defined and that convergence to a valid solution is possible.