Class #26

Tuesday, April 29, 2025 10:23 AM

Reminder: when strong duality holds, we end up with the following conditions (necessary conditions):

① Complementary Slackness

Let x^* and λ^* be optimal primal and dual variables then:

$$\boxed{\lambda_i^* f_i(x^*) = 0 \quad \forall\, i=1,\dots,m}$$

② Lagrangian optimality

Let x^* and (λ^*, ν^*) be optimal primal and dual variables:

Then the Lagrangian evaluated at (λ^*, ν^*) is minimized at x^*.

$$\min_x L(x, \lambda^*, \nu^*) = L(x^*, \lambda^*, \nu^*)$$

Assuming $f_0(x)$, $f_i(x)$, $h_i(x)$, are differentiable
$i=1,\dots,m$ $\quad i=1,\dots,p$

then

$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

Remember:

$$L(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

$$\nabla_x L(x,\lambda,\nu) = \nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x)$$

$$\nabla_x L(x, \lambda, \nu) = \nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) + \sum_{i=1} \nu_i \nabla h_i(x)$$

$$\rightarrow \boxed{\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0}$$

Karush-Kuhn-Tucker (KKT) Conditions for strong duality and optimality

1. Necessary conditions for any constrained optimization problem with strong duality
2. Sufficient conditions for any convex constrained optimization problem for x^* and (λ^*, ν^*) being optimal.
3. When Slater's condition is satisfied for a convex problem $\Rightarrow$ strong duality holds $\Rightarrow$ KKT conditions are both necessary and sufficient for x^* and (λ^*, ν^*) to be primal and dual optimal.

KKT Conditions

Let x^* and (λ^*, ν^*) be primal and dual optimal variables. Then:

1. $f_i(x^*) \le 0$, $i = 1, \dots, m$ (feasibility of x^*)
2. $h_i(x^*) = 0$, $i = 1, \dots, p$ (feasibility of x^*)
3. $\lambda_i^* \ge 0$, $i = 1, \dots, m$ (feasibility of λ^*)

④ $\lambda_i^* f_i(x^*) = 0$, $i=1,\dots,m$ (complementary slackness)

⑤ $\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \boxed{\sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*)} = 0$ (Lagrangian optimality)

$A^T \nu^*$ for convex problems

When the problem is convex:

$h_i(x) = 0$, $i=1,\dots,p$ $\Leftrightarrow$ Linear constraints

$\Leftrightarrow$ $Ax = b$ $A \in \mathbb{R}^{p\times n}$ matrix

$\Leftrightarrow$ $Ax - b = 0$

$\Leftrightarrow$ $\sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = A^T \nu^*$

$\nu^T \begin{bmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{bmatrix} = \nu^T A x \Rightarrow \nabla(\nu^T A x) = A^T \nu$

Remark: All convex optimization solvers are based on the KKT conditions.

(Convex constrained optimization problems have a hierarchy

① Top level problem:

A convex quadratic problem with equality constraints ONLY

The solution of this problem comes from the KKT conditions and is equivalent to simply solving a system of linear equations

The solution of this problem comes from the KKT conditions and is equivalent to simply solving a system of linear equations

$\hookrightarrow$ Numerical linear algebra

② Next level problem:

A general convex objective $f_0(x)$, not quadratic, with equality constraints ONLY

$\hookrightarrow$ Approximate at each iteration $x^{(k)}$, the function $f_0(x^{(k)})$ by a quadratic (requires twice differentiability) subject to $A(x^{(k)} + \Delta x_{nt}^{(k)}) = b$

$$f_0\left(x^{(k)} + \Delta x_{nt}^{(k)}\right) \approx f_0(x^{(k)}) + \nabla f_0(x^{(k)}) \Delta x_{nt}^{(k)} + \frac{1}{2} \Delta x_{nt}^{(k)T} \nabla^2 f_0(x^{(k)}) \Delta x_{nt}^{(k)}$$

Quadratic function w.r.t Δx_{nt}

$$\text{s.t.} \quad A\left(x^{(k)} + \Delta x_{nt}^{(k)}\right) = b$$

Feasible-Start Newton's Method

③ Third level Problem

A general convex problem, with both inequality and equality constraints.

$\hookrightarrow$ At each iteration $x^{(k)}$, we will approximate the general problem with a convex problem with equality constraints ONLY, solve that to obtain the next iterate value and continue.

↳ Interior-point methods

Solution of a Convex Quadratic Problem with Equality Constraints ONLY

$$\min_{x} \quad \frac{1}{2} x^T P x + q^T x + r \Rightarrow \nabla f_0(x) = Px + q$$

$$\text{subject to} \quad Ax = b$$

↳ $A \in \mathbb{R}^{p \times n}$

$P \succeq 0 \ (P \in S^n_+)$.

Slater condition holds (unless the problem is infeasible) $\Rightarrow$ KKT conditions are both necessary and sufficient.

① $Ax^* = b$

② $\nabla f_0(x^*) + A^T \nu^* = 0$

$\nabla f_0(x) = Px + q$

$\Leftrightarrow$ ★ ① $Ax^* = b$ → p linear equation

② $Px^* + A^T \nu^* + q = 0$

↳ n linear equations

System of linear equations

Unknows: $x^* \Rightarrow n$ elements

$\nu^* \Rightarrow p$ elements

$(n+p) \times (n+p)$
$$\begin{bmatrix} P & A^T \\ \cdots \end{bmatrix} \begin{bmatrix} x^* \\ \cdots \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix}}_{(n+p)\times(n+p)\ \text{symmetric matrix}} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} \\ b \end{bmatrix}$$

$(n+p)$ - dimensional vector

KKT equations | KKT System

$$B = \begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+p)}$$

unknown

$$y = \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} \in \mathbb{R}^{n+p}$$

$$z = \begin{bmatrix} -q \\ b \end{bmatrix} \in \mathbb{R}^{n+p}$$

The Solution (primal + dual) of a quadratic problem with equality constraints is given by the Solution of the KKT System:

$$By = z$$

① when $z \notin \mathcal{R}(B) \Rightarrow$ There is no Solution

$\Rightarrow$ Infeasible problem or problem is unbounded below.

② when B is rank deficient (i.e., $\text{rank}(B) < n+p$) then we have infinite many solutions.

③ when B is full rank, there is a unique Solution.

Solution.

$$\begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = B^{-1} z$$

A Sufficient condition for B to be full rank (unique solution) is: $P \succ 0$ (positive definite P)

General Convex optimization with Equality Constraints ONLY

Approach I: Eliminate the equality constraint and work with an unconstrained problem only (chapter 4)

$$\min_x \quad f_0(x)$$

$$\text{s.t.} \quad Ax = b$$

$$\{x : Ax = b\} = \{Fz + x_0 : z \in \mathbb{R}^{n-p}\}$$

Where $F \in \mathbb{R}^{n\times(n-p)}$ is any matrix whose range is the null space of A.

$$\Leftrightarrow \quad \min_z \; f_0(Fz + x_0)$$

↳ unconstrained optimization

Solve using GD, Newton's method.

How to find F and x_0?

QR factorization of $A^T = [Q_1 \; Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}$ → upper triangular

QR factorization of $A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}$ → upper triangular

Q_1: $n \times p$; Q_2: $n \times (n-p)$

$$Q_1^T Q_1 = I \quad ; \quad Q_1^T Q_2 = 0$$

$$Q_2^T Q_2 = I$$

Complexity is $O(np^2)$

Take: $x_0 = Q_1 R^{-T} b$ (verify that $Ax_0 = b$)

$$F = Q_2$$

⇒ Solve $\min_z f_0(Q_2 z + Q_1 R^{-T} b)$

to get z^*

$$x^* = Fz^* + x_0 = Q_2 z^* + Q_1 R^{-T} b$$