Class #22

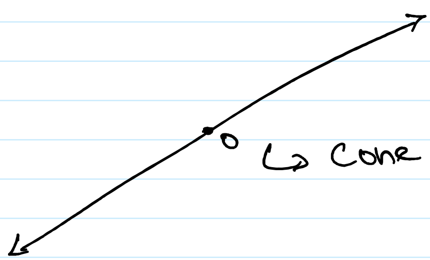
Tuesday, April 15, 2025 12:16 AM

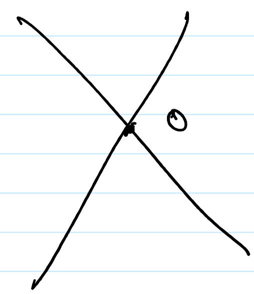
Cone: A set C is called a cone (non-negative homogeneous) if

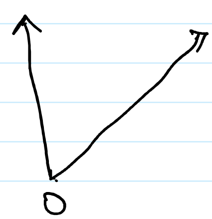
$$\forall x \in C, \text{ and } \forall \theta \geq 0, \ \theta x \in C$$

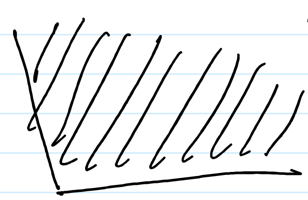
$\Rightarrow$ since θ can be zero $\Rightarrow$ $0 \in$ Cone.

If x is in C then the ray from 0 to x must be in C.







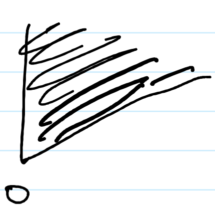


Convex Cone

A set C is a convex cone if it is a cone and convex.

$$\Leftrightarrow \forall \theta_1, \theta_2 \geq 0 \text{ and } \forall x_1, x_2 \in C$$

$$\theta_1 x_1 + \theta_2 x_2 \in C$$



① A line passing through origin is affine, subspace, and a convex cone

② A subspace is always a convex cone.

<u>Space of Positive Semidefinite Matrices</u>

$$S_+^n = \{ X \in \underbrace{S^n}_{\text{Space of symmetric matrices}} : X \succeq 0 \} \qquad \underbrace{S_{++}^n = \{ X \in S^n : X \succ 0 \}}_{0 \notin S_{++}^n}$$

S_+^n is a convex cone.

Let X and $Y \in S_+^n$

Let $\theta_1, \theta_2 \geq 0$

$$v^T (\theta_1 X + \theta_2 Y) v \overset{?}{\geq} 0$$

$$\underbrace{\theta_1}_{\geq 0} \underbrace{v^T X v}_{\geq 0} + \underbrace{\theta_2}_{\geq 0} \underbrace{v^T Y v}_{\geq 0} \geq 0$$

Similar to <u>affine hull</u> and <u>convex hull</u> of a set, we can define <u>conic hull</u> of a set.

Conic Combination of a set of points is:

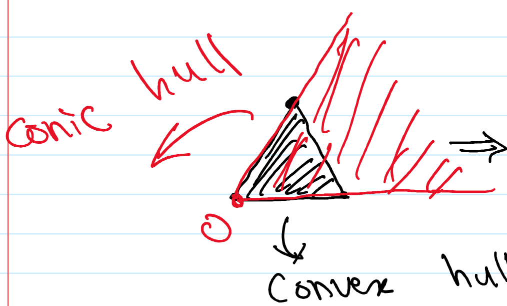
Given $x_1, \ldots, x_k$ and $\theta_1, \ldots, \theta_k \geq 0$

$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$ is called conic Combination

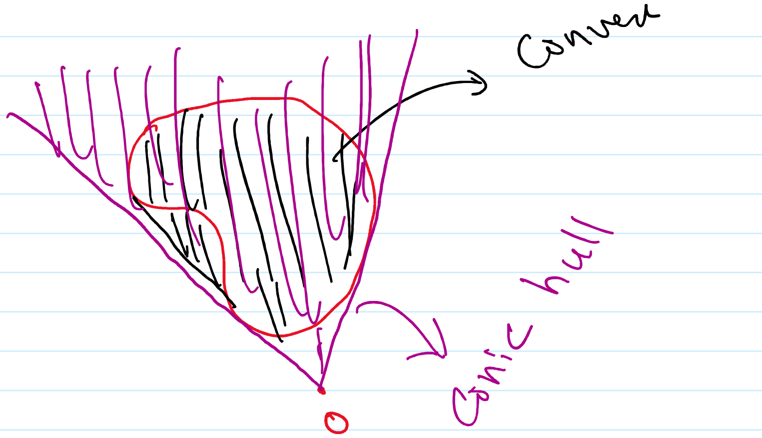
$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$ is called conic combination.

Conic hull of a set C is the set of all conic combinations of C.

$$= \{ \theta_1 x_1 + \cdots + \theta_k x_k ;\ \theta_i \geq 0,\ x_i \in C \}$$



$\Rightarrow$ Affine hull $= \mathbb{R}^2$



Facts :

* Any norm ball is convex

$$\{ x : \|x - x_c\| \leq r \}$$

↳ regardless of the norm

* An ellipsoid is a convex set.

Norm Cone :

$$C = \{ (x,t) : \|x\| \leq t \} \subset \mathbb{R}^{n+1}$$

$$C = \{(x,t) : \|x\| \le t\} \subset \mathbb{R}$$

$t \ge 0$ any norm

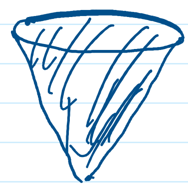
It is a convex cone.

$\underbrace{\text{Second-order cone}}_{\text{ice cream cone}} \Rightarrow$ norm cone corresponding to $\|\cdot\| = \|\cdot\|_2$

$$C = \{(x,t) \in \mathbb{R}^{n+1},\ \|x\|_2 \le t\} \qquad \|z\|_2^2 = z^T z$$

$t \ge 0$

Quadratic Cone



$$\left\{ \begin{bmatrix} x \\ t \end{bmatrix} : \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le 0,\ t \ge 0 \right\}$$

$$\underbrace{z^T R z}_{\text{Quadratic equation}} \le 0$$

Operations that preserve Convexity of Sets

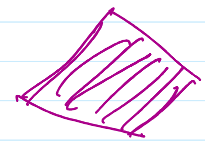
Affine functions and Convex Sets

Let $f(x) = Ax + b$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ be an affine function.

① Let $S \subset \mathbb{R}^n$ be a convex set

$\underbrace{f(S)}_{\text{image of } S \text{ under } f} = \{ f(x) : x \in S \}$ is convex

$S \subset \mathbb{R}^m$

(2) $\underbrace{f^{-1}(S)}_{\text{preimage of } S \text{ under } f} \xrightarrow{\quad f \quad}$ 

The inverse image of any convex set under an affine function is always convex

$$f^{-1}(S) = \{ x \in \mathbb{R}^n : f(x) \in S \}$$

Special Cases

(1) If S is convex then $\alpha S = \{ \alpha x : x \in S \}$ is convex.

↳ why! $f(x) = \alpha x$

image of S under $f(x) = \alpha S$

(2) If S is convex then $S + a = \{ x + a : x \in S \}$ is convex.

↳ why? Take $f(x) = x + a$

image of S under f.

(3) If S_1 and S_2 are convex then

$S_1 + S_2$ is a convex set

$$S_1 + S_2 = \{x+y : x \in S_1, y \in S_2\}$$

$\hookrightarrow f(x,y) = x+y$

④ If $S \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ is convex then projection of S onto some of its coordinates is convex.

⑤ $S = \{x : \underbrace{A(x) \preceq B}\}$ ☆ A_i's and B are symmetric (S^m)

$A(x) - B \preceq 0 \Rightarrow B - A(x) \succeq 0$

where $A(x) = x_1 A_1 + \cdots + x_n A_n$

$\Leftrightarrow x_1 A_1 + \cdots + x_n A_n \preceq B \Rightarrow$ Linear matrix inequality (LMI)

Let $f : \mathbb{R}^n \to S^m$

$$f(x) = B - A(x)$$

$$\underbrace{S} = f^{-1}(S^m_+) = \{x : \underbrace{B - A(x) \succeq 0}_{\text{Convex}}\}$$

$\hookrightarrow$ convex b/c pre-image of a convex set

Perspective function :

n

Perspective function:

$$P : \mathbb{R}^{n+1} \to \mathbb{R}^n \quad \text{with} \quad \operatorname{dom} P = \mathbb{R}^n \times \mathbb{R}_{++}$$

$$P(z,t) = \frac{z}{t}$$

If $C \subseteq \operatorname{dom} P$ is convex then

$$P(C) = \{P(x) : x \in C\}$$

is convex.

Convex functions

Typically, we prove functions are convex in four different ways:

① Jensen's inequality | zeroth-order condition

$$f(\theta x_1 + (1-\theta) x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$

$\forall \ \theta \in [0,1]$

② If f is second-order differentiable then we show that

$$\nabla^2 f(x) \succeq 0 \quad \forall \ x$$

③ f is convex if and only if its restriction to a line is convex

$\tilde{f}(t) = f(z+tv)$ and show it is convex for any z and v

Example: Show that $f(X) = \log\det X$ is concave on $\operatorname{dom} f = S^n_{++}$ (pos. def. matrices)

Let us work with $\tilde{f}(t) = f(Z+tV)$ and show that it is concave in t for any Z and V matrices. Since $t=0 \Rightarrow$ we have $f(Z)$ so $Z \in S^n_{++}$

Since $Z = U\Lambda U^T$

$\Rightarrow Z^{1/2} = U\Lambda^{1/2}U^T \Rightarrow Z = Z^{1/2}Z^{1/2}$

$$
\begin{aligned}
\tilde{f}(t) &= \log\det(Z+tV) \\
&= \log\det\left(Z^{1/2}\left(I+tZ^{-1/2}VZ^{-1/2}\right)Z^{1/2}\right) \\
&= \log\left(\det Z^{1/2} \cdot \det\left(I+tZ^{-1/2}VZ^{-1/2}\right) \cdot \det Z^{1/2}\right) \\
&= \log\left(\det\left(I+tZ^{-1/2}VZ^{-1/2}\right) \cdot \det Z\right) \\
&= \log\det\left(I+tZ^{-1/2}VZ^{-1/2}\right) + \log\det Z
\end{aligned}
$$

Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of $Z^{-1/2}VZ^{-1/2}$

Let $\{\lambda_i\}_{i=1}$ be the eigenvalues of $Z \;\; V Z$

$\Rightarrow$ Eigenvalues of $\left(I + t Z^{-1/2} V Z^{-1/2}\right)$

are $\{1 + t\lambda_i\}_{i=1}^{n}$

$$\det(M) = \prod_{i=1}^{n} \text{eigenvalues}(M)$$

$$\Rightarrow \tilde{f}(t) = \log \prod_{i=1}^{n} (1 + t\lambda_i) + \log\det Z$$

$$= \sum_{i=1}^{n} \log(1 + t\lambda_i) + \log\det Z$$

$\tilde{f}(t)$ is twice-differentiable in t

$\Rightarrow \tilde{f}(t)$ is concave $\Leftrightarrow \tilde{f}''(t) \leq 0$

$$\tilde{f}'(t) = \sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i} + 0$$

$$\tilde{f}''(t) = -\sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + t\lambda_i)^2} \leq 0 \Rightarrow \log\det X$$

is concave $\square$