# Linear Algebra and Applications
# Homework #07

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# Problem 1:

**(a)**

$Q_1$ is orthogonal: $Q_1^T Q_1 = I$

$Q_2$ is orthogonal: $Q_2^T Q_2 = I$

Let, $Q = Q_1 Q_2$

Now,

$$
\begin{aligned}
Q^T Q &= (Q_1 Q_2)^T Q_1 Q_2 \\
&= Q_2^T Q_1^T Q_1 Q_2 \\
&= Q_2^T \, I \, Q_2 \qquad (Q_1^T Q_1 = I) \\
&= Q_2^T Q_2 \\
&= I \qquad (Q_2^T Q_2 = I)
\end{aligned}
$$

So, $Q = Q_1 Q_2$ is orthogonal if $Q_1$ & $Q_2$ are orthogonal.

**(b)** $Q$ is orthogonal: $Q^T Q = I$

$$\Rightarrow \det(Q^T Q) = \det(I)$$

$$\Rightarrow \det(Q^T) . \det(Q) = 1$$

But $\det(Q^T) = \det(Q)$

so, $(\det(Q))^2 = 1$

$\Rightarrow \boxed{\det Q = \pm 1}$

(c)

$Q_1$ : Counter-clockwise rotations by $\theta_1$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

similarly, $Q_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

Now, $Q_1 Q_2 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

similarly,

$$Q_2 Q_1 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

So, $Q_1 Q_2 = Q_2 Q_1 =$ counter-clockwise rotation by $(\theta_1 + \theta_2)$

(d)

No, not all $2\times 2$ orthogonal matrices commute.

Let, $Q_1 =$ counter-clockwise rotation by $\theta$ $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$Q_2 =$ reflection with respect to $x$ axis $= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Now,

$$Q_1 Q_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

$$Q_2 Q_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$$

Here, $Q_1 Q_2 \neq Q_2 Q_1$ unless $\theta = 0$

so, $Q_1$ & $Q_2$ does not commute in general.

(e) Orthogonal matrix with $\det(Q) = 1$ represents a rotation

Because:

(i) $|\det(Q)| = 1$, so it is length-preserving

(ii) $\det(Q) = 1$, it preserves orientation (no reflection)

So, it is a rotation.

Also, a rotation (counter-clockwise) by angle $\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\det(Q) = \cos^2\theta + \sin^2\theta = 1$$

(f) orthogonal matrix with $\det(Q) = -1$ represents a reflection.

Because:

(I) $|\det(Q)| = 1$, so it preserves length of the vector

(II) $\det(Q) = -1$, it reverses the orientation.

So, it represents a reflection.

Also, reflection matrix $M_\phi$ over a line through the origin making angle $\phi$ with the x-axis :

$$M_\phi = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$$

$$\det(M_\phi) = -\cos^2 2\phi - \sin^2 2\phi = -1$$

<u>(g)</u>

$Q_1$ = Reflection $\rightarrow \det(Q_1) = -1$

$Q_2$ = Rotation $\rightarrow \det(Q_2) = 1$

$Q_1 Q_2$ : rotation followed by reflection

<u>Now</u> $\det(Q_1 Q_2) = \det(Q_1) \cdot \det(Q_2)$

$= -1 \cdot 1 = -1$

So, $\boxed{Q_1 Q_2 \text{ Represents a reflection}}$

Now,

$Q_2Q_1$: reflection followed by rotation

Now,

$$\det(Q_2 Q_1) = \det(Q_2) \cdot \det(Q_1)$$
$$= 1 \cdot -1$$
$$= -1$$

So, $\boxed{Q_2Q_1 \text{ represents a reflection}}$

So, $Q_1Q_2$ & $Q_2Q_1$ are both reflections ($\det = -1$) but $Q_1Q_2 \neq Q_2Q_1$ in general (unless no rotation)

So, they don't represent the same reflection, the specific lines of reflection differ.

(h) $Q_1$: reflection

$Q_2$: reflection

<u>$Q_1 Q_2$</u>:

$$\det(Q_1 Q_2) = \det(Q_1)\det(Q_2)$$
$$= -1 \cdot -1$$
$$= 1$$

So, $\boxed{Q_1 Q_2 \text{ represents a rotation}}$

$\det(Q_1 Q_2) = 1 \longrightarrow$ both orientation & length preserving

If,

$Q_1 =$ reflection across a line $L_1$ at angle $\theta_1$ $= \begin{bmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ \sin 2\theta_1 & -\cos 2\theta_1 \end{bmatrix}$

$Q_2 =$ reflection across a line $L_2$ at angle $\theta_2$ $= \begin{bmatrix} \cos 2\theta_2 & \sin 2\theta_2 \\ \sin 2\theta_2 & -\cos 2\theta_2 \end{bmatrix}$

then,

$$S_1 S_2 = \begin{bmatrix} \cos 2(\theta_1-\theta_2) & -\sin 2(\theta_1-\theta_2) \\ \sin 2(\theta_1-\theta_2) & \cos 2(\theta_1-\theta_2) \end{bmatrix}$$

This is rotation by angle $2(\theta_1-\theta_2)$

Problem 2

(a)

Here

$$u_k = \frac{v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j}{\left\| v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j \right\|_2}$$

rearranging,

$$v_k = \overbrace{\left\| v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j \right\|_2}^{\beta_{kk}} u_k$$

$$+ \underbrace{\sum_{j=1}^{k-1} \langle v_k, u_j \rangle}_{\beta_{kj}} u_j$$

Thus,

$$v_k = \sum_{j=1}^{k} \beta_{kj} u_j$$

where,

$$\beta_{kj} = \begin{cases} \langle v_k, u_j \rangle & ; \text{for } j < k \\ \left\| v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j \right\|_2 & ; \text{for } j = k \end{cases}$$

(b) Here,

matrix V: $= [v_1 \ v_2 \ \cdots \ v_n]$ $m \times n$ matrix

matrix Q: $= [u_1 \ u_2 \ \cdots \ u_n]$ $m \times n$ matrix orthonormal columns

From part (a), each $v_k$ is a linear combinations of $u_1 \cdots u_n$:

$$v_k = \sum_{j=1}^{k} \beta_{kj} u_j$$

So, in matrix format:

$(m \times n) \times (n \times n)$

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ 0 & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \beta_{nn} \end{bmatrix}$$

$$V = Q \cdot R$$

Thus $R$ must be upper triangular with

entries $R_{jk} = \beta_{kj}$ for $j \le k$

& $R_{jk} = 0$ for $j > k$

**(c)** From (a), the entries of $R$ are:

$$R_{jk} = \begin{cases} \beta_{kj} = \langle v_k, u_j \rangle & ; \text{ for } j < k \\ \beta_{kk} = \left\| v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j \right\|_2 & ; \text{ for } j = k \\ 0 & ; \text{ for } j > k \end{cases}$$

Structure of $R$:

(i) Upper Triangular: $R_{jk} = 0$ for $j > k$

(ii) Diagonal entries : $R_{kk} = \beta_{kk}$

(iii) off diagonal entries: $R_{jk} = \beta_{kj}$ for $j < k$