Class #23

Friday, April 18, 2025 10:21 AM

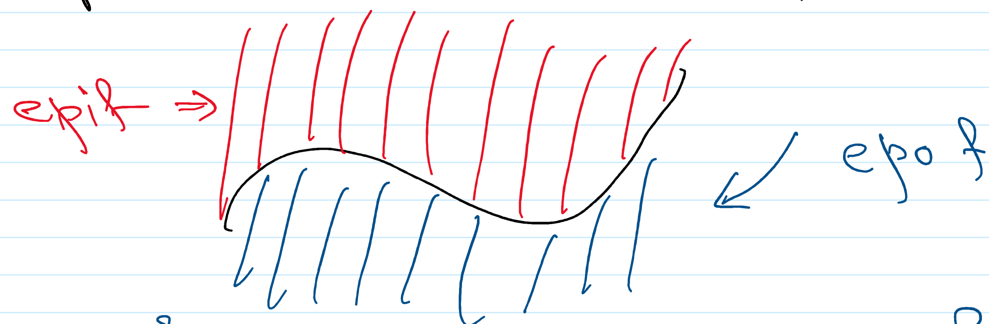
Sublevel set

$$S_\alpha = \{x : f(x) \le \alpha\} \subset \mathbb{R}^n$$

Epigraph of a function

Let $f : \mathbb{R}^n \to \mathbb{R}$

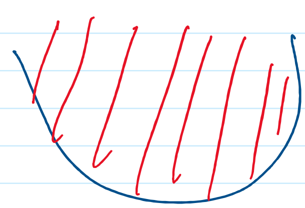
$$\underbrace{\text{epi } f}_{\text{epigraph of } f} = \{(x,t) : x \in \text{dom} f,\ f(x) \le t\} \subset \mathbb{R}^{n+1}$$



$$\text{epo } f = \{(x,t) : x \in \text{dom} f,\ f(x) \ge t\}$$

$f(x)$ is convex if and only if epi f is convex

" " concave " " " " epo f is convex



Example: Let f be convex and $w \ge 0$

Define $g(x) = w f(x)$

Is $g(x)$ convex?

$$\text{epi } g = \{(x,t) : x \in \text{dom} g,\ g(x) \le t\}$$

$$= \{(x,t) : x \in \text{dom} f,\ \omega f(x) \le t\}$$

$$= \{(x,t) : x \in \text{dom} f,\ \underbrace{f(x) \le \frac{t}{\omega}}_{\text{epi } f \text{ wrt } \frac{t}{\omega}}\}$$

identity matrix

$$= \underbrace{\begin{bmatrix} I & 0 \\ 0 & \omega \end{bmatrix}}_{\text{linear transformation}} \underbrace{\text{epi} f}_{\text{convex}} \qquad \left(x, \frac{t}{\omega}\right)$$

$\Rightarrow$ epi g is convex $\Rightarrow$ g is a convex function.

Functions / properties that lead to convex functions

① Non-negative weighted sums of convex functions

Let $f_1, \ldots, f_m$ be convex

$\Rightarrow$ non-negative linear combinations of them are convex

$$g(x) = \omega_1 f_1(x) + \cdots + \omega_m f_m(x)$$

and $\omega_i \geq 0 \Rightarrow g(x)$ is convex.

② Composition with an affine function

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$

If f is convex then

$g(x) = f(Ax+b)$ is convex on $\text{dom}\, g = \{x : Ax+b \in \text{dom} f\}$

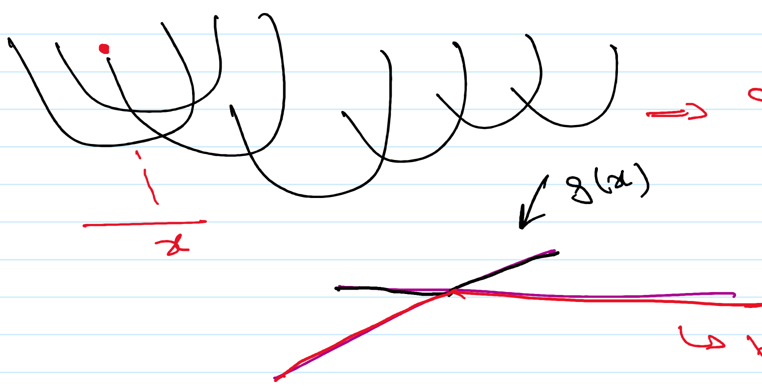
③ Pointwise maximum and supremum of functions

Let f_1, f_2 be convex functions

Let $g(x) = \max\{f_1(x), f_2(x)\}$

Then $g(x)$ is convex

In general, pointwise maximum of any finite number of convex functions is convex.



→ $g(x)$ is convex

↳ pointwise minimum for concave

Let $f_\alpha(x)$ be convex $\forall\, \alpha \in A$

↳ set that is uncountable

$$g(x) = \sup_{\alpha \in A} f_\alpha(x)$$

↳ $g(x)$ is still convex

Proof by epigraph:

$$\text{epi}\, g = \{(x,t) : g(x) \le t\}$$

$$= \{(x,t) : \sup_{\alpha \in A} f_\alpha(x) \le t\}$$

$$\Downarrow$$

$f_\alpha(x) \le t$ for every $\alpha \in A$

$(x,t) \in \text{epi}\, g$ if and only

$(x,t) \in \text{epi}\, f_\alpha \quad \forall\, \alpha \in A$

$$\text{epi}\, g = \bigcap_{\alpha \in A} \underbrace{\text{epi}\, f_\alpha}_{\text{convex sets}}$$

= convex set since intersection of finite or infinite convex sets is convex.

$\Rightarrow g$ is convex.

Example:

Let $f(X) = \lambda_{max}(X)$, $\text{dom}\, f = S^n$

symmetric matrices.

If $f(X)$ a convex function of X?

Remember: Given a matrix X

$$\lambda_{max}(X) = \sup_{v : \|v\|_2 = 1} v^T X v$$

$$= \sup_{v} \frac{v^T X v}{\|v\|_2^2}$$

$$g_v(X) = v^T X v \; ; \; v \in \{v : \|v\|_2 = 1\}$$

↳ Convex function of X for any fixed v

$$\lambda_{max}(X) = \sup_{v : \|v\|_2 = 1} g_v(X)$$

= Convex b/c we are taking a supremum of convex functions.

<u>What about minimization?</u>

Let $f(x, y)$ be convex

<u>Ex:</u> $\lambda_{max}(X) \Rightarrow f(X, v) = v^T X v$

Define $g(x) = \inf_y f(x, y)$

↳ we cannot claim that $g(x)$ is convex

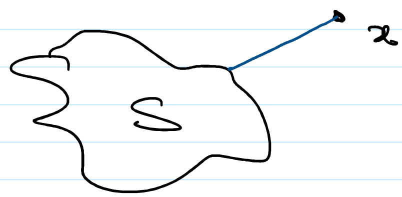
($g(x) = \sup f(x, y)$ is convex)

($g(x) = \sup_y f(x,y)$ is convex)

But Let C be a convex set and $y \in C$. Then

$$\boxed{g(x) = \inf_{y \in C} f(x,y) \text{ is convex.}}$$

Example: Distance of a point x to a set $S \subset \mathbb{R}^n$ using some norm $\|\cdot\|$



$$\text{dist}(x,S) = \inf_{y \in S} \|x-y\|$$

↳ This is a convex function if S is convex.

④ Composition of functions

Ex: ① $f(x) = e^{x}$ is convex

what about $f(x) = e^{g(x)}$?

② $f(x) = \log\left(\sum_{i=1}^{n} e^{x_i}\right)$

What about $f(x) = \log\left(\sum_{i=1}^{n} e^{g_i(x_i)}\right)$?

General Case:

Let $h : \mathbb{R}^k \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^k$

Define: $f(x) = h(g(x))$; $f = h \circ g$

When is f convex?

Look at the special case of k and $n = 1$

$f(x) = h(g(x))$ and assume $h, g \in C^2$

f is convex $\iff f''(x) \geq 0$

$$f''(x) = h''(g(x))\, g'(x)^2 + \underline{h'(g(x))}\, g''(x)$$

Case I: ① h is convex and h is nondecreasing

② g is $\underline{\text{convex}}$

Case II: ① h is convex and h is nonincreasing

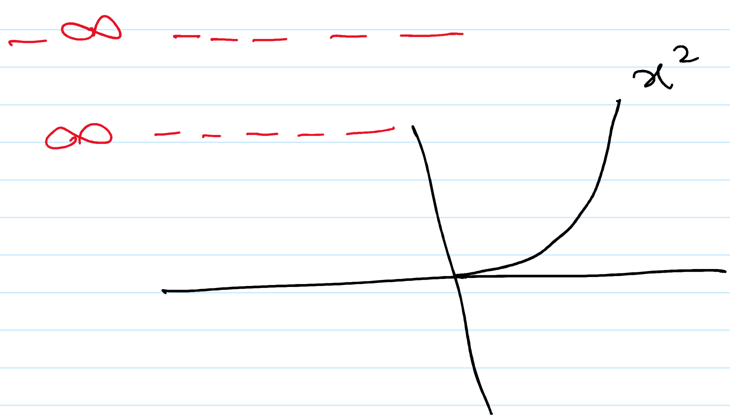
② g is $\underline{\text{concave}}$

When $\operatorname{dom} h \neq \mathbb{R}$ (e.g.; $h(x) = \log(x)$)

then the non-increasing or non-decreasing have to be checked on an extension of $h(x)$

$$\tilde{h}(x) = \begin{cases} h(x), & x \in \operatorname{dom} h \\ \infty, & x \notin \operatorname{dom} h \end{cases}$$

$\log(x)$ $\downarrow$ concave



x^2 for $x \geq 0$

General Case of $K \geq 1$ and $n \geq 1$

$$f(x) = h(g(x)) = h\left(g_1(x), g_2(x), \ldots, g_K(x)\right)$$

Populer Convex optimization Problems

Quadratic Program

- Objective function is quadratic

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- Inequality constraint functions are linear
  (+ Equality constraints are linear)

  ↳ Constraint set is polyhedron

Quadratically Constrained Quadratic Program

↳ Inequality constraints are quadratic

Second-order Cone Program

$$\min \quad a^T x$$

Corresponds to a second-order cone in $\mathbb{R}^{n+1}$

$$\text{s.t.} \quad \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i=1,\dots,m$$

$$Fx = g$$

Semidefinite Program

↳ $\min_x \quad a^T x$

$$\text{s.t.} \quad x_1 F_1 + \dots + x_n F_n + G \succeq 0$$

(LMI)

$F_1, \dots, F_n \in S^k$

$F_1, \ldots, F_n \in S^k$

$Ax = b$