In-Session Discussion

Wednesday, May 7, 2025 4:45 PM

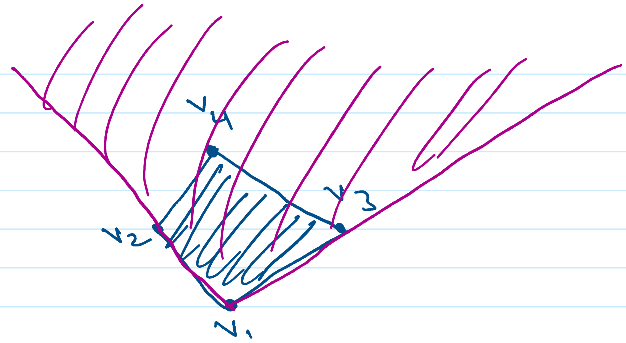
Q1 (Practice Final)

$v_1 = (0,0)$

$v_2 = (-1,1)$

$v_3 = (1,1)$

$v_4 = (0,2)$



Q3: $\{x \in \mathbb{R}^n : \alpha_i \le x_i \le \beta_i \quad i=1,\dots,n\}$? Convex?

$x_i \le \beta_i$, $x_i \ge \alpha_i$, $i=1,\dots,n$

$x_i = e_i^T x$; $e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i^{th}$ location

$\hookrightarrow$ i^{th} standard basis vector

$\Rightarrow$ $\left.\begin{array}{l} e_i^T x - \beta_i \le 0 \\ \boxed{e_i^T x - \alpha_i \ge 0} \end{array}\right\} i=1,\dots,n$

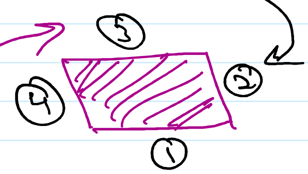
$\Rightarrow$ Rectangle is intersection of $2n$ half spaces.

$\Rightarrow$ Polyhedron $\Rightarrow$ convex

$n=2$
$\beta_i = 1$
$\alpha_i = 0$;

$e_1^T x - 1 \le 0$

$e_2^T x - 1 \le 0$



Q5: (b) Is $\{a \in \mathbb{R}^k,\ p(0)=1,\ |p(t)| \le 1,\ \text{for } \alpha \le t \le \beta\}$

where $p(t) = \underline{a_1 \cdot 1 + a_2 t + \dots + a_k \cdot t^{k-1}}$

convex? $\alpha, \beta \in \mathbb{R}$.

$$p(t) = \underset{\sim}{a}^T \tilde{t} \qquad \tilde{t} = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{k-1} \end{bmatrix}$$

$p(t)$ is a linear function of a for any fixed t.

$$p(0) = \tilde{a}^T \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad |p(t)| \le 1$$

$$\Longleftrightarrow \quad \begin{array}{l} a^T \tilde{t} \le 1 \\ a^T \tilde{t} \ge -1 \end{array}$$

$$C = \underbrace{\left\{a : \tilde{a}^T \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1\right\}}_{\text{hyperplane}} \cap \left(\bigcap_{\alpha \le t \le \beta} \underbrace{\{a : a^T \tilde{t} \le 1\}}_{\text{halfspace}} \right) \cap \left(\bigcap_{\alpha \le t \le \beta} \underbrace{\{a : a^T \tilde{t} \ge -1\}}_{\text{halfspace}} \right)$$

hyperplane

halfspace

Q7: $f(X) = (\det X)^{1/n}$; $X \succ 0$

Let Z and $V \in S^n_{++}$

$$g(t) = f(Z + tV) \; ; \; \mathrm{dom}(g) = \{t : Z + tV \in S^n_{++}\}$$

If we can prove that $g(t)$ is concave

$\Longrightarrow f(X)$ is concave.

$$= \left(\det(Z + tV)\right)^{1/n}$$

$)^{1/n}$

$$= \left(\det\left(Z + tV\right)\right)$$

$$= \left(\det\left(Z^{1/2} \cdot Z^{1/2} + tV\right)\right)^{1/n}$$

$$= \left(\det\left(Z^{1/2}\left(I + tZ^{-1/2}VZ^{-1/2}\right)Z^{1/2}\right)\right)^{1/n}$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i(A)$$

$$\left(\det\left(I + tZ^{-1/2}VZ^{-1/2}\right)\right)^{1/n} = \left(\prod_{i=1}^{n}\left(1 + t\underbrace{\lambda_i\left(Z^{-1/2}VZ^{-1/2}\right)}_{x_i}\right)\right)^{1/n}$$

Q9:

$$\min \; e^{-x} \rightarrow \text{Convex} \qquad D = \{(x,y) : y > 0\}$$

$$\text{subject to} \quad \frac{x^2}{y} \le 0$$

a)

$\hookrightarrow$ Quadratic / linear $\rightarrow$ Convex

$p^* = ?$

$$C = \left\{(x,y) : \frac{x^2}{y} \le 0\right\}$$

$$= \left\{ \{0\} \times (0, \infty) \right\}$$

$$p^* = 1$$

b) $L(x,\lambda) = e^{-x} + \lambda \dfrac{x^2}{y}$, $\quad \lambda \ge 0$

$\{(x,y) : y > 0\}$

$$g(\lambda) = \inf_{(x,y)} \left(e^{-x} + \frac{\lambda x^2}{y}\right)$$

$$\max_{\lambda} \; g(\lambda), \quad \lambda \ge 0$$

$$\max_{\lambda} g(\lambda) \quad , \quad \lambda \geq 0$$

$$\lambda^* \quad , \quad d^*$$

$$g(\lambda) = \inf_{(x,y)} e^{-x} + \lambda \inf_{(x,y)} \frac{x^2}{y}$$

$$= 0 + \lambda . 0 = 0$$

$$d^* = \max_{\lambda} g(\lambda) = 0$$

$$p^* \geq d^* \qquad 1 \geq 0$$