Class #24

Tuesday, April 22, 2025 10:20 AM

Chapter 5

Primal problem

$$\min_x \quad f_0(x) \quad \to x \in \mathbb{R}^n$$

Such that $f_i(x) \le 0$, $i=1,\dots,m$
$h_i(x) = 0$, $i=1,\dots,p$ (P_0)

$$D = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i$$

$p^* \Rightarrow$ optimal value of (P_0) (if solvable)

$x^* \Rightarrow$ one of the optimal (if it exists)

Lagrangian of a Constrained optimization Problem

$$L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$$

$p \to$ # of equality constraints

$m \to$ # of inequality constraints

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

$\lambda \in \mathbb{R}^m$, $\nu \in \mathbb{R}^p$

$$= f_0(x) + \lambda^T F(x) + \nu^T H(x)$$

$$F(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \qquad H(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{bmatrix}$$

$\boxed{\lambda \Rightarrow}$ vector of Lagrange multipliers associated with inequality constraints

$\boxed{\nu \Rightarrow}$ vector of Lagrange multipliers associated with equality constraints

$(\lambda, \nu) \Rightarrow$ dual variables

$x \Rightarrow$ primal variable

Dual function (only function of dual variables)

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$$

Convention: when $L(x, \lambda, \nu)$ is unbounded below in x for certain $(\lambda, \nu) \Rightarrow g(\lambda, \nu) = -\infty$

Lagrangian is a linear function of λ and ν for any x.

linear function

$$\tilde{L}_x(\lambda, \nu) = L(x, \lambda, \nu) \text{ for any } x \in D$$

$$g(\lambda, \nu) = \inf_{x \in D} \tilde{L}_x(\lambda, \nu)$$

Since $g(\lambda, \nu)$ is pointwise infimum of concave (linear functions are concave) functions $\Rightarrow g$ is concave in (λ, ν).

Fact: Dual function is always concave, regardless of the primal problem.

Fact: The dual function provides a uniform lower bound on the optimal value, which is p^*, of the original (primal) problem $\forall \lambda \geq 0$.

$$g(\lambda, \nu) \leq p^* \quad \forall \lambda \succeq 0 \quad \begin{pmatrix} \lambda_1 \geq 0 \\ \lambda_2 \geq 0 \\ \vdots \\ \lambda_m \geq 0 \end{pmatrix}$$

$$\Downarrow$$

$$\underbrace{\max_{\substack{\lambda, \nu: \\ \lambda \succeq 0}} g(\lambda, \nu)}_{} \leq p^*$$

Concave optimization problem

Dual Problem:

$$\max_{\lambda, \nu} \; g(\lambda, \nu)$$ → Constrained optimization

$\Rightarrow$ Always Concave

Subject to $\quad \lambda \succeq 0$

Dual problem when only equality constraints

$\Rightarrow \quad \max_{\nu} g(\nu) \Rightarrow$ Dual problem is unconstrained!

Optimal value of the dual function is

denoted by d^*

λ^* and ν^* denote variables that give us d^*,

Proof that $g(\lambda,\nu) \le p^* \ \forall \lambda \succeq 0$

Let $\tilde{x}$ be any feasible point of the primal problem.

$$\underbrace{\sum_{i=1}^{m} \lambda_i \overbrace{f_i(\tilde{x})}^{\le 0}}_{\le 0 \text{ for } \lambda_i \ge 0} + \underbrace{\sum_{i=1}^{p} \nu_i \overbrace{h_i(\tilde{x})}^{=0}}_{=0}$$

$$L(\tilde{x},\lambda,\nu) = f_0(\tilde{x}) + \underbrace{\sum_{i=1}^{m} \lambda_i f_i(\tilde{x})}_{\le 0 \ \forall \lambda_i \ge 0} + \underbrace{\sum_{i=1}^{p} \nu_i h_i(\tilde{x})}_{=0}$$

$$\le f_0(\tilde{x}) \quad \forall \lambda_i \ge 0$$

$\forall \lambda \succeq 0$

$$g(\lambda,\nu) = \inf_{x \in D} L(x,\lambda,\nu) \le L(\tilde{x},\lambda,\nu)$$

$$\le f_0(\tilde{x})$$

$$\le \min_{\tilde{x}} f_0(\tilde{x})$$

$$\le p^*$$

$$\Rightarrow \forall \lambda \succeq 0, \ g(\lambda,\nu) \le p^*$$

∎

$\Rightarrow \; \lambda \geq 0, \quad g(\lambda, \nu) = f$

(λ, ν) is called dual feasible when:

① $\lambda \succeq 0$

② $(\lambda, \nu) \in \mathrm{dom}\, g$ (i.e., $g(\lambda, \nu) > -\infty$)

Weak duality: Given d^* as the solution of the dual problem, and p^* as the solution of the primal problem:

$$d^* \leq p^*$$

$p^* - d^* \geq 0$ is called duality gap.

when the duality gap is zero $\Leftrightarrow$

$$d^* = p^*$$

we say we have strong duality.

Strong duality $\Rightarrow$ Holds for some nonconvex problems.

In the case of convex primal problems, it can be shown to hold through verification of certain conditions.

Slater's Condition for Strong Duality

Let the primal problem (given by (Po)) be

Slater's Conditions for Strong Duality

Let the primal problem (given by (P_0)) be convex. Then strong duality holds under the following conditions:

There exists some x in the relative interior of D (i.e., it is not on the boundary of D) such that:

① $f_i(x) < 0 \quad \forall\, i = 1, \ldots, m$

② $h_i(x) = 0 \quad \forall\, i = 1, \ldots, p$

$\Longleftrightarrow \quad Ax = b$

Translation: There must be a feasible x in the interior of D such that the inequality constraints are not active.

Weakened form of Slater's Condition when some of the inequality constraints are linear.

Suppose out of $f_i(x)$, $i = 1, \ldots, m$, the first k are linear.

In that case, we only need the inactive inequality constraints on x to be for the nonlinear functions. Problem is convex

① $x \in \text{relint}(D)$

② $f_i(x) \leq 0$, $\forall\, i = 1, \ldots, k$ $\Rightarrow$ linear constraints

③ $f_i(x) < 0$, $\forall\, i = k+1, .., m$

④ $h_i(x) = 0$, $\forall\, i = 1, ..., p$

Special case: All constraint functions are linear $\Rightarrow$ Constraint set is polyhedron

$\Rightarrow$ The only thing to check is that the feasible set has a point in the interior of domain D.

$\Downarrow$

Strong duality holds $\Leftrightarrow d^* = p^*$

$\exists\, (\lambda^*, \nu^*)$ s.t. $g(\lambda^*, \nu^*) = p^*$

Enforcing Concepts through some problems

① $\min_{x} \|x\|_2$

Subject to $Ax = b$

* Trivial Solution when A is full rank

$\Rightarrow$ Only one x satisfies $Ax = b$

$x = A^{-1} b$

* No solution when $b \notin R(A)$

Equivalent problem: $\min_{x} \|x\|_2^2 = x^T x$

Subject to $Ax = b$

→ Quadratic program

$D = \text{dom } f_0 \cap \mathbb{R}^n$

$= \mathbb{R}^n \cap \mathbb{R}^n = \mathbb{R}^n$

As long as $b \in \mathcal{R}(A) \Rightarrow$ we will always have an x s.t. $Ax = b$ and $x \in \text{relint}(D)$

$\Rightarrow$ Slater's Condition is satisfied and we have strong duality.