Functions of a symmetric, real matrix.

A $n \times n$ real, symmetric matrix

$A = Q D Q^T$ where Q is an orthogonal matrix, i.e., $Q = [\underline{x}_1 \underline{x}_2 \cdots \underline{x}_n]$ where $\underline{x}_k \in \mathbb{R}^n$ are mutually orthogonal and of length 1. D is diagonal.

If f is a function $\mathbb{R} \to \mathbb{R}$ then we define

$$f(A) = Q f(D) Q^T$$

where

$$f(D) = f\left(\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right)$$

$$:= \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix}$$

this makes sense since we know the entries of the diagonal matrix D (the eigen values of A) are <u>real</u>.

We note that

$$f(A) = [\underline{x}_1 \cdots \underline{x}_n] \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix} \begin{bmatrix} \underline{x}_1^T \\ \vdots \\ \underline{x}_n^T \end{bmatrix}$$

$$= [\underline{x}_1 \quad \underline{x}_n] \begin{bmatrix} f(\lambda_1)\underline{x}_1^T \\ \\ \\ f(\lambda_n)\underline{x}_n^T \end{bmatrix}$$

$$= f(\lambda_1)\,\underline{x}_1\underline{x}_1^T + f(\lambda_2)\underline{x}_2\underline{x}_2^T + \cdots + f(\lambda_n)\underline{x}_n\underline{x}_n^T$$

(where $A = \lambda_1\underline{x}_1\underline{x}_1^T + \lambda_2\underline{x}_2\underline{x}_2^T \cdots + \lambda_n\underline{x}_n\underline{x}_n^T$)

We note that if the λ_i's are distinct then the unit size eigen vectors $\underline{x}_i$ are unique up to a sign, <u>i.e.</u>, $\pm\underline{x}_i$

but we also note that the formula for $f(A)$ <u>does not</u> change with the sign.

$$\boxed{\text{In general: } f(A) \text{ is independent of the choice of eigenvectors (or the ordering of the eigenvalues)}}$$

<u>Exercise 1:</u>

Calculate A^{500}, e^{A} for the matrix

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

<u>Exercise 2:</u>

For real values x, y we have

$$e^{x+y} = e^{x} e^{y}$$

Does the same hold for symmetric, real matrices A & B?

Exercise 3:

Can you give a geometric interpretation of the matrices $\underline{x}_k \underline{x}_k^T$?

Observation:

Suppose A is $n \times n$, real, symmetric and with distinct eigenvalues $\lambda_1 \cdots \lambda_n$.

Then, if B is $n \times n$ real and

$$AB = BA,$$

it follows that B is symmetric and $B = f(A)$ for some function $f(\cdot)$.

Proof:

Let $\underline{x}_k$ be an eigenvector (of length 1) for A corresponding to eigenvalue λ_k, then

$$AB\underline{x}_k = BA\underline{x}_k = B\lambda_k \underline{x}_k = \lambda_k B\underline{x}_k$$

Therefore

- $\boxed{B\underline{x}_k = 0}$, or
- $B\underline{x}_k$ is an eigenvector for A corresponding to eigenvalue λ_k, in which case $\boxed{B\underline{x}_k = \mu_k \underline{x}_k}$ since the eigenvalues of A are distinct and thus the eigenspace associated to λ_k is <u>one</u>-dimensional.

In summary $\underline{x}_k$ is an eigenvector for B. B therefore satisfies

$$B\,[\underline{x}_1\ \underline{x}_2 \cdots \underline{x}_n] = [\underline{x}_1 \cdots \underline{x}_n] \begin{bmatrix} \mu_1 & & 0 \\ & \mu_2 & \\ 0 & & \mu_n \end{bmatrix}.$$

or

$$B = Q \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix} Q^T \quad \text{with}$$

$$Q = [\underline{x}_1 \cdots \underline{x}_n]$$

(here we use that the $\underline{x}_k$'s are orthogonal and of length 1)

This clearly shows that <u>B is symmetric</u>

Since $\lambda_1 \cdots \lambda_m$ are distinct we may define a function (actually many) so that

$$f(\lambda_k) = \mu_k$$

and with this definition it follows that

$$B = Q \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_m) \end{bmatrix} Q^T$$

$$= f(A)$$

<u>Exercise 4:</u>

Could you take f to be a polynomial ?

Exercise 5:

Show that $f(A)\,g(A) = g(A)\,f(A)$
for any functions f & g.

Exercise 6:

Let A be $n \times n$, symmetric, real
with distinct eigenvalues and
let $p(\lambda) = \det(A - \lambda I)$ denote the
characteristic (real) polynomial. What
can you say about $p(A)$?