Practice Problems (Weeks 13 and 14)

Friday, April 25, 2025 12:42 PM

Q1

A simple example. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

with variable $x \in \mathbf{R}$.

(a) *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.

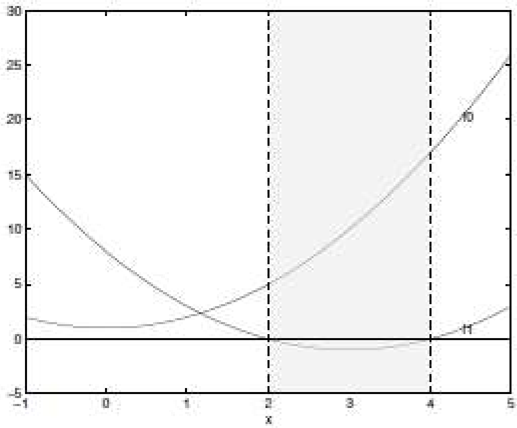
(b) *Lagrangian and dual function.* Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^\star \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution $\lambda^\star$. Does strong duality hold?

Solution.

(a) The feasible set is the interval $[2, 4]$. The (unique) optimal point is $x^\star = 2$, and the optimal value is $p^\star = 5$.

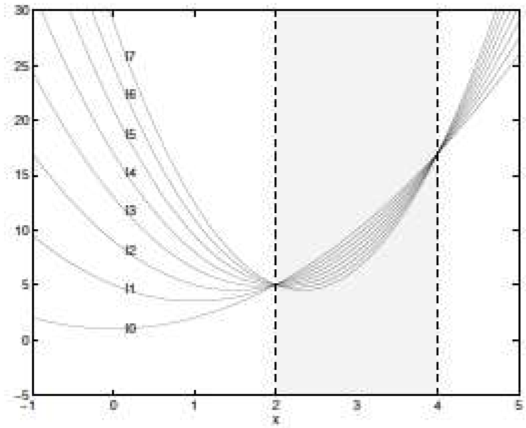
The plot shows f_0 and f_1.



(b) The Lagrangian is

$$L(x, \lambda) = (1+\lambda)x^2 - 6\lambda x + (1+8\lambda).$$

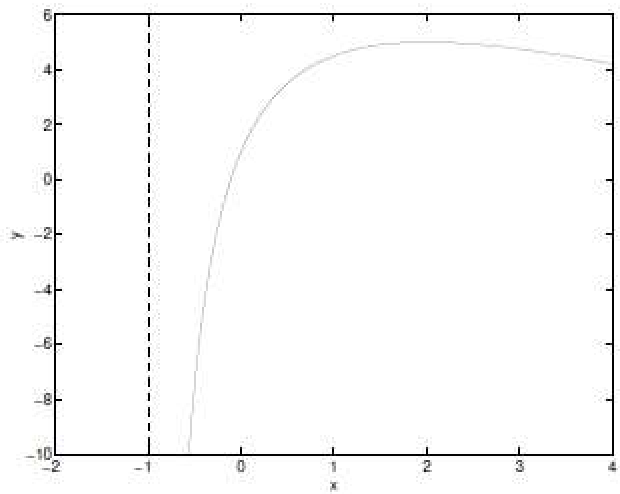
The plot shows the Lagrangian $L(x, \lambda) = f_0 + \lambda f_1$ as a function of x for different values of $\lambda \geq 0$. Note that the minimum value of $L(x, \lambda)$ over x (*i.e.*, $g(\lambda)$) is always less than $p^\star$. It increases as λ varies from 0 toward 2, reaches its maximum at $\lambda = 2$, and then decreases again as λ increases above 2. We have equality $p^\star = g(\lambda)$ for $\lambda = 2$.



For $\lambda > -1$, the Lagrangian reaches its minimum at $\tilde{x} = 3\lambda/(1+\lambda)$. For $\lambda \leq -1$ it is unbounded below. Thus

$$g(\lambda) = \begin{cases} -9\lambda^2/(1+\lambda) + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}$$

which is plotted below.



We can verify that the dual function is concave, that its value is equal to $p^\star = 5$ for $\lambda = 2$, and less than $p^\star$ for other values of λ.

(c) The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & -9\lambda^2/(1+\lambda) + 1 + 8\lambda \\ \text{subject to} & \lambda \geq 0. \end{array}$$

The dual optimum occurs at $\lambda = 2$, with $d^\star = 5$. So for this example we can directly observe that strong duality holds (as it must — Slater's constraint qualification is satisfied).

Q2 *Weak duality for unbounded and infeasible problems.* The weak duality inequality, $d^\star \leq p^\star$, clearly holds when $d^\star = -\infty$ or $p^\star = \infty$. Show that it holds in the other two cases as well: If $p^\star = -\infty$, then we must have $d^\star = -\infty$, and also, if $d^\star = \infty$, then we must have $p^\star = \infty$.

Solution.

Q2 *Weak duality for unbounded and infeasible problems.* The weak duality inequality, $d^\star \le p^\star$, clearly holds when $d^\star = -\infty$ or $p^\star = \infty$. Show that it holds in the other two cases as well: If $p^\star = -\infty$, then we must have $d^\star = -\infty$, and also, if $d^\star = \infty$, then we must have $p^\star = \infty$.

Solution.

(a) $p^\star = -\infty$. The primal problem is unbounded, *i.e.*, there exist feasible x with arbitrarily small values of $f_0(x)$. This means that

$$L(x,\lambda) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x)$$

is unbounded below for all $\lambda \succeq 0$, *i.e.*, $g(\lambda) = -\infty$ for $\lambda \succeq 0$. Therefore the dual problem is infeasible ($d^\star = -\infty$).

(b) $d^\star = \infty$. The dual problem is unbounded above. This is only possible if the primal problem is infeasible. If it were feasible, with $f_i(\tilde{x}) \le 0$ for $i = 1, \ldots, m$, then for all $\lambda \succeq 0$,

$$g(\lambda) = \inf(f_0(x) + \sum_i \lambda_i f_i(x)) \le f_0(\tilde{x}) + \sum_i \lambda_i f_i(\tilde{x}),$$

so the dual problem is bounded above.

Problems with one inequality constraint. Express the dual problem of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & f(x) \le 0, \end{array}$$

with $c \ne 0$, in terms of the conjugate f^*. Explain why the problem you give is convex. We do not assume f is convex.

Solution. For $\lambda = 0$, $g(\lambda) = \inf_x c^T x = -\infty$. For $\lambda > 0$,

$$\begin{aligned} g(\lambda) &= \inf_x (c^T x + \lambda f(x)) \\ &= \lambda \inf_x ((c/\lambda)^T x + f(x)) \\ &= -\lambda f^*(-c/\lambda), \end{aligned}$$

i.e., for $\lambda \ge 0$, $-g$ is the perspective of f^*, evaluated at $-c/\lambda$. The dual problem is

$$\begin{array}{ll} \text{minimize} & -\lambda f^*(-c/\lambda) \\ \text{subject to} & \lambda \ge 0. \end{array}$$

Q3: *Dual of general LP.* Find the dual function of the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \end{array}$$

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Dual of general LP. Find the dual function of the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b. \end{array}$$

Give the dual problem, and make the implicit equality constraints explicit.

Solution.

(a) The Lagrangian is

$$\begin{array}{rcl} L(x,\lambda,\nu) & = & c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b) \\ & = & (c^T + \lambda^T G + \nu^T A)x - \lambda^T h - \nu^T b, \end{array}$$

which is an affine function of x. It follows that the dual function is given by

$$g(\lambda,\nu) = \inf_x L(x,\lambda,\nu) = \begin{cases} -\lambda^T h - \nu^T b & c + G^T\lambda + A^T\nu = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

(b) The dual problem is

$$\begin{array}{ll} \text{maximize} & g(\lambda,\nu) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

After making the implicit constraints explicit, we obtain

$$\begin{array}{ll} \text{maximize} & -\lambda^T h - \nu^T b \\ \text{subject to} & c + G^T\lambda + A^T\nu = 0 \\ & \lambda \succeq 0. \end{array}$$

Q4:

Equality constrained least-squares. Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbf{R}^{m\times n}$ with $\mathbf{rank}\, A = n$, and $G \in \mathbf{R}^{p\times n}$ with $\mathbf{rank}\, G = p$.

Give the KKT conditions, and derive expressions for the primal solution $x^\star$ and the dual solution $\nu^\star$.

Solution.

(a) The Lagrangian is

$$\begin{array}{rcl} L(x,\nu) & = & \|Ax - b\|_2^2 + \nu^T (Gx - h) \\ & = & x^T A^T Ax + (G^T\nu - 2A^T b)^T x - \nu^T h, \end{array}$$

with minimizer $x = -(1/2)(A^TA)^{-1}(G^T\nu - 2A^Tb)$. The dual function is

$$g(\nu) = -(1/4)(G^T\nu - 2A^Tb)^T(A^TA)^{-1}(G^T\nu - 2A^Tb) - \nu^T h$$

(b) The optimality conditions are

$$2A^T(Ax^\star - b) + G^T\nu^\star = 0, \qquad Gx^\star = h.$$

(c) From the first equation,

$$x^\star = (A^T A)^{-1}(A^T b - (1/2)G^T \nu^\star).$$

Plugging this expression for $x^\star$ into the second equation gives

$$G(A^T A)^{-1} A^T b - (1/2)G(A^T A)^{-1} G^T \nu^\star = h$$

i.e.,

$$\nu^\star = -2(G(A^T A)^{-1} G^T)^{-1}(h - G(A^T A)^{-1} A^T b).$$

Substituting in the first expression gives an analytical expression for $x^\star$.

Q5: Derive the KKT conditions for the problem

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}\, X - \log\det X \\ \text{subject to} & Xs = y, \end{array}$$

with variable $X \in \mathbf{S}^n$ and domain $\mathbf{S}^n_{++}$. $y \in \mathbf{R}^n$ and $s \in \mathbf{R}^n$ are given, with $s^T y = 1$. Verify that the optimal solution is given by

$$X^\star = I + yy^T - \frac{1}{s^T s} ss^T.$$

Solution. We introduce a Lagrange multiplier $z \in \mathbf{R}^n$ for the equality constraint. The KKT optimality conditions are:

$$X \succ 0, \qquad Xs = y, \qquad X^{-1} = I + \frac{1}{2}(zs^T + sz^T). \tag{4.30.A}$$

We first determine z from the condition $Xs = y$. Multiplying the gradient equation on the right with y gives

$$s = X^{-1}y = y + \frac{1}{2}(z + (z^T y)s). \tag{4.30.B}$$

By taking the inner product with y on both sides and simplifying, we get $z^T y = 1 - y^T y$. Substituting in (4.30.B) we get

$$z = -2y + (1 + y^T y)s,$$

and substituting this expression for z in (4.30.A) gives

$$\begin{aligned} X^{-1} &= I + \frac{1}{2}(-2ys^T - 2sy^T + 2(1 + y^T y)ss^T) \\ &= I + (1 + y^T y)ss^T - ys^T - sy^T. \end{aligned}$$

Finally we verify that this is the inverse of the matrix $X^\star$ given above:

$$\begin{aligned} &\left(I + (1 + y^T y)ss^T - ys^T - sy^T\right) X^\star \\ &\quad= (I + yy^T - (1/s^T s)ss^T) + (1 + y^T y)(ss^T + sy^T - ss^T) \\ &\qquad - (ys^T + yy^T - ys^T) - (sy^T + (y^T y)sy^T - (1/s^T s)ss^T) \\ &\quad= I. \end{aligned}$$

To complete the solution, we prove that $X^\star \succ 0$. An easy way to see this is to note that

$$X^\star = I + yy^T - \frac{ss^T}{s^T s} = \left(I + \frac{ys^T}{\|s\|_2} - \frac{ss^T}{s^T s}\right)\left(I + \frac{ys^T}{\|s\|_2} - \frac{ss^T}{s^T s}\right)^T.$$

Infeasible start Newton method and initially satisfied equality constraints. Suppose we use the infeasible start Newton method to minimize $f(x)$ subject to $a_i^T x = b_i$, $i = 1, \dots, p$.

(a) Suppose the initial point $x^{(0)}$ satisfies the linear equality $a_i^T x = b_i$. Show that the linear equality will remain satisfied for future iterates, *i.e.*, if $a_i^T x^{(k)} = b_i$ for all k.

(b) Suppose that one of the equality constraints becomes satisfied at iteration k, *i.e.*, we have $a_i^T x^{(k-1)} \neq b_i$, $a_i^T x^{(k)} = b_i$. Show that at iteration k, *all* the equality constraints are satisfied.

Solution.
Follows easily from

$$r^{(k)} = \left(\prod_{i=0}^{k-1} (1 - t^{(i)}) \right) r^{(0)}.$$