Notes about the determinant.

Let f be a function of n vectors in $\mathbb{R}^n$

$$f(\underline{a}_1, \underline{a}_2 \cdots \underline{a}_n)$$

and require that

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- f is multilinear, i.e.,

$$f(\underline{a}_1, \underline{a}_2 \cdots \boxed{s\underline{a}_k' + t\underline{a}_k''}\ \underline{a}_{k+1} \cdots \underline{a}_n)$$
$$= s f(\underline{a}_1, \underline{a}_2 \cdots \underline{a}_k' \cdots \underline{a}_n) + t f(\underline{a}_1, \underline{a}_2 \cdots \underline{a}_k'' \cdots \underline{a}_n)$$

- f is alternating, i.e.,

$$f(\underline{a}_1 \cdots \underline{a}_k \cdots \underline{a}_\ell \cdots \underline{a}_n) = -f(\underline{a}_1 \cdots \underline{a}_\ell \cdots \underline{a}_k \cdots \underline{a}_n)$$

- $$f\left(\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}\right) = 1$$

From the multilinearity it follows that

$$\underset{\substack{\uparrow\\ k\text{th place}}}{f(\underline{a}_1\ \underline{a}_2\ \ \underline{a}_k + t\underline{a}_\ell} \cdots \underline{a}_n) = f(\underline{a}_1, \underline{a}_2 \cdots \underline{a}_k \cdots \underline{a}_n)$$
$$+ t f(\underline{a}_1, \underline{a}_2\ \ \underset{\substack{\uparrow\\ k\text{ th place}}}{\underline{a}_\ell} \cdots \underline{a}_n)$$

If now $l \neq k$ then

$$f(\underline{a}_1\ \underline{a}_2 \quad \underset{\substack{\uparrow \\ k\text{'th place}}}{\underline{a}_l} \quad \underset{\substack{\uparrow \\ l\text{'th place}}}{\underline{a}_l} \quad \underline{a}_n) = 0$$

(since we may interchange the two $\underline{a}_l$'s to pick up a different sign if it was not 0)

in other words

$$\left[\begin{array}{l} f(\underline{a}_1\ \underline{a}_2 \quad \underset{\substack{\uparrow \\ k\text{'th place}}}{\underline{a}_k + t\underline{a}_l} \quad \underline{a}_n) = f(\underline{a}_1\, \underline{a}_2 \cdots \underline{a}_k \cdots \underline{a}_n) \\ \\ \text{if } \boxed{l \neq k} \end{array}\right.$$

Ex: $n=2$

(use that $a_{11} \neq 0$)

$$f\left(\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}\right) = f\left(\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \begin{pmatrix} 0 \\ a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{pmatrix}\right) \quad \text{why?}$$

$$= a_{11}\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right) f\left(\begin{pmatrix} 1 \\ a_{12}/a_{11} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \quad \text{why?}$$

$$= a_{11}\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right) 1 \quad \text{why}$$

$$= a_{11}a_{22} - a_{12}a_{21} = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where I have used your standard def of the 2×2 determinant.

In general ⊛ implies that

$$f(\underline{a}_1\, \underline{a}_2 \quad \underline{a}_m) = \det \begin{bmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \\ \underline{a}_m^T \end{bmatrix}$$

⊛⊛ In other words: the determinant is the only function that satisfies ⊛

consider now the function

$$A \to \frac{\det(AB)}{\det(B)} \qquad (B \text{ fixed}, \det B \neq 0)$$

(A and B $m\times m$ matrices)

This function satisfies ⊛ where $\underline{a}_1^T\ \underline{a}_2^T$ $\underline{a}_m^T$ are the rows of A.

due to (**) we therefore have

$$\frac{\det(AB)}{\det(B)} = \det(A)$$

or $$\det(AB) = \det(A)\det(B)$$

– what if $\det(B) = 0$?