Linear Algebra and Applications
Homework #01

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$\boxed{1}$ $B_1 = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B_{11}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{B_{12}}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_{13}} \right\}$. $B_2 = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{B_{21}}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{B_{22}}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}_{B_{23}} \right\}$

Here,

$$B_{11} = B_{22} - \frac{1}{2} B_{23}$$

$$B_{12} = B_{21} - B_{22}$$

$$B_{13} = \frac{1}{2} B_{23}$$

So,

$$I_{B_1 \to B_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$I_{B_2 \to B_1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

(a)

$$I_{B_2 \to B_1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

(b)

$$I_{B_1 \to B_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(c) Here,

$$I_{B_2 \to B_1} \times I_{B_1 \to B_2}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

So, they are inverse

d

Here B_1 is cannonical Basis.

$$\text{So, } L_{B1} = A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

e

$$L_{B_2} = I_{B_1 \to B_2} \; A \; I_{B_2 \to B_1}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 6 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$L_{B2} = \begin{bmatrix} 15 & 10 & 12 \\ -3 & -6 & -6 \\ 9 & 6 & 6 \end{bmatrix}$$

$\boxed{2}$

(a) $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix}$; $R_2 = R_2 - 5R_1$; $R_3 = R_3 - 9R_1$

$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $R_3 = R_3 - 2R_2$

$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $R_2 = \dfrac{R_2}{-4}$

Now,

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ -9 & 0 & 1 \end{bmatrix}$; $R_2 = R_2 - 5R_1$, $R_3 = R_3 - 9R_1$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \; ; \quad R_3 = R_3 - 2R_2 \text{ on } I$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \quad R_2 = \frac{R_2}{-4} \text{ on } I$$

<u>Now,</u>

$$E_3 E_2 E_1 B = U$$

$$B = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$= LU \qquad [L = E_3^{-1} E_2^{-1} E_1^{-1}]$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix} \; ; \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \; ; \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -4 & 0 \\ 9 & -8 & 1 \end{bmatrix}$$

$$B = LU = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -4 & 0 \\ 9 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & [illegible] \end{bmatrix}$$

(b) From the reduced row echelon form

Rank = Num of linearly independent rows

$\boxed{=2}$

(c) From previous,

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 = R_1 - 2R_2$$

<u>Null space</u>

$$N(A) = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = 0 \}$$

<u>Now,</u>

$$B\underline{x} = 0$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>This implies,</u>

$$x_1 - x_3 - 2x_4 = 0$$

$$x_2 + 2x_3 + 3x_4 = 0$$

$$x_1 = x_3 + 2x_4$$

$$x_2 = -2x_3 - 3x_4$$

So,
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Now, Null space is the subspace of all solutions that leads to $B\underline{x} = 0$

So, $$N(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$R(B)$:-

$$R(B) = \{ y \in \mathbb{R}^3 : \exists\, \underline{x} \in \mathbb{R}^4 \text{ with } y = B\underline{x} \}$$

$B = 3\times 4$ Matrix

From reduced Row echelon form, there are two pivot (leading 1) columns.

So, $$R(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$$

3(a) Given,

$$L\,p(t) = \frac{d}{dt}\left[(1-t^2)\frac{d}{dt}p(t)\right]$$

Also, $P_3 = \{$polynomials in t of degree $\leq 3\}$

Let, $p_1(t), p_2(t) \in P_3$

Now,

$$\mathcal{L}\left[c\,p_1(t) + d\,p_2(t)\right]$$ where c, d scaler

$$= \frac{d}{dt}\left[(1-t^2)\frac{d}{dt}\{c\,p_1(t) + d\,p_2(t)\}\right]$$

$$= \frac{d}{dt}\left[(1-t^2)\left[c\frac{d}{dt}p_1(t) + d\frac{d}{dt}p_2(t)\right]\right]$$

$$= \frac{d}{dt}\left[(1-t^2)\,c\,\frac{d}{dt}p_1(t)\right] + \frac{d}{dt}\left[(1-t^2)\cdot d\cdot\frac{d}{dt}p_2(t)\right]$$

$$= c\cdot\mathcal{L}(p_1(t)) + d\cdot\mathcal{L}\,p_2(t)$$

So $$\mathcal{L}\left[c\,p_1(t) + d\,p_2(t)\right] = c\,\mathcal{L}[p_1(t)] + d\,\mathcal{L}[p_2(t)]$$

So, $\mathcal{L}$ is a linear operator.

b Matrix A in the basis $\{1, t, t^2, t^3\}$

$$\mathcal{L}\{x_1 + x_2 t + x_3 t^2 + x_4 t^3\} = y_1 + y_2 t + y_3 t^2 + y_4 t^3$$

$$\underline{y} = A\,\underline{x}$$

Here

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ; \quad \mathcal{L}(1) = 0$$

$$A \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} ; \quad \mathcal{L}(t) = -2t$$

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -6 \\ 0 \end{bmatrix} \quad \mathcal{L}(t^2) = 2 - 6t^2$$

$$A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ -12 \end{bmatrix} \quad \mathcal{L}(t^3) = \frac{d}{dt}\{(1-t^2)\,3t^2\}$$

$$= 6t - 12t^3$$

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$\underline{\underline{c}}$ Matrix B in the basis $\left\{ 1, t, t^2 - \frac{1}{3}, t^3 - \frac{3}{5}t \right\}$

$$\underline{y} = B\, \underline{x}$$

$$B \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \; \mathcal{L}(1) = 0$$

$$B \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} \quad ; \; \mathcal{L}(t) = -2t$$

$$B \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 0 \end{bmatrix} \quad ; \; \mathcal{L}\left(t^2 - \frac{1}{3}\right) = \frac{d}{dt}\left[(1-t^2) \cdot 2t\right]$$

$$= \frac{d}{dt}(2t - 2t^3)$$

$$= 2 - 6t^2$$

$$= -6\left(t^2 - \frac{1}{3}\right)$$

$$B\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\-12\end{bmatrix} \quad ; \quad L\left(t^3-\frac{3}{5}t\right)$$

$$= \frac{d}{dt}\left[(1-t^2).\left(3t^2-\frac{3}{5}\right)\right]$$

$$= \frac{d}{dt}\left(3t^2-\frac{3}{5}-3t^4+\frac{3}{5}t^2\right)$$

$$= \frac{d}{dt}\left(-\frac{3}{5}+\frac{18}{5}t^2-3t^4\right)$$

$$= \frac{36}{5}t - 12t^3$$

$$= -12\left(t^3-\frac{3}{5}t\right)$$

So,
$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

d) Let,

$$B_2 = \left\{1,\ t,\ t^2-\frac{1}{3},\ t^3-\frac{3}{5}t\right\}$$

$$B_1 = \{1,\ t,\ t^2,\ t^3\}$$

So, the coefficient matrix that relates B_2 to B_1 will be

$$C = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and corresponding,

$$C^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{3}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, $B = C^{-1} A C$