Linear Algebra and Applications
Homework #04

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1(a) Given, $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

Eigenvalues will be the roots of $\det(A - \lambda I) = 0$

Now,

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det \begin{bmatrix} 4-\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda) \det \begin{bmatrix} -\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} - 1 \det \begin{bmatrix} 1 & 2 \\ 0 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 4\lambda - 4) - 1(4-\lambda) = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 4\lambda - 4 - 1) = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 4\lambda - 5) = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 4 \\ \lambda_2 = 5 \\ \lambda_3 = -1 \end{cases} \text{ eigenvalues}$$

eigenvectors:- $\longrightarrow$ $A\underline{v} = \lambda \underline{v}$

$(A - \lambda I)\underline{v} = 0$

for $\lambda = 4$,

$$(A - 4I)\underline{v}_1 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} x_2 = 0 \\ x_1 - 4x_2 + 2x_3 = 0 \\ 2x_2 = 0 \end{cases} \Rightarrow \begin{array}{l} x_2 = 0 \\ x_1 = -2x_3 \end{array}$$

So, $\underline{v}_1$ = scaler multiple of $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$

for $\lambda = 5$

$$(A - 5I)\underline{v}_2 = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -x_1 + x_2 = 0 \\ x_1 - 5x_2 + 2x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases} \Rightarrow \left.\begin{aligned} x_1 &= x_2 \\ -4x_2 + 2x_3 &= 0 \\ 2x_2 - x_3 &= 0 \end{aligned}\right\}$$

$$\Downarrow$$

$$\left.\begin{aligned} x_1 &= x_2 \\ x_3 &= 2x_2 \end{aligned}\right\}$$

So $\underline{v_2}$ = scaler multiple of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \equiv \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$

<u>for $\lambda = -1$,</u>

$$(A + I)\,\underline{v_3} = 0$$

$$\Rightarrow \begin{bmatrix} 5 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} 5x_1 + x_2 = 0 \\ x_1 + x_2 + 2x_3 = 0 \\ 2x_2 + 5x_3 = 0 \end{cases} \Rightarrow \left.\begin{aligned} x_2 &= -5x_1 \\ x_3 &= -\frac{2}{5}x_2 \end{aligned}\right\}$$

$$= +2x_1$$

So $\underline{v_3}$ = scaler multiple of $\begin{bmatrix} 1 \\ -5 \\ +2 \end{bmatrix} \equiv \begin{bmatrix} 1/\sqrt{30} \\ -5/\sqrt{30} \\ +2/\sqrt{30} \end{bmatrix}$

eigen values: $\lambda_1 = 4, \quad \lambda_2 = 5, \quad \lambda_3 = -1$

eigenvectors corresponding: $\underline{v}_1 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1/\sqrt{30} \\ -5/\sqrt{30} \\ -2/\sqrt{30} \end{bmatrix}$

1(b)

From definition of eigenvectors:

$$\cancel{A \lambda_1 = \lambda_1 v_1}$$
$$\cancel{A \lambda_2 = \lambda_2 v_2}$$
$$\cancel{A \lambda_3 =}$$

$$\left.\begin{aligned} A\underline{v}_1 &= \lambda_1 \underline{v}_1 \\ A\underline{v}_2 &= \lambda_2 \underline{v}_2 \\ A\underline{v}_3 &= \lambda_3 \underline{v}_3 \end{aligned}\right\}$$

Written combinely,

$$A \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix}$$

$$\Rightarrow \quad A \, \underline{C}_1 = \Lambda \, \underline{C}_1$$

$$\Rightarrow \quad \underline{C}_1^{-1} A \, \underline{C}_1 = \Lambda$$

where, $\underline{C}_1 = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{6}} & -\frac{5}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \end{bmatrix}$

Calculating C_1^{-1} explicitly:

$$C_1^{-1} = \frac{1}{\det(C_1)} \operatorname{adj}(C_1)$$

$$\det(C_1) = \left(-\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{6}}\cdot\frac{2}{\sqrt{30}} - \left(-\frac{5}{\sqrt{30}}\right)\cdot\frac{2}{\sqrt{6}}\right)$$

$$- \frac{1}{\sqrt{6}}\left(0\cdot\frac{2}{\sqrt{30}} - \left(-\frac{5}{\sqrt{30}}\right)\cdot\frac{1}{\sqrt{5}}\right)$$

$$+ \frac{1}{\sqrt{30}}\left(0\cdot\frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{5}}\right)$$

$$= -1$$

Each element of cofactor matrix C_1 is

$$C_{ij} = (-1)^{i+j}\det(M_{ij}) \longrightarrow \left(C_{11} = +1\cdot\left(\frac{2}{\sqrt{180}} + \frac{10}{\sqrt{180}}\right) = 12/\sqrt{180} \right)$$

M_{ij} is the submatrix obtained by deleting ith row & jth column of C_1

Calculating all (similar to C_{11})

$$\operatorname{Cof}(C_1) = \begin{bmatrix} +\frac{12}{\sqrt{180}} & \frac{5}{\sqrt{150}} & -\frac{1}{\sqrt{30}} \\ 0 & \frac{-5}{\sqrt{150}} & \frac{5}{\sqrt{30}} \\ \frac{-6}{\sqrt{180}} & \frac{-10}{\sqrt{150}} & \frac{-2}{\sqrt{30}} \end{bmatrix}$$

$$\operatorname{adj}(C_1) = [\operatorname{Cof}(C_1)]^T$$

So, $C_1^{-1} = \dfrac{1}{\det(C_1)} \operatorname{adj}(C_1)$

$$= \begin{bmatrix} \frac{-2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{6}} & +\frac{2}{\sqrt{6}} \\ +\frac{1}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & +\frac{2}{\sqrt{30}} \end{bmatrix}$$

Again, As C_1 is an orthogonal matrix, (orthonormal)

$$C_1^{-1} = C_1^T$$

A is real & symmetric $\longrightarrow$ A has a basis of orthonormal eigenvectors

$$\Downarrow$$

$C_1^{-1} = C_1^T \longleftarrow C_1$ orthonormal matrix

So, we can calculate C_1^{-1} as:

$$C_1^{-1} = C_1^T = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & \frac{2}{\sqrt{30}} \end{bmatrix}$$

which is same as before!

1(c) Given,

$$\frac{d}{dt} U(t) = A_1 U(t)$$

$$\Rightarrow C_1^{-1} \frac{d}{dt} U(t) = C_1^{-1} A_1 U(t)$$

$$\Rightarrow \frac{d}{dt}\left(C_1^{-1} U(t)\right) = C_1^{-1} A_1 C_1 C_1^{-1} U(t)$$

$$= \left(C_1^{-1} A_1 C_1\right) \cdot \left(C_1^{-1} U(t)\right) \quad [C_1 C_1^{-1} = I]$$

$$\Rightarrow \frac{d}{dt}\left(v(t)\right) = \Lambda \, v(t)$$

where, $v(t) = C_1^{-1} U(t)$

So, differential equation of $v(t)$:

$$\frac{d}{dt}\left(v(t)\right) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} v(t)$$

$$\Rightarrow \begin{bmatrix} v_1'(t) \\ v_2'(t) \\ v_3'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix}$$

and, $v(0) = C_1^{-1} U(0) = C_1^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{6} \\ 1/\sqrt{30} \end{bmatrix}$

1(d) From 'c'

$$v_1'(t) = \lambda_1 v_1(t)$$
$$= 4 v_1(t)$$

So $v_1(t) = c_1 e^{4t}$

putting $t=0$, $v_1(0) = c_1 \cdot 1 = c_1$

So $v_1(t) = v_1(0) \cdot e^{4t}$

So
$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = v(t) = \begin{bmatrix} e^{4t} & 0 & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \cdot v(0) = \begin{bmatrix} \frac{-2e^{4t}}{\sqrt{5}} \\ e^{5t}/\sqrt{6} \\ e^{-t}/\sqrt{30} \end{bmatrix}$$

1(e) Now, $v(t) = C_1^{-1} U(t)$

So $U(t) = C_1 v(t)$

$$\Rightarrow U(t) = C_1 \begin{bmatrix} e^{4t} & 0 & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} v(0)$$

$$\boxed{U(t) = C_1 \begin{bmatrix} e^{4t} & 0 & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} C_1^{-1} U(0)}$$

explicitly,

$$V(t) = C_1\, v(t)$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{-5}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} \frac{-2e^{4t}}{\sqrt{5}} \\ \frac{e^{5t}}{\sqrt{6}} \\ \frac{e^{-t}}{\sqrt{30}} \end{bmatrix}$$

Problem 2

2(a) $A_2 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix}$

eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det\begin{bmatrix} 4-\lambda & 1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda)\det\begin{bmatrix} -\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix} - 1 \cdot \det\begin{bmatrix} 1 & -1 \\ 0 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 4\lambda + 1) - (4-\lambda) = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 4\lambda) = 0$$

$$\Rightarrow \lambda_1 = 4, \quad \lambda_2 = 4, \quad \lambda_3 = 0$$

eigenvectors:

for $\lambda_1 = 4$,

$$A_2 v_1 = \lambda_1 v_1$$

$$\Rightarrow (A_2 - \lambda_1 I) v_1 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & -1 \\ 0 & 1 & 0 \end{bmatrix} v_1 = 0$$

$$\Rightarrow \begin{cases} v_2 = 0 \\ v_1 - 4v_2 - v_3 = 0 \\ v_2 = 0 \end{cases} \Rightarrow \left. \begin{matrix} v_2 = 0 \\ v_1 = v_3 \end{matrix} \right\}$$

So, $\underline{v_1}$ = scaler multiple of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

For, $\lambda_2 = 4$, similar analysis for $\lambda_1 = 4$

$$\underline{v_2} = \underline{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for, $\underline{\lambda_3 = 0,}$

$$(A_2 - \lambda_3 I)\, v_3 = 0$$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} \underline{x_3} = 0$$

$$\Rightarrow \begin{cases} 4x_1 + x_2 = 0 \\ x_1 - x_3 = 0 \\ x_2 + 4x_3 = 0 \end{cases} \Rightarrow \left. \begin{matrix} x_1 = x_3 \\ x_2 = -4x_1 = -4x_3 \end{matrix} \right\}$$

So, $\underline{v}_3$ = scaler multiple of $\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$

So, we only have two linearly independent eigenvectors, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$, so, A_2 does not have a full basis of eigenvectors.

(A_2 is 3×3, so it needs 3 linearly independent eigenvectors to form a basis of eigenvectors)

2(b)

From 2(a) $\rightarrow$

$$\lambda_3 = 0, \quad \underline{v}_3 = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$$

so,

$$A_2 \underline{v}_3 = \lambda_3 \underline{v}_3$$

$$A_2 \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = 0 \cdot \underline{v}_3 = \underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, from definition of Nullspace;

$$N(A_2) = \text{span}\{ (1 \ -4 \ 1)^T \}$$

Now, $(A_2-4)^2$:-

$$A_2 - 4I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(A_2-4I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -1 \\ -4 & 16 & 4 \\ 1 & -4 & -1 \end{bmatrix}$$

For nullvector $\underline{v}$ of $(A_2-4I)^2$:-

$$(A_2-4I)^2\underline{v} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -4 & -1 \\ -4 & 16 & 4 \\ 1 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} x_1 - 4x_2 - x_3 = 0 \\ -4x_1 + 16x_2 + 4x_3 = 0 \\ x_1 - 4x_2 - x_3 = 0 \end{cases}$$

same

From 1st & 3rd equation $x_1 = 4x_2 + x_3$

Substituting into 2nd, $-4(4x_2 + x_3) + 16x_2 + 4x_3 = 0$

$$\Rightarrow 0 = 0$$

So, these three equations are linearly dependent.

So, General solution:

$$\underline{v} = \begin{bmatrix} 4x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where x_2 & x_3 are scalers.

So, $N((A_2 - 4I)^2) = \text{span}\left\{(1\ 0\ 1)^T, (4\ 1\ 0)^T\right\}$

2(c) $\left\{(1\ 0\ 1)^T, (4\ 1\ 0)^T \ \& \ (1, -4, 1)^T\right\}$ will form a basis in $\mathbb{R}^3$ if they are linearly independent of each other.

$\Downarrow$

This will be the case if the matrix form by them is full Rank. $\longrightarrow$ Non-singular Matrix $\longrightarrow$ (determinant $\neq 0$)

Now, $M = C_{New} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$

$$\det M = 1 \det\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} - 4 \det\begin{bmatrix} 0 & -4 \\ 1 & 1 \end{bmatrix} + 1 \det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= 1 - 4.(-4) + 1.(-1)$$

$$= -16 \neq 0$$

So, Determinat $\neq 0$

↳ Columns are linearly independent

↳ These three vectors form a basis in $\mathbb{R}^3$

Representation of A_2 in M: (A_{new})

$$A_{new} = M^{-1} A_2 M$$

$$= \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 17 & 0 \\ 0 & 4 & 0 \\ 4 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2(d)

Now,
$$\frac{d}{dt} u(t) = A_2 u(t)$$

$$M^{-1} \frac{d}{dt} u(t) = M^{-1} A_2 u(t)$$

$$\frac{d}{dt}\left(M^{-1} u(t)\right) = M^{-1} A_2 M M^{-1} u(t)$$

$$\left\{\begin{array}{l} M M^{-1} = I \\ M^{-1} A_2 M = A_{new} \\ M^{-1} u(t) = v(t) \end{array}\right.$$

let, $v(t) = M^{-1} u(t)$

So,
$$\frac{d}{dt} v(t) = A_{new} \cdot v(t)$$

$$\frac{d}{dt} v(t) = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} v(t)$$

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \\ v_3'(t) \end{bmatrix} = \begin{bmatrix} 4 v_1(t) + v_2(t) \\ 4 v_2(t) \\ 0 \end{bmatrix}$$

So,
$$v_1'(t) = 4 v_1(t) + v_2(t)$$
$$v_2'(t) = 4 v_2(t)$$

and,
$$v(0) = M^{-1} u(0)$$

so, $v_2'(t) = 4 v_2(t)$

$$\therefore v_2(t) = c_2 e^{4t} = v_2(0) e^{4t}$$

and, $v_1'(t) = 4v_1(t) + v_2(t)$

$$= 4v_1(t) + c_2 e^{4t}$$

So, in $v_1(t)$, we wil get e^{4t} and te^{4t} rather than purely $e^{\lambda t}$ (e^{4t}), because Anew is not purely diagonal matrix here. Anew representation helps us here by decoupling the variable (partially)

Compared to previous problem, A_1 could be fully diagonalized there, so each variable was decoupled from each other there. Now, A_2 is not diagonalizable like A_1, so we will get term like $e^{\lambda t}$ & $te^{\lambda t}$ as the variables can't be decoupled fully.