Class #1

Tuesday, January 21, 2025 10:23 AM

What is optimization?

Consider a function $f: \underbrace{\mathbb{R}^n}_{\text{dom } f} \to \mathbb{R}$

e.g., $f(x) = \log(x)$

$$f: \mathbb{R} \to \mathbb{R}$$

$$\text{dom } f = (0, \infty)$$

Optimization is concerned with finding the maximum or minimum value that f takes:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{or} \quad \max_{x \in \mathbb{R}^n} f(x)$$

infimum supremum

If min or max exists

e.g., $f(x) = \log(x)$

$$\min_x f(x) \Rightarrow \text{does not exist}$$

$$f(x) = x^2 \Rightarrow \min_x f(x) = 0$$

$x_n = \frac{1}{n}$; $n = 1, 2, \ldots$

$\min_n x_n$ does not exist

but $\inf_n x_n = 0$

(or) Finding the x that gives the minimum or maximum value of f

$$\boxed{\arg}\min_{x \in \mathbb{R}^n} f(x) \quad \text{or} \quad \boxed{\arg}\max_{x \in \mathbb{R}^n} f(x)$$

$$\boxed{\arg}\min_{x \in \mathbb{R}^n} f(x) \quad \text{or} \quad \boxed{\arg}\max_{x \in \mathbb{R}^n} f(x)$$

e.g.; $f(x) = x^2$

$$\min_x f(x) = 0$$

$$\arg\min_x f(x) = 0$$

$$f(x) = x^2 + 1$$

$$\min_x f(x) = 1$$

$$\arg\min_x f(x) = 0$$

$$f(x) = (x-1)^2$$

$$\min_x f(x) = 0$$

$$\arg\min_x f(x) = 1$$

Practically, the function $f(x)$ represents some real-world quantity that we care about and $x \in \mathbb{R}^n$ corresponds to n parameters of the problem that alter the quantity of interest.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example: Portfolio optimization

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{array}{l} \rightarrow \text{investment in stock 1} \\ \\ \\ \rightarrow \text{investment in stock } n \end{array}$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}$

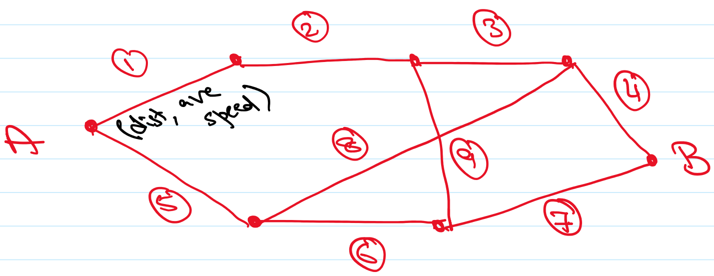
↳ Profit / loss after 6 months

$\arg\max_{x} f(x)$

Constraints: Total budget $\Rightarrow \sum_{i=1}^{n} x_i \le B$

$x_i \ge 0$

Example: Route optimization w/ respect to minimization of time.



option 1:
$① \to ② \to ③ \to ④$

option 2
$① \to ② \to ⑨ \to ⑦$

$f(x)$ = Time needed to go from A to B
↳ encoding the path

$f(x): \{0,1\}^9 \to \mathbb{R}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_9 \end{bmatrix} \Rightarrow \begin{array}{l} x_i = 0 \text{ if } i^{th} \text{ edge is not taken} \\ \quad = 1 \text{ if } i^{th} \text{ edge is taken} \end{array}$$

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{option } 1$$

$$f(x) = \boxed{\frac{dist(1)}{ave.\,speed(1)}} \times x_1 + \frac{dist(2)}{av.\,speed(2)} \times x_2 + \cdots + \frac{dist(9)}{av.\,speed(9)} \times x_9$$

↳ time taken to traverse edge 1

This problem involves optimizing a 9-dimensional binary vector $x \in \{0,1\}^9$

$$\underset{x \in \{0,1\}^n}{\arg\min}\, f(x) \quad \underline{\text{and}} \quad \min_x f(x)$$

This actually is not over all $\{0,1\}^9$

We have constraints that $\underline{x}$ must correspond to a valid route from A to B

$$\mathcal{X} = \{ x : x \text{ corresponds to a valid route from A to B.} \}$$

Much smaller than 2^9

$$\underset{x \in \mathcal{X}}{\arg\min}\, f(x)$$

$\mathcal{X}$ → Constraint set in optimization

This is an example of combinatorial optimization.

<u>Other Examples</u>

① IC design layout and routing of wires

② Machine learning

③ Robot motion planning

Summary → choosing.

Optimization problems are everywhere

① Modeling the problem into an optimization framework that is solvable.

** → Theory

we have combinatorial,intezar,convex ,stochastic optimization

② Recognizing the nature of the optimization framework and manipulating it so that it is efficient to solve.

** → Theory / Implementation

linear, first order ,gradient discent ?

③ Using an appropriate numerical optimization method to help us solve the problem ⇒ Run-time

** → Theory

④ Understanding the 'goodness' of the solution provided by the algorithm.

~ x ~ x ~

Mathematical optimization : Foundational concepts

① Unless otherwise stated, $f(\cdot)$ is assumed to attain its minimum or maximum value.

② Without loss of generality, we will stick with $\min_x f(x)$ or $\max_x f(x)$, rather than $\arg\min_x f(x)$ or $\arg\max_x f(x)$

An optimization problem will be written as

$$\min_{x \in \mathbb{R}^n} f(x)$$

or $$\max_{x \in \mathbb{R}^n} f(x)$$

but when the variable is obvious then we write

$$\min f(x) \quad \text{or} \quad \max f(x)$$

$f \Rightarrow$ Objective function

$x \Rightarrow$ Optimization Variable

Unconstrained optimization

↳ $\min_{\boxed{x \in \mathbb{R}^n}} f(x)$ or $\min_{\boxed{x \in \operatorname{dom} f}} f(x)$

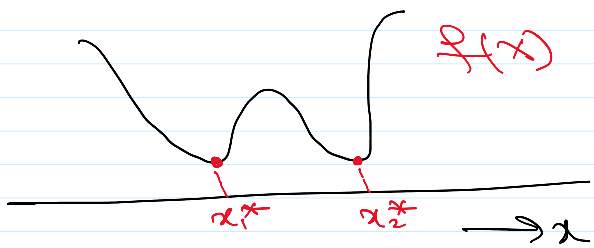
↳ 'x' should be searched over the entire domain of f.

Solution $\Rightarrow$ A solution to an optimization problem $\min_{x \in \mathbb{R}^n} f(x)$ is often denoted by $x^* \in \mathbb{R}^n$ and it satisfies :-

$$\boxed{f(x^*) \le f(x) \;\; \forall \, x \in \mathbb{R}^n}$$

x^* is referred to as a minimizer or an optimal value.

We can have multiple solutions ; x_1^*, x_2^*, ---



We can have 'no' solution.

e.g.; $f(x) : (0, \infty) \to \mathbb{R}$

$$f(x) = -\frac{1}{x}$$

$\min_x f(x) \implies$ no x is going to give us the minimum value.

multiple solutions

$\implies f(x) = \cos(x)$

$$\min_x \cos(x)$$

$$x^* \in \{\pm\pi, \pm 3\pi, \pm 5\pi, \dots\}$$