1. Let A_1 denote the matrix

$$A_1 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

(a) Find the eigenvalues and corresponding eigenvectors for A_1.

(b) Find a matrix C_1 such that $C_1^{-1}AC_1 = \Lambda$, where Λ is a diagonal matrix. Calculate explicitly C_1^{-1}.

Suppose now we want to solve the 3×3 system of first order differential equations

$$\frac{d}{dt}u(t) = A_1 u(t) \ , \text{ with initial condition } u(0) = (1\ 0\ 0)^T \ .$$

(c) Write the corresponding system of differential equations for $v(t) = C_1^{-1}u(t)$. What is the initial condition for $v(0)$.

(d) Find the solution to the problem in (c), *i.e.*, find the explicit formula for $v(t)$.

(e) Use the formula for $v(t)$ to derive a formula for $u(t)$

2. Let A_2 denote the matrix

$$A_2 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of A_2. Can you find a basis of eigenvectors?

(b) Show that

$$N\left((A_2 - 4)^2\right) = \text{span}\{(1\ 0\ 1)^T, (4\ 1\ 0)^T\} \text{ and } N(A_2) = \text{span}\{(1\ \ -4\ 1)^T\} \ .$$

(c) Show that $\{(1\,0\,1)^T, (4\,1\,0)^T, (1\,-4\,1)^T\}$ forms a basis of $\mathbb{R}^3$. Find the representation of A_2 in this basis.

(d) Can you briefly describe how this representation would help you solve $\frac{d}{dt}u(t) = A_2u(t)$ with $u(0) = (1\,0\,0)^T$. What is the main difference when compared to the solution in Problem 1 (d)-(e)?