Convex Optimization
Homework #08

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Date: Apr 25, 2025

Problem 1:

Given, $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$C = \text{Cone}\{a_1, a_2\}$

$C = \{\theta_1 a_1 + \theta_2 a_2 \mid \theta_1, \theta_2 \geq 0\}$

This is all non-negative combinations of standard basis vectors a_1 & a_2

So, C fills the entire first quadrant of $\mathbb{R}^2$.

$D = \text{Cone}\{a_1, a_2, a_3\}$

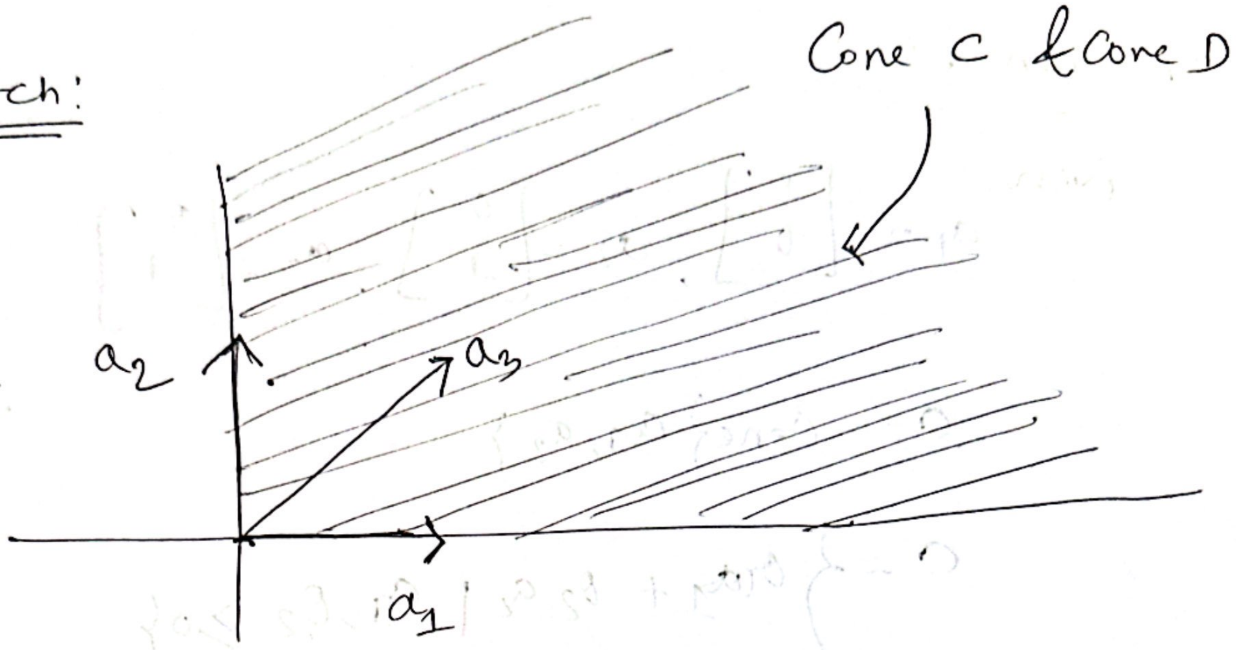
$= \{\theta_1 a_1 + \theta_2 a_2 + \theta_3 a_3 \mid \theta_1, \theta_2, \theta_3 \geq 0\}$

Here, $a_3 = a_1 + a_2$, so a_3 is already inside the Cone generated by a_1 & a_2

So, adding a_3 doesn't change the cone

So, D also fills the first quadrant of $\mathbb{R}^2$

Sketch:



Problem 2

Let's take,

$$(\bar{x}, \bar{y}_1 + \bar{y}_2), (x', y_1' + y_2') \in S$$

with,

$$(\bar{x}, \bar{y}_1) \in S_1 \qquad (x', y_1') \in S_1$$

$$(\bar{x}, \bar{y}_2) \in S_2 \qquad (x', y_2') \in S_2$$

Now, for $\theta \in [0,1]$

$$\theta(\bar{x}, \bar{y}_1 + \bar{y}_2) + (1-\theta)(x', y_1' + y_2')$$

$$= \left(\theta\bar{x} + (1-\theta)x', \left(\theta\bar{y}_1 + (1-\theta)y_1'\right) + \left(\theta\bar{y}_2 + (1-\theta)y_2'\right)\right)$$

This belongs to S

Because by convexity of S_1:

$$\left(\theta\bar{x} + (1-\theta)x', \theta\bar{y}_1 + (1-\theta)y_1'\right) \in S_1$$

Similarly, by convexity of S_2:

$$\left(\theta\bar{x} + (1-\theta)x', \theta\bar{y}_2 + (1-\theta)y_2'\right) \in S_2$$

Hence, S is convex.

Problem 3:

let $y_1, y_2 \in C^\circ$

By definition, $\forall x \in C$,

$$y_1^T x \leq 1 \quad \& \quad y_2^T x \leq 1 \quad \text{--- (i)}$$

Now, Let's take: $y = \theta y_1 + (1-\theta) y_2$, $\theta \in [0,1]$

For any $x \in C$,

$$y^T x = (\theta y_1 + (1-\theta) y_2)^T x$$

$$= \theta y_1^T x + (1-\theta) y_2^T x$$

But, $y_1^T x \leq 1$, $y_2^T x \leq 1$ [From (i)]

& $\theta \in [0,1]$

So $y^T x \leq \theta . 1 + (1-\theta) . 1 = 1$

So, $y^T x \leq 1 \quad \forall x \in C$

So, $y \in C^\circ$

So, C° is convex.

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Problem 4:

hypograph of g: $[hyp(g)]$

$$hyp(g) = \{(y, t) \mid y \in dom(g),\ t \leq g(y)\}$$

This is the ~~set of~~ all the points lying on or below the graph of $g(y)$.

Now, if $hyp(g)$ is convex $\rightarrow$ g is concave.

Now, g is inverse of f

So, $t \leq g(y)$

$\Rightarrow f(t) \leq y$ (f is increasing)

So, $hyp(g) = \{(y, t) \mid f(t) \leq y,\ t \in (a, b),\ y \in (f(a), f(b))\}$

Now, let's take,

$(y_1, t_1), (y_2, t_2) \in hyp(g)$

so, $f(t_1) \leq y_1$ ---(i)

$f(t_2) \leq y_2$

so, for any $\theta \in [0,1]$,

Let, $(y,t) = \theta(y_1, t_1) + (1-\theta)(y_2, t_2)$

$= \left(\theta y_1 + (1-\theta) y_2, \theta t_1 + (1-\theta) t_2\right)$

Since f is convex,

$f(t) = f(\theta t_1 + (1-\theta) t_2) \leq \theta f(t_1) + (1-\theta) f(t_2)$

$\xrightarrow{\text{From (i)}}$ $f(t) \leq \theta y_1 + (1-\theta) y_2 = y$

So, $f(t) \leq y \Longrightarrow (y,t) \in hyp(g)$

So, $hyp(g)$ is convex

Hence, $\boxed{g \text{ is concave}}$

Problem 5:

Restricting to a line :-

Let, $g(t) = f(Z + tV) = tr\left((Z+tV)^{-1}\right)$

where,

$$Z \succ 0 \ (P.D)$$

$$\& \ V \in S^n$$

Now,

$$(Z+tV)^{-1} = Z^{-\frac{1}{2}}\left(I + tZ^{-1/2} V Z^{-1/2}\right)^{-1} Z^{-\frac{1}{2}}$$

$$= Z^{-\frac{1}{2}}\left(I + t\tilde{V}\right)^{-1} Z^{-\frac{1}{2}}$$

where, $\tilde{v} = Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}}$

Now, $\tilde{V}$ is symmetric, so it can be diagonalized.

$$\tilde{V} = Q \Lambda Q^T$$

where, Q is an orthogonal matrix $(QQ^T = I)$

Λ is diagonal matrix

Now,
$$I + t\tilde{V} = I + t\, Q \Lambda Q^T$$
$$= Q(I + t\Lambda)\, Q^T \quad ; \text{ because } Q \text{ is orthogonal}$$

So, $(I + t\tilde{V})^{-1} = Q(I + t\Lambda)^{-1} Q^T$

So:
$$g(t) = tr\left((Z + tV)^{-1}\right) = tr\left(Z^{-1/2} Q (I + t\Lambda)^{-1} Q^T Z^{-\frac{1}{2}}\right)$$

Now, From cyclic property of trace:
$$tr(ABC) = tr(BCA) = tr(CAB)$$

So $g(t) = tr\left(Q^T Z^{-1} Q (I + t\Lambda)^{-1}\right)$

Now, $Q^T Z^{-1} Q$ is symmetric

$(I + t\Lambda)^{-1}$ is diagonal with entries $(1 + t\lambda_i)^{-1}$

So
$$g(t) = \sum_{i=1}^{n} \left(Q^T Z^{-1} Q\right)_{ii} (1 + t\lambda_i)^{-1}$$

This is a weighted sum of functions of the form:

$$(1+t\lambda_i)^{-1}$$

Now, let, $\phi_i(t) = \frac{1}{1+t\lambda_i}$

So, $\phi_i'(t) = -\lambda_i (1+t\lambda_i)^{-2}$

$$\phi_i''(t) = 2\lambda_i^2 (1+t\lambda_i)^{-3}$$

Here, $(1+t\lambda_i)^{-3} > 0$ when $1+t\lambda_i > 0$

This is true ~~iff~~ if t is in a region where

$$Z + tV \succ 0$$

So, $\phi_i''(t) \geq 0 \longrightarrow$ so $\frac{1}{1+t\lambda_i}$ is convex

Now, a positive weighted sum of convex functions is convex

So, $\boxed{g(t) \text{ is convex}}$

Problem 6:-

Here each $\|A^{(i)}x - b^{(i)}\|$ is convex

Because, ⊛ norm $\| \|$ is convex

⊛ $A^{(i)}x - b^{(i)}$ is affine in x

So, the composition $\|A^{(i)}x - b^{(i)}\|$ is convex since convex functions composed with affine function remain convex.

Now, let, $f_i(x) = \|A^{(i)}x - b^{(i)}\|$ is convex

So, $f(x) = \max\{f_1(x), \cdots f_k(x)\}$ is the pointwise maximum of convex functions.

for $\theta \in [0,1]$ then, $\forall x, y \in \mathbb{R}^n$:

$$f(\theta x + (1-\theta)y) = \max_i \|A^{(i)}(\theta x + (1-\theta)y) - b^{(i)}\|$$

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By convexity of each f_i :

$$\| A^{(i)} (\theta x + (1-\theta) y) - b^{(i)} \|$$

$$\leq \theta \| A^{(i)} x - b^{(i)} \| + (1-\theta) \cdot \| A^{(i)} y - b^{(i)} \|$$

Taking maximum over i :

$$f(\theta x + (1-\theta) y) \leq \theta \max_i \| A^{(i)} x - b^{(i)} \|$$

$$+ (1-\theta) \max_i \| A^{(i)} y - b^{(i)} \|$$

$$= \theta f(x) + (1-\theta) f(y)$$

Thus, f is convex.

Problem 7:

Given, $\log \sum_{i=1}^{n} e^{y_i}$ is convex

so, $g(x) = \log\left(\sum_{i=1}^{n} e^{a_i^T x + b_i}\right)$ is convex

because it is a composition of log-sum-exp function & an affine mapping after that

so; $-g(x)$ is concave

Now, $h(y) = -\log y$ is convex & decreasing

Because, $\nabla^2 h(y) = \frac{1}{y^2} > 0$ for $y > 0$

So, $f(x) = h(-g(x))$ is convex,

Because, $-g(x)$ is concave, $h(y)$ is convex & decreasing $\Longrightarrow$ if ϕ is convex & decreasing and if ψ is concave, then $\phi(\psi(x))$ is convex

Problem 8:

f & g are convex.

So, $\forall x, y \in I$ & $\theta \in [0,1]$

$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y) \quad \text{--- (1)}$$

$$g(\theta x + (1-\theta) y) \leq \theta g(x) + (1-\theta) g(y) \quad \text{--- (2)}$$

Now,

$$h(x) = f(x)\, g(x)$$

$$h(\theta x + (1-\theta) y) = f(\theta x + (1-\theta) y) \cdot g(\theta x + (1-\theta) y)$$

$$\leq \left(\theta f(x) + (1-\theta) f(y)\right)\left(\theta g(x) + (1-\theta) g(y)\right)$$

[using (1) & (2)]

So,

$$h(\theta x + (1-\theta) y) \leq \theta^2 f(x)\, g(x) + \theta(1-\theta)[f(x)\, g(y) + f(y)\, g(x)] + (1-\theta)^2 f(y)\, g(y)$$

Now,

if f and g are nondecreasing, then for $x \leq y$:

$$f(x) \leq f(y)$$
$$g(x) \leq g(y)$$

So, $f(x)\,g(y) + f(y)\,g(x) \leq f(x)\,g(x) + f(y)\,g(y)$

for f & g are nonincreasing, the same inequality holds because the signs align.

So,
$$h(\theta x + (1-\theta) y) \leq \theta^2 f(x)\,g(x) + \theta(1-\theta)\left[f(x)g(x) + f(y)\,g(y)\right] + (1-\theta)^2 f(y)\,g(y)$$

$$h(\theta x + (1-\theta) y) \leq \theta\, f(x)\,g(x) + (1-\theta)\, f(y)\,g(y)$$

$$h(\theta x + (1-\theta) y) \leq \theta\, h(x) + (1-\theta)\, h(y)$$

So, $\boxed{h \text{ is convex on } I}$