Let A be an $n \times n$ real matrix. The "Power Method" refers to the following iterative scheme

$$x_{n+1} = Ax_n / \|Ax_n\|_2 \quad \text{starting with } x_0 \neq 0$$

(assuming Ax_n never vanishes). The "QR Algorithm" refers to the following iterative scheme

$$A_{n+1} = R_n Q_n \text{ where } A_n = Q_n R_r \quad \text{starting with } A_0 = A \; ,$$

Q_n is orthogonal and R_n is upper triangular.

Problem 1.

(a) By hand calculations find out what happens when you apply the Power Method and the QR Algorithm to the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} .$$

Briefly explain your results.

(b) By hand calculations find out what happens when you apply the Power Method and the QR Algorithm to the matrix

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} .$$

Briefly explain your results.

Problem 2.

(a) Write a small program that implements the Power Method.

(b) Write a small program that implements the QR Algorithm. For this task you may call a system routine that generates the QR decomposition of an $n \times n$ matrix.

(c) Apply the programs you wrote in (a) and (b) to the symmetric, tridiagonal, 10×10 matrix A, given by $a_{i\,i} = 2$, $i = 1, \ldots, 10$ and $a_{i\,i-1} = -1$, $i = 2, \ldots, 10$.

(d) Apply the programs you wrote in (a) and (b) to the symmetric, full, 10×10 matrix A, given by $a_{i\,i} = 2$, $i = 1, \ldots, 10$ and $a_{i\,j} = -1/(i + j)$ for all entries with $j \neq i$.

(e) Briefly describe the differences and similarities between the results in (c) and (d).