Least squares (revisited)

$m \times n$

$$\|Ax - b\|_2 = \|V\Sigma U^T x - b\|_2$$

SVD

$$= \|V(\Sigma U^T x - V^T b)\|_2$$

Since V is orthogonal

$$= \|\Sigma y - V^T b\|_2$$

with $y = U^T x$

or $x = Uy$

Therefore the problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 \qquad (1)$$

is equivalent to

$$\min_{y \in \mathbb{R}^n} \|\Sigma y - V^T b\|_2 \qquad (2)$$

where x and y are related by $\underline{\underline{x = Uy}}$

Let $\Sigma = \begin{bmatrix} \sigma_1 & & & & & 0 \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_k & & \\ & & & & 0 & \\ 0 & & & & & \ddots \; 0 \end{bmatrix}$ $m \times n$, $m > n > k$, $\sigma_i \neq 0 \quad i = 1 \dots k$

Then

$$\|\Sigma y - V^T b\|_2 = \left[\sum_{i=1}^{k} \left(\sigma_i y_i - (V^T b)_i\right)^2 + \sum_{i=k+1}^{m} (V^T b)_i^2 \right]^{1/2}$$

and so a (the) solution to (2) is given by

$$\boxed{y_i = \frac{(V^T b)_i}{\sigma_i} \quad i = 1 \dots k}$$

and if we take $\boxed{y_i = 0 \quad i = k+1 \dots m}$

then we get the solution with minimal norm $\|y\|_2$

The corresponding $x = Uy$ is the solution to (1) with minimal norm $\|x\|_2$ (since $\|x\|_2 = \|y\|_2$)

why?

$x = Vy$