Linear Algebra and Applications
Homework #03

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Problem 1:

Given, $P_2(t) = \|x + ty\|_2^2$

(a)

For ~~euclev~~ euclidian norm of a vector z,

$$\|z\|_2^2 = z \cdot z$$

So,

$$P_2(t) = \|x + ty\|_2^2 = (x+ty) \cdot (x+ty)$$

$$= x \cdot x + x \cdot ty + x \cdot ty + t^2 (y \cdot y)$$

$$= x \cdot x + 2t\,(x \cdot y) + t^2 (y \cdot y)$$

again,

$$x \cdot x = \|x\|_2^2$$
$$y \cdot y = \|y\|_2^2$$

so,

$$\boxed{P_2(t) = \|x\|_2^2 + 2t\,(x \cdot y) + t^2 \|y\|_2^2}$$

(b) for $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, for $P_2(t) = t^2 \|y\|_2^2 + 2t\, x\cdot y + \|x\|_2^2 = 0$

with, $a = \|y\|_2^2$, $b = 2x\cdot y$, $c = \|x\|_2^2$

$$t = \frac{-2x\cdot y \pm \sqrt{4(x\cdot y)^2 - 4.\|y\|_2^2\|x\|_2^2}}{2.\|y\|_2^2}$$

$$= \frac{-(x\cdot y) \pm \sqrt{(x\cdot y)^2 - \|y\|_2^2\|x\|_2^2}}{\|y\|_2^2}$$

(c) From part (b), t will have two Distinct real roots,

if Determinat (D) $= 4(x\cdot y)^2 - 4\|y\|_2^2\|x\|_2^2 > 0$

$$\Rightarrow (x\cdot y)^2 - \|y\|_2^2\|x\|_2^2 > 0$$

Now, we know,

$x\cdot y = \|x\|_2^2 \|y\|_2^2 \cos\theta$; where θ is the angle between x & y

$$(x\cdot y)^2 = \|x\|_2^2\|y\|_2^2 \cos^2\theta$$

So,

$$(x \cdot y)^2 - \|y\|_2^2 \|x\|^2$$

$$= \|x\|_2^2 \|y\|_2^2 \cos^2\theta - \|y\|_2^2 \|x\|_2^2$$

$$= \|x\|_2^2 \|y\|_2^2 (\cos^2\theta - 1) \leqslant 0$$

Because,

$$\cos\theta \leq 1$$

$$\Rightarrow \cos^2\theta - 1 \leq 0$$

So,

$$D \leqslant 0 \longrightarrow D \not> 0$$

+ will not have two Distinct real values

$\downarrow$

$P_2(t)$ will have two identical real roots for $D = 0$

$$\Rightarrow \cos^2\theta - 1 = 0$$

$$\Rightarrow \cos\theta = 1$$

$$\Rightarrow \theta = 0 \quad \text{(when } x \text{ \& } y \text{ are parallel)}$$

Otherwise, $D < 0$

↳ two complex distinct roots.

d From (B) & (C)

$$D \leq 0$$

$$\Rightarrow 4(x \cdot y)^2 - 4\|x\|_2^2\|y\|_2^2 \leq 0$$

$$\Rightarrow (x \cdot y)^2 \leq \|x\|_2^2\|y\|_2^2$$

So $$\boxed{|x \cdot y| \leq \|x\|_2\|y\|_2}$$

Problem 2

Given,

$$\|B\|_F = \left(\sum_{i,j=1}^{n,d} (b_{ij})^2 \right)^{\frac{1}{2}}$$

(a) $$\|Bx\|_2^2 = Bx \cdot Bx = \sum_{i=1}^{n} \left(\sum_{j=1}^{d} b_{ij} x_j \right)^2$$

From Cauchy-schwarts inequality,

$$\left(\sum_{j=1}^{d} b_{ij} x_j \right)^2 \leq \left(\sum_{j=1}^{d} b_{ij}^2 \right) \left(\sum_{j=1}^{d} x_j^2 \right)$$

$$= \left(\sum_{j=1}^{d} b_{ij}^2 \right) \|x\|_2^2$$

(From definition of euclidian norm)

summing over i,

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{d} b_{ij} x_j \right)^2 \leq \left(\sum_{i=1}^{n} \sum_{j=1}^{d} b_{ij}^2 \right) \|x\|_2^2$$

$$\Rightarrow \boxed{\|Bx\|_2^2 \leq \|B\|_F^2 \|x\|_2^2}$$

$$\|Bx\|_2 \leq \|B\|_F \|x\|$$

(b) Given, $x_\lambda \to x_0$ as $\lambda \to 0$

iff, $\| x_\lambda - x_0 \|_2 \to 0$ as $\lambda \to 0$

From part (a),

$$\| B x_\lambda - B x_0 \|_2 = \| B (x_\lambda - x_0) \|_2$$

$$\boxed{\| B x_\lambda - B x_0 \|_2 \leq \|B\|_F \| x_\lambda - x_0 \|_2} \quad \text{--- (1)}$$

Now, if $x_\lambda \to x_0$ as $\lambda \to 0$

$\Rightarrow \| x_\lambda - x_0 \|_2 \to 0$

$\Rightarrow \|B\|_F \|x_\lambda - x_0\| \to 0$

From (1)

$\Rightarrow \| B x_\lambda - B x_0 \|_2 \to 0$ as $\lambda \to 0$

$\Rightarrow \boxed{B x_\lambda \to B x_0}$

Problem 3:

Given, $$C = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

(a) eigenvalues:

eigenvalues of C are the roots of

$$\det(C - \lambda I) = 0$$

$$\Rightarrow \det \begin{pmatrix} 4-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (4-\lambda)(2-\lambda).(4-\lambda) + 1(0-(2-\lambda)) = 0$$

$$\Rightarrow (2-\lambda)(4-\lambda)^2 - (2-\lambda) = 0$$

$$\Rightarrow (2-\lambda)\left[(4-\lambda)^2 - 1\right] = 0$$

$\lambda = 2$

$4-\lambda = \pm 1$

$\lambda = 4 \pm 1$

$= 3, 5$

eigenvectors: ($\underline{v}$)

for $\lambda=2$,

$$C\underline{v} = \lambda \underline{v}$$

$$\Rightarrow (C - \lambda I)\,\underline{v} = 0$$

$$\Rightarrow \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} 2x + z = 0 \\ x + 2z = 0 \end{cases} \longrightarrow \left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\}$$

so, for $\lambda = 2$, $\underline{v}$ = scaler multiple of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

for $\lambda=3$

$$(C - \lambda I)\underline{v} = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} x + z = 0 \\ -y = 0 \\ x + z = 0 \end{cases} \longrightarrow \left.\begin{matrix} x = -z \\ y = 0 \end{matrix}\right\}$$

So, for $\lambda = 3$,

$\underline{v} =$ scaler multiple of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

(orthonormal)

for $\lambda = 5$,

$(C - \lambda I)\underline{v} = 0$

$$\Rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \left\{ \begin{array}{l} -x + z = 0 \\ -3y = 0 \\ x - z = 0 \end{array} \right. \longrightarrow \left. \begin{array}{l} x = z \\ y = 0 \end{array} \right\}$$

So, for $\lambda = 5$,

$\underline{v} =$ scaler multiple of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

(b) As C is a 'real' and 'symmetric' matrix

$C = Q D Q^T$ where $Q = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= Q D Q^{-1}$

and,

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

As Q consists of ~~three~~ orthonormal vector, $Q^T = Q^{-1}$

\# So,

$$C = Q D Q^T = Q D Q^{-1}$$

$$= \underbrace{\begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q^T = Q^{-1}}$$

(c) Now,

$$C = Q D Q^T$$

So, $C^{100} = (Q D Q^T)^{100}$

$$= Q D \underbrace{Q^T . Q}_{I} D \underbrace{Q^T . Q}_{I} D Q^T \ldots \quad Q D Q^T \text{ (100 times)}$$

Now, $Q^T = Q^{-1}$, so, $Q^T Q = Q Q^T = I$

So, $C^{100} = Q D^{100} . Q^T$

$$C^{100} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 2^{100} & 0 & 0 \\ 0 & 3^{100} & 0 \\ 0 & 0 & 5^{100} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Similarly,

$$C = Q D Q^T$$

So,

$$\sqrt{C} = Q \sqrt{D} Q^T = Q \cdot \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix} \sqrt{Q}$$

This can be verified as:

$$\sqrt{C} \cdot \sqrt{C} = Q \sqrt{D} \underbrace{Q^T \cdot Q}_{I} \sqrt{D} Q^T$$

$$= Q \cdot \sqrt{D} \cdot \sqrt{D} \cdot Q^T$$

$$\boxed{\sqrt{C}\sqrt{C} = Q \cdot D\, Q^T = C}$$

Problem-4.

Given,
$$\|A\|_6 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

if μ is an eigenvalue of A.

then with corresponding eigenvector v

$$Av = \mu v$$

$$\|Av\|_2 = \|\mu v\|_2$$

$$= |\mu| \, \|v\|_2 \quad (\mu \text{ scaler})$$

So,

$$\frac{\|Av\|_2}{\|v\|_2} = |\mu| \quad \text{---} \quad (1)$$

But $\|A\|_6$ is the maximum of $\frac{\|Ax\|_2}{\|x\|_2}$ over all $x \neq 0$

So, any $\frac{\|Ax\|_2}{\|x\|_2} \leq \|A\|_6$

from (1) $\longrightarrow$

So,
$$\boxed{|\mu| \leq \|A\|_6}$$

b if A is symmetric,

A has an orthonormal basis of eigenvectors $\{x^{(1)}, x^{(2)} \ldots x^{(n)}\}$

with corresponding eigenvalues $\{\mu_1, \mu_2 -- \mu_n\}$

So, any vector x can be represented as:

$$x = \sum_{i=1}^{n} c_i \, x^{(i)}$$

Now, $A x(i) = \mu_i \, x(i)$

So, $$A x = \sum_{i=1}^{n} c_i \, \mu_i \, x(i)$$

Now, $$\| A x \|_2^2 = \sum_{i=1}^{n} |c_i|^2 |\mu_i|^2$$

This is because $x^{(i)}$'s are orthonormal.

So, $$\begin{cases} x^{(i)} \cdot x^{(j)} = 0 & \text{for } i \neq j \\ \text{and } \| x^{(i)} \|_2^2 = 1 \end{cases}$$

Again, $x = \sum_{i=1}^{n} c_i \, x^{(i)}$

$$\|x\|_2^2 = \sum_{i=1}^{n} |c_i|^2$$

So, $$\frac{\|Ax\|_2}{\|x\|_2} = \left(\frac{\sum_{i=1}^{n} |c_i|^2 |M_i|^2}{\sum_{i=1}^{n} |c_i|^2} \right)^{\frac{1}{2}} \le \max_{1 \le i \le n} |M_i|$$

Because,

$$\frac{\sum_{i=1}^{n} |c_i|^2 |M_i|^2}{\sum_{i=1}^{n} |c_i|^2} \le \frac{\sum_{i=1}^{n} |c_i|^2 \left(\max_{1 \le j \le n} |M_j|^2 \right)}{\sum_{i=1}^{n} |c_i|^2}$$

$$= \max_{1 \le j \le n} |M_j|^2 \, \frac{\cancel{\sum_{i=1}^{n} |c_i|^2}}{\cancel{\sum_{i=1}^{n} |c_i|^2}}$$

$$= \max_{1 \le j \le n} |M_j|^2$$

So, $$\left(\frac{\sum_{i=1}^{n} |c_i|^2 |M_i|^2}{\sum_{i=1}^{n} |c_i|^2} \right)^{\frac{1}{2}} \le \max_{1 \le i \le n} |M_i|$$

So,

$$\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \max_{1 \leq i \leq n} |M_i|$$

(c)

From part (a),

$$|M| \leq \|A\|_6$$

From part (b),

$$\|A\|_6 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \max_{1 \leq i \leq n} |M_i|$$

Combining these two,

$$\boxed{\|A\|_6 = \max_{1 \leq i \leq n} |M_i|}$$