Recall that the singular value decomposition (SVD) of an $m \times d$ matrix, A. is a factorization of the form

$$A = V \Sigma U^T \ ,$$

where V is an $m \times m$ orthogonal matrix, and U is a $d \times d$ orthogonal matrix. Σ is an $m \times d$ diagonal matrix with non-negative real numbers σ_i on the diagonal.

Alternatively this expresses A as a sum of rank-1 matrices

$$A = \sum_{i=1}^{r} \sigma_i \mathbf{v}_i \mathbf{u}_i^T \ ,$$

where σ_i, $1 \leq i \leq r$, are the positive entries of Σ, and $\mathbf{v}_i$, $\mathbf{u}_i$ are the corresponding column vectors of V and U, respectively.

(a) Show that the columns of U form an orthonormal basis in $\mathbb{R}^d$ of eigenvectors for $A^T A$. Show that the columns of V form an orthonormal basis in $\mathbb{R}^m$ of eigenvectors for AA^T.

(b) Use the fact that $U^T A^T A U = \Sigma^T \Sigma$ to show that

$$U_1^T A^T A U_1 = \Sigma_1^2 \ ,$$

where $U_1 = [\mathbf{u}_1 \mathbf{u}_2 \ldots \mathbf{u}_r]$, and Σ_1 is the $r \times r$ diagonal matrix with all the positive σ_i on the diagonal. How does one find the $m \times r$ matrix $V = [\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_r]$ based on A, U_1 and Σ_1?

(c) Write code (Python or Matlab) that, given a matrix A, forms $A^T A$ and returns U and $\Sigma^T \Sigma$. You may use a system routine to find this spectral decomposition of $A^T A$. Expand the code to find U_1, V_1 and Σ_1.

(d) Use the code from above to find U_1, V_1 and Σ_1 for the three image-files (three-28, columbia-128 and columbia-256) uploaded to canvas (in Modules).

(e) Use the results obtained in (d) to calculate the rank-k approximations

$$A_k = \sum_{i=1}^{k} \sigma_i \mathbf{v}_i \mathbf{u} i^T \quad , k = 4, 20 \text{ and } r$$

for each of the three image-files. Plot the corresponding nine images. For this purpose make sure the σ_i are ordered in decreasing order $\sigma_1 > \sigma_2 > \ldots > \sigma_r > 0$.