Linear Algebra and Applications
Homework #09

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1(a) <u>Linear independence of $\sin^2 t$, $\cos^2 t$ & 1:</u>

we check if there exists constants c_1, c_2, c_3 (all non-zero) such that:

$$c_1 \sin^2 t + c_2 \cos^2 t + c_3 \cdot 1 = 0 \quad \text{for all } t$$

Now,

$$1 = \sin^2 t + \cos^2 t$$

So,

$$c_1 \sin^2 t + c_2 \cos^2 t + c_3(\sin^2 t + \cos^2 t) = 1$$

$$\Rightarrow (c_1 + c_3) \sin^2 t + (c_2 + c_3) \cos^2 t = 1$$

For this to hold for all t:

$$c_1 + c_3 = 0 \quad \& \quad c_2 + c_3 = 0$$

if $c_3 = 1$,

$$c_1 = -1, \quad c_2 = -1$$

So, a non-trivial solution exists

$$-\sin^2 t - \cos^2 t + 1 = 0$$

$\boxed{\text{Not linearly independent}}$

1(iii)

Let, for a,b,c the linear combination is,

$$a\sin^2 t + b\cos^2 t + c\sin t = 0 \quad (\text{for all } t)$$

for $t=0$:

$$0 + b + 0 = 0$$

$$b = 0$$

for $t = \pi/2$

$$a + 0 + c = 0$$

$$a + c = 0$$

for $t = \frac{\pi}{6}$

$$\frac{a}{4} + b\frac{3}{4} + \frac{c}{2} = 0$$

$$\Rightarrow -\frac{c}{4} + \frac{c}{2} = 0 \quad \left[\begin{matrix} b=0 \\ a=-c \end{matrix}\right]$$

$$\Rightarrow \frac{c}{4} = 0$$

$$c = 0$$

$$a + c = 0, \quad a = 0$$

So, only solution exists when $\boxed{a=b=c=0}$

$\boxed{\text{linearly independent}}$

Problem 2

The given $(n+2) \times (n+2)$ matrix is a block diagonal matrix, with

(I) a 2×2 block matrix at first. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(II) a $n \times n$ diagonal matrix later.

The determinat of a block diagonal matrix is the product of the determinants of its diagonal blocks.

So,

$$\det(A) = \det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \det \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 2 & \vdots \\ & & \ddots \\ 0 & \cdots & n \end{bmatrix}$$

$$= (1-4) \cdot (1 \cdot 2 \cdot 3 \cdots n)$$

$$\boxed{\det(A) = (-3)\, n!}$$

Problem 3:-

we can write,

$$\frac{d}{dt}v = Av \quad \text{--- (i)}$$

where, $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$v(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigenvalue of A:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \lambda^2 - \sqrt{2}\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{\sqrt{2} \pm i\sqrt{2}}{2} = \cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4}$$

$$= e^{\pm i\frac{\pi}{4}}$$

we can write:

$$A = C^{-1} \Lambda C$$

$$Av = C^{-1} \Lambda C v$$

where,

$C = [\underline{c}_1, \underline{c}_2]$

$\underline{c}_1, \underline{c}_2$ are the eigenvectors

$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Now, from (1),

$$\cancel{C^{-1}} \quad \frac{d}{dt}V = AV$$

$$\Rightarrow C^{-1}\frac{d}{dt}(V) = C^{-1}AV$$

$$= C^{-1}ACC^{-1}V \qquad \Big|\; CC^{-1} = I$$

$$= (C^{-1}AC)\, C^{-1}V$$

$$= \Lambda\, C^{-1}V \qquad \Big|\; A = C^{-1}\Lambda C, \quad \Lambda = C^{-1}AC$$

So, $\frac{d}{dt}(C^{-1}V) = \Lambda(C^{-1}V)$

Let, $C^{-1}V = U$

then $\frac{d}{dt}(U) = \Lambda U$

Now, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} e^{\frac{i\pi}{4}} & 0 \\ 0 & e^{\frac{-i\pi}{4}} \end{bmatrix}$

so, $\frac{d}{dt}(U) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U$

$$U = \begin{bmatrix} a e^{\lambda_1 t} \\ b e^{\lambda_2 t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

& $C^{-1} V = U$

$V = CU$ ---- (2)

Now, <u>for eigenvectors:</u>

<u>λ_1</u> : $(A - \lambda_1 I) c_1 = 0$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} = 0$$

$$c_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\underline{\underline{\lambda_2:}}$

$$(A - \lambda_2 I)\, c_2 = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1-(1-i) & -1 \\ 1 & 1-(1-i) \end{bmatrix}\begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} = 0$$

$$c_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

So, $C = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ & $C^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

From (2) $\longrightarrow$

$$V = C\,U$$

$$= \begin{bmatrix} 1 & i \\ -i & i \end{bmatrix} \begin{bmatrix} a e^{\lambda_1 t} \\ b e^{\lambda_2 t} \end{bmatrix} \quad \text{--- (3)}$$

Now,

$$V[0] = C\,U[0]$$

$$\Rightarrow U[0] = C^{-1} V[0]$$

$$= \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So,

$$V = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$U[0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

So, $a = \frac{1}{2}$, $b = \frac{1}{2}$

From (3) $\longrightarrow$

Now,

$$v = Cu$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{\lambda_1 t} + e^{\lambda_2 t} \\ -ie^{\lambda_1 t} + ie^{\lambda_2 t} \end{bmatrix}$$

Now,

$$e^{\lambda_1 t} = e^{(\alpha + i\alpha)t} = e^{\alpha t} \cdot e^{i\alpha t}$$

$$e^{\lambda_2 t} = e^{(\alpha - i\alpha)t} = e^{\alpha t} \cdot e^{-i\alpha t}$$

$$\alpha = \frac{1}{\sqrt{2}}$$

$$e^{\lambda_1 t} = e^{\alpha t}\left(\cos(\alpha t) + i\sin(\alpha t)\right)$$

$$e^{\lambda_2 t} = e^{\alpha t}\left(\cos(\alpha t) - i\sin(\alpha t)\right)$$

So,

$$e^{\lambda_1 t} + e^{\lambda_2 t} = 2e^{\alpha t} \cdot \cos \alpha t$$

$$e^{\lambda_1 t} - e^{\lambda_2 t} = 2ie^{\alpha t} \cdot \sin \alpha t$$

So, $v = \frac{1}{2}\begin{bmatrix} 2e^{\alpha t}\cdot\cos\alpha t \\ -i\cdot 2i\cdot e^{\alpha t}\cdot\sin\alpha t \end{bmatrix}$

$= e^{\alpha t}\begin{bmatrix} \cos\alpha t \\ \sin\alpha t \end{bmatrix}$

$\alpha = \frac{1}{\sqrt{2}}$

$$\boxed{V(t) = e^{\frac{t}{\sqrt{2}}}\begin{bmatrix} \cos\frac{t}{\sqrt{2}} \\ \sin\frac{t}{\sqrt{2}} \end{bmatrix}}$$

Problem 4:-

Given, $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Eigenvalues of A:-

$\det(A - \lambda I) = 0$

$\Rightarrow \lambda^2 - \sqrt{2}\lambda + 1 = 0$

$\lambda_{1,2} = \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$

$= \cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4}$

$= e^{\pm i \frac{\pi}{4}}$

eigenvectors:- Let $\underline{c_1}, c_2$ eigenvectors of A

for, $\lambda_1 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, $\frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \underline{c_1} = 0$

$c_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

for $\lambda_2 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$ $\longrightarrow$ $\frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \underline{c}_2 = 0$

$$\underline{c}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$C = [\underline{c}_1 \quad \underline{c}_2] = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

$$C^{-1} = \frac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

Now,

$$A = CDC^{-1} \qquad D = \begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}$$

$$A^N = CD^N C^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-\frac{i\pi}{4}} \end{bmatrix}^N \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\frac{iN\pi}{4}} & 0 \\ 0 & e^{-\frac{iN\pi}{4}} \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{i\left(\frac{N\pi}{4}\right)} & i e^{\left(\frac{N\pi}{4}\right)} \\ e^{-i\left(\frac{N\pi}{4}\right)} & -i e^{-\left(\frac{iN\pi}{4}\right)} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{i\frac{N\pi}{4}} + e^{-i\frac{N\pi}{4}} & i\left(e^{i\frac{N\pi}{4}} - e^{-i\frac{N\pi}{4}}\right) \\ -i\left(e^{i\frac{N\pi}{4}} - e^{-i\frac{N\pi}{4}}\right) & e^{i\frac{N\pi}{4}} + e^{-i\frac{N\pi}{4}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(\frac{N\pi}{4}\right) & -\sin\left(\frac{N\pi}{4}\right) \\ \sin\left(\frac{N\pi}{4}\right) & \cos\left(\frac{N\pi}{4}\right) \end{bmatrix}$$

(Ans)

Using the identity,

$$\begin{cases} \dfrac{e^{it} + e^{-it}}{2} = \cos t \\[2ex] \dfrac{e^{it} - e^{-it}}{2i} = \sin t \end{cases}$$

Problem 5:

Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Let, $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Using Graham-schmidt orthogonalization:

$$u_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$q_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

orthogonalizing a_2: projection of a_2 on q_2:

$$\text{proj}_{q_2} a_2 = (a_2 \cdot q_1)\, q_1$$

$$= \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) q_1$$

$$= \sqrt{2}\, q_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Now,

$$u_2 = a_2 - \text{proj}_{q_1} a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Orthogonalizing a_3: projection of a_3 on q_1 & q_2:

$$\text{proj}_{q_1} a_3 = (a_3 \cdot q_1) \cdot q_1$$

$$= \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) q_1$$

$$= \frac{1}{2} q_1$$

$$\text{proj}_{q_2} a_3 = \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) q_2$$

$$= q_2$$

$$v_3 = a_3 - \text{proj}_{q_1} a_3 - \text{proj}_{q_2} a_3$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

So, $q_3 = \dfrac{v_3}{\|v_3\|} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

So, $Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$

For R:

$R_{ij} = a_j \cdot q_i$ (for $i \leq j$), otherwise 0

$R_{11} = a_1 \cdot q_1 = \sqrt{2}$

$R_{12} = a_2 \cdot q_1 = \sqrt{2}$

$R_{13} = a_3 \cdot q_1 = \frac{1}{\sqrt{2}}$

$R_{22} = 1$

$R_{23} = 1$

$R_{33} = \frac{1}{\sqrt{2}}$

So, $R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

So, $A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

Problem 6:

(a) $A = \begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$

Characteristic polynomial:

$$\det(A - \lambda I)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix}$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 6$$

$$\Rightarrow \boxed{\lambda^2 - 3\lambda - 4}$$

(b)

Now, $A^2 = \begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 3\sqrt{6} \\ 3\sqrt{6} & 10 \end{bmatrix}$$

$$A^2 - 3A - 4I = \begin{bmatrix} 7 & 3\sqrt{6} \\ 3\sqrt{6} & 10 \end{bmatrix} - \begin{bmatrix} 3 & 3\sqrt{6} \\ 3\sqrt{6} & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This result is expected because the characteristic polynomomial $\lambda^2 - 3\lambda - 4$ implies $A^2 - 4I - 4I = 0$. This is called Cayley-Hamilton theorem: every square matrix A satisfy it's own characteristic theorem.