Class #27

Thursday, May 1, 2025 9:45 PM

Approach II Solve the dual problem first

since we only have equality constraints, the dual problem is unconstrained

$$\nu^* = \max_{\nu} g(\nu)$$

↳ Solve using any solver (GD, NM)

Next: Assuming strong duality holds

find x^* by minimizing $L(x, \nu^*)$

Reminder: $$\boxed{g(\nu)} = \inf_x \left(f_0(x) + \nu^T(Ax - b) \right)$$

$$= \inf_x \left(f_0(x) + \nu^T A x - \nu^T b \right)$$

$$= \underbrace{\boxed{\inf_x} \left(f_0(x) + \nu^T A x \right)}_{\text{what is this ?}} - \nu^T b$$

concept of Conjugate function of $f(x)$

$$f^*(\underline{\underline{y}}) = \sup_x \left(\underline{\underline{y}}^T x - f(x) \right)$$

↳ linear function

$$(f^*)^* = f$$; Conjugate of a quadratic is a quadratic

a quadratic

$$f(x) = \frac{1}{2} x^T Q x \quad ; \quad f^*(y) = \frac{1}{2} y^T Q^{-1} y$$

$Q \succ 0.$

$$g(\nu) = \inf_x \left(f_0(x) + \nu^T A x \right) - \nu^T b$$

$$= -\sup_x \left(-f_0(x) - \nu^T A x \right) - \nu^T b$$

$$= -\underbrace{\sup_x \left(\left(-A^T \nu \right)^T x - f_0(x) \right)}_{f_0^*(-A^T \nu)} - \nu^T b$$

$$g(\nu) = -f_0^*(-A^T \nu) - \nu^T b$$

↳ If we know the conjugate of our objective function, we know exactly $g(\nu)$.

<u>Example</u>: Suppose

$$\min_x \; \frac{1}{2} x^T Q x \; ; \; Q \succ 0$$

$$\text{s.t.} \quad Ax = b$$

$$f_0^*(y) = \boxed{\frac{1}{2} y^T Q^{-1} y}$$

$$g(\nu) = -\boxed{\frac{1}{2} \left(-A^T \nu \right)^T Q^{-1} \left(-A^T \nu \right)} - \nu^T b$$

$$= -\frac{1}{2} \nu^T A Q^{-1} A^T \nu - \nu^T b$$

$$= -\frac{1}{2}\nu^T \underbrace{P}\nu - \nu^T b$$

$$P = AQ^{-1}A^T$$

Example 10.2 $\Rightarrow$ Read from text

Approach 3 : Directly solve the primal method by approximating our objective as a quadratic function in each iteration and solving a system of linear equations given by the KKT System.

$$\min_x \quad f(x) \quad \longrightarrow f \in C^2(\mathbb{R}^n)$$

$$\text{s.t.} \quad Ax = b$$

Idea: Approximate $f(x)$ by its second-order approximation where $Ax^{(k)} = b$

When we are at $x^{(k)}$, the goal is to take a step in some direct $\Delta x_{nt}^{(k)}$ s.t. $A(x^{(k)} + \Delta x_{nt}^{(k)}) = b$

$\Downarrow$

This step corresponds to solving the following KKT System

$$A\Delta x_{nt} = 0 \Leftarrow \begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{nt} \\ \omega \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Special case: $A = 0 \iff$ No constraints

$$\nabla^2 f(x)\, \Delta x_{nt} = -\nabla f(x)$$

If $\nabla^2 f(x) \succ 0$

$$\Longrightarrow \quad \Delta x_{nt} = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$$

$\hookrightarrow$ regular Newton's method

Assume $x^{(k)}$ satisfied the constraint:

$$A x^{(k)} = b$$

Then we must have $A\left(x^{(k)} + \overset{t^{(k)}}{\Delta x_{nt}^{(k)}}\right) = b$

$$A x^{(k)} + t^{(k)} A \Delta x_{nt}^{(k)} = b + 0 = b$$

If we start with a feasible $x^{(k)}$, we end up with a feasible $x^{(k+1)}$

Feasible Start Newton's Method

Requires assumptions of strong duality (Slater's condition is enough for that). A sufficient condition is the f is twice differentiable and the problem is feasible.

Algorithm:

Initialize: $x^{(0)}$: $x^{(0)} \in \operatorname{dom} f$ and $\underline{A x^{(0)} = b}$ feasible start

$\epsilon > 0$

$K \leftarrow 0$

$$\alpha \in (0, 1/2)$$
$$\beta \in (0, 1)$$

<u>Repeat</u>

① Compute the search direction $\Delta x_{nt}^{(k)}$ by solving

$$\begin{bmatrix} \nabla^2 f(x^{(k)}) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{nt}^{(k)} \\ w^{(k)} \end{bmatrix} = \begin{bmatrix} -\nabla f(x^{(k)}) \\ 0 \end{bmatrix}$$

② $$\lambda(x^{(k)}) = \left(\Delta x_{nt}^{(k)T} \nabla^2 f(x^{(k)}) \Delta x_{nt}^{(k)}\right)^{1/2}$$

Quit if $\dfrac{\lambda^2(x^{(k)})}{2} \leq \epsilon$

③ Choose step size $t^{(k)}$ by doing backtracking line search:

while $f(x^{(k)} + t\,\Delta x_{nt}^{(k)}) > f(x^{(k)}) + \alpha t \nabla f(x^{(k)})^T \Delta x_{nt}^{(k)}$
$$t \leftarrow \beta t$$

④ Update $x^{(k+1)} \leftarrow x^{(k)} + t^{(k)} \Delta x_{nt}^{(k)}$

⑤ $k \leftarrow k+1$

Challenge: what if $\mathrm{dom}\, f \neq \mathbb{R}^n$ and using QR decomposition of A^T to get x s.t.

using QR decomposition of A^T to get x s.t. $Ax = b$ does not yield an x that belongs to dom f.

$\Downarrow$

Finding a feasible x to start our method may not always be possible.

$\Downarrow$

Solution: Infeasible Start Newton's method.

↳ Primal-dual Newton's method

↳ Updates both primal variable x and dual variable ν.

we have two search directions now:

$$\Delta x_{nt} \quad \text{and} \quad \Delta \nu_{nt}$$

we find them by solving the KKT system:

$$(*) \qquad \begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{nt} \\ \underbrace{\nu + \Delta \nu_{nt}}_{w} \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ \underbrace{Ax - b}_{0} \end{bmatrix}$$

Note: If x ever ends up satisfying the constraint $Ax = b$, then

$A(x + \Delta x_{nt}) = b$ and effectively then we start satisfying the equality constraint.

Stopping criterion in infeasible start Newton's method

Stopping criterion in infeasible start Newton's method is based on the residual vector:

$$r(x,\nu) = \left(r_{dual}(x,\nu),\ r_{pri}(x,\nu)\right)$$

$$r_{dual}(x,\nu) = \nabla f(x) + A^T\nu \longrightarrow 0 \text{ as } x \to x^*,\ \nu \to \nu^*$$

$$r_{pri}(x,\nu) = Ax - b \longrightarrow 0 \text{ as } x \to x^*$$

Infeasible Start Newton's Method

Initialize: $x^{(0)} \in \text{dom } f$, $\nu^{(0)}$

$\epsilon > 0$

$\alpha \in (0, 1/2)$

$\beta \in (0,1)$

$k \leftarrow 0$

Repeat:

① Compute $\Delta x_{nt}^{(k)}$ and $\Delta \nu_{nt}^{(k)}$ by solving the KKT system in (*) above and setting

$$\Delta \nu_{nt}^{(k)} = w^{(k)} - \nu^{(k)}$$

② Backtracking line search on the residual vector

$t := 1$

while $\|r(x^{(k)} + t\Delta x_{nt}^{(k)}, \nu^{(k)} + t\Delta \nu_{nt}^{(k)})\|_2$

while $\|r(x^{(k)} + t\Delta x_{nt}^{(k)}, \nu^{(k)} + t\Delta \nu_{nt}^{(k)})\|_2$

$$> (1-\alpha t)\|r(x^{(k)}, \nu^{(k)})\|_2$$

$$t \leftarrow \beta t$$

③ Update

$$x^{(k+1)} \longleftarrow x^{(k)} + t^{(k)} \Delta x_{nt}^{(k)}$$

$$\nu^{(k+1)} \longleftarrow \nu^{(k)} + t^{(k)} \Delta \nu_{nt}^{(k)}$$

④ $K \leftarrow K+1$

<u>until</u>: $Ax^{(k)} = b$ and $\|r(x^{(k)}, \nu^{(k)})\|_2 \leq \epsilon$