Class #19

Friday, April 4, 2025 10:21 AM

Standard form of Constrained optimization

$$\min_{x} f_0(x)$$

$f_0(x) \to$ objective function

$x \to$ optimization

(P_0)

subject to

$f_i(x) \le 0$, $i=1,\dots,m \to$ Inequality constraints

$h_i(x) = 0$, $i=1,\dots,p \to$ Equality constraints

$f_0(x) \Rightarrow \text{dom } f_0$

$f_i(x) \Rightarrow \text{dom } f_i$

$h_i(x) \Rightarrow \text{dom } h_i$

Domain of the problem $\mathcal{D} = \bigcap_{i=0}^{m} \text{dom} f_i \cap \bigcap_{i=1}^{p} \text{dom} h_i$

A point $x \in \mathcal{D}$ is feasible if

$f_i(x) \le 0$, $i=1,\dots,m$
$h_i(x) = 0$, $i=1,\dots,p$

and infeasible otherwise.

Set of all feasible points
Constraint set
Feasible set

$C = \{x \in \mathcal{D} : f_i(x) \le 0, i=1,\dots,m, \; h_i(x)=0, i=1,\dots,p\}$

Another way of writing (P_0) is:

$$\min_{x \in C} f_0(x) \longrightarrow \text{Implicitly Constrained}$$

$$\min_{x \in C} f_0(x) \longrightarrow \text{Implicitly Constrained Optimization problem}$$

(P0) $\Longrightarrow$ Explicitly Constrained Optimization problem

E.g.,

$$\min_{x \in [-1,1]} x^2 + 2x$$

$$\min_{x} x^2 + 2x$$

subject to

$$x \le 1 \iff \underbrace{x-1}_{f_1(x)} \le 0$$

$$x \ge -1 \qquad \underbrace{-x-1}_{f_2(x)} \le 0$$

optimal value p^*

$$p^* = \inf \{ f_0(x) : x \in C \}$$

e.g.,

$$\min_{x \in (-1,1)} -x^2$$

$$p^* = -1$$

what if $C = \phi$?

$$p^* = \infty$$

what if $p^* = -\infty$? $\Longrightarrow$ $f_0(x)$ is unbounded below for $x \in C$

for $x \in C$

$\Leftrightarrow \exists\ x_k$ s.t. $f(x_k) \to -\infty$

Optimal point: A point x^* is called optimal if

(i) $x^* \in C \Rightarrow$ it satisfies constraints $\Rightarrow$ feasible

(ii) $f_0(x^*) = p^* \Rightarrow$ It attains the optimal value,

we say x^* solves (P_0).

Optimal Set: $X_{opt} = \{x : x \in C;\ f_0(x^*) = p^*\}$

(P_0) is solvable if $X_{opt} \neq \phi$

In constraint optimization problems, one of the first things to check is if the feasible set is non-empty. Because if $C = \phi \Rightarrow X_{opt} = \phi$

Feasibility Problem: Meant to determine whether $C = \phi$ and if not, may be find some feasible x's.

$$\text{find } x$$

$$\begin{aligned} \text{s.t.}\quad & f_i(x) \le 0, \quad i = 1, \dots, p \\ & h_i(x) = 0, \quad i = 1, \dots, m \end{aligned}$$

Redundant Constraints: A constraint is redundant if removing it does not change the feasible set.

e.g.;
$$\min_{\substack{x \succeq 0 \\ x \in [-1,1]}} f_0(x)$$

$$\left.\begin{array}{l} -x \le 0 \\ x-1 \le 0 \\ \boxed{-x-1 \le 0} \end{array}\right] \quad C = [0,1]$$

$\boxed{-x-1 \le 0} \rightarrow$ redundant

ϵ- Suboptimal Solution

If $x \in C$ and $f_0(x) \le p^* + \epsilon$

↳ we call x ϵ- Suboptimal Solution

$$X_{opt}^{\epsilon} = \{x : x \in C ;\ f_0(x) \le p^* + \epsilon\}$$

$$X_{opt} \mid X_{opt}^{\epsilon} \Rightarrow \text{Global Solution}$$

Local solution: A feasible x^* is called locally optimal if $\exists\ R > 0$ such that

$$f_0(x^*) = \inf \{f_0(z) : z \in C ,\ \|z - x^*\|_2 \le R\}$$

Active vs. Inactive Constraints

↳ Inequality Constraints

An inequality constraint $f_i(x) \le 0$ is called active for a feasible 'x' if

$$f_i(x) = 0 .$$

Else it is called inactive.

Constrained Optimization

Step 1

↳ write in standard form

Step 2

↳ Translate to an 'equivalent problem' that is amenable to be solved by an optimization algorithm.

Equivalent Problems

Two optimization problems, say (P_1) and (P_2), are called equivalent if the solution for (P_1) can be easily obtained from the solution for (P_2) and vice versa.

↳ The two solutions are not necessarily the same.

$$\min_x f(x) \quad \text{vs.} \quad \min_x f(x-4)$$

Example: $\min_x \|Ax-b\|_2$

$$\min_x f_0(x) \quad \text{where } f_0(x) = \|Ax-b\|_2$$

Issue: $f_0(x)$ is not differentiable

$$\min_x \; \varphi_0\big(f_0(x)\big) \quad \text{where} \quad \varphi_0(z) = z^2$$

$$\Longrightarrow \min_x \; \|Ax-b\|_2^2$$

Equivalency under Change of Variables

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be bijection (i.e., ϕ^{-1} exists) and image of $\phi \supseteq D$.

Redefine the problem as:

$x = \phi(z)$
↳ change of variables.

$$\tilde{f}_0(z) = f_0(\phi(z))$$

$$\tilde{f}_i(z) = f_i(\phi(z)), \quad i=1,\dots,m$$

$$\tilde{h}_i(z) = h_i(\phi(z)), \quad i=1,\dots,p$$

Then:

$$\left.\begin{aligned} &\min_z \; \tilde{f}_0(z) \\ &\text{s.t.} \; \tilde{f}_i(z) \le 0, \quad i=1,\dots,m \\ &\phantom{\text{s.t.}} \; \tilde{h}_i(z) = 0, \quad i=1,\dots,p \end{aligned}\right\} (P_z)$$

is equivalent to (P_0).

If x solves (P_0) $\Longleftrightarrow$ $z = \phi^{-1}(x)$ solves the new problem.

If z solves $(P_z) \Leftrightarrow x = \phi(z)$ solves the (P_0) problem.

Equivalency under transformation of objective and constraint functions

Suppose $\varphi_0 : \mathbb{R} \to \mathbb{R}$ is monotonically increasing on $\operatorname{dom} f_0$.

Then map $f_0(x)$ to $\tilde{f}_0(x) = \varphi_0(f_0(x))$

↳ we end up with an equivalent problem.

In the case of inequality constraint functions, let $\varphi_1, \dots, \varphi_m : \mathbb{R} \to \mathbb{R}$ and they satisfy

$$\varphi_i(u) \le 0 \Leftrightarrow u \le 0$$

In the case of equality constraint functions, let $\varphi_{m+1}, \dots \varphi_{m+p} : \mathbb{R} \to \mathbb{R}$ and they satisfy

$$\varphi_{m+j}(u) = 0 \Leftrightarrow u = 0$$

Problem: $\quad \min_x \ \varphi_0(f_0(x))$

s.t. $\quad \psi_i(f_i(x)) \leq 0 \quad , i=1,\dots,m$

$\quad \psi_{m+i}(h_i(x)) = 0, \quad i=1,\dots,p$

is equivalent to (P_0).

Slack Variables

Suppose an x solves (P_0).

Given that x, $f_i(x) \leq 0$

$\Leftrightarrow \exists\, s_i \geq 0$ such that $f_i(x) + \underline{s_i} = 0$ ↳ slack variable

If $s_i > 0 \Leftrightarrow f_i(x)$ is inactive

If $s_i = 0 \Leftrightarrow f_i(x)$ is active

Slack variables help turn all inequality constraints for x into equality constraints.

New Equivalent to (P_0) Problem:

$$\min_{\substack{x \in \mathbb{R}^n \\ s \in \mathbb{R}^m}} \quad f_0(x)$$

non-negativity constraints

subject to $\quad \underline{s_i \geq 0} \quad , i=1,\dots,m \qquad (P_s)$

$\quad f_i(x) + s_i = 0, \quad i=1,\dots,m$

$\quad h_i(x) = 0, \quad i=1,\dots,p$

(P_S) is equivalent to (P_D).

* If (x, s) is feasible for (P_S) then x is feasible for (P_D)

* If (x, s) is optimal for (P_S) then x is optimal for (P_D)