Problem 1.

In the following Q denotes a real $n \times n$ matrix.

(a) Prove that if Q_1 and Q_2 are orthogonal then so is the product Q_1Q_2.

(b) Prove that if Q is orthogonal then $\det Q = \pm 1$.

From now on suppose $n = 2$.

(c) Let Q_1 and Q_2 be 2×2 matrices representing counter-clockwise rotations by angle θ_1 and θ_2, respectively. What mappings do the products Q_1Q_2 and Q_2Q_1 represent?

(d) Do all 2×2 orthogonal matrices commute? Justify your answer.

(e) Suppose Q is a 2×2 orthogonal matrix with $\det Q = 1$. Can you determine what kind of mapping Q represents and why?

(f) Suppose Q is a 2×2 orthogonal matrix with $\det Q = -1$. Can you determine what kind of mapping Q represents and why?

(g) Let Q_1 be a 2×2 reflection matrix and let Q_2 be a 2×2 rotation matrix. What mappings do the products Q_1Q_2 and Q_2Q_1 represent and why?

(h) Let Q_1 and Q_2 be 2×2 reflection matrices. What mapping does the product Q_1Q_2 represent and why?

Problem 2.

Given a set of n linearly independent vectors $\{\mathbf{v}_1\ \mathbf{v}_2\ \ldots \mathbf{v}_n\}$ in $\mathbb{R}^m$, the Gram-Schmidt process creates a set of **orthonornal** vectors $\{\mathbf{u}_1\ \mathbf{u}_2\ \ldots \mathbf{u}_n\}$ with $\text{span}\{\mathbf{u}_1\ \mathbf{u}_2\ \ldots \mathbf{u}_n\} = \text{span}\{\mathbf{v}_1\ \mathbf{v}_2\ \ldots \mathbf{v}_n\}$. The process is the following:

$$\textbf{step 1}: \mathbf{u}_1 = \mathbf{v}_1/\|\mathbf{v}_1\|_2$$

$$\textbf{step 2}: \mathbf{u}_2 = [\mathbf{v}_2 - < \mathbf{v}_2, \mathbf{u}_1 > \mathbf{u}_1] / \|\mathbf{v}_2 - < \mathbf{v}_2, \mathbf{u}_1 > \mathbf{u}_1\|_2$$

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$$\textbf{step k}: \mathbf{u}_k = \left[\mathbf{v}_k - \sum_{j=1}^{k-1} < \mathbf{v}_k, \mathbf{u}_j > \mathbf{u}_j\right] / \|\mathbf{v}_k - \sum_{j=1}^{k-1} < \mathbf{v}_k, \mathbf{u}_j > \mathbf{u}_j\|_2$$

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$$\textbf{step n}: \mathbf{u}_n = \left[\mathbf{v}_n - \sum_{j=1}^{n-1} < \mathbf{v}_n, \mathbf{u}_j > \mathbf{u}_j\right] / \|\mathbf{v}_n - \sum_{j=1}^{n-1} < \mathbf{v}_n, \mathbf{u}_j > \mathbf{u}_j\|_2$$

(a) Show that $\mathbf{v}_k = \sum_{j=1}^{k} \beta_{kj}\mathbf{u}_j$ for some coefficients β_{kj}, $1 \le k \le n,\ 1 \le j \le k$. Give precise formulas for β_{kj}.

(b) Show that the Gram-Schmidt process leads to a decomposition

$$V = QR ,$$

where V is the $m \times n$ matrix $V = [\mathbf{v}_1\ \mathbf{v}_2\ \dots\ \mathbf{v}_n]$, Q is the $m \times n$ matrix $Q = [\mathbf{u}_1\ \mathbf{u}_2\ \dots\ \mathbf{u}_n]$ (with **orthonormal columns**), and R is an **upper triangular** $n \times n$ matrix. This is referred to as the QR-decomposition of V.

(c) Give the formula for the entries of R in terms of the coefficients β_{kj} from (a).