Class #4

Friday, January 31, 2025 10:51 AM

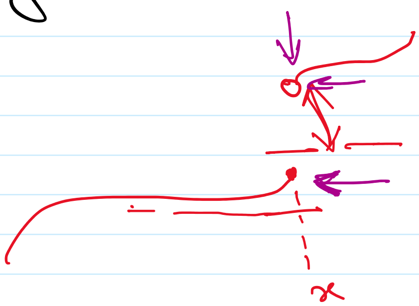
Calculus Concepts

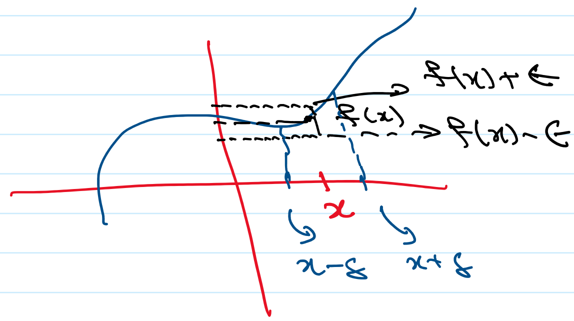
Continuous functions

A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x \in \mathbb{R}^n$ if for any $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$\|y - x\|_2 \leq \delta \implies \|f(y) - f(x)\|_2 \leq \epsilon$$

for all $y \in \operatorname{dom} f$.





$\iff$ Given any sequence $x_1, x_2, \dots$

Let $\lim_{n\to\infty} x_n = x$

f is continuous at x if and only if

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x)$$

f is a continuous function if it is continuous at all $x \in \operatorname{dom} f$

Derivative of a function $f: \mathbb{R}^n \to \mathbb{R}^m$

Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ and $x \in \text{int}(\text{dom} f)$

The function f is differentiable at $x \in \mathbb{R}^n$ if $\exists$ a matrix $Df(x) \in \mathbb{R}^{m \times n}$ that satisfies:

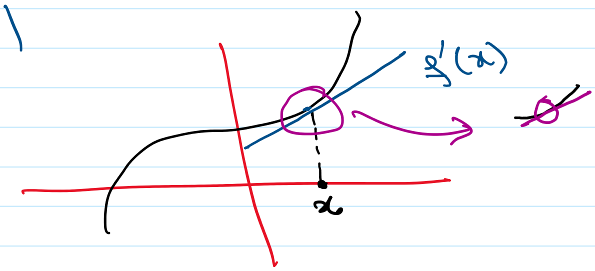
$$\lim_{\substack{z \in \text{dom} f,\, z \neq x \\ z \to x}} \frac{\| f(z) - f(x) - Df(x)(z-x) \|_2}{\| z - x \|_2} = 0$$

$Df(x)$ is called derivative of f at x

↳ when $m > 1 \Rightarrow$ Jacobian of f.

Special case: $n = 1$, $m = 1$

$$\lim_{\substack{z \in \text{dom} f,\, z \neq x \\ z \to x}} \frac{| f(z) - f(x) - f'(x)(z-x) |}{| z - x |} = 0$$

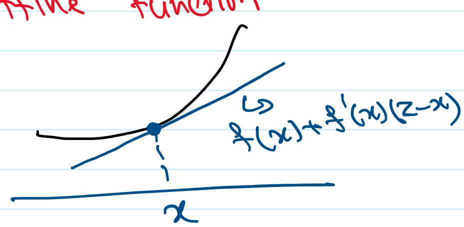


The function f is differentiable if $\text{dom} f$ is open and f is differentiable at every $x \in \text{dom} f$.

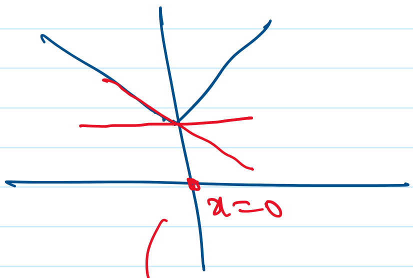
Derivatives provide first-order (or linear) approximation of f at x

$$\boxed{\hat{f}(z) = f(x) + Df(x)(z-x)}$$ → Affine function

↳ linear function of z



$Df(x)$ is unique.



first-order approximation of a function

Not differentiable at $x=0$

$$f(x) = 3x + 2$$

$$f'(x) = 3 \implies \hat{f}(z) = 3x + 2 + \boxed{3}(z - x)$$
$$= 3z + 2 + \cancel{3x} - \cancel{3x}$$
$$= 3z + 2$$

$$[Df(x)]_{ij} = \frac{\partial f_i(x)}{\partial x_j} \; ; \quad i = 1, \ldots, m \quad j = 1, \ldots, n$$

$$f(x) : \mathbb{R}^2 \to \mathbb{R} \; ; \quad f(x) = x_1^2 + x_2^2$$

$$Df(x) : \mathbb{R}^{1 \times 2} \; ; \quad [Df(x)]_{11} = \frac{\partial f(x)}{\partial x_1} = 2x_1$$

$$[Df(x)]_{12} = \frac{\partial f(x)}{\partial x_2} = 2x_2$$

$$Df(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}$$

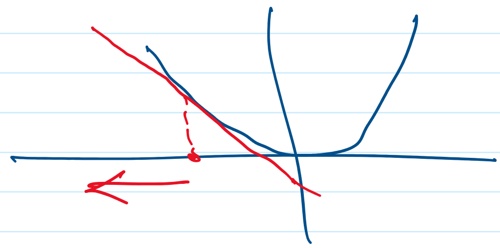
when $m = 1 \implies Df(x)$ is a row vector of length n

T

$$Df(x)^T = \nabla f(x) \Longrightarrow \text{Gradient of the function at } x$$

$$[\nabla f(x)]_i = \frac{\partial f(x)}{\partial x_i}, \quad i = 1, \ldots, n$$

$f(x) = x^2$



Scaler-valued functions

$$\hat{f}(z) = f(x) + \nabla f(x)^T (z - x)$$

$$\boxed{\hat{f}(z) \longrightarrow f(z) \quad \text{as} \quad z \longrightarrow x}$$