Problem 1. For $x \in \mathbb{R}^d$ let $\|x\|_2$ denote the Euclidean norm $\|x\|_2 = \left(\sum_{j=1}^d (x_j)^2\right)^{1/2}$, and consider the function

$$\mathbb{R} \ni t \to p_2(t) = \|x + ty\|_2^2 \ ,$$

where x, y are two fixed vectors in $\mathbb{R}^d \setminus \{0\}$.

(a) Show that

$$p_2(t) = t^2\|y\|_2^2 + 2tx \cdot y + \|x\|_2^2 \ .$$

(b) Admitting complex t, find the formula for the roots of $p_2(\cdot)$.

(c) From the formula $p_2(t) = \|x + ty\|_2^2$, argue that p_2 can have at most one distinct real root (actually it has no real roots, except when x and y are parallel).

(d) By a combination of (b) and (c) what can you conclude about the (discriminant) expression $4(x \cdot y)^2 - 4\|x\|_2^2\|y\|_2^2$? Use this fact to derive the inequality

$$|x \cdot y| \leq \|x\|_2\|y\|_2 \ ,$$

for all $x, y \in \mathbb{R}^d$. This is called the Cauchy-Schwarz inequality.

Problem 2. Let B be an $n \times d$ real matrix, and define

$$\|B\|_F = \left(\sum_{i,j=1}^{n,d} (b_{ij})^2\right)^{1/2} \ .$$

This is called the Frobenius norm of the matrix B.

(a) Show that

$$\|Bx\|_2 \leq \|B\|_F\|x\|_2 \ , \quad \text{for all } x \in \mathbb{R}^d \ .$$

Here $\|\cdot\|_2$ denotes the Euclidean norm, $\|x\|_2 = \left(\sum_{j=1}^d (x_j)^2\right)^{1/2}$.

Let $x_\lambda \in \mathbb{R}^d$ be a family of vectors indexed on $\lambda > 0$. We say that

$$x_\lambda \to x_0 \quad \text{as } \lambda \to 0 \ , \quad \text{or equivalently} \quad x_0 = \lim_{\lambda \to 0} x_\lambda \ ,$$

if and only if

$$\|x_\lambda - x_0\|_2 \to 0 \quad \text{as} \quad \lambda \to 0 \ .$$

(b) Show that

$$x_\lambda \to x_0 \quad \text{as} \quad \lambda \to 0 \quad \text{implies that } Bx_\lambda \to Bx_0 \quad \text{as} \quad \lambda \to 0$$

Problem 3. Let C denote the matrix

$$C = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix} .$$

(a) Find the eigenvalues and corresponding eigenvectors for C.

(b) Use your results from (a) to diagonalize C, *i.e.*, to write C as $C = QDQ^T = QDQ^{-1}$, where Q is orthogonal and D is diagonal.

(c) Based on the decomposition you found in (b) find a simple formula for C^{100}. Similarly find a symmetric matrix $\sqrt{C}$ with the property that $\sqrt{C}\sqrt{C} = C$.

Problem 4. Given a real $n \times n$ matrix A define

$$\|A\|_\sigma = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} .$$

(a) Show that if μ is an eigenvalue for A then $|\mu| \leq \|A\|_\sigma$.

(b) Now assume A is symmetric and as a consequence it has an orthonormal basis of eigenvectors $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ with corresponding eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_n\}$. In that case show that

$$\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \max_{1 \leq i \leq n} |\mu_i| .$$

(c) Based on (a) and (b) show that $\|A\|_\sigma = \max_{1 \leq i \leq n} |\mu_i|$ for any real symmetric matrix A.