ECE 509 (Spring 2024) – Final Exam

May 7, 2024

Name: Solution

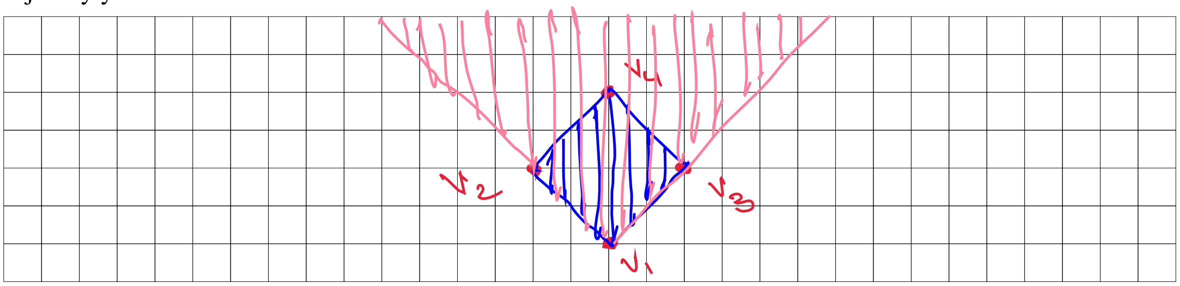
By writing my name, I affirm *on my honor that I have neither received nor given any unauthorized assistance on this examination*.

Read (and comply with) all of the following information before starting:

- The exam is open book, open notes, and open to any other material, provided it is in non-electronic format. However, an exception is made for paper-like e-ink devices such as the reMarkable tablet and e-ink Kindle. The use of electronic devices, including cell phones, smart watches, tablets, laptops, etc., is strictly forbidden during the exam, with the exception of the specified e-ink devices. Please ensure that you only have the permitted items on your desk before starting the exam.
- Show all work, clearly and in order, if you want to get full credit. In addition, ***justify your answers*** to ensure full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Pages are provided at the end of the answer book for rough work and additional space for answers. If your answer spills over into these pages or other unused pages in the exam booklet, please clearly indicate the relevant page numbers to facilitate correct marking.
- This exam has 10 questions, for a total of 85 points and 10 bonus points. You have 3 hours to complete it.
- Good luck!

Page:	1	2	3	4	5	6	7	9	Total
Points:	16	19	8	8	10	14	10	0	85
Bonus Points:	0	0	0	0	0	0	0	10	10

1. Consider four vectors in $\mathbb{R}^2$, given by $\mathbf{v}_1 = (0,0)$, $\mathbf{v}_2 = (-1,1)$, $\mathbf{v}_3 = (1,1)$, and $\mathbf{v}_4 = (0,2)$.
   (a) (3 points) Sketch the convex hull of these four vectors.
   (b) (3 points) Sketch the conic hull of these four vectors.
   (c) (4 points) Determine whether the convex hull and the conic hull of these vectors are polyhedra. Briefly justify your answer.



(a) Blue shaded region is the convex hull.
(b) Pink shaded region is the conic hull.

(c) Both the convex hull and the conic hull are polyhedra. The convex hull is the intersection of four half spaces, which makes it a polyhedron. It is also a polytope, because it is bounded. The conic hull is the intersection of two half spaces.

2. Determine if each set below is convex. Justify your answers.
   (a) (3 points) $\left\{(x,y) \in \mathbb{R}^2_+ \mid \frac{x}{y} \le 1\right\}$ $\mathbb{R}^2_{++}$
   (b) (3 points) $\left\{(x,y) \in \mathbb{R}^2_+ \mid \frac{x}{y} \ge 1\right\}$ $\mathbb{R}^2_{++}$

(a) $\frac{x}{y} \le 1$; $(x,y) \in \mathbb{R}^2_{++}$

$\Leftrightarrow$ $x > 0$; $y > 0$; $\frac{x}{y} \le 1 \Leftrightarrow x - y \le 0$
$\Leftrightarrow \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le 0$

So $\frac{x}{y} \le 1$; $(x,y) \in \mathbb{R}^2_{++}$ is the intersection of three half spaces $\Rightarrow$ It is a polyhedron / convex.

(b) $\frac{x}{y} \ge 1$; $(x,y) \in \mathbb{R}^2_{++}$
$\Leftrightarrow$ $x > 0$; $y > 0$; $x - y \ge 0 \Leftrightarrow \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ge 0$

So $\frac{x}{y} \ge 1$; $(x,y) \in \mathbb{R}^2_{++}$ is also the intersection of three half spaces $\Rightarrow$ It is a polyhedron / convex.

3. Which of the following sets are convex? Justify your answers.
   (a) (4 points) A slab, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \alpha \leq \mathbf{a}^\top \mathbf{x} \leq \beta\}$, where $\alpha, \beta \in \mathbb{R}$.
   (b) (4 points) A rectangle, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$, where $\forall i, \alpha_i, \beta_i \in \mathbb{R}$.
   (c) (4 points) A wedge, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_1^\top \mathbf{x} \leq b_1, \mathbf{a}_2^\top \mathbf{x} \leq b_2\}$, where $b_1, b_2 \in \mathbb{R}$.

(a) It is intersection of two half spaces: $a^T x \leq \beta$ and $a^T x \geq -\alpha$ $\Rightarrow$ It is convex (a polyhedron).

(b) $\alpha_i \leq x_i \leq \beta_i \iff \alpha_i \leq a_i^T x \leq \beta_i$, where a_i is the i^{th} standard basis vector: $e_i^T = [0 \cdots \underset{i^{th}\text{ position}}{1} \cdots 0]$

So it's intersection of n slabs $\Rightarrow$ convex.

(c) It is the intersection of two half spaces, hence it is a polyhedron / convex.

4. For each of the following functions, determine whether it is convex, concave, or neither. Justify your answers.
   (a) (3 points) $f(x) = e^x - 1$ on $\mathbb{R}$.
   (b) (4 points) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

(a) $f'(x) = e^x$; $f''(x) = e^x > 0 \ \forall x$

$\Rightarrow$ convex.

(b) $\nabla f(x) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$; $\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$x^T \nabla^2 f(x) x = [x_1 \ x_2] \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = x_1 x_2 + x_2 x_1$

when $x = [1 \ 1] \Rightarrow x^T \nabla^2 f(x) x > 0$

when $x = [-1 \ 1] \Rightarrow x^T \nabla^2 f(x) x < 0$

$\Rightarrow \nabla^2 f(x)$ is neither positive semidefinite, nor negative semidefinite. It is neither convex, nor concave.

5. Which of the following sets are convex? Justify your answers.
   (a) (4 points) The polar of a set C in $\mathbb{R}^n$, defined as $C^\circ = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^\top \mathbf{x} \le 1 \text{ for all } \mathbf{x} \in C\}$. Note that C cannot be assumed to be convex.
   (b) (4 points) The set $\{\mathbf{a} \in \mathbb{R}^k \mid p(0) = 1, |p(t)| \le 1 \text{ for } \alpha \le t \le \beta\}$, where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$ and $\alpha, \beta \in \mathbb{R}$.

(a) $C^\circ = \bigcap_{x \in C} \{y \in \mathbb{R}^n : y^T x \le 1\} = \bigcap_{x \in C} C^\circ_x$

where $C^\circ_x = \{y \in \mathbb{R}^n : y^T x \le 1\}$

For a fixed x, C°_x is a half space, so convex. C° is an infinite or finite intersection of convex sets, so C° is convex.

(b) Let us define $w_t = [1 \ t \ \_\_ \ t^{k-1}]^T \in \mathbb{R}^k$

Then $p(t) = a^T w_t$; $p(0) = a^T w_0$

$\Rightarrow \{a \in \mathbb{R}^k : p(0) = 1, |p(t)| \le 1 \text{ for } \alpha \le t \le \beta\}$

$= \{a \in \mathbb{R}^k : a^T w_0 = 1\} \bigcap_{\alpha \le t \le \beta} \{a \in \mathbb{R}^k : a^T w_t \le 1\}$

$\bigcap_{\alpha \le t \le \beta} \{a \in \mathbb{R}^k : a^T w_t \ge -1\}$.

This is intersection of a hyperplane and halfspaces. Hence, it is convex.

6. (8 points) Let $C \subset \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c \leq 0\},$$

with $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Show that C is convex if $\mathbf{A} \succeq 0$.

Hint: ~~Recall that a set is convex if and only if its intersection with an arbitrary line $\{\widehat{\mathbf{x}} + t\mathbf{u} \mid t \in \mathbb{R}\}$ is convex. Consider the intersection of C with such a line and analyze the resulting inequality.~~

The function $f(x) = x^T A x + b^T x + c$ is convex if $A \succeq 0$. $C = \{x \in \mathbb{R}^n : x^T A x + b^T x + c \leq 0\}$ is sublevel set of $f(x)$. Sublevel sets of all convex functions are convex, so C is convex.

7. (10 points) Adapt the proof of concavity of the log-determinant function to show that the function $f(\mathbf{X}) = (\det \mathbf{X})^{1/n}$ is concave on the domain $\mathbb{S}^n_{++}$, where $\mathbb{S}^n_{++}$ denotes the set of $n \times n$ positive definite symmetric matrices.

We show that $f(Z+tV)$ is concave for any $t \in \mathbb{R}$, for $Z, V \in S^n_{++}$ and fixed.

$$f(Z+tV) = \left(\det(Z+tV)\right)^{1/n}$$

$$= \left(\det\left(Z^{1/2}\left(I + tZ^{-1/2} V Z^{-1/2}\right) Z^{1/2}\right)\right)^{1/n}$$

$$= \left(\det(Z) \det\left(I + tZ^{-1/2} V Z^{-1/2}\right)\right)^{1/n}$$

$$= \left(\prod_{i=1}^{n} \lambda_i(Z)\right)^{1/n} \left(\prod_{i=1}^{n} \left(1 + t\lambda_i\left(Z^{-1/2} V Z^{-1/2}\right)\right)\right)^{1/n}$$

$$= a \left(\prod_{i=1}^{n} q_i(t)\right)^{1/n}, \text{ where}$$

$$a := \left(\prod_{i=1}^{n} \lambda_i(Z)\right)^{1/n} > 0, \text{ since } Z \in S^n_{++}$$

and $q_i(t) = 1 + t\lambda_i\left(Z^{-1/2} V Z^{-1/2}\right)$

$\left(\prod_{i=1}^{n} q_i(t)\right)^{1/n}$ is a concave function of t because it is composition of a concave function (geometric mean) with a linear function

$$t \longmapsto \begin{bmatrix} 1+q_1(t) \\ \vdots \\ 1+q_n(t) \end{bmatrix} \Rightarrow f(Z+tV) \text{ is concave in } t \Rightarrow f(x) = (\det(x))^{1/n}$$

is concave.

8. Determine whether the following functions are convex and justify your answers.
   (a) (6 points) Consider the function $f(\mathbf{x}) = \operatorname{tr}\left((\mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n)^{-1}\right)$ on the domain

   $$\{\mathbf{x} \mid \mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n \succ 0\},$$

   where each $\mathbf{A}_i$ is an element of $\mathbb{S}^m$, the set of $m \times m$ symmetric matrices. **Hint:** Recall that $\operatorname{tr}(\mathbf{X}^{-1})$ is convex on the set of positive definite symmetric matrices, $\mathbb{S}^m_{++}$.
   (b) (8 points) Consider the function $f(\mathbf{x}, u, v) = -\log(uv - \mathbf{x}^\top\mathbf{x})$ on the domain

   $$\{(\mathbf{x}, u, v) \mid uv > \mathbf{x}^\top\mathbf{x}, u, v > 0\}.$$

   Hint: Express $-\log(uv - \mathbf{x}^\top\mathbf{x})$ as $-\log u - \log(v - \mathbf{x}^\top\mathbf{x}/u)$.

(a) $f(x)$ is a composition of two functions

$g(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$

with $\operatorname{dom} g = \{x : A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$

and $h(Z) = \operatorname{tr}(Z^{-1})$, with $\operatorname{dom} h = \{Z : Z \succ 0\}$

$f(x) = h(g(x))$

h is convex and g is an affine function of $x \Rightarrow f(x)$ is convex.

(b) we take the hint and write

$$-\log(uv - x^T x) = -\log(u) - \log\left(v - \frac{x^T x}{u}\right)$$

$-\log(u)$ is convex in (x, u, v)

$-\log\left(v - \frac{x^T x}{u}\right)$ is composition of two functions:

$h(g((x,u,v)))$, where $h(z) = -\log(z)$

and $g((x,u,v)) = v - \frac{x^T x}{u}$

g is a concave function because it is sum of a linear function and $-\frac{x^T x}{u}$, which is quadratic over linear form that is concave.

h is convex and nonincreasing.
$\Rightarrow -\log\left(v - \frac{x^T x}{u}\right)$ is convex
$\Rightarrow f(x)$ is convex due to it being non-negative sum of two convex functions.

9. Consider the optimization problem:

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & \frac{x^2}{y} \leq 0 \end{array}$$

with variables $x \in \mathbb{R}$ and $y \in \mathbb{R}$, and domain $\mathcal{D} = \{(x, y) \mid y > 0\}$.

(a) (4 points) Verify that this is a convex optimization problem. Find the optimal value.

(b) (4 points) Give the Lagrange dual problem, and find the optimal solution λ^* and optimal value d^* of the dual problem. What is the optimal duality gap?

(c) (2 points) Does Slater's condition hold for this problem? Justify your answer.

(a) $f_0(x) = e^{-x}$ is a convex function

$f_1(x) = \frac{x^2}{y}$ is quadratic-over-linear form, which is convex for $y > 0$

$\Rightarrow$ This is a convex optimization problem, with one convex inequality constraint.

Since $\frac{x^2}{y} \leq 0$ and $y > 0$, the constraint set is simply $C = \{(x,y) : x = 0 \text{ and } y > 0\}$.

$\Rightarrow p^* = 1$.

(b) $\mathcal{L}((x,y),\lambda) = e^{-x} + \lambda \frac{x^2}{y}$, $y>0$

$$g(\lambda) = \inf_{(x,y)} \left(e^{-x} + \lambda \frac{x^2}{y} \right), \quad y>0$$

$$= \inf_{x} \inf_{y} \left(e^{-x} + \lambda \frac{x^2}{y} \right), \quad y>0$$

$$= \inf_{x} \left(e^{-x} + \lambda \cdot 0 \right) = \inf_{x} e^{-x} = 0$$

$$d^* = \sup_{\lambda \geq 0} g(\lambda) = 0$$

$$p^* - d^* = 1 - 0 = 1$$

(c) we do not have strong duality, since $p^* \neq d^* \Rightarrow$ Slater's condition does not hold.

It is also clear that we do not have any (x,y) in constraint set such that

$\frac{x^2}{y} < 0$ (strict feasibility).

10. **(Bonus)** Consider the function $f(\mathbf{p}) = \max_{i=1,\ldots,n} \left|\log(\mathbf{a}_i^\top \mathbf{p})\right|$, where $\mathbf{p} \in \mathbb{R}_+^m$ and $\mathbf{a}_i \in \mathbb{R}^m, i = 1, \ldots, n$.

   (a) (5 points (bonus)) Show that $\exp(f(\mathbf{p}))$ is convex on the domain $\{\mathbf{p} \mid \mathbf{a}_i^\top \mathbf{p} > 0, i = 1, \ldots, n\}$.
   Hint: Recall that $|\log(z)| = \max\{\log(z), \log(1/z)\} = \log(\max\{z, 1/z\})$.

   (b) (5 points (bonus)) Show that the following optimization problem is convex. Justify your answer.

$$\begin{array}{ll} \text{minimize} & \exp(f(\mathbf{p})) \\ \text{subject to} & \sum_{i=1}^{l} p_{[i]} \le 0.5 \sum_{i=1}^{m} p_i \end{array}$$

   where l is a positive integer less than or equal to m and $p_{[i]}$ is the i-th largest component of $\mathbf{p}$.

(a) $f(p) = \max_{i=1,\ldots,n} |\log(a_i^T p)| = \max_{i=1,\ldots,n} \log\left(\max\{a_i^T p, \frac{1}{a_i^T p}\}\right)$

$a_i^T p$ is a convex function.

$\frac{1}{a_i^T p}$ is a convex function for $a_i^T p > 0$.

(Composition of a convex function with a linear function; $\frac{1}{z}$, $z > 0$ and $a_i^T p$)

max of convex functions is convex

$\Rightarrow \log\left(\max\{a_i^T p, \frac{1}{a_i^T p}\}\right)$ is convex for each fixed i.

$f(p)$ is max of 'n' such convex functions, which is convex.

Finally: $\exp(f(p))$ is convex since $\exp(\cdot)$ is a nondecreasing convex function and $f(p)$ is a convex function, so $\exp(f(p))$ is convex.

—Scratch Pages—

(b) We already know that $\exp(f(p))$ is convex. We only need to verify that the constraint, which is an inequality constraint, corresponds to a convex inequality constraint.

Let $\sum_{i=1}^{l} p[i] = g(p)$

Then $g(p)$ is a convex function of p (see Example 3.6).

$$\Rightarrow \underbrace{g(p)}_{\text{convex}} - 0.5\, \mathbf{1}^T p \le 0$$

$-\mathbf{1}^T p$ is linear $\Leftrightarrow$ convex / concave

non-negative weighted combination of convex functions is convex

$$\Rightarrow f_1(p) = g(p) - 0.5\, \mathbf{1}^T p$$

is convex.

$$\Rightarrow \min_p \ \exp(f(p))$$
$$\text{s.t. } f_1(p) \le 0 \quad \text{is}$$

convex optimization.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	6	12	7	8	8	10	14	10	0	85
Score:											