Class #20

Monday, April 7, 2025 7:21 PM

Hyperplane in a vector Space

Let's look at $\mathbb{R}^n$

$\mathcal{H} = \{x : a^T x = b\}$ for fixed $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$

Hyperplane

It is an affine set. Hence it is convex.

Solution set of a nontrivial linear equation

Affine set has associated with it a subspace.

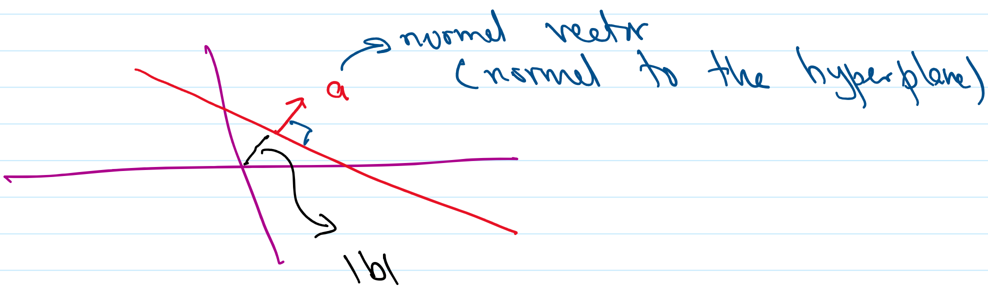
Subspace is $\boxed{a^T x = 0}$ $\Rightarrow$ Null Space of a^T.

$\dim(\mathcal{H}) = n-1$

A hyperplane in $\mathbb{R}$ is a point.

" " " $\mathbb{R}^2$ is a line

" " " $\mathbb{R}^3$ is a plane



If $b > 0 \Rightarrow$ offset is $\dfrac{|b|}{\|a\|_2}$

in the direction of the a vector

If $b < 0 \Rightarrow$ offset is in the opposite direction of the a vector.

Half Space: $\{x : a^T x \boxed{\leq} b\}$ is a closed half space.

$\hookrightarrow$ if $> b \Rightarrow$ open half space.

$\{x : a^T x \boxed{\geq} b\}$ is a closed half space

A hyperplane cuts $\mathbb{R}^n$ into two half spaces

Half space is a convex set.

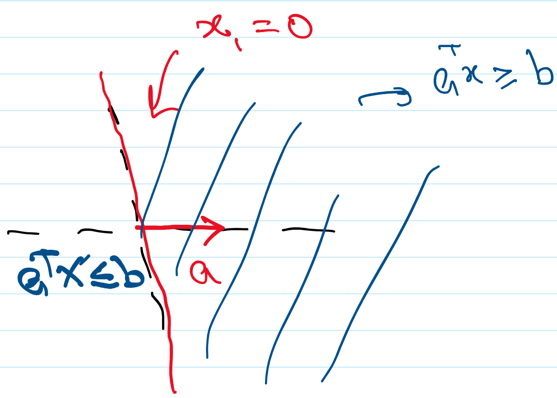
E.g.: $\mathcal{H} = \{x : a^T x = b\}$

Ex1: $b = 0$

$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

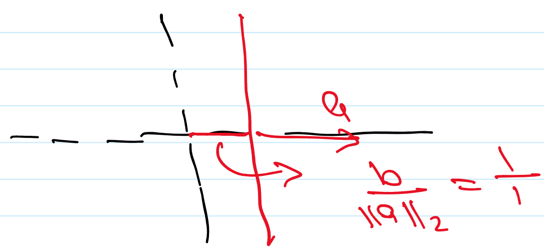
$1 \cdot x_1 + 0 \cdot x_2 = 0$

$\Leftrightarrow x_1 = 0$



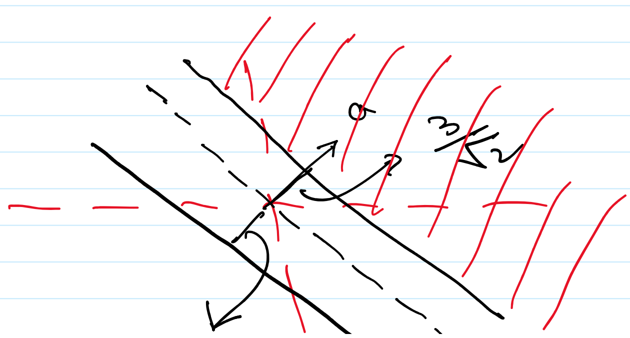
$b = 1$

$\Longrightarrow$

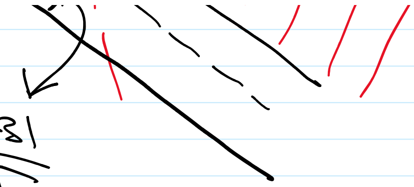


Ex2: $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$b = 3$



$$\frac{3}{\sqrt{2}}$$



$-\frac{3}{\sqrt{2}}$

$b = -3$

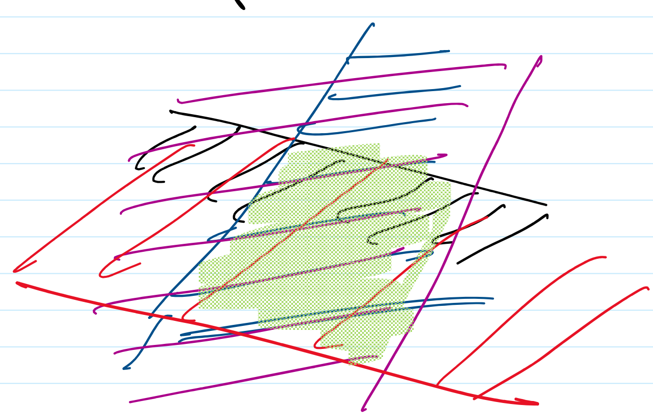
Ex 3: $a = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a^T x \geq b$

Polytope / Polyhedron

Solution set of a finite number of linear equations and inequalities

$$P = \{ x : \underbrace{a_j^T x \leq b_j}_{\text{one of this is a half space}} , j = 1, \dots, m, \underbrace{c_j^T x = d_j}_{\text{one of this is a hyperplane}}, j = 1, \dots, p \}$$

Intersection of a finite number of half spaces and hyperplanes.



Bounded polyhedron are called polytopes.

Polyhedra are convex sets.

Every convex hull is nothing but a polyhedron

Fact: Intersection of convex sets (finite or infinite) gives rise to a convex set.

Let $\alpha \in A$ and let S_α be convex
$\hookrightarrow$ indexing set

$$\bigcap_{\alpha \in A} S_\alpha \text{ is convex.}$$

Polyhedron $\Rightarrow$ It is convex because it is intersection of hyperplanes (affine; convex) and half spaces (convex).

Ex1: $S = \left\{ x \in \mathbb{R}^m : \left| \sum_{k=1}^{m} x_k \cos(kt) \right| \le 1 ;\ |t| \le \pi/3 \right\}$

Is S a convex set?

$$S = \left\{ x : -1 \le \sum_{k=1}^{m} x_k \cos(kt) \le 1 ;\ |t| \le \pi/3 \right\}$$

$$S = \left\{ x : \underbrace{\sum_{k=1}^{m} x_k \cos(kt) \le 1}, \ \underbrace{\sum_{k=1}^{m} x_k \cos(kt) \ge -1}_{a_t^T x \ge -1},\ |t| \le \pi/3 \right\}$$

for a fixed t, this a half space

$\Downarrow$

$a_t^T x \leq 1$

where $a_t = [\cos(t), \cos(2t), \ldots, \cos(mt)]$

$S_t^+ = \{x : a_t^T x \leq 1\}$ → convex

$\Rightarrow S = \bigcap_{-\pi/3 \leq t \leq \pi/3} S_t^+ \cap \bigcap_{-\pi/3 \leq t \leq \pi/3} S_t^-$

$S_t^- = \{x : a_t^T x \geq -1\}$

↳ convex

<u>Ex2</u>: Vector space of symmetric matrices S^n

$\{X \in S^n : z^T X z \geq 0\}$ → for a fixed z $\Rightarrow$ linear function of X in S^n $\Rightarrow$ halfspace in S^n

$\bigcap_{z \neq 0} \{X \in S^n : z^T X z \geq 0\}$ → space of positive semi definite matrices

$\Rightarrow$ Set of PSD matrices is convex

<u>Convex optimization in standard form:</u>

$$\min_x \; f_0(x) \qquad (P_0)$$

Subject to

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$$f_i(x) \leq 0 \quad , \quad i=1,\dots,m$$

$$h_i(x) = 0 \quad , \quad i=1,\dots,p$$

(P_0) is a convex optimization problem if and only if

① $f_0(x)$ is convex

② Domain of the problem

$$\mathcal{D} = \operatorname{dom} f_0 \cap \left(\bigcap_{i=1}^{m} \operatorname{dom} f_i\right) \cap \left(\bigcap_{i=1}^{p} \operatorname{dom} h_i\right)$$

is convex

$\Leftrightarrow$ dom f_0 is convex
dom f_i is convex
dom h_i is convex

③ Constraint set

$$C = \{x : f_i(x) \leq 0, i=1,\dots,m, h_i(x)=0, i=1,\dots,p\}$$

is convex

This requires that
(i) The equality constraints are linear
<u>i.e.</u> $h_i(x) = a_i^T x$ for some a_i

$\iff \quad Ax = 0$ where $A = \begin{bmatrix} — a_1^T — \\ — a_2^T — \\ \vdots \\ — a_p^T — \end{bmatrix}$

(ii) The inequality constraint functions must be convex.

Remember: $\underbrace{S_i = \{x : f_i(x) \le 0\}}_{\text{sublevel set of } f_i(x) \text{ at } \alpha = 0}$ and $f_i(x)$ is convex

Fact: Sublevel sets of convex functions are convex.

Summary: (P_0) is convex if

1. f_0 is convex
2. f_i's are convex
3. h_i's are linear $\iff Ax = 0$

Ex: $\min_x \quad x_1^2 + x_2^2$

s.t. $\quad \dfrac{x_1}{(1+x_1^2)} \le 0 \longrightarrow$ Not convex

$(x_1 + x_2)^2 = 0$

$\hookrightarrow$ Not linear

But note that

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$$\frac{x_1}{(1+x_1^2)} \leq 0 \iff x_1 \leq 0$$

$$(x_1+x_2)^2 = 0 \iff x_1+x_2 = 0$$

$$\left.\begin{aligned} \min_{x} \quad & x_1^2 + x_2^2 \\ & x_1 \leq 0 \\ & x_1+x_2 = 0 \end{aligned}\right\} \Longrightarrow \text{Convex}$$