Linear Algebra

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Preface

This document is essentially a summary of the main results from *Linear Algebra: Ideas and Applications, Fourth Edition* by Richard C. Penney in LaTeX.

The reason I selected this book to summarize the main results of linear algebra is twofold: first of all, this is the book that was used in my undergraduate education when I took linear algebra, so I am used to its style. Secondly, the subject is presented rigorously in Penney's book, but not in a dry, terse manner; there's a fair amount of examples, and proofs are constructed carefully without the confusion of missing steps.

Topics that are in Penney's book but that aren't usually covered in a first introductory rigorous course on linear algebra will not be discussed here; these excluded topics include the various computer projects and applications, least squares, and numerical techniques to compute eigenvalues.

Definitions end with a $\triangle$, and proofs of theorems, propositions and lemmas

Chapter 1

Systems of linear equations

1.1 The vector space of $m \times n$ matrices

A matrix is a rectangular array of numbers arranged into rows and columns. A matrix has **size** $m \times n$ if it has m rows and n columns.

The following are examples of matrices

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 9 \\ 5 & 15 & -7 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 7 & 1 \\ -2 & 5 & 10 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -2 \\ 0 & 7 \\ 6 & 6 \end{pmatrix}$$

We note that matrices are usually denoted by an uppercase Latin letter. If the matrix has an equal number of rows and columns (such as matrix A), we say that the matrix is **square**.

Definition 1.1. The set of all $m \times n$ matrices is denoted $M(m, n)$. △

Each row of an $m \times n$ matrix may be thought of as a $1 \times n$ matrix. Similarly, each column of an $m \times n$ matrix may be thought of as an $m \times 1$ matrix. Any matrix with only one row is called a **row vector**. Similarly, a matrix with only one column is called a **column vector**.

The entry in the ith row and jth column of the matrix A may be denoted by either A_{ij} or a_{ij}. a_{ij} is referred to as the "(i, j) entry of A". The notation $A = [a_{ij}]$ is also used, with the meaning "A is the matrix whose (i, j) entry is a_{ij}."

If A and B are $m \times n$ matrices, then $A + B$ is the $m \times n$ matrix defined by

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \tag{1.1}$$

that is, each entry in the (i, j) position of each matrix is added.

For example,

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 9 \\ 5 & 15 & -7 \end{pmatrix},\ B = \begin{pmatrix} -5 & 2 & 3 \\ 0 & 3 & -2 \\ -1 & 8 & -6 \end{pmatrix} \Rightarrow A + B = \begin{pmatrix} -3 & 2 & 4 \\ -1 & 6 & 7 \\ 4 & 23 & -13 \end{pmatrix} \tag{1.2}$$

Addition of matrices of different sizes is not defined.

If c is a number or scalar and $A = [a_{ij}]$ is an $m \times n$ matrix, we define **scalar multiplication** as

$$cA = c[a_{ij}] = [ca_{ij}] = [a_{ij}]c = Ac \tag{1.3}$$

Subtraction of matrices is defined analogously to matrix addition.

1.1.1 The space $\mathbb{R}^n$

A 2×1 column vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$ may be thought of as a point in the plane with coordinates (x, y). X may also be thought of as the vector from the origin to (x, y). When thought of as points in two-dimensional space, the set of 2×1 matrices will be denoted by $\mathbb{R}^2$.

Like with matrices, addition/subtraction of vectors and multiplication of vectors by scalars can be readily defined; in particular, if X and Y are vectors with the same initial point, then $X + Y$ is the diagonal of the parallelogram with sides X and Y beginning at the same initial point. If $c > 0$, then cX is a vector with the same direction as that of X but whose length has been stretched or contracted by a factor of c. If $c < 0$, then the direction of the vector is reversed.

All of this also applies to 3×1 matrices such as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This matrix may be thought to represent either the point (x, y, z) in three-dimensional space or the vector from $(0, 0, 0)$ to (x, y, z). Matrix addition and scalar multiplication can be defined as before for the two-dimensional case. When thought of as points in three-dimensional space, the set of 3×1 matrices will be denoted by $\mathbb{R}^3$.

This can be generalized to n dimensions.

Definition 1.2. $\mathbb{R}^n$ is the set of all $n \times 1$ matrices. △

1.1.2 Linear combinations and linear dependence

Consider the matrix C, defined as the sum of A and B of 1.2.

$$C = A + B = \begin{pmatrix} -3 & 2 & 4 \\ -1 & 6 & 7 \\ 4 & 23 & -13 \end{pmatrix} \tag{1.4}$$

The matrix C depends on A and B; any change in either A or B means C will generally change. This leads to the following concept.

Definition 1.3. Let $S = \{A_1, A_2, \ldots, A_k\}$ be a set of elements of $M(m, n)$. An element C of $M(m, n)$ is **linearly dependent on** S if there are scalars b_i such that

$$C = b_1 A_1 + b_2 A_2 + \cdots + b_k A_k \tag{1.5}$$

We also say that "C is a linear combination of the A_i". △

Chapter 2

Linear independence and dimension

Chapter 3

Linear transformations