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# 'On the permissible curvature of space' by K Schwarzschild

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If I presume to present a few remarks that have neither any real practical applicability nor any pertinent mathematical meaning, my excuse is that the topic we are considering has a particular attraction for many of you because it presents an extension of our view of things way beyond that due to our accessible experience, and opens the most strange prospects for later possible experiences. That it requires a total break with the astronomers' deeply entrenched views cannot but seem a further advantage to anyone convinced that all knowledge is relative.

We are considering the possibility of curvature of space. You already know that besides the usual Euclidean geometry other—non-Euclidean—geometries have been developed during this century, in particular the so-called spherical and psueudospherical spaces, with which we shall be principally concerned. One could present, down to the finest detail, how the world would appear as a spherical or pseudospherical, curved, possibly finite and self-intersecting space. I need only to remind you in this context of Helmholtz' essay 'On the origin and significance of geometrical axioms'. One finds oneself there—if one wants to—in a geometrical fairyland, but the best thing about these fairy stories is that one does not know whether they will indeed turn out to be true. The questions as to how far we have pushed back the boundaries of this fairyland can now be asked: *how small is the curvature of space? and what is a lower bound for its radius of curvature?*‡.

As usual an inadequate answer will be given, at least for the astronomers. Although in Euclidean geometry the sum of the angles in a triangle amounts to $2\pi$§, this is not true in the non-Euclidean case, and the difference becomes more marked as the triangle gets bigger. Now one could say that even for the largest measured triangles, whose apex is a fixed star and whose base is a diameter of the Earth's orbit, the sum of the angles deviates, at most, only slightly from $2\pi$ and so the curvature of space must be extraordinarily small. However this overlooks the fact that the angle at the star is not measured, and the distance of the star and the size of the triangle are calculated using Euclidean geometry whose validity needs to be checked. Also it is insufficient for an astronomer to know that the curvature of space, out to the nearest fixed stars with measurable parallaxes, can be ignored; he needs to take into consideration the distances to the faintest furthest stars if he is going to be able to sketch a picture of the formation of star systems.

I am approaching this problem from a starting point which allows one to assess, to some

† Translated from the original German by John M Stewart (Department of Applied Mathematics & Theoretical Physics, Silver Street, Cambridge CB3 9EW, UK) and Mary E Stewart (Robinson College, Cambridge CB3 9AN, UK).

‡ The traditional manner of emphasizing printed German text, doubling the interletter spacing, has been replaced by italicization.

§ The original has $2R$.

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extent, the theoretical status of non-Euclidean geometry. A triangle between three points will be defined by the paths of light rays from one point to the next, the side lengths $a$, $b$ and $c$ and the times light needs to traverse them; the angles $\alpha$, $\beta$ and $\gamma$ are to be measured with a standard optical instrument. Experience teaches us the validity, within observational accuracy, of plane trigonometry for all triangles for which more than three elements have been measured. We shall assume however that the usual trigonometry is not absolutely exact and that in reality the following equations govern the relations between the sides and angles

$$\sin\alpha : \sin\beta : \sin\gamma = \sin\frac{a}{R} : \sin\frac{b}{R} : \sin\frac{c}{R}, \tag{a}$$

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R} + \sin\frac{a}{R}\sin\frac{b}{R}\cos\gamma. \tag{b}$$

Here $R$ is a certain very large distance that we shall call the 'radius of curvature of space', without looking for too close an analogy to the radius of curvature of a surface. These equations agree with the basic formulae of spherical trigonometry, and as is known, these reduce to the usual trigonometric formulae when the triangle sides are small in comparison with the sphere radius $R$. If one chooses $R$ to be sufficiently large, then the sides of an arbitrary measured triangle will appear to be small compared with $R$; thus by enlarging $R$ one can arrange that, within the accuracy of observations, the formulae (a) and (b) agree with the usual trigonometric formulae, i.e. *the formulae (a) and (b) can never disagree with experience provided one chooses $R$ sufficiently large.*

We do not need to consider here the question, of a purely mathematical nature, whether we can assume that formulae (a) and (b) are valid for any triangle in space without internal contradiction, for the answer is known to be affirmative, and the requirement of the validity of spherical trigonometry for all triangles in space does not fix uniquely the connectivity properties of the space. The simplest and best known of the possible space types which admit spherical trigonometry are the so-called 'spherical space' and 'elliptic space'. The following properties hold both for spherical and elliptical spaces: the space is finite and the formula for the volume depends on the radius of curvature. If one follows a light ray along its path then after a certain distance one returns to the starting point. The conditions in a plane of the space are very similar to those on the surface of a sphere in the usual picture. Thus a plane in curved space is defined pretty naturally, as usual, through all the straight lines or light rays which travel between two intersecting light rays. Each straight line in a plane of a curved space corresponds to a great circle on the sphere. If one constructs two parallel lines, that is lines which intersect a third line with the same angle, e.g. a right angle, this is analogous to two meridians which intersect the Equator orthogonally. And just as meridians meet at a pole, so also parallel lines meet in a curved space after a distance $\frac{1}{2}\pi R$. One might say that two straight lines in a plane in curved space must intersect twice, like great circles on the sphere; this view leads to the concept of spherical space. However it is also possible that two straight lines intersect only once and this condition typifies 'elliptic' space. For one can picture a plane of the curved space on the surface of the sphere in such a way that a single point of the plane corresponds not to a radius but a diameter of the sphere. Great circles through a point on the sphere intersect in the diametrically opposite point and these correspond to a single point in the plane of a curved space, and the corresponding lines intersect only at this point. It follows from this that if one travels along a line a length $\pi R$ (not $2\pi R$) then one returns to the starting point, and that the maximum distance between two points is given by a quadrant, $\frac{1}{2}\pi R$. We shall be concerned in the following with this *elliptic space*, the simplest of the spaces with spherical trigonometry. (In the preliminary discussion above we spoke of spherical rather than elliptic space because the expression is more familiar and has richer associations.)

Next we need to mention another very simple generalization of Euclidean geometry. If we replace $R$ by the pure imaginary quantity i$R$ in (a) and (b) we obtain†

$$\sin\alpha : \sin\beta : \sin\gamma = \sinh\frac{a}{R} : \sinh\frac{b}{R} : \sinh\frac{c}{R}, \tag{a$'$}$$

$$\cosh\frac{c}{R} = \cosh\frac{a}{R}\cosh\frac{b}{R} + \sinh\frac{a}{R}\sinh\frac{b}{R}\cos\gamma. \tag{b$'$}$$

As $R$ increases these formulae also reduce to the formulae of plane trigonometry. Also there exist quite a lot of spaces in which the trigonometry is determined by (a′) and (b′). The simplest and best known of these spaces is the so called *pseudospherical* or *hyperbolic space*. It is infinite, through each point there is a whole bundle of lines which do not intersect a given line; the geometry in one of its planes is analogous to geometry on a so-called pseudosphere, a surface of constant negative curvature.

We now want to treat the problem of parallax measurement for the two cases of elliptic and hyperbolic spaces. Every measurement of parallax boils down to the measurement of the angle between the directions of two stars twice, half a year apart. To simplify the discussion we shall assume that one of the stars $S_1$ lies exactly in the direction of the corresponding diameter of the Earth's orbit, and the other star $S_2$ is approximately at right angles to it. Let $E_1$ and $E_2$ be the position of the Earth at the two times, so that $2r = E_1E_2$‡ denotes the diameter of the Earth's orbit. Observations produce the two angles $S_1E_1S_2 = \alpha$ and $S_1E_2S_2 = \beta$, and the quantity $p = \frac{1}{2}(\alpha - \beta)$ is usually called the parallax of star $S_2$. The problem is then to determine from the three quantities $\alpha$, $\beta$ and $r$, the distances $E_2S_2 = a$ and $E_1S_2 = b$ of the the star $S_2$ from the Earth's two positions, once on the basis of spherical trigonometry and again using pseudospherical trigonometry. Taking account of the fact that the direction of $S_2$ is approximately at right angles to $E_1E_2S_1$, we may set $a = b = d$, where $d$ is the distance of the star, noting further that the parallax $p$ is always a very small angle, and that the radius of curvature of space must be assumed, without doubt, to be large compared to the diameter of the Earth's orbit, one easily obtains the formulae for the distance calculation, in the case of an elliptic space

$$\cot\frac{d}{R} = \frac{R}{r}.p \qquad \text{or} \qquad \sin\frac{d}{R} = \frac{r}{\sqrt{p^2R^2 + r^2}}, \tag{c}$$

and in the case of a hyperbolic space

$$\coth\frac{d}{R} = \frac{R}{r}.p \qquad \text{or} \qquad \sinh\frac{d}{R} = \frac{r}{\sqrt{p^2R^2 - r^2}}. \tag{d}$$

The hyperbolic space can be taken care of very easily using the last formula. Note that for any real distance $d$ the inequality $pR > r$ must hold. Therefore there exists a minimum parallax $p = r/R$ which each star, no matter how distant, can exhibit. However it is known that many stars have a parallax smaller than $0''.05$, so that the value of the minimum parallax must be smaller than $0''.05$, which gives a lower limit for the radius of curvature of a hyperbolic space§:

$$R > \frac{r}{\text{arc } 0''.05} \qquad \text{or} \qquad R > 4 \times 10^6 \text{ au}.$$

† Here and henceforth, modern notation has been used for standard special functions, e.g. Cos → cosh, cotg → cot.

‡ The original has $r = E_1E_2$.

§ In the original, distances are always given in units of the radius of the Earth's orbit. Nowadays the term 'astronomical unit', abbreviated to au, is always used. One au is about $1.5 \times 10^8$ km.

Thus the curvature of a hyperbolic space is so insignificant that it cannot be observed via solar system measurements, and because hyperbolic space is infinite, like Euclidean space, no unusual appearances will be observed on looking at fixed star systems.

Before treating elliptic space I need to interject a more general remark. As has recently been shown by Professor Seeliger, the most rational view of the arrangement of stars that one can build on the basis of current observations, is that all visible stars, whose number can be estimated to be no greater than 40 million, can be considered to lie within a space of a few hundred au, and that outside this is a relative void. Even if this view is reassuring, by offering us a significant step in our understanding of the Universe through a complete investigation of this limited stellar system, this reassurance, so satisfying to reason, would be experienced to an even greater degree if we could conceive of space as being closed and finite and filled, more or less completely by this stellar system. If this were the case, then a time will come when space will have been investigated like the surface of the Earth, where macroscopic investigations are complete and only the microscopic ones need continue. A major part of the interest for me inherent in the hypothesis of an elliptic space derives from this far reaching view.

Let us check the parallax determination in an elliptic space. From the formula obtained above

$$\cot\frac{d}{R} = \frac{R}{r}.p,$$

one obtains for each measured parallax $p$ a real and not necessarily impossible value for the distance of the star *whatever value one chooses for the radius of curvature $R$*. We recognize immediately that it is a mistake to believe that a limit for $R$ can be found simply from measurements of the parallax of fixed stars. On the basis of these measurements space could be so strongly curved that one would come after travelling a distance—let us say—one thousand times the separation between the Earth and Sun (which would take light a few days) back to one's starting point. It is only physical rather than pure metrical arguments which would lead to the conclusion of a larger radius of curvature.

Too small a radius of curvature would of course lead to measurable discrepancies in the solar system. Since we will in any case immediately find that we would need a higher limit, it suffices here to note that for a radius of curvature of say 30, 000 au the effect of spatial curvature even on triangles which extend as far as Neptune's orbit would be unmeasurable. This radius of curvature corresponds to a distance which is less than a tenth of the distance that we usually ascribe to the fixed stars.

Suppose we take $R = 30,000$ au and compute from (c) the distance of stars for a given parallax then we obtain

$$p = 1''.0: \qquad d = 0.908\frac{R\pi}{2} = 42,800 \text{ au}$$

$$p = 0''.1: \qquad d = 0.991\frac{R\pi}{2} = 46,700 \text{ au}$$

$$p = 0''.0: \qquad d = 1.000\frac{R\pi}{2} = 47,100 \text{ au.}$$

It is immediately obvious that this result turns out to be pretty absurd. There could be 100 stars with a parallax $p > 0''.1$. These hundred stars would have to be scattered over distances out to 46, 700 au while for the remaining million stars only the remaining 400 au are available. Thus the Sun would exist in a region of extremely small density of stars, while outside a given distance from it an extraordinarily large amount of stars would have to occur. In order to make clear the amount of compression, I have calculated the volume of space with a radius of 46,700 au as well as that of the remainder and, on the assumption that there are about $10^8$ stars, I have computed the mean distance between two stars. It turns out that for the relatively empty neighbourhood of the Sun, the allowed mean separation is 15,000 au while for the densely filled remainder of space the corresponding figure is 40 au. It is out of the question that the stars could be so closely packed without their mutual interaction being evident, and it therefore follows unconditionally that a radius of curvature of 30,000 au is too small.

It is clear that one can overcome these difficulties by increasing $R$, because if $R = \infty$ these difficulties disappear, according to the standard view, and indeed it suffices to increase $R$ sufficiently so that the assumed hundred million stars with a parallax under $0''.1$ occupy a volume one million times that of the hundred stars with parallax greater than $0''.1$. A simple calculation shows that this is the case for

$$R = 1.6 \times 10^8 \text{ au.}$$

For such a radius, light could cross the 'path round the world $\pi R$' in about 8,000 years. The corresponding elliptic space would still not be as large as the usual estimates for the size of the stellar system; however it would agree with the assumptions that Seeliger has made for the finite fixed star system of his hypothesis. One could also reduce $R$ by a factor 2 or 3 without producing an abnormal void in the neighbourhood of the Sun and an overpopulated region far from it.

We may conclude therefore that there is no contradiction with experience provided one assumes that $R \approx 10^8$ au. For $R$ of this order the finite space populated by the stars visible to us would seem to be more or less homogeneously filled.

However there remains one point in particular to be considered. Each light ray returns, after circumnavigating an elliptic space, to its starting point. Some of the light rays that emerge into space from the opposite side of the Sun to us will arrive at the Earth after crossing the entire space, and will produce an identical image, in the opposite direction of the real Sun, whose luminosity will not be less, because the light rays on returning to their origin will refocus to the same extent as if they had come by the direct route from the source. Because such a second image is not visible we must conclude that the light while travelling around space has suffered sufficient absorption to wipe out the secondary image, and for this an absorption of about 40 magnitudes is needed. Further considerations do not rule out such an absorption which is negligible for terrestrial distances.

In summary we have obtained the result: *one may, without coming into contradiction with experience, assume that the world is contained in a hyperbolic (pseudospherical) space with a radius of curvature greater than $4 \times 10^6$ au, or in a finite elliptic space with a radius of curvature greater than $10^8$ au, where, in the last case, it is assumed that the intensity of light circumnavigating the world has been reduced by 40 magnitudes.*

And for the time being this is where matters stand. I for one do not see how a significant breakthrough can be made with older and current research methods, e.g. showing that the volume of space is large in comparison with that of the star system visible to us, or showing that space really has a positive or negative curvature. However I would like to advance some observations, which although unable to lead to a firm conclusion, may suggest a preferred value for $R$ within the limits obtained above.

As is well known attempts to clarify the spatial ordering of the fixed stars are based on making the simplest and most reasonable assumptions about their mean luminosity and then distributing them at various distances from the Sun in such a way that one obtains the correct, observed proportions for stars of each magnitude. Such a study, whose main results have already been mentioned above, has been undertaken recently by Professor Seeliger, and could be carried out in similar fashion for either a pseudospherical or elliptic space. As a simple example I have computed for myself in these two types of space how the number of stars depends on their magnitude, making the assumptions that the the luminosity of all stars is the same, and that their mean (number) density is the same in all parts of space, and I have found that the number grows with magnitude more slowly in pseudospherical space, and more quickly in elliptic space, than under the same assumptions in Euclidean space. In reality it is known that the number of stars grows more slowly than predicted from these simple hypotheses in Euclidean space, and one might be persuaded to conclude that space is pseudospherical. Of course this exercise has no real significance, because the assumptions of equal luminosity and density of stars are known not to be satisfied. However, as stated, the theory could be developed for curved space on the basis of the same general principles that Professor Seeliger has used for Euclidean space, and comparison with experience could show that the assumption of a space with a particular curvature leads to the simplest model of the distribution of stars. It has to be admitted that there is little prospect that the decision could be made with any great confidence, and so we are led to the unfortunate conclusion that there is hardly any hope that we will soon be in a position of certainty about the finiteness of space.

## Postscript

Of all of the spaces in which 'free motion of rigid bodies' can take place, only the standard types (in the nomenclature of F Klein) have been taken into account in the previous discussion. In order to complete the discussion we should compare the properties of the remaining spaces with astronomical experience. However I would like to exclude 'spherical space' and in particular the so-called 'double space' in which all of the light emitted at a point collects again at a second point, because one would not consider such complicated assumptions unless it was really necessary. All that remains are the so-called *Clifford–Klein spaces.*

A particular peculiarity of the Clifford–Klein spaces is that they demonstrate in the simplest manner that, contrary to what is normally supposed, the validity of Euclidean geometry does not imply that space is infinite. One could imagine that as a result of enormously extended astronomical experience, the entire Universe consists of countless identical copies of our Milky Way, that the infinite space can be partitioned into cubes each containing an exactly identical copy of our Milky Way. Would we really cling on to the assumption of infinitely many identical repetitions of the same world? In order to see how absurd this is consider the implication that we ourselves as observing subjects would have to be present in infinitely many copies. We would be much happier with the view that these repetitions are illusory, that in reality space has peculiar connection properties so that if we leave any one cube through a side, then we immediately reenter it through the opposite side. The space that we have posited here is nothing other than the simplest Clifford–Klein space, a finite space with Euclidean geometry. One recognizes immediately the sole condition that astronomical experience imposes on this Clifford–Klein space: because visible repetitions of the Milky Way have not yet been observed, the volume of the space must be much greater than the volume we ascribe to the Milky Way on the basis of Euclidean geometry.

The other simple Clifford–Klein spaces can be dealt with briefly because their mathematical study is incomplete. They all arise in the same way through apparent identical copies of the same world, be it now in a Euclidean, elliptic or hyperbolic space, and our experience imposes the condition that their volume must be bigger than that of the visible star system.