CMB-slow, or How to Estimate Cosmological Parameters by Hand

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Abstract

I derive analytically the spectrum of the CMB fluctuations. The final result for C_l is presented in terms of elementary functions with an explicit dependence on the basic cosmological parameters. This result is in a rather good agreement with CMBFAST for a wide range of parameters around concordance model. This allows us to understand the physical reasons for dependence of the particular features of the CMB spectrum on the basic cosmological parameters and to estimate the possible accuracy of their determination. I also analyse the degeneracy of the spectrum with respect to certain combinations of the cosmological parameters.

1 Introduction

After recombination the primordial radiation doesn't interact anymore with the matter and most of the photons come to us without further scattering. Since the radiation is extremely isotropic in nearly all angular scales we conclude that at the moment of recombination the universe was extremely homogeneous and its temperature could not vary from place to place more than about few times in thousandth of the percent.

On the other hand, the origin of the large scale structure requires the presence of small inhomogeneities in the distribution of the matter and therefore the temperature of CMB should also vary a little bit. These variations are observed today as the angular fluctuations of the CMB temperature [1]. The expected fluctuations in a given angular scale are basically determined by the inhomogeneities on the spatial scales having today an appropriate angular size if placed at the distance corresponding to the recombination redshift.

The Hubble scale at recombination epoch plays especially important role, distinguishing the inhomogeneities which are still frozen from those ones which already entered the horizon and therefore could be amplified by gravitational instability. At the scales bigger than the Hubble size, the perturbations generated during inflation remain unchanged. Therefore, observing the fluctuations on the angular scales $\theta > 1^\circ$, corresponding to super-Hubble scales at recombination, we directly probe the primordial inflationary spectrum not influenced by the evolution. The perturbations which entered the horizon before recombination evolve in a rather complicated way. The transfer functions relating the initial spectrum to the resulting one strongly depends on the major cosmological parameters, and the shape of the CMB fluctuations spectrum at $\theta < 1^\circ$ is very sensitive to the exact values of these parameters. Therefore by measuring the fluctuations at small angular scales we can determine these parameters.

The recent observations of the CMB fluctuations [1] give us a hope that finally we will be able to determine the cosmological parameters with a very high precision. One of the most important parameters among them is the spectral index n_s characterizing the initial perturbations. According to inflationary paradigm n_s should deviate from $n_s = 1$ and be in the range $0.92 < n_s < 0.97$ depending on the particular scenario of the simple[1] inflation [2], [3]. It is very important to find these deviations to confirm or disprove inflationary paradigm. The accuracy of the current observations is not yet high enough to conclude about the deviations of the spectral index from the flat one [4]. However the future measurements will be able to reach the needed precision.

The CMB spectrum depends on the various cosmological parameters in a rather complicated way. It is very important to clarify this dependence to be sure which features of the spectrum are most sensitive to the particular combinations of cosmological parameters. The usual approach using the computer code CMBFAST [5] is very helpful, but it does not completely solve the problem since the parameter space has too many dimensions. There are various semi-analytical and analytical approaches to this problem [6],[7]. However I was not able to find in the literature elementary analytical expression which would explicitly describe the dependence of the CMB spectrum on the cosmological parameters and would be in a reasonably good agreement with numerics. In this paper I derive such expressions. The main results of this paper are the equations (92)-(100).

I start with a pedagogical introduction reminding the derivation of the Sachs-Wolfe effect in the conformal Newtonian coordinate system [3] and first make the calculations assuming the instantaneous recombination. In this approximation the radiation can be well described in a perfect fluid approximation before recombination and as an ensemble of free photons immediately after that. This is well justified by causality only when we consider the fluctuations corresponding to the superhorizon scales. At small angular scales the delayed recombination is quite important and leads to an extra damping of the fluctuations. Therefore as a next step I show how the formulae obtained in the instantaneous recombination approximation should be modified to account for this effect. Finally the spectrum for the small angular scales is derived and the precision of

[1]Under simple inflation I mean the scenarios with the minimal number of free parameters.

the determination of cosmological parameters and degeneracy of the spectrum with respect to certain combinations of these parameters is discussed.

One important simplification I make is that I nearly always consider the most observationally favored case of a flat universe. The modifications of the most important features of the CMB spectrum due to the spatial curvature are rather obvious.

In Appendix A I derive the analytical formulae describing non-instantaneous recombination which I use in the section on the finite thickness effects. In Appendix B the transfer functions in short and longwave limits are derived in the conformal Newtonian gauge.

2 Sachs-Wolfe effect

Before recombination the radiation is strongly coupled to the matter and it can be well described by a perfect fluid approximation. When the hydrogen becomes neutral, most of the photons do not interact anymore with the matter and therefore to describe them we need the kinetic equation.

The free propagating photons are described by the distribution function f defined via

$$dN = f(x^i, p_j, \eta) d^3x d^3p \tag{1}$$

where dN is the number of particles at time η in the appropriate element of the phase volume $d^3xd^3p \equiv dx^1dx^2dx^3dp_1dp_2dp_3$, so that f is the particle density in the one-particle phase space. I assume that the indices $\alpha, \beta ...$ run always over $0, ..., 3$ while i, k take only spatial values $1, 2, 3$.The phase volume is invariant under coordinate transformations and hence the distribution function f is a space-time scalar. Since the phase volume is conserved along the trajectory, the distribution function *in the absence of the scatterings* should obey the collisionless Boltzmann equation

$$\frac{Df\left(x^i\left(\eta\right), p_i\left(\eta\right), \eta\right)}{D\eta} \equiv \frac{\partial f}{\partial \eta} + \frac{dx^i}{d\eta}\frac{\partial f}{\partial x^i} + \frac{dp_i}{d\eta}\frac{\partial f}{\partial p_i} = 0 \tag{2}$$

where $dx^i/d\eta$ and $dp_i/d\eta$ are the appropriate derivatives calculated on the photon's geodesic.

Temperature and its transformation properties. The energy (frequency) of the photon with the 4-momentum p_α measured by an observer having the 4-velocity u^α is equal to the scalar product of these vectors: $\omega = p_\alpha u^\alpha$. This can be easily understood by going to the local inertial coordinate frame of the observer. If the spectrum of the quanta coming to an observer from a particular direction on the sky, characterized by the vector $n_i = -p_i/p$, where $p = \left(\Sigma p_i^2\right)^{1/2}$, is the Planckian one then the temperature, defined via

$$f = \bar{f}\left(\frac{\omega}{T}\right) \equiv \frac{2}{\exp(\omega/T(x^\alpha, n_i)) - 1} \tag{3}$$

generically depends not only on the direction n_i but also on the moment of time η and the position of the observer x^i. The factor two here accounts for two possible polarizations of the photons. From now on I consider the Universe where the fluctuations of the temperature are very small and therefore one can write

$$T(x^\alpha, l_i) = T_0(\eta) + \delta T(x^\alpha, n_i) \tag{4}$$

where δT is much smaller that homogeneous component T_0. If the observer is at rest with respect to a certain coordinate system then taking into account that $g_{\alpha\beta}u^\alpha u^\beta = g_{00}\left(u^0\right)^2 = 1$ we find that the photon frequency measured by this observer is equal to $\omega = p_0/\sqrt{g_{00}}$. If one goes to the other coordinate system $\widetilde{x}^\alpha = x^\alpha + \xi^\alpha$ infinitesimally different from the old one, then the frequency of the same photon, measured by a different observer, who is at rest with respect to this new coordinate system, changes. From the transformation properties of the metric and the 4-momentum one gets that

$$\omega \Rightarrow \tilde{\omega} = \tilde{p}_0/\sqrt{\tilde{g}_{00}} = \omega(1 + \xi^{i\prime} n_i) \tag{5}$$

where I used the eq. $p_\alpha p^\alpha = 0$ and kept only the first order terms in ξ and metric perturbations; prime denotes the derivative with respect to time η. Taking into account that the distribution function is a scalar, one easily finds that the small temperature fluctuations measured by an observer (at rest) in the new coordinate system are given by

$$\widetilde{\delta T} = \delta T - T_0'\xi^0 + T_0\xi^{i\prime} n_i \tag{6}$$

Hence, we see that only the monopole and dipole components depend on the particular coordinate system. The monopole component can always be removed by a redefinition of the background temperature and can not be measured locally. The dipole component depends on the motion of the observer with respect to the "new ether" defined by the background radiation and measuring it we can find how the Earth moves relative to CMB. Both of these components are not very interesting regarding the spectrum of the initial fluctuations. The higher multipoles depend neither on the particular observer or coordinate system we use to calculate them. Therefore I perform the calculations in conformal Newtonian coordinate system where these calculations look especially simple.

Let us solve the Boltzmann's equation for the free propagating radiation in a flat universe with the metric

$$ds^2 = a^2\left\{(1 + 2\Phi)\, d\eta^2 - (1 - 2\Phi)\, \delta_{ik} dx^i dx^k\right\} \tag{7}$$

where $\Phi \ll 1$ is the gravitational potential. Using the geodesic equations

$$\frac{dx^\alpha}{d\lambda} = p^\alpha, \qquad \frac{dp_\alpha}{d\lambda} = \frac{1}{2}\frac{\partial g_{\gamma\delta}}{\partial x^\alpha} p^\gamma p^\delta, \tag{8}$$

where λ is an affine parameter, the Boltzmann's equation (2) takes the form

$$\frac{\partial f}{\partial \eta} + n^i (1 + 2\Phi)\frac{\partial f}{\partial x^i} + 2p\frac{\partial \Phi}{\partial x^j}\frac{\partial f}{\partial p_j} = 0. \tag{9}$$

Taking into account that

$$\omega = p_0/\sqrt{g_{00}} = (1+\Phi)\,p/a, \tag{10}$$

and using the Planck the ansatz (3), (4) one can easily get that in the lowest order in perturbations the Boltzmann's equation reduces to

$$(T_0 a)' = 0, \tag{11}$$

while the first order terms lead to

$$\left(\frac{\partial}{\partial\eta} + n^i\frac{\partial}{\partial x^i}\right)\left(\frac{\delta T}{T} + \Phi\right) = 2\frac{\partial\Phi}{\partial\eta}. \tag{12}$$

In the most interesting case when the universe after recombination is dominated by dust, a nondecaying mode of the gravitational potential is constant and therefore the right hand side of the equation (12) vanishes. The operator in the left hand side is a total time derivative and therefore

$$\left(\frac{\delta T}{T} + \Phi\right) = const, \tag{13}$$

along a null geodesics. The influence of the gravitational potential on the CMB fluctuations is known as Sachs-Wolfe effect. In the case when the gravitational potential is time dependent the combination $(\delta T/T + \Phi)$ is not constant anymore. As it is clear from (12) its change is given by the integral from the partial time derivative of the potential along geodesics. This effect is usually called the integrated Sachs-Wolfe effect. If at late stages the universe is dominated by quintessence , the integrated SW effect can be essential, changing the resulting amplitudes of the fluctuations by $10 \div 20$ percent in big angular scales $\theta > 1^\circ$. The accounting of this effect is rather obvious and therefore to avoid the overcomplication of the final formulae I consider only the case of the constant potential.

As it follows from the geodesics equations the photons arriving at present time η_0 to observer located at x_0^i from the direction n^i propagate along geodesics

$$x^i(\eta) \simeq x_0^i + n^i(\eta - \eta_0). \tag{14}$$

Therefore, from (13) we get that $\delta T/T$ in the direction n^i on the sky is equal today to

$$\frac{\delta T}{T}(\eta_0, x_0^i, n^i) = \frac{\delta T}{T}(\eta_r, x^i(\eta_r), n^i) + \left[\Phi(\eta_r, x^i(\eta_r)) - \Phi\left(\eta_0, x_0^i\right)\right] \tag{15}$$

where η_r is the recombination moment and $x^i(\eta_r)$ is given by (14). Since we live in a particular place we are only interested in n^i -dependence of the temperature fluctuations. Therefore, the last term in (15), contributing only to the monopole component, which is not measurable locally anyway, can be ignored. As we see the angular dependence of $(\delta T/T)_0$ is determined by two factors: a) by the "initial value" of $(\delta T/T)_r$ in $\mathbf{n}-$direction in a place from where the photons arrive and b) by the value of the gravitational potential Φ in this place. The appropriate temperature fluctuations at the moment of recombination $(\delta T/T)_r$ can be easily expressed in terms of the gravitational potential and the fluctuations of the photon energy density $\delta_\gamma \equiv \delta\varepsilon_\gamma/\varepsilon_\gamma$ at this time. With this purpose let us use the matching conditions for the hydrodynamical energy momentum tensor (EMT), which describes the radiation before decoupling, and the kinetic EMT (see, for instance,[8])

$$T^{\alpha}_{\beta} = \frac{1}{\sqrt{-g}} \int d^3 p f \frac{p^{\alpha} p_{\beta}}{p^0}, \tag{16}$$

characterizing the gas of the free photons after decoupling. Substituting the expression for the metric into (16) and using for the distribution function ansatz (3) we get (up to the linear in perturbations terms):

$$T^0_0 \simeq \frac{(1+2\Phi)}{a^4} \int d^3 p \bar{f}\left(\frac{\omega}{T}\right) p_0 \simeq T_0^4 \int \left(1 + 4\frac{\delta T}{T_0}\right) \bar{f}\left(y\right) y^3 dy d^2 l, \tag{17}$$

where $y \equiv \omega/T$ and we have expressed p_0 and p through ω. The integral over y from the Planckian function $\bar{f}$ can be explicitly calculated and give the numerical factor, which, being combined with $4\pi T_0^4$, is equal to the energy density of the unperturbed radiation. Right before recombination the appropriate component of hydrodynamical EMT for the radiation is equal to $T^0_0 = \varepsilon_\gamma \left(1 + \delta_\gamma\right)$. This component doesn't jump at the moment when the universe becomes transparent and hence

$$\delta_\gamma = 4 \int \frac{\delta T}{T} \frac{d^2 n}{4\pi}. \tag{18}$$

Similar by, one can derive from (16) that for the kinetic EMT

$$T^i_0 \simeq 4\varepsilon_\gamma \int \frac{\delta T}{T} n^i \frac{d^2 n}{4\pi}. \tag{19}$$

On the other hand as it follows from the conservation law for the coupled photon-baryon plasma (132) (see appendix B) the appropriate divergence for the hydrodynamical components of T^i_0 can be expressed in terms of δ_γ and Φ; hence

$$\delta'_\gamma = -4 \int n^i \nabla_i \left(\frac{\delta T}{T}\right) \frac{d^2 n}{4\pi}. \tag{20}$$

where I have assumed that at recombination the cold matter dominates and therefore neglected the time derivative of the potential: $\Phi'\left(\eta_r\right) = 0$. Going to the Fourier space we easily infer that

$$\left(\frac{\delta T}{T}\right)_k \left(\eta_r\right) = \frac{1}{4}\left(\delta_k + \frac{3i}{k^2}\left(k_m n^m\right) \delta'_k\right) \tag{21}$$

satisfies both matching conditions (18) and (20). Here and later on we skip the index γ assuming that δ always denotes the fluctuations of the radiation energy density. Taking into account the initial conditions (21) and skipping the monopole term in (15), we obtain the following expression for the temperature fluctuations in the direction $\mathbf{n} \equiv (n^1, n^2, n^3)$ at location $\mathbf{x}_0 \equiv (x^1, x^2, x^3)$

$$\frac{\delta T}{T}(\eta_0, \mathbf{x}_0, \mathbf{n}) = \int \left(\left(\Phi + \frac{\delta}{4}\right)_{\mathbf{k}} - \frac{3\delta'_{\mathbf{k}}}{4k^2}\frac{\partial}{\partial \eta_0}\right)_{\eta_r} e^{i\mathbf{k}\cdot(\mathbf{x}_0 + \mathbf{n}(\eta_r - \eta_0))} \frac{d^3 k}{(2\pi)^{3/2}} \tag{22}$$

where $k \equiv |\mathbf{k}|$, $\mathbf{k} \cdot \mathbf{n} \equiv k_m n^m$ and $\mathbf{k} \cdot \mathbf{x}_0 \equiv k_n x_0^n$. Since $\eta_r/\eta_0 \lesssim 1/30$ we can neglect here η_r compared to η_0. It is clear that the first term under the integral represents the combined result from the initial inhomogeneities in the radiation energy density and Sachs-Wolfe effect, while the second term is related to the velocities of the baryon-radiation plasma at recombination and therefore, is called Doppler contribution to the fluctuations.

3 Correlation function and multipoles

In the experiments one usually measures the temperature difference of the photons received by two antennae separated by a given angle θ and this squared difference is averaged over the substantial part of the sky. The obtained quantity can be expressed in terms of the correlation function

$$C(\theta) = \left\langle \frac{\delta T}{T_0}(\mathbf{n}_1) \frac{\delta T}{T_0}(\mathbf{n}_2) \right\rangle \tag{23}$$

where the brackets $\langle\rangle$ denote the averaging over all $\mathbf{n}_1$ and $\mathbf{n}_2$, satisfying the condition $\mathbf{n}_1 \bullet \mathbf{n}_2 = \cos(\theta)$. Actually,

$$\left\langle \left(\frac{\delta T}{T_0}(\theta) \right)^2 \right\rangle \equiv \left\langle \left(\frac{T(\mathbf{n}_1) - T(\mathbf{n}_2)}{T_0} \right)^2 \right\rangle = 2\left(C(0) - C(\theta)\right) \tag{24}$$

On the other hand, for a given perturbation spectrum the correlation function $C(\theta)$ can be easily expressed in terms of the expectation values of the Fourier components of the quantities characterizing these perturbations at the moment of recombination.

The Universe is homogeneous and isotropic in big scales and therefore the averaging over the sky for a particular observer and a spatial averaging over the positions $\mathbf{x}_0$ should give for small angles (or big multipoles) nearly the same results. Therefore, the problem of averaging is finally reduced to the averaging of the products of Fourier components for the random Gaussian field. Substituting (22) into (23) and taking into account that, $\langle \Phi_{\mathbf{k}} \Phi_{\mathbf{k}'} \rangle = |\Phi_k|^2 \delta(\mathbf{k} + \mathbf{k}')$, after integrating over the angular part of $\mathbf{k}$ we obtain:

$$C = \int \left(\Phi_k + \frac{\delta_k}{4} - \frac{3\delta_k'}{4k^2} \frac{\partial}{\partial \eta_1} \right) \left(\Phi_k + \frac{\delta_k}{4} - \frac{3\delta_k'}{4k^2} \frac{\partial}{\partial \eta_2} \right)^* \frac{\sin(k|\mathbf{n}_1\eta_1 - \mathbf{n}_2\eta_2|)}{k|\mathbf{n}_1\eta_1 - \mathbf{n}_2\eta_2|} \frac{k^2 dk}{2\pi^2}, \tag{25}$$

where after differentiation with respect to η_1 and η_2 we have to put $\eta_1 = \eta_2 = \eta_0$. Now using the formula (see (10.1.45) in [9]):

$$\frac{\sin(k|\mathbf{n}_1\eta_1 - \mathbf{n}_2\eta_2|)}{k|\mathbf{n}_1\eta_1 - \mathbf{n}_2\eta_2|} = \sum_{l=0}^{\infty} (2l+1) j_l(k\eta_1) j_l(k\eta_2) P_l(\cos\theta) \tag{26}$$

where $P_l(\cos\theta)$ and $j_l(k\eta)$ are, respectively, the Legendre polynomials and spherical Bessel functions of order l, we can rewrite the expression for the correlation function in the following form

$$C(\theta) = \frac{1}{4\pi}\sum_{l=2}^{\infty}(2l+1)\,C_l P_l(\cos\theta) \tag{27}$$

where the monopole and dipole components $(l = 0, 1)$ were excluded and

$$C_l = \frac{2}{\pi}\int\left|\left(\Phi_k(\eta_r) + \frac{\delta_k(\eta_r)}{4}\right)j_l(k\eta_0) - \frac{3\delta'_k(\eta_r)}{4k}\frac{dj_l(k\eta_0)}{d(k\eta_0)}\right|^2 k^2dk. \tag{28}$$

The coefficients C_l are directly related to the coefficients a_{lm} in the expansion of $\delta T/T$ in terms of spherical harmonics, namely $C_l = \left\langle |a_{lm}|^2\right\rangle$, and therefore they characterize the contribution of the multipole component l to the correlation function. If $\theta \ll 1$ the main contribution to $C(\theta)$ give the multipoles with $l \sim 1/\theta$.

The resulting spectrum of CMB-fluctuations depends of the various cosmological parameters. First of all, these are the amplitude and the index of the primordial spectrum of inhomogeneities, generated by inflation. The rather generic prediction of inflation is that in the interesting for us scales[2]: $|\Phi_k^2 k^3| = Bk^{n-1}$, with $1 - n_s \sim 0.03 \div 0.08$ [2], [3]. The amplitude B is not predicted and should be normalized to fit the observations. The other parameters on which the shape of the CMB-spectrum depends are the baryon density, characterized by Ω_b, the contribution of the clustered cold matter to the total energy density Ω_m ($\Omega_m = \Omega_b + \Omega_{cdm}$), the Hubble constant h_{75} (I normalize it on 75 $km/sec \cdot Mpc$) and the cosmological constant (quintessence) characterized by Ω_Λ. The present data are best fitted assuming that the universe is flat with $\Omega_{tot} = \Omega_m + \Omega_\Lambda \simeq 1$ and the total energy density is dominated by the dark cold matter and quintessence with only a few percent of baryons. Below I concentrate mostly on the models, which deviate from the "concordance model" not too much. Our purpose is to clarify how the variation of the parameters influences the observed CMB spectrum and to get an idea up to what extent the CMB determination of the cosmological parameters is robust. I will consider the different angular scales separately.

4 Anisotropies in big angular scales

The formula (28) was derived in the approximation of the instantaneous recombination. Because of causality this approximation is rather good when we consider big angular scales, where the CMB fluctuations are mainly determined by inhomogeneities exceeding the horizon scale at recombination. Moreover, the perturbations spectrum in these scales is not much influenced by

[2]Most of the calculations will be done here for a flat spectrum ($n = 1$). The consideration can be easily generalized for an arbitrary spectral index n and in the case of small deviations from the flat spectrum the modification of the final results is obvious (it will be briefly discussed later).

the evolution. Hence the CMB fluctuations in big angular scales deliver us the undisturbed information about primordial inhomogeneities, which are characterized by the amplitude of the primordial spectrum B and by the spectral index n_s. The horizon at recombination is about the Hubble scale $H_r^{-1} = 1.5t_r$, which in flat universe has the angular size 0.87° on today's sky. Therefore, the fluctuations which we will consider in this section refer to the angles $\theta \gg 1^\circ$ or, to the multipoles $l \ll 1/\theta_H \sim 200$.

For the superhorizon adiabatic perturbations with $k\eta_r \ll 1$ we have (see Appendix B):

$$\delta_k\left(\eta_r\right) \simeq -\frac{8}{3}\Phi_k, \quad \delta_k'\left(\eta_r\right) \simeq 0. \tag{29}$$

As it follows from (22) their contribution to the temperature fluctuations is equal to

$$\frac{\delta T}{T}\left(\eta_0, \mathbf{x}_0, \mathbf{n}\right) \simeq \frac{1}{3}\Phi\left(\eta_r, \mathbf{x}_0 - \mathbf{n}\eta_0\right), \tag{30}$$

that is the observed fluctuations constitute one third of the gravitational potential in a place from where the photons arrived. Taking into account that after equality the potential in supercurvature scales drops compared to its initial value Φ_k^0 by factor 9/10 [3], substituting (29) into (28) and calculating the integral with the help of the standard formula

$$\int_0^\infty s^{m-1} j_l^2\left(s\right) = 2^{m-3}\pi \frac{\Gamma\left(2-m\right)\Gamma\left(l+\frac{m}{2}\right)}{\Gamma^2\left(\frac{3-m}{2}\right)\Gamma\left(l+2-\frac{m}{2}\right)} \tag{31}$$

for the flat initial spectrum ($\left|\left(\Phi_k^0\right)^2 k^3\right| = B$) we obtain well known result:

$$\left(l\left(l+1\right)C_l\right)_{l<30} = \frac{9B}{100\pi} = const. \tag{32}$$

Deriving this formula I used in the integrand the flat spectrum everywhere, assuming that the main contribution for small l comes from the scales exceeding the horizon, where the primordial spectrum is not modified by evolution. This is a rather good approximation for l up to $20 \div 30$. Nonetheless, when we are interested in the precise normalization we need to take into account the corrections coming from the modified spectrum of the perturbations at big k. This can be well traced only in numerical calculations.

Unfortunately, the accuracy of the direct information about the statistical properties of the primordial perturbations spectrum gained from the measurements in big angular scales is restricted by the *cosmic variance*. In fact, within cosmic horizon there are only $2l+1$ samples of the statistical realization for every particular multipole component l. This leads to the minimal inevitable typical "statistical fluctuations" in C_l

$$\frac{\Delta C_l}{C_l} \simeq \left(2l+1\right)^{-1/2}. \tag{33}$$

Hence, the statistical properties of the spectrum in the scales corresponding to the multipole l can be determined in observations only up to an inevitable "error" given by (33). For the quadrupole ($l = 2$) this "typical error" is about 50% and therefore it can not be used for the normalization of the spectrum. For $l \sim 20$ the error constitutes 15%. Therefore, if we want to get a better accuracy in determining the spectrum of primordial inhomogeneities we are forced to go to smaller angular scales, where the spectrum is distorted by evolution. On the one hand it is bad news, since we lose the "pristine information". However, on the other hand, the distortions of the spectrum depend on the other cosmological parameters involving them "directly in the game" and, therefore, allowing us to determine these parameters under condition of having precise enough measurements.

On small angular scales we can not ignore anymore the effect of the delayed recombination and the obtained above formulae should be corrected. Therefore before I proceed with calculations of the fluctuations in small scales I will find how the basic formulae should be modified to account for the effect of delayed recombination.

5 Delayed recombination and finite thickness effect

The delayed (non-instantaneous) recombination is important because of two reasons. First of all, the finite duration of recombination makes the moment when a specific photon decouples to be not very definite. As a result the information about the place from where this photon arrives is "smeared out". This leads to a suppression of the CMB-fluctuations in small angular scales, known as finite thickness effect. The delayed recombination leads also to an extra dissipation of the inhomogeneities increasing the Silk damping scale and hence changing the conditions in the places where the photons decouple. First we consider the finite thickness effect.

Let us consider a particular photon arriving to us from the direction $\mathbf{n}$. With non-negligible probability this photon could decouple at any value of the redshift in the interval: $1200 > z > 900$ and propagate without further scatterings afterwards. If this happens at the moment η_L then the photon arrives to us from the place $\mathbf{x}(\eta_L) = \mathbf{x}_0 + \mathbf{n}(\eta_L - \eta_0)$ without further scatterings and brings the information about conditions in this particular place. Since we do not know exactly when and where the particular photon decouples, a set of the photons arriving from a definite direction brings us only "smeared" information about the conditions within the layer of width $\Delta x \sim \Delta\eta_L$, where $\Delta\eta_L$ is the duration of recombination. It is clear that if the perturbation has a scale smaller than $\Delta\eta_L$ then as a result of this smearing the information about the structure of this perturbation will be lost and we expect that the contribution of these scales to the temperature fluctuations will be strongly suppressed.

Let us calculate the probability that the photon was scattered last time within the time interval Δt_L at the moment of physical time t_L (corresponding to the conformal time η_L) and then avoided further scatterings until present time t_0. With this purpose we divide the time interval $t_0 > t > t_L$ into N small pieces of duration Δt, so that, $t_j = t_L + j\Delta t$ and $N > j > 1$. It is obvious that the required probability is

$$\Delta P = \frac{\Delta t_L}{\tau(t_L)}\left(1-\frac{\Delta t}{\tau(t_1)}\right)...\left(1-\frac{\Delta t}{\tau(t_j)}\right)...\left(1-\frac{\Delta t}{\tau(t_N)}\right), \tag{34}$$

where $\tau(t_j) = (\sigma_T n_t(t_j) X(t_j))^{-1}$ is the mean free time due to the Thompson scattering at t_j and n_t, X are, respectively, the total number density of all (bounded and free) electrons and the degree of ionization. Taking limit $N \to \infty$ ($\Delta t \to 0$) and going back from the physical time t to conformal time η we obtain:

$$dP(\eta_L) = \mu'(\eta_L)\exp(-\mu(\eta_L))\, d\eta_L, \tag{35}$$

where prime, as usually, denotes the derivative with respect to conformal time and the optical depth

$$\mu(\eta_L) \equiv \int_{t_L}^{t_0} \frac{dt}{\tau(t)} = \int_{\eta_L}^{\eta_0} \sigma_T n_t X_e a(\eta)\, d\eta. \tag{36}$$

was introduced. Now, taking into account that in the formula (22) the recombination moment η_r should be replaced by η_L weighted with the probability (35,) we conclude that this formula should be modified as:

$$\frac{\delta T}{T} = \int \left\{\Phi + \frac{\delta}{4} - \frac{3\delta'}{4k^2}\frac{\partial}{\partial \eta_0}\right\}_{\eta_L} e^{i\mathbf{k}\cdot(\mathbf{x}_0+\mathbf{n}(\eta_L-\eta_0))}\mu' \exp(-\mu)\, d\eta_L \frac{d^3k}{(2\pi)^{3/2}} \tag{37}$$

I would like to stress that in distinction from (22) here one can not neglect η_L compared to η_0 anymore since when we integrate over η_L the appropriate argument of the exponent changes very much for $k > \eta_L^{-1}$.

It is easy to see that the visibility function $\mu' \exp(-\mu)$ vanishes at very small η_L (because $\mu \gg 1$) and at big η_L ($\mu' \to 0$) and reaches the maximum at η_r determined by the condition

$$\mu'' = \mu'^2 \tag{38}$$

Since in the case of non-instantaneous recombination the moment when the photons decouple from the matter become smeared over rather substantial time interval we reserve from now on the notation η_r for the time when the visibility function takes its maximum value. This maximum is located within thin layer $1200 > z > 900$. During this short time interval the scale factor and the total number density n_t do not change very substantially and therefore we neglect their time dependence, estimating the appropriate values at $\eta = \eta_r$. On the contrary, the ionization degree X changes by few orders of magnitude. Taking this into account we can rewrite the condition (38) as:

$$X_r' \simeq -(\sigma_T n_t a)_r X_r^2 \tag{39}$$

where index r means that the appropriate quantities are estimated at η_r. At $1200 > z > 900$ the ionization degree X is well described by the formula (115) in Appendix A. The change of X is mainly due to the exponential factor there; hence

$$X' \simeq -\frac{1.44 \times 10^4}{z} \mathcal{H} X \tag{40}$$

where $\mathcal{H} \equiv (a'/a)$. Substituting this relation in (39) we get

$$X_r \simeq \mathcal{H}_r \kappa \left(\sigma_T n_t a\right)_r^{-1} \tag{41}$$

where $\kappa \equiv 14400/z_r$. Together with (115) this equation determines when the visibility function takes its maximum value. It is easy to see that this happens in the "middle" of the recombination layer at $z_r \simeq 1050$ irrespective of the values of the cosmological parameters. At this time the ionization degree X_r is still $\kappa \simeq 13.7$ times bigger than the ionization degree at the moment of decoupling determined by condition $t \sim \tau_\gamma$ (see (119)). Near its maximum the visibility function can be well approximated by the Gaussian one:

$$\mu' \exp\left(-\mu\right) \propto \exp\left(-\frac{1}{2}\left(\mu - \ln \mu'\right)_r' \left(\eta_L - \eta_r\right)^2\right) \tag{42}$$

Calculating the derivatives with the help of (40), (41) we obtain

$$\mu' \exp\left(-\mu\right) \simeq \frac{\left(\kappa \mathcal{H} \eta\right)_r}{\sqrt{2\pi}\eta_r} \exp\left(-\frac{1}{2}\left(\kappa \mathcal{H} \eta\right)_r^2 \left(\frac{\eta_L}{\eta_r} - 1\right)^2\right) \tag{43}$$

where the pre-exponential factor was taken to satisfy the normalization condition $\int \mu' \exp\left(-\mu\right) d\eta_L = 1$.

We can use this formula to perform the explicit integration over η_L in (37). The gravitational potential and the slowly varying contribution to δ_γ practically do not change during recombination. Therefore, they can be approximated by their values at η_r. The only term inside the curly brackets in (37) which could incur a very substantial change is the Silk damping scale. Keeping in mind that the main contribution to the integral comes from the region near η_r we estimate this scale also at η_r. Of course this is a rather rough estimate which nevertheless reproduces the results of the numerics with rather good accuracy . Thus, ignoring η_L-dependence of the expression in curly brackets in (37) and taking its value at η_r, after substitution (43) in (37) and integration over η_L we obtain

$$\frac{\delta T}{T} = \int \left\{\Phi + \frac{\delta}{4} - \frac{3\delta'}{4k^2}\frac{\partial}{\partial \eta_0}\right\}_{\eta_r} \exp\left(-\left(\sigma k \eta_r\right)^2\right) e^{i\mathbf{k}\cdot\left(\mathbf{x}_0 + \mathbf{n}\left(\eta_r - \eta_0\right)\right)} \frac{d^3k}{\left(2\pi\right)^{3/2}} \tag{44}$$

where

$$\sigma \equiv \frac{1}{\sqrt{6}\left(\kappa \mathcal{H} \eta\right)_r} \tag{45}$$

In deriving (44) I replaced $\left(\mathbf{k}\cdot\mathbf{n}\right)^2$ by $k^2/3$, keeping in mind the isotropy of the perturbations. Note that now we can neglect in the exponent η_r compared to η_0 . To find how σ depends on the cosmological parameters we have to calculate $(\mathcal{H}\eta)_r$. At recombination and before the cosmological term can be ignored and the behavior of the scale factor is well described by solution for the matter-radiation universe (see, for instance, [3]),

$$a(\eta) = a_m\left(\left(\frac{\eta}{\eta_*}\right)^2 + 2\left(\frac{\eta}{\eta_*}\right)\right), \tag{46}$$

Note, that η_* is a bit different from the equality time η_{eq} when the energy densities of radiation and matter are exactly equal. The relation $\eta_{eq} = \left(\sqrt{2}-1\right)\eta_*$ follows from the equation $a(\eta_{eq}) = a_m$. Hence

$$(\mathcal{H}\eta)_r = 2\frac{1+(\eta_r/\eta_*)}{2+(\eta_r/\eta_*)} \tag{47}$$

where (η_r/η_*) can be expressed through the ratio of redshifts at equality and recombination if we use obvious relation

$$\left(\frac{\eta_r}{\eta_*}\right)^2 + 2\left(\frac{\eta_r}{\eta_*}\right) \simeq \frac{z_{eq}}{z_r} \tag{48}$$

Substituting (47) in (45) and taking into account that $\kappa \simeq 13.7$ we obtain:

$$\sigma \simeq 1.49\times 10^{-2}\left(1+\left(1+\frac{z_{eq}}{z_r}\right)^{-1/2}\right) \tag{49}$$

The exact value of z_{eq} depends on the cold matter contribution to the total energy density and from the number of the ultrarelativistic species. Assuming three types of neutrino z_r/z_{eq} can be estimated as

$$\frac{z_{eq}}{z_r} \simeq 12.8\left(\Omega_m h_{75}^2\right) \tag{50}$$

The value of σ depends on the amount of the cold matter not very sensitively; if $\Omega_m h_{75}^2 \simeq 0.3$ then $\sigma \simeq 2.2\times 10^{-2}$, while for $\Omega_m h_{75}^2 \simeq 1$, $\sigma \simeq 1.9\times 10^{-2}$.

Now let us find how non-instantaneous recombination influences the Silk dissipation scale. As we mentioned above, at $\eta = \eta_r$ the ionization degree is $\kappa \sim 13.7$ times bigger than at the decoupling and the mean free path of the photon is correspondingly smaller than the horizon scale. Therefore we can try to use the result (141) of Appendix B, obtained in imperfect fluid approximation, to estimate the corrections to the formula (142) due to noninstantaneous recombination. Using the approximate formula (115) which is valid when the ionization drops below unity we obtain

$$(k_D\eta)_r^{-2} \simeq 0.36\left(\Omega_m h_{75}^2\right)^{1/2}\left(\Omega_b h_{75}^2\right)^{-1} z_r^{-3/2} + \frac{12}{5}c_S^2\sigma^2 \tag{51}$$

The first term here is the same as dissipation scale (142) derived for the case of instantaneous recombination. It accounts the dissipation until the moment when recombination begins. The second term is due to an extra dissipation which happens in the process of recombination. Note that the second term in (51) corresponds to the scale which at η_r smaller that the mean free path τ_γ and barely can be trusted literally. However, within the time interval $\Delta\eta \sim \eta_r\sigma$ when the visibility function is different from zero the free propagating photons have enough time only to go at the (comoving) distance $\lambda \sim \eta_r\sigma$ which roughly corresponds to the second term in (51). Hence, although the imperfect fluid interpretation of the second term becomes questionable, it can be nevertheless used to make an estimate of the damping scale. For the realistic values of the dark matter and baryon densities, $\Omega_m h_{75}^2 \simeq 0.3$ and $\Omega_b h_{75}^2 \simeq 0.04$, this term is nearly twice bigger than the first term; hence the extra Silk dissipation due to delayed recombination is rather important. At very low baryon density the first term in (51) dominates and most of the dissipation happens before ionization significantly drops.

Thus, we found that the delayed recombination can be taken into account in a simple way. First, there occurs the extra dissipation of the perturbations and the dissipation scale can increase in few times compared to the case of instantaneous recombination. Second, it leads to an uncertainty when the photons decouple from the matter and as a result to an extra suppression of the CMB-fluctuations in small angular scales. Although both effects are interconnected they have different nature and should not be confused.

The formulae derived in the approximation of instantaneous recombination are modified in an obvious way. Namely, the formula (22) should be replaced by (44). Repeating the steps which lead to the key formula (28) we conclude that the expression under the integral there should be just multiplied by a general factor $\exp\left(-2\left(\sigma k\eta_r\right)^2\right)$.

6 Small angular scales

At big l, corresponding to small angular scales, the main contribution to C_l give the perturbations which being placed at recombination have an angular size $\theta \sim 1/l$ on today's sky . The multipole moment $l \sim 200$ corresponds to the sound horizon scale at recombination. Hence the perturbations responsible for the fluctuations with $l > 100 \div 200$ should have the wavenumbers $k > \eta_r^{-1}$, that is, they enter horizon before recombination. These perturbations evolve in a rather complicated way and the primordial spectrum is strongly modified at $k > \eta_r^{-1}$. In realistic models, the transfer functions relating the initial spectrum of gravitational potential Φ_k^0 with the resulting spectra for Φ and δ_γ at η_r can be analytically derived only in two limiting cases: a) for the perturbations which entered horizon *well* before equality and b) *much later* after equality (when the gravitational field of radiation can be ignored). In the limit of very big k the result is given by the formulae (152), (153), while for very small k by (143) (see Appendix B). Unfortunately, for the realistic values of the cosmological parameters none of these results can be directly used to calculate the CMB fluctuations in the most interesting region of first few acoustic peaks. Actually, the derived shortwave asymptotic is applicable only for those perturbations which have chance for at least one oscillation before equality ($k\eta_{eq} > 2\sqrt{3}\pi \sim 10$). At the same time the longwave asymptotic (143) can be literally applied only to the perturbations which entered the horizon when the radiation was already negligible compared to the matter. If $\Omega_m h_{75}^2 \simeq 0.3$ then as it follows from (50) $z_r/z_{eq} \simeq 4$, and the radiation still constitute about 20% of the energy density at the recombination time. Hence, the formula (143) is not trustable for those perturbations which enter the horizon in between equality and recombination and responsible for the fluctuations in the region of first few acoustic peaks.

6.1 Transfer functions

To describe the perturbations in these intermediate region we have to modify the derived formulae. Taking into account the time behavior of the asymptotic WKB-solutions of Appendix B we conclude that at the moment of recombination:

$$\Phi_k + \frac{\delta_k}{4} \simeq \left[T_p \left(1 - \frac{1}{3c_s^2} \right) + T_o \sqrt{c_s} \cos \left(k \int_0^{\eta_r} c_s d\eta \right) e^{-(k/k_D)^2} \right] \Phi_k^0 \tag{52}$$

and respectively

$$\delta_k' \simeq -4 T_o k c_s^{3/2} \sin \left(k \int_0^{\eta_r} c_s d\eta \right) e^{-(k/k_D)^2} \Phi_k^0 \tag{53}$$

where the transfer functions T_p and T_o should depend on the wavenumber k, equality time η_{eq} and baryon density Ω_b. To simplify the consideration we will restrict ourselves by the case when the baryon density is small compared to the total density of the cold matter, that is, $\Omega_b \ll \Omega_m$. This will allow us to neglect the baryon contribution to the gravitational potential compared to the contribution of the cold *dark* matter, which interacts with the radiation only gravitationally. However even in this case the baryons influence the speed of sound and we have to take this into account. This is the situation for the concordance model and one can use analytical results, which I will derived below, only to study the dependence of the fluctuations on the values of the major cosmological parameters within some "window" around this model. If the contribution of the baryons to the gravitational potential is negligible the transfer functions T_p and T_o depend only on k and η_{eq}, which on dimensional grounds can enter T_p and T_o only in combination $k\eta_{eq}$. Their asymptotic behavior can be easily inferred from (143), (153). For the longwave perturbations with $k\eta_{eq} \ll 1$,

$$T_p \to \frac{9}{10}; \quad T_o \to \frac{9}{10} \cdot 3^{-3/4} \simeq 0.4, \tag{54}$$

while in the shortwave limit for $k\eta_{eq} \gg 1$

$$T_p \to \frac{\ln (0.15 k\eta_{eq})}{(0.27 k\eta_{eq})^2} \to 0; \quad T_o \to \frac{3^{5/4}}{2} \simeq 1.97, \tag{55}$$

where the factor 10/9 accounts for the change of the gravitational potential for superhorizon perturbations after matter-radiation equality. Unfortunately, in the most interesting intermediate range of scales $1 < k\eta_{eq} < 10$ which is responsible for the fluctuations in the region of first few acoustic peaks, the transfer functions can be calculated only numerically. In the interval: $1 < k\eta_{eq} < 10$, one can approximate T_p with good accuracy by[10]

$$T_p \simeq 0.25 \ln \left(\frac{14}{k\eta_{eq}} \right), \tag{56}$$

and, respectively[3],

$$T_o \simeq 0.36 \ln (5.6 k\eta_{eq}) . \tag{57}$$

The transfer functions are monotonic; as $k\eta_{eq}$ increases the function T_p decreases and approaches zero, while T_o increases and reaches its asymptotic value $T_o \simeq 1.97$. For perturbations which enter horizon well before equality, the function T_o is about five times bigger than for the perturbations which cross the horizon late after equality. The physical origin of this difference is rather transparent. Before equality the gravitational field of the radiation can not be neglected. Therefore when perturbation enters horizon the gravity field of the radiation extra boosts the generated sound wave and its amplitude will be five times bigger than the amplitude in the case when the radiation can be neglected.

6.2 Calculating the spectrum

To calculate C_l we should substitute (52), (53) into formula (28), which should be appropriately corrected for the finite thickness effect. However, the obtained integrals are not very transparent and before we proceed with their calculation, it makes sense to simplify these integrals using the advantage of considering $l \gg 1$. With this purpose we first get rid of the derivatives of the spherical Bessel function in (28). Using the Bessel function equation one can easily verify that

$$j_l'^2 (z) = \left[1 - \frac{l(l+1)}{z^2} \right] j_l^2 (z) + \frac{\left(z j_l^2 (z) \right)''}{2z} \tag{58}$$

where prime denotes the derivative with respect to the Bessel function argument. Substituting this into (28) and integrating by parts we get

$$C_l = \frac{2}{\pi} \int \left(\left| \Phi + \frac{\delta}{4} \right|^2 k^2 + \frac{9 \left| \delta' \right|^2}{16} \left(1 - \frac{l(l+1)}{(k\eta_0)^2} \right) \right) (1 + O)\, e^{-2(\sigma k \eta_r)^2} j_l^2 (k\eta_0)\, dk \tag{59}$$

where by O I denoted the corrections of order η_r/η_0 and $(k\eta_0)^{-1}$, which were estimated taking into account the general structure of the expressions in (52), (53). The corrections η_r/η_0 can be neglected compared to unity since $\eta_r/\eta_0 \lesssim z_r^{-1/2} \sim 1/30$. At big l only those k give a substantial

[3]I would like to thank A. Makarov for numerical calculations of T_o function in the limit of vanishing baryon density.

contribution to the integral for which $k\eta_0 \geq l$. Actually, as $l \to \infty$ we can use the following approximation for the Bessel functions

$$j_l(z) \to \begin{cases} 0, & z < \nu, \\ z^{-1/2}\left(z^2-\nu^2\right)^{-1/4}\cos\left(\sqrt{z^2-\nu^2} - \nu\arccos\left(\nu/z\right) - \pi/4\right), & z > \nu, \end{cases} \tag{60}$$

where $\nu/z \neq 1$ is held fixed and $\nu \equiv l + 1/2$; hence the correction $1/k\eta_0 \sim 1/l \ll 1$ can also be skipped.

Now I will use (60) in the integrand of (59). Keeping in mind that the argument of $j_l^2(k\eta_0)$ changes with k much faster that the argument of the oscillating part of the WKB solutions for δ_k let us replace the cosine squared, coming from (60), by its average value 1/2. The result reads

$$C_l \simeq \frac{1}{16\pi}\int_{l\eta_0^{-1}}^{\infty}\left(\frac{\left|4\Phi+\delta\right|^2 k^2}{(k\eta_0)\sqrt{(k\eta_0)^2 - l^2}} + \frac{9\sqrt{(k\eta_0)^2-l^2}}{(k\eta_0)^3}\delta_k'^2\right)e^{-2(\sigma k\eta_r)^2}dk, \tag{61}$$

where using the advantage of considering only big multipoles I replaced $l+1$ with l. This result was first derived in [7].

Let us consider the flat initial spectrum: $\left|\left(\Phi_k^0\right)^2 k^3\right| = B$. Substituting (52), (53) into (61) and changing the integration variable to $x \equiv k\eta_0/l$ after elementary calculations we arrive to the following result:

$$l^2 C_l \simeq \frac{B}{\pi}\left(O+N\right) \tag{62}$$

where keeping in mind $l-$dependence of l^2C_l I have written it as a sum of different terms. Namely,

$$O \equiv O_1 + O_2, \tag{63}$$

is the oscillating contribution to the spectrum given by two terms with twice different periods:

$$O_1 = 2\sqrt{c_s}\left(1 - \frac{1}{3c_s^2}\right)\int_1^{\infty}\frac{T_pT_oe^{\left(-\frac{1}{2}\left(l_f^{-2}+l_S^{-2}\right)^2l^2x^2\right)}\cos\left(l\varrho x\right)}{x^2\sqrt{x^2-1}}dx, \tag{64}$$

and

$$O_2 = \frac{c_s}{2}\int_1^{\infty}T_o^2\frac{\left(1-9c_s^2\right)x^2+9c_s^2}{x^4\sqrt{x^2-1}}e^{-(l/l_S)^2x^2}\cos\left(2l\varrho x\right)dx, \tag{65}$$

These terms modulate the spectrum, leading to the peaks and valleys. I have introduced here the ratio

$$\varrho \equiv \frac{1}{\eta_0}\int_0^{\eta_r} c_s(\eta)\,d\eta, \tag{66}$$

which determines the period of oscillations and location of the peaks. The scales l_f and l_S characterizing the damping of the fluctuations because of the Silk dissipation and finite thickness effect are equal to:

$$l_f^{-2} \equiv 2\sigma^2\left(\frac{\eta_r}{\eta_0}\right)^2; \quad l_S^{-2} \equiv 2\left(\sigma^2 + (k_D\eta_r)^{-2}\right)\left(\frac{\eta_r}{\eta_0}\right)^2, \tag{67}$$

where σ is given in (49). The analytical estimate for the Silk scale $k_D\eta_r$ is not very accurate, however one still can use the estimate (51) for $k_D\eta_r$.

In turn, the nonoscillating contribution I_c can be written as a sum of three integrals

$$N = N_1 + N_2 + N_3, \tag{68}$$

where

$$N_1 = \left(1 - \frac{1}{3c_s^2}\right)^2 \int_1^\infty \frac{T_p^2 e^{-\left(l/l_f\right)^2 x^2}}{x^2\sqrt{x^2-1}} dx \tag{69}$$

is proportional to the baryon density and vanishes in the absence of baryons when $c_s^2 = 1/3$. The other two integrals are:

$$N_2 = \frac{c_s}{2} \int_1^\infty \frac{T_o^2 e^{-(l/l_S)^2 x^2}}{x^2\sqrt{x^2-1}} dx, \tag{70}$$

and

$$N_3 = \frac{9c_s^3}{2} \int_1^\infty T_o^2 \frac{\sqrt{x^2-1}}{x^4} e^{-(l/l_S)^2 x^2} dx. \tag{71}$$

Before we proceed further with the calculation of the integrals let us express the parameters entering (62), namely, c_s, l_f, l_S, ϱ and transfer functions T_o, T_p through the basic cosmological parameters $\Omega_b, \Omega_m, h_{75}$ and $\Omega_\Lambda = 1 - \Omega_m$.

6.3 Parameters

The speed of sound c_s at recombination depends only on the baryon density, which determines how it deviates from the speed of sound in purely ultrarelativistic medium. To characterize these deviations it is convenient instead of the baryon density to introduce the parameter ξ defined as

$$\xi \equiv \frac{1}{3c_s^2} - 1 = \frac{3}{4}\left(\frac{\varepsilon_b}{\varepsilon_\gamma}\right)_r \simeq 17\left(\Omega_b h_{75}^2\right), \tag{72}$$

Then c_s^2 can be expressed through ξ as

$$c_s^2 = \frac{1}{3(1+\xi)}$$

For the realistic value of the baryon density $\Omega_b h_{75}^2 \simeq 0.035$ one gets $\xi \simeq 0.6$.

The damping scales l_f, l_S are given by (67). It is clear that to express them through the cosmological parameters we first have to calculate the ratio η_r/η_0, which also depends on the cosmological term. To calculate this ratio let us consider an auxiliary moment of time $\eta_0 > \eta_x > \eta_r$, when the radiation is already negligible and the cosmological term is still not relevant for dynamics. Then to determine η_x/η_0 we can use the exact solution describing a flat universe filled by the matter and cosmological constant:

$$a(t) = a_0 \left(\sinh \frac{3}{2} H_0 t\right)^{2/3} \tag{73}$$

As a result we obtain:

$$\eta_x/\eta_0 \simeq I_\Lambda z_x^{-1/2} = 3 \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/6} \left(\int_0^y \frac{dx}{(\sinh x)^{2/3}}\right)^{-1} z_x^{-1/2}, \tag{74}$$

with the upper limit of integration $y \equiv \sinh^{-1}(\Omega_\Lambda/\Omega_m)^{1/2}$. Taking into account that $\Omega_\Lambda = 1 - \Omega_m$ one can use the following numerical fit for I_Λ in (74):

$$I_\Lambda \simeq \Omega_m^{-0.09} \tag{75}$$

which approximates the exact result with the accuracy better that 1% everywhere within the interval $0.1 < \Omega_m < 1$.

The ratio η_x/η_r can be calculated with the help of (46) and is equal to

$$\frac{\eta_r}{\eta_x} \simeq \left(\frac{z_x}{z_r}\right)^{1/2} \left(1 + 2\frac{\eta_*}{\eta_r}\right)^{-1/2} = \left(\frac{z_x}{z_{eq}}\right)^{1/2} \left(\left(1 + \frac{z_{eq}}{z_r}\right)^{1/2} - 1\right), \tag{76}$$

where we used the equation (48) to express η_*/η_r in terms of z_{eq}/z_r. Combining this formula with (74) we obtain

$$\frac{\eta_r}{\eta_0} = \frac{1}{\sqrt{z_r}} \left(\left(1 + \frac{z_r}{z_{eq}}\right)^{1/2} - \left(\frac{z_r}{z_{eq}}\right)^{1/2}\right) I_\Lambda \tag{77}$$

Substituting this together with the expression (49) for σ into (67) one gets

$$l_f \simeq 1530 \left(1 + \frac{z_r}{z_{eq}}\right)^{1/2} I_\Lambda^{-1} \tag{78}$$

where the ratio of the redshifts at recombination and equality for three neutrino types (see (50)) is equal to

$$\frac{z_r}{z_{eq}} \simeq 7.8 \times 10^{-2} \left(\Omega_m h_{75}^2\right)^{-1} \tag{79}$$

The scale l_f characterizes the damping of CMB-fluctuations because of finite thickness effect. It depends on both cosmological term and $\Omega_m h_{75}^2$ not very sensitively; for instance, if $\Omega_m h_{75}^2 \simeq 0.3$ and $\Omega_\Lambda h_{75}^2 \simeq 0.7$ we have $l_f \simeq 1580$, while for $\Omega_m h_{75}^2 \simeq 1$ and $\Omega_\Lambda h_{75}^2 \simeq 0$ one gets $l_f \simeq 1600$.

The scale l_S describing the combined effect from the finite thickness and Silk damping can be calculated similar by. Using the estimate (51) for Silk dissipation scale one can easily find that

$$l_S \simeq 0.7 l_f \left(\frac{1+0.56\xi}{1+\xi} + \frac{0.8}{\xi\,(1+\xi)} \frac{\left(\Omega_m h_{75}^2\right)^{1/2}}{\left(1+\left(1+z_{eq}/z_r\right)^{-1/2}\right)^2} \right)^{-1/2} \tag{80}$$

This formula is not as reliable as the estimate for l_f since first we neglected the contribution of the heat conductivity to Silk dissipation scale and second we calculated it using imperfect fluid approximation which surely breaks down when the visibility function reaches its maximum. Nevertheless it is still trustable within 10% accuracy and an exact result is a bit smaller than given by (80). In distinction from l_f the damping scale l_S depends not only on the matter density and cosmological term but also on the baryon density, characterized by ξ. However, this dependence is very strong only for $\xi \ll 1$ when the second term inside the bracket in (80) dominates. For $\xi = 0.6$ we get $l_S \simeq 1100$ if $\Omega_m h_{75}^2 \simeq 0.3$ and $l_S \simeq 980$ for $\Omega_m h_{75}^2 \simeq 1$. The dissipation scale in the universe with more cold matter is bigger (correspondingly l_S is smaller) because in this case the recombination happens at later cosmic time t_r and hence the perturbations get an extra time to be washed out.

The parameter ρ, which determines the location of the peaks, can be easily calculated if one substitutes the speed of sound

$$c_s(\eta) = \frac{1}{\sqrt{3}} \left(1 + \xi \left(\frac{a(\eta)}{a(\eta_r)}\right)\right)^{-1/2} \tag{81}$$

where $a(\eta)$ is given by (46), into (66) and performs an explicit integration there. The result is

$$\varrho \simeq \frac{I_\Lambda}{\sqrt{3 z_r \xi}} \ln \left(\frac{\sqrt{\left(1+z_r/z_{eq}\right)\xi} + \sqrt{(1+\xi)}}{1+\sqrt{\xi\,(z_r/z_{eq})}} \right) \tag{82}$$

It is clear that ϱ depends on both baryon and matter densities. However, it is not very transparent how ϱ behaves when we change these parameters. Therefore it is worthwhile to find a simple numerical fit for (82), which would reproduce the parameter dependence of ϱ within reasonable range of change of ξ and $\Omega_m h_{75}^2$. The fit[4]

$$\varrho \simeq 0.014\,(1+0.13\xi)^{-1} \left(\Omega_m h_{75}^2\right)^{1/4} I_\Lambda \tag{83}$$

reproduces the exact result (82) with the accuracy about $5 \div 7\%$ or better everywhere in the region $0 < \xi < 5$, $0.1 < \Omega_m h_{75} < 1$ where the function ϱ itself changes in about three times. Combining this with the numerical fit for I_Λ in (75) we have

$$\varrho \simeq 0.014\,(1+0.13\xi)^{-1} \left(\Omega_m h_{75}^{3.1}\right)^{0.16} \tag{84}$$

[4] I am very grateful to P. Steinhardt for helping me to check numerically the accuracy of this fit and the fits (97)-(99).

The transfer functions T_p, T_o depend only on $k\eta_{eq}$ and can be expressed as the functions of variable $x = k\eta_0/l$:

$$k\eta_{eq} = \frac{\eta_{eq}}{\eta_0} lx \simeq 0.72 \left(\Omega_m h_{75}^2\right)^{-1/2} I_\Lambda l_{200} x \tag{85}$$

where $l_{200} \equiv l/200$. As we will see the contributions to the integrals defining the fluctuations in the region of the first few acoustic peaks comes from $O(1) > x \geq 1$. Therefore for $200 < l < 1000$ the transfer functions in the relevant range of $k\eta_{eq}$ can be approximated by (56), (57); hence

$$T_p(x) = 0.74 - 0.25 (P + \ln x) \tag{86}$$

where

$$P(l, \Omega_m, h_{75}) \equiv \ln \left(\frac{I_\Lambda l_{200}}{\sqrt{\Omega_m h_{75}^2}} \right), \tag{87}$$

and, respectively,

$$T_o(x) = 0.5 + 0.36 (P + \ln x) \tag{88}$$

6.4 Calculating the spectrum (continuation)

Now I will proceed with the calculations of the fluctuations. The main contribution to the integrals from the oscillating functions (64), (65) gives the vicinity of the singular point $x = 1$. These integrals have the form

$$\int_1^\infty \frac{f(x)\cos(ax)}{\sqrt{x-1}} dx \tag{89}$$

and after making substitution $x = y^2 + 1$ can be calculated using stationary (saddle) point method. The result is

$$\int_1^\infty \frac{f(x)\cos(ax)}{\sqrt{x-1}} dx \approx \frac{f(1)}{(1+B^2)^{1/4}} \sqrt{\frac{\pi}{a}} \cos\left(a + \frac{\pi}{4} + \frac{1}{2}\arcsin\frac{D}{\sqrt{1+D^2}}\right), \tag{90}$$

where $D \equiv (d\ln f/adx)_{x=1}$. For big a we can put $D \approx 0$ and the above formula simplifies to

$$\int_1^\infty \frac{f(x)\cos(ax)}{\sqrt{x-1}} dx \approx f(1) \sqrt{\frac{\pi}{a}} \cos\left(a + \frac{\pi}{4}\right) \tag{91}$$

Using (91) to calculate the integrals in (64), (65) we obtain:

$$O \simeq \sqrt{\frac{\pi}{\varrho l}} \left(\mathcal{A}_1 \cos(l\varrho + \pi/4) + \mathcal{A}_2 \cos(2l\varrho + \pi/4)\right) \tag{92}$$

where the coefficients

$$\mathcal{A}_1 \equiv -\left(\frac{4}{3\left(1+\xi\right)}\right)^{1/4} \xi \left(T_p T_o\right)_{x=1} e^{\left(\frac{1}{2}\left(l_S^{-2}-l_f^{-2}\right)l^2\right)}, \quad \mathcal{A}_2 \equiv \frac{\left(T_o^2\right)_{x=1}}{4\sqrt{3\left(1+\xi\right)}} \tag{93}$$

are slowly varying functions of l. They also depend on the basic cosmological parameters and the spectrum of the fluctuations at $l>200$ is rather sensitive to the variation of these parameters. It is worth to mention that in this approximation the contribution of the Doppler term to the oscillating part of the spectrum drops out. One can check that actually this contribution at $l>200$ do not exceed few percent of the total amplitude. If $\Omega_b \ll \Omega_m$ the transfer functions for the most interesting range $200<l<1000$ can be approximated by (86) and (88). In this case we have

$$\mathcal{A}_1 \simeq 0.1 \frac{\left(\left(P-0.78\right)^2-4.3\right)\xi}{\left(1+\xi\right)^{1/4}} e^{\left(\frac{1}{2}\left(l_S^{-2}-l_f^{-2}\right)l^2\right)}, \quad \mathcal{A}_2 \simeq 0.14 \frac{\left(0.5+0.36P\right)^2}{\left(1+\xi\right)^{1/2}} \tag{94}$$

where P is given by (87)

Substituting (86) in the expression (69) for non-oscillating contribution N_1 we get

$$N_1 \simeq \xi^2 \left[\left(0.74-0.25P\right)^2 I_0 - \left(0.37-0.125P\right) I_1 + \left(0.25\right)^2 I_2\right] \tag{95}$$

where the integrals

$$I_m\left(l/l_f\right) \equiv \int_1^\infty \frac{\left(\ln x\right)^m}{x^2\sqrt{x^2-1}} e^{-\left(l/l_f\right)^2 x^2} dx \tag{96}$$

can be calculated in terms of the hypergeometric functions. However the obtained expressions are not very transparent and therefore it makes sense to find the numerical fits for them. The final result is

$$N_1 \simeq 0.063\xi^2 \frac{\left(P-0.22\left(l/l_f\right)^{0.3}-2.6\right)^2}{1+0.65\left(l/l_f\right)^{1.4}} e^{-\left(l/l_f\right)^2}. \tag{97}$$

Similar by, we obtain

$$N_2 \simeq \frac{0.037}{\left(1+\xi\right)^{1/2}} \frac{\left(P-0.22\left(l/l_S\right)^{0.3}+1.7\right)^2}{1+0.65\left(l/l_S\right)^{1.4}} e^{-\left(l/l_S\right)^2} \tag{98}$$

The Doppler contribution to nonoscillating part of the spectrum is comparable to N_2 and is equal to

$$N_3 \simeq \frac{0.033}{\left(1+\xi\right)^{3/2}} \frac{\left(P-0.5\left(l/l_S\right)^{0.55}+2.2\right)^2}{1+2\left(l/l_S\right)^2} e^{-\left(l/l_S\right)^2} \tag{99}$$

The numerical fits (97)-(99) reproduce the exact result in the most interesting range of multipoles with a few percent accuracy for a wide range of cosmological parameters. The extra dependence on l/l_S and l/l_f is due to the fact that the exponent in the integrals from nonoscillating functions can not be just simply estimated at $x=1$. When the expression under the integral is monotonic function the substantial contribution to it comes not only from the vicinity of $x = 1$ but also from $x \sim O(1)$. The nonoscillating contribution of the Doppler term given by N_3 is rather essential and can not be ignored.

It is convenient to normalize $l(l+1)C_l$ for big l to the amplitude of fluctuations for small l, given by (32), so that finally we obtain

$$\frac{l(l+1)C_l}{(l(l+1)C_l)_{l<30}} = \frac{100}{9}(O + N_1 + N_2 + N_3). \tag{100}$$

where O, N_1, N_2, N_3 are respectively given by (92), (97), (98), (99). In the case of the concordance model ($\Omega_m = 0.3, \Omega_\Lambda = 0.7, \Omega_b = 0.04$ and $H = 70km/sec \cdot Mpc$) the result is presented in Fig.1, where I have separately shown by the dashed and thin solid lines, respectively, the overall nonoscillating and oscillating contributions. The total resulting fluctuations are shown by the thick solid line.

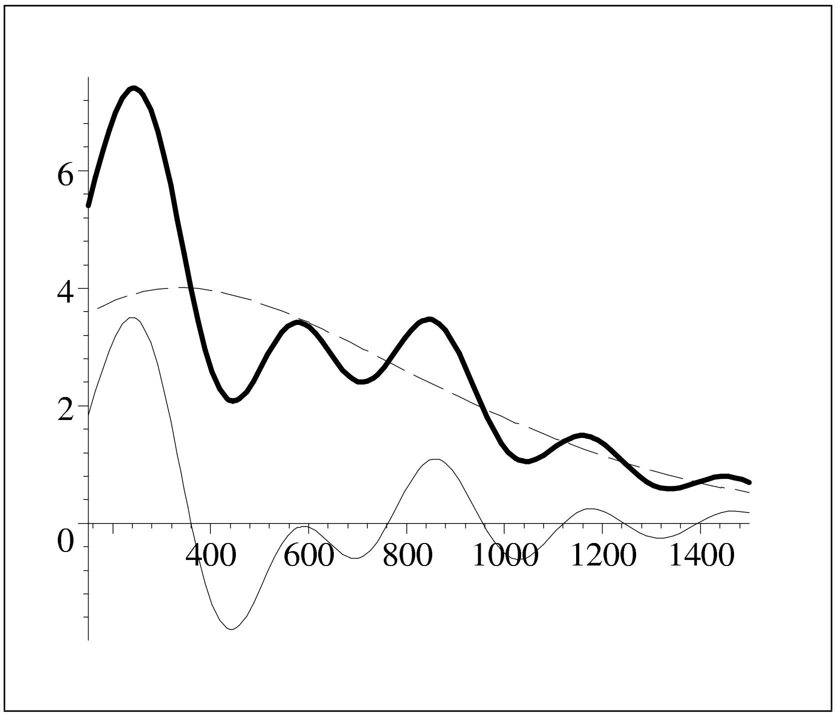

figure 1

About accuracy. Comparing (100) to CMBFAST runs[5] one can easily check that the analytical approximation works rather well reproducing the numerical results with good accuracy in a rather wide range of the cosmological parameters around concordance model. Namely, for

[5] I am very grateful to P. Steinhardt, S. Bashinsky and U.Seljak for performing numerous CMBFAST runs necessary for this work.

$\Omega_m = 0.3$ the agreement is still very good up to $\Omega_b \simeq 0.08$, when the baryons constitute already about 30% percent of the total cold matter density. At higher Ω_b the contribution of the baryons to the gravitational potential which we neglected becomes very essential and one can not use anymore the analytical formula (100). This formula was derived under assumption $\Omega_b \ll \Omega_m$ and is not trustable when the baryons constitute a very substantial fraction of the total amount of cold matter. It is also worth to mention that at high $\Omega_m h_{75}^2$ the expected accuracy in the region of the first peak is not as good as in the region of the second and third peaks . This is because the used approximations for transfer functions become not so accurate on the border of the interval corresponding to $1 < k\eta_{eq} < 10$. In particular, if $\Omega_m h_{75}^2 = 1$ the main contribution to the first acoustic peak located at $l \sim 200$ give the perturbations with the wavenumbers $k\eta_{eq} \sim 0.7$ (see (85)) where the approximations (56) and (57) are not very accurate. Hence, although for the model with $\Omega_m h_{75}^2 = 1$ and $\Omega_b = 0.04$ the analytical result is still in a fair agreement with the numerics, its accuracy in the region of the first peak is not as good as for concordance model. Also note that the peaks given by (100) are shifted by about 10% compared to the numerical results. One of the reasons for that is that with the purpose to simplify the final expression we neglected in (90) an extra phase shift proportional to D. The other reason is that we underestimated the parameter ϱ which was derived in the assumption of instantaneous recombination. In reality the recombination takes place within about quarter of the cosmological time and in the process of recombination the baryons decouple from the radiation. As a result the sound speed increases and ϱ should be a bit bigger compared to (82).

However the main value of the analytical result is not in its competitive accuracy with the numerics, but because it allows us to understand the main features of the CMB spectrum and study explicitly how they depend on the cosmological parameters. In turn it opens a possibility to understand the degeneracy of the spectrum with respect to the certain combinations of the cosmological parameters which could lead to a "cosmic confusion".

7 Determining the cosmological parameters

Let us discuss how the main features of the spectrum change when the cosmological parameters vary. These parameters are: the amplitude B and the slope n_s of the primordial spectrum, the baryon density characterized by Ω_b, the total cold matter density Ω_m, the cosmological constant Ω_Λ and the Hubble constant h_{75}. The amplitude and the slope of the primordial spectrum can already be determined with a reasonably good accuracy when we consider only the measurements in big angular scales. From these observations it follows that the spectrum does not deviate too much from the scale invariant ($n_s = 1$). The cosmic variance, which is important in the big scales which are not much disturbed by the transfer functions does not allow us to conclude anything about small deviations from scale invariant spectrum predicted by inflation on basis of only these observations.

The results for the fluctuations in small angular scales were derived assuming a flat universe, where $\Omega_m + \Omega_\Lambda = 1$, and a scale invariant spectrum with $n_s = 1$. How do they change when the spectrum deviates from the scale invariant will be pointed out below. First, I would like to concentrate on flat models with scale invariant spectrum ($n_s = 1$) and find how the characteristic features of the CMB spectrum depend on the cosmological parameters Ω_b, Ω_m and h_{75} (the cosmological constant is fixed by the flatness condition to be $\Omega_\Lambda = 1 - \Omega_m$).

The location of the peaks and the flatness of the universe. The most interesting feature of the spectrum is the presence of the peaks and valleys, the height and location of which very sensitively depend of the major cosmological parameters. At $l > 1000$ the fluctuations are strongly suppressed and therefore the most interesting part of the spectrum is those one where the first three peaks are located. These peaks arise as a result of superimposing of the oscillating contribution to the fluctuations O, given by (92), on the "hill" $N(l) = N_1 + N_2 + N_3$ representing a nonoscillating part of the spectrum (see Fig.1). It is clear that the locations and the heights of the peaks depend not only on the oscillating part, but also on the shape of the "hill". Let us neglect for a moment the effect of the "hill shape". In this case the location of the peaks would be determined by the superposition of two cosines in (92). If $|\mathcal{A}_1| \ll \mathcal{A}_2$ the peaks should be located at

$$l_n = \pi\varrho^{-1}\left(n - \frac{1}{8}\right) \tag{101}$$

where $n = 1, 2, 3...$ and ϱ is given by (83). The first term in (92) has twice bigger period than the second and its amplitude $\mathcal{A}_1$ is negative. Therefore it participates in the constructive interference for the odd peaks ($n = 1, 3, ...$) and in destructive interference for the even peaks ($n = 2, 4, ...$). Moreover, because of the shift of the arguments of two cosines, the maxima of these cosines do not coincide and, as a result, first and third peaks (for which the interference is constructive) should be located in between the appropriate maxima of these two cosines, that is, at

$$l_1 \simeq \left(\frac{6}{8} \div \frac{7}{8}\right)\pi\varrho^{-1},\ l_3 \simeq \left(2\frac{6}{8} \div 2\frac{7}{8}\right)\pi\varrho^{-1}, \tag{102}$$

where the symbol $\div$ denotes the appropriate interval. If $|\mathcal{A}_1| \gg \mathcal{A}_2$ the peaks move closer to the lower bounds of the intervals in (102). In fact, the situation is more complicated because the nonoscillating contribution N is not constant but is represented by "hill". As it is clear from Fig.1, this leads to the further shift of the peaks to the "top of the hill". For instance, for concordance model first peak moves a bit to the right, while the third peak to the left. Substituting $\xi \simeq 0.6$ and $\Omega_m h_{75}^2 \simeq 0.26$ into (82) we find that for this model the first peak should be located at $l_1 \simeq 225 \div 265$, that is, somewhere in between 225 and 265. For the third peak $l_3 \simeq 825 \div 865$. Because of the reasons I mentioned above, this result should be corrected by about 10% shifting the first peaks to the left.

In the region of the odd peaks one has destructive interference of the oscillating terms. The first term in (92) which takes the minimal (negative) value tries to annihilate these peaks. The second peak (if it exists), should be located at

$$l_2 \simeq \left(2\frac{6}{8} \div 2\frac{7}{8}\right)\pi\varrho^{-1} \tag{103}$$

or in the concordance model at $l_2 \simeq 525 \div 565$.

How sensitive is the peaks location to the variation of the cosmological parameters? According to (82) (see also (84)) ϱ changes when the baryon and cold matter densities vary and therefore (see (102), (103)) the peaks location should also depend on these parameters. The parameter ϱ is not very sensitive function of Ω_m, h_{75} and ξ. Therefore, the location of the first peak in a flat universe is relatively stable when we vary these parameters. In particular, when the baryon density increases in two times ($\xi \simeq 0.6 \rightarrow \xi \simeq 1.2$) the first peak moves to the right by $\Delta l_1 \sim +20$ and the shift of the second and third peaks are, respectively, $\Delta l_2 \sim +40$ and $\Delta l_3 \sim +60$. When determining the location of the peaks, the baryon density always enters in combination $\xi \propto \Omega_b h_{75}^2$ with the Hubble constant. The cold matter density comes together with h_{75} as $\Omega_m h_{75}^{3.1}$. The increase of the cold matter density has an effect opposite to the increase of the baryon density, namely, if for given $\xi \simeq 0.6$ the cold matter density increases twice ($\Omega_m h_{75}^{3.1} \simeq 0.3 \rightarrow \Omega_m h_{75}^{3.1} \simeq 0.6$), the first peak goes to the left by $\Delta l_1 \sim -20$ and respectively $\Delta l_2 \sim -40$ and $\Delta l_3 \sim -60$. Thus we see that even in a flat universe, one can shift the location of the first peak quite substantially ($\Delta l_1 \sim 40$) increasing the baryon density twice and simultaneously decreasing the cold matter density by the same factor.

Why in this case can we be sure that the first peak location is a good indicator of the universe curvature? Fortunately, if we will fix the height of the first peak, then its location becomes "stable" with respect to the admitted variations of the cosmological parameters. The height of the first peak sensitively depends of the cold matter and baryon density. Given the height of this peak we can still vary the baryon and cold matter densities together. However if the cold matter density would increase and we would like still to keep the height of the peak to be the same, we have simultaneously change the baryon density, namely, it also should increase. Since the change of the baryon and cold matter densities have opposite effects on the peak location, it will be shifted not very much if both of them will increase simultaneously. For instance, if both densities increase by a factor two around concordance model, one can expect that $\Delta l_1 \sim 0$. This explains the stability of the location of the first peak for the acceptable range of change of the cosmological parameters in a flat universe. The obtained result on the location of the first Doppler peak and its relative stability to the variation of the unknown cosmological parameters is a fair agreement with numerical calculations. The stability of the first peak location makes it an irreplaceable indicator of the total energy density of the universe. Actually, the peak location is incomparably more sensitive to the total energy density (in the open universe without cosmological constant $l_1 \propto \Omega_{tot}^{-1/2}$). The present observations strongly favor a flat universe ($\Omega_{tot} = 1$), as predicted by inflation.

Height of the peaks and the baryon and cold matter densities. In concordance model the amplitude of the first acoustic peak is in about $7 \div 8$ times bigger than the amplitude of the fluctuations in big angular scales. Substituting l_n, given by (102), (103), into (87) and using the formula (82) for ϱ we see that the factor I_Λ is cancelled in the expression for P and therefore the height of the peaks given by (100) estimated at l_n can depend only on $\Omega_m h_{75}^2$ and $\Omega_b h_{75}^2$ (or ξ). If, for fixed $\Omega_m h_{75}^2$, one increases the baryon density, the height $H_1\left(\Omega_m h_{75}^2, \xi\right)$ also increases. In the concordance model the increase of the baryon density by factor two (from $\xi \simeq 0.6$ to $\xi \simeq 1.2$) leads to the increase of the amplitude H_1 in $1,5$ times. This increase in the amplitude is mostly due to two terms in (100), N_1 (proportional to ξ^2) and O (since $\mathcal{A}_1 \propto \xi$). In turn, the increase of the cold matter density (at fixed ξ) suppresses the height of the first peak H_1. It becomes clear why this happens if we note that for fixed l, the function P entering the formulae for fluctuations decreases when $\Omega_m h_{75}^2$ increases. As a result the overall amplitude of the first peak decreases (mainly because N_2 and N_3 contributions decrease when $\Omega_m h_{75}^2$ increases). Therefore the height of the first peak is degenerate with respect to a certain combination of the baryon and cold matter densities. In a certain range of parameters the increase of the height due to the baryon density can be compensated if we simultaneously increase the cold matter density. However, if the baryon density would be too high, the increase of the height of first peak could not be anymore compensated by increase in $\Omega_m h_{75}^2$ because $\Omega_m h_{75}^2$ can not much exceed unity. (Moreover, for big $\Omega_m h_{75}^2$ the transfer functions responsible for $\Omega_m h_{75}^2-$dependence of H_1 reach their asymptotic values for those values of $k\eta_{eq}$ which mainly contribute to the fluctuations in the region of the first peak). *Hence, just relying on the result about the height of the first peak one can safely conclude that the baryon density can not be more than* $15 \div 20\%$ *of the total critical density.*

The degeneracy in determining $\Omega_m h_{75}^2$ and ξ parameters can be easily resolved if we consider the second peak, which results mostly from the destructive interference of the oscillating terms in (92) superimposed on the "hill" given by N-contribution. In the concordance model this peak is strongly suppressed in $O-$contribution and partially recovered only in the resulting spectrum because of the $N-$contribution (as one can see in Fig.1 the "hill" has a sufficiently steep decline in this region). The presence of the second peak sensitively depends on the ratio of the amplitudes $\mathcal{A}_1$ and $\mathcal{A}_2$. Since the amplitude $\mathcal{A}_1$ of the first term in (92), which tries to "kill" the peak is proportional to the baryon density ξ, while $\mathcal{A}_2$ slightly decreases when ξ grows, one can expect that the presence of large amount of baryons should diminish and may be even completely remove the second peak. Actually the "$O-$contribution" to the peak disappears when the baryon density increases only twice compared to its value in concordance model. However, in the resulting spectrum this peak still survives. This is because the growing amount of baryons simultaneously amplifies N_1-contribution to nonoscillating part of the spectrum and in turn this significantly steepens the "hill" in the region where the second peak is located. The analytical formulae become inapplicable at very high baryon densities. However the numerical calculations show that for $\Omega_m h_{75}^2 \simeq 0.26$ the second peak is still present and has nearly the same amplitude as the third peak even if baryons constitute about 70% of all cold matter. Hence the presence of the second peak can not *alone* be considered as the indication of the low baryon density. Nevertheless, *in combination with the observed height of the first peak the second peak is a very sensitive indicator not only for the baryon density, but also for total cold matter density.* Given the height of the first peak, we can still vary the baryon and cold matter densities increasing or decreasing them simultaneously, since they "act in opposite directions". However they influence differently the second peak. Namely, the simultaneous increase of the baryon and cold matter densities tries to "annihilate this peak". Actually, the amplitude of the second peak depends on the amplitudes $\mathcal{A}_1\left(\xi, \Omega_m h_{75}^2\right)$ and $\mathcal{A}_2\left(\xi, \Omega_m h_{75}^2\right)$ in superposition of two cosines in (92). The increase of the baryon density tends to "kill" this second acoustic peak. The increase of the cold matter density at fixed ξ has a similar effect. This is because $\mathcal{A}_2 \propto (T_o^2)_{x=1}$ decreases faster than $\mathcal{A}_1 \propto (T_oT_p)_{x=1}$ when $\Omega_m h_{75}^2$ increases. At big $\Omega_m h_{75}^2$ the term which "kills the peak" dominates. Hence the height of the second peak depends simultaneously on the baryon and total cold matter densities and is very sensitive to the independent variation of both of them. Fixing the relation between ξ and $\Omega_m h_{75}^2$ from the height of the first peak we can find the particular values of these parameters measuring the height of the second acoustic peak. For instance, if $\Omega_m h_{75}^2 = 1$, then ξ should be about unity if one want to get the height of the first peak to be in agreement with observations. In this case second peak completely disappears. Hence *the experimental detection of the second peak proves that the total density of the cold matter is smaller that the critical one and the baryon density is smaller than* $6 \div 8\%$.This is in an excellent agreement with nucleosynthesis bounds. Shortly both of the results could be formulated as "too much baryons would destroy all deuterium and kill the second acoustic peak". In combination with the location and height of the first peak the presence of the second peak is also a strong independent indicator of the dark energy in the universe. In fact, *from the location of the first peak it follows that the total density in the universe is critical and the presence of the second peak means that the cold matter can constitute only the fraction of it.*

Since the heights and locations of the peaks depend on the different combination of Ω_m and h_{75} this allows us to resolve the degeneracy in determining the Hubble constant. As we have seen for a given $\Omega_b h_{75}^2$ the location of the peaks depends on $\Omega_m h_{75}^{3.1}$, while their heights is determined by $\Omega_m h_{75}^2$. Therefore keeping $\Omega_b h_{75}^2$ and $\Omega_m h_{75}^2$ to be fixed by the heights of the peaks we can still vary the Hubble parameter h_{75} shifting the position of the peaks. As it follows from (102), (84), for the given $\Omega_b h_{75}^2 \simeq 0.04$ and $\Omega_m h_{75}^2 \simeq 0.3$ the increase of the Hubble constant by 20% (say from 70 $km/sec \cdot Mpc$ to 85 $km/sec \cdot Mpc$) moves the peaks to the left by 3%, that is, $\Delta l_1 \simeq 7$ and $\Delta l_2 \simeq 15$. Hence if we want to get an accurate determination of the Hubble constant from CMB spectrum *alone* we have to know the location of the peaks with very high accuracy. If the location of the peaks will be determined with 1% accuracy then the expected accuracy of the Hubble constant will be about 7%.

Up to now we were assuming that the primordial spectrum of the inhomogeneities is scale invariant, that is the spectral index is $n_s = 1$. The inflation predicts that there should be deviations from the scale invariant spectrum and we expect that $n_s \simeq 0.92 \div 0.97$. The above derivation for the CMB fluctuations can be easily modified to account for these deviations.

If $n_s \neq 1$ the obtained amplitudes of the fluctuations at given l should be just multiplied by the factor proportional to l^{1-n}. To resolve the degeneracy in determining the cosmological parameters in this case the heights and location of the first two peaks are not sufficient. Actually for a given n one can always find the combination of the $\Omega_b h_{75}^2$ and $\Omega_m h_{75}^2-$ parameters to fit the heights of the first two peaks. The location of these peaks is also not very sensitive to the deviations of the spectral index from unity. Therefore one needs extra information. With this purpose we can use for instance the height of the third acoustic peak. As one can check the height of this peak is not so sensitive to $\Omega_b h_{75}^2$ and $\Omega_m h_{75}^2$ as for the first two peaks. Fixing these parameters and varying the spectral index n_s for a given unchanged height of the first peak (this can always be done if together with n_s we vary the amplitude of the spectrum B) we find that the relative height of the third peak changes as

$$\frac{\Delta H_3}{H_3} \sim \left(\frac{l_3}{l_1}\right)^{1-n_s} - 1 \tag{104}$$

For instance, if $n_s \simeq 0.95$ the height of the third peak increases by about 5% compared to the case of $n_s = 1$. From this estimate one can get a rough idea about necessary accuracy of the measurements to find the expected deviations from the scale invariant spectrum.

8 Conclusions

In this paper I have shown that if we assume that the main ingredients of the cosmological model are known, then we can completely resolve the degeneracy and determine the main cosmological parameters from the CMB spectrum. For that we just need to know the main features of the spectrum, namely, the heights and location of the peaks. Of course, the accuracy of the determination is different for different parameters and seems to be the worst for the Hubble constant. The information we gain in the observations exceeds the discussed features of the spectrum. Namely, one measures also the entire shape of the spectrum, which, of course, also depends on the cosmological parameters. The necessity to fit this shape restricts the possible values of the parameters even in the case when we have the measurements only in the region of the first peak. This shape (as well as the heights and location of the peaks) also depends on the dissipation scales l_f and l_S, which in turn slightly depends on the cosmological parameters. For the concordance model $l_S \sim 1000$, and it is clear that the dissipation does not influence very much the first peak and becomes very essential in the region of the second peak and at high l. In particular, at $l > 1000$ this effect entirely dominates, leading to the exponential falloff of the spectrum at very high multipoles. This falloff is very sensitive to the parameters and, being measured, can give us extra information about them. The measurements of the polarization provides additional valuable information about the cosmological parameters. When we vary the parameters the detailed behavior of the spectrum is, of course, more complicated than I described above (I also neglected here the primordial gravity waves which can give rather substantial contribution at $l < 30$). However, the above consideration correctly reflects the main features of this behavior and gives the physical understanding why the CMB spectrum so sensitively depends on the cosmological parameters.

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A Hydrogen recombination

The equilibrium description of recombination by Saha's formula fails nearly immediately after the beginning of recombination when only a few percent of hydrogen becomes neutral. Therefore one has to use the kinetic approach to describe the noninstantaneous (delayed) recombination[11].

The direct recombination to the ground state with the emission of one energetic photon is not very efficient. The emitted photon has enough energy to immediately ionize the first neutral hydrogen atom it meets. One can easily check that the two competing processes, direct recombination to the ground state and ionization, occur with a very high rate leaving no net contribution. More efficient is the cascade recombination when the neutral hydrogen is first formed in the excited state and then goes to the ground state. However, even in the cascade recombination at least one very energetic photon is emitted. Its energy corresponds to the energy difference between $2p-$ and $1s-$states. This Lyman-alpha photon (L_α) has the energy $3B_H/4 \Rightarrow 117000^\circ K$ and a rather big resonance absorption cross-section which at the recombination temperature is about $\sigma_\alpha \simeq 10^{-17} \div 10^{-16}\ cm^2$. The $L_\alpha-$photons are reabsorbed in $\tau_\alpha \simeq (\sigma_\alpha n_H)^{-1} \sim 10^3 \div 10^4$ sec after emission. This time has to be compared to the cosmological time. During the matter dominated epoch the cosmological time can be easily expressed through the temperature if we just equate the energy density of the cold particles to the critical energy density $\varepsilon^{cr} = 1/(6\pi t^2)$ and note that $T = T_{\gamma 0}(1+z)$; hence

$$t_{sec} \simeq 2.75 \times 10^{17} \left(\Omega_m h_{75}^2\right)^{-1/2} \left(\frac{T_{\gamma 0}}{T}\right)^{3/2}, \tag{105}$$

where h_{75} is the Hubble constant normalized on 75 $km/sec \cdot Mpc$, $T_{\gamma 0} \simeq 2.72\ K$ and Ω_m is the contribution of cold matter to the critical density. At the moment of recombination $\tau_\alpha \ll t_c \sim 10^{13}$ sec and the $L_\alpha-$quanta are not significantly redshifted before they are reabsorbed. Therefore below I will neglect the redshifts of these quanta, which could in principle take them outside of the resonance line. The presence of the big number of L_α photons leads to an overabundance of the electrons (e), protons (p) and $2s, p-$ states of the neutral hydrogen, compared to what is predicted by the equilibrium Saha's formula. In turn, this delays the recombination and for a given temperature the actual degree of ionization exceeds its equilibrium value given by the equilibrium Saha's formula. The full system of kinetic equations describing the recombination is rather complicated and can be solved only numerically. Here I will present the useful approximate treatment, which is in a very good agreement with the results of the numerical calculations.

I neglect all highly excited hydrogen states so that the main "players left in the game" will be $1s$, $2s$, $2p-$states together with the electrons, protons, thermal photons, L_α- and other energetic quanta emitted during recombination. The main processes, involving these components, are symbolically shown in Fig.2. They are responsible for the "converting elements" and bring them from one "reservoir" to the other changing the appropriate concentrations. First note that the recombination acts directly to the ground state do not lead to any net change in the system and leave the concentrations in the appropriate "reservoirs" in Fig.2 without change.

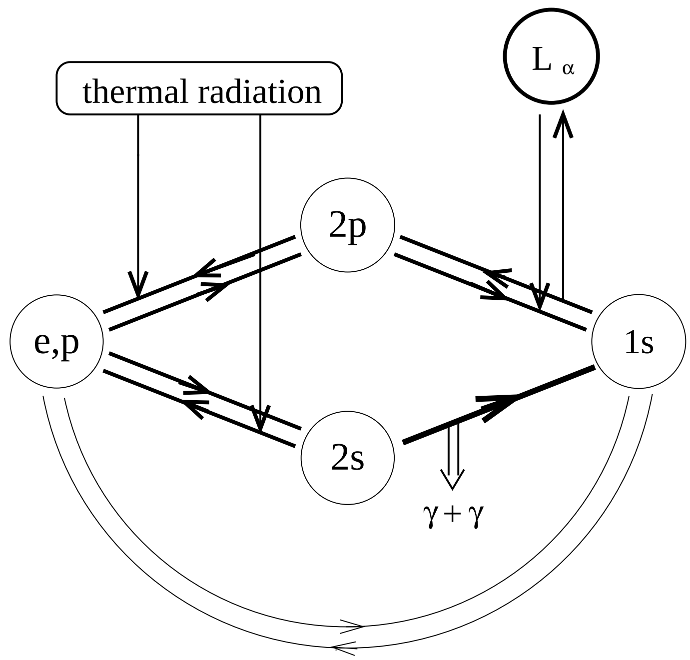

figure 2

Second, the thermal radiation is still very efficient and plays the dominant role in the ionization of the excited hydrogen atom $2s, p-$states (at least, at the beginning of recombination). Actually to ionize the excited hydrogen atom the energy of the photon should be only one quarter of the binding energy B_H. The number of such photons is still bigger that the number of highly energetic photons emitted in recombination acts and therefore considering the ionization of the excited atoms one can safely ignore the distortions of the thermal radiation spectrum. On the contrary, these thermal quanta do not play any essential role in the transitions between $1s$ and $2p, s-$states. The transitions $1s \to 2p$ are mostly due to $L_\alpha-$quanta which at the beginning are present in the same number as the neutral atoms in the ground state. After the degree of ionization significantly drops the free electrons and the excited states are overabundant compared to what one would expect according to the equilibrium Saha's formula. This is why we can neglect $1s+\gamma+\gamma\rightarrow 2s$ transitions compared to the two-photon decay of $2s-$state: $2s\rightarrow 1s+\gamma+\gamma$. The probability of this process ($W_{2s\rightarrow 1s}\simeq 8.23\ sec^{-1}$) is much smaller than the probability of $2p\rightarrow 1s+L_\alpha-$decay ($W_{2s\rightarrow 1s}\simeq 4\times 10^8\ sec^{-1}$). Nevertheless, it plays the main role in the nonequilibrium recombination being, in fact, responsible for the net change of the concentrations of all "elements".

The L_α quanta emitted in $2p\rightarrow 1s$ transitions are fast reabsorbed by the hydrogen atoms in the ground state, and these atoms go back to the "$2p-$reservoir". Therefore, the main source of the irreversible "leakage" from "$e,p-$ to $1s-$reservoir" is the two quanta decay via $2s-$levels and the net change of the electron concentration is mainly due to this process. All other processes return the "escaped" electrons very fast back to "$e,p-$reservoir". Hence, the rate of the overall decrease of the electron concentration (which is equal to the increase of the neutral atoms in the ground state) due to the two-photon decay of $2s-$states is:

$$\frac{dX_e}{dt}=-\frac{dX_{1s}}{dt}=-W_{2s}X_{2s}, \tag{106}$$

where the relative concentrations $X_e\equiv n_e/n_t$, $X_{2s}\equiv n_{2s}/n_t$ have been introduced; here n_t is the total number density of all neutral atoms plus electrons. I would like to stress once more that the equation (106) ignores all other irreversable processes, besides of $2s\rightarrow 1s+\gamma+\gamma$ decay, which could lead as a final outcome to the neutral hydrogen atoms in the ground state . As we will see later this assumption is valid until the degree of the ionization drops to rather small values. After that, at the end of recombination, when some other irreversable processes (in addition to two quanta decay) become important, I will correct the main equations to account for them.

To express X_{2s} through X_e let us use the quasi-equilibrium condition for "$2s-$reservoir". The rates of the reactions depicted in the Fig. 2 are very high compared to the rate of the expansion. Therefore the concentrations of the elements in the "intermediate reservoirs" quickly adjust their quasi-equilibrium values which are determined by condition that the "net flux" for an appropriate "reservoir" should be equal to zero. For "$2s-$reservoir" this condition takes the following form:

$$\langle\sigma v\rangle_{ep\rightarrow\gamma 2s}n_en_p-\langle\sigma\rangle_{\gamma 2s\rightarrow ep}n_\gamma^{eq}n_{2s}-W_{2s\rightarrow 1s}n_{2s}=0, \tag{107}$$

where $\langle\sigma v\rangle$ are the effective rates of the appropriate reactions and n_γ^{eq} is the number density of the thermal photons. The relation between the cross-sections of the direct and inverse reactions can be easily found if one notes that in the state of equilibrium these reactions should compensate each other; hence

$$\frac{\langle\sigma\rangle_{\gamma 2s\rightarrow ep}n_\gamma^{eq}}{\langle\sigma v\rangle_{ep\rightarrow\gamma 2s}}=\frac{n_e^{eq}n_p^{eq}}{n_{2s}^{eq}}=\left(\frac{Tm_e}{2\pi}\right)^{3/2}\exp\left(-\frac{B_H}{4T}\right) \tag{108}$$

where in the second equality I used the Saha's formula and took into account that the binding energy of $2s-$state is $B_H/4$. Using this relation we can express X_{2s} from (107) as

$$X_{2s}=\left(\frac{W_{2s}}{\langle\sigma v\rangle_{ep\rightarrow 2s}}+\left(\frac{Tm_e}{2\pi}\right)^{3/2}\exp\left(-\frac{B_H}{4T}\right)\right)^{-1}n_tX_e^2 \tag{109}$$

Substituting this expression into (106) we obtain

$$\frac{dX_e}{dt} = -W_{2s}\left(\frac{W_{2s}}{\langle\sigma v\rangle_{ep\to 2s}} + \left(\frac{Tm_e}{2\pi}\right)^{3/2}\exp\left(-\frac{B_H}{4T}\right)\right)^{-1} n_t X_e^2 \tag{110}$$

When the first term inside the bracket is small compared to the second one the electron and *excited states* of hydrogen atoms are in equilibrium with each other and with thermal radiation. In this case the last term in the equation (107) is small compared to the other terms and the relative concentrations of e, p and $2s-$states still satisfy the appropriate Saha's relation (r.h.s. equality in (108)). Of course, it does not mean that the ionization degree in this case is given by the equilibrium Saha's formula, which is derived under assumption that $1s-$state is also in thermal equilibrium with the other states. As I mentioned, the ground state drops out of equilibrium with the other levels soon after recombination begins and there is an overabundance of the atoms in the excited states compared to what one would expect according to the equilibrium Saha's formula[6].

The rate of the recombination to $2s-$level is well approximated by the formula (see, for instance, [12]):

$$\langle\sigma v\rangle_{ep\to\gamma 2s} \simeq 6.3\times 10^{-14}\left(\frac{B_H}{4T}\right)^{1/2}\ \frac{cm^3}{sec} \tag{111}$$

and one can easily verify that two terms inside the brackets in (109) becomes comparable at the temperature $\simeq 2450°K$. Hence only at the temperatures higher than $2450°K$ the $e-p$ recombination processes are faster than the two photon decay and thermal radiation is efficient in keeping the chemical equilibrium between e, p and $2s-$states.

As long as the temperature drops below this value the photoionization of $2s-$states becomes less efficient than their two quantum decay. The thermal radiation does not play essential role after that and the quasi-equilibrium concentration of $2s-$ states is regulated by the balance of the recombination rate to $2s-$levels and their two quanta decay rate (the second term in the equation (107) can be neglected compared to the third one). In this case the second term inside the brackets in (110) is small compared to the first one and the rate of recombination due to the leakage of the electron from "$e, p-$reservoir" through $2s$ reservoir is proportional to $\langle\sigma v\rangle_{ep\to\gamma 2s} n_t X_e^2$ and does not depend on W_{2s} . It is entirely determined by the rate of the recombination to $2s$ level. At the same time $2p-$ states also drop out equilibrium with electrons, protons and thermal radiation and most of L_α are destroyed in two quanta decays. As a result the "$e, p \to 2p \to 1s-$channel" becomes also efficient in converting the free electrons and protons into the neutral hydrogen and increases the "leakage of the electrons from $e, p-$reservoir". Moreover, nearly every recombination act in one of the excited states lead to the formation of the neutral

[6] At the beginning of recombination the rate of change of the hydrogen atoms in $1s-$state is proportional to $W_{2s\to 1s} n_{2s}$.This rate is much smaller than the rate of the reactions $ep \leftrightarrows \gamma 2s$, determining the concentration of $2s-$states.

hydrogen atom. This effect is relevant only at the late stages of recombination and can be easily taken into account if we substitute in (110) instead of $\langle\sigma v\rangle_{ep\to\gamma 2s}$ the rate for recombination to all *excited* states, which is well approximated by the fitting formula (see, for instance, [12])

$$\langle\sigma v\rangle_{rec} \simeq 8.7\times 10^{-14}\left(\frac{B_H}{4T}\right)^{0.8}\ \frac{cm^3}{sec} \tag{112}$$

It is convenient to rewrite the equation (110) using instead of temperature and cosmological time, related to the temperature via (105), the redshift parameter $z+1 = T/T_{\gamma 0}$. Substituting the numerical values for the reaction rates and the number density n_t in the obtained equation after elementary calculations we get:

$$\frac{dX_e}{dz} = 15.\,3\frac{\Omega_b h_{75}}{\sqrt{\Omega_m}}\left(0.72\left(\frac{z}{14400}\right)^{0.3} + 10^4 z\exp\left(-\frac{14400}{z}\right)\right)^{-1} X_e^2, \tag{113}$$

when we neglect unity compared to z. This equation can be easily integrated:

$$X_e(z) = 6.53\times 10^{-2}\frac{\sqrt{\Omega_m}}{\Omega_b h_{75}}\left(\int_z \frac{dz}{\left(0.72\left(z/\left(1.44\times 10^4\right)\right)^{0.3} + 10^4 z\exp\left(-1.44\times 10^4/z\right)\right)}\right)^{-1} \tag{114}$$

One can verify that the solution $X_e(z)$ is not very sensitive to the "initial conditions" which could be taken at $z_{in} > z$ (for instance, at $T \simeq 3500^\circ K$), when $X_e(z_{in}) \gg X_e(z)$. The main contribution to the integral in this case give $z < z_{in}$. At $z > 900$ (appropriately at the temperature $T > 2450^\circ K$) the first term inside the bracket in the integrand can be neglected. In this case, the expression (114) is well approximated by the formula[13]:

$$X_e(z) \simeq 9.\,1\times 10^6\frac{\sqrt{\Omega_m}}{\Omega_b h_{75}} z^{-1}\exp\left(-\frac{14400}{z}\right) \tag{115}$$

and the overall rate of the recombination is completely determined by the rate of two quanta decay. It is clear from the derivation of (114) that this formula and, correspondingly (115), are applicable only when the degree of ionization drops significantly below unity and the deviations from the full equilibrium become quite essential. As a rough criteria for the applicability of these formulae let us take the moment when the concentration of the neutral hydrogen reaches about ten percent. According to (115) for the realistic values of the cosmological parameters: $\Omega_m h_{75}^2 \simeq 0.3$ and $\Omega_b h_{75}^2 \simeq 0.03$ this happens at $z \sim 1220$ (appropriately $T \sim 3300 \div 3400^\circ K$). Therefore in this case the range of applicability of (115) is not very big, namely, $1200 > z > 900$. However during this time when the temperature drops only from $3400^\circ K$ to $2450^\circ K$ the degree of ionization decrease very substantially; at $T \simeq 2450^\circ K$ it constitutes $X_e(900) \simeq 2\times 10^{-2}$. It is interesting to compare this result to the prediction of the equilibrium Saha's formula. According to the equilibrium Saha's formula $X_e(3400^\circ K) \sim 10^{-1}$ and $X_e(2450^\circ K) \sim 10^{-5}$, that is, at $z \simeq 900$ the ionization degree exceed the equilibrium one more than in thousand times. Hence, the deviation from the equilibrium very essentially delay the recombination process. The other interesting thing is that the equilibrium ionization degree depends only on the baryon number density, while in (114) enters also the density of the cold matter. It is not surprising, since the cold matter determines the rate of the cosmological expansion which is very important for kinetics when the deviations from equilibrium become essential.

When the temperature drops below $2450°K$ at $z < 900$ the approximate formula (115) is not valid anymore and we have to use (114). The degree of ionization first continues to drop and finally freezes-out; for instance, for $\Omega_m h_{75}^2 \simeq 0.3$ and $\Omega_b h_{75}^2 \simeq 0.03$ the formula (114) gives: $X_e\,(z=800) \simeq 5\times10^{-3}$, $X_e\,(400) \simeq 7\times10^{-4}$ and $X_e\,(100) \simeq 4\times10^{-4}$. To calculate the freeze-out concentration we note that the integral in (114) converges for $z=0$ and is about 4×10^3; hence

$$X_e^f \simeq 1.6\times10^{-5}\frac{\sqrt{\Omega_m}}{\Omega_b h_{75}} \tag{116}$$

After ionization degree drops below unity the approximate results given (114) and (115) are in very good agreement with the numerical solutions of the kinetic equations, while the Saha's approximation do not reproduce the ionization behavior even roughly.

At the beginning of recombination most of the neutral hydrogen atoms were formed as a result of the cascade transitions and the number of $L_\alpha-$photons was about the same as a number of hydrogen atoms. What happens with all these $L_\alpha-$photons afterwards? Will they survive and, if so, could we observe them today as an appropriately redshifted narrow line in the spectrum of CMB? During the recombination the number density of the $L_\alpha-$quanta n_α is determined by the quasi-equilibrium condition for "$L_\alpha-$reservoir"

$$W_{2p\to1s}n_{2p} = \langle\sigma_\alpha\rangle\, n_\alpha n_{1s}. \tag{117}$$

Since $n_{1s} \approx n_t$ and $n_{2p} \propto X_e^2$ we see that the number of these quanta drops proportionally to the ionization degree squared. Thus, nearly all $L_\alpha-$photons which emerged at the beginning disappear because they are "de facto" destroyed due to the two-photons decay of $2s-$states. Therefore there will be no sharp line in the primordial radiation spectrum. Nevertheless as a result of recombination this spectrum will be significantly warped in the Wien region. Unfortunately, the spectrum distortions lie in those part of the spectrum, where they are strongly saturated by the radiation from the other astrophysical sources and one can not observationally verify this important consequence of the hydrogen recombination.

Finally let us find when the universe becomes transparent for the radiation. It happens when the typical time between the photon scattering begins to exceed the cosmological time. The Raleigh's cross-section for the scattering on the neutral hydrogen is negligibly small and in spite of their low concentration, the main role in opaqueness play the free electrons . The cross-section of the scattering on free electron is equal to $\sigma_T \simeq 6.65\times10^{-25}\ cm^2$ and the equation defining the moment when the radiation completely decouples form the matter takes the form:

$$\frac{1}{\sigma_T n_t X_e} \sim t_{cosm}, \tag{118}$$

This equation can be rewritten as

$$X_e^{dec} \sim 40 \frac{\sqrt{\Omega_m}}{\Omega_b h_{75}} \left(\frac{T_{\gamma 0}}{T_{dec}} \right)^{3/2} \tag{119}$$

By "try-out" one can easily check that the decoupling happens at $T_{dec} \sim 2500^{\circ} K$ (the corresponding redshift $z_{dec} \sim 900$) irrespective how big are the values of the cosmological parameters. If $\Omega_m h_{75}^2 \simeq 0.3$ and $\Omega_b h_{75}^2 \simeq 0.03$ the ionization degree at this moment is about[7] 2×10^{-2}.

B Asymptotic behavior of the transfer functions

The resulting fluctuations of the background radiation depend on the gravitational potential Φ and the radiation energy density fluctuations $\delta_\gamma \equiv \delta\varepsilon_\gamma / \varepsilon_\gamma$ at the moment of recombination. To determine these quantities we have to study the gravitational instability in two component medium consisting of the coupled baryon-radiation plasma and the cold dark matter. Because these components interact only gravitationally their energy-momentum tensors conserve separately. In the cosmological conditions the shear viscosity can not be neglected for the baryon-radiation plasma and leads to the dissipation of perturbations in small scales (Silk damping). For imperfect fluid with the energy density ε and the pressure p one can use the energy-momentum tensor given in[8] [7]. Then one can easily find that in a homogeneous universe with small perturbations described by the metric (7) the conservation laws $T^{\alpha}_{\beta;\alpha} = 0$ in the first order in perturbations reduce to

$$\delta\varepsilon' + 3\mathcal{H} \left(\delta\varepsilon + \delta p \right) - 3 \left(\varepsilon + p \right) \Phi' + a \left(\varepsilon + p \right) u^i{}_{,i} = 0. \tag{120}$$

$$\frac{1}{a^4} \left(a^5 \left(\varepsilon + p \right) u^i{}_{,i} \right)' - \frac{4}{3} \boldsymbol{\eta} \Delta u^i{}_{,i} + \Delta\delta p + \left(\varepsilon + p \right) \Delta\Phi = 0. \tag{121}$$

where $\delta\varepsilon, \delta p$ are, respectively, the perturbations of the energy density and pressure ; u^i is the peculiar 3-velocity and $\boldsymbol{\eta}$ is the shear viscosity coefficient. Note that the first equations which follows from $T^{\alpha}_{0;\alpha} = 0$ does not contain the shear viscosity. The second equation was obtained by taking the divergence of the equations $T^{\alpha}_{i;\alpha} = 0$. As it was already noted, these two equations are *separately* valid for the dark matter and for the baryon-radiation plasma.

Dark matter. For dark matter, the pressure p and the shear viscosity $\boldsymbol{\eta}$ are both equal to zero. Taking into account that $\varepsilon_d a^3 = const$ we obtain from (120) that the fractional perturbations in the energy of dark matter component $\delta_d \equiv \delta\varepsilon_d / \varepsilon_d$ satisfy the equation

$$\left(\delta_d - 3\Phi \right)' + a u^i{}_{,i} = 0. \tag{122}$$

[7] It is rather interesting to note that this time coincides with the moment when e, p and $2s-$levels come out of equilibrium and the approximate formula (115) becomes inapplicable.

[8] I will neglect the heat conduction since it does not change substantially the Silk damping scale.

If we express $u^i_{,i}$ in terms of δ_d and Φ and substitute it into (121) then the resulting equation takes the following form:

$$\left(a\left(\delta_d - 3\Phi\right)'\right)' - a\Delta\Phi = 0. \qquad (123)$$

Radiation-baryon plasma. The baryons and radiation are tightly coupled before recombination and, therefore, generically only the sum of their energy-momentum tensors satisfies the conservation laws (120) and (121). Nevertheless, in particular case when the baryons are nonrelativistic, the equation (120) is still valid separately by baryons and radiation, because the energy conservation law for the baryons, $T^{\alpha}_{0;\alpha} = 0$, reduces in this case to the conservation law for the total baryon number. (Of course this is not true for (121)) since baryons and radiation "move together" and there exist momentum exchange between these components.) Hence, the fractional density fluctuations in baryons, $\delta_b \equiv \delta\varepsilon_b/\varepsilon_b$, satisfy the equation similar to (122):

$$(\delta_b - 3\Phi)' + au^i{}_{,i} = 0. \qquad (124)$$

As it follows from (120) the corresponding equation for the perturbations in the radiation component, $\delta_\gamma \equiv \delta\varepsilon_\gamma/\varepsilon_\gamma$ takes the form

$$(\delta_\gamma - 4\Phi)' + \frac{4}{3}au^i{}_{,i} = 0. \qquad (125)$$

Since the photons and baryons are tightly coupled their velocities are the same. Therefore multiplying (125) by 3/4 and subtracting equation (124) we obtain

$$\frac{\delta s}{s} \equiv \frac{3}{4}\delta_\gamma - \delta_b = const \qquad (126)$$

where $\delta s/s$ are the fractional entropy fluctuations in the baryon-radiation plasma. For adiabatic perturbations, $\delta s = 0$ and, therefore, we have

$$\delta_b = \frac{3}{4}\delta_\gamma. \qquad (127)$$

If we express $u^i_{,i}$ in terms of δ_γ and Φ from (125) and substitute into (121), we obtain

$$\left(\frac{\delta'_\gamma}{c_s^2}\right)' - \frac{3\boldsymbol{\eta}}{\varepsilon_\gamma a}\Delta\delta'_\gamma - \Delta\delta_\gamma = \frac{4}{3c_s^2}\Delta\Phi + \left(\frac{4\Phi'}{c_s^2}\right)' - \frac{12\boldsymbol{\eta}}{\varepsilon_\gamma a}\Delta\Phi', \qquad (128)$$

where Δ is the Laplacian and c_s^2 is the squared speed of sound in the baryon-radiation plasma, which is equal to:

$$c_s^2 \equiv \frac{\delta p}{\delta\varepsilon} = \frac{\delta p_\gamma}{\delta\varepsilon_\gamma + \delta\varepsilon_b} = \frac{1}{3}\left(1 + \frac{3}{4}\frac{\varepsilon_b}{\varepsilon_\gamma}\right)^{-1}. \qquad (129)$$

Without taking into account the polarization effects the shear viscosity coefficient entering (128) is given by[7]:

$$\boldsymbol{\eta} = \frac{4}{15}\varepsilon_\gamma\tau_\gamma \qquad (130)$$

where τ_γ is the mean free time for the photons.

Thus we derived two perturbation equations (123) and (128), which being supplemented by 0-0 component of the Einstein equations [3]

$$\Delta\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi = 4\pi G a^2\left(\varepsilon_d\delta_d + \frac{1}{3c_s^2}\varepsilon_\gamma\delta_\gamma\right) \tag{131}$$

form a closed system of equations for three unknown variables δ_d, δ_γ and Φ (we used (127) to express δ_b in terms of δ_γ).

¿From (125) it follows the useful relation for *only the radiation contribution* to the divergence of $0-i$ components of the energy-momentum tensor,

$$T^i_{0,i} = \frac{4}{3}\varepsilon_\gamma u_0 u^i_{,i} = (4\Phi - \delta_\gamma)'\,\varepsilon_\gamma \tag{132}$$

which is used in (20)

Longwave perturbations: ($k \ll \eta_r^{-1}$) The behavior of perturbations strongly depends on how big is their scales compared to the horizon. First I consider the long wavelength perturbations with $k\eta_r \ll 1$ (k is comoving wavenumber) which cross the horizon only after recombination. Knowing the gravitational potential we can easily find δ_γ. In fact for this longwave perturbations one can neglect the velocity term in the equation (125), which after that can be easily integrated with the result

$$\delta_\gamma - 4\Phi = C, \tag{133}$$

where C is the constant of integration. To determine C, we note that, during the radiation dominated epoch, the gravitational potential is mostly due to the fluctuations in the radiation component and does not change on supercurvature scales. At early times, $\delta_\gamma \simeq -2\Phi\,(\eta \ll \eta_{eq}) \equiv -2\Phi^0$ (see [3]) ; hence $C = -6\Phi^0$. After equality, when the dark matter overtakes the radiation, the gravitational potential Φ changes its value by factor of 9/10 and then remains constant, that is, $\Phi\,(\eta \gg \eta_{eq}) = (9/10)\,\Phi^0$. Therefore, if cold dark matter dominates at recombination, it follows from (133) that

$$\delta_\gamma\,(\eta_r) = -6\Phi^0 + 4\Phi\,(\eta_r) = -\frac{8}{3}\Phi\,(\eta_r)\,. \tag{134}$$

One arrives at the same conclusion by noting that, for the adiabatic perturbations, $\delta_\gamma = 4\delta_d/3$ and $\delta_d \simeq -2\Phi\,(\eta_r)$ at recombination.

Intermediate scales ($\eta_r^{-1} < k < \eta_{eq}^{-1}$) Next I consider the scales which enter horizon in between the equality (η_{eq}) and recombination (η_r). The perturbations which enter horizon within this rather short time interval are especially interesting since they are responsible for the first few acoustic peaks in the CMB spectrum. Unfortunately, for the realistic values of the cosmological parameters the solution for these perturbations cannot be found analytically with needed accuracy because in the realistic models the condition $\eta_{eq} \ll \eta_r$ is not satisfied. Nevertheless to gain an intuition about the behavior of perturbations, it is very useful to consider the models where $\eta_{eq} \ll$

η_r and derive the appropriate asymptotic expressions for the perturbations with $\eta_r^{-1} \ll k \ll \eta_{eq}^{-1}$. To simplify the consideration I also assume that the contribution of baryons to the gravitational potential is negligible compared to the contribution of the cold dark matter.

In general, there exist four instability modes in two component medium. The set of equations is rather complicated and they can not be solved analytically without making further assumptions. However in our case, the problem can be simplified if we note that if the perturbation enters horizon sufficiently late after equality ($\eta \gg \eta_{eq}$), the appropriate gravitational potential, which is mainly due to the perturbations in the cold dark matter component, remains unchanged and stays constant afterwards ($\Phi_k(\eta) = const$) [3]. The baryons do not contribute much to the gravitational potential, however they can still significantly influence the speed of sound after equality.

Under assumption we have made, the gravitational potential Φ can be considered as an external source in equation (128). Therefore, the general solution of this equation is given by the sum of a general solution of homogeneous equation (with $\Phi = 0$) and a particular solution of (128). Introducing the variable x, defined by $dx = c_s^2 d\eta$, and taking into account that the time derivatives of the potential (128) are equal to zero ($\Phi = const$), we reduce the equation (128) to

$$\frac{d^2\delta_\gamma}{dx^2} - \frac{4\tau_\gamma}{5a}\Delta\frac{d\delta_\gamma}{dx} - \frac{1}{c_s^2}\Delta\delta_\gamma = \frac{4}{3c_s^4}\Delta\Phi. \tag{135}$$

where the second term is due to the viscosity. If the speed of sound is slowly varying, this equation has an obvious approximate solution

$$\delta_\gamma \simeq -\frac{4}{3c_s^2}\Phi. \tag{136}$$

The general solution of the homogeneous equation (135) can be obtained in the WKB approximation. Let us consider the plane wave perturbation with the comoving wavenumber k. Introducing instead of δ_γ the new variable

$$y \equiv \delta_\gamma \exp\left(\frac{2}{5}k^2 \int \frac{\tau_\gamma}{a}dx\right), \tag{137}$$

we find from (135) that it satisfies the equation

$$\frac{d^2y}{dx^2} + \frac{k^2}{c_s^2}\left(1 - \frac{4c_s^2}{25}\left(\frac{k\tau_\gamma}{a}\right)^2 - \frac{2c_s^2}{5}\left(\frac{\tau_\gamma}{a}\right)'\right) y = 0. \tag{138}$$

For the perturbations with the scale ($\lambda_{ph} \sim a/k$) bigger than the mean free path of the photons[9] ($\sim \tau_\gamma$), the second term inside the brackets is negligible. The third term which is about $\tau_\gamma/a\eta \sim \tau_\gamma/t \ll 1$ can be also skipped. Therefore the WKB solution for y is

$$y \simeq \sqrt{c_s}\left(C_1 \cos\left(k\int\frac{dx}{c_s}\right) + C_2 \sin\left(k\int\frac{dx}{c_s}\right)\right). \tag{139}$$

[9] In fact, the imperfect fluid approximation can be used only in this case

Returning back to δ_γ (see (137)) and combining this solution with (136), we obtain

$$\delta_\gamma \simeq -\frac{4}{3c_s^2}\Phi_k + \sqrt{c_s}\left(C_1 \cos\left(k\int c_s d\eta\right) + C_2 \sin\left(k\int c_s d\eta\right)\right) e^{-(k/k_D)^2}. \tag{140}$$

Here we have introduced the dissipation scale characterized by the comoving wavenumber:

$$k_D(\eta) \equiv \left(\frac{2}{5}\int_0^\eta c_s^2 \frac{\tau_\gamma}{a} d\eta\right)^{-1/2}. \tag{141}$$

In the limit of constant speed of sound and vanishing viscosity the solution (140) is exact and valid also in the limit $k \to 0$.

¿From (140), it is clear that the viscosity efficiently damps the perturbations on comoving scales $\lambda \leq 1/k_D$. Using the formula (141) with $c_s^2 = 1/3$ and, assuming instantaneous recombination we obtain the following estimate for the dissipation scale

$$(k_D\eta_r)^{-1} \simeq 0.6\left(\Omega_m h^2\right)^{1/4}\left(\Omega_b h^2\right)^{-1/2} z_r^{-3/4}. \tag{142}$$

The constants of integration C_1 and C_2 in (140) can be determined if we note that at earlier stages when the speed of sound does not change too much the solution (140) is also valid when the scale of perturbation still exceeds the horizon scale. As we have found before the amplitude of the longwave perturbations ($k\eta \ll 1$) is equal to $\delta_\gamma \simeq -8\Phi_k/3 = const$ at $\eta \gg \eta_{eq}$. Assuming that at the moment when the perturbation enters the horizon the speed of sound is still not very different from $1/\sqrt{3}$ we find that $C_1 = 4/3^{3/4}$ and $C_2 = 0$; hence

$$\delta_\gamma(\eta) = \left[-\frac{4}{3c_s^2} + \frac{4\sqrt{c_s}}{3^{3/4}} e^{-(k/k_D)^2} \cos\left(k\int_0^\eta c_s d\eta'\right)\right]\left(\frac{9}{10}\Phi_k^0\right) \tag{143}$$

for $k \ll \eta_{eq}^{-1}$. Here we took into account that $\Phi_k = 9\Phi_k^0/10$ and expressed the result in terms of the initial gravitational potential on superhorizon scales before equality Φ_k^0. The result (143) coincides with the result obtained by S. Weinberg [7] in synchronous coordinate system.

Shortwave perturbations ($k \gg \eta_{eq}^{-1}$) Finally I consider the perturbations which enter the curvature scale before equality. At $\eta \ll \eta_{eq}$, the radiation dominates and in this case the appropriate expressions for Φ and δ_γ were derived, for instance, in [3]. Neglecting the decaying mode we find that after perturbation entered the horizon, that is, at $k^{-1} \ll \eta \ll \eta_{eq}$

$$\delta_\gamma \simeq 6\Phi_k^0 \cos\left(k\eta/\sqrt{3}\right), \quad \Phi_k(\eta) \simeq -\frac{9\Phi_k^0}{(k\eta)^2}\cos\left(k\eta/\sqrt{3}\right). \tag{144}$$

The dissipation, which becomes important only before recombination, can be treated similar to how it was done above. Therefore I neglect the dissipation term here and restore the damping factors only in the final expressions.

After inhomogeneity entered the horizon, the cold dark matter starts "to slide" with respect to the radiation. To get an idea about the behavior of the inhomogeneities in the cold dark matter component itself we can use the equation (123), which after integration becomes

$$\delta_d = 3\Phi + \int \frac{d\eta'}{a} \int a\Delta\Phi d\eta''. \tag{145}$$

Note that this is an exact relation which is always valid for any k. During the radiation-dominated epoch, the main contribution to the gravitational potential is due to the radiation, and, therefore, the gravitational potential in the equation (145) can be treated as an external source. We can fix the constant of integration in (145) substituting there the exact solution for the radiation dominated universe (see (5.45)-(5.46) in [3]) and noting that, at earlier times on superhorizon scales one has to match the well-known result for the longwave perturbations: $\delta_d \simeq 3\delta_\gamma/4 \simeq -3\Phi_k^0/2$. As a result we obtain that after entering the horizon, but before equality

$$\delta_d \simeq -9\left(\mathbf{C}-\frac{1}{2}+\ln\left(k\eta/\sqrt{3}\right)+O\left(\left(k\eta\right)^{-1}\right)\right)\Phi_k^0, \tag{146}$$

where $\mathbf{C} = 0.577...$ is the Euler constant. That is the perturbations in the cold matter component are "frozen" (they grow only logarithmically)

¿From (131) it is easy to see that before equality the contribution of dark matter perturbations to the gravitational potential is suppressed by the factor $\varepsilon_d/\varepsilon_\gamma$ compared to the contribution from the radiation component. At equality, the dark matter begins to dominate and the density perturbation δ_d grow as $\propto \eta^2$(see, [3]). As a result, the appropriate gravitational potential "freeze out" at the value

$$\Phi_k\left(\eta > \eta_{eq}\right) \sim -\frac{4\pi G a^2 \varepsilon}{k^2}\delta_d\bigg|_{\eta_{eq}} \sim O\left(1\right)\frac{\ln\left(k\eta_{eq}\right)}{\left(k\eta_{eq}\right)^2}\Phi_k^0 \tag{147}$$

and stays constant until the recombination. One can get the exact numerical coefficients in this formula in the following way. For shortwave perturbations, the time derivatives of the gravitational potential in (123), (131) can be neglected compared to the spatial derivatives. Then from these equations it follows that

$$\left(a\delta_d'\right)' - 4\pi G a^3\left(\varepsilon_d\delta_d + \frac{1}{3c_s^2}\varepsilon_\gamma\delta_\gamma\right) = 0. \tag{148}$$

The second term here induces the corrections to the solution (146) which become significant only near equality. These corrections are mostly due to $\varepsilon_d\delta_d-$term. The term $\propto \varepsilon_\gamma\delta_\gamma$ does not lead to essential corrections to the solution (146) before equality and it is also negligible compared to $\varepsilon_d\delta_d-$term after equality; hence it can be skipped in (148). As a result the obtained equation can be rewritten in the following form

$$x\left(1+x\right)\frac{d^2\delta_d}{dx^2} + \left(1+\frac{3}{2}x\right)\frac{d\delta_d}{dx} - \frac{3}{2}\delta_d = 0. \tag{149}$$

where $x \equiv a/a_{eq}$. The general solution of this equation is (see, for instance, [7])

$$\delta_d = C_1 \left(1 + \frac{3}{2}x\right) + C_2 \left[\left(1 + \frac{3}{2}x\right) \ln \frac{\sqrt{1+x}+1}{\sqrt{1+x}-1} - 3\sqrt{1+x}\right] \tag{150}$$

At $x \ll 1$ it should coincide with (146). Comparing (150) with (146) at $x \ll 1$ we find

$$C_1 \simeq -9 \left(\ln\left(\frac{2k\eta_*}{\sqrt{3}}\right) + \mathbf{C} - \frac{7}{2}\right) \Phi_k^0, \qquad C_2 \simeq 9\Phi_k^0 \tag{151}$$

where $\eta_* = \eta_{eq}/\left(\sqrt{2}-1\right)$. During the matter dominated epoch ($x \gg 1$), the second term in (150) corresponds to the decaying mode. Neglecting this mode, assuming that the baryon contribution to the potential is negligible compared to the dark matter and using the relation between the gravitational potential and δ_d (see (131)) one finally gets

$$\Phi_k\left(\eta \gg \eta_{eq}\right) \simeq \frac{\ln\left(0.15k\eta_{eq}\right)}{\left(0.27k\eta_{eq}\right)^2}\Phi_k^0 \tag{152}$$

in agreement with [7]. The fluctuations in the radiation δ_γ after equality continue to behave as sound waves in the external gravitational potential given by (152). Therefore, they are described by (140), where we have to substitute the potential (152) instead of Φ_k^0. The constant of integration can be fixed by comparing the oscillating part of this solution to the result in (144) at $\eta \sim \eta_{eq}$. Then, we find that at $\eta \gg \eta_{eq}$

$$\delta_\gamma \simeq \left[-\frac{4}{3c_s^2}\frac{\ln\left(0.15k\eta_{eq}\right)}{\left(0.27k\eta_{eq}\right)^2} + 3^{5/4}\sqrt{4c_s}\cos\left(k\int_0^\eta c_s d\eta\right)e^{-(k/k_D)^2}\right]\Phi_k^0 \tag{153}$$

for $k \gg \eta_{eq}^{-1}$. We have restored here the Silk damping factor. During the radiation dominated epoch, the damping scale, which is proportional to the photon mean free path, is very small. However, it increases just before the recombination and therefore the oscillating contribution to δ_γ is exponentially suppressed on small scales.

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