Fibonacci Sequence Closed Form

The Fibonacci sequence F_n is defined as follows.

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n.$$

Theorem. The Fibonacci sequence has closed closed formula, dubbed h_n, defined as

$$h_n = \frac{1}{\sqrt{5}}\left(\phi_+^{n+1} - \phi_-^{n+1}\right) = \frac{1}{\sqrt{5}}\left(\phi_+^{n} - \phi_-^{n}\right).$$

where $\phi_+ = \frac{1+\sqrt{5}}{2}$ and $\phi_- = \frac{1+\sqrt{5}}{2}$. where

$$\phi_+ = \frac{1+\sqrt{5}}{2}, \qquad \phi_- = \frac{1+\sqrt{5}}{2}.$$

Proof. Consider the infinite x-power series

$$f = \sum_{i=0}^{\infty} F_{i+1} x^i \tag{$\mathtt{f_0}$}$$

and suppose $0 < x < 1/2$ (e.g. 1/4). Then

$$\begin{aligned}
(1-x-x^2)f &= \sum_{i=0}^{\infty}(1-x-x^2)F_{i+1}x^i && (\mathtt{g_1})\\
&= \sum_{i=0}^{\infty} F_{i+1}(x^i - x^{i+1} - x^{i+2})\\
&= (F_1x^0 - F_1x^1 - F_1x^2) + (F_2x^1 - F_2x^2 - F_2x^3) + (F_3x^2 - F_3x^3 - F_3x^4) + \cdots\\
&= F_1x^0 + (F_2 - F_1)x^1 + (F_3 - F_1 - F_2)x^2 + (F_4 - F_2 - F_3)x^3 + \cdots && (\mathtt{g_2})\\
&= 1x^0 + 0x^1 + ((F_1 + F_2) - F_1 - F_2)x^2 + ((F_2 + F_3) - F_2 - F_3)x^3 + \cdots && (\mathtt{g_3})\\
&= 1 + 0 + 0x^2 + 0x^3 + \cdots\\
&= 1 && (\mathtt{g_4})\\
\implies f &= \frac{1}{1-x-x^2} && (\mathtt{f_1})\\
&= \frac{1}{\sqrt{5}}\left(\frac{\phi_+}{1-\phi_+} - \frac{\phi_-}{1-\phi_-}\right). && (\mathtt{f_2})
\end{aligned}$$

Since for $|x| < 1/2$, $\frac{1}{(1-ax)} = \sum_{i=0}^{\infty}(ax)^n$, we have

$$\begin{aligned}
f = \frac{1}{\sqrt{5}}\left(\frac{\phi_+}{1-\phi_+} - \frac{\phi_-}{1-\phi_-}\right) &= \frac{1}{\sqrt{5}}\left(\left(\phi_+\sum_{i=0}^{\infty}\phi_+^i x^i\right) - \left(\phi_-\sum_{i=0}^{\infty}\phi_-^i x^i\right)\right) && (\mathtt{f_3})\\
&= \sum_{i=0}^{\infty}\left(\frac{1}{\sqrt{5}}\left(\phi_+^{i+1} - \phi_-^{i+1}\right)x^i\right) && (\mathtt{f_4})\\
\implies \sum_{i=0}^{\infty} F_{i+1}x^i &= \sum_{i=0}^{\infty}\left(\frac{1}{\sqrt{5}}\left(\phi_+^{i+1} - \phi_-^{i+1}\right)x^i\right) && (\mathtt{f_0} \approx \mathtt{f_4})\\
\implies \forall n, F_{n+1} &= \frac{1}{\sqrt{5}}\left(\phi_+^{n+1} - \phi_-^{n+1}\right) = h_{n+1} && (F_{n+1} \equiv h_{n+1})\\
\implies \forall n, F_n &= h_n. && (F_n \equiv h_n)
\end{aligned}$$

$\square$