Fibonacci Sequence Closed Form

The Fibonacci sequence F_n is defined as follows:

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n.$$

Theorem. The Fibonacci sequence has the following closed formula

$$F_n = \frac{1}{\sqrt{5}} \left(\phi_+^i - \phi_-^i \right).$$

where $\phi_+ = \frac{1+\sqrt{5}}{2}$ and $\phi_- = \frac{1+\sqrt{5}}{2}$.

Proof. Consider the infinite x-power series f defined as

$$f = \sum_{i=0}^{\infty} F_{i+1} x^i,$$

and suppose $0 < x < 1/2$ (e.g. $1/4$). Then

$$\begin{aligned}
(1 - x - x^2) f &= \sum_{i=0}^{\infty} (1 - x - x^2) F_{i+1} x^i \\
&= \sum_{i=0}^{\infty} F_{i+1} (x^i - x^{i+1} - x^{i+2}) \\
&= (F_1 x^0 - F_1 x^1 - F_1 x^2) + (F_2 x^1 - F_2 x^2 - F_2 x^3) + (F_3 x^2 - F_3 x^3 - F_3 x^4) + \cdots \\
&= F_1 x^0 + (F_2 - F_1) x^1 + (F_3 - F_1 - F_2) x^2 + (F_4 - F_2 - F_3) x^3 + \cdots \\
&= 1x^0 + 0x^1 + ((F_1 + F_2) - F_1 - F_2) x^2 + ((F_2 + F_3) - F_2 - F_3) x^3 + \cdots \\
&= 1 + 0 + 0x^2 + 0x^3 + \cdots \\
&= 1 \\
\implies f &= \frac{1}{1 - x - x^2} \\
&= \frac{1}{\sqrt{5}} \left(\frac{\phi_+}{1 - \phi_+} - \frac{\phi_-}{1 - \phi_-} \right).
\end{aligned}$$

Since for $|x| < 1/2$, $\frac{1}{(1-ax)} = \sum_{i=0}^{\infty} (ax)^n$, we have

$$\begin{aligned}
f &= \frac{1}{\sqrt{5}} \left(\frac{\phi_+}{1 - \phi_+} - \frac{\phi_-}{1 - \phi_-} \right) \\
&= \frac{1}{\sqrt{5}} \left(\left(\phi_+ \sum_{i=0}^{\infty} \phi_+^i x^i \right) - \left(\phi_- \sum_{i=0}^{\infty} \phi_-^i x^i \right) \right) \\
&= \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{5}} \left(\phi_+^{i+1} - \phi_-^{i+1} \right) x^i \right).
\end{aligned}$$

Then finally,

$$\begin{aligned}
f = \sum_{i=0}^{\infty} F_{i+1} x^i &= \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{5}} \left(\phi_+^{i+1} - \phi_-^{i+1} \right) x^i \right) \\
\implies F_{i+1} &= \frac{1}{\sqrt{5}} \left(\phi_+^{i+1} - \phi_-^{i+1} \right) \\
\implies F_i &= \frac{1}{\sqrt{5}} \left(\phi_+^i - \phi_-^i \right).
\end{aligned}$$

$\square$