1.4. n-Dimensional Packings

Definition. The **fcc** lattice is the face-centered cubic lattice.
Definition. A lattice Λ has a **dual** lattice Λ^* given by

$$\Lambda^* := \{x : \forall u \in \Lambda, x \cdot u \in \mathbb{Z}\} .$$

For example, the dual of the fcc lattice is the **body-centered cubic** lattice (bcc lattice). If A is a Gram matrix for Λ, then Λ^* has Gram matrix A^{-1}.

Why is is finding dense packings in n-dimensions interesting?

1. Interesting problem in pure geometry. Hilbert mentioned it in 1900 in his ist of open problems [Hil1], [Mil5].

2. Has (sometimes unexpected) connections to other branches of mathematics. For example, the densest lattice packings in up to 8 dimensions belong to the families A_n, D_n, E_n and the corresponding Coxeter-Dynkin diagrams turn in several seemingly unrelated areas.

3. The Leech lattice in 24 dimensions, Λ_{24}, has mysterious connections with hyperbolic geometry, Lie algebras, and the Monster simple group.

4. There are direct applications of lattice packings to number theory e.g. solving Diophantic equations and the "geometry of numbers." [Cas2], [Gru1], [Gru1a], [Han3], [Hla1], [Hla3], [Kel1], [Min4], [Min6]. (See Section 2.3.)

5. There are practical applications of sphere packings to digital communications (see Chapter 3).

6. 2- and 3-dimensional spheres have many practical applications in general e.g. positioning optical fibers in the cross-section of a cable [Kin1], chemistry, physics, antenna design, X-ray tomography, and statistical analysis (on spheres).

7. n-dimensional packings may be used in the numerical evaluation of integrals, either on the surface of a sphere in $\mathbb{R}^n$ or its interior. (See Section 3.2.)

8. Dual theory and superstring theory in physics have made use of E_8 and Λ_{24} lattices and their related Lorentzian lattices in dimensions 10 and 26 discussed in Chapters 26 and 27.