1 Introduction

This course covers electricity and magnetism. At the heart of the topics are the four famous equations known as the *Maxwell's* equations,

$$\nabla.\vec{D} = \rho \qquad \text{Gauss law} \quad (1.1)$$

$$\nabla.\vec{B} = 0 \qquad \text{No magnetic monopoles} \quad (1.2)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{Ampere's law} \quad (1.3)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday's law} \quad (1.4)$$

At the end of this course you will understand what each of these equations stand for. These equations would lead to the unification of electromagnetism with light and are responsible to explain many wonderful phenomena you see in daily life.

Warning: This course is extremely math intensive and it would build upon the materials taught during the lectures. So you have to keep up or else you will be lost. Even sparing a week will put you in trouble. We would try to explain phenomena seen in life and connect them to electricity and magnetism. For example, lightening, magnetic levitation, northern lights, why is the sky blue, rainbows, etc. Our goal is to make you see through these equations, to make you see and understand the beauty around you and in doing so fell in love with physics. In general, the lectures will not be a repeat of your book but will be complementary to the material. The notes are accessible online but they do not stand on their own. Please remember that they should be used in combination with the lectures.

2 Electrostatics

2.1 Introduction - *What holds the world together?*

Electricity and magnetism is all around us. Some examples are lights, television, computers, calculators, cell phones and etc. Light itself is an electromagnetic phenomena as radio waves are. The colors of the rainbows in the sky is due to electricity. Cars, planes and trains can only run because of electricity. Even horses need electricity due to the fact that muscle contraction is caused by electrical signals. In fact your nervous system is driven by electricity and you may not be able to think if there was no electricity.

A modern picture of an atom is illustrated in Fig. 2.1. Electrons are in a cloud around the core (nucleus) and they are negatively charged. The nucleus includes of protons which are positively charged ,p^+, and neutrons which are neutral. Their masses are,

$$m_p \approx m_n \approx 1.7 \times 10^{-27} kg \quad (2.1)$$

$$m_e \approx 9.1 \times 10^{-31} kg \quad (2.2)$$

For a neutral atom, the number of electrons and protons are the same meaning the cumulative charge of the hole atom results in zero. Thus, there are two types of charges which are the positive and the negative charges. Protons are very difficult to remove from an atom but electrons are different. Adding an extra electron would result in a negative ion where as removing an electron would result in a positive ion. Therefore, charge can be redistributed but never destroyed.

In 600 B.C., people knew that if you rub amber, it attracts pieces of dry leaves. The Greek word for amber is "electron". In the mid-sixteenth century, when people got bored at parties, they would rub amber jewelry and stroke frogs with it causing them to jump in despair due to static electricity shock. In eighteenth century, it was discovered that there are two types of electricity when you rub amber versus glass. It was also discovered that the same types repelled each other while the opposites attracted each other. Benjamin Franklin introduced the idea that all substances are penetrated with what he called electrical fluid or electric fire. He stated that if you get too much

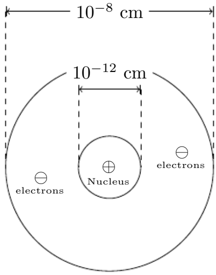


Figure 2.1: Electrons are in a cloud around the nucleus. The nucleus is consist of protons, which are positive charged, and neutrons, which have no net electric charge. So the nucleus is positive. Electrons are negative charged.

of the fire, you would get positively charged and if you have a deficiency, you would get negatively charged. Hence, he introduced the sign convention which is still in use.

If a positive charge is created in a medium by removing its electrons,then the electrons should have moved to another medium. Therefore, the second medium becomes negatively charged.Hence, you cannot create any charge. You can only "*distribute*" it!

The following observations could be done from Franklin's experiments:

1. Same charges repel while opposites attract. So "+" and "+" or "-" and "-" repel; "+" and "-" attract.

2. More the charges, the more the force.

3. Closer the charged objects, higher the force.

4. Some materials "conduct" the charges and others do not. Thus there are conductors and insulators. Conductors are materials in which electrons are weakly bonded and can be stripped off easily from the atoms.

In an experiment, lets take a glass rod and positively charge it and bring it close to a conducting balloon (as shown in Fig. 2.2). In a conductive balloon, electrons are free to move around and they will be attracted to the rod. This is called "induction".

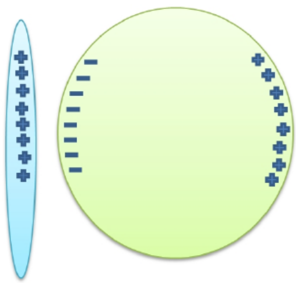


Figure 2.2: Induction of poles using a glass rod.

In this situation, the balloon should get attracted to the rod. Now if we let the balloon to touch the rod, some electrons will move into the glass rod. Hence, the balloon gets positively charged and it should repel the glass rod. So it is clear that there are two charge types.

Question: What happens if we choose a non-conductive balloon? Would a similar situation happen? It sure does. While electrons are not free to move, they are pulled towards one side. This is called "polarization". We will spend a lecture on this later on.

Friction can cause "induction". An interesting experiment for you to try is when you wear a nylon shirt. Go to a pitch dark room in front of a mirror try taking the shirt off. You may see blue sparks coming off. You see this effect everyday. Try combing dry hair into one place! or how many of you have been zapped by door knobs!. Why doesn't the "saran" wrap come-off so easily?

2.2 Coulomb's law

Charge is a fundamental quantity in electricity and magnetism. There are two kinds of electric charges: positive and negative. The interaction of charges obeys *Coulomb's law*,

$$\vec{F}_{Qq} = k\frac{Qq}{r^2}\hat{r}_{Qq} \tag{2.3}$$

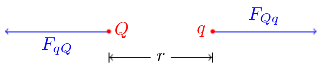


Figure 2.3: the interaction of two point charges

If Q, q are both positive or both negative, $F > 0$, and the interaction is repulsive; otherwise, if Q, q are oppositely charged, $F < 0$, the interaction is attractive. Hence the equation is sign sensitive.

In SI units, q, Q are in Coulomb C, r is in meter m, and F is in Newton N. k, which is called *Coulomb's constant*, is given by $k = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$. Note that the unit of charge, C, is derived from the unit of current (Ampere). One Coulomb is a horrendously large amount of charge and we normally work in micro Coulombs. To put it in perspective, one Coulomb is the charge transported by a steady current of one ampere in one second,

$$1\ \mathrm{C} = 1\ \mathrm{A} \times 1\ \mathrm{s} \tag{2.4}$$

The Coulomb's law can be written in an alternative form,

$$|F| = |\vec{F}_{Qq}| = |\vec{F}_{qQ}| = \left|\frac{Qq}{4\pi\epsilon_0 r^2}\right| \tag{2.5}$$

where ϵ_0 is called *vacuum permittivity* or *electric constant*. $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2} = 8.85 \times 10^{-12}\ \mathrm{F/m}$ (F is known as *Farad*, which will be discussed when studying capacitance). If the charges were immersed in a dielectric (insulator) medium, the Coulomb's law in dielectric medium is used,

$$F = \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{r^2} = \frac{1}{4\pi(\epsilon_r\epsilon)}\frac{q_1 q_2}{r^2} \tag{2.6}$$

In the equation, ϵ is known as the *permittivity of the dielectric medium*; ϵ_r is the *relative permittivity of the medium.* For vacuum, $\epsilon_r = 1$.

Considering the interaction between nucleus and electrons in the atom, the electric force is about 39 orders bigger than the gravitation force. This can be shown by the following calculation,

$$\frac{F_e}{F_g} = \left(\frac{1}{4\pi\epsilon_0}\frac{e^+e^-}{r^2}\right) / \left(G\frac{m_p m_e}{r^2}\right) = \frac{-8.1 \times 10^{-8}}{3.7 \times 10^{-47}} \approx 2 \times 10^{39} \tag{2.7}$$

Conservation of charge: charge can neither be created nor destroyed (globally). However, locally, we can create an imbalance and redistribute the charge.

Example 1 Three charges are at the corners of an equilateral triangle. Find the net force on charge 1.

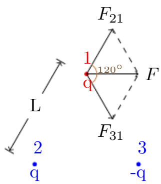


Figure 2.4: Example 1

$$
\begin{aligned}
F_{21} =& \frac{kq^2}{L^2} \rightarrow \text{repulsive} && (2.8)\\
F_{31} =& -\frac{kq^2}{L^2} \rightarrow \text{attractive} && (2.9)\\
F^2 =& F_{21}^2 + F_{31}^2 + 2|F_{21}||F_{31}| \cos 120^\circ && (2.10)\\
=& 2\left(\frac{kq^2}{L^2}\right)^2 + 2 \cdot \left(\frac{kq^2}{L^2}\right)^2 \left(-\frac{1}{2}\right)\\
=& \left(\frac{kq^2}{L^2}\right)^2
\end{aligned}
$$

So, $F = \frac{kq^2}{L^2}$ and directing to $+x$. Remember that the net force comes from a vector addition. Here are some questions to think about:

- Is it obvious that "superposition" principle should work? $\rightarrowtail$ No.
- Do you believe in it? Why? $\rightarrowtail$ Yes, all experiments confirm it.
- How do electric forces compare with gravity? $\rightarrowtail$ They are way more powerful than gravity as calculated above.

Example 2 An electron is placed a distance d under a proton. The proton is fixed in place. When the electron is let go it is found that it does not move. Find d.

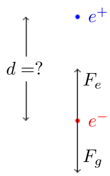


Figure 2.5: Example 2

$$
\begin{aligned}
&\because \text{ the electron does not move} \\
&\therefore \sum \vec{F} = 0 \qquad (2.11)
\end{aligned}
$$

Note that direction of $\vec{F_e}$ and $\vec{F_g}$ are anti-parallel. We have

$$
\begin{aligned}
F_e =& mg \qquad (2.12) \\
\frac{kq^2}{d^2} =& m_e g \\
d^2 =& \frac{kq^2}{m_e g} \\
=& \frac{8.99 \times 10^9 \times (1.6 \times 10^{-19})^2}{9.11 \times 10^{-31} \times 9.8} \\
d =& 5.077 = 5.08 \text{ (m)}
\end{aligned}
$$

Example 3 Find the charge q on each sphere (point mass) so the system is in equilibrium.

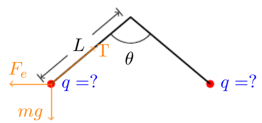


Figure 2.6: Example 3

Because the system is in equilibrium, the net force applied on the sphere should equals zero. If we consider the forces on the left sphere,

$$
\begin{cases}
\sum \vec{F_x} = (T\sin(\theta/2) - F_e)\hat{x} = 0 \\
\sum \vec{F_y} = (T\cos(\theta/2) - mg)\hat{y} = 0
\end{cases} \qquad (2.13)
$$

$$
\begin{aligned}
F_e =& mg \times \tan(\theta/2) \qquad (2.14) \\
\frac{kq^2}{(2L\sin(\theta/2))^2} =& mg\tan(\theta/2) \\
q =& \sqrt{\frac{4mgL^2\sin^3(\theta/2)}{k\cos(\theta/2)}}
\end{aligned}
$$

Example 4 Two charges Q_1 and Q_2 are fixed in space as shown, where would a third have to be placed so that the net force on it is zero?

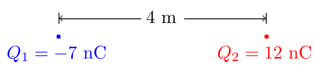
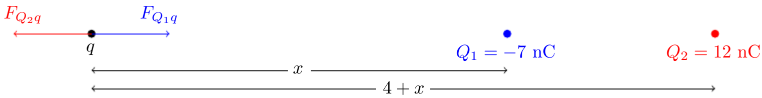


Figure 2.7: Example 4

1. it should be apparent that the third charge will have to be placed along the line connecting Q_1 and Q_2. (We are not saying between Q_1 and Q_2, but along the line).

2. it should also be apparent that the third charge cannot be placed in between.
3. if the above two points are not clear, so back and make sure you know what is going on.
4. placing the charge to the right side of the 12 nC charge will also not work, why?
5. the only place for this to work has to be the left of Q_1. Assume the distance between the third charge and Q_1 is x, and the charge of the third q is positive.



$$
\begin{aligned}
\sum \vec{F}_x &= 0 \\
\frac{kqQ_1}{x^2} + \frac{kqQ_2}{(x+4)^2} &= 0 \\
(x+4)^2 Q_1 + x^2 Q_2 &= 0 \\
x &\approx 12.9
\end{aligned} \tag{2.15}
$$

3 Electric Field Model

3.1 Electric field of discrete charges

We saw that if we assume a test charge of q is located near a charge of Q, a force of $\vec{F}_Q$ will act on it as

$$\vec{F}_Q = \frac{1}{4\pi\epsilon_0} \frac{Qq}{r^2} \hat{r} \tag{3.1}$$

Electric Field is a vector field, or a vector function. It has a magnitude and direction which is, in general, a function of space (position x,y,z) and time (t) as shown in Fig. 3.1.

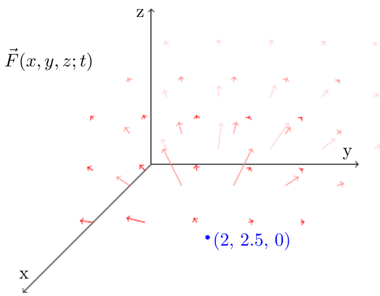


Figure 3.1: The electric field of a charge point lies at (2, 2.5, 0)

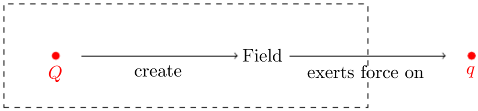


Figure 3.2: Two charges interacts through an electric field.

The *vector field* can be used to describe water flow, atmospheric wind, gravitation force, electromagnetic force, etc. A *scalar field* or *scalar function* has a magnitude only (no direction), and, in general, is a function of space and/or time, e.g. temperature distribution, potential distribution, etc.

We no longer regard the interaction between the charges directly. Instead, charges interact through a field. From Coulomb's law (Eq. 3.1) we saw that $\hat{r} = \vec{r}/r$ is the unit vector in the direction of interacting r. Next, consider q to be a *test-charge*; move it around Q and note the force experienced by it. We are finding the *force field* established by Q around itself. This force field, on unit test-charge is called the *electric field* $\vec{E}$. It is defined as,

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q}{q} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \quad [\mathrm{N/C}] \tag{3.2}$$

It looks similar to $\vec{F}_Q$ but does not have q any more. In establishing this model, we assume that the test-charge (q) is so small that does not disturb the primary charge responsible for the field. Mathematically,

$$\vec{E}(\vec{r}) = \lim_{q\to 0} \frac{\vec{F}_Q}{q} \tag{3.3}$$

Note: at any point, magnitude of $\vec{E}$, $|\vec{E}| = q_1/4\pi\epsilon_0 r^2$; direction of $\vec{E}$ is $\hat{r}$.

Again $\hat{E}$ (direction of $\vec{E}$) depends on polarity of Q. If $Q > 0$, $\hat{E}$ is in $\hat{r}$ and pointing away from Q while if $Q < 0$, $\hat{E}$ is in $-\hat{r}$ and pointing towards Q. We adopt the convention that $\vec{E}$ is always in the direction of a positive test charge (q). So what does $\vec{E}$ represent? It represents something that happens around the charge Q. Hence, if we know $\vec{E}$ for a complex charge distribution, then we could easily calculate $\vec{F}_q$ in the field as $\vec{F}_q = q\vec{E}$. The distribution of $\vec{E}$ can be shown as in Fig. 3.3 for two oppositely charged spheres. As shown, the field lines are pointing away for the positive charge and it is pointing towards the negative charge. The vector lengths present the field strength which is much stronger for the positive charge which is 5 times the negative charge.

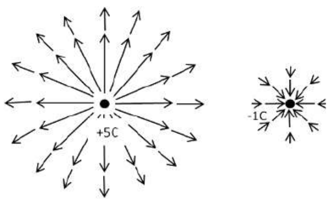


Figure 3.3: Electric field lines of two positively charged and negatively charged spheres.

If there are more than one charge, considering i number of charges, to calculate $\vec{E}$ at O we can use the superposition principle as follows,

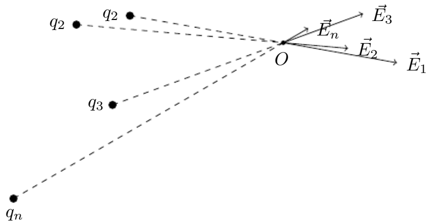


Figure 3.4: The total field at some point is the sum of all the field at that point

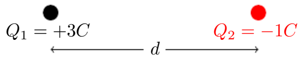
1. find $\vec{E}$ due to each charge at O
2. add all $\vec{E}$ vectorially to find the resultant field, i.e.,

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + ... + \vec{E}_N = \sum_{n=1}^{N} \vec{E}_n \tag{3.4}$$

If a charge q is placed at O, the force acting on q will be $\vec{F}_q = q\vec{E}_q$.

Question: Does electric field represents the direction in which charge q would move if it was allowed to move freely?

Question: Let's consider the following case. What will be the $\vec{E}$ field due to this situation? Without doing maths, can you physically think of what is going to happen to the electric field?

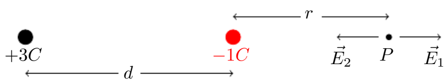


If we are very close to the $-1C$ charge, then due to $1/r^2$ variation, -1 is going to win over and the $\vec{E}$ should be pointing towards it.

If we go very far away from the two charges, what would happen?

$$\frac{1}{r^2} \approx \frac{1}{(r+d)^2} \qquad\qquad r \gg d \tag{3.5}$$

So it would look as the two charges are one and they will appear as one $+2C$ charge. The $\vec{E}$ field should be pointing outwards since the cumulative charge is positive. So somewhere in between, the $\vec{E}$ field would change direction and hence it becomes zero. Let us find that point!



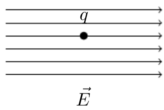
$$\begin{aligned} \frac{Q_1}{4\pi\epsilon_0(r+d)^2} &= \frac{Q_2}{4\pi\epsilon_0 r^2} \\ (r+d)^2 Q_2 &= Q_1 r^2 \end{aligned} \tag{3.6}$$

from which r can be calculated.

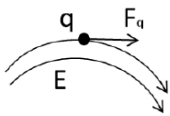
$\underline{\vec{E}\ \textit{field lines}}$: It is convenient to visualize the electric field pattern by lines of force. Properties of these force lines are:

1. $\vec{F}$ is tangential to the $\vec{E}$ field lines for any charge distribution.
2. croess-sectional density of the lines (number of field lines per unit cross-section perpendicular to the line) is proportional to the magnitude of electric field at that point.

Question: If the $\vec{E}$ field lines are straight and we release a charge, how will it move?



Assuming q is released at rest, then q will experience a force in the direction of $\vec{E}$ and the charge will move in that direction. However, this would only happen in this very special case where the field lines are uniform and straight. If the field lines are curved, it would experience the force $\vec{F_q}$ and it will accelerate tangentially to $\vec{E}$. Then the direction of force will change and the motion of charge would be complex. So keep in mind that field lines are not trajectory of motion.



Now consider two positive charges with different values such as shown. As seen you will get a hair blower effect. Where does $\vec{E}$ equals zero?

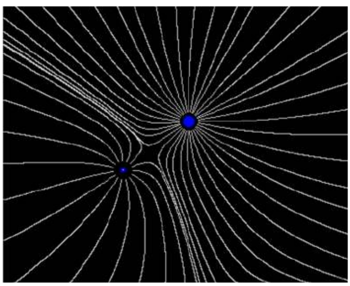


Figure 3.5: Lines of force between two same polarity charges, one is stronger than the other.

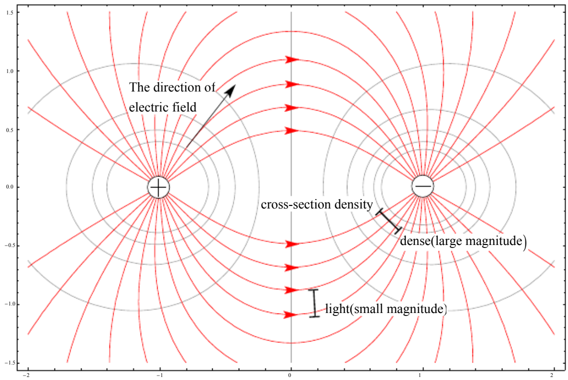


Figure 3.6: Lines of force of a dipole

Example 1 Electric dipole.

Two equal but opposite charges are shown in Fig. 3.6. As it can be seen, the field lines seem to be symmetric near the charges. What do you think it would happen if we move far away from the dipole? Does $+q$ and $-q$ cancel each other and you do not get any $\vec{E}$ field? **No, this will not happen!**

So for a single charge, $\vec{E}$ decreases as $\frac{1}{r^2}$ with increasing r. Would this rate be faster for a dipole or slower? i.e. if the $\vec{E}$ field falls as $\frac{1}{r^n}$, will $n > 2$ or $n < 2$ or $n = 2$? Dipoles are infact very common in physics. We should calculate the electric field due to a it to answer the aforementioned questions.

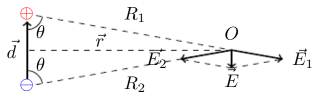


Figure 3.7: Example 1. Electric Dipole.

We first consider the point O locating at a distance r along the perpendicular bisector of the dipole.

$$R_1^2 + R_2^2 = (d/2)^2 + r^2 \tag{3.7}$$

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{(d/2)^2 + r^2} \tag{3.8}$$

$$E_2 = \frac{1}{4\pi\epsilon_0}\frac{q}{(d/2)^2 + r^2} \tag{3.9}$$

Note that the direction of the two fields are different. We need to perform the vector sum $\vec{E}_1 + \vec{E}_2$ to find the net field $\vec{E}$.

$$
\begin{aligned}
E =& |\vec{E}| = E_1 \cos\theta + E_2 \cos\theta \\
=& 2E_1 \cos\theta \\
=& \frac{2(d/2)q}{4\pi\epsilon_0((d/2)^2 + r^2)^{3/2}}
\end{aligned} \tag{3.10}
$$

The product $2aq$ is called the *dipole moment*, also known as $\vec{p} = q\vec{d}$. When $d \ll r$, we can write

$$
E = \frac{p}{4\pi\epsilon_0 r^3} \tag{3.11}
$$

If the point we considering is *arbitrary*, then we have:

$$
\begin{aligned}
\vec{E} =& \vec{E}_1 + \vec{E}_2 \\
=& \frac{1}{4\pi\epsilon_0}\left(\frac{(+q)(-\vec{d}/2 + \vec{r})}{|-\vec{d}/2 + \vec{r}|^3} + \frac{(-q)(-\vec{d}/2 + \vec{r})}{|-\vec{d}/2 + \vec{r}|^3}\right)
\end{aligned} \tag{3.12}
$$

If $d \ll r$, then,

$$
\begin{aligned}
& |-\vec{d}/2 + \vec{r}|^3 \\
=& \left(\left(-\vec{d}/2 + \vec{r}\right)^2\right)^{3/2} \\
=& \left((d/2)^2 + r^2 + 2 \cdot -\vec{d}/2 \cdot \vec{r}\right)^{3/2} \\
\approx& r^3 - \frac{3}{2} r \cdot (\vec{d} \cdot \vec{r})
\end{aligned} \tag{3.13}
$$

So we have,

$$
\begin{aligned}
& \frac{(+q)(-\vec{d}/2 + \vec{r})}{|-\vec{d}/2 + \vec{r}|^3} \\
\approx& \frac{(+q)(-\vec{d}/2 + \vec{r})}{r^3 - \frac{3}{2} r \cdot (\vec{d} \cdot \vec{r})} \\
\approx& \frac{(+q)}{r^3}\left((-\vec{d}/2 + \vec{r})(1 + \frac{3}{2}\frac{\vec{d} \cdot \vec{r}}{r^2})\right)
\end{aligned} \tag{3.14}
$$

Use the same approximation to simplify the $-q$ term and combine them together,

$$
\begin{aligned}
\vec{E} =& \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left((-\vec{d}/2 + \vec{r})(1 + \frac{3}{2}\frac{\vec{d} \cdot \vec{r}}{r^2}) - (\vec{d}/2 + \vec{r})(1 - \frac{3}{2}\frac{\vec{d} \cdot \vec{r}}{r^2})\right) \\
=& \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left(\frac{3\vec{r} \cdot \vec{d}}{r^2}\vec{r} - \vec{d}\right)
\end{aligned} \tag{3.15}
$$

The $\vec{E}$ field drops as $\frac{1}{r^3}$ for the dipole instead of $\frac{1}{r^2}$ for a single charge. We can define a term called "electric dipole moment", $\vec{p}$ as charge multiplied by displacement ($\vec{p} = q\vec{d}$), and rewrite $\vec{E}$ as:

$$
\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left(3(\vec{r} \cdot \vec{p})\hat{r} - \vec{p}\right) \tag{3.16}
$$

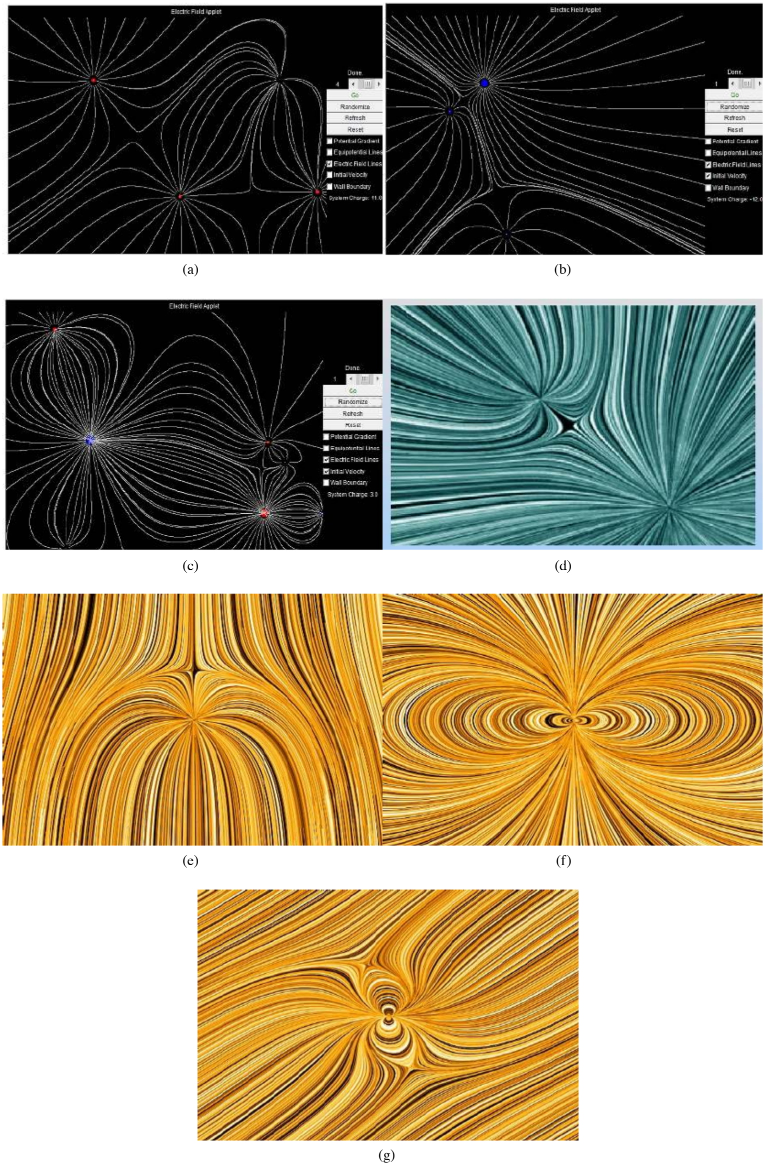
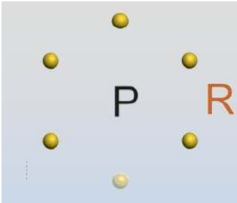


Figure 3.8: Explain each of the above situations. ©CC-By MITOCW

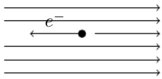
Additional examples of electrical field lines: Here there are some additional examples provided to help you visualize the electrical field distribution around discrete charges in different scenarios. In each case, try to explain the situation.

In another example, let us consider six equal positive charges of q that are positioned at vertices of a regular hexagon with sides of length R. If we remove the bottom charge, what is the electric field at point P in the center of the hexagon?



1. $\vec{E} = \frac{2kq}{R^2}\hat{\mathbf{j}}$
2. $\vec{E} = -\frac{2kq}{R^2}\hat{\mathbf{j}}$
3. $\vec{E} = \frac{kq}{R^2}\hat{\mathbf{j}}$
4. $\vec{E} = -\frac{kq}{R^2}\hat{\mathbf{j}}$
5. $\vec{E} = 0$
6. I don't know!

Motion of charge in $\vec{E}$ field: Calculate the acceleration of charge (an electron) in uniform $\vec{E}$ field.

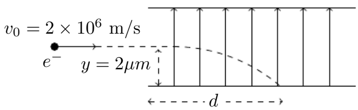


$\vec{E} = E_0\hat{x}$

$$\vec{F} = q\vec{E} = -eE_0\hat{x} \tag{3.17}$$
$$\vec{a} = \frac{\vec{F}}{m} = \frac{-eE_0}{m}\hat{x}$$

Will the direction of motion be the same as the E-lines in this case?

An electron enters a space with uniform electric field as shown. Where does the electron hit the ground? At what angle? At what speed? Neglect gravity. Does anything happen at region 1 before entering the plates area?



$\vec{E} = 1.3 \times 10^5 \hat{y}$ N/C

In region 2 in the plates area:

$$a = \frac{F}{m} = \frac{qE}{m}$$

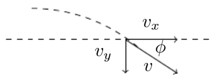
time it takes to drop "y"

$$y = 1/2at^2 \ \rightarrow \ t = \sqrt{\frac{2y}{a}}$$

then $d = v_{ox}t$

$$d = v_0\sqrt{\frac{2y}{a}} = v_0\sqrt{\frac{2ym}{eE_0}}$$

At impact:



$$v_x = v_0 \text{ Nothing happens in "x"}$$

$$v_y = v_{0y} + at = 0 + a\sqrt{\frac{2y}{a}} = \sqrt{2ya}$$

$$v_y = \sqrt{\frac{2yeE_0}{m}}$$

$$\therefore tan(\phi) = \frac{v_y}{v_x} = \sqrt{\frac{2yeE_0}{m}} \times \frac{1}{v_0}$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(v_0^2 + \sqrt{\frac{2yeE_0}{m}}\right)}$$

3.2 Continuous Charge Distribution

As seen in previous section, the electric field due to a small number of charges can be computed using the superposition principle. So what happens if we have a large number of charges distributed in a region in space?

Volume charge density

Suppose we are going to find the electric field due to a volume of charges at some point P which is at a distance r away from the volume charge. Let's consider a small volume element ΔV_i which contains a small amount of charge Δq_i. The distances between the volume elements ΔV_i is much smaller than r which is the distance between ΔV_i and P. When ΔV_i becomes infinitesimally small, we can define the volume charge density $\rho_v(\vec{r})$ as,

$$\rho_v(\vec{r}) = \lim_{\Delta V_i \to 0} \frac{\Delta q_i}{\Delta V_i} = \frac{dq}{dV} \qquad [C/m^3] \tag{3.18}$$

Then, the total amount of charge in the volume V can be written as,

$$Q = \sum_i \Delta q_i = \int_V \rho(\vec{r})dV \tag{3.19}$$

The concept of charge density is analogous to mass density. When the atoms are tightly packed into a volume, the total mass becomes the integral over the density of those atoms.

Surface charge density

In a similar manner, the charge can be distributed over a surface S of area A with surface charge density σ_s,

$$\sigma_s(\vec{r}) = \frac{dq}{dA} \qquad [C/m^2] \tag{3.20}$$

Then, the total amount of charge in the surface S can be written as,

$$Q = \iint_S \sigma_s(\vec{r}) dA \tag{3.21}$$

Line charge density

If the charge is distributed over a line of length l, then the linear charge density λ_l is,

$$\lambda_l(\vec{r}) = \frac{dq}{dl} \qquad [C/m] \tag{3.22}$$

Then, the total amount of charge over the line can be written as,

$$Q = \int_l \lambda_l(\vec{r}) dl \tag{3.23}$$

Therefore when the charge is distributed continuously, we can devise the following strategy to find the integral field at point P:

1. divide the charge distribution into infinitesimal elements dq.
2. consider each element as a point charge and calculate the vector field $d\vec{E}$ due to each at the observation point $\vec{r}$
3. add the individual vector fields to obtain the resultant field $\vec{E}$.

If the charge is distributed uniformly, the charge density is independent of position,

1. charge volume density
$$\rho_V = \frac{Q}{V} \tag{3.24}$$
2. charge area density
$$\sigma_S = \frac{Q}{A} \tag{3.25}$$
3. charge linear density
$$\lambda_l = \frac{Q}{L} \tag{3.26}$$

Example 1 Uniform ring source. The radius of the ring is R. The observe point O lies on the axis of the ring. The total charge of the ring is Q. The radius of the ring is R.

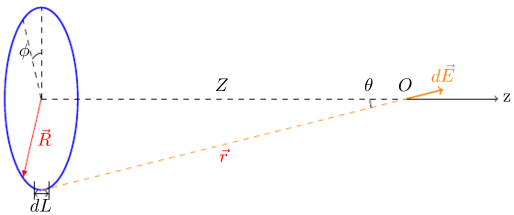


Figure 3.9: Example 1

The total charge of the ring is Q. Since the charge is uniformly distributed on the ring, so the charge linear density is

$$\lambda = Q/L = Q/(2\pi R) \tag{3.27}$$

Therefore,

$$dq = \lambda \cdot dL = \frac{Q}{2\pi R} R d\phi = \frac{Q}{2\pi} d\phi \tag{3.28}$$

From Coulomb's law, the contribution form dq at ϕ is

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} \tag{3.29}$$

Because O is on the axis, the total field other than z-direction must be 0. The element field in z-direction,

$$d\vec{E} \cdot \hat{z} = \frac{dq}{4\pi\epsilon_0} \frac{r \cdot \hat{z}}{r^3} \tag{3.30}$$

$$= \frac{dq}{4\pi\epsilon_0} \frac{r \times \cos\theta}{r^3} \hat{z} \tag{3.31}$$

$$= \frac{dq}{4\pi\epsilon_0} \frac{Z}{(Z^2 + R^2)^{3/2}} \hat{z} \tag{3.32}$$

The total field,

$$\vec{E}_{tot} = \vec{E}_{tot} \cdot \hat{z} \tag{3.33}$$

$$= \int d\vec{E}\hat{z} \tag{3.34}$$

$$= \int \frac{dq}{4\pi\epsilon_0} \frac{Z}{(Z^2 + R^2)^{3/2}} \hat{z} \tag{3.35}$$

$$= \int_0^{2\pi} \frac{d\phi Q}{2\pi} \frac{Z}{4\pi(Z^2 + R^2)^{3/2}\epsilon_0} \hat{z} \tag{3.36}$$

$$= \frac{ZQ}{4\pi(Z^2 + R^2)^{3/2}\epsilon_0} \hat{z} \tag{3.37}$$

When $Z \gg R$,

$$\vec{E}_{tot} = \frac{Q}{4\pi\epsilon_0 Z^2} \hat{z} \tag{3.38}$$

From a large distance the ring appears to be like a point of charge Q at the origin.

Example 2 Uniform disk source. The radius of the disk is R. The observe point O lies on the axis of the disk. The total charge of the disk is Q. The radius of the disk is R.

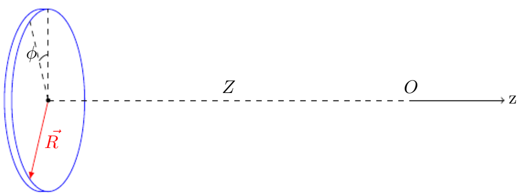


Figure 3.10: Example 2

A disk is consist of infinity rings,

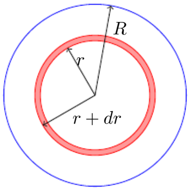


Figure 3.11: Example 2. Front view.

The charge of a ring with radius r and width dr is

$$dq(r) = \frac{Q}{\pi R^2} A_{ring} \tag{3.39}$$
$$= \frac{Q}{\pi R^2} \left(\pi (r + dr)^2 - \pi r^2 \right) \tag{3.40}$$
$$= \frac{2rQ}{R^2} dr \tag{3.41}$$

So the total electric field at O is,

$$\vec{E}_{tot} = \int d\vec{E} \tag{3.42}$$
$$= \int \frac{Z dq}{4\pi (Z^2 + r^2)^{3/2} \epsilon_0} \hat{z} \tag{3.43}$$
$$= \int_0^R dr \frac{2rZQ}{4\pi (Z^2 + r^2)^{3/2} R^2 \epsilon_0} \hat{z} \tag{3.44}$$
$$= \frac{ZQ\hat{z}}{4\pi R^2 \epsilon_0} \int_0^R \frac{2r dr}{(Z^2 + r^2)^{3/2}} \tag{3.45}$$
$$= \frac{ZQ\hat{z}}{4\pi R^2 \epsilon_0} \int_0^{R^2} \frac{dt}{(Z^2 + t)^{3/2}} \tag{3.46}$$
$$= \frac{ZQ\hat{z}}{4\pi R^2 \epsilon_0} \times 2 \left(\frac{1}{Z} - \frac{1}{\sqrt{Z^2 + R^2}} \right) \tag{3.47}$$
$$= \frac{Q\hat{z}}{2\pi R^2 \epsilon_0} \left(1 - \frac{Z}{\sqrt{Z^2 + R^2}} \right) \tag{3.48}$$

If we substitute $Q/\pi R^2$ with the charge area density σ,

$$E_z = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{Z}{\sqrt{Z^2 + R^2}} \right) \tag{3.49}$$

When $R \gg Z$, We have $\lim_{R/Z \to +\infty} \frac{Z}{\sqrt{Z^2+R^2}} = 0$,

$$\lim_{R/Z \to +\infty} E_z = \frac{\sigma}{2\epsilon_0} \tag{3.50}$$

So the electric field on a infinite large, uniformly charged plane is

$$E = \frac{\sigma}{2\epsilon_0} \tag{3.51}$$

is a function only depends on the charge density.

Example 3 Field to a line of charge. The observe point O lies on the z axes as shown in the figure. The charge of the line is Q. The length of the line is L.

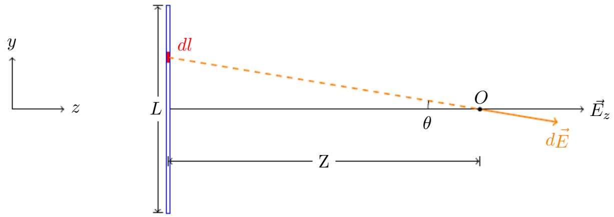


Figure 3.12: Example 3

From the symmetry of the line, $\vec{E}$ is along z direction.

$$\vec{E}_{tot} = E_z\hat{z} \tag{3.52}$$

$$= \int d\vec{E}\hat{z} \tag{3.53}$$

$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dlQ}{L}\frac{1}{4\pi\epsilon_0}\frac{1}{Z^2+l^2}\cos\theta\hat{z} \tag{3.54}$$

$$= \frac{Q}{4\pi\epsilon_0 L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{Z\,dl}{(Z^2+l^2)^{3/2}}\hat{z} \tag{3.55}$$

The expression in the integration is symmetry to $L = 0$. Therefore the integration can be simplified to be

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{Z\,dl}{(Z^2+l^2)^{3/2}} = 2\int_0^{\frac{L}{2}} \frac{Z\,dl}{(Z^2+l^2)^{3/2}} \tag{3.56}$$

To obtain this integration, substitute $l \to Z\tan\alpha$, where $\pi/2 > \alpha \geq 0$. Then $dl \to (Z/\cos^2\alpha)d\alpha$ and $dl \to (Z/\cos^2\alpha)d\alpha$.

$$\int \frac{Z\,dl}{(Z^2+l^2)^{3/2}} \tag{3.57}$$

$$= \int \frac{Z^2 d\alpha}{Z^3(1+\tan^2\alpha)^{3/2}\cos^2\alpha} \tag{3.58}$$

$$= \int \frac{\cos\alpha}{Z}d\alpha \tag{3.59}$$

$$= \frac{\sin\alpha}{Z} \tag{3.60}$$

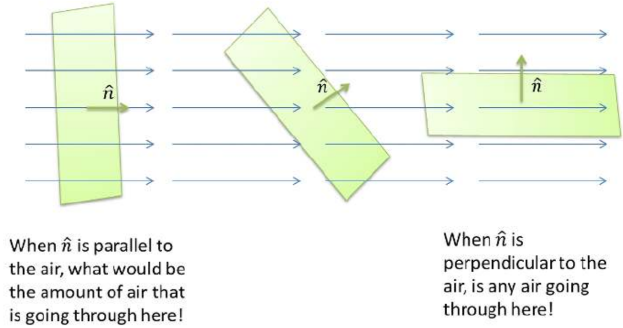
$$= \frac{1}{Z}\sqrt{\frac{\tan^2\alpha}{\tan^2\alpha+1}} \tag{3.61}$$

$$= \frac{1}{Z}\sqrt{\frac{l^2}{l^2+Z^2}} \tag{3.62}$$

Note when $\alpha < 0$, the sign in the last equation will be minus. So the field is,

$$\vec{E}_{tot} = \frac{Q}{4\pi\epsilon_0 L}\frac{2}{Z}\sqrt{\frac{l^2}{l^2+Z^2}}\bigg|_0^{\frac{L}{2}}\hat{z} \tag{3.63}$$

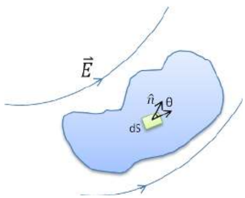
$$= \frac{Q}{2\pi\epsilon_0 Z\sqrt{L^2+4Z^2}}\hat{z} \tag{3.64}$$



4 Gauss's Law

4.1 Concept of Flux

Suppose there is an electric field and we bring in the electric field an open surface like a handkerchief. We can carve the handkerchief into small pieces. Let us call each piece dS. A normal vector $\hat{n}$ pointing perpendicularly out of the surface.



we carve the surface into small pieces. Lets call each piece ds. We can define a normal vector $\hat{n}$ pointing perpendicularly out of the surface. The electric flux at that element, $d\phi$, can be defined as:

$$d\phi = \vec{E} \cdot \hat{n} ds = \vec{E} \cdot d\vec{s} \tag{4.1}$$

The vector $d\vec{s}$ is always perpendicular to the small surface ds. In case there is an angle between the field and the normal surface vector we can write:

$$d\phi = Edscos\theta \tag{4.2}$$

Remember for two vectors $\vec{A}$ and $\vec{B}$ making an angle θ between them,

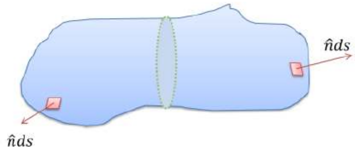
$$\vec{A} \cdot \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| cos\theta \tag{4.3}$$

If we were to find ϕ over the whole surface, we can write:

$$\phi = \int d\phi = \int\limits_{surface} \vec{E} \cdot \hat{n} ds \qquad [\frac{N}{C} m^2] \tag{4.4}$$

Let us consider an air flow to get an intuition of flux. The arrows represent air flow over a surface that is lying in it. Now think of the blue air vectors as electric field. What is $\vec{E} \cdot S\hat{n}$ in the first case? What about the last case?

So far we have done an open surface. Now let us fully close this surface (i.e. like a cube or a sphere). you cannot get in or out of the surface without penetrating through it.



In this case, the normal to the surface is always from the inside towards the outside. It is always uniquely defined (by convention). Now the whole flux becomes,

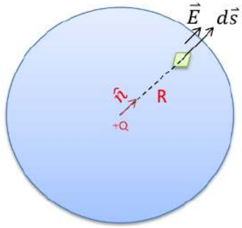
$$\Phi = \int d\Phi = \oint_{surface} \vec{E} \cdot \hat{n} ds \tag{4.5}$$

The circle reminds us that the integral is a closed surface.

What does either case of $\phi > 0$, $\phi < 0$ or $\phi = 0$ mean physically?

What does $\phi = 0$ represents? It would mean that what ever flows in, flows out. *Should there be a charge inside?* If $\phi > 0$, more flows out than flowing into the closed surface. Now, do you think there should be a charge inside? If yes, of what kind?

Now, let us consider a point charge. Lets surround it by a sphere of radius R.

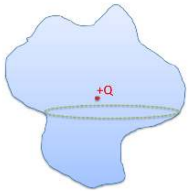


$\vec{E}$ and $d\vec{s}$ are in the same direction! Is $\left|\vec{E}\right|$ same anywhere on the surface? Does $\vec{E} \cdot d\vec{s}$ change anywhere on the surface?

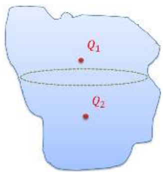
$$\begin{aligned}
\therefore \Phi &= \oint_{surface} \vec{E} \cdot d\vec{s} = \oint_{surface} E ds \\
&= E \oint_{surface} ds = 4\pi R^2 E \\
\text{But} \quad \vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \vec{r} \\
\text{therefore} \quad \Phi &= \frac{Q}{\epsilon_0}
\end{aligned} \tag{4.6}$$

Does this vary with distance R? If you consider flux to be air flow, it is not surprising, is it?

If instead of a sphere we have an arbitrarily shaped surface, even then, $\Phi = \frac{Q}{\epsilon_0}$.



We could have any type of a funny surface and the result would still be the same.
If we have more than one charge, even then,



$$\Phi = \frac{Q_1 + Q_2}{\epsilon_0} \tag{4.7}$$

This brings us to the first Maxwell's equation i.e. *Gauss Law.*

$$\Phi = \oint_{surface} \vec{E} \cdot d\vec{s} = \frac{\sum Q_{enc}}{\epsilon_0} \tag{4.8}$$

A surface over which E is constant is called a Gaussian surface. Such a surface can only be thought of for very symmetrical distributions. There are three types of symmetries, *spherical, cylindrical and flat plane.* We may also be able to use Gauss law using superposition.

Now, let us use the symmetry arguments;

Example 1: Spherical symmetry

We have a very thin spherical shell of radius R with charge Q uniformly distributed over its surface. Remember that uniform distribution is crucial else there is no symmetry!

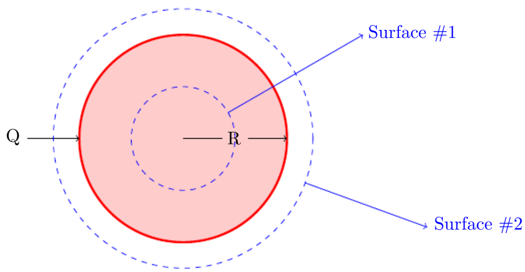


Figure 4.1: Example 4.

So what symmetry does exists? What kind of surface can be used here?
Because of spherical symmetry, we choose a Gaussian surface that is a sphere of radius r, $\vec{E} \parallel d\vec{A}$.

Since the charge distribution looks the same anywhere on a conducting sphere, when we look the $\vec{E}$ field at A or B or from any point on the surface of the sphere it should be the same.

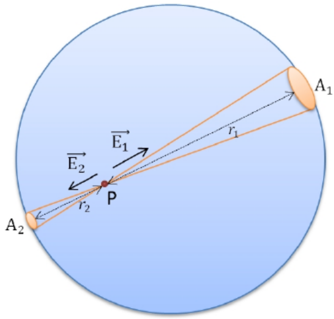
Second symmetry argument is that the electric field should be radially pointing. But that is the same direction as $\hat{n}$,

$$\therefore \vec{E} \cdot d\vec{s} = E\hat{n} \cdot \hat{n}ds = Eds$$

So for $r < R$

$$\oint_{surface} Eds = \frac{Q_{enc}}{\epsilon_0}$$

But for $r < R$, $Q_{enc} = 0$ inside the conductive sphere (i.e. there is no charge inside) $\rightarrow E = 0$ $\rightarrow$ amazing result! Anywhere inside the surface $\vec{E} = 0$! But how can this be true? Consider this,



while $r_2 < r_1$, $A_2 < A_1$ by the same amount.

$\therefore \left|\vec{E}_1\right| = \left|\vec{E}_2\right|$ but opposite. Beautiful result!

Now for $r > R$,

$$\oint_{surface} = \frac{Q_{enc}}{\epsilon_0}$$

$$E \oint ds = \frac{Q_{enc}}{\epsilon_0} \rightarrow Eds \text{ is constant over the sphere}$$

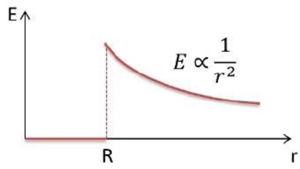
$$\therefore E = \frac{Q_{enc}}{4\pi r^2 \epsilon_0}$$

But now $Q_{enc} = Q$,

$$\therefore \vec{E} = \frac{Q}{4\pi r^2 \epsilon_0}\hat{r} \tag{4.9}$$

This result is as if the charge Q on the surface is at a point in the center. For $r > R \rightarrow$ it does not matter whether the charge has spherical distribution or a point charge! Again a very surprising result!

So for $r > R$, the sphere shell is equivalent to a point charge. So if we were to plot $\vec{E}$ versus r,



Gauss law will only hold true if $E \propto \frac{1}{r^2}$. It will not hold true even if r^2 is actually $r^1.999$ and 2 is not an experimentally significant number. So the question we asked in the first week of the class is answered here.

Example2 Consider an infinite line along $\hat{z}$ with linear charge density ρ, we had solved this previously using integrals. Calculate $\vec{E}$ at point P which is at a distance r away from the line. What symmetry does exists? If we look anywhere around a cylinder, does the problem look the same?

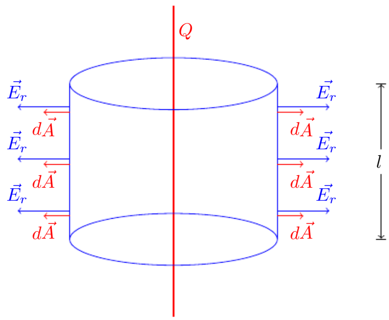
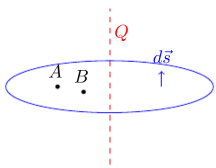


Figure 4.2: Example 2.

So let us consider a cylinder of radius r. Anywhere on the cylinder surface $\vec{E}$ is in the direction of $\hat{n}$ i.e. $\hat{r}$.

$$\therefore \int \vec{E} \cdot d\vec{s} = \int E ds$$

Is this a closed surface? No, we can get in from top and bottom. So let us put some covers on top and bottom as shown here. Will this cause any problems? Will $\left|\vec{E}\right|$ be the same at A and B?



What is $d\vec{s}$ on top cover? $\quad \rightarrow ds\hat{n}$
What is $\vec{E} \cdot d\vec{s}$ at any point?

$$E \perp ds \qquad \therefore \vec{E}\cdot d\vec{s} = 0$$

There is no flux on the top and bottom! Now Gauss law states,

$$\oint \vec{E}\cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$
$$\int\limits_{\substack{cylindrical\\surface}} Eds + \int\limits_{\substack{topor\\bottom}} \vec{E}\cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$

The second integral in the above equation is zero so there is no need to worry about it. What is Q_{enc} ?

$$Q_{enc} = \rho L$$
$$\therefore \int\limits_{\substack{cylindrical\\surface}} Eds = \frac{\rho L}{\epsilon_0}$$
$$E\int ds = \frac{\rho L}{\epsilon_0}$$
$$E 2\pi r L = \frac{\rho L}{\epsilon_0}$$
$$E = \frac{\rho}{2\pi\epsilon_0 r}$$
$$\text{or} \quad \vec{E} = \frac{\rho}{2\pi\epsilon_0 r}\hat{r} \tag{4.10}$$

Check it with the calculation done previously for a continuous charge distribution. Which one is easier? Can we do this method if the line charge was of finite length "L"? Will symmetry hold in that case?

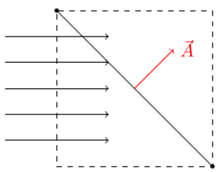
Note on Gauss Law: As mentioned earlier, laws of force are also helpful in visualizing the field patterns (electric, magnetic or any other field).

1. flux (Φ_e) of the electric field $\rightarrow$ "flow" of the field.
2. flux density ($\vec{D}$) (flux or "flow" per unit area).
3. flux density ($\vec{D}$) is related to the electric field intensity ($\vec{E}$) at the point:

$$\vec{D} = \epsilon\vec{E} \tag{4.11}$$

in free-space, $\epsilon = \epsilon_0$; in dielectric medium $\epsilon = \epsilon_0\epsilon_r$.

For example, a fluid flow characterized by a constant flow vector $\vec{v}$

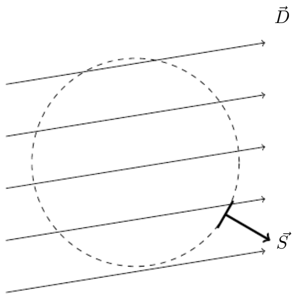


Vector area: it is a vector perpendicular to the surface area, and if it is a planar surface, the magnitude of this vector is equal to the area of the surface; otherwise (in general) its magnitude represents the "differential" surface elements area (dS). The mass flux Φ_v kg/sec is

$$\Phi_v = \rho\vec{v} \cdot \vec{A} \tag{4.12}$$

If in the water flow analysis we replace $\vec{v}$ by the electric flux density $\vec{D}$, we have the concept of flow of electric field.

$$d\Phi_E = \vec{D} \cdot d\vec{S} = \epsilon\vec{E} \cdot d\vec{S} \tag{4.13}$$



$$\Phi_E = \int_S \vec{D} \cdot d\,\vec{S} = \int_S \epsilon\vec{E} \cdot d\vec{S} \tag{4.14}$$

Hence, in the general case the Gauss's law would become:

$$\Phi_E = Q_{enc} \tag{4.15}$$

or

$$\epsilon_0 \oint \vec{E} \cdot d\vec{S} = Q_{enc} \tag{4.16}$$

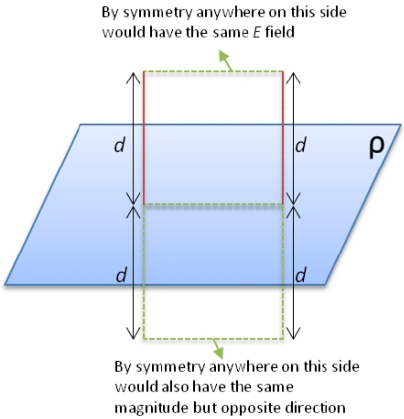
Gauss's law and Coulomb's law are consistent with each other, i.e.,

1. if a electric field solution is obtained by using Coulomb's Law, it will satisfy Gauss's Law automatically everywhere.

2. if the electric field calculation is one by some judicious was of the Gauss's Law, the solution will be identical to that obtained from Coulomb's Law.

Example 3 Field due to an infinite plane of charge. Lets help you set up the problem. The charge distribution is ρ and the plane is infinite.

What should the Gauss surface be? Can we choose a sphere? Sure we can but we would get lost. Now, lets consider the following paths:



So what should the Gaussian surface be?

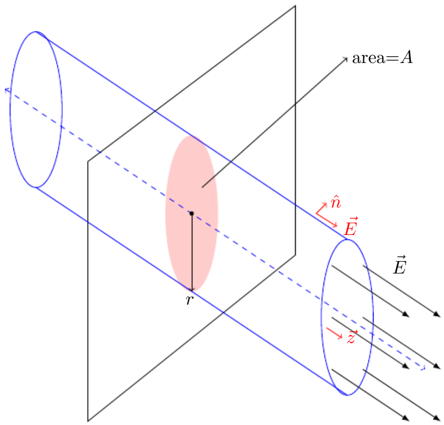


Figure 4.3: Example 3.

Considering the above cylindrical Gaussian surface, what is $\vec{E} \cdot d\vec{s}$ on top, bottom and sides of the cylinder?

Now, can you calculate the $\vec{E}$ field?

I want you to show that,

$$\begin{aligned} \vec{E} &= \frac{\rho}{2\epsilon_0}\hat{z} \qquad\qquad z > 0 \\ &= \frac{\rho}{2\epsilon_0}(-\hat{z}) \qquad z < 0 \end{aligned}$$

Answer: To solve this consider the following symmetries:

1. $\vec{E}$ in front of plane has same magnitude.
2. $\vec{E} \parallel d\vec{A}$ of cylinder top and bottom.
3. $\vec{E}$ points along z.

Using Gauss's law:

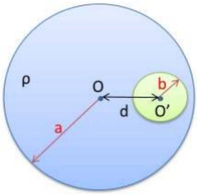
$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q \tag{4.17}$$

$$2(E\pi r^2) = \frac{\pi r^2 \rho}{\epsilon_0} \tag{4.18}$$

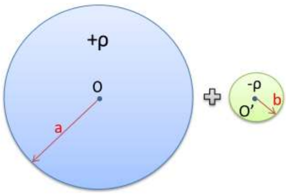
$$E = \frac{\rho}{2\epsilon_0} \tag{4.19}$$

the factor 2 in the second equation is for the bottom and the top of the cylinder.

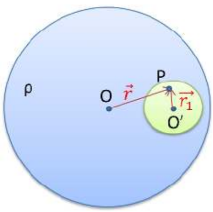
Now let us see if you understand Gauss's law. Charge distribution is shown below. It consists of a uniform charge on a sphere of radius "a" except for a small sphere "b" where it is 0. The sphere centers are d apart. Calculate $\vec{E}$ at any point in the small sphere!



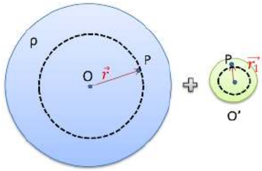
Is there any symmetry? Can we use Gauss law?
So what should we do? By superposition, can we do this,



When we add those two we can get the exact same charge distribution as in the problem. Let's choose an arbitrary point "p".



By decomposing the above in a similar fashion we would get,



In the above condition, is there individual symmetry? So what should the Gaussian surface be? For the bigger sphere (referred to as sphere 1):

$$\oint \vec{E}_i \cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$
$$E_1 4\pi r^2 = \frac{\rho V_{enc}}{\epsilon_0}$$
$$E_1 4\pi r^2 = \frac{\rho}{\epsilon_0}\frac{4}{3}\pi r^3$$
$$E_1 = \frac{\rho r}{3\epsilon_0}$$
$$\text{or in vector format } E_1 = \frac{\rho r}{3\epsilon_0}\hat{r}$$

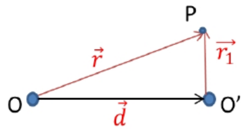
Similarly, for the smaller sphere (referred to as sphere 2)':

$$E_2 = \frac{-\rho r}{3\epsilon_0}\vec{r}_1 = \frac{-\rho r}{3\epsilon_0} r_1 \hat{r_1}$$

$\therefore$ $\vec{E}$ at point P is the superposition of $\vec{E_1}$ and $\vec{E}_2$,

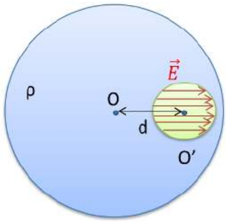
$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho}{3\epsilon_0}(\vec{r} - \vec{r}_1)$$

Now see that,



$$\vec{r} - \vec{r}_1 = \vec{d}$$
$$\therefore \vec{E} = \frac{\rho}{3\epsilon_0}\vec{d}$$

This means that for any point "P" in the hole, $\vec{E}$ is same in magnitude and pointing along $\vec{d}$.

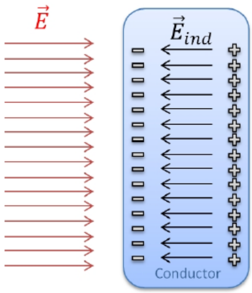


So, a uniform $\vec{E}$ field is achieved! Very surprising result!

Remember: If you find symmetry, Gauss's law is your best choice!

4.2 Conductors in Electrostatic Fields

As discussed before, conductor have "free" electrons. If a field exists inside a conductor, electrons will move, so that the conductor will not be in electrostatic equilibrium. So lets review what happens if we put the conductor inside an external field.



Electrons will feel a force equal to $-e\vec{E}$, so they will travel to one side. The other side has lack of electrons and therefore become positively charged. This would create an induced electric field within the conductor which will oppose the external field $\vec{E}$.

How long will this continue? → Till $E_{ind} = E$ in magnitude.

So what happens to the total electric field inside the conductor? → It should be 0!

$$\therefore \vec{E}_{ind} + \vec{E} = 0$$

No electric field exists inside a conductor! However, a surface charge is created. We will learn how we could calculate that later on!

So when the $\vec{E}$ inside a conductor must be zero, the Q inside must also be zero.
Why? If Q inside the conductor was not zero, let us choose a Gaussian surface enclose some charges.

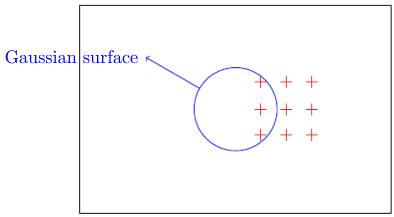


Figure 4.4: Gaussian surface in a conductor.

From Gauss's law we know $\oint \vec{E} \cdot d\vec{A} = q \neq 0$, the field somewhere on the surface must be none zero, which is conflict to the assumption $\vec{E} = 0$ in the conductor.
Here we conclude that the Q inside the conductor must be zero; charge in solid conductor migrates to the surface.

Example 1 Conducting thick sphere shell with charge Q.

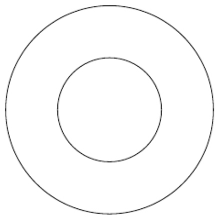


Figure 4.5: Example 1.

Rules:

1. charge cannot be inside conductor.
2. charge cannot be on internal surface. why?
3. charge must go to outside surface.

Why 2? Consider the surface as follows:

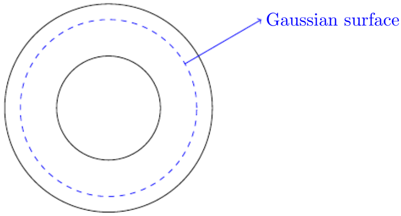
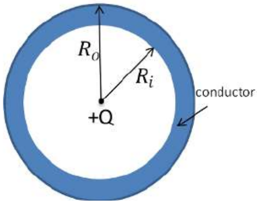


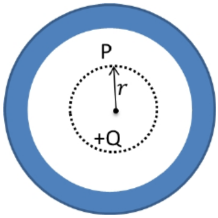
Figure 4.6: Gaussian surface in a sphere shell conductor.

If the charge on inner surface is not zero, the field on the Gaussian surface in not zero somewhere, which can not happen.

Example 2 A thin spherically conductive shell is covering a charge Q inside at its center. Calculate $\vec{E}$ anywhere in space!

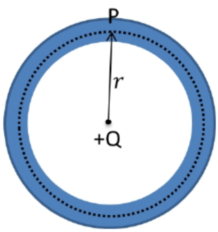


Does this problem present spherical symmetry? Can we use Gauss law?
Case I: Point P is located inside R_i i.e. $r < R_i$. So by Gauss law:



$$E \times 4\pi r^2 = \frac{Q}{\epsilon_0}$$
$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$
$$\text{or in vector form : } \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

Case II: Point P is located within the conductor. $\therefore R_i \leq r \leq R_o$
Can electric field exist inside the conductor? $\therefore E = 0$
But Gauss law states that:



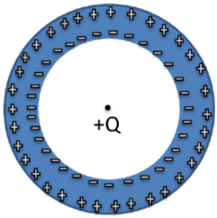
$$E \oint ds = \frac{Q_{enc}}{\epsilon_0}$$

but $E = 0$, therefore $Q_{enc} = 0$. Hence, $-Q$ charge should be induced on the inside boundary of the conductor. This is the surface charge.

$$\text{charge density } \rho = \frac{-Q}{\text{surface area of the sphere}}$$

$$\rho = \frac{-Q}{4\pi R_i^2} \qquad \text{at } r = R_i$$

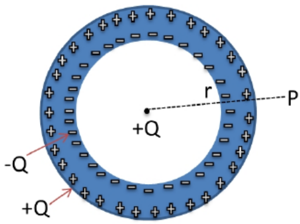
But these electrons have moved within the conductor. So outside boundary of the conductor should also have a charge $+Q$. Hence, the situation looks like this:



So the total charge on the inside surface is $-Q$ and the total charge on the outside surface is $+Q$. Now, what is the charge density ρ_0 on the outside surface?

$$\rho_0 = \frac{Q}{4\pi R_o^2}$$

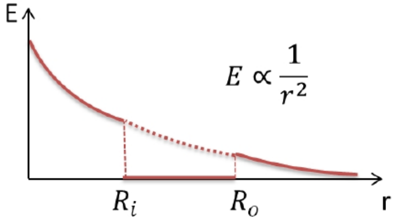
<u>Case III</u>: What is the electric field outside the conductor ($r > R_o$)?



Again, by Gauss law:

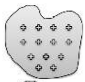
$$\begin{aligned} E 4\pi r^2 &= \frac{Q_{enc}}{\epsilon_0} \\ &= \frac{Q + (-Q) + Q}{\epsilon_0} \\ \therefore E &= \frac{Q}{4\pi\epsilon_0 r^2} \end{aligned}$$

Does it seem as if there was a point charge only?
If we were plotting $\vec{E}$ as r is changed from 0 to ∞, we would get:

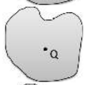


Has the conductor changed anything beyond its boundaries? Doesn't look like it, does it?
Well it has changed a few things which will be apparent later. Let us consider another case.
Example 3: We have a conductor which is arbitrarily shaped. If we touch it with a charged rod, we would release a charge Q on it. Where would the charge go?
Possibilities are here:

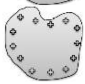
a) charge uniformly distributed over the conductor



b) charge is accumulated in the center



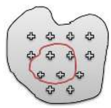
c) charge is uniformly distributed on the surface



d) none of the above!

option (b) is ridiculous! Charges will repel each other. So if free to move, why should they accumulate in the center? in general, (a) and (b) are ruled out by Gauss law. If we draw a closed surface inside the conductor as seen here:

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$



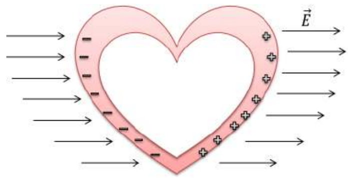
So if the charge was within the conductor, Q_{enc} would have a finite value and then $\vec{E} \cdot d\vec{s}$ will have to exist and thus $\vec{E}$ would exist. But $\vec{E} = 0$ inside the conductor. So the answer cannot be (a) or (b).
So the Gauss law would leave us with no possibility that there is charge outside the conductor. So remember $\rightarrow$ *charge can only distribute over the surface of a conductor; not its volume!*
Therefore, the charge has to be on the surface of the conductor, but would it be uniform?
Before we try to answer this question, let us take another case. Lets look at a heart with a finite thickness. If we release a charge on the outside surface, will there be any charge on the inside part?

Let's see! Using Gauss law,

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$

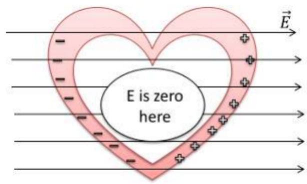
Well, $\vec{E} = 0$ inside the conductor. Hence, $Q_{enc} = 0$. Therefore, there will be no charge on the inside surface! All the charge has to be on the outer surface. Any shape of a conductor would have no charge on the inside surface.

So let us look at a more complex situation. If we take a hollow conducting object and put it in an external electric field, what happens in the space inside? What would be the charge inside that surface?



The charge inside the object has to be zero. What is the field inside there? $\rightarrow$ it has to be zero.

As you bring the hollow object in the electric field, you may get a negative charge on the *outside* surface on one side and a positive charge on the *outside* surface on the other side of the object. But you have *no charge* on the inside surfaces.



E is zero inside the conductor; It is zero inside the cavity. So if you are inside the cavity, you will be electrically isolated from the outside world. This is called electrostatic shielding and such conductor cavities are called *Faraday Cage*.

Example 4 What if uniformly charged sphere was placed inside a thick spherical conducting shell?

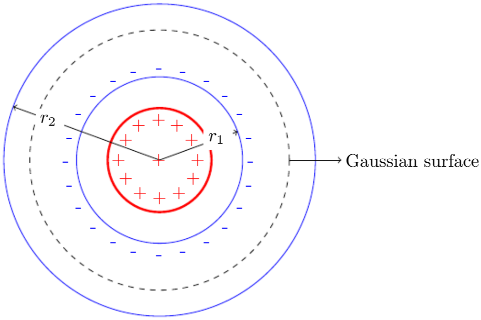


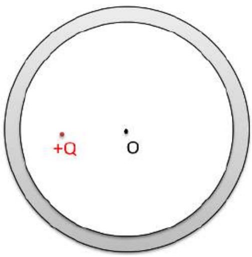
Figure 4.7: Example 2.

Consider the Gaussian surface above. Because of the spherical symmetry, there was a field along the Gaussian surface, $|\vec{E}|$ will be uniform along surface and $\vec{E} \parallel d\vec{A} \Rightarrow \Phi_e = \frac{Q}{\epsilon_0}$. Because the field in the conductor must be zero, $Q = 0$, the charge on internal surface must be zero as well. The surface charge density is

$$
\begin{aligned}
\oint_S \sigma dS =& Q \\
\sigma \cdot (4\pi r_1^2) =& Q \\
\sigma =& \frac{Q}{4\pi r_1^2}
\end{aligned}
$$

Is this similar to having just a point charge in the center instead of a charged sphere?

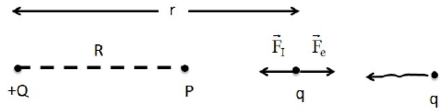
Here is a challenge problem for you! Charge $+Q$ is placed asymmetrically inside a conducting shell! Draw the $\vec{E}$ field lines everywhere in the space.



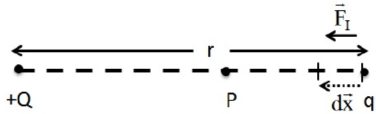
5 The Electric Potential

5.1 Electrostatic Potential Energy (U)

Let us consider two charges q and Q separated by a distance R. In presence of Q, is it required to do some work to bring q to P. Do Q and q repel each other (assuming q is positive)? So, if they repel each other, isn't it like pushing in a spring? If you release the spring you get some energy back.
Similarly, when bringing q to P, there is some energy stored in the system. This energy is Electrostatic Potential Energy.



Let us calculate this energy:
How much work is done to bring q from ∞ to P?
Moving q to a distance r away, there is the force $\vec{F_e}$ acting and we have to apply a force $\vec{F_I}$ apposite to $\vec{F_e}$.
Now, we move q a small distance, dx, as shown below:



Since $\vec{F_I}$ and $\vec{dx}$ are in same direction, it is very clear that we have to do positive work. This work can be obtained as follow:

$$dw = \vec{F}.\vec{dx} = F.dr$$

So, the work done to bring q from ∞ to P can be calculated as:

$$W_I = \int_{\infty}^{R} \vec{F_I}.\vec{dx} = \int_{\infty}^{R} F_I.dx$$

But, $\vec{F_I} = -\vec{F_e}$. So,

$$W_I = -\int_{\infty}^{R} F_e.dx$$

Should we be worried about the path we take? Is electric force a conservative force?

$$\begin{aligned} W_I &= -\int_{\infty}^{R} F_e.dx \\ &= -\int_{\infty}^{R} \frac{Qq}{4\pi\epsilon_0}\frac{dx}{r^2} \\ &= \frac{Qq}{4\pi\epsilon_0}\left[-\frac{-1}{r}\right]_{\infty}^{R} \\ \Rightarrow W_I &= \frac{Qq}{4\pi\epsilon_0 R} \end{aligned} \tag{5.1}$$

So, the work we have done to bring q to $+R$ is given by $\frac{Qq}{4\pi\epsilon_0 R}$. This work has been stored as energy. What is the work done by the electric force, W_e?

$$W_e = -W_I = \frac{-Qq}{4\pi\epsilon_0 R}$$

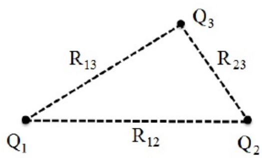
From ECE 105, you know that

$$\Delta u = -W_c$$

Change in potential is negative of the work done by a conservative force.

$$\Delta u = \frac{Qq}{4\pi\epsilon_0 R}$$

Is there a similarity to gravitational potential energy?
If both Q and q are either positive or negative $\Rightarrow$ ΔU is a positive value $\Rightarrow$ the force is repulsive $\Rightarrow$ it acts like a compressed spring.
If one of Q or q is positive or negative $\Rightarrow$ ΔU is a negative value $\Rightarrow$ the force is attractive $\Rightarrow$ it acts like a stretched spring.
What happens if there are more than two charges? In this case, we can add the work done by considering one charge at a time.
Consider the three charges shown here:



Assume that only Q_1 exists and we bring Q_2 to its position. In this case:

$$\Delta u_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$$

Now we bring Q_3 to its position. In this case, work has to be done to move Q_3 against the force from Q_1 and Q_2.

$$\Delta u_3 = \frac{Q_1 Q_3}{4\pi\epsilon_0 R_{13}} + \frac{Q_2 Q_3}{4\pi\epsilon_0 R_{23}}$$

Thus, the total potential energy will be (based on pair-wise addition):

$$\Delta u = \Delta u_2 + \Delta u_3$$
$$\Delta u = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}} + \frac{Q_1 Q_3}{4\pi\epsilon_0 R_{13}} + \frac{Q_2 Q_3}{4\pi\epsilon_0 R_{23}}$$

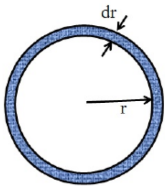
Did you notice that ΔU has no vector form? This is obvious since energy is a scalar quantity.
So, if we have a charge distribution, does the integral for energy become easier?

5.1.1 Example

Energy stored in a uniformly charged sphere e.g. a nucleolus.
A charge Q has been uniformly distributed over a radios R. What is the energy stored in the system?
Charge density, ρ, is:

$$\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3}$$

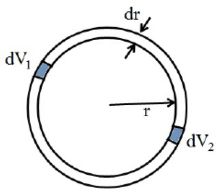
Let us assume that we have already assembled a sphere of radios r and now we want to add a thin shell with thickness of dr.



Charge within this already assembled sphere is:

$$Q' = \rho\frac{4}{3}\pi r^3$$

From Gauss Law we know $\vec{E}$ due to Q' outside of r is the same as if the charge Q' is at the center. Assume two small volume dV_1 and dV_2:



Charge in dV_1 is equal to ρdV_1.
In presence of Q', how much work do we need to assemble ρdV_1?

$$dW_1 = \frac{\rho dV_1}{4\pi\epsilon_0 r} \times Q'$$

Is this work different for dV_2?
What is the differential work we need to assemble a shell with thickness of dr around Q'?

$$dW = \frac{charge\ in\ shell}{4\pi\epsilon_0 r} \times Q'$$

Area of Spherical shell is 4(pi)r^2 needs to be fixed Firas

$$\begin{aligned} Charge\ in\ shell =& \rho \times Volume\ of\ shell \\ =& \rho \times (2\pi r)dr \end{aligned}$$

So:

$$dW = \frac{\rho(2\pi r)dr}{4\pi\epsilon_0 r} \times Q'$$

But, $Q' = \rho\frac{4}{3}\pi r^3$

$$\begin{aligned} dW =& \frac{\rho dr}{2\epsilon_0} \times \rho\frac{4}{3}\pi r^3 \\ dW =& \frac{4\pi}{6\epsilon_0}\rho^2 r^3 dr \end{aligned} \tag{5.2}$$

What is the total work we need to do? We need to change r from 0 to R.

$$\begin{aligned} W =& \frac{4\pi}{6\epsilon_0}\rho^2 \int_0^R r^3 dr \\ W =& \frac{4\pi}{6\epsilon_0}\rho^2 \frac{R^4}{4} = \frac{\pi\rho^2 R^4}{6\epsilon_0} \end{aligned} \tag{5.3}$$

Energy stored in the system is:

$$\Delta U = W = \frac{\pi\rho^2 R^4}{6\epsilon_0}$$

We know:

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

So,

$$\begin{aligned} \Delta U =& \pi(\frac{Q}{\frac{4}{3}\pi R^3})^2 \frac{R^4}{6\epsilon_0} \\ \Rightarrow \Delta U =& \frac{9}{96\pi\epsilon_0 R^2} Q^2 \end{aligned}$$

See how much energy stored in the electron cloud in a typical atom.

Hence to summerize, in Electric field just like in gravity field, we have the concept of potential energy,

1. potential at any point is equivalent to the height of a point in the gravity field.
2. the electric potential energy stored in a system of charges is equal to the work required to assemble the system by bringing the charges to their position from *infinity* or *reference point*.

Electric potential difference between the points is defined as the work done by an external agent to move a unit positive charge from one point to the other in a given electric field. If the potential difference of B with respect to A $(V_B - V_A)$ is:

1. negative $\rightarrow$ electric field will do the work ti complete the movement from A to B.
2. positive $\rightarrow$ external agent will do the work.
3. zero $\rightarrow$ no net work is done.

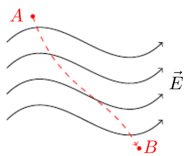


Figure 5.1: The electric field around A and B

We define:

$$(V_B - V_A) = -\frac{W_{AB}}{q} \tag{5.4}$$

W_{AB} is work done by the electric field in moving q_0 from A to B.

$$\Delta W = \vec{F_e} \cdot d\vec{l} = (q_0\vec{E}) \cdot d\vec{l} \tag{5.5}$$

$$W_{AB} = \int_A^B q_0 \vec{E} \cdot d\vec{l} = q_0 \int_A^B (\vec{E} \cdot d\vec{l}) \tag{5.6}$$

hence, $(V_B - V_A) = -\frac{W_{AB}}{q} = -\int_A^B \vec{E} \cdot d\vec{l}$.
Unit: If 1 joule of work is done to move a unit positive charge from A to B, then the potential difference between these two points is said to be 1 volt, i.e.,

$$1 \text{ V} = \frac{1 \text{ J}}{1 \text{ C}} \tag{5.7}$$

5.2 Electric Potential

Using point A as a *reference point* and *arbitrarily* assign a value of zero to it ($V_A \equiv 0$), then we define potential at any other point (B) as:

$$V_B = -\frac{W_B}{q_0} \tag{5.8}$$

1. typically infinity is taken as the reference point of zero potential (for most of the theoretical work at least, explain ground potential as reference in experimental works).
2. potential at any point is defined as the work done (be external agent) to move a unit charge from infinity (or reference point) to that point in the presence of the given electric field.
3. note: potential distribution (or potential map) is a scalar field (i.e., it has magnitude only).

Therom: Path independence. The potential defference $(V_B - V_A)$ is independent of the path followed in moving the test charge from A to B.
Proof: Considering a point charge field firstly.

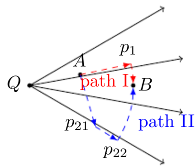


Figure 5.2: Path I is along the electric field direction in the first segment, then perpendicular to the electric field direction. Path II is an arbitrary path other than path I.

Now we prove $\int \vec{E} \cdot d\vec{l}$ doest not depend on path.

$$\int_{\text{II}} \vec{E}(r) \cdot d\vec{l}_{\text{II}} \tag{5.9}$$

$$= \int_{\text{II}} E(r)\hat{r} \cdot \left(dr\hat{r} + rd\theta\hat{\theta}\right) \tag{5.10}$$

$$= \int_{\text{II}} E(r)dr \tag{5.11}$$

$$= \int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon r^2} dr \tag{5.12}$$

$$= \frac{Q}{4\pi\epsilon}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \tag{5.13}$$

We can obtain the integral of path I by simply substitute I with II, and the result does not depend on path.
The above consequence can be generalized to any electric field distribution, since such a distribution may be considered as due to an assembly of many point charges.

1. in a riven electric field, locus of all the points that have the same potential is called an equipotential surface.
2. work done to move a charge on an equipotential surface is zero.
3. two equipotential surface of a given field distribution cannot intersect.
   Example equipotential surfaces of the electric field due to a point charge are concentric spherical surfaces with the charge at the origin.
4. Lines of force (electric field lines) are always directed normal to the equipotential surfaces.

From the example of point charge we know that

$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l} = -\frac{Q}{4\pi\epsilon}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \tag{5.14}$$

In the limit $r_A \to \infty$, potential at a general point $r = r_B$:

$$V = \frac{Q}{4\pi\epsilon_0 r} \tag{5.15}$$

Thus, the equipotential surface of point source field are concentric spheres.
In a multiple charge distributed system, the potential can be obtained by using superposition:

$$V = \sum_{n=1}^{N} V_n = \int dV \tag{5.16}$$

Example 1 Electric Dipole

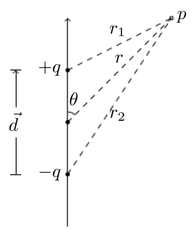
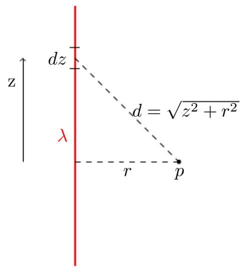


Figure 5.3: Example 1. Electric Dipole Again.

$$\begin{aligned} V =& V_{+q} + V_{-q} & (5.17)\\ =& \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) & (5.18)\\ =& \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r} - \vec{d}/2|} - \frac{1}{|\vec{r} + \vec{d}/2|}\right) & (5.19)\end{aligned}$$



For $r \gg d$,

$$\frac{1}{|\vec{r} \pm \vec{d}/2|} = \frac{1}{\sqrt{(\vec{r} \pm \vec{d}/2)^2}} \tag{5.20}$$

$$\approx \frac{1}{\sqrt{r^2 \pm \vec{r} \cdot \vec{d}/2}} \tag{5.21}$$

$$\approx \frac{1}{r \pm (d\cos\theta)/2} \tag{5.22}$$

$$\approx \frac{r \mp (d\cos\theta)/2}{r^2} \tag{5.23}$$

So the potential is

$$V(r) = \frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2} \tag{5.24}$$

$$= \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \tag{5.25}$$

The equipotential surface of dipole is shown in Figure **??** as the grey lines.

Example 2 Infinity long uniform charged line.

$$V = \int_{-L\to-\infty}^{L\to\infty} dV \tag{5.26}$$

$$= 2\int_0^\infty \frac{\lambda dz}{4\pi\epsilon_0\sqrt{z^2 + r^2}} \tag{5.27}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\int_0^\infty \frac{dz}{\sqrt{z^2 + r^2}} \tag{5.28}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln(z + \sqrt(z^2 + r^2))\Big|_{z=0}^{z=L\to\infty} \tag{5.29}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln(r) + C \tag{5.30}$$

where $C = \frac{\lambda\ln(2L)}{2\pi\epsilon_0}$ is a constant. The potential difference of two point is

$$V_B - V_A = \frac{\lambda}{2\pi\epsilon_0}(-\ln(r_B) + C + \ln(r_A) - C) \tag{5.31}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln(\frac{r_A}{r_B}) \tag{5.32}$$

If $r_B > r_A$, that is, point B is further to the charged line than point A, then $V_B - V_A$ will be a negative quantity for $\lambda > 0$.The field will do the work and the external agent will gain, as expected (unit positive charge will get pushed by the electric field from r_A to r_B). An alternative way to calculate the potential is by the definition. From equation(3.64) we know $\vec{E} = \lambda/(2\pi\epsilon_0 r)$ when $L \to \infty$.

$$\begin{aligned}
&V_B - V_A && (5.33)\\
&= -\int_A^B \vec{E} d\vec{r} && (5.34)\\
&= -\frac{\lambda}{2\pi\epsilon_0}\int_A^B \frac{dr}{r} && (5.35)\\
&= \frac{\lambda}{2\pi\epsilon_0}\ln(\frac{r_A}{r_B}) && (5.36)
\end{aligned}$$

Example 3 The electric potential on the axis of a charged ring.

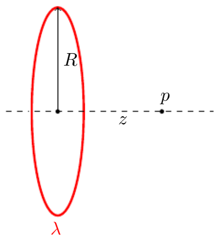


Figure 5.4: Example 3.

Method I:

$$\begin{aligned}
V(z) &= \int dV && (5.37)\\
&= \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi} \frac{Rd\theta}{\sqrt{R^2+z^2}} && (5.38)\\
&= \frac{\lambda}{2\epsilon_0}\frac{R}{\sqrt{R^2+z^2}} && (5.39)
\end{aligned}$$

Method II:

$$\begin{aligned}
V(z) &= V(z) - V(\infty) && (5.40)\\
&= -\int_z^\infty E(z')dz' && (5.41)\\
&= -\frac{\lambda R}{2\epsilon_0}\int_z^\infty \frac{z'dz'}{(z'^2+R^2)^{3/2}} && (5.42)\\
&= \frac{\lambda R}{2\epsilon_0}\frac{1}{\sqrt{z^2+R^2}} && (5.43)
\end{aligned}$$

Example 4 The potential on the axis due to disk.

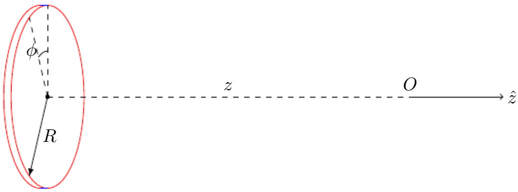


Figure 5.5: Example 4.

Method I:

$$
\begin{aligned}
V(z) &= \int dV && (5.44)\\
&= \int_0^R dr \int_0^{2\pi} r d\phi \frac{\sigma}{4\pi\epsilon_0} \frac{1}{\sqrt{r^2+z^2}} && (5.45)\\
&= \frac{\sigma}{2\epsilon_0} \int_0^R \frac{rdr}{\sqrt{r^2+z^2}} && (5.46)\\
&= \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2+z^2} - z\right) && (5.47)
\end{aligned}
$$

Method II:

$$
\begin{aligned}
V(z) =& V(z) - V(\infty) && (5.48)\\
=& -\int_z^\infty E(z')dz' && (5.49)\\
=& -\frac{\sigma}{2\epsilon_0} \int_z^\infty \left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) dz && (5.50)\\
=& -\frac{\sigma}{2\epsilon_0} \left(z - \sqrt{z^2+R^2}\right)\Bigg|_z^\infty && (5.51)\\
=& \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2+z^2} - z\right) && (5.52)
\end{aligned}
$$

The last equal is from

$$
\begin{aligned}
& \lim_{z\to\infty} \left(z - \sqrt{z^2+R^2}\right) && (5.53)\\
=& -\lim_{z\to\infty} \frac{R^2}{z+\sqrt{z^2+R^2}} && (5.54)\\
=& 0 && (5.55)
\end{aligned}
$$

Example 5 The potential of uniformly charged insulating sphere By applying Gauss's law, we can easily have the function of $\vec{E} = E(r)\hat{r}$

$$
E(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} & \text{r}\geq\text{R} \\ \frac{rQ}{4\pi R^3\epsilon_0} & \text{r<R} \end{cases} \tag{5.56}
$$

where R is the radius of the sphere. The potential is (the reference point is at infinity)

$$
V(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0}\frac{1}{r} & \text{r}\geq\text{R} \\ \frac{Q}{8R\pi\epsilon_0}(3 - \frac{r^2}{R^2}) & \text{r<R} \end{cases} \tag{5.57}
$$

Plot the potential,

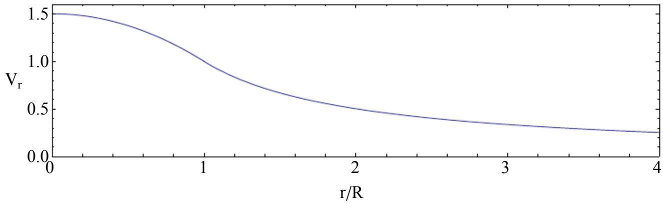
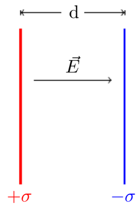


Figure 5.6: The potential of a uniformly charged sphere.

Example 6 Parallel plate capacitor



newline

Figure 5.7: Example 6.

Generally a capacitor satisfies $d \ll S$, where S is the area of the capacitor. Then we have

$$V_B - V_A = -\int_A^B \vec{E} d\vec{r} \tag{5.58}$$

$$= -\frac{\sigma d}{\epsilon_0} \tag{5.59}$$

Example 7 Potential as related to work done. Find the work done if you bring a charge q from infinity to within r of Q.

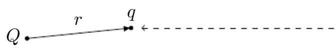


Figure 5.8: Example 7

The work done,

$$W = \int_\infty^{\vec{r}} \vec{F} \cdot d\vec{r} \tag{5.60}$$

$$= \int_\infty^r \frac{Qq \cdot dr}{4\pi\epsilon_0 r^2} \tag{5.61}$$

$$= -\frac{Qq}{4\pi\epsilon_0 r} \tag{5.62}$$

On the other hand,

$$\Delta V q + W = 0 \tag{5.63}$$

So we have

$$V(\vec{r}) - V(\infty)(= 0) = -W/q = \frac{Q}{4\pi\epsilon_0 r} \tag{5.64}$$

5.3 Calculation of E from V

From the examples in the above section, we know that the electric potential can be obtained without learning the $\vec{E}$. If the potential of a system is known, the field can be calculated by solving $\vec{E}$ from

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{l} \tag{5.65}$$

Consider a point $\vec{r}_0$ and another point in its neighborhood $\vec{r}_0 + \lambda\hat{a}$, where λ is parameter, and $\hat{a}$ is a unit vector pointing to an arbitrary direction.

$$V(\vec{r}_0 + \lambda\hat{a}) - V(\vec{r}_0) \tag{5.66}$$

$$= -\int_{\infty}^{\vec{r}_0+\lambda\hat{a}} \vec{E} \cdot d\vec{l} + \int_{\infty}^{\vec{r}_0} \vec{E} \cdot d\vec{l} \tag{5.67}$$

$$= -\int_{\vec{r}_0}^{\vec{r}_0+\lambda\hat{a}} \vec{E} \cdot d\vec{l} \tag{5.68}$$

$$\tag{5.69}$$

in the limit $\lambda \to 0$,

$$\lim_{\lambda\to 0} V(\vec{r}_0 + \lambda\hat{a}) - V(\vec{r}_0) \tag{5.70}$$

$$= -\lim_{\lambda\to 0} \int_{\vec{r}_0}^{\vec{r}_0+\lambda\hat{a}} \vec{E} \cdot d\vec{l} \tag{5.71}$$

$$= -\lambda\vec{E} \cdot \hat{a} \tag{5.72}$$

If we choose $\hat{a}$ as $\hat{x}$,

$$\vec{E} \cdot \hat{x} = E_x = -\frac{V(\vec{r}_0 + \lambda\hat{x}) - V(\vec{r}_0)}{\lambda} \tag{5.73}$$

$$= -\frac{\partial V(x,y,z)}{\partial x} \tag{5.74}$$

and so are the condition of $\hat{a} = \hat{y}$ and $\hat{a} = \hat{z}$.

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} = -\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) V \tag{5.75}$$

The symbol ∇, known as "nabla", is used to substitute $\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)$.

$$\vec{E} = -\nabla V \tag{5.76}$$

6 Capacitors and Dielectrics

6.1 Capacitor

A device for storing electric energy. This is done by storing electric charges of equal and opposite magnitude on two conductors, separated by an insulating medium (dielectric).

1. capacitors are one of the basic passive components of any electronic or electric circuit (other basic passive components are inductors and resistors).

2. if a potential difference (V_{AB}) is impressed between the two plates of a capacitor, $\pm q$ amount of charge will be induced on the plates.

3. it can be shown that for a given structure (conductors surrounded by medium), the ratio (q/V_{AB}) remains constant. i.e.

$$\frac{q}{V_{AB}} = C \tag{6.1}$$

4. this constant is defined as the capacitance of the structure.

5. the unit of capacitance is called Farad,

$$C \text{ Farad} = \frac{q}{V} \frac{\text{Coulomb}}{\text{Volt}} \tag{6.2}$$

6. recall for a point charge

$$V = \frac{q}{4\pi\epsilon_0 r} \Rightarrow \epsilon_0 = \frac{1}{4\pi} \frac{q/V}{r} \frac{F}{m} \tag{6.3}$$

   In general, we use this unit for ϵ_0 (F/m). Generally in practical work, we use the subunits $\mu F = 10^{-6}F$, $nF = 10^{-9}F$, and $pF = 10^{-12}F$

Example 1 Co-axial cable transmission line capacitance.

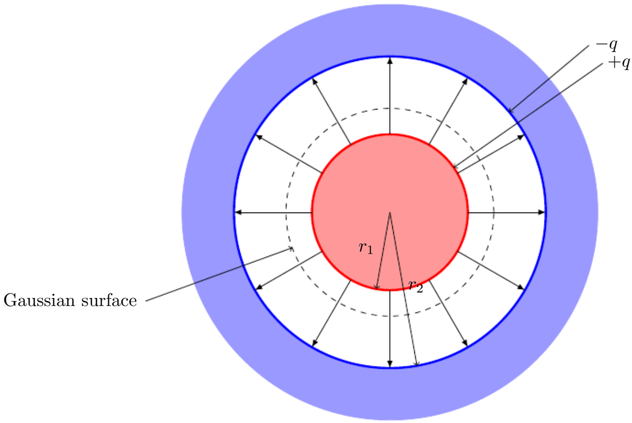


Figure 6.1: Example 1.

Use Gauss's law to find $\vec{E}$ field.

$$\epsilon_0 \oint \vec{E} \cdot d\vec{S} = q \tag{6.4}$$

From the symmetry we see that $\vec{E}$ has a constant magnitude on Gaussian surface and it is normal to the surface.

$$\epsilon_0 E(2\pi r \cdot l) = q \tag{6.5}$$

$$E = \frac{q}{2\pi\epsilon_0 l r} \tag{6.6}$$

Potential difference between the conductors,

$$V_B - V_A = -\int_a^b \vec{E} \cdot d\vec{r} \tag{6.7}$$

$$= -\int_a^b \frac{q}{2\pi\epsilon_0 l} \frac{dr}{r} \tag{6.8}$$

$$= -q2\pi\epsilon_0 \ln(b/a) \tag{6.9}$$

Therefore,

$$C_l = \frac{q}{|V_{AB}|} = \frac{2\pi\epsilon_0}{\ln b/a} \cdot l \tag{6.10}$$

Example 2 Parallel plate capacitor. $S \gg d$. 1) Find $V_B - V_A$. 2) What is the maximum speed a proton will have if accelerated through the gap from rest. $\sigma = 16$ nC/m^2, $p^+ = 1.6 \times 10^{-19}$ C, $m_p = 1.67 \times 10^{-27}$ kg, $\epsilon_0 = 8.85 \times 10^{-12}$ F/m, $d = 4$ mm.

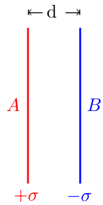


Figure 6.2: Example 2.

1.

$$V_B - V_A = -\vec{E} \cdot \Delta\vec{x} = \frac{\sigma}{\epsilon_0} d \tag{6.11}$$

$$= -\frac{16 \times 10^{-9}}{8.85 \times 10^{-12}} \times (4 \times 10^{-3}) \tag{6.12}$$

$$= -7.23 \tag{6.13}$$

2.

$$\Delta K = -\Delta U \tag{6.14}$$

$$\frac{1}{2} m_p v^2 = -q^+ \delta V \tag{6.15}$$

$$v^2 = \frac{2(1.6 \times 10^{-19} \times 7.23)}{1.67 \times 10^{-27}} v = 3.7 \times 10^4 \tag{6.16}$$

6.2 Energy of capacitor

The energy of capacitor can be calculate by analysing the following process,

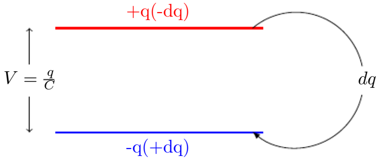


Figure 6.3: Moving dq from one plate to another changes the energy.

Work done to move dq from upper plate to lower plate is

$$dW = V dq = \frac{q}{C} dq \tag{6.17}$$

Hence, the work done by the capacitor to discharge from Q to 0 is,

$$W = \int_0^Q dW \tag{6.18}$$

$$= \frac{1}{C} \int_0^Q q dq \tag{6.19}$$

$$= \frac{1}{2} \frac{Q^2}{C} \tag{6.20}$$

So the energy stored in a capacitor with capacitance C is,

$$E_C = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C V^2 \tag{6.21}$$

or more generally,

$$E_C = \int_V \frac{1}{2} \epsilon E(r)^2 d\vec{r} \tag{6.22}$$

This equation also holds for *any* electric field distribution.

Example 1 Energy stored around an isolated charged conducting sphere.

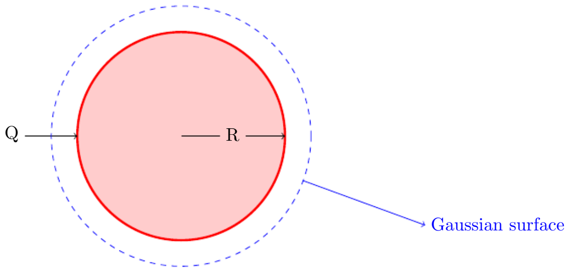


Figure 6.4: Example 1.

From the symmetry we know $\vec{E}$ and $\hat{r}$ are parallel.

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \tag{6.23}$$

The energy

$$
\begin{aligned}
U &= \int\limits_V \frac{1}{2}\epsilon_0 E(r)^2 d\vec{r} && (6.24)\\
&= \int_R^\infty \int_0^{2\pi} \int_0^\pi r^2 \sin\theta \frac{1}{2}\epsilon_0 \frac{Q^2}{(4\pi\epsilon_0 r^2)^2} \cdot d\phi d\theta dr && (6.25)\\
&= \int_R^\infty \frac{Q^2}{8\pi\epsilon_0} \frac{dr}{r^2} && (6.26)\\
&= \frac{Q^2}{8\pi\epsilon_0 R} && (6.27)
\end{aligned}
$$

6.3 Capacitor in Circuit

Link a capacitor to a voltage source.

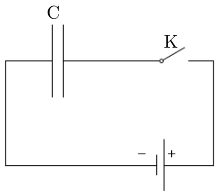


Figure 6.5: A capacitor in a circuit

Close K. Initially, electron will flow until the fully charged capacitor becomes equilibrium. At equilibrium, the voltage difference across the capacitor should equal the voltage of the source. Therefore, the charge on the capacitor is $q = E/C$.

Consider multi-capacitor in a circuit

1. Capacitors in series,

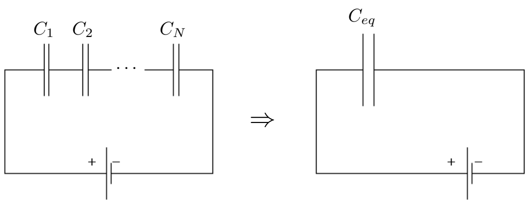


Figure 6.6: Capacitors in series.

$$
\begin{aligned}
V &= V_1 + V_2 + \ldots + V_N && (6.28)\\
&= \frac{q}{C_1} + \frac{q}{C_2} + \ldots + \frac{q}{C_N} && (6.29)\\
&= \frac{q}{C_{eq}} && (6.30)
\end{aligned}
$$

where $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}$.
The seconde equality comes from the fact that the charge on every capacitor equals. This is because the island between two capacitors is isolated from rest of the circuit:

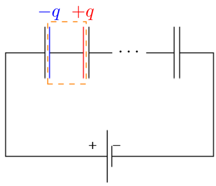


Figure 6.7: Charges on an island is zero.

The total charge of the island is always 0. Therefore neighbor capacitors must have equal charge.

2. Capacitors in parallel.

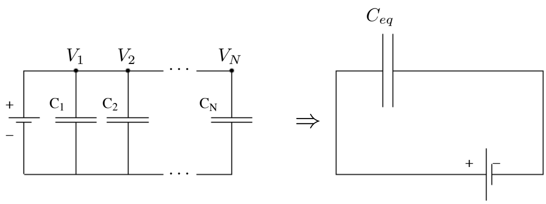


Figure 6.8: Capacitors in parallel.

$$V_{eq} = V_1 = V_2 = \ldots = V_N \tag{6.31}$$
$$q_{eq} = q_1 + q_2 + \ldots + q_N \tag{6.32}$$
$$C_{eq} = \frac{q}{V} = \frac{C_1 V + C_2 V + \ldots + C_N V}{V} = C_1 + C_2 + \ldots + C_N \tag{6.33}$$

6.4 Dielectrics

If the medium is not vacuum, we have to substitute ϵ_0 with ϵ.
Example 1 The capacitance of parallel plate capacitance.

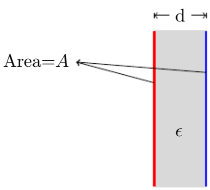


Figure 6.9: Example 1.

$$C = \frac{q}{\Delta V} = \frac{\sigma A}{\frac{\sigma}{\epsilon} d} = \frac{\epsilon A}{d} = \frac{\epsilon_0 \epsilon_r A}{d} = \epsilon_r C_0 \tag{6.34}$$

Experimentally, one can observe that parallel plate whose capacitance is $C_0 = \epsilon_0 A/d$ in free space. Increases its capacitance to $C = \epsilon A/d$ when a material of dielectric constant (or relative permittivity) ϵ_r fills the space between the plate. Since by definition, $C = q/V$, it follows that for the same charge q on the plates, the voltage between the plates,

$$V_{dia} = \frac{V_{vac}}{\epsilon_r} \tag{6.35}$$

We know that the electric field and the potential in the inter plate region

$$E = \frac{V}{d} \tag{6.36}$$

So we can write $E_{dia} = E_{vac}/\epsilon_r$, i.e. the field intensity is reduced inside the dielectric.
There is no free electrons in an ideal dielectric. In real life, lossy dielectric have some free electrons, but negligible compared to the number in conductors. In dielectrics, electrons are strongly bound to the parent atoms (material is electrically neutral).

Figure 6.10: polarization

This phenomenon of "charge-center" displacement is called *polarization.*

At macroscopic level polarization can be observed due to these microscopic level phenomena,

1. electronic polarization ($\propto$ Electric field strength)

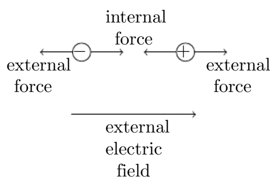


Figure 6.11: Electronic polarization.

Equilibrium: internal force=external force.

2. ionic polarization ($\propto$ Electric field strength)
   Charge center displacement occurs in polar molecules composed of positively and negatively charged ions.

3. orientation polarization.

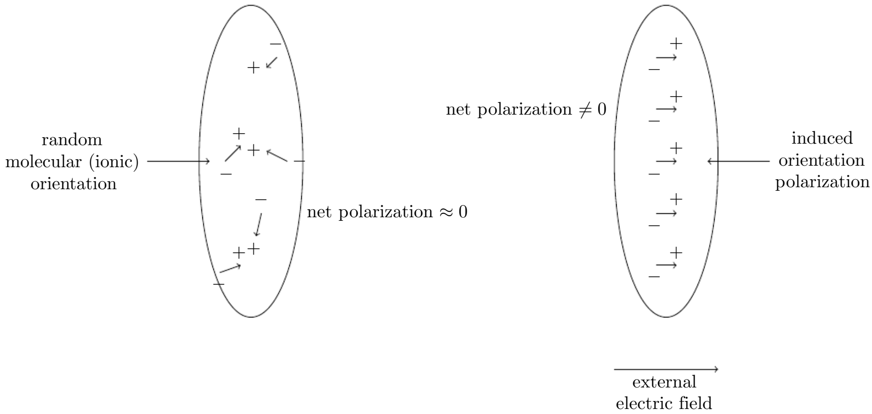


Figure 6.12: Orientation polarization.

> In most dielectrics, when subjected to an external electric field, a combination of these polarization effects take place.

Consider a block of dielectric subjected to an externally applied field, E_0:

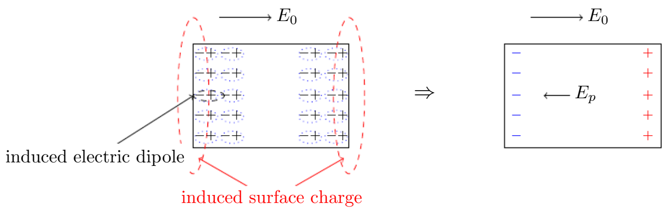
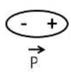


Figure 6.13: A block of dielectric in an external electric field/

In this case, while the electrons are not free, the atoms will still get polarized creating induced electric dipoles as shown in the above figure. Slab as a whole remains electrically neutral. Shift (stretch/orientation) in charge centers (polarization effects) causes a "pick-up" of induced surface charge (note: however these are bound charges and normally they cannot leave the dielectric material). Inside the material, the bound charge layer can be seen as setting up the opposing polarization field $\vec{E_p}$. Hence, $\vec{E}_{diel} = \vec{E}_0 + \vec{E}_p$. Thus, the net effect of polarization is to weaken the field inside a dielectric: $E_{diel} = E_0.\epsilon_r$.

Consider one atom of dipole:



There will be a dipole moment, $\vec{p}$, created which is a quantum mechanical property. We define a macroscopic property called polarized vector, $\vec{P}$, as:

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{P}}{\Delta v} \quad (\frac{C}{m^2}) \tag{6.37}$$

The above formula express the average of numbers and strength of the dipoles which presents per unit volume and show how well they align.
Every material has a unique $\vec{P}$ vector. $\vec{P}$ is created due to polarization by external electric field, $\vec{E}$. Hence, at least to first order, $\vec{P}$ should be proportional to $\vec{E}$. Stronger $\vec{E}$ should lead to a higher polarization.
Typically we have,

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{6.38}$$

where χ_e is the electrical susceptibility. (How susceptible the material is to be polarized)
We introduces a new quantity, called displacement vector or the electric flux density, $\vec{D}$, to model the polarization phenomenon mathematically,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{6.39}$$

$\vec{P}$ is the polarization vector, representing the polarization effects.

$$\begin{aligned}
\vec{D} =& \epsilon_0 \vec{E} + \vec{P} && (6.40)\\
=& \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} && (6.41)\\
=& \epsilon_0 (1 + \chi_e) \vec{E} && (6.42)\\
=& \epsilon_0 \epsilon_r \vec{E} && (6.43)
\end{aligned}$$

where $\epsilon_r = 1 + \chi_e$ is called relative permittivity and give by the data sheet of different material.
We can write:

$$\begin{aligned}
\vec{D} =& \epsilon_0 \epsilon_r \vec{E} && (6.44)\\
=& \epsilon \vec{E} && (6.45)\\
& && (6.46)
\end{aligned}$$

where $\epsilon = \epsilon_0 \epsilon_r$.

6.5 Gauss's Law for Dielectrics and Dielectric-Filled Capacitors

New Gauss's law writes,

$$\oint \vec{D} \cdot d\vec{s} = Q \tag{6.47}$$

or

$$\epsilon \oint \vec{E} \cdot d\vec{s} = Q \tag{6.48}$$

This is the actual general Gauss's law.

$$\begin{aligned}
\oint \vec{E} \cdot d\vec{s} =& \frac{Q_{enc}}{\epsilon} && (6.49)\\
\oint \vec{E} \cdot d\vec{s} =& \frac{Q_{force} + Q_{induced}}{\epsilon} && (6.50)\\
& && (6.51)
\end{aligned}$$

where, Q_{force} is the charge we put in the system, and $Q_{induced}$ is the charge get gets induced.
In point form:

$$\bigtriangledown \cdot \vec{E} = \frac{\rho}{\epsilon} + \frac{\rho_v}{\epsilon} \tag{6.52}$$

But, $\rho_v = - \bigtriangledown \cdot \vec{P}$, So:

$$\bigtriangledown \cdot \vec{E} = \frac{\rho}{\epsilon} - \frac{\bigtriangledown \cdot \vec{P}}{\epsilon} \tag{6.53}$$

$$\bigtriangledown \cdot (\epsilon \vec{E} + \vec{P}) = \rho \tag{6.54}$$

But, $\vec{D} = \epsilon \vec{E} + \vec{P}$. So, Gauss Law becomes:

$$\bigtriangledown \cdot \vec{D} = \rho \tag{6.55}$$

Or,

$$\oint \vec{D} \cdot d\vec{s} = Q_{enc} \tag{6.56}$$

this charge is now the force charge.

$$\vec{D} = \epsilon \vec{E} \tag{6.57}$$

Effects of induced charge is observed here.
$\vec{D}$ is material independent. Effects of material on $\vec{E}$ is accumulated for in ϵ.
We can calculated $\vec{D}$ without worrying about the material and then calculate as $\frac{\vec{D}}{\epsilon}$.

- What is ϵ_r for a perfect conductor?

$\epsilon_r = \infty \qquad E = \frac{D}{\epsilon_0 \epsilon_r} = \frac{D}{\infty} = 0$

Example 1
Parallel plate capacitors with a dielectric slab between two plates. (situation 1)

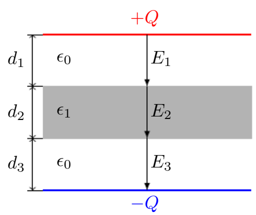


Figure 6.14: Example 1.

And the cross section is A m^2.
Charge density (on the plates) $= \frac{Q}{A}$. Hence, by using Gauss's law,

$$E_1 = \frac{Q}{\epsilon_0 A} \tag{6.58}$$

$$E_2 = \frac{E_1}{\epsilon_r} = \frac{Q}{\epsilon_0 \epsilon_r A} \tag{6.59}$$

$$E_3 = E_1 \tag{6.60}$$

Potential difference,

$$V_{diff} = V_+ - V_- \tag{6.61}$$
$$= -\int_{x=0}^{d} \vec{E} \cdot d\vec{d} \tag{6.62}$$
$$= \int_0^{d_1} E_1 dx + \int_0^{d_2} E_2 dx + \int_0^{d_3} E_3 dx \tag{6.63}$$
$$= \frac{Q}{A}\left(\frac{d_1}{\epsilon_0} + \frac{d_2}{\epsilon} + \frac{d_3}{\epsilon_0}\right) \tag{6.64}$$

Capacitance,

$$C_{eq} = \frac{Q}{V_{diff}} \tag{6.65}$$
$$= \frac{Q}{\frac{Q}{A}\left(\frac{d_1}{\epsilon_0} + \frac{d_2}{\epsilon} + \frac{d_3}{\epsilon_0}\right)} \tag{6.66}$$
$$\frac{1}{C_{eq}} = \frac{1}{\frac{\epsilon_0 A}{d_1}} + \frac{1}{\frac{\epsilon A}{d_2}} + \frac{1}{\frac{\epsilon_0 A}{d_3}} \tag{6.67}$$
$$= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \tag{6.68}$$

Example 2
Parallel plate capacitors with a dielectric slab between two plates. (situation 2)

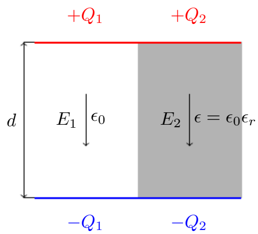


Figure 6.15: Example 2.

And the cross section for the left part is A_1 and for the right part is A_2
The potential of the left part is

$$V_1 = \int_0^d E_1 dx = d \cdot E_1 \tag{6.69}$$
$$V_2 = \int_0^d E_2 dx = d \cdot E_2 \tag{6.70}$$

The potential of the same plate of course is the same. Therefore we must have $E_1 = E_2$. On the other hand,

$$E_1 = \frac{Q}{\epsilon_0 A_1} \Rightarrow Q_1 = \epsilon_0 E_1 A_1 \tag{6.71}$$
$$E_2 = \frac{Q}{\epsilon A_2} \Rightarrow Q_2 = \epsilon E_2 A_2 = \epsilon E_1 A_2 \tag{6.72}$$
$$\tag{6.73}$$

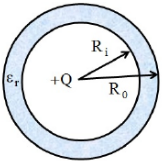
Net charge on the plates,

$$Q = Q_1 + Q_2 = (\epsilon_0 A_1 + \epsilon A_2)E_1 \tag{6.74}$$

Capacitance

$$C_{eq} = \frac{Q}{V} = \frac{\epsilon_0 A_1 + \epsilon A_2}{d} \tag{6.75}$$

newline *Example 3*
A point charge is surrounded by a dielectric spherical shell of ϵ_r with an inner radios R_i and outer radios R_0. Calculate $\vec{D}$, $\vec{E}$, V and $\vec{P}$ as a function of R.



For $R \geq R_0$

$$\oint \vec{D} \cdot d\vec{s} = Q_{enc} = Q$$

$$D = \frac{Q}{4\pi R^2}$$

$$\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{R}$$

$$\vec{D} = \frac{Q}{4\pi R^2}\hat{R}$$

$$\vec{P} = \vec{D} - \epsilon_0\vec{E} = 0$$

$$V = -\int_{-\infty}^{R} \vec{E} \cdot d\vec{R} = \frac{Q}{4\pi\epsilon_0 R}$$

For $R_i < R < R_0$

$$\oint \vec{D} \cdot d\vec{s} = Q_{enc} = Q$$

$$D = \frac{Q}{4\pi R^2}\hat{R}$$

$$\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{\vec{D}}{\epsilon_0\epsilon_r} = \frac{Q}{4\pi\epsilon_0\epsilon_r R^2}\hat{R}$$

$$\vec{P} = \vec{D} - \epsilon_0\vec{E} = (1 - \frac{1}{\epsilon_r})\frac{Q}{4\pi R^2}$$

$$\begin{aligned} V &= -\int_{-\infty}^{R} \vec{E} \cdot d\vec{R} \\ &= -\int_{-\infty}^{R_0} \vec{E} \cdot d\vec{R} - \int_{R_0}^{R} \vec{E} \cdot d\vec{R} \\ &= V_1|_{R=R_0} - \frac{Q}{4\pi\epsilon}\int_{R_0}^{R} \frac{1}{R^2} \cdot d\vec{R} \\ &= \frac{Q}{4\pi\epsilon_0 R_0} + \frac{Q}{4\pi\epsilon_0\epsilon_r}[\frac{1}{R} - \frac{1}{R_0}] \\ V &= \frac{Q}{4\pi\epsilon_0}[(1 - \frac{1}{\epsilon_r})\frac{1}{R_0} + \frac{1}{\epsilon_r R}] \end{aligned} \tag{6.76}$$

For $R < R_i$

$$\oint \vec{D} \cdot d\vec{s} = Q$$

$$D = \frac{Q}{4\pi R^2}$$

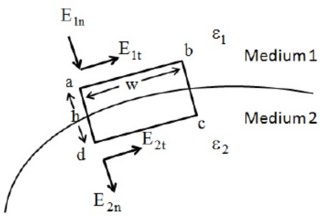
$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$\vec{P} = \vec{D} - \epsilon\vec{E}$$

$$\begin{aligned} V &= -\int_{-\infty}^{R} \vec{E} \cdot d\vec{R} \\ &= -\int_{-\infty}^{R_i} \vec{E} \cdot d\vec{R} - \int_{R_i}^{R_0} \vec{E} \cdot d\vec{R} \\ &= \frac{Q}{4\pi\epsilon_0}[(1 - \frac{1}{\epsilon_r})\frac{1}{R_0} - (1 - \frac{1}{\epsilon_r})\frac{1}{R_i} + \frac{1}{R}] \end{aligned} \tag{6.77}$$

As can be seen $\vec{E}$ has not changes from point source but V has changed. Why ?

6.6 Boundary Condition for Electrostatic Fields



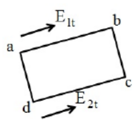
$$\int_{abcda} \vec{E} \cdot d\vec{l} = V_a - V_b = 0$$

We want to calculate E_{2t}, E_{2n} given E_{1n}, E_{1t} as we go from medium 1 with ϵ_1 to medium 2 with ϵ_2.

$$\int_{abcda} \vec{\ }$$

this has been changed to va-va was vb -va this is now ok Firas

$$E_{1t}w + E_{2t}(-w) = 0$$



$$\Rightarrow E_{1t} = E_{2t}$$

The tangential component of the field remains the same.

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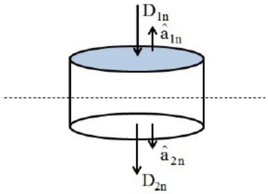
text was ok, but diagram off
diagram now ok
Firas



$$\oint \vec{D} \cdot d\vec{s} = \rho_s \Delta s$$

$$D_{2n} - D_{1n} = 0$$
$$\epsilon_2 E_{2n} = \epsilon_1 E_{1n}$$

So the boundary conditions are:

$$E_{1t} = E_{2t}$$
$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

There is a discontinuity in the second equation.
In general,

$$D_{2n} - D_{1n} = \rho_s$$

Example 1
In medium 1, E_1 makes an angle α_1 with the normal axis. Calculate E_2 and α_2.

Decompose E_1 into E_{1n} and E_{1t}.

$$\epsilon_2 E_{2n} = \epsilon_1 E_{1n}$$
$$\epsilon_2 E_2 cos(\alpha_2) = \epsilon_1 E_1 cos(\alpha_1) \qquad (1)$$

Also,

$$E_{2t} = E_{1t}$$
$$E_2 sin(\alpha_2) = E_1 sin(\alpha_1) \qquad (2)$$

So, $\frac{(2)}{(1)}$ gives us:

$$\frac{tan(\alpha_2)}{\epsilon_2} = \frac{tan(\alpha_1)}{\epsilon_1}$$
$$\Rightarrow tan(\alpha_2) = \frac{\epsilon_2}{\epsilon_1} tan(\alpha_1)$$

$$E_2 = \sqrt{E_{2t}^2 + E_{2n}^2} = \sqrt{(E_2 sin\alpha_2)^2 + (E_2 cos\alpha_2)^2}$$
$$= \left[(E_1 sin\alpha_1)^2 + (\frac{\epsilon_1}{\epsilon_2} E_1 cos\alpha_1)^2\right]^{\frac{1}{2}}$$
$$= E_1 \left[sin^2\alpha_1 + (\frac{\epsilon_1}{\epsilon_2})^2 cos^2\alpha_1\right]^{\frac{1}{2}}$$

7 Current, Current Density and Resistance

Electric current in static case (time independent or DC) is defined as

$$I = \frac{Q}{t} \tag{7.1}$$

unit:

$$\text{Ampere(A)} = \frac{\text{Coulomb}}{\text{second}}\left(\frac{C}{s}\right) \tag{7.2}$$

For a time dependent situation (dynamic case),

$$i = \frac{dq}{dt} \tag{7.3}$$

Note: Dynamic current and AC current are not the same. AC is a special case of dynamic current that involves only periodic oscillating change with time, which leads to frequency idea. Current (I or i) is a scalar quantity, while current density ($\vec{J}$) is a vector concept and it gives us the microscopic model of the "rate of flow" of charge.

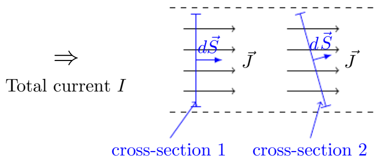


Figure 7.1: Current through different cross section.

1. for area A_1, $I = JA$ or $J = \frac{I}{A}$ where $J = |\vec{J}|$
2. for area A_2,

$$I = \int_{A_2} (\vec{J} \cdot d\vec{S}) \tag{7.4}$$

$$= \int_{A_2} JdS\cos\theta \tag{7.5}$$

Direction of $\vec{J}$ is defined as the direction in which a positive charge (if available) will move. That is,

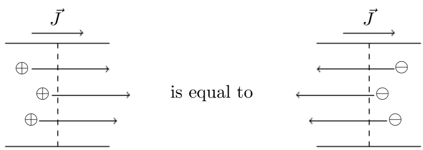


Figure 7.2: Moving a negative charge is equal to moving a costive charge in opposite direction.

Note: Postiche charges move in the direction of $\vec{E}$ field (decreasing potential), and negative charges move in the direction opposite to the $\vec{E}$ direction (increasing potential).
In a material with both positive and negative free charges (carriers) subjected to an electric field, we will have

$$\vec{J}_{tot} = \vec{J}_+ + \vec{J}_- \tag{7.6}$$

In a perfect conductor (metal) there is only one type of carrier free electrons ($-ine$ charge). Total charge in the sample of length l,

$$Q = (nAl)e \tag{7.7}$$

If this amount of charge goes through the cross-section in t sec, then by definition (static case),

$$I = \frac{Q}{t} = \frac{(nAl)e}{t} \tag{7.8}$$

$$= nAe \cdot v_d \tag{7.9}$$

Note:$fraclt = v_d$ is the *drift-speed* of the electron in a given conducting material in the presence of a given electric field (a potential difference). Typical drift speed is $\sim 10^4$ m/s^2, while electron's random motion speed $\sim 10^6$ m/s^2. In a general vector form,

$$\vec{v}_d = \frac{\vec{J}}{ne} \tag{7.10}$$

7.1 Resistance and resistivity

Between the surface A and B of the sample, we have V and I (macroscopic). At a point in the sample, we have $\vec{E}$ and $\vec{J}$ (microscopic). Resistance (Ohm's law):

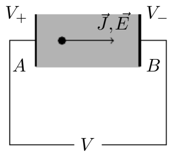


Figure 7.3: Current through a resistor.

$$R = \frac{V}{I} \tag{7.11}$$

$$\text{Ohm} = \frac{\text{Volt(V)}}{\text{Ampere(A)}} \tag{7.12}$$

Resistivity (ρ) is the microscopic or the point-form of the resistance. It is a characteristic property of a material. For an isotropic material, by definition:

$$\rho = \frac{\vec{E}}{\vec{J}} \tag{7.13}$$

$$\sigma = \frac{1}{\rho} \tag{7.14}$$

unit (ρ):

$$\Omega \cdot \text{m} = \frac{\text{V/m}}{\text{A/m}^2} \tag{7.15}$$

$$\tag{7.16}$$

in vector form

$$\vec{J} = \frac{1}{\rho}\vec{E} = \sigma\vec{E} \tag{7.17}$$

Consider a thin cylinder of length l, and cross-section A

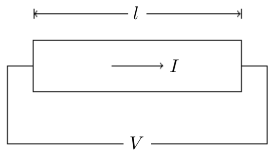


Figure 7.4: Current through a resistor.

If A is small, we can make uniform current density and electric field assumption $E = V/l$, and $J = I/A$, hence,

$$\rho = \frac{E}{J} = \frac{V/l}{I/A} = \frac{V}{I}\frac{A}{l} \tag{7.18}$$

$$\Rightarrow R = \frac{V}{I} = \frac{\rho l}{A} \tag{7.19}$$

In general, if $\vec{E}$ and $\vec{J}$ were not uniform,

$$R = \frac{V}{I} = \frac{-\int_a^b \vec{E}\cdot d\vec{l}}{\int_S \vec{J}\cdot d\vec{S}} \tag{7.20}$$

7.2 Temperature Dependence

Resistivity of any material is a function of temperature, i.e., the second order approximation of resistivity ρ is a linear function of T,

$$\rho(T) = \rho_0 \left[1 + \alpha_T (T - T_0)\right] \tag{7.21}$$

here, ρ_0 is resistivity of the material at the reference temperature (T_0); α_T is the average temperature coefficient of resistivity (typically a constant over a range of temperature). Similarly,

$$R(T) = \rho(T)\frac{l}{A} = R_0 \left[1 + \alpha_T (T - T_0)\right] \tag{7.22}$$

7.3 Energy and Power Consumption

Say in time dt, dq amount of charge is transferred from the + electrode to the − electrode through the power consuming device.

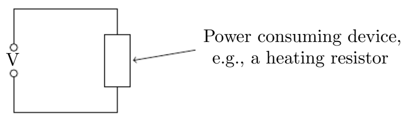


Figure 7.5: Resistor can consume energy.

Potential energy lost by the source in this process must have been consumed by the device (conservation of energy), i.e.,

$$dU = dq \cdot V_{ab} = (idt) \cdot V_{ab} \tag{7.23}$$

or

$$\frac{dU}{dt} = iV_{ab} \tag{7.24}$$

By definition, power is consumed (or flow) per unit time, i.e. $P = \frac{dU}{dt} = iV_{ab}$. If we use resister for power consumption

$$P = iV_{ab} = i(iR) = i^2 R \tag{7.25}$$

$$= \frac{V_{ab}}{R} V_{ab} = \frac{(V_{ab})^2}{R} \tag{7.26}$$

8 The Magnetic Field

8.1 Magnetic Field Produced by Electric Current

Recall, magnetic charges do not exist.

- Any electric current (moving electric charge) produces magnetic field ($\vec{B}$) around itself.
- Similar to Coulomb's law, for magnetic fields we have Biot-Savart law which can be used to calculate magnetic field ($\vec{B}$) produced by a current element (a source concept, similar to a point charge).

8.2 Biot-Savart Law

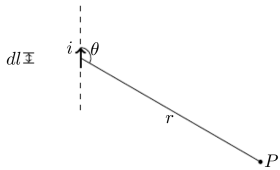


Figure 8.1: Magnetic field generated by dl.

At the observation point P,

$$dB = \frac{\mu_0}{4\pi} \frac{idl}{r^2} \sin\theta \tag{8.1}$$

Direction of $\vec{B}$ has to be specified. In vector form Biot-Savart law,

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{id\vec{l} \times \hat{r}}{r^2} \text{ T, tesla} \tag{8.2}$$

where μ_0 is the permeability of free-space (vacuum). It is a parallel concept to ϵ_0 permittivity.

$$\mu_0 = 4\pi \times 10^{-7} \ \frac{\text{T} \cdot \text{m}}{\text{A}} \tag{8.3}$$

For a general current carrying conductor,

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{id\vec{l} \times \hat{r}}{r^2} \tag{8.4}$$

here the integral is over the entire carrying conductor.

Example 1 Magnetic field due to a straight long current carrying.

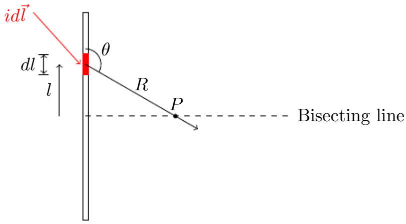


Figure 8.2: Example 1.

Note: This direction will be the same for every current element located along the wire. So the vector summation of magnetic field contribution at P from all the current elements (l from $-\infty$ to $+\infty$) will be simply a scalar sum (this is not a general rule). From Biot-Savart Law,

$$dB = \frac{\mu_0}{4\pi}\frac{idl\sin\theta}{R^2} \tag{8.5}$$

$$B = \frac{\mu_0 i}{4\pi}\int_{-\infty}^{+\infty}\frac{dl\sin\theta}{R^2} \tag{8.6}$$

Note: θ and R are functions of l, since l is the integration variable, $\sin\theta$ and R must be represented in terms of l. From the geometry, $R = \sqrt{l^2 + r^2}$ and $\sin\theta = \frac{r}{\sqrt{l^2+r^2}}$,

$$B = \frac{\mu_0 i}{4\pi}\int_{-\infty}^{+\infty}\frac{rdl}{(l^2+r^2)^{3/2}} \tag{8.7}$$

$$= \frac{\mu_0 i}{4\pi}2\int_{0}^{\infty}\frac{rdl}{(l^2+r^2)^{3/2}} \tag{8.8}$$

$$= \frac{\mu_0 i}{2\pi r} \tag{8.9}$$

Note: From the direction of the magnetic field in this example, it is seen that the magnetic field lines close in on itself. This fact turns out to be true in general.

Example 2 Magnetic field due to a current loop.

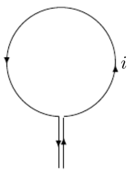


Figure 8.3: Example 2. Overhead view.

A current loop is also a good model for orbiting electrons in atomic structures of material.

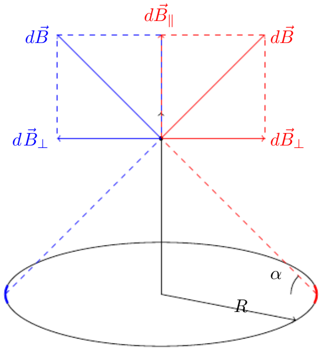


Figure 8.4: Example 2.

Biot-Savart law:

$$dB_1 = |d\vec{B}_1|^2 = \frac{\mu_0}{4\pi}\frac{idl}{r^2} \sim \theta \tag{8.10}$$

In this case, $\theta = \pi/2$ and $\sin\theta = 1$.

$$dB_1 = \frac{\mu_0}{4\pi}\frac{idl}{r^2} = \frac{mu_0}{4\pi}\frac{idl}{R^2+z^2} \tag{8.11}$$

From symmetry, the two diametrically opposite elements (red and blue) will produce $d\vec{B}_1$ and $d\vec{B}_2$, such that $dB_{1\perp}$ and $dB_{2\perp}$ will cancel each other and $d\vec{B}_{1\parallel}$ and $d\vec{B}_{2\parallel}$ will add to produce the net field

$$dB = dB_{1\parallel} + dB_{2\parallel} = 2dB_1\cos\alpha \tag{8.12}$$

$$\cos\alpha = \frac{R}{(R^2+z^2)^{1/2}} \tag{8.13}$$

$$B(z) = \int_{ring} dB \tag{8.14}$$

$$= \frac{\mu_0 i}{4\pi}\frac{R}{(R^2+z^2)^{3/2}}\int dl \tag{8.15}$$

$$= \frac{\mu_0 i}{4\pi}\frac{2R\cdot\pi R}{(R^2+z^2)^{3/2}} \tag{8.16}$$

$$= \frac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}} \tag{8.17}$$

Two interesting results:

1. when the loop radius is very small (loop is called magnetic dipole), i.e. $R \ll z$

$$B \approx \frac{\mu_0 i R^2}{2z^3} = \frac{\mu_0 m}{2\pi z^3} \tag{8.18}$$

where $m = Ai = \pi R^2 i$. m is called magnetic dipole moment. In vector form,

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{\vec{m}}{z^3} \tag{8.19}$$

Recall, electric dipole gave $\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\vec{p}}{z^3}$

2. For $z = 0$.

$$B = \frac{\mu_0}{2} i \cdot R^2 R^3 = \frac{\mu_0 i}{2R} \tag{8.20}$$

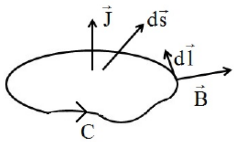
If we have a N-turn coil (tightly wound),

$$B = \frac{\mu_0}{2} \frac{NI}{R} \tag{8.21}$$

This will be used when we calculate the inductance of solenoids.

9 Ampere's Law

In magnetostatics, Biot-Savart Law provides general solution to calculation of magnetic flux density, $\vec{B}$. In the more general case analogous to Gauss's Law, we have Ampere's Law.
Assume that we have current density $\vec{J}$. Then,



$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

where C is a closed path and $I_{enclosed}$ is the total current enclosed within the surface of C.

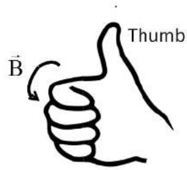
$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

It basically states that the line integral of $\vec{B}_{||}$ over the path which is **closed** (has no beginning and no end) will be proportional to the total current cutting through the surface of the closed path C.
Like Gauss's Law, Ampere's Law can be used to solve for the value of $\vec{B}$ for symmetrical problems. The method will be as follows:

Step 1:
By symmetry, can you assume that B over the closed path C is constant everywhere?
The closed path C has to be perpendicular plane to the current (this makes sure you do not have worry about the dot product $\vec{J} \cdot d\vec{s}$) Step 2:
If you can find such C, then $|\vec{B}|$ comes out of intergral and you can calculate B as:

$$B = \frac{\mu_0 I_{enc}}{\oint_C d\vec{l}} = \frac{\mu_0 I_{enc}}{circum\ force}$$

Step 3:
Direction of $\vec{B}$ is governed by Right Hand Thumb Rule. Thumb of right hand points in direction of current; fingers are along direction of $\vec{B}$.

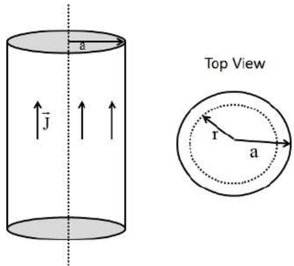


Example 1

Magnetic field of a cylindrical conductor.

An infinite long cylindrical conductor is carrying current I and has radius a. Calculate $\vec{B}$ for r$<a$ and r$\geq a$.

$$J = \frac{I}{\pi a^2}$$

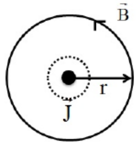


Cylindrical symmetry. So the closed loop is as shown.
r$<a$
By Ampere's Law,

$$\begin{aligned}\oint \vec{B}\cdot d\vec{l} =& \mu_0 I_{enclosed}\\ B\times 2\pi r =& \mu_0 \int Jds\\ =& \mu_0 \int \frac{I}{\pi a^2} ds\\ B\times 2\pi r =& \mu_0 \frac{I}{\pi a^2}\pi r^2\\ B =& \frac{\mu_0 I}{2\pi a^2} r\end{aligned}$$

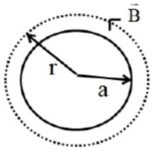
Direction as given by Right Hand Thumb Rule.

$$B = \frac{\mu_0 I r}{2\pi a^2}\hat{\phi}$$



For r$\geq a$

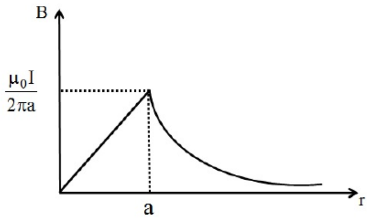
$$B \times 2\pi r = \mu_0 I_{enc} = \mu_0 I$$
$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$



The conductor outside behaves like a thin line.

$$\vec{B} = \frac{\mu_0 I r}{2\pi a^2}\hat{\phi} \qquad r < a$$
$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} \qquad r \geq a$$

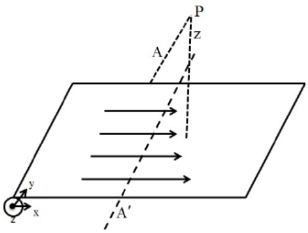
Plot the magnitude:



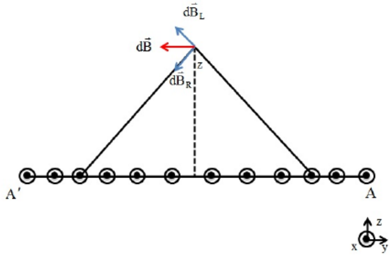
Same behavior as an infinite line of charge.

Example 2

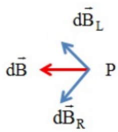
Infinite plane carrying current density, $\vec{J_0}$. Find B at point P.



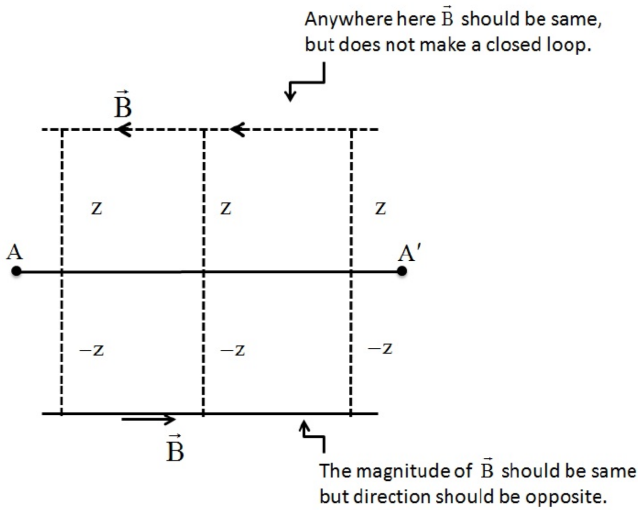
Let us consider the line AA',



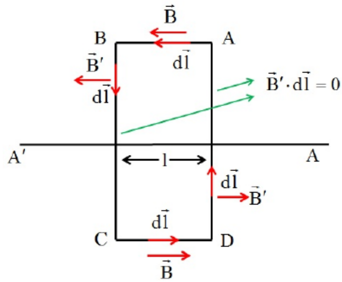
$d\vec{B}$ due to element on left and right are shown.
So at P,



$d\vec{B}$ is in $(-\hat{y})$ direction.



So, consider this:



By Ampere's Law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

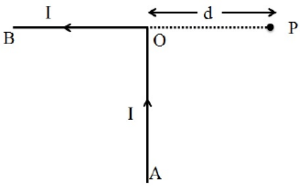
$$\begin{aligned} Bl_{from\ AB} + 0_{from\ BC} + Bl_{from\ CD} + 0_{from\ DA} =& \mu_0 I_0 l \\ 2Bl =& \mu_0 J_0 l \end{aligned}$$

$$\Rightarrow B = \frac{\mu_0 J_0}{2}$$

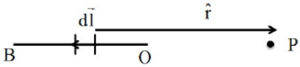
$$\begin{aligned} \vec{B} =& \frac{\mu_0 J_0}{2}(-\hat{y}) \quad z > 0 \\ \vec{B} =& \frac{\mu_0 J_0}{2}(\hat{y}) \qquad z < 0 \end{aligned}$$

Example 3

An infinite wire has been bend. Calculate $\vec{B}$ at point P.

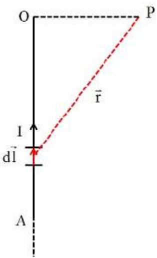


Has no symmetry. We cannot use Ampere's Law.
Let us see what happens for OB,



$$|d\vec{l} \times \hat{r}| = |d\vec{l}||\hat{r}|sin180° = 0$$

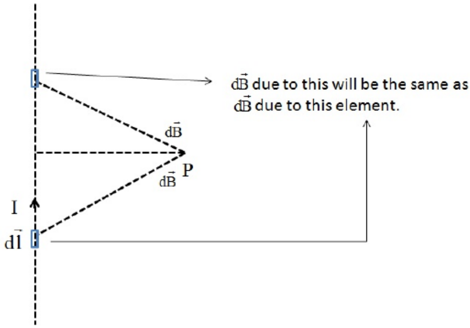
There is no field due to OB at point P.
So the problem is just:



Let us take a small element $d\vec{l}$:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

Now, let us put an imaginary wire on top:



So for semi-infinite wire AO,

$$d\vec{B} = \int_{-\infty}^{0} \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

is nothing but equal to

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

But for an infinite wire we have $B = \frac{\mu_0 I}{2\pi r}$ from Ampere's Law.
$\vec{B}$ for semi-infinite wire is:

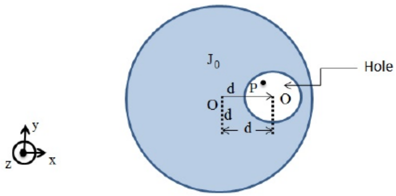
$$\begin{aligned} \vec{B} &= \frac{1}{2} \vec{B}_{infinite} \\ &= \frac{1}{2} \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{4\pi r} \end{aligned}$$

for $r = d$

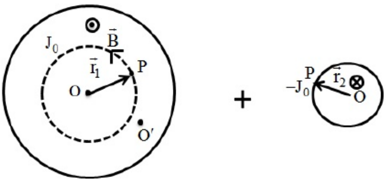
$$\vec{B} = \frac{\mu_0 I}{4\pi d}$$

Example 4

A very long cylinder has a hole drilled in it. It contains current density J_0. Calculate $\vec{B}$ in the hole.



By Superposition,



From example 1,

$$\vec{B}_1 = \frac{\mu_0 J_0}{2} r_1 \hat{\phi}_1$$
$$\vec{B}_2 = \frac{-\mu_0 J_0}{2} r_2 \hat{\phi}_2$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2$$
$$\vec{B} = \frac{\mu_0 J_0}{2} r_1 \hat{\phi}_1 - \frac{\mu_0 J_0}{2} r_2 \hat{\phi}_2$$

But $\hat{\phi}_1$ and $\hat{\phi}_2$ are vectors on two different origins.

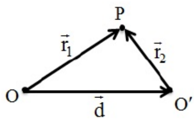
$$\hat{\phi} = \hat{z} \times \hat{r}$$

This is same on both origins O and O'.

$$\hat{\phi}_1 = \hat{z} \times \hat{r}_1$$
$$\hat{\phi}_2 = \hat{z} \times \hat{r}_2$$

$$\vec{B} = \frac{\mu_0 J_0}{2} r_1 (\hat{z} \times \hat{r}_1) - \frac{\mu_0 J_0}{2} r_2 (\hat{z} \times \hat{r}_2)$$
$$= \frac{\mu_0 J_0}{2} \hat{z} \times \{r_1 \hat{r}_1 - r_2 \hat{r}_2\}$$

$$\vec{B} = \frac{\mu_0 J_0}{2} \hat{z} \times \{\vec{r_1} - \vec{r_2}\}$$



But $\vec{r}_1 - \vec{r}_2 = \vec{d} = d\hat{x}$

$$\vec{B} = \frac{\mu_0 J_0}{2} (\hat{z} \times d\hat{x})$$
$$\vec{B} = \frac{\mu_0 J_0}{2} d(\hat{z} \times \hat{x})$$
$$\vec{B} = \frac{\mu_0 J_0}{2} d\hat{y}$$

Because $\hat{z} \times \hat{x} = \hat{y}$. Therefore $\vec{B}$ is: