THE UNIVERSITY OF WARWICK

FIRST YEAR EXAMINATION: June 2019

GEOMETRY AND MOTION

Time Allowed: **2 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. When answering the questions below, you may cite any statement proved during the lectures without giving a proof.

a) Let $\mathbf{r}(t)$ be a parametrisation of a smooth curve $\mathcal{C}$ in $\mathbb{R}^3$ such that $\mathbf{r}(0) = (R, -R, R)$, where $R \in \mathbb{R}$. Suppose $\mathbf{r}(t) \neq \mathbf{0}$ and $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ for all points of $\mathcal{C}$. Show that *any* such $\mathcal{C}$ must lie on the surface of a sphere. Find the position of the sphere's centre and determine its radius. **[4]**

b) Suppose that the plane curve C is parameterised in terms of polar coordinates $(r = R(t), \theta = T(t)), t \in [a, b]$, where $R, T : [a, b] \to \mathbb{R}$ are continuously differentiable functions. Find an integral expression for the length of C in terms of R and T. **[4]**

c) Sketch the logarithmic spiral

$$\mathbf{r}(t) = e^t \cos t \cdot \mathbf{i} + e^t \sin t \cdot \mathbf{j}, \ t \in \mathbb{R}.$$

 Question 1 continued overleaf

Compute the unit tangent and principle normal vectors at $t = 0$ and plot them on your sketch. Using the alternative formula for curvature, compute the curvature for all t's. What is the radius of curvature at $t = 0$? Determine the centre of the osculating circle at $t = 0$ and show the circle on your sketch. **[7]**

d) Find and classify the critical points of $f(x, y) = 4 + x^3 + y^3 - 3xy$. Suppose this f was used to generate a system of ODE's

$$\dot{x} = -\frac{\partial f}{\partial x}(x, y), \; \dot{y} = -\frac{\partial f}{\partial y}(x, y).$$

Based on the above analysis of critical points, what is the large time limit of the solution for the initial condition $x = y = 1/2$? **[7]**

e) Evaluate $\int\int\int_\Omega \sqrt{x^2 + y^2} dV$, where Ω is the region inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$. **[5]**

f) Parameterise the conical surface $z^2 = x^2 + y^2$ for $0 \leq z \leq 1$. Find the element of surface area dS. Find the surface area of a solid circular cone with height $h = 1$ and radius $R = 1$. **[5]**

g) Find the flux of $\mathbf{v}(x, y, z) = (0, 0, x^2 + y^2)$ through the disk $z = 0, x^2 + y^2 \leq 1$ in the $+\mathbf{k}$ direction. **[4]**

h) Let $\mathbf{F} : \mathbb{R}^3 \setminus \mathbf{0} \to \mathbb{R}^3$ be the following vector field:

$$\mathbf{F}(\mathbf{r}) = k \frac{\mathbf{r}}{||\mathbf{r}||^3}, \; \mathbf{r} \neq \mathbf{0},$$

where $k \in \mathbb{R}$ is a constant. Calculate $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$, where $C \subset \mathbb{R}^3 \setminus \mathbf{0}$ is any closed regular curve not containing the origin. **[4]**

OPTIONAL QUESTIONS

2. (Analysis of a cycloid.) In two dimensions, mark a point on the outer rim of a wheel of radius 1 rolling on a horizontal surface $y = 0$ without slippage. Assume that the centre of the wheel is moving with velocity $\mathbf{v} = V_0\mathbf{i}$, $V_0 > 0$ and that the marked point is at $(0, 0)$ at time $t = 0$.

a) Find $\mathbf{r}(t)$, the position of the marked point at time t. The curve traced out by the marked point is called a *cycloid*. **[5]**

b) Compute the the velocity $\mathbf{v}(t)$, speed $c(t)$, and acceleration $\mathbf{a}(t)$ of the marked point. **[5]**

c) Sketch the path of the point $\mathbf{r}(t)$ (over some reasonable range of t). Identify all points at which $\mathbf{v}(t) = 0$ and mark them on your sketch. **[5]**

d) Compute the length of the path between two consecutive points where the path touches $y = 0$. **[5]**

---

3. (Planar curves with constant positive curvature.) Consider a regular planar curve $\Gamma \in \mathbb{R}^2$. Let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ be its natural parameterisation, which is assumed to be twice continuously differentiable.

a) For a point of Γ corresponding to the value of natural parameter s , define the curvature $\kappa(s)$ of the curve and the unit tangent and the principle normal vectors $\mathbf{T}(s)$ and $\mathbf{N}(s)$ to the curve. Express $\mathbf{T}'$ in terms of $\mathbf{N}$ and κ. **[5]**

b) Calculate $\mathbf{N}'(s)$ in terms of the curvature $\kappa(s)$ and the tangent vector $\mathbf{T}(s)$. **[5]**

c) Use the Frenet-Serret formulae derived in a), b) to describe all closed smooth planar curves with constant curvature $\kappa > 0$. **[10]**

Hint. Calculate $\frac{d}{ds}(\mathbf{r}(s) + \frac{1}{\kappa}\mathbf{N}(s))$ remembering that κ is a constant. Use your answer to determine the shape of Γ.

---

4. (Surface area of a torus)

a) Let $S \in R^3$ be a surface defined parametrically, $S = \{\mathbf{r}(u, v), (u, v) \in D \subset \mathbb{R}^2\}$, where $\mathbf{r} : D \to \mathbb{R}^3$ has continuous partial derivatives. Let $f : S \to \mathbb{R}$ be a continuous function. Express the surface integral $\int\int_S f dS$ in terms of a double integral over D. **[2]**

b) Use the formula you stated in part (a) to express the surface area of S as a double integral over D. **[2]**

c) Let T be the torus parameterised by

$$\mathbf{r}(u, v) = \big((A+a\cos u)\cos v, (A+a\cos u)\sin v, a\sin u\big), \quad 0 \le u \le 2\pi, 0 \le v \le 2\pi.$$

Here $A > a > 0$ are constants. Sketch T. Include coordinate axes in your sketch. **[6]**

d) Use the formula stated in part (b) to find the surface area of T. **[10]**

---

5. (The inverse square laws.)

a) Let $\mathbf{v} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Define the flux of $\mathbf{v}$ through a closed surface $S \subset \Omega$ in the outward direction. **[3]**

b) Suppose $S = \{\mathbf{r}(u, v), (u, v) \in D \subset \mathbb{R}^2)\}$, where $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ is a map with continuous partial derivatives. Express the flux of $\mathbf{v}$ through S as a double integral over D. **[5]**

c) Let $\mathbf{f}(\mathbf{r}) = F(||\mathbf{r}||)\frac{\mathbf{r}}{||\mathbf{r}||}$ be a radial vector field defined on $\mathbb{R}^3 \setminus \mathbf{0}$, where $F : (0, \infty) \to \mathbb{R}$ is an unknown continuous function. Determine F from the condition that the flux of $\mathbf{f}$ in the outward direction through the sphere of radius R centred on the origin is independent of R and is equal to some constant $\Phi \in \mathbb{R}$. Depending on the interpretation of Φ, you have calculated the gravitational (or electric) field created by a point mass (or charge) placed at the origin. **[12]**

MA 1340

Remark. All exam questions are taken from parts A and B of homework assignments.

MA 1340

MATHEMATICS DEPARTMENT
FIRST YEAR UNDERGRADUATE EXAMS – June 2019

Course Title: GEOMETRY AND MOTION

Model Solution No: 1

a) $\mathbf{r}(t) \neq \mathbf{0}$ gives $\|\mathbf{r}(t)\| \neq 0$. So can divide $\mathbf{r}(t) \cdot \mathbf{r}'(t)$ by $\|\mathbf{r}(t)\|$ to obtain

$$\frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|} = 0.$$

Therefore $\frac{d\|\mathbf{r}(t)\|}{dt} = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\|\mathbf{r}(t)\|} = 0$ so that for all times $\|\mathbf{r}(t)\| = \|\mathbf{r}(0)\| = \sqrt{3}|R|$. This is enough to conclude that $\mathbf{r}(t)$ belongs to a sphere of radius $\sqrt{3}|R|$ centred on the origin.

b) $x(t) = R(t)\cos T(t)$, $y(t) = R(t)\sin T(t)$, $t \in [a, b]$, is a Cartesian parameterisation of C. Therefore,

$$Length(C) = \int_a^b ||\mathbf{r}'(t)||dt = \int_a^b \sqrt{x'^2(t) + y'^2(t)}dt$$

$$= \int_a^b \sqrt{(R'\cos T - RT'\sin T)^2 + (R'\sin T + RT'\cos T)^2}dt$$

$$= \int_a^b \sqrt{(R'^2(t) + R^2(t)T'^2(t)}dt$$

c) The sketch is shown in Fig. 1 below.

$$\mathbf{r}'(t) = e^t\left((\cos t - \sin t)\cdot\mathbf{i} + (\cos t + \sin t)\cdot\mathbf{j}\right),$$

$$\mathbf{r}''(t) = 2e^t\left(-\sin t\cdot\mathbf{i} + \cos t\cdot\mathbf{j}\right).$$

Therefore, $\mathbf{r}(0) = (1, 0)$, $\mathbf{r}'(0) = (1, 1)$. So,

$$\mathbf{T}(0) = (1, 1)/\sqrt{2}, \mathbf{N}(0) = (-1, 1)/\sqrt{2}.$$

$\mathbf{N}(0)$ is determined from the condition $\mathbf{N}(0)\cdot\mathbf{T}(0) = 0$ and from the sketch using the fact that it must point in the direction of the centre of the osculating circle. The curvature at the point $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{||\mathbf{r}'(t)\times\mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3} = \frac{2e^{2t}}{e^{3t}}\frac{|(\cos t - \sin t)\cos t + (\sin t + \cos t)\sin t|}{((\cos t - \sin t)^2 + (\cos t + \sin t)^2)^{3/2}}$$

$$= 2e^{-t}\frac{1}{2^{3/2}} = \frac{e^{-t}}{\sqrt{2}}.$$

Therefore, the radius of curvature at $\mathbf{r}(0)$ is $\rho(0) = \sqrt{2}$. Therefore, the centre of the osculating circle to the curve at $\mathbf{r}(0)$ is $(1, 0) + \sqrt{2}\mathbf{N}(0) = (0, 1)$, see Figure 1 for the osculating circle.

d) The critical point equations are

$$\partial_x f(x,y) = 3x^2 - 3y = 0,\ \partial_y f(x,y) = 3y^2 - 3x = 0.$$

Therefore there are two critical points - $(0,0)$ and $(1,1)$. The set of second derivatives of f is

$$\partial_x^2 f(x,y) = 6x,\ \partial_y^2 f(x,y) = 6y,\ \partial_x\partial_y f(x,y) = \partial_y\partial_x f(x,y) = -3.$$

Therefore, $\partial_x^2 f(0,0)\partial_y^2 f(0,0) - (\partial_x\partial_y f(0,0))^2 = -9 < 0$ and the critical point $(0,0)$ is a saddle.

Similarly, $\partial_x^2 f(1,1)\partial_y^2 f(1,1) - (\partial_x\partial_y f(1,1))^2 = 36 - 9 > 0$ and $\partial_x^2 f(1,1) > 0$. Therefore, the point $(1,1)$ is a local minimum.

On the diagonal, $(x,y) = t(1,1)$, $\nabla f(x,y) = 3t(t-1)(1,1)$ is parallel to $(1,1)$. Therefore, a solution to $\dot{\mathbf{r}} = -\nabla f(\mathbf{r})$ with initial conditions on the diagonal stays on the diagonal and moves in the direction of the descent of f. If in particular $\mathbf{r}_0$ is the solution for the initial condition $(1/2, 1/2)$, it will move in the direction of the local minimum at $(1,1)$. Conclusion:

$$\lim_{t\to\infty} \mathbf{r}_0(t) = (1,1).$$

e) The calculation is best done in cylindrical coordinates, $x = r\cos\phi, y = r\sin\phi, z = z$, $r \geq 0$, $\phi \in [0, 2\pi]$, $z \in \mathbb{R}$. The integration region Ω in cylindrical coordinates is the interior of a cuboid, $r \in [0,4]$, $\phi \in [0,2\pi]$, $z \in [-5,4]$. The volume element in cylindrical coordinates is $dV = r dr d\phi dz$. Therefore,

$$\int\int\int_\Omega \sqrt{x^2+y^2} dV = \int_0^{2\pi} d\phi \int_{-5}^4 dz \int_0^4 dr r \cdot r$$

$$= 18\pi \int_0^4 dr r^2 = 6\pi 4^3 = (360+24)\pi = 384\pi.$$

f) One possible parameterisation is

$$x(r,\phi) = r\cos\phi,\ y(r,\phi) = r\sin\phi, z(r,\phi) = r, \phi \in [0,2\pi], r \in [0,1].$$

Therefore,

$$dS = ||\partial_r \mathbf{r}(r,\phi) \times \partial_\phi \mathbf{r}(r,\phi)|| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi & \sin\phi & 1 \\ -r\sin\phi & r\cos\phi & 0 \end{vmatrix} \right\| dr d\phi = \sqrt{2} r dr d\phi.$$

Therefore, the surface area of the solid circular cone is

$$\pi + \int_0^1 dr \int_0^{2\pi} d\phi \sqrt{2} r = \pi(1+\sqrt{2}).$$

Notice that the first term on the left hand side is the area of the circular base of radius 1.

g) The normal to the disk D in the required direction is $\mathbf{n} = \mathbf{k}$ at any point of the disk. Therefore,

$$Flux(\mathbf{v}) = \int\int_D \mathbf{v}(\mathbf{r})\cdot\mathbf{n}(\mathbf{r})dS = \int_0^{2\pi}\int_0^1 drr(x^2(r,\phi)+y^2(r,\phi)) = 2\pi\int_0^1 drr^3 = \pi/2.$$

h) The vector field in question is gradient with potential $U(\mathbf{r}) = -k/||\mathbf{r}||$. Therefore, for any closed curve $C \subset \mathbb{R}^3 \setminus \mathbf{0}$,

$$\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$$

by the fundamental theorem of line integrals.

MA 1340

MATHEMATICS DEPARTMENT
FIRST YEAR UNDERGRADUATE EXAMS – June 2019

Course Title: GEOMETRY AND MOTION

Model Solution No: 2

a) The wheel's centre at time t is at position $\mathbf{a}(t) = (V_0 t, 1)$. To an observer positioned at the centre of the wheel, it appears that the marked point on the rim is rotating clockwise (as viewed from the positive half of z-axis) at a constant rate. One revolution happens in the time it takes the centre of the wheel to travel the distance equal to the circumference of the wheel equal to 2π. The period of the revolution is therefore $T = 2\pi/V_0$, the rotation rate $2\pi/T = V_0$ (rad/sec). Therefore, the position of the marked point with respect to the centre of the wheel is

$$\mathbf{b}(t) = (\cos(-\pi/2 - V_0 t), \sin(-\pi/2 - V_0 t)).$$

The position of the marked point as measured from the origin is therefore

$$\mathbf{r}(t) = \mathbf{a}(t) + \mathbf{b}(t) = (V_0 t - \sin(V_0 t), 1 - \cos(V_0 t)).$$

b) The point's velocity is :

$$\mathbf{v}(t) = V_0\,(1 - \cos V_0 t)\,\mathbf{i} + V_0 \sin V_0 t\,\mathbf{j},$$

the speed is:

$$c(t) = \|\mathbf{v}(t)\| = V_0\left((1 - \cos V_0 t)^2 + \sin^2 V_0 t\right)^{1/2} = V_0\,(2 - 2\cos V_0 t)^{1/2} = 2V_0\,|\sin V_0 t|\,.$$

The acceleration:

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = V_0^2\,(\sin V_0 t\,\mathbf{i} + \cos V_0 t\,\mathbf{j})$$

c) See Figure 2 below. The velocity $\mathbf{v}(\tau) = 0$, where $\tau = 2\pi n/V_0$ for integer n's. At these points the cycloid loses its regularity.

d) The curve touches $y = 0$ when $t = 2n\pi/V_0$, for integer n's (at these points $\mathbf{v} = 0$ as well). Length of the curve between two consecutive touches is

$$\ell = \int_0^{2\pi/V_0} \|\mathbf{r}'(t)\|dt = \int_0^{2\pi/V_0} c(t)dt = \int_0^{2\pi} 2\sin\frac{t}{2}dt = -4\cos\frac{t}{2}\Big|_0^{2\pi} = 8$$

MA 1340

MATHEMATICS DEPARTMENT
FIRST YEAR UNDERGRADUATE EXAMS – June 2019

Course Title: GEOMETRY AND MOTION

Model Solution No: 3

a) By definition, the unit tangent vector is $\mathbf{T}(s) = \mathbf{r}'(s)/||\mathbf{r}'(s)||$, the principal normal vector is $\mathbf{N}(s) = \mathbf{T}'(s)/||\mathbf{T}'(s)||$. As the parameterisation at hand is natural, the curvature is $\kappa(s) = ||\mathbf{T}'(s)||$. Therefore,

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s).$$

b) As $||\mathbf{N}|| = 1$, $\mathbf{N}'(s)$ is perpendicular to $\mathbf{N}(s)$, hence parallel to $\mathbf{T}(s)$. Therefore, $\mathbf{N}'(s) = b(s)\mathbf{T}(s)$ for some scalar function b. To find b, differentiate the relation $\mathbf{N}(s) \cdot \mathbf{T}(s) = 0$ with respect to s:

$$\frac{d}{ds}\left(\mathbf{N} \cdot \mathbf{T}\right)(s) = \mathbf{N}'(s) \cdot \mathbf{T}(s) + \mathbf{N}(s) \cdot \mathbf{T}'(s) = b(s) + \kappa(s) = 0$$

Therefore, $b(s) = -\kappa(s)$ and we conclude that

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s).$$

c) Using the hint,

$$\frac{d}{ds}(\mathbf{r}(s) + \frac{1}{\kappa}\mathbf{N}(s)) = \mathbf{T}(s) + \frac{1}{\kappa}\mathbf{N}'(s) = \mathbf{T}(s) - \frac{1}{\kappa}(\kappa\mathbf{T}(s)) = 0$$

The first equality uses the definition of tangent in natural parametrisation, the second - Frenet-Serret's equations. Therefore, $\mathbf{r}(s) + \frac{1}{\kappa}\mathbf{N}(s) = \mathbf{a}$ for some constant vector $\mathbf{a}$. Since $\mathbf{N}$ is a unit vector, this implies

$$||\mathbf{r}(s) - \mathbf{a}|| = 1/\kappa,$$

meaning that Γ is a segment of a circle of radius $1/\kappa$. As the curve is closed, it must coincide with the circle. We conclude that a closed planar curve with constant curvature κ is a circle of radius $1/\kappa$ centred on an arbitrary point $\mathbf{a} \in \mathbb{R}^2$.

MA 1340

MATHEMATICS DEPARTMENT
FIRST YEAR UNDERGRADUATE EXAMS – June 2019

Course Title: GEOMETRY AND MOTION

Model Solution No: 4

a)
$$\int\int_S f(\mathbf{r})dS = \int_D f(\mathbf{r}(u,v)) \left\|\frac{\partial \mathbf{r}}{\partial u}(u,v) \times \frac{\partial \mathbf{r}}{\partial v}(u,v)\right\| dudv,$$

b) The surface area of S is
$$\int\int_S dS = \int_D \left\|\frac{\partial \mathbf{r}}{\partial u}(u,v) \times \frac{\partial \mathbf{r}}{\partial v}(u,v)\right\| dudv,$$

c) See Figure 3 below.

d) First we need to express dS in terms of $du\,dv$:
$$dS = \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| du\,dv$$

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= a\big(-\sin u\cos v, -\sin u\sin v, \cos u\big)\\ \frac{\partial \mathbf{r}}{\partial v} &= \big(-(A+a\cos u)\sin v, (A+a\cos u)\cos v, 0\big)\end{aligned}$$
$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = a(A+a\cos u)\big(-\cos u\cos v, -\cos u\sin v, -\sin u\big)$$
so
$$dS = a(A+a\cos u)\,du\,dv$$
Then area is given by
$$\iint_T dS = \int_0^{2\pi}\int_0^{2\pi} a(A+a\cos u)\,du\,dv = 4\pi^2 Aa$$

MATHEMATICS DEPARTMENT
FIRST YEAR UNDERGRADUATE EXAMS – June 2019

Course Title: GEOMETRY AND MOTION

Model Solution No: 5

a) Let $\mathbf{n} : S \to \mathbb{R}^3$ be the outward unit normal to S. Then the flux of $\mathbf{v}$ through S in in the outward direction is

$$\int\int_S \mathbf{v}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) dS$$

b) The answer is

$$\pm \int_D \mathbf{v}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u}(u,v) \times \frac{\partial \mathbf{r}}{\partial v}(u,v) \right) dudv,$$

where the sign should be chosen in such a way that $\mathbf{n} = \pm \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is an outward normal to S.

c) Spheres in $\mathbb{R}^3$ are level surfaces for any radially symmetric function of three variables $g(\mathbf{r}) = G(||\mathbf{r}||)$, where G is a function of one variable, $G' > 0$. Therefore, $\nabla g(\mathbf{r})$ is normal to the sphere passing through $\mathbf{r}$. (Part I of the Course.) But

$$\nabla g(\mathbf{r}) = G'(||\mathbf{r}||) \frac{\mathbf{r}}{||\mathbf{r}||}.$$

We conclude that the outward unit normal to the sphere of radius R at point $\mathbf{r}$ is $\mathbf{n}(\mathbf{r}) = \frac{\mathbf{r}}{||\mathbf{r}||}$. Therefore, the outward flux of $\mathbf{f}$ through the sphere S_R of radius R centred at the origin is

$$\Phi = \int\int_{S_R} \mathbf{f} \cdot \mathbf{n} dS = \int\int_{S_R} F(||\mathbf{r}||) dS = F(R) \int\int_{S_R} dS = 4\pi R^2 F(R).$$

Therefore,

$$F(R) = \frac{\Phi}{4\pi R^2}.$$

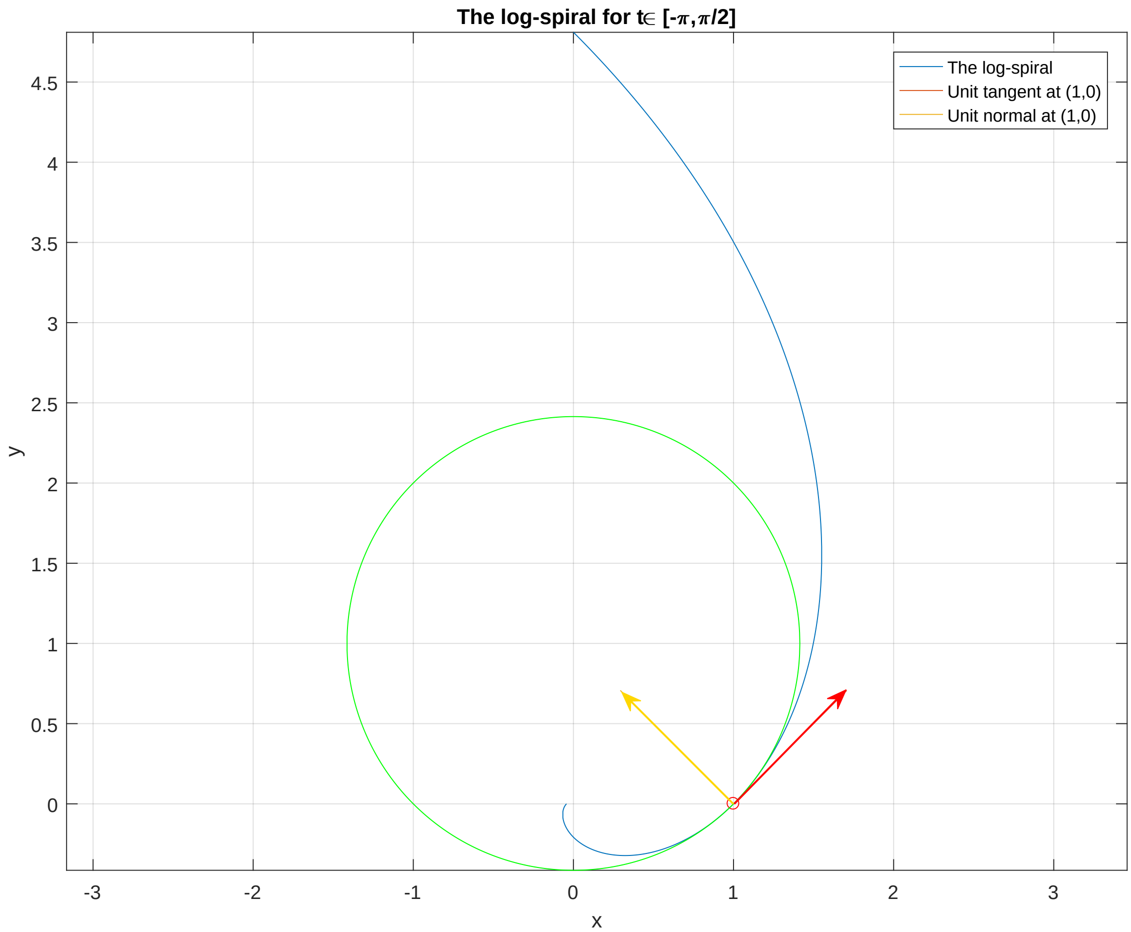

Figure 1: Question 1c. The marked point corresponds to $t = 0$. The osculating circle at $t = 0$ is shown in green.

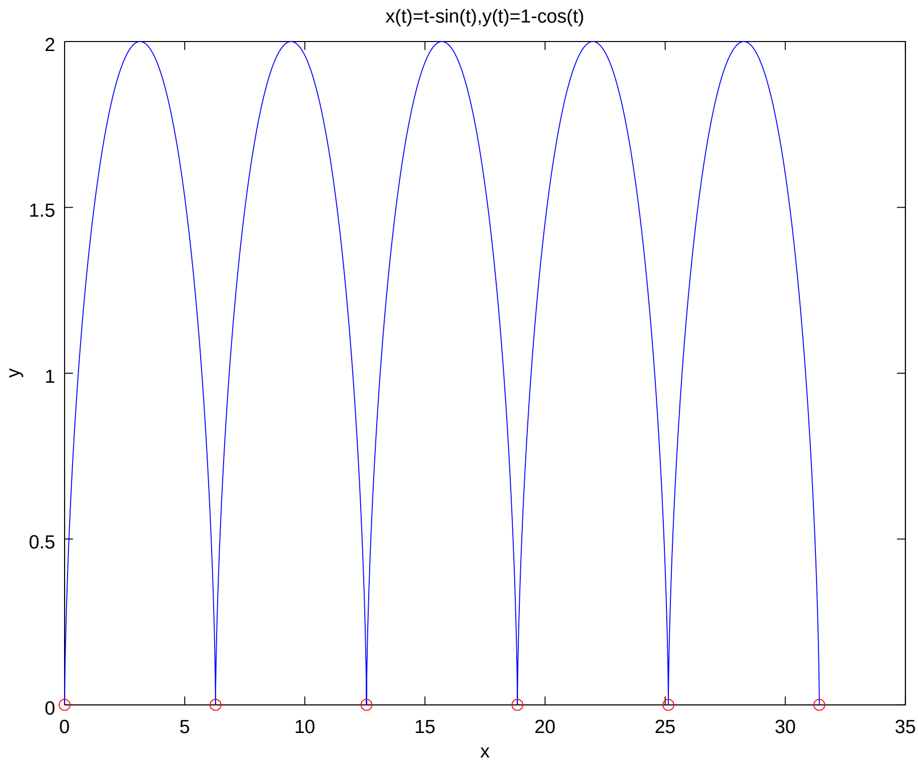

Figure 2: Question 2. The cycloid for $V_0 = 1$. The points corresponding to zero velocity are marked in red.

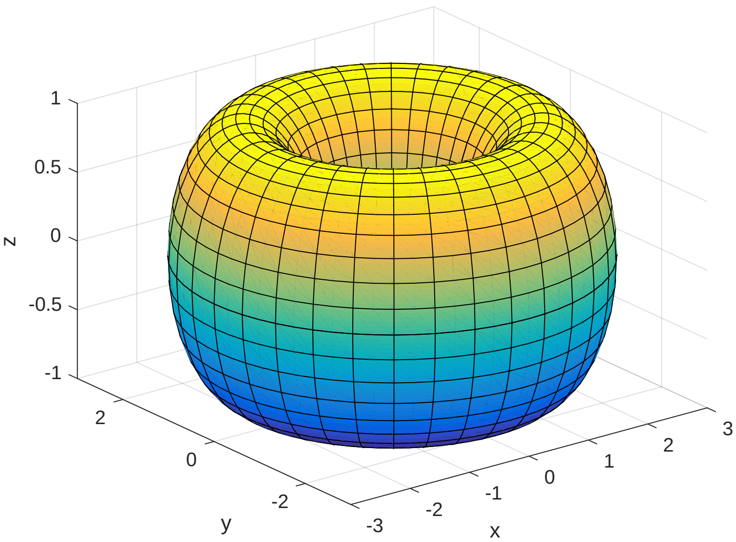

Figure 3: Question 4c. The torus for $A = 2, a = 1$.