D

Classical Logic

The following is not intended as an introduction to classical logic, but rather as a review of the concepts and a setting of notation. We start with propositional calculus and then move to first-order logic. (We do the latter for completeness, but in fact first-order logic plays almost no role in this book.)

D.1 Propositional calculus

Syntax

Given a set P of propositional symbols, the set of sentences in the propositional calculus is the smallest set $\mathcal{L}$ containing P such that if $\varphi, \psi \in \mathcal{L}$ then also $\neg\varphi \in \mathcal{L}$ and $\varphi \wedge \psi \in \mathcal{L}$. Other connectives such as $\vee$, $\rightarrow$, and $\equiv$ can be defined in terms of $\wedge$ and $\neg$.

Semantics

interpretation

model

A propositional *interpretation* (or a *model*) is a set $M \subset P$, the subset of true primitive propositions. The satisfaction relation $\models$ between models and sentences is defined recursively as follows.

- For any $p \in P$, $M \models p$ iff $p \in M$.
- $M \models \varphi \wedge \psi$ iff $M \models \varphi$ and $M \models \psi$.
- $M \models \neg\varphi$ iff it is not the case that $M \models \varphi$.

validity

entailment

We overload the $\models$ symbol. First, it is used to denote *validity*; $\models \varphi$ means that φ is true in all propositional models. Second, it is used to denote *entailment*; $\varphi \models \psi$ means that any model that satisfies φ also satisfies ψ.

Axiomatics

The following axiom system is sound and complete for the class of all propositional models:

A1. $A \rightarrow (B \rightarrow A)$

A2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

A3. $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

R1 (Modus Ponens). $A, A \rightarrow B \vdash B$.

D.2 First-order logic

This book makes very little reference to first-order constructs, but we include the basic material on the first-order logic for completeness.

Syntax

Given a set C of constant symbols, V of variables, F of function symbols each of a given arity, and R of relation (or predicate) symbols each of a given arity. The set of terms is the smallest set T such that $C \cup V \subset T$, and if $f \in F$ is an n-ary functions symbol and $t_1, \ldots, t_n \in T$ then also $f(t_1, \ldots, t_n) \in T$. The set of sentences is the smallest set $\mathcal{L}$ satisfying the following conditions.

If r is an n-ary relation symbol and $t_1, \ldots, t_n \in T$ then $r(t_1, \ldots, t_n) \in \mathcal{L}$. These are the atomic sentences.
If $\varphi, \psi \in \mathcal{L}$ then also $\neg\varphi \in \mathcal{L}$ and $\varphi \wedge \psi \in \mathcal{L}$.
If $\varphi \in \mathcal{L}$ and $v \in V$ then $\forall v \varphi \in L$.

Semantics

interpretation

model

A first-order *interpretation* (or a *model*) is a tuple $M = (D, G, S, \mu)$. D is the domain of M, a set. G is a set of functions from the domain onto itself, each of a given arity. S is a set of relations over the domain, each of a given arity. μ is an (overloaded) interpretation function: $\mu : C \cup V \mapsto D$, $\mu : F \mapsto G$, $\mu : R \mapsto S$ (we assume that μ respects the arities of the function and relations). We can lift μ to apply to any term by the recursive definition $\mu(f(t_1, \ldots, t_n)) = \mu(f)(\mu(t_1), \ldots, \mu(t_n))$.

The satisfaction relation $\models$ between models and sentences is defined recursively.

- For any atomic sentence $\varphi = r(t_1, \ldots, t_n)$, $M \models \varphi$ iff $(\mu(t_1), \ldots, \mu(t_n)) \in \mu(r)$.
- $M \models \varphi \wedge \psi$ iff $M \models \varphi$ and $M \models \psi$.
- $M \models \neg\varphi$ iff it is not the case that $M \models \varphi$.
- $M \models \forall v \varphi$ iff $M \models \varphi[v/t]$ for all terms t, where $\varphi[v/t]$ denotes φ with all free instances of v replaced by t.

We overload the $\models$ symbol as before.

Axiomatics

We omit the axiomatics of first-order logic since they play no role at all in this book.