11

Protocols for Multiagent Resource Allocation: Auctions

In this chapter we consider the problem of allocating (discrete) resources among selfish agents in a multiagent system. Auctions—an interesting and important application of mechanism design—turn out to provide a general solution to this problem. We describe various different flavors of auctions, including single-good, multiunit, and combinatorial auctions. In each case, we survey some of the key theoretical, practical, and computational insights from the literature.

The auction setting is important for two reasons. First, auctions are widely used in real life, in consumer, corporate, as well as government settings. Millions of people use auctions daily on Internet consumer Web sites to trade goods. More complex types of auctions have been used by governments around the world to sell important public resources such as access to electromagnetic spectrum. Indeed, all financial markets constitute a type of auction (one of the family of so-called *double auctions*). Auctions are also often used in computational settings, to efficiently allocate bandwidth and processing power to applications and users.

The second—and more fundamental—reason to care about auctions is that they provide a general theoretical framework for understanding resource allocation problems among self-interested agents. Formally speaking, an auction is any protocol that allows agents to indicate their interest in one or more resources and that uses these indications of interest to determine both an allocation of resources and a set of payments by the agents. Thus, auctions are important for a wide range of computational settings (e.g., the sharing of computational power in a grid computer on a network) that would not normally be thought of as auctions and that might not even use money as the basis of payments.

11.1 Single-good auctions

It is important to realize that the most familiar type of auction—the ascending-bid, English auction—is a drop in the ocean of auction types. Indeed, since auctions are simply mechanisms (see Chapter 10) for allocating goods, there is an infinite number of auction types. In the most familiar types of auctions there is one good for sale, one seller, and multiple potential buyers. Each buyer has his own valuation for the good, and each wishes to purchase it at the lowest possible price. These auctions are called *single-sided*, because there are multiple agents on only one side of the market. Our task is to design a protocol for this auction

single-sided auction

that satisfies certain desirable global criteria. For example, we might want an auction protocol that maximizes the expected revenue of the seller. Or, we might want an auction that is economically efficient; that is, one that guarantees that the potential buyer with the highest valuation ends up with the good.

Given the popularity of auctions, on the one hand, and the diversity of auction mechanisms, on the other, it is not surprising that the literature on the topic is vast. In this section we provide a taste for this literature, concentrating on auctions for selling a single good. We explore richer settings later in the chapter.

11.1.1 *Canonical auction families*

To give a feel for the broad space of single-good auctions, we start by describing some of the most famous families: English, Japanese, Dutch, and sealed-bid auctions. We end the section by presenting a unifying view of auctions as structured negotiations.

English auctions

English auction

The *English auction* is perhaps the best-known family of auctions, since in one form or another such auctions are used in the venerable, old-guard auction houses, as well as most of the online consumer auction sites. The auctioneer sets a starting price for the good, and agents then have the option to announce successive bids, each of which must be higher than the previous bid (usually by some minimum increment set by the auctioneer). The rules for when the auction closes vary; in some instances the auction ends at a fixed time, in others it ends after a fixed period during which no new bids are made, in others at the latest of the two, and in still other instances at the earliest of the two. The final bidder, who by definition is the agent with the highest bid, must purchase the good for the amount of his final bid.

Japanese auctions

Japanese auction

The *Japanese auction*[1] is similar to the English auction in that it is an ascending-bid auction but is different otherwise. Here the auctioneer sets a starting price for the good, and each agent must choose whether or not to be "in," that is, whether he is willing to purchase the good at that price. The auctioneer then calls out successively increasing prices in a regular fashion,[2] and after each call each agent must announce whether he is still in. When an agent drops out it is irrevocable, and he cannot reenter the auction. The auction ends when there is exactly one agent left in; the agent must then purchase the good for the current price.

Dutch auctions

Dutch auction

In a *Dutch auction* the auctioneer begins by announcing a high price and then proceeds to announce successively lower prices in a regular fashion. In practice,

1. Unlike the terms *English* and *Dutch*, the term *Japanese* is not used universally; however, it is commonly used, and there is no competing name for this family of auctions.
2. In the theoretical analyses of this auction the assumption is usually that the prices rise continuously.

the descending prices are indicated by a clock that all of the agents can see. The auction ends when the first agent signals the auctioneer by pressing a buzzer and stopping the clock; the signaling agent must then purchase the good for the displayed price. This auction gets its name from the fact that it is used in the Amsterdam flower market; in practice, it is most often used in settings where goods must be sold quickly.

Sealed-bid auctions

open-outcry auction

sealed-bid auction

first-price auction

second-price auction

kth-price auction

All the auctions discussed so far are considered *open-outcry* auctions, in that all the bidding is done by calling out the bids in public (however, as we will discuss shortly, in the case of the Dutch auction this is something of an optical illusion). The family of *sealed-bid auctions*, probably the best known after English auctions, is different. In this case, each agent submits to the auctioneer a secret, "sealed" bid for the good that is not accessible to any of the other agents. The agent with the highest bid must purchase the good, but the price at which he does so depends on the type of sealed-bid auction. In a first-price sealed-bid auction (or simply *first-price auction*) the winning agent pays an amount equal to his own bid. In a *second-price auction* he pays an amount equal to the next highest bid (i.e., the highest rejected bid). The second-price auction is also called the *Vickrey auction*. In general, in a *kth-price auction* the winning agent purchases the good for a price equal to the k^{th} highest bid.

Auctions as structured negotiations

elimination auction

While it is useful to have reviewed the best-known auction types, this list is far from exhaustive. For example, consider the following auction, consisting of a sequence of sealed bids. In the first round the lowest bidder drops out; his bid is announced and becomes the minimum bid in the next round for the remaining bidders. This process continues until only one bidder remains; this bidder wins and pays the minimum bid in the final round. This auction, called the *elimination auction*, is different from the auctions described earlier, and yet makes perfect sense. Or consider a procurement reverse auction, in which an initial sealed-bid auction is conducted among the interested suppliers, and then a reverse English auction is conducted among the three cheapest suppliers (the "finalists") to determine the ultimate supplier. This two-phase auction is not uncommon in industry.

Indeed, a taxonomical perspective obscures the elements common to all auctions, and thus the infinite nature of the space. What is an auction? At heart it is simply a structured framework for negotiation. Each such negotiation has certain rules, which can be broken down into three categories.

1. *Bidding rules:* How are offers made (by whom, when, what can their content be)?
2. *Clearing rules:* When do trades occur, or what are those trades (who gets which goods, and what money changes hands) as a function of the bidding?
3. *Information rules:* Who knows what when about the state of negotiation?

The different auctions we have discussed make different choices along these three axes, but it is clear that other rules can be instituted. Indeed, when viewed this way, it becomes clear that what seem like three radically different commerce mechanisms—the hushed purchase of a Matisse at a high-end auction house in London, the mundane purchase of groceries at the local supermarket, and the one-on-one horse trading in a Middle Eastern *souk*—are simply auctions that make different choices along these three dimensions.

11.1.2 *Auctions as Bayesian mechanisms*

We now move to a more formal investigation of single-good auctions. Our starting point is the observation that choosing an auction that has various desired properties is a mechanism design problem. Ordinarily we assume that agents' utility functions in an auction setting are quasilinear. To define an auction as a quasilinear mechanism (see Definition 10.3.2) we must identify the following elements:

- set of agents N;
- set of outcomes $O = X \times \mathbb{R}^n$;
- set of actions A_i available to each agent $i \in N$;
- choice function χ that selects one of the outcomes given the agents' actions; and
- payment function $\wp$ that determines what each agent must pay given all agents' actions.

In an auction, the possible outcomes O consist of all possible ways to allocate the good—the set of choices X—and all possible ways of charging the agents. The agents' actions will vary in different auction types. In a sealed-bid auction, each set A_i is an interval from $\mathbb{R}$ (i.e., an agent's action is the declaration of a bid amount between some minimum and maximum value). A Japanese auction is an extensive-form game with chance nodes (see Section 5.2), and so in this case the action space is the space of all policies the agent could follow (i.e., all different ways of acting conditioned on different observed histories). As in all mechanism design problems, the choice and payment functions χ and $\wp$ depend on the objective of the auction, such as achieving an efficient allocation or maximizing revenue.

A Bayesian game with quasilinear preferences includes two more ingredients that we need to specify: the common prior and the agents' utility functions. We will say more about the common prior—the distribution from which the agents' types are drawn—later; here, just note that the definition of an auction as a Bayesian game is incomplete without it. Considering the agents' utility functions, first note that the quasilinearity assumption (see Definition 10.3.1) allows us to write $u_i(o, \theta_i) = u_i(x, \theta_i) - f_i(p_i)$. The function f_i indicates the agent's risk attitude, as discussed in Section 10.3.1. Unless we indicate otherwise, we will commonly assume risk neutrality.

We are left with the task of describing the agents' valuations: their utilities for different allocations of the goods $x \in X$. Auction theory distinguishes between a number of different settings here. One of the best-known and most extensively studied is the *independent private value* (IPV) setting. In this setting all agents' valuations are drawn independently from the same (commonly known) distribution, and an agent's type (or "signal") consists only of his own valuation, giving him no information about the valuations of the others. An example where the IPV setting is appropriate is in auctions consisting of bidders with personal tastes who aim to buy a piece of art purely for their own enjoyment. In most of this section we will assume that agents have independent private values, though we will explore an alternative, the common-value assumption, in Section 11.1.10.

independent private value (IPV)

11.1.3 *Second-price, Japanese, and English auctions*

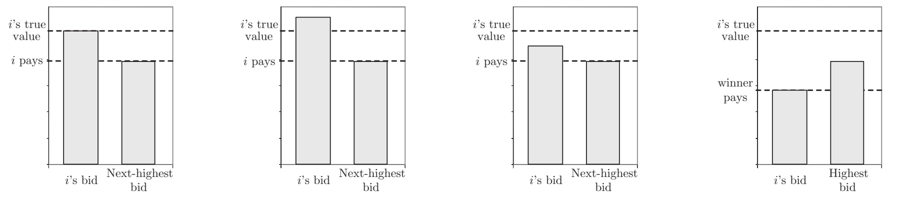
Let us now consider whether the second-price sealed-bid auction, which is a direct mechanism, is truthful (i.e., whether it provides incentive for the agents to bid their true values). The following, very conceptually straightforward proof shows that in the IPV case it is.

Theorem 11.1.1 *In a second-price auction where bidders have independent private values, truth telling is a dominant strategy.*

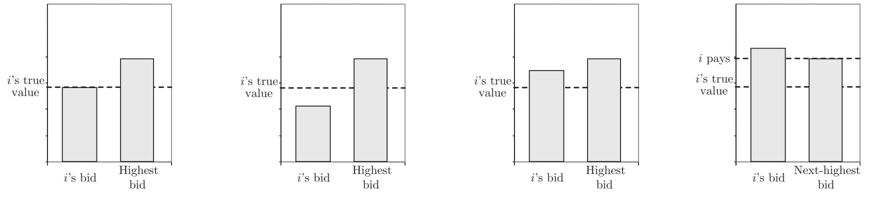
The second-price auction is a special case of the VCG mechanism, and hence of the Groves mechanism. Thus, Theorem 11.1.1 follows directly from Theorem 10.4.2. However, a proof of this narrower claim is considerably more intuitive than the general argument.

Proof. Assume that all bidders other than i bid in some arbitrary way, and consider i's best response. First, consider the case where i's valuation is larger than the highest of the other bidders' bids. In this case i would win and would pay the next-highest bid amount, as illustrated in Figure 11.1a. Could i be better off by bidding dishonestly in this case? If he bid higher, he would still win and would still pay the same amount, as illustrated in Figure 11.1b. If he bid lower, he would either still win and still pay the same amount (Figure 11.1c) or lose and pay zero (Figure 11.1d).[3] Since i gets nonnegative utility for receiving the good at a price less than or equal to his valuation, i cannot gain, and would sometimes lose by bidding dishonestly in this case. Now consider the other case, where i's valuation is less than at least one other bidder's bid. In this case i would lose and pay zero (Figure 11.1e). If he bid less, he would still lose and pay zero (Figure 11.1f). If he bid more, either he would still lose and pay zero (Figure 11.1g) or he would win and pay more than his valuation (Figure 11.1h), achieving negative utility. Thus again, i cannot gain, and would sometimes lose by bidding dishonestly in this case. ■

3. Figure 11.1d is oversimplified: the winner will not always pay i's bid in this case. (Do you see why?)



(a) Bidding honestly, i has the highest bid.
(b) i bids higher and still wins.
(c) i bids lower and still wins.
(d) i bids even lower and loses.



(e) Bidding honestly, i does not have the highest bid.
(f) i bids lower and still loses.
(g) i bids higher and still loses.
(h) i bids even higher and wins.

Figure 11.1 A case analysis to show that honest bidding is a dominant strategy in a second-price auction with independent private values.

Notice that this proof does not depend on the agents' risk attitudes. Thus, an agent's dominant strategy in a second-price auction is the same regardless of whether the agent is risk neutral, risk averse or risk seeking.

In the IPV case, we can identify strong relationships between the second-price auction and Japanese and English auctions. Consider first the comparison between second-price and Japanese auctions. In both cases the bidder must select a number (in the sealed-bid case the number is the one written down, and in the Japanese case it is the price at which the agent will drop out); the bidder with highest amount wins, and pays the amount selected by the second-highest bidder. The difference between the auctions is that information about other agents' bid amounts is disclosed in the Japanese auction. In the sealed-bid auction an agent's bid amount must be selected without knowing anything about the amounts selected by others, whereas in the Japanese auction the amount can be updated based on the prices at which lower bidders are observed to drop out. In general, this difference can be important (see Section 11.1.10); however, it makes no difference in the IPV case. Thus, Japanese auctions are also dominant-strategy truthful when agents have independent private values.

Obviously, the Japanese and English auctions are closely related. Thus, it is not surprising to find that second-price and English auctions are also similar. One connection can be seen through *proxy bidding*, a service offered on some online auction sites such as eBay. Under proxy bidding, a bidder tells the system the maximum amount he is willing to pay. The user can then leave the site, and the system bids as the bidder's proxy: every time the bidder is outbid, the system will respond with a bid one increment higher, until the bidder's maximum is

proxy bidding

reached. It is easy to see that if all bidders use the proxy service and update it only once, what occurs will be identical to a second-price auction (excepting that the winner's payment may be one bid increment higher).

The main complication with English auctions is that bidders can place so-called *jump bids*: bids that are greater than the previous high bid by more than the minimum increment. Although it seems relatively innocuous, this feature complicates analysis of such auctions. Indeed, when an ascending auction is analyzed it is almost always the Japanese variant, not the English.

11.1.4 *First-price and Dutch auctions*

Let us now consider first-price auctions. The first observation we can make is that the Dutch auction and the first-price auction, while quite different in appearance, are actually the same auction (in the technical jargon, they are *strategically equivalent*). In both auctions each agent must select an amount without knowing about the other agents' selections; the agent with the highest amount wins the auction, and must purchase the good for that amount. Strategic equivalence is a very strong property: it says the auctions are exactly the same no matter what risk attitudes the agents have, and no matter what valuation model describes their utility functions. This being the case, it is interesting to ask why both auction types are held in practice. One answer is that they make a trade-off between time complexity and communication complexity. First-price auctions require each bidder to send a message to the auctioneer, which could be unwieldy with a large number of bidders. Dutch auctions require only a single bit of information to be communicated to the auctioneer, but requires the auctioneer to broadcast prices.

Of course, all this talk of equivalence does not help us to understand anything about how an agent should actually *bid* in a first-price or Dutch auction. Unfortunately, unlike the case of second-price auctions, here we do not have the luxury of dominant strategies, and must thus resort to Bayes–Nash equilibrium analysis. Let us assume that agents have independent private valuations. Furthermore, in a first-price auction, an agent's risk attitude also matters. For example, a risk-averse agent would be willing to sacrifice some expected utility (by increasing his bid over what a risk-neutral agent would bid), in order to increase his probability of winning the auction. Let us assume that agents are risk neutral and that their valuations are drawn uniformly from some interval, say $[0, 1]$. Let s_i denote the bid of player i, and v_i denote his true valuation. Thus if player i wins, his payoff is $u_i = v_i - s_i$; if he loses, it is $u_i = 0$. Now we prove in the case of two agents that there is an equilibrium in which each player bids half of his true valuation. (This also happens to be the *unique* symmetric equilibrium, but we do not demonstrate that here.)

Proposition 11.1.2 *In a first-price auction with two risk-neutral bidders whose valuations are drawn independently and uniformly at random from the interval* $[0, 1]$, $(\frac{1}{2}v_1, \frac{1}{2}v_2)$ *is a Bayes–Nash equilibrium strategy profile.*

Proof. Assume that bidder 2 bids $\frac{1}{2}v_2$. From the fact that v_2 was drawn from a uniform distribution, all values of v_2 between 0 and 1 are equally likely. Now consider bidder 1's expected utility, in order to write an expression for his best response.

$$E[u_1] = \int_0^1 u_1 dv_2 \tag{11.1}$$

The integral in Equation (11.1) can be broken up into two smaller integrals that describe cases in which player 1 does and does not win the auction.

$$E[u_1] = \int_0^{2s_1} u_1 dv_2 + \int_{2s_1}^1 u_1 dv_2$$

We can now substitute in values for u_1. In the first case, because 2 bids $\frac{1}{2}v_2$, 1 wins when $v_2 < 2s_1$ and gains utility $v_1 - s_1$. In the second case 1 loses and gains utility 0. Observe that we can ignore the case where the agents tie, because this occurs with probability zero.

$$\begin{aligned} E[u_1] &= \int_0^{2s_1} (v_1 - s_1)dv_2 + 0 \\ &= (v_1 - s_1)v_2 \Big|_0^{2s_1} \\ &= 2v_1s_1 - 2s_1^2 \end{aligned} \tag{11.2}$$

We can find bidder 1's best response to bidder 2's strategy by taking the derivative of Equation (11.2) and setting it equal to zero.

$$\begin{aligned} \frac{\partial}{\partial s_1}(2v_1s_1 - 2s_1^2) &= 0 \\ 2v_1 - 4s_1 &= 0 \\ s_1 &= \frac{1}{2}v_1 \end{aligned}$$

Thus when player 2 is bidding half her valuation, player 1's best strategy is to bid half his valuation. The calculation of the optimal bid for player 2 is analogous, given the symmetry of the game and the equilibrium. ■

This proposition was quite narrow: it spoke about the case of only two bidders, and considered valuations that were drawn uniformly at random from a particular interval of the real numbers. Nevertheless, this is already enough for us to be able to observe that first-price auctions are not incentive compatible (and hence, unsurprisingly, are not equivalent to second-price auctions).

Somewhat more generally, we have the following theorem.

Theorem 11.1.3 *In a first-price sealed-bid auction with n risk-neutral agents whose valuations are independently drawn from a uniform distribution on the same bounded interval of the real numbers, the unique symmetric equilibrium is given by the strategy profile* $(\frac{n-1}{n}v_1, \ldots, \frac{n-1}{n}v_n)$.

In other words, the unique equilibrium of the auction occurs when each player bids $\frac{n-1}{n}$ of his valuation. This theorem can be proved using an argument similar to that used in Proposition 11.1.2, although the calculus gets a bit more involved (for one thing, we must reason about the fact that each of several opposing agents may place the high bid). However, there is a broader problem: that proof only showed how to *verify* an equilibrium strategy. How do we identify one in the first place? Although it is also possible to do this from first principles (at least for straightforward auctions such as first-price), we will explain a simpler technique in the next section.

11.1.5 *Revenue equivalence*

Of the large (in fact, infinite) space of auctions, which one should an auctioneer choose? To a certain degree, the choice does not matter, a result formalized by the following important theorem.[4]

Theorem 11.1.4 (Revenue equivalence theorem) *Assume that each of n risk-neutral agents has an independent private valuation for a single good at auction, drawn from a common cumulative distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \overline{v}]$. Then any efficient[5] auction mechanism in which any agent with valuation $\underline{v}$ has an expected utility of zero yields the same expected revenue, and hence results in any bidder with valuation v_i making the same expected payment.*

Proof. Consider any mechanism (direct or indirect) for allocating the good. Let $u_i(v_i)$ be i's expected utility given true valuation v_i, assuming that all agents including i follow their equilibrium strategies. Let $P_i(v_i)$ be i's probability of being awarded the good given (i) that his true type is v_i; (ii) that he follows the equilibrium strategy for an agent with type v_i; and (iii) that all other agents follow their equilibrium strategies.

$$u_i(v_i) = v_i P_i(v_i) - E[\text{payment by type } v_i \text{ of player } i] \tag{11.3}$$

From the definition of equilibrium, for any other valuation $\hat{v}_i$ that i could have,

$$u_i(v_i) \geq u_i(\hat{v}_i) + (v_i - \hat{v}_i)P_i(\hat{v}_i). \tag{11.4}$$

To understand Equation (11.4), observe that if i followed the equilibrium strategy for a player with valuation $\hat{v}_i$ rather than for a player with his (true) valuation v_i, i would make all the same payments and would win the good with the same probability as an agent with valuation $\hat{v}_i$. However, whenever he wins the good, i values it $(v_i - \hat{v}_i)$ more than an agent of type $\hat{v}_i$ does. The inequality must hold because in equilibrium this deviation must

4. What is stated, in fact, is the revenue equivalence theorem for the private-value, single-good case. Similar theorems hold for other—though not all—cases.
5. Here we make use of the definition of economic efficiency given in Definition 10.3.6. Equivalently, we could require that the auction has a symmetric and increasing equilibrium and always allocates the good to an agent who placed the highest bid.

be unprofitable. Consider $\hat{v}_i = v_i + dv_i$, by substituting this expression into Equation (11.4):

$$u_i(v_i) \geq u_i(v_i + dv_i) + dv_i P_i(v_i + dv_i). \tag{11.5}$$

Likewise, considering the possibility that i's true type could be $v_i + dv_i$,

$$u_i(v_i + dv_i) \geq u_i(v_i) + dv_i P_i(v_i). \tag{11.6}$$

Combining Equations (11.5) and (11.6), we have

$$P_i(v_i + dv_i) \geq \frac{u_i(v_i + dv_i) - u_i(v_i)}{dv_i} \geq P_i(v_i). \tag{11.7}$$

Taking the limit as $dv_i \to 0$ gives

$$\frac{du_i}{dv_i} = P_i(v_i). \tag{11.8}$$

Integrating up,

$$u_i(v_i) = u_i(\underline{v}) + \int_{x=\underline{v}}^{v_i} P_i(x)dx. \tag{11.9}$$

Now consider any two efficient auction mechanisms in which the expected payment of an agent with valuation $\underline{v}$ is zero. A bidder with valuation $\underline{v}$ will never win (since the distribution is atomless), so his expected utility $u_i(\underline{v}) = 0$. Because both mechanisms are efficient, every agent i always has the same $P_i(v_i)$ (his probability of winning given his type v_i) under the two mechanisms. Since the right-hand side of Equation (11.9) involves only $P_i(v_i)$ and $u_i(\underline{v})$, each agent i must therefore have the same expected utility u_i in both mechanisms. From Equation (11.3), this means that a player of any given type v_i must make the same expected payment in both mechanisms. Thus, i's *ex ante* expected payment is also the same in both mechanisms. Since this is true for all i, the auctioneer's expected revenue is also the same in both mechanisms. ■

Thus, when bidders are risk neutral and have independent private valuations, all the auctions we have spoken about so far—English, Japanese, Dutch, and all sealed-bid auction protocols—are revenue equivalent. The revenue equivalence theorem is useful beyond telling the auctioneer that it does not much matter which auction she holds, however. It is also a powerful analytic tool. In particular, we can make use of this theorem to identify equilibrium bidding strategies for auctions that meet the theorem's conditions.

For example, let us consider again the n-bidder first-price auction discussed in Theorem 11.1.3. Does this auction satisfy the conditions of the revenue equivalence theorem? The second condition is easy to verify; the first is harder, because it speaks about the outcomes of the auction under the equilibrium bidding strategies. For now, let us assume that the first condition is satisfied as well.

The revenue equivalence theorem only helps us, of course, if we use it to compare the revenue from a first-price auction with that of another auction that we already understand. The second-price auction serves nicely in this latter role: we already know its equilibrium strategy, and it meets the conditions of the theorem. We know from the proof that a bidder of the same type will make the same expected payment in both auctions. In both of the auctions we are considering, a bidder's payment is zero unless he wins. Thus a bidder's expected payment conditional on being the winner of a first-price auction must be the same as his expected payment conditional on being the winner of a second-price auction. Since the first-price auction is efficient, we can observe that under the symmetric equilibrium agents will bid this amount all the time: if the agent is the high bidder then he will make the right expected payment, and if he is not, his bid amount will not matter.

We must now find an expression for the expected value of the second-highest valuation, given that bidder i has the highest valuation. It is helpful to know the formula for the k^{th} *order statistic*, in this case of draws from the uniform distribution. The k^{th} order statistic of a distribution is a formula for the expected value of the k^{th}-largest of n draws. For n IID draws from $[0, v_{max}]$, the k^{th} order statistic is

order statistic

$$\frac{n+1-k}{n+1} v_{max}. \tag{11.10}$$

If bidder i's valuation v_i is the highest, then there are $n-1$ other valuations drawn from the uniform distribution on $[0, v_i]$. Thus, the expected value of the second-highest valuation is the first-order statistic of $n-1$ draws from $[0, v_i]$. Substituting into Equation (11.10), we have $\frac{(n-1)+1-(1)}{(n-1)+1}(v_i) = \frac{n-1}{n} v_i$. This confirms the equilibrium strategy from Theorem 11.1.3. It also gives us a suspicion (that turns out to be correct) about the equilibrium strategy for first-price auctions under valuation distributions other than uniform: each bidder bids the expectation of the second-highest valuation, conditioned on the assumption that his own valuation is the highest.

A caveat must be given about the revenue equivalence theorem: this result makes an "if" statement, not an "if and only if" statement. That is, while it is true that all auctions satisfying the theorem's conditions must yield the same expected revenue, it is *not* true that all strategies yielding that expected revenue constitute equilibria. Thus, after using the revenue equivalence theorem to identify a strategy profile that one believes to be an equilibrium, one must then prove that this strategy profile is indeed an equilibrium. This should be done in the standard way, by assuming that all but one of the agents play according to the equilibrium and show that the equilibrium strategy is a best response for the remaining agent.

Finally, recall that we assumed above that the first-price auction allocates the good to the bidder with the highest valuation. The reason it was reasonable to do this (although we could instead have proved that the auction has a symmetric, increasing equilibrium) is that we have to check the strategy profile derived using the revenue equivalence theorem anyway. Given the equilibrium strategy, it is

Risk-neutral, IPV	Jap	=	Eng	=	2nd	=	1st	=	Dutch
Risk-averse, IPV		=		=		$<$		=	
Risk-seeking, IPV		=		=		$>$		=	

Table 11.1 Relationships between revenues of various single-good auction protocols.

easy to confirm that the bidder with the highest valuation will indeed win the good.

11.1.6 *Risk attitudes*

One of the key assumptions of the revenue equivalence theorem is that agents are risk neutral. It turns out that many of the auctions we have been discussing cease to be revenue-equivalent when agents' risk attitudes change. Recall from Section 10.3.1 that an agent's risk attitude can be understood as describing his preference between a sure payment and a gamble with the same expected value. (Risk-averse agents prefer the sure thing; risk-neutral agents are indifferent; risk-seeking agents prefer to gamble.)

To illustrate how revenue equivalence breaks down when agents are not risk-neutral, consider an auction environment involving n bidders with IPV valuations drawn uniformly from [0, 1]. Bidder i, having valuation v_i, must decide whether he would prefer to engage in a first-price auction or a second-price auction. Regardless of which auction he chooses (presuming that he, along with the other bidders, follows the chosen auction's equilibrium strategy), i knows that he will gain positive utility only if he has the highest utility. In the case of the first-price auction, i will always gain $\frac{1}{n}v_i$ when he has the highest valuation. In the case of having the highest valuation in a second-price auction i's *expected* gain will be $\frac{1}{n}v_i$, but because he will pay the second-highest actual bid, the amount of i's gain will vary based on the other bidders' valuations. Thus, in choosing between the first-price and second-price auctions and conditioning on the belief that he will have the highest valuation, i is presented with the choice between a sure payment and a risky payment with the same expected value. If i is risk averse, he will value the sure payment more highly than the risky payment, and hence will bid more aggressively in the first-price auction, causing it to yield the auctioneer a higher revenue than the second-price auction. (Note that it is i's behavior in the *first-price* auction that will change: the second-price auction has the same dominant strategy regardless of i's risk attitude.) If i is risk seeking he will bid *less* aggressively in the first-price auction, and the auctioneer will derive greater profit from holding a second-price auction.

The strategic equivalence of Dutch and first-price auctions continues to hold under different risk attitudes; likewise, the (weaker) equivalence of Japanese, English, and second-price auctions continues to hold as long as bidders have IPV valuations. These conclusions are summarized in Table 11.1.

A similar dynamic holds if the bidders are all risk neutral, but the *seller* is either risk averse or risk seeking. The variations in bidders' payments are greater in second-price auctions than they are in first-price auctions, because the former depends on the two highest draws from the valuation distribution, while the latter depends on only the highest draw. However, these payments have the same expectation in both auctions. Thus, a risk-averse seller would prefer to hold a first-price auction, while a risk-seeking seller would prefer to hold a second-price auction.

11.1.7 *Auction variations*

In this section we consider three variations on our auction model. First, we consider reverse auctions, in which one buyer accepts bids from a set of sellers. Second, we discuss the effect of entry costs on equilibrium strategies. Finally, we consider auctions with uncertain numbers of bidders.

Reverse auctions

request for quote

reverse auction

So far, we have considered auctions in which there is one seller and a set of buyers. What about the opposite: an environment in which there is one buyer and a set of sellers? This is what occurs when a buyer engages in a *request for quote* (RFQ). Broadly, this is called a *reverse auction*, because in its open-outcry variety this scenario involves prices that descend rather than ascending.

It turns out that everything that we have said about auctions also applies to reverse auctions. Reverse auctions are simply auctions in which we substitute the word "seller" for "buyer" and vice versa and furthermore, negate all numbers indicating prices or bid amounts. Because of this equivalence we will not discuss reverse auctions any further; note, however, that our concentration on (nonreverse) auctions is without loss of generality.

Auctions with entry costs

entry cost

A second auction variation *does* complicate things, though we will not analyze it here. This is the introduction of an *entry cost* to an auction. Imagine that a first-price auction cost $1 to attend. How should bidders decide whether or not to attend, and then how should they decide to bid given that they're no longer sure how many other bidders will have chosen to attend? This is a realistic way of augmenting our auction model: for example, it can be used to model the cost of researching an auction, driving (or navigating a Web browser) to it, and spending the time to bid. However, it can make equilibrium analysis much more complex.

Things are straightforward for second-price (or, for IPV valuations, Japanese and English) auctions. To decide whether to participate, bidders must evaluate their expected gain from participation. This means that the equilibrium strategy in these auctions now *does* depend on the distribution of other agents' valuations and on the number of these agents. The good news is that, once they have decided to bid, it remains an equilibrium for bidders to bid truthfully.

In first-price auctions (and, generally, other auctions that do not have a dominant-strategy equilibrium) auctions with entry costs are harder—though certainly not impossible—to analyze. Again, bidders must make a trade-off between their expected gain from participating in the auction and the cost of doing so. The complication here is that, since he is uncertain about other agents' valuations, a given bidder will thus also be uncertain about the number of agents who will decide that participating in the auction is in their interest. Since an agent's equilibrium strategy given that he has chosen to participate depends on the number of other participating agents, this makes that equilibrium strategy more complicated to compute. And that, in turn, makes it more difficult to determine the agent's expected gain from participating in the first place.

Auctions with uncertain numbers of bidders

Our standard model of auctions has presumed that the number of bidders is common knowledge. However, it may be the case that bidders are uncertain about the number of competitors they face, especially in a sealed-bid auction or in an auction held over the internet. The preceding discussion of entry costs gave another example of how this could occur. Thus, it is natural to elaborate our model to allow for the possibility that bidders might be uncertain about the number of agents participating in the auction.

It turns out that modeling this scenario is not as straightforward as it might appear. In particular, one must be careful about the fact that bidders will be able to update their *ex ante* beliefs about the total number of participants by conditioning on the fact of their own selection, and thus may lead to a situation in which bidders' beliefs about the number of participants may be asymmetric. (This can be especially difficult when the model does not place an upper bound on the number of agents who can participate in an auction.) We will not discuss these modeling issues here; interested readers should consult the notes at the end of the chapter. Instead, simply assume that the bidders hold symmetric beliefs, each believing that the probability that the auction will involve j bidders is $p(j)$.

Because the dominant strategy for bidding in second-price auctions does not depend on the number of bidders in the auction, it still holds in this environment. The same is not true of first-price auctions, however. Let $F(v)$ be a cumulative probability density function indicating the probability that a bidder's valuation is greater than or equal to v, and let $b^e(v_i, j)$ be the equilibrium bid amount in a (classical) first-price auction with j bidders, for a bidder with valuation j. Then the symmetric equilibrium of a first-price auction with an uncertain number of bidders is

$$b(v_i) = \sum_{j=2}^{\infty} \frac{F^{j-1}(v_i)p(j)}{\sum_{k=2}^{\infty} F^{k-1}(v_i)p(k)} \, b^e(v_i, j).$$

Interestingly, because the proof of the revenue equivalence theorem does not depend on the number of agents, that theorem applies directly to this environment. Thus, in this stochastic environment the seller's revenue is the same when she runs a first-price and a second-price auction. The revenue equivalence theorem can thus be used to derive the strategy above.

11.1.8 *"Optimal" (revenue-maximizing) auctions*

So far in our theoretical analysis we have considered only those auctions in which the good is allocated to the high bidder and the seller imposes no reserve price. These assumptions make sense, especially when the seller wants to ensure *economic efficiency*—that is, that the bidder who values the good most gets it. However, we might instead believe that the seller does not care who gets the good, but rather seeks to maximize her expected revenue. In order to do so, she may be willing to risk failing to sell the good even when there is an interested buyer, and furthermore might be willing sometimes to sell to a buyer who did not make the highest bid, in order to encourage high bidders to bid more aggressively. Mechanisms that are designed to maximize the seller's expected revenue are known as *optimal auctions*.

optimal auction

Consider an IPV setting where bidders are risk neutral and each bidder i's valuation is drawn from some strictly increasing cumulative density function $F_i(v)$, having probability density function $f_i(v)$. Note that we allow for the possibility that $F_i \neq F_j$: bidders' valuations can come from different distributions. Such interactions are called *asymmetric auctions*. We do assume that the seller knows the distribution from which each individual bidder's valuation is drawn and hence is able to distinguish strong bidders from weak bidders.

asymmetric auction

virtual valuation

Define bidder i's *virtual valuation* as

$$\psi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)},$$

and assume that the valuation distribution is such that each ψ_i is increasing in v_i. Also define an agent-specific reserve price r_i^* as the value for which $\psi_i(r_i^*) = 0$. The optimal (single-good) auction is a sealed-bid auction in which every agent is asked to declare his true valuation. These declarations are used to compute a virtual (declared) valuation for each agent. The good is sold to the agent i whose virtual valuation $\psi_i(\hat{v}_i)$ is the highest, as long as this value is positive (i.e., the agent's declared valuation v_i exceeds his reserve price r_i^*). If every agent's virtual valuation is negative, the seller keeps the good and achieves a revenue of zero. If the good is sold, the winning agent i is charged the smallest valuation that he could have declared while still remaining the winner: $\inf\{v_i^* : \psi_i(v_i^*) \geq 0 \text{ and } \forall j \neq i,\ \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$.

How would bidders behave in this auction? Note that it can be understood as a second-price auction with a reserve price, held in virtual valuation space rather than in the space of actual valuations. However, since neither the reserve prices nor the transformation between actual and virtual valuations depends on the agent's declaration, the proof that a second-price auction is dominant-strategy truthful applies here as well, and hence the optimal auction remains strategy-proof.

We began this discussion by introducing a new assumption: that different bidders' valuations could be drawn from different distributions. What happens when this does not occur, and instead all bidders' valuations come from the same distribution? In this case, the optimal auction has a simpler interpretation: it is simply a second-price auction (without virtual valuations) in which the seller sets a reserve price r^* at the value that satisfies $r^* - \frac{1-F_i(r^*)}{f_i(r^*)} = 0$. For this reason, it is common to hear the claim that optimal auctions correspond to setting reserve prices optimally. It is important to recognize that this claim holds only in the case of *symmetric* IPV valuations. In the asymmetric case, the virtual valuations can be understood as artificially increasing the amount of weak bidders' bids in order to make them more competitive. This sacrifices efficiency, but more than makes up for it on expectation by forcing bidders with higher expected valuations to bid more aggressively.

detail-free auction

Wilson doctrine

Although optimal auctions are interesting from a theoretical point of view, they are rarely to never used in practice. The problem is that they are not *detail free*: they require the seller to incorporate information about the bidders' valuation distributions into the mechanism. Such auctions are often considered impractical; famously, the *Wilson doctrine* urges auction designers to consider only detail free mechanisms. With this criticism in mind, it is interesting to ask the following question. In a symmetric IPV setting, is it better for the auctioneer to set an optimal reserve price (causing the auction to depend on bidders' valuation distribution) or to attract one additional bidder to the auction? Interestingly, the auctioneer is better off in the latter case. Intuitively, an extra bidder is similar to a reserve price in the sense that his addition to the auction increases competition among the other bidders, but differs because he can also buy the good himself. This suggests that trying to attract as many bidders as possible (by, among other things, running an auction protocol with which bidders are comfortable) may be more important than trying to figure out the bidders' valuation distributions in order to run an optimal auction.

11.1.9 *Collusion*

collusion

Since we have seen that an auctioneer can increase her expected revenue by increasing competition among bidders, it is not surprising that bidders, conversely, can reduce their expected payments to the auctioneer by reducing competition among themselves. Such cooperation between bidders is called *collusion*. Collusion is usually illegal; interestingly, however, it is also notoriously difficult for agents to pull off. The reason is conceptually similar to the situation faced by agents playing the Prisoner's Dilemma (see Section 3.4.3): while a given agent is better off if everyone cooperates than if everyone behaves selfishly, he is *even* better off if everyone else cooperates and he behaves selfishly himself. An interesting question to ask about collusion, therefore, is which collusive protocols have the property that agents will gain by colluding while being unable to gain further by deviating from the protocol.

Second-price auctions

cartel

bidding ring

ring center

First, consider a protocol for collusion in second-price (or Japanese/English) auctions. We assume that a set of two or more colluding agents is chosen exogenously; this set of agents is called a *cartel* or a *bidding ring*. Assume that the agents are risk neutral and have IPV valuations. It is sometimes necessary (as it is in this case) to assume the existence of an agent who is not interested in the good being auctioned, but who serves to run the bidding ring. This agent does not behave strategically, and hence could be a simple computer program. We will refer to this agent as the *ring center*. Observe that there may be agents who participate in the main auction and do not participate in the cartel; there may even be multiple cartels. The protocol follows.

1. Each agent in the cartel submits a bid to the ring center.
2. The ring center identifies the maximum bid that he received, $\hat{v}_1^r$; he submits this bid in the main auction and drops the other bids. Denote the highest dropped bid as $\hat{v}_2^r$.
3. If the ring center's bid wins in the main auction (at the second-highest price in that auction, $\hat{v}_2$), the ring center awards the good to the bidder who placed the maximum bid in the cartel and requires that bidder to pay $\max(\hat{v}_2, \hat{v}_2^r)$.
4. The ring center gives every agent who participated in the bidding ring a payment of k, regardless of the amount of that agent's bid and regardless of whether or not the cartel's bid won the good in the main auction.

How should agents bid if they are faced with this bidding ring protocol? First of all, consider the case where $k = 0$. Here it is easy to see that this protocol is strategically equivalent to a second-price auction in a world where the bidder's cartel does not exist. The high bidder always wins, and always pays the globally second-highest price (the max of the second-highest prices in the cartel and in the main auction). Thus the auction is dominant-strategy truthful, and agents have no incentive to cheat each other in the bidding ring's "preauction." At the same time, however, agents also do not gain by participating in the bidding ring: they would be just as happy if the cartel disbanded and they had to bid directly in the main auction.

Although for $k = 0$ the situation with and without the bidding ring is equivalent from the bidders' point of view, it is different from the point of view of the ring center. In particular, with positive probability $\hat{v}_2^r$ will be the globally second-highest valuation, and hence the ring center will make a profit. (He will pay $\hat{v}_2$ for the good in the main auction, and will be paid $\hat{v}_2^r > \hat{v}_2$ for it by the winning bidder.) Let $c > 0$ denote the ring center's expected profit. If there are n_r agents in the bidding ring, the ring center could pay each agent up to $k = \frac{c}{n_r}$ and still budget balance on expectation. For values of k smaller than this amount but greater than zero, the ring center will profit on expectation while still giving agents a strict preference for participation in the bidding ring.

How are agents able to gain in this setting—doesn't the revenue equivalence theorem say that their gains should be the same in all efficient auctions? Observe that the agents' expected payments are in fact unchanged, although not all of this amount goes to the auctioneer. What does change is the unconditional payment that every agent receives from the ring center. The second condition of the revenue-equivalence theorem states that a bidder with the lowest possible valuation must receive zero expected utility. This condition is violated under our bidding ring protocol, in which such an agent has an expected utility of k.

First-price auctions

The construction of bidding ring protocols is much more difficult in the first-price auction setting. This is for a number of reasons. First, in order to make a lower expected payment, the winner must actually place a lower bid. In a second-price auction, a winner can instead persuade the second-highest bidder to leave the auction and make the same bid he would have made anyway. This difference matters because in the second-price auction the second-highest bidder has no incentive to renege on his offer to drop out of the auction; by doing so, he can only make the winner pay more. In the first-price auction, the second-highest bidder could trick the highest bidder into bidding lower by offering to drop out, and then could still win the good at less than his valuation. Some sort of enforcement mechanism is therefore required for punishing cheaters. Another problem with bidding rings for first-price auctions concerns how we model what noncolluding bidders know about the presence of a bidding ring in their auction. In the second-price auction we were able to gloss over this point: the noncolluding agents did not care whether other agents might have been colluding, because their dominant strategy was independent of the number of agents or their valuation distributions. (Observe that in our previous protocol, if the cumulative density function of bidders' valuation distribution was F, the ring center could be understood as an agent with a valuation drawn from a distribution with CDF F^{n_r}.) In a first-price auction, the number of bidders and their valuation distributions matter to bidders' equilibrium strategies. If we assume that bidders know the true number of bidders, then a collusive protocol in which bidders are dropped does not make much sense. (The strategies of other bidders in the main auction would be unaffected.) If we assume that noncolluding bidders follow the equilibrium strategy based on the number of bidders who actually bid in the main auction, bidder-dropping collusion does make sense, but the noncolluding bidders no longer follow an equilibrium strategy. (They would gain on expectation if they bid more aggressively.)

For the most part, the literature on collusion has sidestepped this problem by considering first-price auctions only under the assumption that all n bidders belong to the cartel. In this setting, two kinds of bidding ring protocols have been proposed.

The first assumes that the same bidders will have repeated opportunities to collude. Under this protocol all bidders except one are dropped, and this bidder bids zero (or the reserve price) in the main auction. Clearly, other bidders could gain by cheating and also placing bids in the main auction; however, they are dissuaded from doing so by the threat that if they cheat, the cartel will be disbanded and they will lose the opportunity to collude in the future. Under appropriate assumptions about agents' discount rates (their valuations for profits in the future), their number, their valuation distribution, and so on, it can be shown that it constitutes an equilibrium for agents to follow this protocol. A variation on the protocol, which works almost regardless of the values of these variables, has the other agents forever punish any agent who cheats, following a grim trigger strategy (see Section 6.1.2).

The second protocol works in the case of a single, unrepeated, first-price auction. It is similar to the protocol introduced in the previous section.

1. Each agent in the cartel submits a bid to the ring center.
2. The ring center identifies the maximum bid that he received, $\hat{v}_1$. The bidder who placed this bid must pay the full amount of his bid to the ring center.

3. The ring center bids in the main auction at 0. Note that the bidding ring always wins in the main auction as there are no other bidders.
4. The ring center gives the good to the bidder who placed the winning bid in the preauction.
5. The ring center pays every bidder other than the winner $\frac{1}{n-1}\hat{v}_1$.

Observe that this protocol can be understood as holding a first-price auction for the right to bid the reserve price in the main auction, with the profits of this preauction split evenly among the losing bidders. (We here assume a reserve price of zero; the protocol can easily be extended to work for other reserve prices.) Let $b^{n+1}(v_i)$ denote the amount that bidder i would bid in the (standard) equilibrium of a first-price auction with a total of $n+1$ bidders. The symmetric equilibrium of the bidding ring preauction is for each bidder i to bid

$$\hat{v}_i = \frac{n-1}{n} b^{n+1}(v_i).$$

Demonstrating this fact is not trivial; details can be found in the paper cited at the end of the chapter. Here we point out only the following. First, the $\frac{n-1}{n}$ factor has nothing to with the equilibrium bid amount for first-price auctions with a uniform valuation distribution; indeed, the result holds for any valuation distribution. Rather, it can be interpreted as meaning that each bidder offers to pay everyone else $\frac{1}{n}b^{n+1}(v_i)$, and thereby also to gain utility of $\frac{1}{n}b^{n+1}(v_i)$ for himself. Second, although the equilibrium strategy depends on b^{n+1}, there are really only n bidders. Finally, observe that this mechanism is budget balanced (i.e., not just on expectation).

11.1.10 *Interdependent values*

So far, we have only considered the independent private values (IPV) setting. As we discussed earlier, this setting is reasonable for domains in which the agents' valuations are unrelated to each other, depending only on their own signals—for example, because an agent is buying a good for his own personal use. In this section, we discuss different models, in which agents' valuations depend on both their own signals and other agents' signals.

Common values

common value

First of all, we discuss the *common value* (CV) setting, in which all agents value the good at exactly the same amount. The twist is that the agents do not know this amount, though they have (common) prior beliefs about its distribution. Each agent has a private signal about the value, which allows him to condition his prior beliefs to arrive at a posterior distribution over the good's value.[6]

6. In fact, most of what we say in this section also applies to a much more general valuation model in which each bidder may value the good differently. Specifically, in this model each bidder receives a signal drawn independently from some distribution, and bidder i's valuation for the good is some arbitrary function of all of the bidders' signals, subject to a symmetry condition that states that i's valuation does not depend on *which* other agents received which signals. We focus here on the common value model to simplify the exposition.

For example, consider the problem of buying the rights to drill for oil in a particular oil field. The field contains some (uncertain but fixed) amount of oil, the cost of extraction is about the same no matter who buys the contract, and the value of the oil will be determined by the price of oil when it is extracted. Given publicly available information about these issues, all oil drilling companies have the same prior distribution over the value of the drilling rights. The difference between agents is that each has different geologists who estimate the amount of oil and how easy it will be to extract, and different financial analysts who estimate the way oil markets will perform in the future. These signals cause agents to arrive at different posterior distributions over the value of the drilling rights based on which, each agent i can determine an expected value v_i. How can this value v_i be interpreted? One way of understanding it is to note that if a single agent i was selected at random and offered a take-it-or-leave-it offer to buy the drilling contract for price p, he would achieve positive expected utility by accepting the offer if and only if $p < v_i$.

Now consider what would happen if these drilling rights were sold in a second-price auction among k risk-neutral agents. One might expect that each bidder i ought to bid v_i. However, it turns out that bidders would achieve negative expected utility by following this strategy.[7] How can this be—didn't we previously claim that i would be happy to pay any amount up to v_i for the rights? The catch is that, since the value of the good to each bidder is the same, each bidder cares as much about *other* bidders' signals as he does about his own. When he finds out that he won the second-price auction, the winning bidder also learns that he had the most optimistic signal. This information causes him to downgrade his expectation about the value of the drilling rights, which can make him conclude that he paid too much! This phenomenon is called the *winner's curse*.

winner's curse

Of course, the winner's curse does not mean that in the CV setting the winner of a second-price auction always pays too much. Instead, it goes to show that truth telling is no longer a dominant strategy (or, indeed, an equilibrium strategy) of the second-price auction in this setting. There is still an equilibrium strategy that bidders can follow in order to achieve positive expected utility from participating in the auction; this simply requires the bidders to consider how they would update their beliefs on finding that they were the high bidder. The symmetric equilibrium of a second-price auction in this setting is for each bidder i to bid the amount $b(v_i)$ at which, if the second-highest bidder also happened to have bid $b(v_i)$, i would achieve zero expected gain for the good, conditioned on the two highest signals both being v_i.[8] We do not prove this result—or even state it more formally—as doing so would require the introduction of considerable notation.

What about auctions other than second-price in the CV setting? Let us consider Japanese auctions, recalling from Section 11.1.3 that the this auction can be used

7. As it turns out, we can make this statement only because we assumed that $k > 2$. For the case of exactly two bidders, bidding v_i is the right thing to do.

8. We do not need to discuss how ties are broken since i achieves zero expected utility whether he wins or loses the good.

as a model of the English auction for theoretical analysis. Here the winner of the auction has the opportunity to learn more about his opponents' signals, by observing the time steps at which each of them drops out of the auction. The winner will thus have the opportunity to condition his strategy on each of his opponents' signals, unless all of his opponents drop out at the same time. Let us assume that the sequence of prices that will be called out by the auctioneer is known: the t^{th} price will be p_t. The symmetric equilibrium of a Japanese auction in the CV setting is as follows. At each time step t, each agent i computes the expected utility of winning the good v_{i,t_i}, given what he has learned about the signals of opponents who dropped out in previous time steps, and assuming that all remaining opponents drop out at the current time step. (Bidders can determine the signals of opponents who dropped out, at least approximately, by inverting the equilibrium strategy to determine what opponents' signals must have been in order for them to have dropped out when they did.) If $v_{i,t_i} > p_{t+1}$, then if all remaining agents actually did drop out at time t and made i the winner at time $t + 1$, i would gain on expectation. Thus, i remains in the auction at time t if $v_{i,t_i} > p_{t+1}$, and drops out otherwise.

Observe that the stated equilibrium strategy is different from the strategy given above for second-price auctions: thus, while second-price and Japanese auctions are strategically equivalent in the IPV case, this equivalence does not hold in CV domains.

Affiliated values and revenue comparisons

affiliated values

The common value model is generalized by another valuation model called *affiliated values*, which permits correlations between bidders' signals. For example, this latter model can describe cases where a bidder's valuation is divided into a private-value component (e.g., the bidder's inherent value for the good) and a common-value component (e.g., the bidder's private, noisy signal about the good's resale value). Technically, we say that agents have affiliated values when a high value of one agent's signal increases the probability that other agents will have high signals as well. A thorough treatment is beyond the scope of this book; however, we make two observations here.

First, in affiliated values settings generally—and thus in common-value settings as a special case—Japanese (and English) auctions lead to higher expected prices than sealed-bid second-price auctions. Even lower is the expected revenue from first-price sealed-bid auctions. The intuition here is that the winner's gain depends on the privacy of his information. The more the price paid depends on others' information (rather than on expectations of others' information), the more closely this price is related to the winner's information, since valuations are affiliated. As the winner loses the privacy of his information, he can extract a smaller "information rent," and so must pay more to the seller.

linkage principle

Second, this argument leads to a powerful result known as the *linkage principle*. If the seller has access to any private source of information that she knows is affiliated with the bidders' valuations, she is better off precommitting to reveal it honestly. Consider the example of an auction of used cars, where the quality of each car is a random variable about which the seller, and each bidder, receives some information. The linkage principle states that the seller is better off committing to declare everything she knows about each car's defects before the auctions, even though this will sometimes lower the price at which she will be able to sell an individual car. The reason the seller gains by this disclosure is that making her information public also reveals information about the winner's signal and hence reduces his ability to charge information rent. Note that the seller's "commitment power" is crucial to this argument. Bidders are only affected in the desired way if the seller is able to convince them that she will always tell the truth, for example, by agreeing to subject herself to an audit by a trusted third party.

11.2 Multiunit auctions

We have so far considered the problem of selling a single good to one winning bidder. In practice there will often be more than one good to allocate, and different goods may end up going to different bidders. Here we consider *multiunit auctions*, in which there is still only one *kind* of good available, but there are now multiple identical copies of that good. (Think of new cars, tickets to a movie, MP3 downloads, or shares of stock in the same company.) Although this setting seems like only a small step beyond the single-item case we considered earlier, it turns out that there is still a lot to be said about it.

multiunit auctions

11.2.1 *Canonical auction families*

In Section 11.1.1 we surveyed some canonical single-good auction families. Here we review the same auctions, explaining how each can be extended to the multiunit case.

Sealed-bid auctions

Overall, sealed-bid auctions in multiunit settings differ from their single-unit cousins in several ways. First, consider payment rules. If there are three items for sale, and each of the top three bids requests a single unit, then each bid will win one good. In general, these bids will offer different amounts; the question is what each bidder should pay. In the pay-your-bid scheme (the so-called *discriminatory pricing rule*) each of the three top bidders pays a different amount, namely, his own bid. This rule therefore generalizes the first-price auction. Under the *uniform pricing rule* all winners pay the same amount; this is usually either the highest among the losing bids or the lowest among the winning bids.

discriminatory pricing rule

uniform pricing rule

Second, instead of placing a single bid, bidders generally have to provide a price offer for every number of units. If a bidder simply names one number of units and is unwilling to accept any fewer, we say he has placed an *all-or-nothing bid*. If he names one number of units but will accept any smaller number at the same price-per-unit we call the bid *divisible*. We investigate some richer ways for bidders to specify multiunit valuations towards the end of Section 11.2.3.

all-or-nothing bid

divisible bid

Finally, tie-breaking can be tricky when bidders place all-or-nothing bids. For example, consider an auction for 10 units in which the highest bids are as follows, all of them all-or-nothing: 5 units for \$20/unit, 3 units for \$15/unit, 5 units for \$15/unit, and 1 unit for \$15/unit. Presumably, the first bid should be satisfied, as well as two of the remaining three—but which? Here one sees different tie-breaking rules—by quantity (larger bids win over smaller ones), by time (earlier bids win over later bids), and combinations thereof.

English auctions

When moving to the multiunit case, designers of English auctions face all of the problems discussed above. However, since bidders can revise their offers from one round to the next, multiunit English auctions rarely ask bidders to specify more than one number of units along with their price offer. Auction designers still face the choice of whether to treat bids as all-or-nothing or divisible. Another subtlety arises when you consider minimum increments. Consider the following example, in which there is a total of 10 units available, and two bids: one for 5 units at \$1/unit, and one for 5 units at \$4/unit. What is the lowest acceptable next bid? Intuitively, it depends on the quantity—a bid for 3 units at \$2/unit can be satisfied, but a bid for 7 units at \$2/unit cannot. This problem is avoided if the latter bid is divisible, and hence can be partially satisfied.

Japanese auctions

Japanese auctions can be extended to the multiunit case in a similar way. Now after each price increase each agent calls out a number rather than the simple in/out declaration, signifying the number of units he is willing to buy at the current price. A common restriction is that the number must decrease over time; the agent cannot ask to buy more at a high price than he did at a lower price. The auction is over when the supply equals or exceeds the demand. Different implementations of this auction variety differ in what happens if supply exceeds demand: all bidders can pay the last price at which demand exceeded supply, with some of the dropped bidders reinserted according to one of the tie-breaking schemes above; goods can go unsold; one or more bidders can be offered partial satisfaction of their bids at the previous price; and so on.

Dutch auctions

In multiunit Dutch auctions, the seller calls out descending per unit prices, and agents must augment their signals with the quantity they wish to buy. If that is not the entire available quantity, the auction continues. Here there are several options—the price can continue to descend from the current level, can be reset to a set percentage above the current price, or can be reset to the original high price.

11.2.2 *Single-unit demand*

Let us now investigate multiunit auctions more formally, starting with a very simple model. Specifically, consider a setting with k identical goods for sale and risk-neutral bidders who want only 1 unit each and have independent private

Bidder	Bid amount
1	\$25
2	\$20
3	\$15
4	\$8

Table 11.2 Example valuations in a single-unit demand multiunit auction.

values for these single units. Observe that restricting ourselves to this setting gets us around some of the tricky points above such as complex tie breaking.

We saw in Section 11.1.3 that the VCG mechanism can be applied to provide useful insight into auction problems, yielding the second-price auction in the single-good case. What sort of auction does VCG correspond to in our simple multiunit setting? (You may want to think about this before reading on.) As before, since we will simply apply VCG, the auction will be efficient and dominant-strategy truthful; since the market is one-sided it will also satisfy *ex post* individual rationality and weak budget balance. The auction mechanism is to sell the units to the k highest bidders for the same price, and to set this price at the amount offered by the highest losing bid. Thus, instead of a second-price auction we have a $k + 1^{\text{st}}$-price auction.

One immediate observation that we can make about this auction mechanism is that a seller will not necessarily achieve higher profits by selling more units. For example, consider the valuations in Table 11.2.

If the seller were to offer only a single unit using VCG, he would receive revenue of \$20. If he offered two units, he would receive \$30: less than before on a per unit basis, but still more revenue overall. However, if the seller offered three units he would achieve total revenue of only \$24, and if he offered four units he would get no revenue at all. What is going on? The answer points to something fundamental about markets. A dominant-strategy, efficient mechanism can use nothing but losing bidders' bids to set prices, and as the seller offers more and more units, there will necessarily be a weaker and weaker pool of losing bidders to draw upon. Thus the per unit price will weakly fall as the seller offers additional units for sale, and depending on the bidders' valuations, his total revenue can fall as well. What can be done to fix this problem? As we saw for the single-good case in Section 11.1.8, the seller's revenue can be increased on expectation by permitting inefficient allocations, for example, by using knowledge of the valuation distribution to set reserve prices. In the preceding example, the seller's revenue would have been maximized if he had been lucky enough to set a \$15 reserve price. (To see how the auction behaves in this case, think of the reserve price simply as k additional bids placed in the auction by the seller.) However, these tactics only go so far. In the end, the law of supply and demand holds—as the supply goes up, the price goes down, since competition between bidders is reduced. We will return to the importance of this idea for multiunit auctions in Section 11.2.4.

The $k+1^{\text{st}}$-price auction can be contrasted with another popular payment rule, used for example in a multiunit English auction variant by the online auction site eBay. In this auction bidders are charged the lowest winning bid rather than the highest losing bid.[9] This has the advantage that winning bidders always pay a nonzero amount, even when there are fewer bids than there are units for sale. In essence, this makes the bidders' strategic problem somewhat harder (the lowest winning bidder is able to improve his utility by bidding dishonestly, and so overall, bidders no longer have dominant strategies) in exchange for making the seller's strategic problem somewhat easier. (While the seller can still lower his revenue by selling too many units, he does not have to worry about the possibility of giving them away for nothing.)

Despite such arguments for and against different mechanisms, as in the single-good case, in some sense it does not matter what auction the seller chooses. This is because the revenue equivalence theorem for that case (Theorem 11.1.4) can be extended to cover multiunit auctions.[10] The proof is similar, so we omit it.

Theorem 11.2.1 (Revenue equivalence theorem, multiunit version) *Assume that each of n risk-neutral agents has an independent private valuation for a single unit of k identical goods at auction, drawn from a common cumulative distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \overline{v}]$. Then any efficient auction mechanism in which any agent with valuation $\underline{v}$ has an expected utility of zero yields the same expected revenue, and hence results in any bidder with valuation v_i making the same expected payment.*

Thus all of the payment rules suggested in the previous paragraph must yield the same expected revenue to the seller. Of course, this result holds only if we believe that bidders are correctly described by the theorem's assumptions (e.g., they are risk neutral) and that they will play equilibrium strategies. The fact that auction houses like eBay opt for non-dominant-strategy mechanisms suggests that these beliefs may not always be reasonable in practice.

We can also use this revenue equivalence result to analyze another setting: repeated single-good auction mechanisms, or so-called *sequential auctions*. For example, imagine a car dealer auctioning off a dozen new cars to a fixed set of bidders through a sequence of second-price auctions. With a bit of effort, it can be shown that for such an auction there is a symmetric equilibrium in which bidders' bids increase from one auction to the next, and in a given auction bidders with higher valuations place higher bids. (To offer some intuition for the first of these claims, bidders still have a dominant strategy to bid truthfully in the final auction. In previous auctions, bidders have positive expected utility after losing the auction, because they can participate in future rounds. As the number of future rounds decreases, so does this expected utility; hence in equilibrium bids rise.) We

sequential auction

9. Confusingly, this multiunit English auction variant is sometimes called a *Dutch auction*. This is a practice to be discouraged; the correct use of the term is in connection with the descending open-outcry auction.

10. As before, we state a more restricted version of this revenue equivalence theorem than necessary. For example, revenue equivalence holds for *all* pairs of auctions that share the same allocation rule (not just for efficient auctions) and does not require our assumption of single-unit demand.

can therefore conclude that the auction is efficient, and thus by Theorem 11.2.1 each bidder makes the same expected payment as under VCG. Thus there exists a symmetric equilibrium in which bidders bid honestly in the final auction k, and in each auction $j < k$, each bidder i bids the expected value of the k^{th}-highest of the other bidders' valuations, conditional on the assumption that his valuation v_i lies between the j^{th}-highest and the $j+1^{\text{st}}$-highest valuations. This makes sense because in each auction the bidder who is correct in making this assumption will be the bidder who places the second-highest bid and sets the price for the winner. Thus, the winner of each auction will pay an unbiased estimate of the overall $k+1^{\text{st}}$-highest valuation, resulting in an auction that achieves the same expected revenue as VCG.

Very similar reasoning can be used to show that a symmetric equilibrium for k sequential *first-price* auctions is for each bidder i in each auction $j \leq k$ to bid the expected value of the k^{th}-highest of the other bidders' valuations, conditional on the assumption that his valuation v_i lies between the $j-1^{\text{st}}$-highest and the j^{th}-highest valuations. Thus, each bidder conditions on the assumption that he is the highest bidder remaining; the bidder who is correct in making this assumption wins, and hence pays an amount equal to the expected value of the overall $k+1^{\text{st}}$-highest valuation.

11.2.3 *Beyond single-unit demand*

Now let us investigate how things change when we relax the restriction that each bidder is only interested in a single unit of the good.

VCG for general multiunit auctions

How does VCG behave in this more general setting? We no longer have something as simple as the $k+1^{\text{st}}$-price auction we encountered in Section 11.2.2. Instead, we can say that all winning bidders who won the same number of units will pay the same amount as each other. This makes sense because the change in social welfare that results from dropping any one of these bidders will be the same. Bidders who win different numbers of units will not necessarily pay the same per unit prices. We can say, however, that bidders who win larger numbers of units will pay at least as much (in total, though not necessarily per unit) as bidders who won smaller numbers of units, as their impact on social welfare will always be at least as great.

VCG can also help us notice another interesting phenomenon in the general multiunit auction case. For all the auctions we have considered in this chapter so far, it has always been computationally straightforward to identify the winners. In this setting, however, the problem of finding the social-welfare-maximizing allocation is computationally hard. Specifically, finding a subset of bids to satisfy that maximizes the sum of bidders' valuations for them is equivalent to a weighted knapsack problem, and hence is NP-complete.

winner determination problem, general multiunit auction

Definition 11.2.2 (Winner determination problem (WDP)) *The* winner determination problem *(WDP) for a general multiunit auction, where m denotes the* *total number of units available and $\hat{v}_i(k)$ denotes bidder i's declared valuation for being awarded k units, is to find the social-welfare-maximizing allocation of goods to agents. This problem can be expressed as the following integer program.*

$$\text{maximize} \quad \sum_{i \in N} \sum_{1 \le k \le m} \hat{v}_i(k) x_{k,i} \tag{11.11}$$

$$\text{subject to} \quad \sum_{i \in N} \sum_{1 \le k \le m} k \cdot x_{k,i} \le m \tag{11.12}$$

$$\sum_{1 \le k \le m} x_{k,i} \le 1 \qquad \forall i \in N \tag{11.13}$$

$$x_{k,i} = \{0, 1\} \qquad \forall 1 \le k \le m, i \in N \tag{11.14}$$

This integer program uses a variable $x_{k,i}$ to indicate whether bidder i is allocated exactly k units, and then seeks to maximize the sum of agents' valuations for the chosen allocation in the objective function (11.11). Constraint (11.13) ensures that no more than one of these indicator variables is nonzero for any bidder, and constraint (11.12) ensures that the total number of units allocated does not exceed the number of units available. Constraint (11.14) requires that the indicator variables are integral; it is this constraint that makes the problem computationally hard.

Representing multiunit valuations

We have assumed that agents can communicate their complete valuations to the auctioneer. When a large number of units are available in an auction, this means that bidders must specify a valuation for every number of units. In practice, it is common that bidders would be provided with some *bidding language* that would allow them to convey this same information more compactly.

bidding language

Of course, the usefulness of a bidding language depends on the sorts of underlying valuations that bidders will commonly want to express. A few common symmetric valuations are the following.

- *Additive valuation:* The bidder's valuation of a set is directly proportional to the number of goods in the set, so that $v_i(S) = c|S|$ for some constant c.
- *Single item valuation:* The bidder desires any single item, and only a single item, so that $v_i(S) = c$ for some constant c for all $S \neq \emptyset$.
- *Fixed budget valuation:* Similar to the additive valuation, but the bidder has a maximum budget of B, so that $v_i(S) = \min(c|S|, B)$.
- *Majority valuation:* The bidder values equally any majority of the goods, so that

$$v_i(S) = \begin{cases} 1 & \text{if } |S| \ge m/2; \\ 0 & \text{otherwise.} \end{cases}$$

We can generalize all of these valuations to a general symmetric valuation.

- *General symmetric valuation:* Let $p_1, p_2, \ldots, p_m$ be arbitrary nonnegative prices, so that p_j specifies how much the bidder is willing to pay of the j^{th} item won. Then

$$v_i(S) = \sum_{j=1}^{|S|} p_j.$$

- *Downward sloping valuation:* A downward sloping valuation is a symmetric valuation in which $p_1 \geq p_2 \geq \cdots \geq p_m$.

11.2.4 *Unlimited supply: random sampling auctions*

Earlier, we suggested that MP3 downloads serve as a good example of a multiunit good. However, they differ from the other examples we gave, such as new cars, in an important way. This difference is that a seller of MP3 downloads can produce additional units of the good at zero marginal cost, and hence has an effectively unlimited supply of the good. This does not mean that the units have no value or that the seller should give them away—after all, the *first* unit may be very expensive to produce, requiring the seller to amortize this cost across the sale of multiple units. What it does mean is that the seller will not face any supply restrictions other than those she imposes herself.

We thus face the following multiunit auction problem: how should a seller choose a multiunit auction mechanism for use in an unlimited supply setting if she cares about maximizing her revenue? The goal will be finding an auction mechanism that chooses among bids in a way that achieves good revenue without artificially picking a specific number of goods to sell in advance, and also without relying on distributional information about buyers' valuations. We also want the mechanism to be dominant-strategy truthful, individually rational, and weakly budget balanced. Clearly, it will be necessary to artificially restrict supply (and thus cause allocative inefficiency), because otherwise bidders would be able to win units of the good in exchange for arbitrarily small payments. Although this assumption can be relaxed, to simplify the presentation we will return to our previous assumption that bidders are interested in buying at most one unit of the good.

The main insight that allows us to construct a mechanism for this case is that, if we *knew* bidders' valuations but had to offer the goods at the same price to all bidders, it would be easy to compute the optimal single price.

optimal single price

Definition 11.2.3 (Optimal single price) *The* optimal single price *is calculated as follows.*

1. *Order the bidders in descending order of valuation; let v_i denote the i^{th}-highest valuation.*
2. *Calculate $opt \in \arg\max_{i \in \{1,\ldots,n\}} i \cdot v_i$.*
3. *The optimal single price is v_{opt}.*

Simply offering the good to the agents at the optimal single price is not a dominant-strategy truthful mechanism: bidders would have incentive to misstate their valuations. However, this procedure can be used as a building block to construct a simple and powerful dominant-strategy truthful mechanism.

random sampling optimal price auction

Definition 11.2.4 (Random sampling optimal price auction) *The* random sampling optimal price auction *is defined as follows.*

1. *Randomly partition the set of bidders N into two sets, N_1 and N_2 (i.e., $N = N_1 \cup N_2$; $N_1 \cap N_2 = \emptyset$; each bidder has probability 0.5 of being assigned to each set).*
2. *Using the procedure above find p_1 and p_2, where p_i is the optimal single price to charge the set of bidders N_i.*
3. *Then set the allocation and payment rules as follows:*
   - *For each bidder $i \in N_1$, award a unit of the good if and only if $b_i \geq p_2$, and charge the bidder p_2;*
   - *For each bidder $j \in N_2$, award a unit of the good if and only if $b_j \geq p_1$, and charge the bidder p_1.*

Observe that this mechanism follows the Wilson doctrine: it works even in the absence of distributional information. Random sampling optimal price auctions also have a number of other desirable properties.

Theorem 11.2.5 *Random sampling optimal price auctions are dominant-strategy truthful, weakly budget balanced and* ex post *individually rational.*

The proof of this theorem is left as an exercise to the reader. The proof of truthfulness is essentially the same as the proof of Theorem 11.1.1: bidders' declared valuations are used only to determine whether or not they win, but beyond serving as a maximum price offer do not affect the price that a bidder pays. Of course the random sampling auction is not efficient, as it sometimes refuses to sell units to bidders who value them. The random sampling auction's most interesting property concerns revenue.

Theorem 11.2.6 *The random sampling optimal price auction always yields expected revenue that is at least a ($\frac{1}{4.68}$) constant fraction of the revenue that would be achieved by charging bidders the optimal single price, subject to the constraint that at least two units of the good must be sold.*

A host of other auctions have been proposed in the same vein, for example covering additional settings such as goods for which there is a limited supply of units. (Here the trick is essentially to throw away low bidders so that the number of remaining bidders is the same as the number of goods, and then to proceed as before with the additional constraint that the highest rejected bid must not exceed the single price charged to any winning bidder.) In this limited supply case, both the random sampling optimal price auction and its more sophisticated counterparts can achieve revenues much higher than VCG, and hence also higher than all the other auctions discussed in Section 11.2.2. Other work considers the *online auction* case, where bidders arrive one at a time and the auction must decide whether each bidder wins or loses before seeing the next bid.

online auction

11.2.5 *Position auctions*

position auction

The last auction type we consider in this section goes somewhat beyond the multiunit auctions we have defined previously. Like multiunit auctions in the case of single-unit demand, *position auctions* never allocate more than one item per bidder and ask bidders to specify their preferences using a single real number. The wrinkle is that these auctions sell a set of goods among which bidders are not indifferent: one of a set of ordered positions. The motivating example for these auctions is the sale of ranked advertisements on a page of search results. (For this reason, these auctions are also called *sponsored search auctions*.) Since the goods are not identical, we cannot consider them to be multiple units of the same good. In this sense position auctions can be understood as combinatorial auctions, the topic of the Section 11.3. Nevertheless, we choose to present them here because they have been called multiunit auctions in the literature, and because their bidding and allocation rules have a multiunit flavor.

Regardless of how we choose to classify them, position auctions are very important. From a theoretical point of view they are interesting and have good properties both in terms of incentives and computation. Practically speaking, major search engines use them to sell many billions of dollars worth of advertising space annually, and indeed did so even before much was known about the auctions' theoretical properties. In these auctions, search engines offer a set of keyword-specific "slots"—usually a list on the right-hand side of a page of search results—for sale to interested advertisers. Slots are considered to be more valuable the closer they are to the top of the page, because this affects their likelihood of being clicked by a user. Advertisers place bids on keywords of interest, which are retained by the system. Every time a user searches for a keyword on which advertisers have bid, an auction is held. The outcome of this auction is a decision about which ads will appear on the search results page and in which order. Advertisers are required to pay only if a user clicks on their ad. Because sponsored search is the dominant application of position auctions, we will use it as our motivating example here.

How should position auctions be modeled? The setting can be understood as inducing an infinitely repeated Bayesian game, because a new auction is held every time a user searches for a given keyword. However, researchers have argued that it makes sense to study an unrepeated, perfect-information model of the setting. The single-shot assumption is considered reasonable because advertisers tend to value clicks additively (i.e., the value derived from a given user clicking on an ad is independent of how many other users clicked earlier), at least when advertisers do not face budget constraints. The perfect-information assumption makes sense because search engines allow bidders either to observe other bids or to figure them out by probing the mechanism.

We now give a formal model. As before, let N be the set of bidders (advertisers), and let v_i be i's (commonly known) valuation for getting a click. Let $b_i \in \mathbb{R}_+$ denote i's bid, and let $b_{(j)}$ denote the j^{th}-highest bid, or 0 if there are fewer than j bids. Let $G = \{1, \ldots, m\}$ denote the set of goods (slots), and let α_j denote the expected number of clicks (the *click-through rate*) that an ad will receive if it is listed in the i^{th} slot. Observe that we assume that α does not depend on the bidder's identity.

click-through rate

The generalized first-price auction was the first position auction to be used by search engine companies.

generalized first-price auction (GFP)

Definition 11.2.7 (Generalized first-price auction) *The* generalized first-price auction *(GFP) awards the bidder with the j^{th}-highest bid the j^{th} slot. If bidder i's ad receives a click, he pays the auctioneer b_i.*

Unfortunately, these auctions do not always have pure-strategy equilibria, even in the unrepeated, perfect-information case. For example, consider three bidders $1, 2,$ and 3 who value clicks at \$10, \$4, and \$2 respectively, participating in an auction for two slots, where the probability of a click for the two slots is $\alpha_1 = 0.5$ and $\alpha_2 = 0.25$, respectively. Bidder 2 needs to bid at least \$2 to get a slot; suppose he bids \$2.01. Then bidder 1 can win the top slot for a bid of \$2.02. But bidder 2 could get the top slot for \$2.03, increasing his expected utility. If the agents bid by best responding to each other—as has indeed been observed in practice—their bids will increase all the way up to bidder 2's valuation, at which point bidder 2 will drop out, bidder 1 will reduce his bid to bidder 3's valuation, and the cycle will begin again.

The instability of bidding behavior under the GFP led to the introduction of the generalized second-price auction, which is currently the dominant mechanism in practice.

generalized second-price auction (GSP)

Definition 11.2.8 (Generalized second-price auction) *The* generalized second-price auction *(GSP) awards the bidder with the j^{th}-highest bid the j^{th} slot. If bidder i's ad is ranked in slot j and receives a click, he pays the auctioneer $b_{(j+1)}$.*

The GSP is more stable than the GFP. Continuing the example from above, if all bidders bid truthfully, then bidder 1 would pay \$4 per click for the first slot, bidder 2 would pay \$2 per click for the second slot, and bidder 3 would lose. Bidder 1's expected utility would be $0.5(\$10 - \$4) = \$3$; if he bid less than \$4 but more than \$2 he would pay \$2 per click for the second slot and achieve expected utility of $0.25(\$10 - \$2) = \$2$, and if he bid even less then his expected utility would be zero. Thus bidder 1 prefers to bid truthfully in this example. If bidder 2 bid more than \$10 then he would win the top slot for \$10, and would achieve negative utility; thus in this example bidder 2 also prefers honest bidding.

This example suggests a connection between the GSP and the VCG mechanism. However, these two mechanisms are actually quite different, as becomes clear when we apply the VCG formula to the position auction setting.

Definition 11.2.9 (VCG) *In the position auction setting, the* VCG mechanism *awards the bidder with the j^{th}-highest bid the j^{th} slot. If bidder i's ad is ranked in slot j and receives a click, he pays the auctioneer $\frac{1}{\alpha_j}\sum_{k=j+1}^{m+1} b_{(k)}(\alpha_{k-1}-\alpha_k)$.*

Intuitively, the key difference between the GSP and VCG is that the former does not charge an agent his social cost, which depends on the differences between click-through rates that other agents would receive with and without his presence. Indeed, truthful bidding is not always a good idea under the GSP. Consider the same bidders as in our running example, but change the click-through rate of slot 2 to $\alpha_2 = 0.4$. When all bidders bid truthfully we have already shown that bidder 1 would achieve expected utility of \$3 (this argument did not depend on α_2). However, if bidder 1 changed his bid to \$3, he would be awarded the second slot and would achieve expected utility of $0.4(\$10 - \$2) = \$3.2$. Thus the GSP is not even truthful in equilibrium, let alone in dominant strategies.

What *can* be said about the equilibria of the GSP? Briefly, it can be shown that in the perfect-information setting the GSP has many equilibria. The dynamic nature of the setting suggests that the most stable configurations will be *locally envy free*: no bidder will wish that he could switch places with the bidder who won the slot directly above his. There exists a locally envy-free equilibrium of the GSP that achieves exactly the VCG allocations and payments. Furthermore, all other locally envy-free equilibria lead to higher revenues for the seller, and hence are worse for the bidders.

locally envy free

What about relaxing the perfect information assumption? Here, it is possible to construct a generalized *English* auction that corresponds to the GSP, and to show that this English auction has a unique equilibrium with various desirable properties. In particular, the payoffs under this equilibrium are again the same as the VCG payoffs, and the equilibrium is *ex post* (see Section 6.3.4), meaning that it is independent of the underlying valuation distribution.

11.3 Combinatorial auctions

We now consider an even broader auction setting, in which a whole variety of different goods are available in the same market. This differs from the multiunit setting because we no longer assume that goods are interchangeable. Switching to a multi*good* auction model is important when bidders' valuations depend strongly on which set of goods they receive. Some widely studied practical examples include governmental auctions for the electromagnetic spectrum, energy auctions, corporate procurement auctions, and auctions for paths (e.g., shipping rights; bandwidth) in a network.

More formally, let us consider a setting with a set of bidders $N = \{1, \ldots, n\}$ (as before) and a set of goods $G = \{1, \ldots, m\}$. Let $v = (v_1, \ldots, v_n)$ denote the true *valuation functions* of the different bidders, where for each $i \in N$, $v_i : 2^G \mapsto \mathbb{R}$. There is a substantive assumption buried inside this definition: that there are *no externalities*. (Indeed, we have been making this assumption almost continuously since introducing quasilinear utilities in the previous chapter; however, this is a good time to remind the reader of it.) Specifically, we have asserted that a

valuation function

no externalities

bidder's valuation depends only on the set of goods he wins. This assumption is quite standard; however, it does not allow us to model a bidder who also cares about the allocations and payments of the other agents.

nonadditive valuation functions

We will usually be interested in settings where bidders have *nonadditive valuation functions*, for example valuing bundles of goods more than the sum of the values for single goods. We identify two important kinds of nonadditivity. First, when two items are *partial substitutes* for each other (e.g., a Sony TV and a Toshiba TV, or, more partially, a CD player and an MP3 player), their combined value is less than the sum of their individual values. Strengthening this condition, when two items are *strict substitutes* their combined value is the same as the value for either one of the goods. For example, consider two nontransferable tickets for seats on the same plane. Sets of strictly substitutable goods can also be seen as multiple units of a single good.

partial substitutes

strict substitutes

Definition 11.3.1 (Substitutability) *Bidder i's valuation v_i exhibits* substitutability *if there exist two sets of goods $G_1, G_2 \subseteq G$, such that $G_1 \cap G_2 = \emptyset$ and $v(G_1 \cup G_2) < v(G_1) + v(G_2)$. When this condition holds, we say that the valuation function v_i is* subadditive.

substitutability

The second form of nonadditivity we will consider is *complementarity*. This condition is effectively the opposite of substitutability: the combined value of goods is greater than the sum of their individual values. For example, consider a left shoe and a right shoe, or two adjacent pieces of real estate.

Definition 11.3.2 (Complementarity) *Bidder i's valuation v_i exhibits* complementarity *if there exist two sets of goods $G_1, G_2 \subseteq G$, such that $G_1 \cap G_2 = \emptyset$ and $v(G_1 \cup G_2) > v(G_1) + v(G_2)$. When this condition holds, we say that the valuation function v_i is* superadditive.

complementarity

How should an auctioneer sell goods when faced with such bidders? One approach is simply to sell the goods individually, ignoring the bidders' valuations. This is easy for the seller, but it makes things difficult for the bidders. In particular, it presents them with what is called the *exposure problem*: a bidder might bid aggressively for a set of goods in the hopes of winning a bundle, but succeed in winning only a subset of the goods and therefore pay too much. This problem is especially likely to arise in settings where bidders' valuations exhibit strong complementarities, because in these cases bidders might be willing to pay substantially more for bundles of goods than they would pay if the goods were sold separately.

exposure problem

The next-simplest method is to run essentially separate auctions for the different goods, but to connect them in certain ways. For example, one could hold a multiround (e.g., Japanese) auction, but synchronize the rounds in the different auctions so that as a bidder bids in one auction he has a reasonably good indication of what is transpiring in the other auctions of interest. This approach can be made more effective through the establishment of constraints on bidding that span all the auctions (so-called activity rules). For example, bidders might be allowed to increase their aggregate bid amount by only a certain percentage from one round to the next, thus providing a disincentive for bidders to fail to participate in early rounds of the auction and thus improving the information transfer between auctions. Bidders might also be subject to other constraints: for example a budget constraint could require that a bidder not exceed a certain total commitment across all auctions. Both of these ideas can be seen in some government auctions for electromagnetic spectrum (where the so-called *simultaneous ascending auction* was used) as well as in some energy auctions. Despite some successes in practice, however, this approach has the drawback that it only mitigates the exposure problem rather than eliminating it entirely.

simultaneous ascending auction

A third approach ties goods together in a more straightforward way: the auctioneer sells all goods in a single auction, and allows bidders to bid directly on bundles of goods. Such mechanisms are called *combinatorial auctions*. This approach eliminates the exposure problem because bidders are guaranteed that their bids will be satisfied "all or nothing." For example a bidder may be permitted to offer \$100 for the pair (TV, DVD player), or to make a disjunctive offer "either \$100 for TV1 or \$90 for TV2, but not both." However, we will see that while combinatorial auctions resolve the exposure problem they raise many other questions. Indeed, these auctions have been the subject of considerable recent study in both economics and computer science, some of which we will describe in the remainder of this section.

combinatorial auction

11.3.1 *Simple combinatorial auction mechanisms*

The simplest reasonable combinatorial auction mechanism is probably the one in which the auctioneer computes the allocation that maximizes the social welfare of the declared valuations (i.e., $\chi = \max_{x\in X} \sum_{i\in N} \hat{v}_i(x)$), and charges the winners their bids (i.e., for all $i \in N$, $\wp_i = \hat{v}_i$). This is a direct generalization of the first-price sealed-bid auction, and like it this naive auction is not incentive compatible. Consider the following simple valuations in a combinatorial auction setting.

Bidder 1	**Bidder 2**	**Bidder 3**
$v_1(x, y) = 100$ $v_1(x) = v_1(y) = 0$	$v_2(x) = 75$ $v_2(x, y) = v_2(y) = 0$	$v_3(y) = 40$ $v_3(x, y) = v_3(x) = 0$

This example makes it easy to show that the auction is not incentive compatible: for example, if agents 1 and 2 bid truthfully, agent 3 is better off declaring, for example, $v_3(y) = 26$. Unfortunately, it is not apparent how to characterize the equilibria of this auction using the techniques that worked in the single-good case: we do not have a simple analytic expression that describes when a bidder wins the auction, and we also lack a revenue equivalence theorem.

An obvious alternative is the method we applied most broadly in the multigood case: VCG. In the example above, VCG would award x to 2 and y to 3. Bidder 2 would pay 60; without him in the auction bidder 1 would have gotten both goods, gaining 100 in value, while with bidder 2 in the auction the other bids only net a total value of 40 (from good x assigned to 3). Similarly, bidder 3 would pay 25; the difference between 100 and 75. The reader can verify that no bidder can gain by unilaterally deviating from truthful bidding, and that bidders strictly lose from some deviations.

As in the multiunit case, VCG has some attractive properties when applied to combinatorial auctions. Specifically, it is dominant-strategy truthful, efficient, *ex post* individual rational and weakly budget balanced (the latter by Theorems 10.4.8 and 10.4.10). The VCG combinatorial auction mechanism is not without shortcomings, however, as we already discussed in Section 10.4.5. (Indeed, though we did not discuss them above, most of these shortcomings also affect the use of VCG in the multiunit case, and some even impact second-price single-good auctions.) For example, a bidder who declares his valuation truthfully has two main reasons to worry—one is that the seller will examine his bid before the auction clears and submit a fake bid just below, thus increasing the amount that the agent would have to pay if he wins. (This is a so-called *shill bid*.) Another possibility is both his competitors and the seller will learn his true valuation and will be able to exploit this information in a future transaction. Indeed, these two reasons are often cited as reasons why VCG auctions are rarely seen in practice. Other issues include the fact that VCG is vulnerable to collusion among bidders, and, conversely, to one bidder masquerading as several different ones (so-called *pseudonymous bidding* or *false-name bidding*). Perhaps the biggest potential hurdle, however, is computational, and it is not specific to VCG. This is the subject of the next section.

shill bid

pseudonymous bidding

false-name bidding

11.3.2 *The winner determination problem*

Both our naive first-price combinatorial auction and the more sophisticated VCG version share an element in common: given the agents' individual declarations $\hat{v}$, they must determine the allocation of goods to agents that maximizes social welfare. That is, we must compute $\max_{x\in X}\sum_{i\in N}\hat{v}_i(x)$. In single-good and single-unit demand multiunit auctions this was simple—we just had to satisfy the agent(s) with the highest valuation(s). In combinatorial auctions, as in the general multiunit auctions we considered in Section 11.2.3, determining the winners is a more challenging computational problem.

winner determination problem, combinatorial auction

Definition 11.3.3 (Winner determination problem (WDP)) *The* winner determination problem *(WDP) for a combinatorial auction, given the agents' declared valuations $\hat{v}$, is to find the social-welfare-maximizing allocation of goods to agents. This problem can be expressed as the following integer program.*

$$\text{maximize} \quad \sum_{i\in N}\sum_{S\subseteq G}\hat{v}_i(S)x_{S,i} \tag{11.15}$$

$$\text{subject to} \quad \sum_{S\ni j}\sum_{i\in N}x_{S,i}\leq 1 \qquad \forall j\in G \tag{11.16}$$

$$\sum_{S\subseteq G}x_{S,i}\leq 1 \qquad \forall i\in N \tag{11.17}$$

$$x_{S,i}=\{0,1\} \qquad \forall S\subseteq G, i\in N \tag{11.18}$$

In this integer programming formulation, the valuations $\hat{v}_i(S)$ are constants and the variables are $x_{S,i}$. These variables are boolean, indicating whether bundle S is allocated to agent i. The objective function (11.15) states that we want to maximize the sum of the agents' declared valuations for the goods they are allocated. Constraint (11.16) ensures that no overlapping bundles of goods are allocated, and constraint (11.17) ensures that no agent receives more than one bundle. (This makes sense since bidders explicitly assign a valuation to *every* subset of the goods.) Finally, constraint (11.18) is what makes this an *integer program*[11] rather than a linear program: no subset can be partially assigned to an agent.

The fact that the WDP is an integer program rather than a linear program is bad news, since only the latter are known to admit a polynomial-time solution. Indeed, a reader familiar with algorithms and complexity may recognize the combinatorial auction allocation problem as a *set packing problem* (SPP). Unfortunately, it is well known that the SPP is NP-complete. This means that it is not likely that a polynomial-time algorithm exists for the problem. Worse, it so happens this problem cannot even be approximated uniformly, meaning that there does not exist a polynomial-time algorithm and a fixed constant $k > 0$ such that for all inputs the algorithm returns a solution that is at least $\frac{1}{k}s^*$, where s^* is the value of the optimal solution for the given input.

set packing problem

There are two primary approaches to getting around the computational problem. First, we can restrict ourselves to a special class of problems for which there is guaranteed to exist a polynomial-time solution. Second, we can resort to heuristic methods that give up the guarantee of polynomial running time, optimality of solution, or both. In both cases, *relaxation methods* are a common approach. One instance of the first approach is to relax the integrality constraint, thereby transforming the problem into a linear program, which is solvable by known methods in polynomial time. In general the solution results in "fractional" allocations, in which fractions of goods are allocated to different bidders. If we are lucky, however, our solution to the LP will just happen to be integral.

relaxation method

Polynomial methods

There are several sets of conditions under which such luck is assured. The most common of these is called *total unimodularity* (TU). In general terms, a constraint matrix A (see Appendix B) is TU if the determinant of every square submatrix is 0, 1, or -1. In the case of the combinatorial auction WDP, this condition (via constraint (11.16)) amounts to a restriction on the subsets that bidders are permitted to bid on. How do we find out if a particular matrix is TU? There are many ways. First, there exists a polynomial-time algorithm to decide whether an arbitrary matrix is TU. Second, we can characterize important subclasses of TU matrices. While many of these subclasses defy an intuitive interpretation, in the following discussion we will present a few special cases that are relevant to combinatorial auctions.

total unimodularity

11. Integer programs and linear programs are defined in Appendix B.

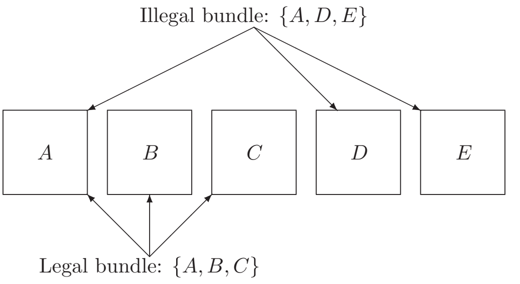


Figure 11.2 Example of legal and illegal bundles for contiguous pieces of lands when we demand the *contiguous ones property*.

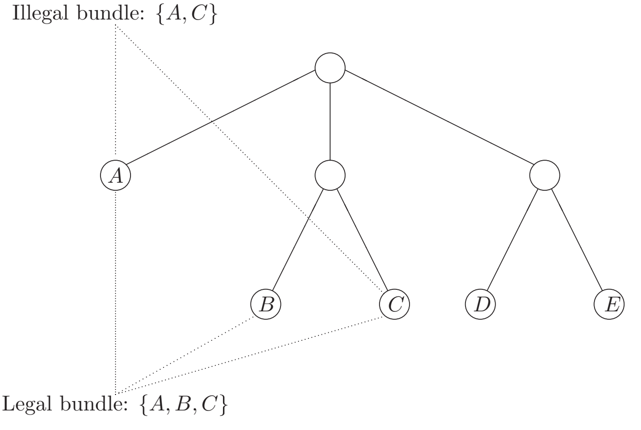


Figure 11.3 Example of a tree-structured bid.

One important subclass of TU matrices is the class of 0–1 matrices with the *consecutive ones property*. In this subclass, all nonzero entries in each column must appear consecutively. This corresponds roughly to contiguous single-dimensional goods, such as time intervals or parcels of land along a shoreline (as shown in Figure 11.2), where bids can only be made on bundles of contiguous goods.

consecutive ones property

Another subclass of auction problems that have integral polyhedra, and thus can be easily solved using linear programming, corresponds to the set of *balanced matrices*. A 0–1 matrix is balanced if it has no square submatrix of odd order with exactly two 1's in each row and column. One class of auction problems that is known to have a balanced matrix are those that allow only *tree-structured bids*, as illustrated in Figure 11.3. Consider that the set of goods for sale are the vertices of a tree, connected by some set of edges. All bids must be on bundles of the form (j, r), which represents the set of vertices that are within distance r of item j. The constraint matrix for this set of possible bundles is indeed balanced, and so the corresponding polyhedron is integral, and the solution can be found using linear programming.

balanced matrix

tree-structured bid

Yet another subclass that can be solved efficiently restricts the bids to be on bundles of no more than two items. The technique here uses dynamic programming, and the algorithm runs in cubic time. Finally, there are a number of positive results covering cases where bidders' valuation functions are subadditive.

Heuristic methods

In many cases the solutions to the associated linear program will not be integral. In these cases we must resort to using *heuristic methods* to find solutions to the auction problem.

Heuristic methods come in two broad varieties. The first is *complete* heuristic methods, which are guaranteed to find an optimal solution if one exists. Despite their discouraging worst-case guarantees, in practice complete heuristic methods are able to solve many interesting problems within reasonable amounts of time. This makes such algorithms the tool of choice for many practical combinatorial auctions. One drawback is that it can be difficult to anticipate how long such algorithms will require to solve novel problem instances, as in the end their performance depends on a problem instance's combinatorial structure rather than on easily-measured parameters like the number of goods or bids. Complete heuristic algorithms tend to perform tree search (to guarantee completeness) along with some sort of pruning technique to reduce the amount of search required (the heuristic).

The second flavor of heuristic algorithm is *incomplete* methods, which are not guaranteed to find optimal solutions. Indeed, as was mentioned earlier, in general there does not even exist a tractable algorithm that can guarantee that you will reach an approximate solution that is within a *fixed fraction* of the optimal solution, no matter how small the fraction. However, methods do exist that can guarantee a solution that is within $1/\sqrt{k}$ of the optimal solution, where k is the number of goods. More importantly, like their complete cousins, incomplete heuristic algorithms often perform very well in practice despite these theoretical caveats. One example of an incomplete heuristic method is a greedy algorithm, a technique that builds up an allocation by adding one bid at a time, and never reconsidering a bid once it has been allocated. Another example is a local search algorithm, in which states in the search space are complete—but possibly infeasible—allocations, and in which the search progresses by modifying these allocations either randomly or greedily.

11.3.3 *Expressing a bid: bidding languages*

We have so far assumed that bidders will specify a valuation for every subset of the goods at auction. Since there are an exponential number of such subsets, this will quickly become impossible as the number of goods grows. If we are to have any hope of finding tractable mechanisms for general combinatorial auctions, we must first find a way for bidders to express their bids in a more succinct manner. In this section we present a number of bidding languages that have been proposed for encoding bids.

As we will see, these languages differ in the ways that they express different classes of bids. We can state some desirable properties that we might like to have in a bidding language. First, we want our language to be *expressive* enough to represent all possible valuation functions. Second, we want our language to be *concise*, so that expressing commonly-used bids does not take space that is exponential in the number of goods. Third, we want our language to be *natural* for humans to both understand and create; thus the structure of the bids should reflect the way in which we think about them. Finally, we want our language to be *tractable* for the auctioneer's algorithms to process when computing an allocation.

In the discussion that follows, for convenience we will often speak about bids as valuation functions. Indeed, in the most general case a bid will contain a valuation for every possible combination of goods. However, be aware that the bid valuations may or may not reflect the players' true underlying valuations. We also limit the scope of our discussion to valuation functions in which the following properties hold:

- *Free disposal:* Goods have nonnegative value, so that if $S \subseteq T$ then $v_i(S) \leq v_i(T)$.
- *Nothing-for-nothing:* $v_i(\emptyset) = 0$ (In other words, a bidder who gets no goods also gets no utility.)

We already discussed multiunit valuations in Section 11.2.3 (additive; single item; fixed budget; majority; general). Combinatorial auctions are different, however, because bidders are expected to value the different goods asymmetrically. For example, there may be different classes of goods, and valuations for sets of goods may be a function of the classes of goods in the set. Imagine that our set G consists of two classes of goods: some red items and some green items, and the bidder requires only items of the same color. Alternatively, it could be the case that the bidder wants exactly one item from each class.

Atomic bids

Let us begin to build up some languages for expressing such bids. Perhaps the most basic bid requests just one particular subset of the goods. We call such a bid an *atomic bid*. An atomic bid is a pair (S, p) that indicates that the agent is willing to pay a price of p for the subset of goods S; we denote this value by $v(S) = p$. Note that an atomic bid implicitly represents an AND operator between the different goods in the bundle. We stated an atomic bid above when we wanted to bid on the TV *and* the DVD player for \$100.

atomic bid

OR bids

Of course, many simple bids cannot be expressed as an atomic bid; for example, it is easy to verify that an atomic bid cannot represent even the additive valuation defined earlier. In order to represent this valuation, we will need to be able to bid on disjunctions of atomic valuations. An *OR bid* is a disjunction of atomic bids $(S_1, p_1) \vee (S_2, p_2) \vee \cdots \vee (S_k, p_k)$ that indicates that the agent is willing to pay a price of p_1 for the subset of goods S_1, or a price of p_2 for the subset of goods S_2, etc.

OR bid

To define the semantics of an OR bid precisely, we interpret OR as an operator for combining valuation functions. Let V be the space of possible valuation functions, and $v_1, v_2 \in V$ be arbitrary valuation functions. Then we have that

$$(v_1 \vee v_2)(S) = \max_{R,T \subseteq S, R \cap T = \emptyset} (v_1(R) + v_2(T)).$$

It is easy to verify that an OR bid can express the additive valuation. As the following result shows, its power is still quite limited; for example, it cannot express the single item valuation described earlier.

Theorem 11.3.4 *OR bids can express all valuation functions that exhibit no substitutability, and only these.*

For example, in the consumer auction setting described earlier, we may have wanted to bid on either the TV and the DVD player for $100, or the TV and the satellite dish for $150, but not both. It is not possible for us to express this using OR bids.

XOR bids

XOR bid

XOR bids do not have this limitation. An XOR bid is an exclusive OR of atomic bids $(S_1, p_1) \oplus (S_2, p_2) \oplus \cdots \oplus (S_k, p_k)$ that indicates that the agent is willing to accept one but no more than one of the atomic bids.

Once again, the XOR operator is actually defined on the space of valuation functions. We can define its semantics precisely as follows. Let V be the space of possible valuation functions, and $v_1, v_2 \in V$ be arbitrary valuation functions. Then we have that

$$(v_1 \oplus v_2)(S) = \max(v_1(S), v_2(S)).$$

We can use XOR bids to express our example from above:

$$(\{\text{TV}, \text{DVD}\}, 100) \oplus (\{\text{TV}, \text{Dish}\}, 150).$$

It is easy to see that XOR bids have unlimited representational power, since it is possible to construct a bid for an arbitrary valuation using an XOR of the atomic valuations for every possible subset $S \subseteq G$.

Theorem 11.3.5 *XOR bids can represent all possible valuation functions.*

However, this does not imply that XOR bids represent every valuation function efficiently. In fact, as the following result states, there are simple valuations that can be represented by short OR bids but that require XOR bids of exponential size.

Theorem 11.3.6 *Additive valuations can be represented by OR bids in linear space, but require exponential space if represented by XOR bids.*

Note that for the purposes of the present discussion, we consider the size of a bid to be the number of atomic formulas that it contains.

Combining the OR and XOR operators

We can also create bidding languages by combining the OR and XOR operators on valuation functions. Consider a language that allows bids that are of the form of an OR of XOR of atomic bids. We call these bids *OR-of-XOR bids*. An *OR-of-XOR bid* is a set of XOR bids, as defined above, such that the bidder is willing to obtain any number of these bids.

OR-of-XOR bid

Like XOR bids, OR-of-XOR bids have unlimited representational power. However, unlike XOR bids, they can specialize to plain OR bids, which affords greater simplicity of expression, as we have seen above. As a specific example, OR-of-XOR bids can express any downward sloping symmetric valuation on m items in size of only m^2. However, this language's compactness is still limited. For example, even simple asymmetric valuations can require size of at least $2^{m/2+1}$ to express in the OR-of-XOR language.

It is also possible to define a language of XOR-of-OR bids, and even a language allowing arbitrary nesting of OR and XOR statements here (we refer to the latter as generalized OR/XOR bids). These languages vary in their compactness.

The OR* bidding language

Now we turn to a slightly different sort of bidding language that is powerful enough to simulate all of the preceding languages with a relatively succinct representation. This language results from the insight that it is possible to simulate the effect of an XOR by allowing bids to include *dummy* (or *phantom*) items. The only difference between an OR and an XOR is that the latter is exclusive; we can enforce this exclusivity in the OR by ensuring that all of the sets in the disjunction share a common item. We call this language *OR**.

OR* bid

Definition 11.3.7 (OR* bid) *Given a set of dummy items G_i for each agent $i \in N$, an* OR* bid *is a disjunction of atomic bids $(S_1, p_1) \vee (S_2, p_2) \vee \cdots \vee (S_k, p_k)$, where for each $l = 1, \ldots, k$, the agent is willing to pay a price of p_l for the set of items $S_l \subseteq G \cup G_i$.*

An example will help make this clearer. If we wanted to express our TV bid from above using dummy items, we would create a single dummy item D, and express the bid as follows.

$$(\{\text{TV}, \text{DVD}, \text{D}\}, 100) \vee (\{\text{TV}, \text{Dish}, \text{D}\}, 150)$$

Any auction procedure that does not award one good to two people will select at most one of these disjuncts. The following results show us that the OR* language is surprisingly expressive and simple.

Theorem 11.3.8 *Any valuation that can be represented by OR-of-XOR bids of size s can also be represented by OR* bids of size s, using at most s dummy items.*

Theorem 11.3.9 *Any valuation that can be represented by OR/XOR bids of size s can also be represented by OR* bids of size s, using at most s^2 dummy items.*

By the definition of OR/XOR bids, we have the following corollary.

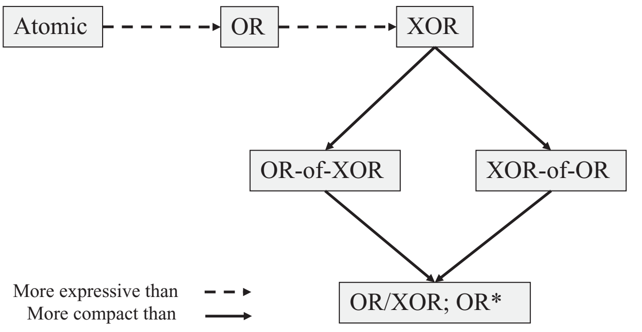


Figure 11.4 Relationships between different bidding languages.

Corollary 11.3.10 *Any valuation that can be represented by XOR-of-OR bids of size s can also be represented by OR* bids of size s, using at most s^2 dummy items.*

Let us briefly review the properties of the languages we have discussed. The XOR, OR-of-XORs, XOR-of-OR and OR* languages are all powerful enough to express all valuations. Second, the efficiencies of the OR-of-XOR and XOR-of-OR languages are incomparable: there are bids that can be expressed succinctly in one but not the other, and vice-versa. Third, the OR* language is strictly more compact than both the OR-of-XOR and XOR-of-OR languages: it can efficiently simulate both languages, and can succinctly express some valuations that require exponential size in each of them. These properties are summarized in Figure 11.4.

Interpretation and verification complexity

Recall that in the auction setting these languages are used for communicating bids to the auctioneer. It is the auctioneer's job to first interpret these bids, and then calculate an allocation of goods to agents. Thus it is natural to be concerned about the computational complexity of a given bidding language. In particular, we may want to know how difficult it is to take an arbitrary bid in some language and compute the valuation of some arbitrary subset of goods according to that bid. We call this the *interpretation complexity*. The interpretation complexity of a bidding language is the minimum time required to compute the valuation $v(S)$, given input of an arbitrary subset $S \subseteq G$ and arbitrary bid v in the language.

interpretation complexity

Not surprisingly, the atomic bidding language has interpretation complexity that is polynomial in the size of the bid. To compute the valuation of some arbitrary subset S, one need only check whether all members of S are in the atomic bid. If they are, the valuation of S is just that given in the bid (because of free disposal); and if they are not, then the valuation of S is 0. The XOR bidding language also has interpretation complexity that is polynomial in the size of the bid; just perform the above procedure for each of the atomic bids in turn. However, all of the other bidding languages mentioned above have interpretation complexity that is exponential in the size of the bid. For example, given the OR bid $(S_1, p_1) \vee (S_2, p_2) \vee \cdots \vee (S_k, p_k)$, computing the valuation of S requires checking all possible combinations of the atomic bids, and there are 2^k such possible combinations.

One might ask why we even consider bidding languages that have exponential interpretation complexity. Simply stated, the answer is that languages with only polynomial interpretation complexity are either not expressive enough or not compact enough. This brings us to a more relaxed criterion. It may be sufficient to require that a given claim about a bid's valuation for a set is *verifiable* in polynomial time. We define the *verification complexity* of a bidding language as the minimum time required to verify the valuation $v(S)$, given input of an arbitrary subset $S \subseteq G$, an arbitrary bid v in the language, and a proof of the proposed valuation $v(S)$. All of the languages we have presented in this section have polynomial verification complexity.

verification complexity

11.3.4 *Iterative mechanisms*

We have argued that in a combinatorial auction setting, agents' valuations can be so complex that they cannot be tractably communicated. The idea behind bidding languages (Section 11.3.3) is that this communication limitation can be overcome if an agent's valuation can be succinctly represented. In this section we consider another, somewhat independent idea: replacing the sealed-bid mechanism we have discussed so far with an indirect mechanism that probes agents' valuations only as necessary.

Intuitively, the use of indirect mechanisms in combinatorial auctions offers the possibility of several benefits. Most fundamentally, allowing the mechanism to query bidders selectively can reduce communication. For example, if the mechanism arrived at the desired outcome (say the VCG allocation and payments) after a small number of queries, other agents could realize that they were unable to make a bid that would improve the allocation, and could thus quit the auction without communicating any more information to the auctioneer. This sort of reduction in communication is still useful even if the auction is small enough that agents' full valuations *could* be tractably conveyed. First, reducing communication can benefit bidders who want to reveal as little as possible about their valuations to their competitors and/or to the auctioneer. Second, iterative mechanisms can help in cases where it is difficult for bidders to determine their own valuations. For example, in a logistics domain a bidder might have to solve a traveling salesman problem to determine his valuation for a given bundle and thus could face a computational barrier to determining his whole valuation function.

Indirect mechanisms also have benefits that go beyond reducing communication. First, they can be easier for bidders to understand than complex direct mechanisms like VCG, and so can be seen as more transparent. This matters, for example, in government auctions of public assets like radio spectrum, where taxpayers want to be assured that the auction was conducted fairly. Finally, while no general result shows this formally, experience with single-good auctions in the common and affiliated values case (Section 11.1.10) suggests that allowing bidders to iteratively exchange partial information about their valuations may lead to improvements in both revenue and efficiency.

Of course, considering iterative mechanisms also invites new challenges. Most of all, such mechanisms are tremendously complicated, and hence can require extensive effort to design. Furthermore, small flaws in this design can lead to huge problems. For example, iterative mechanisms can give rise to considerably richer strategy spaces than direct mechanisms do: an agent can condition his actions on everything he has learned about the actions taken previously by other agents. Beyond potentially making things complicated for the agents, this strategic flexibility can also facilitate undesirable behavior. For example, agents can bid insincerely in order to signal each other (e.g., "do not bid on my bundle and I will not bid on yours"), and thus collude against the seller. Another problem is that agents can often gain by waiting for others to reveal information, especially in settings where determining one's own valuation is costly. The auction must therefore be designed in a way that gives agents some reason to bid early. One approach is to establish activity rules that restrict an agent's future participation in the auction if he does not remain sufficiently active. This idea was already discussed at the beginning of Section 11.3 in the context of decoupled auctions for combinatorial settings such as the simultaneous ascending auction.

Because iterative mechanisms can become quite complex, we will not formally describe any of them here. (As always, interested readers should consult the references cited at the end of the chapter.) However, we will note some of the main questions addressed in this literature, and some of the general trends in the answers to these questions. The first question is what social choice functions to implement, and thus what payments to impose. Here a popular choice is to design mechanisms that converge to an efficient allocation, and that elicit enough information to guarantee that agents pay the same amounts that they would under VCG. Even if an indirect mechanism mimics VCG, it does not automatically inherit its equilibrium properties—the revelation principle only covers the transformation of indirect mechanisms *into* direct mechanisms. Indeed, under indirect mechanisms that mimic VCG, answering queries honestly is no longer a dominant strategy. For example, if agent i knows that agent j will overbid, agent i may also want to do so, as dishonest declarations by j can affect the queries that i will receive in the future. Nevertheless, it can be shown that it is an *ex post* equilibrium (see Section 6.3.4) for agents to cooperate with any indirect mechanism that achieves the same allocation and payment as VCG when all bidders but some bidder i bid truthfully and i bids arbitrarily. In some cases mechanism designers have considered mechanisms that do *not* always converge to VCG payments. Here they usually also assume that agents will bid "straightforwardly"—that is, that they will answer queries truthfully even if it is not in their interest to do. This assumption is typically justified by demonstrations that agents can gain very little by deviations even in the worst case (i.e., straightforward bidding is an ϵ-equilibrium for small ϵ), claims that determining a profitable deviation would be computationally intractable, and/or an appeal to a complete-information Nash analysis. As long as bidders do behave in a way consistent with this assumption, these iterative mechanisms are able to avoid some of the undesirable properties of VCG discussed in Section 10.4.5.

A second question is what sorts of queries the indirect mechanisms should ask. The most popular are probably *value queries* and *demand queries*. A mechanism asks a value query when it suggests a bundle and asks how much it is worth to a bidder. Demand queries are in some sense the reverse: the mechanism asks which bundle the bidder would prefer at given prices. Demand queries come in various different forms: the simplest have prices only on single goods and offer the same prices to all bidders, while the most complicated attach bidder-specific prices to bundles. When only demand queries are used and prices (of whatever form) are guaranteed to rise as the auction progresses, we call the auction an *ascending combinatorial auction*. (Observe that a Japanese auction is a single-good special case of such an auction.) Of course, all sorts of other query families are also possible. For example, bidders can be asked *order queries* (state which of two bundles is preferred) and *bounding queries* (answer whether a given bundle is worth more or less than a given amount). Some mechanisms even allow *push-pull queries*: bidders can answer questions they weren't asked, and can decline to answer questions they were asked.

value query

demand query

ascending combinatorial auction

order query

bounding query

The final general lesson to convey from the literature on iterative combinatorial auctions is that in the worst case, any mechanism that achieves an efficient or approximately efficient[12] allocation in equilibrium must receive an amount of information equal in size to a single agent's complete valuation. Since the size of an agent's valuation is exponential in the number of goods, this is discouraging. However, this result speaks only about the worst case and only about bidders with unrestricted valuations. Researchers have managed to show theoretically that worst-case communication requirements are polynomial under some restricted classes of valuations or when the query complexity is parameterized by the minimal representation size in a given bidding language, and to demonstrate empirically that iterative mechanisms can terminate after reasonable amounts of communication when used with real bidders.

11.3.5 *A tractable mechanism*

tractability

Recall that a *tractable* mechanism is one that can determine the allocation and payments using only polynomial-time computation (Definition 10.3.10). We have seen that such a mechanism can easily be achieved by restricting bidders to expressing valuations from a set that makes the winner determination problem tractable (as discussed in Section 11.3.2), and then using VCG. Here we look beyond such bidding restrictions, seeking more general mechanisms that nevertheless remain computationally feasible. The idea here is to build a mechanism around an optimization algorithm that is guaranteed to run in polynomial time regardless of its inputs.

We will give one example of such a mechanism: a dominant-strategy truthful mechanism for combinatorial auctions that is built around a greedy algorithm.

12. Formally, this result holds for any auction that always finds allocations that achieve more than a $\frac{1}{n}$-fraction of the optimal social welfare.

This mechanism only works for bidders with a restricted class of valuations, called *single minded*.

single-minded

Definition 11.3.11 (Single-minded bidder) *A bidder is* single-minded *if he has the valuation function:*

$$\forall s \in 2^G, \qquad v_i(s) = \begin{cases} v_i > 0 & \text{if } s \supseteq b_i; \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, a bidder is single-minded if he is only interested in a single bundle; he values this bundle and all supersets of it[13] at the same amount, v_i, and values all other bundles at zero. Although this is a severe restriction on agents' valuations, it does not make the winner determination problem any easier. Intuitively, this is because agents remain free to choose *any* bundle s from the set of possible bundles 2^G, and so the auctioneer can receive bids on any set of bundles.

In a direct mechanism for single-minded bidders, every bidder i places a bid $b_i = (\hat{s}_i, \hat{v}_i)$ indicating a bundle of interest $\hat{s}_i$ and an amount $\hat{v}_i$. (Observe that we have assumed that the auctioneer does not know bidder i's bundle of interest s_i; this is why i's bid must have two parts.) Let $apg_i = \hat{v}_i / |\hat{s}_i|$ denote bidder i's declared amount per good, and as before let n be the number of bidders. Now consider the following greedy algorithm.

greedy allocation scheme

Definition 11.3.12 (Greedy allocation scheme) *The* greedy allocation scheme *is defined as follows.*

store the bidders' bids in a list L, sorted in decreasing order of apg
let $L(j)$ denote the j^{th} element of L
$a \leftarrow \emptyset$
$j \leftarrow 1$
while $j \leq n$ **do**
    if $a \cap \hat{s}_j = \emptyset$ **then**
        bid b_j wins
        $a \leftarrow a \cup \hat{s}_j$
foreach *winning bidder* i **do**
    look for a bidder $inext$, the first bidder whose bid follows i's in L, whose bid does not win, and whose bid does win if the greedy allocation scheme is run again with i's bid omitted
    if *a bidder* $inext$ *exists* **then**
        bidder i's payment is $p_i \leftarrow \frac{|\hat{s}_i| \cdot \hat{v}_{inext}}{|\hat{s}_{inext}|}$
    else
        $p_i \leftarrow 0$
foreach *losing bidder* i **do**
    $p_i = \leftarrow 0$

13. Implicitly, this amounts to an assumption of free disposal, as defined in Section 11.3.3.

Intuitively, the greedy allocation scheme ranks all bids in decreasing order of apg, and then greedily allocates bids starting from the top of L. The payment of a winning bidder i is the apg of the highest-ranked bidder that would have won but for i's participation, multiplied by the number of goods allocated to i. Bidder i pays zero if he does not win or if there is no bidder $inext$.

Theorem 11.3.13 *When bidders are single minded, the greedy allocation scheme is dominant-strategy truthful.*

We leave the proof of this theorem as an exercise; however, it is not hard. Observe that, because it is based on a greedy algorithm, this mechanism does not select an efficient allocation. It is natural to wonder whether this mechanism can come close. It turns out that the best that can be achieved comes from modifying the algorithm, replacing apg_i with the ratio $\widehat{v}_i/\sqrt{|\widehat{s}_i|}$. This can be shown to preserve dominant-strategy truthfulness, and to achieve the $1/\sqrt{k}$-bound on efficiency discussed above.

11.4 Exchanges

So far, we have surveyed single-good, multiunit, and combinatorial auctions. Despite the wide variety within these families, we have not yet exhausted the space of auctions. We now briefly discuss one last, important category of auctions. These are *exchanges*: auctions in which agents are able to act as both buyers and sellers. We discuss two varieties, which differ more in their purposes than in their mechanics. The first is intended to allocate goods; the second is designed to aggregate information.

exchange

11.4.1 *Two-sided auctions*

two-sided auction

double auction

In *two-sided auctions*, otherwise known as *double auctions*, there are many buyers and sellers. A typical example is the stock market, where many people are interested in buying or selling any given stock. It is important to distinguish this setting from certain marketplaces (such as popular consumer auction sites) in which there are multiple separate single-sided auctions. We will not have much to say about double auctions, in part because the relative dearth of theoretical results about them. However, let us mention two primary models of single-dimensional double markets, that is, markets in which there are many potential buyers and sellers of many units of the same good (e.g., the shares of a given company). We distinguish here between two kinds of markets, the *continuous double auction* (CDA) and the *periodic double auction* (otherwise known as the *call market*).

continuous double auction (CDA)

periodic double auction

call market

order book

In both the CDA and the call market agents bid at their own pace and as many times as they want. Each bid consists of a price and quantity, where the quantity is either positive (signifying a "buy" order) or negative (signifying a "sell" order). There are no constraints on what the price or quantity might be. Also in both cases, the bids received are put in a central repository, the *order book*. Where the

before	Sell:	5@\$1	3@\$2	6@\$4	2@\$6	4@\$9	
	Buy:	6@\$9	4@\$5	6@\$4	3@\$3	5@\$2	2@\$1

$\Downarrow$

after	Sell:	2@\$6	4@\$9		
	Buy:	2@\$4	3@\$3	5@\$2	2@\$1

Figure 11.5 A call-market order book, before and after market clears.

CDA and call market diverge is on the question of when a trade occurs. In the CDA, as soon as the bid is received, an attempt is made to match it with one or more bids on the order book; for example, a new sell order for 10 units may be matched with one existing buy bid for 4 units and another buy bid for 6 units, so long as both the buy-bid prices are higher than the sell price. In cases of partial matches, the remaining units (either of the new bid or of one of order-book bids) is put back on the order book. For example, if the new sell order is for 13 units and the only buy bids on the order book with a higher price are the ones described (one buy bid for 4 units and another buy bid for 6 units), two trades are arranged—one for 4 units, and one for 6—and the remaining 3 units of the new bid are put on the order book as a sell order. (We have not mentioned the price of the trades arranged; obviously, they must lie in the interval between the price in the buy bid and the price in the sell bid—the so called bid-ask spread—but are unconstrained otherwise. Indeed, the amount paid to the seller could be less than the amount charged to the buyer, allowing a commission for the exchange or broker.)

In contrast, when a bid arrives in the call market, it is simply placed in the order book. No trade is attempted. Then, at some predetermined time, an attempt is made to arrange the maximal amount of trade possible (called clearing the market). In this case this is done simply by ranking the sell bids in ascending order, the buy bids in descending order, and finding the point at which supply meets demand. Figure 11.5 depicts a typical call market. In this example 14 units are traded when the market clears, after the remaining bids are left on the order book awaiting the next market clear.

11.4.2 *Prediction markets*

prediction market

information market

A *prediction market*, also called an *information market*, is a double-sided auction that is used to aggregate agents' beliefs rather than allocating goods. For example, such a market could be used to assess different candidates' chances in an upcoming presidential election. To set up a prediction market, the market designer establishes contracts $(c_1, \ldots, c_k)$, where each contract c_i is a commitment to pay the bearer \$1 if candidate i wins the election, and \$0 otherwise. However, the market designer does not actually sell such contracts himself. Instead, he simply opens up his market, and allows interested agents to both buy and sell contracts with each other. The reason that such a market is interesting is that a risk-neutral bidder should value a contract c_i at exactly the probability that i will win the election. If a bidder believes that he has information about the election that other bidders do not have, and consequently believes that the probability of i winning is greater than the current asking price for contract c_i then i will want to buy contracts. Conversely, if a bidder believes the true probability of i winning is less than the current price, he will want to sell contracts. In equilibrium, prices reflect bidders' aggregate beliefs. This is similar to the idea that price of a company's stock in a stock market will reflect the market's belief about the company's future profitability. Indeed, an even tighter collection is to futures markets, in which traders can purchase delivery of a commodity (e.g., oil) at a future date; prices in these markets are commonly treated as forecasts about future commodity prices. The key distinction is that a prediction market is designed primarily to elicit such information, rather than doing so as a side effect of solving an allocation problem. On the other hand, futures markets are used primarily for hedging risk (e.g., an airline company might buy oil futures in order to defend itself against the risk that oil prices will rise, causing it to lose money on tickets it has already sold).

Prediction markets have been used in practice for a wide variety of belief aggregation tasks, including political polling, forecasting the outcomes of sporting events, and predicting the box office returns of new movies. Of course, there are other methods that can be used for all of these tasks. In particular, opinion polls and surveys of experts are common approaches. However, there is mounting empirical evidence that prediction markets can outperform both of these methods. This is interesting because prediction markets do not need to work with unbiased samples of the population: bidders are not asked what *they* think, but what they think *the whole population* thinks, and they are given an economic incentive to answer correctly. Similarly, prediction markets do not need to explicitly identify or score experts: participants are self-selecting, and weight their input themselves by the size of the position they take in the market. Finally, prediction markets are able to update their forecasts in real time, for example, immediately updating prices to reflect the consequences of a political scandal on an election.

A variety of different mechanisms can be used to implement prediction markets. One straightforward approach is to use continuous double auctions or call markets, as described in the previous section. These mechanisms have the drawback that they can suffer from a lack of liquidity: trades can occur only when agents want contracts on both sides of the market at the same time. Liquidity can be introduced by a market maker. However, such a market maker therefore assumes financial risk, which means the mechanism is not budget balanced. Another approach is a parimutuel market: bidders place money on different outcomes, and when the true outcome is observed, the winning bidders split the pot in proportion to the amount each one gambled. These markets are budget balanced and do not suffer from illiquidity; however, agents have no incentive to place bets before the last moment, and so the market loses the property of aggregating beliefs in real time. Yet more complicated mechanisms, such as dynamic parimutuel markets and market scoring rules, are able to achieve real-time belief aggregation, bound the market-maker's worst-case loss, and still provide substantial liquidity. These markets are too involved to describe briefly here; the reader is referred to the references for details.

11.5 History and references

Krishna [2002] is an excellent book that provides a formal introduction to auction theory. Klemperer [1999b] is a large edited collection of many of the most important papers on the theory of auctions, preceded by a thorough survey by the editor; this survey is reproduced in Klemperer [1999a]. Earlier surveys include Cassady [1967], Wilson [1987a], and McAfee and MacMillan [1987]. These texts cover most of the canonical single-good and multi-unit auction types we discuss in the chapter. (One exception is the elimination auction, which we gave as an example of the diversity of auction types; it was introduced by Fujishima et al. [1999b].)

Vickrey's seminal contribution [Vickrey, 1961] is still recommended reading for anyone interested in auctions. In it Vickrey introduced the second-price auction and argued that bidders in such an auction do best when they bid sincerely. He also provided the analysis of the first-price auction under the independent private value model with the uniform distribution described in the chapter. He even proved an early version of the revenue-equivalence theorem (Theorem 11.1.4), namely that in the independent private value case, the English, Dutch, first-price, and second-price auctions all produce the same expected revenue for the seller. For his work, Vickrey received a Nobel Prize in 1996.

The more general form of the revenue-equivalence theorem, Theorem 11.1.4, is due to Myerson [1981] and Riley and Samuelson [1981], who also investigated optimal (i.e., revenue-maximizing) auctions. Our proof of the theorem follows Klemperer [1999a]. The "auctions as structured negotiations" point of view was advanced by Wurman et al. [2001]. McAfee and McMillan [1987] introduced the notion of auctions with an uncertain number of bidders, and Harstad et al. [1990] analyzed its equilibrium. The so-called Wilson doctrine was articulated in Wilson [1987b]. The result that one additional bidder yields more revenue than an optimal reserve price is due to Bulow and Klemperer [1996]. The most influential theoretical studies of collusion were by Graham and Marshall [1987] for second-price auctions and McAfee and McMillan [1992] for first-price auctions. Early important results on the common-value (CV) model include Wilson [1969] and Milgrom [1981]. The former showed that when bidders are uncertain about their values their bids are not truthful, but rather are lower that their assessment of that value. Milgrom [1981] analyzed the symmetric equilibrium for second-price auctions under common values. The affiliated value model was introduced by Milgrom and Weber [1982].

The equilibrium analysis of sequential auctions is due to Milgrom and Weber [2000], from a seminal paper written in 1982 but only published recently. The random sampling optimal price auction and the first proof that this auction achieves a constant fraction of optimal fixed-price revenue is due to Goldberg et al. [2006]; the $\frac{1}{4.68}$ bound on revenue is due to Saeed et al. [2008]. Our discussion of position auctions generally follows Edelman et al. [2007]; see also Varian [2007].

Combinatorial auctions are covered in depth in the edited collection Cramton et al. [2006], which probably provides the best single-source overview of the area. Several chapters of this book are especially worthy of note here. The computational complexity of the WDP is discussed in a chapter by Lehmann et al. Algorithms for the WDP have an involved history, and are reprised in chapters by Müller and Sandholm. A detailed discussion of bidding languages can be found in a chapter by Nisan; the OR* language is due to Fujishima et al. [1999a]. Iterative mechanisms are covered in chapters by Parkes, Ausubel and Milgrom, and Sandholm and Boutilier; the worst-case communication complexity analysis appears in a chapter by Segal. Finally, the tractable greedy mechanism is due to Lehmann et al. [2002].

There is a great deal of empirical evidence that prediction markets can effectively aggregate beliefs; two prominent examples of this literature consider election results [Berg et al., 2001] and movie box office revenues [Spann and Skiera, 2003]. Dynamic parimutuel markets are due to Pennock [2004], and the market scoring rule is described by Hanson [2003].