

3-7 定積分の例題

不定積分が分かれないと場合の特殊な定積分の一例。

[1] $I = \int_0^\infty e^{-x^2} dx$

$$\begin{aligned} x &= \sqrt{z} \\ dz &= z dy \quad \int_0^\infty e^{-x^2} dx \end{aligned}$$

よければ、

$$\begin{aligned} I &= \int_0^\infty e^{-\sqrt{z}^2} z dz \\ &= \int_0^\infty z e^{-z^2} dz \end{aligned}$$

また

$$I = \int_0^\infty e^{-z^2} dz$$

よからぬ（積分変数の文字をかきかえてR'にする）

$$\begin{aligned} I^2 &= \int_0^\infty e^{-z^2} dz \int_0^\infty z e^{-z^2} dz \\ &= \int_{z=0}^\infty \int_{z=0}^\infty z e^{-(z^2+z^2)} dz dz \\ &= \int_{y=0}^\infty \int_{z=0}^\infty z e^{-(1+y^2)} dz dy \\ &= -\left[\frac{e^{-(1+y^2)}}{2(1+y^2)} \right]_{y=0}^\infty \\ &= \frac{1}{2(1+y^2)} \\ &= \int_{y=0}^\infty \frac{1}{2(1+y^2)} dy \\ &= \frac{1}{2} \int_{y=0}^\infty \frac{dy}{1+y^2} \quad \int_{y=0}^\infty \frac{dy}{1+y^2} \\ &= \frac{\pi}{4} \quad \left[\tan^{-1} y \right]_0^\infty \\ I^2 &= \frac{\pi}{4} \quad I > 0 \text{ です} \\ I &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

(別解)

$$I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$$

ゆえに

$$\begin{aligned} I^2 &= \int_{x=0}^\infty \int_{y=0}^\infty e^{-(x^2+y^2)} dy dx \\ x &= r \cos \theta, \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad \left(\because \frac{\partial(x, y)}{\partial(r, \theta)} dr d\theta = r dr d\theta \right) \\ I^2 &= \int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-r^2} \cdot r dr d\theta \\ &= \int_{r=0}^\infty \int_{\theta=0}^{\pi/2} r e^{-r^2} dr d\theta \\ &= \int_{r=0}^\infty r e^{-r^2} dr \int_{\theta=0}^{\pi/2} d\theta \\ &= \frac{\pi}{2} \int_{r=0}^\infty r e^{-r^2} dr \\ &= \frac{\pi}{2} \cdot \left[-\frac{e^{-r^2}}{2} \right]_0^\infty \\ &= \frac{\pi}{2} \cdot \frac{1}{2} \\ &= \frac{\pi}{4}. \end{aligned}$$

∴ $I > 0$ です $I = \frac{\sqrt{\pi}}{2}$

いきやく計算 $\frac{\partial(x, y)}{\partial(r, \theta)} =$

$A(x, y) \approx \Delta x \Delta y$

$B(I + \frac{\partial x}{\partial \theta} d\theta, y + \frac{\partial y}{\partial \theta} d\theta)$

$D(x + \frac{\partial x}{\partial r} dr, y + \frac{\partial y}{\partial r} dr)$

$C(x + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial r} dr, y + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial r} dr)$

□ ABCD の面積

$$\begin{aligned} &= |\vec{AB} \times \vec{AD}| \\ &= \left| \left(\frac{\partial x}{\partial \theta} d\theta, \frac{\partial y}{\partial \theta} d\theta, 0 \right) \times \left(\frac{\partial x}{\partial r} dr, \frac{\partial y}{\partial r} dr, 0 \right) \right| \\ &= \left| \left(\frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r} - \frac{\partial y}{\partial \theta} \frac{\partial x}{\partial r} \right) dr d\theta \right| \\ &= \left| -r \sin \theta \cdot \sin \theta - r \cos \theta \cos \theta \right| dr d\theta \\ &= \left| r (\sin^2 \theta + \cos^2 \theta) \right| dr d\theta \\ &= r dr d\theta, \end{aligned}$$

(応用)

$$\int_0^\infty e^{-ax^2} dx = \frac{\sqrt{\pi}}{2} \quad (3.17)$$

たとえば $x = \sqrt{a} t$ ($a > 0$) とおき $\frac{dx}{dt} = \sqrt{a}$

$$\begin{aligned} \int_0^\infty e^{-ax^2} \frac{dx}{dt} dt &\quad (x: 0 \rightarrow \infty, t: 0 \rightarrow \infty) \\ &= \int_0^\infty \sqrt{a} e^{-a(\sqrt{a}t)^2} dt \\ &= \left[\frac{-e^{-at^2}}{2\sqrt{a}} \right]_0^\infty \\ &= \frac{1}{2} \sqrt{\frac{\pi}{a}} \quad (3.18) \end{aligned}$$

(3.18) は a が正のとき、1回微分です。

$$\int_0^\infty e^{-ax^2} x^{2n} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} \quad (n=1, 2, 3, \dots)$$

よって結果が得られます。

$$\begin{aligned} \frac{d}{da} (a^{-\frac{1}{2}}) &\quad Q. (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} \\ &= -\frac{1}{2} a^{-\frac{3}{2}} \quad \uparrow = n (-1)^n \text{ が} \# \text{ で } 2 \cdot 3 \cdots \\ \frac{d}{da} \left(-\frac{1}{2} a^{-\frac{3}{2}} \right) &\\ &= +\frac{1 \cdot 3}{2^2} a^{-\frac{5}{2}} \\ \frac{d}{da} \left(\frac{1 \cdot 3}{2^2} a^{-\frac{5}{2}} \right) &\\ &= \left(\frac{1 \cdot 3 \cdot 5}{2^3} a^{-\frac{7}{2}} \right) \end{aligned}$$