3-7 定積分の例題【4】Fresnel積分（フレネル積分）

[4]
式 (3.18)

$$\int_0^\infty e^{-at^2}dt = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad \text{—— (3.18)}$$

両辺 $\times \dfrac{2}{\sqrt{\pi}}$

$$\frac{1}{\sqrt{a}} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-at^2}dt$$

tの代わりに x 用いる（文字変えただけ）

両辺に $\sin\alpha$ をかけ α について，$0 \sim \infty$ まで積分する．

ゆえに

$$\int_0^\infty \frac{\sin\alpha}{\sqrt{\alpha}}d\alpha = \frac{2}{\sqrt{\pi}}\int_0^\infty dx\int_0^\infty e^{-\alpha x^2}\sin\alpha\, d\alpha$$

しかるに，

$$\int_0^\infty e^{-\alpha x^2}\sin\alpha\, d\alpha = \frac{1}{1+x^4},$$

また，

$$\int \frac{1}{1+x^4}dx = \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\left\{\tan^{-1}(\sqrt{2}x+1)+\tan^{-1}(\sqrt{2}x-1)\right\}$$

それゆえ

$$\int_0^\infty \frac{\sin x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}} \quad \text{—— (3.29)}$$

同様に

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}} \quad \text{—— (3.30)}$$

(3.29)，(3.30) において，xの代わりに x^2 とかけば

$$\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{—— (3.31)}$$

(3.29) ～ (3.31) は Fresnelの積分と名づけられている．

この式はどこから？

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

〔既知とされている不定積分の公式〕

$x = \alpha$

$a = -x^2$

$b = 1$

とおくと

$$\int_0^\infty e^{-\alpha x^2}\sin\alpha\, d\alpha = \left[\frac{e^{-x^2\alpha}}{(-x^2)^2+1^2}(-x^2\sin\alpha - \cos\alpha)\right]_{\alpha=0}^{\alpha=\infty}$$

$$= 0 - \frac{1}{1+x^4}(-x^2\cdot 0 - 1)$$

$$= \frac{1}{1+x^4}.$$

この不定積分は？

STEP.1. 分母の因数分解

$$x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

$$\therefore \frac{1}{1+x^4} = \frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}$$

STEP.2 部分分数分解

$$\frac{1}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

通分し，係数比較する

$$\frac{(Ax+B)(x^2-\sqrt{2}x+1)+(Cx+D)(x^2+\sqrt{2}x+1)}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}$$

$$\begin{cases} x^3: & A+C & = 0 & \text{—①} \\ x^2: & -\sqrt{2}A+B+D+\sqrt{2}C & = 0 & \text{—②} \\ x: & A-\sqrt{2}B+C+\sqrt{2}D & = 0 & \text{—③} \\ 1: & B+D & = 1 & \text{—④} \end{cases}$$

をみたす A～D が必要

① $A = -C$

② $-2\sqrt{2}A + B + D = 0$ —②'

③ $-\sqrt{2}B + \sqrt{2}D = 0$

$\therefore B = D$

④ $2B = 1$

$B = D = \dfrac{1}{2}$

②' $A = \dfrac{1}{2\sqrt{2}}$ $\therefore A = \dfrac{1}{2\sqrt{2}},\ B = \dfrac{1}{2},\ C = -\dfrac{1}{2\sqrt{2}},\ D = \dfrac{1}{2}.$

よって

$$\frac{1}{1+x^4} = \frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} + \frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}$$

STEP.3 各項を積分

$$\int\frac{1}{x^4+1}dx$$

$$= \frac{1}{2\sqrt{2}}\int\left(\underbrace{\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}}_{前} + \underbrace{\frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}}_{後}\right)dx$$

前 $\displaystyle\int\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx$

$$= \frac{1}{2}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx + \frac{1}{\sqrt{2}}\int\frac{dx}{x^2+\sqrt{2}x+1}$$

$$= \frac{1}{2}\int\frac{(x^2+\sqrt{2}x+1)'}{x^2+\sqrt{2}x+1}dx + \frac{1}{\sqrt{2}}\int\frac{dx}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}$$

$$= \frac{1}{2}\log(x^2+\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x+1)$$

後 も同様に

$$= -\frac{1}{2}\log(x^2-\sqrt{2}x+1) - \tan^{-1}(-\sqrt{2}x+1)$$

※ $\tan^{-1}(-\sqrt{2}x+1) = -\tan^{-1}(\sqrt{2}x-1)$

$$\therefore \int\frac{1}{x^4+1}dx = \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\left\{\tan^{-1}(\sqrt{2}x+1)+\tan^{-1}(\sqrt{2}x-1)\right\}$$

$\displaystyle\int\frac{1}{1+x^4}dx$ について

定積分 $\displaystyle\int_0^\infty\frac{1}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}$ となる。

これがわかると，ここの計算は $\frac{2}{\sqrt{\pi}}$ をかけた結果として $\sqrt{\frac{\pi}{2}}$ と (3.29) (3.30) でなることがわかる。

$\displaystyle\int_0^\infty\frac{1}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}$ を示せれば ok!

$$\int_0^\infty\frac{1}{x^4+1}dx = \left[\frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\left\{\tan^{-1}(\sqrt{2}x+1)+\tan^{-1}(\sqrt{2}x-1)\right\}\right]_0^\infty$$

$x=0$ ときは，$\log\frac{1}{1} = 0$，$\tan^{-1}1 + \tan^{-1}(-1) = 0$ で合計 0.

$x=\infty$ のとき $\displaystyle\lim_{x\to\infty}\log\frac{x^2}{x^2} = \log\frac{1}{1} = 0$

$\tan^{-1}(\infty) = \dfrac{\pi}{2}$

$\therefore$ π がのこる

$$= \frac{\pi}{2\sqrt{2}}.$$

$$\int_0^\infty\frac{\sin x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$$

$x = t^2$ とおく，$x: 0\to\infty$

$dx = 2t\,dt$ $t: 0\to\infty$

$$\int_0^\infty\frac{\sin x}{\sqrt{x}}dx$$

$$= \int_0^\infty\frac{\sin t^2}{t}\cdot 2t\,dt$$

$$= 2\int_0^\infty\sin t^2 dt$$

$$\therefore \int_0^\infty\sin t^2 dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty\frac{\cos x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$$

$x = t^2$ とおく

$x: 0\to\infty$

$t: 0\to\infty$

$dx = 2t\,dt$

$$\int_0^\infty\frac{\cos x}{\sqrt{x}}dx$$

$$= \int_0^\infty\frac{\cos t^2}{t}\cdot 2t\,dt$$

$$= 2\int_0^\infty\cos t^2 dt$$

$$\therefore \int_0^\infty\cos t^2 dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$