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Typical states of a supersymmetric black hole with finite area

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Abstract

Providing a gravitational description of the microstates responsible for the entropy of a black hole is one of the most pressing open problems in theoretical physics. A complete solution has been provided only for a class of degenerate supersymmetric black holes, whose horizon has zero area. For a finite area horizon, only a subset of microstates is known, whose number is insufficient to reproduce the entire black hole entropy. In this thesis we characterize a class of states that, in the macroscopic charge limit, could reproduce the correct trend of the entropy, at least in an appropriate regime. We then construct the simplest sub-class of this family of states by applying a large diffeomorphism to the vacuum. Finally, for these states, we verify the consistency between the gravity and the CFT descriptions by using a holographic approach.

Introduction

During the last century, two of the most important theories of all physics have been developed: Quantum Mechanics and General Relativity.
Quantum Mechanics is used to describe matter and its interactions at the atomic and subatomic level, where classical mechanics breaks down. This theory led to a huge number of discoveries, both theoretical and practical, that radically improved our lives. For example, almost every electronic device contains components that work because of the laws of Quantum Mechanics. The theory also led to the discovery of new subatomic particles, and ultimately to the development of the Standard Model of particles, which nowadays is the most complete theory known, since it describes three of the four fundamental interactions that are known. The discovery of new subatomic particles also led to the development of new technology, in fact some of them are even used to cure cancer (hadron therapy). Due to all their theoretical and practical applications and the significant number of experimental confirmations received during the years, Quantum Mechanics and the Standard Model are believed to be extremely reliable theories. This is actually true, but we know that Quantum Mechanics is not the end of the story, because it breaks down under some circumstances. However, before talking about this, we must introduce General Relativity.
General Relativity has been first described by Albert Einstein in an article published in 1916. The theory is a generalization of Special Relativity, which was again proposed by him in 1905. General Relativity describes the fundamental interaction that is not described by the Standard Model, which is gravity. At the current time, General Relativity has been tested thoroughly, and its major predictions have been verified: we have observed black holes, gravitational lensing and, in recent times, even gravitational waves. Plus, we have important technological applications deriving from it, such as the GPS system. This is all great, but unfortunately there is a regime in which it breaks down too, together with Quantum Mechanics.
Both theories solved a lot of problems, nevertheless they are fundamentally incompatible. In fact, there exist phenomena in which both theories are involved, and when we try to use them, they just stop working. When does this happen?
By means of dimensional analysis, we can take the fundamental constants c, G, $\hbar$ and form a length, an energy and a time, which are called Planck length, Planck energy and Planck time:

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \cdot 10^{-35} \text{ m} \tag{1}$$

$$E_P = \sqrt{\frac{\hbar c^5}{G}} \approx 1.956 \cdot 10^{9} \text{ J} \approx 1.22 \cdot 10^{19} \text{ GeV} \tag{2}$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \approx 5.391 \cdot 10^{-44} \text{ s} \tag{3}$$

If some phenomenon happens at the Planck scale, then we cannot use General Relativity, nor Quantum Mechanics to describe it: we must develop some new theory of Quantum Gravity, which must include both General Relativity and Quantum Mechanics, in some limit. Moreover, it must contain the Planck constants we just defined as fundamental scales of length, time and energy. One may think that not many phenomena exist that take place at the Planck scale. However, this is not the case, in fact we think that a theory of quantum gravity is relevant when describing the very first moments after the Big Bang, up until the Planck time, for example. Moreover, there are some objects in our universe that are so extreme that they involve both gravitational effects and quantum effects: we are talking about neutron stars, magnetars and black holes. Finally, going back to the Standard Model, there is strong evidence coming from the neutrino physics that the theory is an effective theory, therefore there must be more to discover[1].

From this point on, we will focus on black holes, which are the main topic of this thesis. First of all, we observe that for such objects, quantum gravity effects become relevant if the energies involved are close to E_P. In fact, the Schwarzschild radius of the black hole

$$r_s = \frac{2GM}{c^2} \tag{4}$$

and its Compton wavelength

$$\lambda_c = \frac{\hbar}{Mc} \tag{5}$$

coincide when

$$M = \sqrt{\frac{\hbar c}{2G}} \propto \frac{E_P}{c^2} \tag{6}$$

meaning that, at that energy scale, both gravitational and quantum effects are involved. However, when trying to include quantum effects in the description of black holes, many paradoxes arise. First of all, Bekenstein [1] argued that black holes should be assigned an entropy

$$S_{BH} = \frac{A_{Hor}}{4G} \tag{7}$$

where A_{Hor} is the area of the event horizon, and to avoid paradoxes, we must modify the second law of thermodynamics in order to add the black hole entropy to the total entropy of the universe. Therefore, when we deal with a process involving a black hole, we have that

$$\frac{d}{dt}(S_{BH} + S_{Universe}) \geq 0 \tag{8}$$

a typical example that illustrates this is the following: if someone throws a box full of entropy inside a black hole, it causes its entropy to increase, therefore making the horizon area bigger. The entropy of the rest of the universe decreases, but the sum of the two increases, so there is no violation of the second law. Like in all thermodynamic systems, in order for this whole picture to be fully consistent, there is one last piece missing, that is: black holes must be able to emit radiation. It turns out that this is the case, in fact Hawking [2] discovered that particle pairs are produced across the horizon, causing the black hole to irradiate and eventually disappear. This new discovery ultimately lead to two more serious paradoxes, that are still unsolved: the microstate problem and the information paradox.

If we assign the entropy S_{BH} to the black hole, then the principles of statistical mechanics tell us that there must be

$$\Omega = e^{S_{BH}} \tag{9}$$

[1] There ia a possibility that new physics could also arise at energy scales lower than E_P.

microstates responsible for it. If we look at the General Relativity description of the black hole, we find no trace whatsoever of them.
The second serious problem is the information paradox. Let us say we have a star that eventually dies and collapses into a black hole, which after a long time emits radiation and evaporates. At the start of the process, the matter making up the star is found in a pure state. After the black hole has completely evaporated, we are left with radiation in a mixed state: the matter evolved from a pure state into a mixed state, violating unitarity. Actually, Quantum Mechanics forbids this. Various proposals have been advanced in order to solve the theoretical issues. It is clear at this point that we need some theory of quantum gravity in order to tackle the problem. In this thesis we work with Samir Mathur's fuzzball proposal [3], which makes use of the framework of String Theory.
String Theory is a theory that has been developed at the end of the last century. The basic idea behind it is to postulate that the fundamental objects that make up our universe are not zero-dimensional points, but instead one-dimensional strings. This leads to a quite complex theory that could predict all the elementary particles we know, and some completely new ones. Among them, there is the graviton, which is a remarkable result, since its presence allows String Theory to be a candidate for theory of quantum gravity. Along with the new particles, the theory predicts the presence of extra spatial dimensions, which are not detectable at the energy scales reached by the current particle accelerators, making the theory difficult to verify experimentally.
Another reason for which String Theory is difficult to check is related to the string length, which is of the order of l_P, therefore at the current time it is impossible to observe a stringy behaviour, because we would need much more energy. Currently, the LHC accelerator at CERN can explore energy scales up to 10^4 Gev. It is still a long way to reach the Planck energy, so long that some people say that it will never be possible. This fact, together with other issues that make the theory difficult to test, make String Theory quite controversial. However, there are two good reasons for which we should care about it: first, it might actually turn out to be true in a distant future. Second, the maths involved is undeniably true, and the theory allowed to make new important discoveries in this field, such as the AdS/CFT correspondence, which has applications in other branches of physics, that are way less theoretical, such as condensed matter physics or QCD, when studying the quark-gluon plasma.
It is clear that String Theory contains both the Standard Model and General Relativity. Another theory that it contains, which is obtained by taking a low-energy limit, is Supergravity. Supergravity will be briefly introduced in the thesis because we will use it a lot. We now turn our attention to the fuzzball proposal.
It is possible to deal with black holes in Supergravity. In fact, starting from a higher-dimensional version of the Schwarzschild solution, it is possible to construct solutions describing black holes, by making use of some symmetries present in Supergravity. However, the solutions that we obtain do not contain any trace of the black hole microstates that we seek, which is to be expected, since we start from a solution that contains no information about any microscopical configuration. Actually, the solutions in Supergravity describe the macroscopic ensemble generated by the microstates, since they produce the correct[2] entropy S_{BH}. However, to obtain some information about the microstates, we have to use a different approach and start from a microscopic configuration, which is what Mathur and his collaborators did. The basic idea is to look for the geometries sourced by elementary constituents of the theory (strings and D-branes), which macroscopically look like black holes, as we will see.

[2]Actually, not all of them do, but we can fix this problem by including higher curvature corrections in the action.

The solutions that were found have some nice properties. First of all, they are parametrized by a vibration profile: each profile corresponds to a different microstate. Each microstate differs from each other in a sphere of radius $\sim l_P$ centered at the origin of the coordinates. Outside of this sphere, the microstates look all the same because the vibration profile can be ignored, and we recover the solutions obtained from Supergravity, describing the collection of microstates. In this sense macroscopically the solutions look like black holes. This is a similar procedure to coarse graining in statistical mechanics. These solutions do not have horizons, nor singularities: they end in a smooth cap. The absence of the horizon is crucial to the solution of the information paradox, as we will see.

Nowadays, only a subset of microstates is known explicitly, and only a fraction of the total entropy S_{BH} is captured by them. The aim of this thesis is to lay the foundations to build a new class of microstates. In fact, what we will do is just a preliminary calculation, that if continued will hopefully lead to the discovery of a new solution. We are interested in the so-called typical microstates, which are the ones whose physical properties are closest to the ones of the statistical ensemble, and therefore represent it better. The states that we are going to build are not typical, and capture only a small fraction of the entropy. However, our hope is that we can use them to build more complex microstates, that are typical.

In order to perform our calculations, we need to use the AdS/CFT correspondence, which is a duality that connects (d+1)-dimensional string theories on the Anti-de Sitter spacetime to d-dimensional conformal field theories. It is a very powerful tool because it simplifies calculations by a lot. The metrics describing the black holes we consider, when studied in some limit, contain a AdS factor, and thus they admit a dual description in terms of a CFT. The CFT which is dual to our gravitational description is called D1-D5 CFT, and will be introduced during the work.

We will start from the vacuum state in the CFT, that corresponds to $AdS_3 \times S_3 \times T_4$. Then, we will apply operators in order to excite it. This generates new solutions on the gravity side. We will parametrize these solutions with a continuous parameter χ. We will see that for two particular values of χ we obtain known solutions. What is new are the solutions interpolating between them. We will apply only a restricted class of operators that correspond via AdS/CFT to coordinate transformations on the vacuum that do not vanish at infinity. All the solutions considered in this thesis can be obtained by means of some coordinate transformation of the vacuum. However, more general solutions do not have this property. In this sense, the calculation that we are going to perform is just a preliminary step. The final goal is to obtain clues to see what happens when applying more general operators, which will hopefully lead to more general solutions that encode a more substantial fraction of the entropy, hopefully the whole thing.

The work is organized as follows:

- Chapter 1: We briefly review black hole thermodynamics and see how the classical description leads to the information paradox.
- Chapter 2: We introduce the main topics in String Theory and Supergravity.
- Chapter 3: We review the fuzzball proposal and see how it solves both the microstate problem and the information paradox.
- Chapter 4: We introduce the main topics in CFT. Then, we introduce the AdS/CFT correspondence and the D1-D5 CFT.
- Chapter 5: This is where the original contribution to the thesis resides. We introduce a microscopic configuration in the D1-D5 CFT, calculate its entropy, and then construct the

corresponding microstates. Every calculation will be explained in detail.

- Finally, Appendix A contains a brief review of some concepts from General Relativity that can be useful to follow the discussion.

Conventions:
We work in units where $c = \hbar = G = 1$, except from the part of the work from the start of section 2.1.1 up until equation (2.10), and Appendix A. We may occasionally restore the G factor in some formulas because it will be related to some important quantities.
The signature of the Minkowski metric is $(-, +, +, +)$.

Chapter 1

Black hole thermodynamics

In this chapter we provide a brief review of black hole thermodynamics. It consists in four laws that turn out to be analogous to the four laws of thermodynamics. We introduce them, and show how they end up leading to a theoretical inconsistency: the information paradox. In Appendix A some concepts in General Relativity that can be useful to follow the discussion in this chapter are reviewed.

1.1 Surface gravity

Before diving into the four principles of black hole thermodynamics we need to define and understand an important quantity: the surface gravity κ, which can be defined only for black holes whose event horizon is also a killing horizon.
Consider a Killing vector χ^μ, with associated Killing horizon Σ. Since χ^μ is normal to Σ, it obeys the geodesic equation along it.

$$\chi^\nu \nabla_\nu \chi^\mu = -\kappa \chi^\mu \tag{1.1}$$

We immediately see that there is a subtlety in this definition: the surface gravity may not be uniquely defined. That's because we can always rescale a Killing vector by a real number to get another Killing vector. How do we fix this issue? In a spacetime that is static and asymptotically flat, we may normalize the Killing vector associated with time translations $K = \partial_t$ by setting

$$K_\mu K^\mu (r \to \infty) = -1 \tag{1.2}$$

This will fix the surface gravity of any associated Killing horizon uniquely. If the spacetime is not static, but only stationary, the Killing horizon is associated to a linear combination of time translations and rotations. If we fix the normalization of $K = \partial_t$, the linear combination is fixed as well, and the surface gravity is again unique. In the future we will deal with spacetimes that are asymptotically flat and at least stationary, so we will not concern ourselves with this issue anymore.
We can derive a nice formula for the surface gravity[4]:

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^n u)\Big|_\Sigma \tag{1.3}$$

It is important to discuss the physical interpretation of κ. It turns out that in a static, asymp-

totically flat spacetime, the surface gravity is the acceleration of a static observer near the event horizon, as measured by a static observer at spatial infinity.
All stationary black holes have an horizon that is Killing. What about non-stationary black holes? In recent years, various attempts have been made by different authors of defining the surface gravity of dynamical black holes whose spacetime does not admit a Killing vector. As of today, there is no consensus or agreement of which definition, if any, is correct. [5]

1.2 The four principles

We can now proceed to state the four principles of black hole thermodynamics.

1.2.1 The zeroth law

The zeroth law states that

If the stress-energy tensor $T_{\mu\nu}$ satisfies the dominant energy condition, then κ is constant along the event horizon.

We say $T_{\mu\nu}$ satisfies the dominant energy condition if:

1. It satisfies the weak energy condition: that is, for every timelike vector X^μ, the energy density measured by any observer on a timelike curve, is always non-negative:

$$T_{\mu\nu}X^\mu X^\nu \geq 0 \tag{1.4}$$

2. The quantity $-T^\mu_\nu X^\nu$ is a future-pointing vector field.

These conditions express the requirement that the flux of energy-momentum measured by an observer is causal and points towards the direction of its proper time. This means that superluminal motion is forbidden. Equivalently, the dominant energy condition can be expressed as the requirement that in any orthonormal frame the energy density component of the stress energy tensor dominates over all other components.

$$T_{00} \gg |T_{\mu\nu}| \tag{1.5}$$

Conditions 1 and 2 are required to prove the 0th law, since they show up during the calculations [4].
We notice a suggestive analogy between the surface gravity κ of a black hole and the temperature T of a system in thermal equilibrium. The 0th law of thermodynamics states that the temperature of a system is thermal equilibrium is constant throughout it. In an analogous fashion, stationary black holes have constant surface gravity on the whole event horizon.

1.2.2 The first law

The first law can be expressed as follows

If a stationary black hole of mass M, charge Q and angular momentum J, with event horizon of surface gravity κ, electric surface potential Φ_H and angular velocity Ω_H, is perturbed such that it *settles down to another black hole with mass $M + \delta M$ charge $Q + \delta Q$ and angular momentum $J + \delta J$, then*

$$dM = \frac{\kappa}{8\pi} dA_{Hor} + \Omega_H dJ + \Phi_H dQ \tag{1.6}$$

This law is quite general: it works for the Kerr-Newman black hole, for which $Q \neq 0$ and $J \neq 0$, but it is possible to derive it from first principles, in all generality, without relying to a particular class of black holes.
The first principle of thermodynamics states that for a thermal system, the variation of internal energy can be expressed as

$$dU = TdS + \sum_i A_i dB_i \tag{1.7}$$

Where the A_i's are generalized forces and the B_i's are generalized displacements. For example we can have $-PdV$ for a hydrostatic system or $\mu_0 \vec{H} \cdot d\vec{M}$ for a paramagnetic solid.
There is a striking analogy between (1.6) and (1.7), in fact it is quite evident that the two laws possess the same structure, and this prompts us to compare the various terms.

- The mass of the black hole dM plays the role of the internal energy of a thermodynamic system, dU.

- The term $\frac{\kappa}{8\pi} dA_{Hor}$ represents the heat flow. Since we already identified κ with the temperature T, we are lead to think that the black hole must possess an entropy proportional to the horizon area A_{Hor}. At first sight, this looks completely nonsensical. How can an area correspond to an entropy? Actually, this interpretation turns out to be correct. We will see why is that when we discuss the second principle and Hawking radiation.

- The term $\Omega_H dJ$ describes a change in rotational energy: let us consider a rigid body revolving around a fixed axis. The rotational energy is $K_{Rot} = \frac{1}{2} I\omega^2$, and its variation is $dK_{Rot} = I\omega d\omega = \omega dL$. A Kerr black hole behaves similarly, in fact we can identify ω with the angular velocity of the horizon Ω_H and L with its angular momentum J.

- The term $\Phi_H dQ$ describes a change in electrostatic energy. If we have some charge configuration with electrostatic potential V and total charge q, the potential energy stored in it is $U_{Pot} = qV$, and its variation is $dU_{pot} = Vdq$. This suggests an identification between V, q and Φ_H, Q respectively. Thus, a charged black hole can store electrostatic energy.

Equation (1.6) is not the end of the story. If stationary matter (other than the electromagnetic field) is present around the black hole, then there are additional matter terms on the right side. For our purpose, this statement of the first law is more than enough.

1.2.3 The second law

Like the first law, the second law can be proven in all generality. It is a statement of Hawking's area theorem

If $T_{\mu\nu}$ satisfies the weak energy condition, and assuming that the cosmic censorship hypothesis is true, then the area of the event horizon of an asymptotically flat spacetime is a non-decreasing function of time.

$$\frac{dA_{Hor}}{dt} \geq 0 \tag{1.8}$$

This statement is another suggestion that we should interpret the area of the horizon as a form of entropy. In fact, the second law of thermodynamics states that the variation of entropy of an isolated system is greater than or equal to zero for a spontaneous process. Once again we have a strong resemblance between black holes and thermodynamics.
Now, imagine filling a box with electromagnetic radiation. This radiation carries some entropy. Now, throw it inside a black hole. The box surpasses the event horizon, and at the end of the process it gets destroyed. This is a spontaneous process that lowers the entropy of the universe[1].
A way to fix this problem has been proposed by Bekenstein in 1974 [6]. Instead of trying to modify (1.8), he proposed a generalized version of the second law of thermodynamics, in order to include black holes in it. This is done by stating that the sum of the entropy of the universe and the black holes must be greater than or equal to zero.

$$d(S_{BH} + S_{Universe}) \geq 0 \qquad \text{with } S_{BH} \propto A_{Hor} \tag{1.9}$$

This is the so-called Generalized Second Law of Thermodynamics. This inequality tells us that if we throw something inside the black hole, it gets a little bit bigger and its horizon's area increases. Therefore, the total entropy $S_{BH} + S_{Universe}$ increases and there is no contradiction anymore.
Unfortunately, a theoretical problem still remains. In order for this whole picture to be consistent, we must assume that black holes are able to not only absorb, but also emit radiation. Let us suppose we have a black hole with temperature T_{BH} immersed in a thermal bath with temperature T_{Bath}, with $T_{BH} > T_{Bath}$. If we assume that black holes can only absorb, we are lead to a contradiction: from the conservation of energy we can write

$$T_{BH} dS_{BH} + T_{Bath} dS_{Bath} = 0 \tag{1.10}$$

Based on our assumption that black holes can only absorb, we must have $dS_{BH} > 0$. This implies $dS_{Bath} < 0$. When the system reaches thermal equilibrium at temperature $T_{BH} = T_{Bath} = T$ we have

$$T(dS_{BH} + dS_{Bath}) < 0 \tag{1.11}$$

Which is a plain violation the generalized second law (1.9). If we believe in what we conjectured so far, we must accept the fact that black holes emit radiation. This is the line of reasoning that led to the theorization of a new phenomenon that was not originally predicted by General Relativity: the Hawking Radiation. We will discuss it in more depth in section 1.3.
Based on the three laws we have discussed so far, we can write $T_{BH} = c\kappa$ and $S_{BH} = c' A_{Hor}$. We still don't know know the individual values of c and c', but only their product. From the first law (1.6) we get $cc' = \frac{1}{8\pi}$. Hawking calculated the constant c' exactly [2]. It turns out that $c = \frac{1}{2\pi}$ and $c' = \frac{1}{4}$. This important result allows us to define[2] the Bekenstein-Hawking temperature and entropy with no ambiguity:

$$T_{BH} \coloneqq \frac{\kappa}{2\pi} \tag{1.12}$$

$$S_{BH} \coloneqq \frac{A_{Hor}}{4} \tag{1.13}$$

1.2.4 The third law

The third law is not strictly necessary to what follows, but we state it nonetheless for completeness.

[1] To be specific, with the term "universe" we refer to the region outside the horizon, which is where, at this stage, the second law of thermodynamics holds.

[2] If we do not set $G = 1$ there is a G in the denominator of (1.13).

It is impossible to form a black hole with surface gravity $\kappa = 0$ in a finite number of steps.

This statement is analogous to the third principle of thermodynamics, which states that in a thermodynamic system it is impossible to reach the temperature $T = 0K$ in a finite number of steps. To understand why, let us look at an example involving the Kerr-Newman black hole.
It is fairly easy to calculate κ and A_{Hor}, the results are

$$\kappa = \frac{4\pi\mu}{A_{Hor}} \tag{1.14}$$

$$A = 4\pi\big[2M(M+\mu) - Q^2\big] \tag{1.15}$$

where

$$\mu = \sqrt{M^2 - Q^2 - \frac{J^2}{M^2}} \tag{1.16}$$

In order to achieve $\kappa = 0$ we need to have $\mu = 0$. This means $M^2 = Q^2 + \frac{J^2}{M^2}$. If we study the Kerr-Newman metric in detail, we can see that this condition defines an extremal black hole, and the two horizons perfectly overlap [7].
For a Schwarzschild black hole, for which $Q = J = 0$, we have $\kappa = \frac{1}{4M}$ and to reach $\kappa = 0$ we would need to throw an infinite amount of mass inside it. This is clearly impossible in a finite number of steps.

1.3 Hawking radiation

As previously anticipated, in order for the theory to be consistent, black holes must be able to emit radiation. This was proven in detail in 1975 by Hawking [2]. We report a general idea of the proof, which makes what happens phisically clear. The first step is to show that an accelerated observer in Minkowski spacetime with acceleration a sees radiation at temperature $T_u = \frac{a}{2\pi}$: this is called Unruh effect. Hawking proved that an observer in free fall near the event horizon of a black hole sees the exact same phenomenon. This happens because when we go near the horizon, the Schwarzschild metric (in E-F coordinates) approaches the Minkowski metric in Rindler coordinates, which are used to describe accelerated observers. This limit is called the Near Horizon Limit, and we will make large use of it later. The temperature measured by the static observer at spatial infinity, analogous to the Unruh temperature T_u, is the Bekenstein-Hawking temperature that we previously defined $T_{BH} = \frac{\kappa}{2\pi}$.

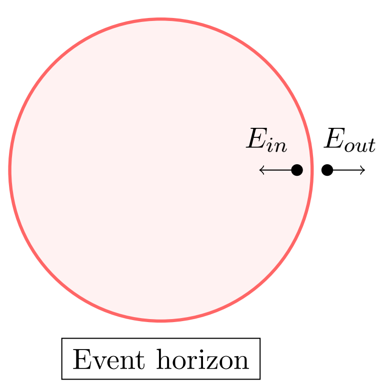

Figure 1.1: A really simplified scheme depicting a pair of particles being created across the horizon. We have $E_{in} < 0$ and $E_{out} > 0$.

Where does this radiation come from? Quantum field theory predicts that, in vacuum, pairs of virtual particles are continuously created and immediately destroyed. However, if a pair were to be created across the horizon, the particles can be separated by the gravitational potential, causing them to become real. This results in a net radiation that can be measured from the outside region. This phenomenon is schematized in figure 1.1.
The ingoing particles are produced with $E_{in} < 0$. Since they cannot escape the horizon, they will eventually hit the singularity of the black hole, decreasing its mass and making it slowly shrink until it completely disappears. This is a really long process: Page estimated [8] an evaporation time of $\tau \approx 10^{66}$ y for stellar black holes, and up to $\tau \approx 10^{106}$ y for supermassive black holes with $M \approx 10^{14} M_{\odot}$.
One could argue that this process violates the area theorem, since the black hole shrinks in size. This is true, however, what ultimately really matters is the generalized second principle (1.9), which still holds true. The black hole entropy decreases, but the total entropy (black hole+universe) does not, since the emitted radiation carries entropy.

1.4 Theoretical problems

So far, we have discussed the four principles, then argued that the second law of thermodynamics has to be modified for consistency, which led to the discovery of a new phenomenon: black hole evaporation. This is not the end of the story: this whole picture seems to be working, but it actually leads to two huge additional problems that, unlike the ones we pointed out so far, are still unsolved: the microstate problem and the information paradox.

1.4.1 The microstate problem

We convinced ourselves that it is possible to associate an entropy $S_{BH} = \frac{A_{Hor}}{4}$ to a black hole. This result can be derived from first principles in General Relativity, but that is not the only way: in fact, there is an alternative derivation that makes use of the euclidean Path Integral, which gives information about the action S, but does not say where the degrees of freedom are [4]. The same result can be found in two independent ways, so we are more likely to believe in it. The problem arises when we try to interpret the entropy from a microscopical point of view. The entropy must have some kind of statistical significance, that is, we must have $S = \log(W)$, where W is the number of microstates corresponding to the macrostate the black hole is in.
In thermodynamics, the entropy is usually a function of some macroscopic variables of a given system, such as the energy E, the volume V, the number of particles N, the temperature T or even the chemical potentials μ_i. After fixing a macrostate by assigning some values to the macroscopic quantities, we count how many different microstates produce it. Then, we take the log and we find the entropy. For a black hole, the macroscopic variables are M, Q and J. If we proceed by analogy, we are prompted to think that maybe there could be different metrics that correspond to the same values of M, Q, J, and we could find the entropy by simply counting how many there are, allowing us to interpret them as the microstates we are looking for. Unfortunately, this does not work. The No Hair theorem states that there is only one such metric, which is the Kerr-Newman metric. It turns out that there can only be one microstate, giving $S = \log(1) = 0$.
It is clear that something went wrong. However, we could have expected this. After all, microstates usually exist in theories somehow involving quantum mechanics. General Relativity is not a quantum theory at all. In order to construct and count the microstates we must try a dif-

ferent approach, that is we must find a way to construct a quantum version of black holes. This necessarily requires some theory of quantum gravity, so we have to choose one. In this work we try to discuss this problem using Supergravity. We will be back on this matter in chapter 3, however we anticipate that some classes of microstates can be found, but only for a particular class of black holes that possess some Supergravity symmetries. Currently, it is still unknown how to apply this formalism to the Schwarzschild black hole because it does not possess such symmetries, making the process extremely complicated.

1.4.2 The information paradox

Let us turn our attention back to the Hawking radiation. Hawking proved that particle pairs are produced in a maximally entangled state. This is a pure state and can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle_{\text{In}} \otimes |0\rangle_{\text{Out}} + |1\rangle_{\text{In}} \otimes |1\rangle_{\text{Out}} \Big) \tag{1.17}$$

The ket $|n\rangle_{\text{In/Out}}$ represents a state with n ingoing/outgoing particles. We ignore the states with $n \geq 2$ because they correspond to processes that have a negligible chance of happening: it is way more frequent to produce a particle pair, compared to 4 particles, or even more.
Let us now suppose we are an observer in the region outside the horizon and we want to measure the state of the outgoing particle. Since we do not have access to the inner region, we are forced to describe the state by means of a density matrix, tracing out the $|n\rangle_{\text{In}}$ states. This yields

$$\rho_{\text{Out}} = \operatorname*{tr}_{|n\rangle_{\text{In}}} \Big(|\Psi\rangle \langle\Psi| \Big) = \frac{1}{2} |0\rangle_{\text{Out}} \langle 0|_{\text{Out}} + \frac{1}{2} |1\rangle_{\text{Out}} \langle 1|_{\text{Out}} \tag{1.18}$$

This means that if we put some detector just outside the horizon and measure the state (1.18), we find a particle half of the times. We can compute the entropy associated to this state using the von Neumann formula

$$S_{\text{Out}} = -\text{tr}\big(\rho_{\text{Out}} \log \rho_{\text{Out}}\big) \tag{1.19}$$

The computation is straightforward and gives $S_{\text{Out}} = \log 2$. Every time the black hole emits a particle, the outgoing radiation's entropy is increased by a factor $\log 2$. This keeps happening until the value S_{BH} is reached, as shown in figure 1.2. At this point the black hole evaporates, halting the process, at time $t = \tau$. At times $t < \tau$, the radiation is entangled with particles that found themselves inside the horizon, making the total state pure. After the evaporation at $t = \tau$, we are left with just a cloud of radiation, which is not entangled to anything, so it cannot be in a pure state anymore. This is a huge problem, since we are left with a violation of the unitarity of quantum mechanics: pure states cannot evolve into mixed states.

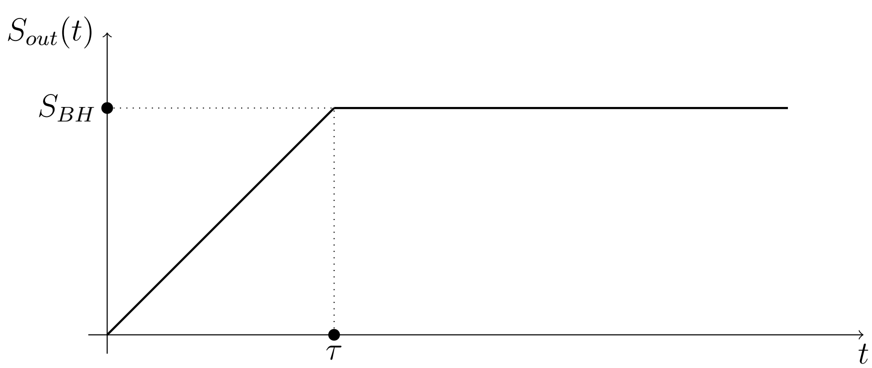

Figure 1.2: The trend of the entropy in the outer region of an evaporating black hole as a function of time.

Let us now focus for a moment on a completely different system: a burning piece of coal. It is possible to follow a similar reasoning in order to study how the radiation emitted behaves. This was done by Page, who found out that the state remains pure and there is no information loss. He obtained the famous Page curve [9] [10], shown in figure 1.3.

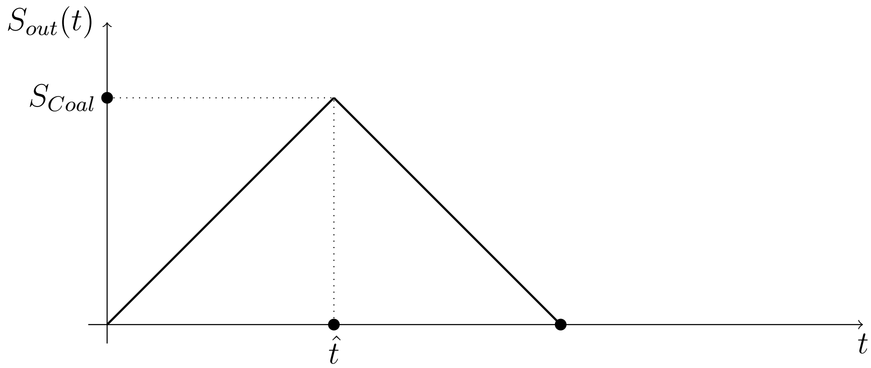

Figure 1.3: The Page curve for coal.

It is possible to factor the total Hilbert space into the coal Hilbert space times the radiation Hilbert space $\mathbb{H}_{tot} = \mathbb{H}_{coal} \otimes \mathbb{H}_{rad}$. Page did so and found a formula for the entropy that explains why the curve has a triangular shape

$$S = \log\left[min\Big(dim(\mathbb{H}_{coal}), dim(\mathbb{H}_{rad})\Big)\right] \tag{1.20}$$

The entropy of the radiation keeps going up, until half of the coal has been burnt at $t = \hat{t}$. Then, the process starts reversing, since the dimension of $\mathbb{H}_{rad}$ gets bigger than the dimension of $\mathbb{H}_{coal}$. In the end we go back to zero entropy, and there is no information loss. Why is that black holes behave so differently? Let us look at figure 1.4. A are ingoing particles, B are outgoing particles and C are outgoing particles that have been emitted some time in the past.
Hawking proved [2] that A and B are entangled. For the state $|\Psi\rangle$ in (1.17) to be pure, A and C need to be entangled as well, but this is not possible due to the Monogamy of Entanglement: A cannot be entangled with two different systems at the same time. Ultimately, it is the horizon's presence that leads to the information paradox: if it was not there, there would be no paradox at all, and black holes would behave like burning pieces of coal.
Different solutions to the information paradox have been proposed, such as the Quantum Extremal Islands [11] [12] [13], or the Black Hole Remnants [14]. However, in this work we work with Samir Mathur's fuzzball proposal [3]. In this picture both problems are solved, as we will see in chapter 3. However, to fully understand Mathur's proposal, some knowledge of String Theory and Supergravity is required. Therefore, we spend the next chapter developing these topics.

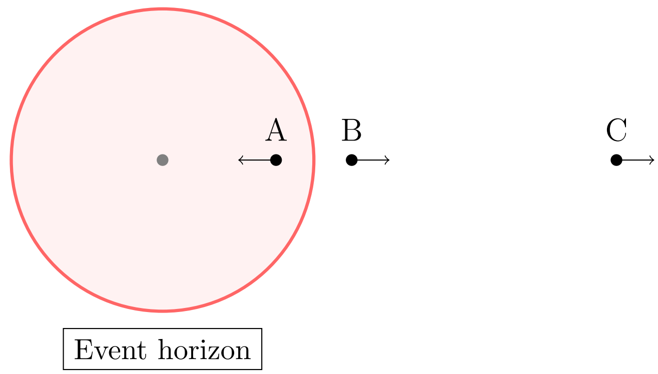

Figure 1.4: A really simplified scheme depicting what leads to the information paradox.

Chapter 2

String Theory and Supergravity

As anticipated, if we wish to find a solution to the problems pointed out in the previous chapter, we have to rely on some theory of quantum gravity. Different theories have been developed in recent times, such as String Theory, Loop Quantum Gravity and many more. Our theory of choice is Supergravity, which can be derived from the framework of String Theory. Our task in this chapter is to introduce String Theory and from that to outline how Supergravity is obtained. We will focus on the basic aspects, as we will not need any advanced topics in order to understand the following chapters. For a more exhaustive analysis, the reader can check out the really good textbooks [15], [16], [17], [18].

2.1 The relativistic string

What is more complex than a point?
This is the fundamental question that leads to the formulation and developing of String Theory. The answer is: a line. In fact, in String Theory, we postulate that elementary particles are not really particles. Instead of working with zero-dimentional objects we decide that the fundamental objects that make up our universe are one-dimentional lines, called strings. What makes String Theory so powerful is that it describes many different particles as many different modes of vibration of the strings. This includes gravitons.
Unfortunately, developing this new theoretical framework is not so straightforward. Together with gravitons, String Theory carries along some features that are bizarre, to say the least. The most prominent one is the fact that many extra spatial dimensions exist. If we want to believe in String Theory, we have to accept these new features and find a way to deal with them.

2.1.1 The relativistic point particle

We begin our analysis by writing an action for a free relativistic point particle. This action will be written in a form that is easy to generalize for a string.
Let us guess the action for the point particle. First of all, we require this object to possess Lorentz-invariance, since it must yield Lorentz-invariant equations of motion. Imagine a particle whose spacetime trajectory starts from point A and ends in point B. There are many possible worldlines that connect A and B. In order to implement Lorentz-invariance, we require that, for every such worldline, every observer computes the same action. We now implement this idea mathematically.

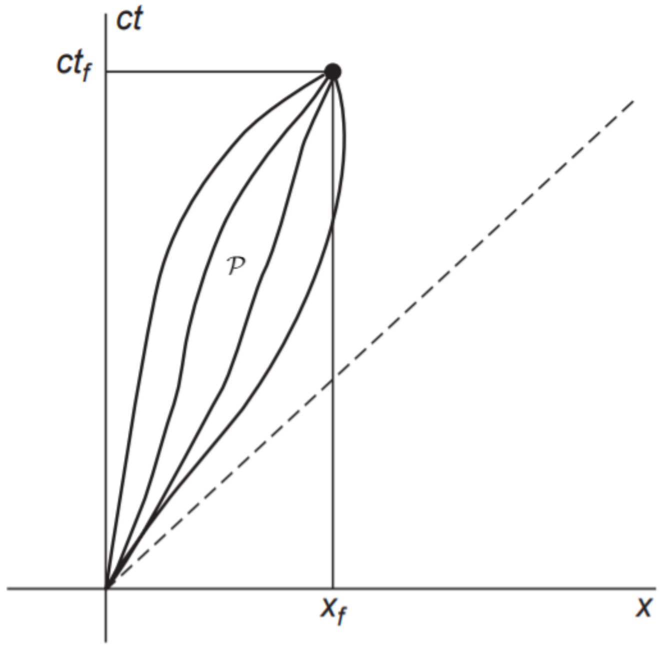

Figure 2.1: A spacetime diagram with a series of worldlines connecting the origin to the spacetime point (ct_f, x_f). [15]

Let $\mathcal{P}$ denote a worldline. What quantity do all observers agree on? The elapsed proper time, which is given by integrating the quantity $\frac{ds}{c}$ over $\mathcal{P}$, where[1]

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = -\eta_{\mu\nu}dx^\mu dx^\nu \tag{2.1}$$

The action must have the dimensions of an angular momentum, so we have to multiply the proper time by some energy. The most natural form of energy we can think of is the rest mass of the particle, times c^2. This is correct up to a numerical factor, which turns out to be -1 in order to yield the correct sign to the hamiltonian and the energy [15]. The action for the free particle is therefore

$$S = -mc\int_{\mathcal{P}} ds \tag{2.2}$$

This action is not only Lorentz-invariant. It has a really important property that will be crucial in String Theory: reparametrization invariance. To make it explicit we re-express the integrand ds in terms of the coordinates of the parametrized world line $x^\mu(\tau)$[2]. We have $ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$, giving

$$S = -mc\int_{\mathcal{P}} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau \tag{2.3}$$

It is easy to check that (2.3) is indeed reparametrization-invariant. Suppose we change the parameter from τ to τ'.

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau} \tag{2.4}$$

The differential $d\tau$ transforms as

$$d\tau = \frac{d\tau}{d\tau'}d\tau' \tag{2.5}$$

[1]We work in SI units for the time being.
[2]For simplicity, we omit the τ dependence in x^μ from now on.

Now, if we substitute (2.4) and (2.5) in (2.3), we see that (2.3) is indeed left unchanged.
The equations of motion for action (2.3) read

$$\frac{dp^\mu}{d\tau} = 0 \tag{2.6}$$

indeed confirming what we already knew: for a free particle, the four-momentum is conserved along its worldline.

2.1.2 The Nambu-Goto action

We can now generalize (2.3) to a string, by proceeding by analogy. As a particle traces out a worldline, a string traces out a 2-dimensional surface called worldsheet, obtained by combining together the worldlines traced out by the points of the string. The string action will be proportional to the surface of the worldsheet.

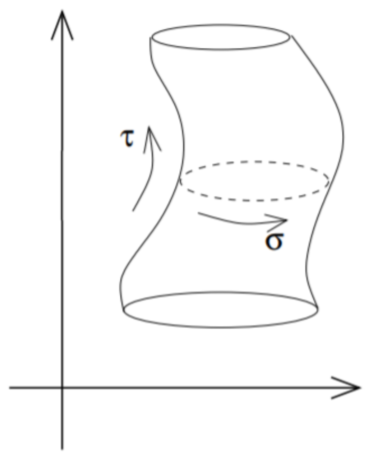

Figure 2.2: A closed string worldsheet. [15]

Let us introduce some notation. We call the two coordinates that parametrize the worldsheet τ and σ. τ spans the whole real line, while σ is restricted to a compact interval $[0, \sigma_1]$, since the string has finite length. We call X^μ the string coordinates, and finally, we introduce the derivatives

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \qquad\qquad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \tag{2.7}$$

With this notation, the area of the worldsheet reads

$$A = \iint \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}\, d\sigma d\tau \tag{2.8}$$

It can be shown that the radicand is always positive or zero, so the area is well-defined [15].
We have to multiply this area by some appropriate quantity in order to fix the dimensions. Since the action must have the dimension of an angular momentum, we multiply the area by some mass divided by some time, or equivalently by some force divided by some velocity. There is a really natural choice, which is $\frac{T_0}{c}$, where T_0 is the string tension. This was the route that was originally followed. However, nowadays we prefer to re-express the string tension in terms of the parameter

α', which has a nice physical interpretation in terms of the string tension T_0, in fact it can be shown that

$$\alpha' = \frac{1}{2\pi\hbar c T_0} \tag{2.9}$$

Moreover, α' is linked to the string length l_s as follows

$$l_s = \hbar c\sqrt{\alpha'} \tag{2.10}$$

So, putting all of this together, and picking the correct numerical factor, which is again -1, we get [15] the famous Nambu-Goto action[3]

$$S = -\frac{1}{2\pi\alpha'}\iint \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}\; d\sigma d\tau \tag{2.11}$$

It is crucial that this action be reparametrization invariant, since this is the mathematical way to describe an elementary string. We can rewrite (2.11) in a manifestly reparametrization invariant way. Let us introduce $\xi^1 := \tau$ and $\xi^2 := \sigma$. We can now define an induced metric $\gamma_{\alpha\beta}$ on the worldsheet:

$$\gamma_{\alpha\beta} := \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta} = \frac{\partial X}{\partial \xi^\alpha}\cdot\frac{\partial X}{\partial \xi^\beta} \tag{2.12}$$

More explicitly

$$\gamma_{\alpha\beta} = \begin{bmatrix} (\dot{X})^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & (X')^2 \end{bmatrix} \tag{2.13}$$

We can now write the Nambu-Goto action in a manifestly reparametrization invariant form

$$S = -\frac{1}{2\pi\alpha'}\iint \sqrt{-\gamma}\; d\xi^1 d\xi^2 \qquad \gamma = \det(\gamma_{\alpha\beta}) \tag{2.14}$$

In this form, it is very easy to generalize this action to describe $2D$ objects and beyond. We now turn to the equations of motion. It is possible to vary the Nambu-Goto action with respect to X^μ, thus getting

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0 \tag{2.15}$$

Where we defined

$$\mathcal{P}^\tau_\mu := \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \tag{2.16}$$

$$\mathcal{P}^\sigma_\mu := \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \tag{2.17}$$

The equations of motion are enormously complicated. In the next section we will choose a particular parametrization in order to re-express them in a more treatable way. We conclude this section by listing the possible boundary conditions that the solutions can satisfy. First of all, we must make a distinction between open and closed strings: open strings need boundary conditions, whereas closed strings do not. Therefore, when dealing with open strings we fix μ and select an endpoint. Let $\sigma_\star$ denote the σ coordinate of an endpoint: $\sigma_\star$ can be either 0 or σ_1. The first

[3] From this point on we use natural units $\hbar = c = 1$

possible condition we can impose is the Dirichlet boundary condition, in which the endpoint of the string remains fixed during the motion (of course we exclude $\mu = 0$ because the endpoint must move forward in time).

$$\textit{Dirichlet boundary condition}: \quad \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_\star) = 0 \qquad \mu \neq 0 \tag{2.18}$$

The second possible condition is the so-called free endpoint condition, which is related to the Neumann boundary condition

$$\textit{Free endpoint condition}: \quad \mathcal{P}^\sigma_\mu(\tau, \sigma_\star) = 0 \tag{2.19}$$

It is a direct consequence of the boundary conditions that the open string endpoints, when subject to Dirichlet boundary conditions, are forced to move inside multidimensional objects. These objects are described by hyperplanes (or portions of them), and are called D-branes. D-branes play a crucial role in String Theory because they are dynamic objects that can be assigned an energy. Plus, we will see in section 2.4.6 that they carry electric and magnetic charges. D-branes are extended along the directions with Neumann boundary conditions, as shown in figure 2.3:

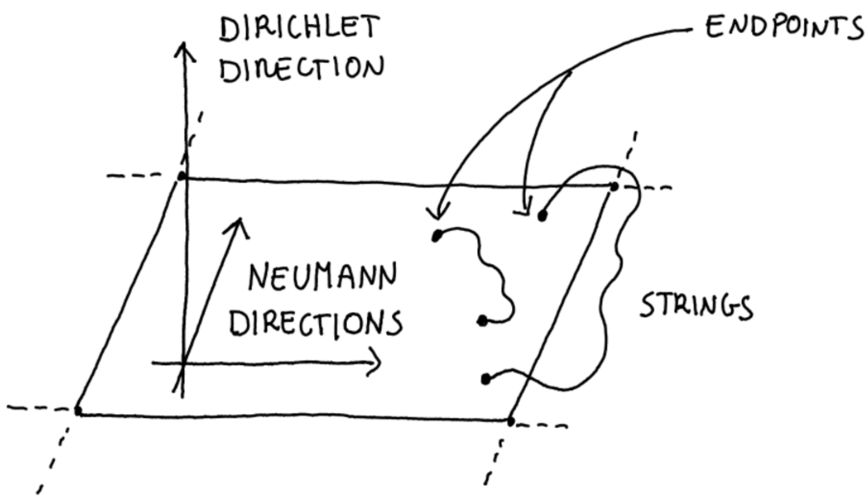

Figure 2.3: An example of a D2-brane in an universe with three spatial directions, with strings attached on it.

Finally, it is crucial to point out that due to reparametrization invariance, the motion of the strings along the longitudinal direction is unphysical. This is a consequence of the fact that the strings are fundamental objects. If this were not the case, they would be able to somehow compress themselves, which is not acceptable. This fact will have consequences regarding the quantization, in fact we will define creation and annihilation operators only along the transverse directions.

2.1.3 The light-cone gauge

The equations of motion (2.15) are too complicated to handle. Since the Nambu-Goto action is reparametrization invariant, we can make use of this property in order to pick a particular parametrization to simplify our calculations. This will turn (2.15) in a familiar wave equation.

Let us begin by changing coordinates. We move to light-cone coordinates

$$\begin{aligned} X^+ &:= \frac{1}{\sqrt{2}}(X^0 + X^1) \\ X^- &:= \frac{1}{\sqrt{2}}(X^0 - X^1) \end{aligned} \tag{2.20}$$

In the old coordinates the indices ran from 0 to $D-1$, where D is the dimensionality of the spacetime. The new indices take the values $+, -, 2, 3, 4, ..., D-1$. From now on, whenever we encounter a set of coordinates X^μ we understand them as $X^\mu = (X^+, X^-, X^I)$, where $I = 2, ..., D-1$. In the light-cone coordinates, The Minkowski metric $\hat{\eta}_{\mu\nu}$ becomes

$$\begin{aligned} &\hat{\eta}_{+-} = \hat{\eta}_{-+} = -1 \\ &\hat{\eta}_{++} = \hat{\eta}_{--} = 0 \\ &\hat{\eta}_{+I} = \hat{\eta}_{-I} = 0 \\ &\hat{\eta}_{II} = 1 \end{aligned} \tag{2.21}$$

The new Minkowski product, that we need later, is

$$a \cdot b = \hat{\eta}_{\mu\nu} a^\mu b^\nu = -a^- b^+ - a^+ b^- + a^I b^I \tag{2.22}$$

Choosing light-cone coordinates does not correspond to choosing a gauge: doing so is a more substantial step, which involves picking parametrizations for τ and σ. Let us start with the proper time τ.
τ is set equal to a linear combination of the string coordinates:

$$n_\mu X^\mu(\tau, \sigma) = \lambda \tau' \tag{2.23}$$

where n^μ is a vector and λ is a constant, to which we are free to assign some fixed value. This condition corresponds to picking a family of gauges. With this choice, along with the choice of the σ parametrization, we find the conditions [15]:

$$\sigma \text{ parametrization:} \qquad n \cdot \mathcal{P}^\sigma = 0 \qquad \text{(Open and closed strings)} \tag{2.24}$$

$$\tau \text{ parametrization:} \qquad \begin{aligned} n \cdot X(\tau, \sigma) &= \beta \alpha' (n \cdot p) \tau \\ n \cdot p &= \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau \end{aligned} \tag{2.25}$$

where $\beta = 2$ for open strings and $\beta = 1$ for closed strings.
The choices made so far simplify the equations of motion in a significant way, as we will see. However, choosing this gauge has also the effect of generating two constrains that the quantities $\dot{X}$ and X' must satisfy. Let us start by computing $n \cdot \mathcal{P}^\sigma$ using (2.17). This is straight-forward and yields

$$n \cdot \mathcal{P}^\sigma = -\frac{1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')\partial_\tau (n \cdot X)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{2.26}$$

Since $\partial_\tau (n \cdot X)$ is a nonvanishing constant (see (2.25)), the above equation yields the condition

$$\dot{X} \cdot X' = 0 \tag{2.27}$$

Now, using (2.23) to simplify the expression (2.16) for $\mathcal{P}^{\tau\mu}$, and taking the product with n we get

$$n \cdot p = \frac{1}{\beta\alpha'} \frac{X'^2 (n \cdot \dot{X})}{\sqrt{-\dot{X}^2 X'^2}} \tag{2.28}$$

Since $n \cdot \dot{X} = \beta\alpha(n \cdot p)$ we obtain

$$1 = \frac{X'^2}{\sqrt{-\dot{X}^2 X'^2}} \tag{2.29}$$

and then

$$\dot{X}^2 + X'^2 = 0 \tag{2.30}$$

Conditions (2.27) and (2.30) can be packed together as

$$(\dot{X} \pm X')^2 = 0 \tag{2.31}$$

What is left is to simplify the equations of motion. We can use the constraints (2.31) to simplify expressions (2.16) and (2.17) considerably. We get

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu \tag{2.32}$$

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X'^\mu \tag{2.33}$$

Substituting all of this in the field equation (2.15) we finally find

$$\ddot{X}^\mu - X''^\mu = 0 \tag{2.34}$$

As anticipated, in this parametrization, the equations of motion turn into more familiar wave equations. For open strings with free endpoints, the wave equations are supplemented by the requirement that $\mathcal{P}^{\sigma\mu}$, and therefore X'^μ, vanish at the endpoints.

We now select the light-cone gauge by imposing conditions (2.25) with a vector n^μ such that $n \cdot X = X^+$. This will simplify the constraints. The correct n^μ is

$$n_\mu = \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, ...\Big) \tag{2.35}$$

we indeed find

$$n \cdot X = \frac{X^0 + X^1}{\sqrt{2}} = X^+ \qquad n \cdot p = \frac{p^0 + p^1}{\sqrt{2}} = p^+ \tag{2.36}$$

using these relations back in (2.25) we have

$$X^+(\tau, \sigma) = \beta\alpha' p^+ \tau \qquad p^+ = \frac{2\pi}{\beta} \mathcal{P}^{\tau +} \tag{2.37}$$

Why did we choose the light-cone gauge? The strategy behind it is to use the simple form of X^+ to show that there is no dynamics in X^-, and that all the dynamics resides in the transverse coordinates X^I. This is a crucial fact when we perform the quantization. Let us start by rewriting the constraint (2.31) using the scalar product (2.22)

$$-2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)^2 = 0 \tag{2.38}$$

Where $(a^I)^2 = a^I a^I$ and there is an implicit sum over the I's. Since $X'^+ = 0$ and $\dot{X}^+ = \beta\alpha' p^+$ we have

$$\dot{X}^- \pm X'^- = \frac{1}{\beta\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X'^I)^2 \tag{2.39}$$

These are two equations that determine X^- in terms of the X^I up to a single integration constant[4]. If we know the value of X^- at some point on the worldsheet, then we can fix the integration constant, and therefore obtain X^-. Our analysis shows that the full evolution of the string is determined by the following set of objects:

$$X^I(\tau,\sigma), \quad p^+, \quad x_0^- \tag{2.40}$$

where x_0^- is the integration constant needed for X^-. These are the quantities that will be promoted to operators in the quantum theory, together with $\mathcal{P}^{\tau I}(\tau,\sigma)$.

2.2 Quantum bosonic strings

We now turn to the string quantization. It would take a lot of space to explain the procedure in full detail. Standard references for it are [15] and [16]. The basic idea is to expand the solutions of the equations of motion in terms of Fourier modes, which will allow us to define a set of creation and annihilation operators. This is the same procedure we do when quantizing the harmonic oscillator. The difference in string theory is that we now have as many oscillators as string modes, which means a numerable infinity. We directly report the end results and briefly comment them.

2.2.1 Quantizing the open strings

For open strings, we can find a general solution to equation (2.34), and expand it in terms of Fourier modes. Note that this solution does not obey the constraints (2.31) because we used them to determine x^+, x^- in terms of the x^I.

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{2\alpha}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma) \tag{2.41}$$

It is useful to define the zero-mode as

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu \tag{2.42}$$

Where p^μ is the momentum of the center of mass of the string. The Fourier modes are promoted to operators, along with the quantities we listed in (2.40). For $n \geq 1$, α_n^I are annihilation operators and α_{-n}^I are creation operators, and we have

$$(\alpha_n^I)^\dagger = \alpha_{-n}^I \tag{2.43}$$

We need to consider only the transverse indices, since X^+ and X^- are proportional to the objects in (2.40) due to the constraints. We have the commutation relations

$$[\alpha_m^I, \alpha_n^J] = m\delta_{m,n}\eta^{IJ} = m\delta_{m,n}\delta^{IJ} \tag{2.44}$$

[4]Euqation (2.39) is only valid for $p^+ \neq 0$. The vanishing of p^+ is really uncommon, therefore we do not worry about it. For more details see [15].

It is often useful to work with another set of operators

$$a_n^I := \frac{\alpha_n^I}{\sqrt{n}} \qquad\qquad a_n^{I\dagger} := \frac{\alpha_n^{I\dagger}}{\sqrt{n}} \tag{2.45}$$

we call these new operators oscillators, since they satisfy the familiar commutation relations

$$[a_m^I, a_n^J] = \delta_{m,n}\delta^{IJ} \tag{2.46}$$

Another set of useful operators are the Virasoro transverse operators. It can be proven that they are the oscillator modes of the energy-momentum tensor. They are defined by

$$L_n := \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I \tag{2.47}$$

Where there is an implicit sum over the indices I. These operators satisfy the following relation, that defines the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D-2}{12}(m^3 - m)\delta_{m+n,0} \tag{2.48}$$

There is another reason why the Virasoro operators are important. They enter the definition of the Lorentz generators. These generators must satisfy the commutation relations that define the Lorentz algebra in order for the quantum theory to be consistent. The only way to make sure these commutation relations are satisfied is to have $D = 26$. This is one of the most important results of the whole theory, since it predicts the dimensionality of the spacetime. This results may appear weird because we live in a four-dimensional world. Actually, we can take care of the extra dimensions by performing a process of compactification, that will be briefly discussed in section 2.4.2. These dimensions are not detectable at the energy scales reached by the current particle accelerators. This is a prediction of the theory, which states that these extra dimensions have no impact in the low-energy effective theory.

2.2.2 Open string state space

We start by defining the ground states of the theory, that we can label with p^+ and the transverse momentum $\vec{p}_T$[5], which are a maximal commuting subset of operators [15]. The ground states are annihilated by definition by all the a_n^I:

$$a_n^I \left|p^+, \vec{p}_T\right\rangle = 0, \qquad n \geq 1, \qquad I = 2, ..., 25 \tag{2.49}$$

In order to create new states from $\left|p^+, \vec{p}_T\right\rangle$ we simply act with creation operators $a_n^{I\dagger}$. The general basis state $|\lambda\rangle$ of the state space can be written as

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{25} (a_n^{I\dagger})^{\lambda_{n,I}} \left|p^+, \vec{p}_T\right\rangle \tag{2.50}$$

where the integer number $\lambda_{n,I}$ denotes how many times the operator $a_n^{I\dagger}$ acts on the ground states. It is possible to define a mass-squared operator

$$M^2 = \frac{1}{\alpha'}(-1 + \hat{N}) \qquad\qquad \hat{N} := \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I \tag{2.51}$$

[5]We do not label the states with p^- because it is fixed by (2.39).

where $\hat{N}$ is the number operator, and counts how many worldsheet excitations[6] are in the state (2.50), its eigenvalue is

$$N = \sum_{n=1}^{\infty} \sum_{I=2}^{25} n\lambda_{n,I} \tag{2.52}$$

In the ground states we have $N = 0$, in the first excited levels we have $N = 1$, and so on. Let us briefly describe them.

- **States with** $N = 0$

Actually, there is only one such state: the ground state $|p^+, \vec{p}_T\rangle$, which has imaginary mass! In fact $M^2 |p^+, \vec{p}_T\rangle = -\frac{1}{\alpha} |p^+, \vec{p}_T\rangle$. This state corresponds to a scalar field with negative mass-squared. Such a field is called a tachyon, and one can think it is problematic. However, this is not the case when dealing with open strings, in fact a tachyonic field denotes an instability of the vacuum of the theory. This field is also related to a maximum in the potential, which tells us that D25-branes are unstable [15]. Tachyonic states could be potentially problematic in superstring theories, but when we add fermions to the theory, they disappear. Therefore, they do not really pose a problem.

- **States with** $N = 1$

These states are obtained by acting with $a_1^{I\dagger}$ on the ground state. Since $I = 2, ..., 25$ there are 24 such states.

$$a_1^{I\dagger} |p^+, \vec{p}_T\rangle \tag{2.53}$$

We have $M^2 |p^+, \vec{p}_T\rangle = 0$, so the states are massless. By taking a linear combination we have

$$\sum_{I=2}^{25} \xi_I a_1^{I\dagger} |p^+, \vec{p}_T\rangle \tag{2.54}$$

where ξ^I is an arbitrary transverse vector. States of the form (2.54) are photon states, and ξ^I is analogous to the more familiar polarization vector. This result is quite remarkable: String Theory predicts photons! This result is not obvious at all, since the Nambu-Goto action (2.11) has no hint whatsoever of electromagnetic gauge invariance.

2.2.3 Quantizing the closed strings

Once we quantize the open string, the quantization of the closed string follows naturally, since it consists in basically two copies of the open string theory, with small differences.
Let us start by constructing a closed string. The coordinates X^μ of this object must satisfy the string equations of motion (2.34). Plus, we must impose the constraint

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \qquad \text{for every } \tau \text{ and } \sigma \tag{2.55}$$

Where σ ranges now in the interval $[0, 2\pi]$. It is clear that this constraint describes a closed string, since the σ coordinate is periodic. The closed string possesses right-moving and left-moving waves,

[6] It is important to avoid confusion between worldsheet excitations and spacetime particles. String theory describes one string at a time, therefore we cannot talk about states describing multiple spacetime particles. Developing a second quantized version of String Theory which describes many strings at once is a very difficult task. Nowadays, different String Field theories (SFT) have been developed, but none of them is fully exhaustive.

in fact the most general solution to (2.34) that also satisfies (2.55) can be decomposed in a left-moving and a right-moving part. After introducing new variables

$$u \coloneqq \tau + \sigma \qquad\qquad v \coloneqq \tau - \sigma \tag{2.56}$$

we write

$$X^\mu = X^\mu_L(u) + X^\mu_R(v) \tag{2.57}$$

It turns out that the left and right-moving parts can be decomposed as follows

$$\begin{aligned} X^\mu_L(u) &= \frac{1}{2}x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^\mu u + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-inu} \\ X^\mu_R(v) &= \frac{1}{2}x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu v + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-inv} \end{aligned} \tag{2.58}$$

The operators α_n^μ and $\bar{\alpha}_n^\mu$ are defined in the same way as in the open string case (and so are a_n^I and $a_n^{I\dagger}$), with only one difference regarding the zero modes, for which we impose, due to (2.55)

$$\alpha_0^\mu = \bar{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu \tag{2.59}$$

There are also two sets of Virasoro operators, that are again defined in analogy with equations (2.47) and (2.48). However, the constraint (2.59) imposes the level-matching condition

$$L_0 = \bar{L}_0 \tag{2.60}$$

Since L_0 and $\bar{L}_0$ appear in the hamiltonian, this condition ensures that the energy levels in the right sector and left sector match.

2.2.4 Closed string space state

We are now ready to build the state space of the quantum closed string theory. The ground state is again $|p^+, \vec{p}_T\rangle$ and is annihilated by both the right-moving and the left-moving annihilation operators

$$\begin{aligned} a_n^I\,|p^+, \vec{p}_T\rangle &= 0, \qquad n \geq 1, \qquad I = 2, ..., 25 \\ \bar{a}_n^I\,|p^+, \vec{p}_T\rangle &= 0, \qquad n \geq 1, \qquad I = 2, ..., 25 \end{aligned} \tag{2.61}$$

The general candidate basis vector is

$$|\lambda, \bar{\lambda}\rangle = \Big[\prod_{n=1}^{\infty}\prod_{I=2}^{25}(a_n^{I\dagger})^{\lambda_{n,I}}\Big] \times \Big[\prod_{m=1}^{\infty}\prod_{J=2}^{25}(a_m^{J\dagger})^{\bar{\lambda}_{m,J}}\Big]\,|p^+, \vec{p}_T\rangle \tag{2.62}$$

Where the integers $\lambda_{n,I}$ and $\bar{\lambda}_{m,J}$ again count the number of right-moving and left-moving excitations. Why are we using the term candidate? Actually, there is an additional constraint that proper states must satisfy: a basis vector $|\lambda, \bar{\lambda}\rangle$ belongs to the state space if and only if it satisfies the level matching condition (due to (2.60))

$$\hat{N}\,|\lambda, \bar{\lambda}\rangle = \hat{\bar{N}}\,|\lambda, \bar{\lambda}\rangle \tag{2.63}$$

where $\hat{N}$ and $\hat{\bar{N}}$ are again number operators that count the right-moving and left-moving particles respectively. Their eigenvalues are

$$N = \sum_{n=1}^{\infty} \sum_{I=1}^{25} n\lambda_{n,I}, \qquad \bar{N} = \sum_{m=1}^{\infty} \sum_{J=1}^{25} m\bar{\lambda}_{m,J} \tag{2.64}$$

The masses of the states are given by

$$M^2 = \frac{2}{\alpha'}(\hat{N} + \hat{\bar{N}} - 2) \tag{2.65}$$

In the ground state we have $N = \bar{N} = 0$, in the first excited we have $N = \bar{N} = 1$ and so on. Let us look at them in detail, since something really interesting arises.

- **States with** $N = \bar{N} = 0$

This is again just the ground state $|p^+, \vec{p}_T\rangle$. We have $M^2 |p^+, \vec{p}_T\rangle = -\frac{4}{\alpha} |p^+, \vec{p}_T\rangle$. Again, we encounter tachyonic states. However, as in the case of the open string, we get rid of them by adding fermions to the theory.

- **States with** $N = \bar{N} = 1$

These states are extremely interesting, and deserve more attention. First of all, the basis states are all of the form

$$a_1^{I\dagger} \bar{a}_1^{J\dagger} |p^+, \vec{p}_T\rangle \tag{2.66}$$

In analogy with (2.54) the general state at fixed momentum is a linear combination of the basis states

$$\sum_{I=2}^{25} \sum_{J=2}^{25} R_{IJ} a_1^{I\dagger} \bar{a}_1^{J\dagger} |p^+, \vec{p}_T\rangle \tag{2.67}$$

where R_{IJ} is a 24×24 square matrix. Since any square matrix can be decomposed into a symmetric part, plus an antisymmetric part we can decompose R_{IJ} as follows

$$R_{IJ} = A_{IJ} + S_{IJ} \tag{2.68}$$

The symmetric part S_{IJ} can be further decomposed into a traceless part, plus a multiple of the identity matrix

$$S_{IJ} = \hat{S}_{IJ} + S'_{IJ} \tag{2.69}$$

where we define

$$\hat{S}_{IJ} := \big(S_{IJ} - \frac{1}{24}\delta_{IJ} S\big) \tag{2.70}$$

$$S'_{IJ} := \frac{1}{24}\delta_{IJ} S \tag{2.71}$$

here S denotes the trace of the matrix, $S := \delta^{IJ} S_{IJ}$. Putting all together we can write

$$R_{IJ} = A_{IJ} + \hat{S}_{IJ} + \bar{S}\delta_{IJ}, \qquad \bar{S} := \frac{S}{24} \tag{2.72}$$

This allows us to divide the states in (2.67) into three different irreducible representations of the Lorentz group

$$\sum_{I=2}^{25} \sum_{J=2}^{25} A_{IJ} a_1^{I\dagger} \bar{a}_1^{J\dagger} |p^+, \vec{p}_T\rangle \tag{2.73}$$

$$\sum_{I=2}^{25}\sum_{J=2}^{25}\hat{S}_{IJ}a_1^{I\dagger}\bar{a}_1^{J\dagger}\left|p^+,\vec{p}_T\right\rangle \tag{2.74}$$

$$\sum_{I=2}^{25}\bar{S}a_1^{I\dagger}\bar{a}_1^{I\dagger}\left|p^+,\vec{p}_T\right\rangle \tag{2.75}$$

We now make a remarkable claim: the states (2.74) are one-particle graviton states! In fact, since $\hat{S}_{IJ}$ is symmetric and traceless, this is the general form of a graviton state [7]. At the beginning of the chapter we claimed that String Theory is a good candidate for a theory of quantum gravity. This is a good reason to believe it. Graviton states do not disappear from the theory when we add fermions, which is good.
The states (2.73) represent one-particle states of an antisymmetric tensor field $B_{\mu\nu}$, called the Kalb-Ramond field. We will briefly discuss this field later on when we talk about Supergravity.
Finally, the state (2.75) describes a one-particle state of a massless scalar field, called the dilaton. The dilaton is extremely important because it can be shown that the string coupling g_s is not a constant, but is proportional to the dilaton field $\phi(x)$ as

$$g_s \sim e^{\phi} \tag{2.76}$$

This fact has important consequences.

2.3 Superstrings and superstring theories

Up to this point, we studied bosonic strings. However, nature is made up by fermions as well, so we have to somehow add them to the theory. This results in the development of superstring theories, which contain both bosons and fermions, and therefore are way more useful to describe the real world. Five different superstring theories are known. We start developing open superstring theory, and then move to closed superstrings.

2.3.1 Worldsheet fermions and sectors

So far, we dealt with bosonic fields X^I only. We define a new fermionic field on the worldsheet, for $\sigma \in [-\pi, \pi]$. We call this field $\Psi^I[\tau, \sigma]$. This object[7] satisfies some equations of motion analogous to the Dirac equation, and has to satisfy some boundary conditions. It turns out that we can choose this fermionic field to be periodic or antiperiodic. This defines two different subspaces of the theory, called sectors: the Neveu-Schwarz sector corresponds to an antiperiodic field Ψ^I, and the Ramond sector corresponds to a periodic field Ψ^I:

$$\begin{aligned} \Psi^I(\tau,\pi) &= +\Psi^I(\tau,-\pi) \qquad \text{Ramond (R)} \\ \Psi^I(\tau,\pi) &= -\Psi^I(\tau,-\pi) \qquad \text{Neveu-Schwarz (NS)} \end{aligned} \tag{2.77}$$

Any superstring theory is made up of a bosonic part and a fermionic part. The bosonic part is the same we developed in the previous section. We now briefly discuss the fermionic part. We can choose between the NS sector and the R sector, thus creating two different sectors for fermions. An extremely important fact that must be pointed out is that in the case of supersymmetric string theories, the spacetime is found to have $D = 10$, instead of $D = 26$.

[7] One can also choose to work with two different fermionic fields: one defined for $\sigma \in [-\pi, 0]$, the other for $\sigma \in [0, \pi]$. However, it is easier to work with an unique field defined for $\sigma \in [-\pi, \pi]$.

- **NS sector**

In this case we can expand the fermionic field as follows

$$\Psi^I(\tau,\sigma) \sim \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^I_r e^{-ir(\tau-\sigma)} \tag{2.78}$$

the b^I_r can be promoted to fermionic operators that satisfy the anticommutation relations

$$\{b^I_r, b^I_s\} = \delta_{r+s,0}\eta^{IJ} \tag{2.79}$$

Following our previous notation, the negatively moded coefficients $b^I_{-\frac{1}{2}}, b^I_{-\frac{3}{2}}, b^I_{-\frac{5}{2}}, \ldots$ are creation operators, while $b^I_{\frac{1}{2}}, b^I_{\frac{1}{2}}, b^I_{\frac{1}{2}}, \ldots$ are annihilation operators. If we denote $|NS\rangle$ the NS vacuum we can construct the general state of the theory by tensoring bosonic and fermionic states as follows

$$|\lambda\rangle = \prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha^I_{-n})^{\lambda_{n,I}} \prod_{J=2}^{9}\prod_{r=-\frac{1}{2},-\frac{3}{2},\ldots}(b^J_{-r})^{\rho_{r,J}}\,|NS\rangle \otimes |p^+, \vec{p}_T\rangle \tag{2.80}$$

Where $\rho_{r,J}$ is either zero or one because of the exclusion principle. The mass-squared operator is

$$M^2 = \frac{1}{\alpha'}(\frac{1}{2}+\hat{N}), \qquad \hat{N} \coloneqq \sum_{p=1}^{\infty}\alpha^I_{-p}\alpha^I_p + \sum_{r=-\frac{1}{2},-\frac{3}{2},\ldots} r b^I_{-r} b^I_r \tag{2.81}$$

$\hat{N}$ is again a number operator. It can be shown that states with integer N are fermionic, and states with half-integer N are bosonic.

- **R sector**

With Ramond boundary conditions we can expand the fermionic field as follows

$$\Psi^I(\tau,\sigma) \sim \sum_{n\in\mathbb{Z}} d^I_n e^{-in(\tau-\sigma)} \tag{2.82}$$

Again, the modes can be promoted to fermionic operators that satisfy the anticommutation relation

$$\{d^I_m, d^I_n\} = \delta_{m+n,0}\eta^{IJ} \tag{2.83}$$

Ramond fermions are more complicated than NS fermions and the eight fermionic zero modes d^I_0 must be treated with care. It turns out that these eight operators can be re-arranged by simple linear combinations into four creation operators and four annihilation operators. We call the creation operators $\xi_1, \xi_2, \xi_3, \xi_4$. Postulating a unique vacuum $|0\rangle$ we can therefore build 16 degenerate Ramond ground states, that we denote with $|R_A\rangle$, $A = 1, 2, \ldots, 16$. The general Ramond state now takes the form

$$|\lambda\rangle = \prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha^I_{-n})^{\lambda_{n,I}} \prod_{J=2}^{9}\prod_{m=1}^{\infty}(d^J_{-m})^{\rho_{m,J}}\,|R_A\rangle \otimes |p^+, \vec{p}_T\rangle \tag{2.84}$$

The states $|R_A\rangle$ can be split into two sets: the states $|R_a\rangle$, $a = 1, \ldots, 8$ are built with an even number of ξ_i's and are fermionic on the worldsheet. The states $|R_{\bar{a}}\rangle$, $\bar{a} = 1, \ldots, 8$ are built with an odd number of ξ_i's and are bosonic on the worldsheet.

$$\begin{array}{ll} |R_a\rangle: & |0\rangle, \xi_1\xi_2|0\rangle, \xi_1\xi_3|0\rangle, \xi_1\xi_4|0\rangle, \xi_2\xi_3|0\rangle, \xi_2\xi_4|0\rangle, \xi_3\xi_4|0\rangle, \xi_1\xi_2\xi_3\xi_4|0\rangle \\ |R_{\bar{a}}\rangle: & \xi_1|0\rangle, \xi_2|0\rangle, \xi_3|0\rangle, \xi_4|0\rangle, \xi_1\xi_2\xi_3|0\rangle, \xi_1\xi_2\xi_4|0\rangle, \xi_1\xi_3\xi_4|0\rangle, \xi_2\xi_3\xi_4|0\rangle \end{array} \tag{2.85}$$

The mass-squared operator is

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty}\left(\alpha_{-n}^I\alpha_n^I + nd_{-n}^I d_n^I\right) \tag{2.86}$$

2.3.2 Open superstrings

We need to perform one last step before having a working theory of open superstrings, which consists in selecting the correct bosonic and fermionic states from the theories we developed so far. These states must describe spacetime bosons and fermions correctly. This is no easy matter, and we will only report the correct result.

We have seen that we can split the R sector into two different subsectors: R_+, which contains bosonics states, and R_-, which contains fermionic states. In a similar way we split the NS sector into NS_+, containing bosonic states and NS_- containing fermionic states.

Open superstring theory is built by selecting spacetime fermions from the R_- sector and spacetime bosons from the NS_+ sector. This process of selection is called Gliozzi-Scherk-Olive (GSO) projection [19]. The resulting theory has a supersymmetric spectrum: each bosonic state has a matching fermionic state. For example, the eight photon states $b^I_{-\frac{1}{2}}|NS\rangle \otimes |p^+, \vec{p}_T\rangle$ match the eight states $|R_a\rangle$, and so on. This is a very remarkable result, because there is no a priori reason for which the indices a and I should both run from 1 to 8.

2.3.3 Type IIA and IIB superstring theories

We saw that closed strings are roughly obtained by tensoring left-moving and right-moving copies of an open string theory. The same holds true for closed superstring theories. Since an open superstring theory has two sectors, closed superstring sectors can be formed in four ways by combining a left-moving sector (R or NS) with a right-moving sector (R or NS). We obtain four possible closed superstring sectors: (NS,NS), (R,NS), (NS,R), (R,R). In order to get a closed string theory with supersymmetry we have to apply the GSO projection again, therefore, only some subsectors are acceptable. It turns out that only two choices lead to supersymmetric theories.

- **Type IIA superstrings**

We pick the subsectors NS_+ and R_- from the left sector, and NS_+ and R_+ from the right sector. We can combine these to get the four sectors that make up type IIA closed superstring theory:

$$(NS_+, NS_+),\ (NS_+, R_+),\ (R_-, NS_+),\ (R_-, R_+) \tag{2.87}$$

The sectors $(NS_+, NS_+), (R_-, R_+)$ are found to contain spacetime bosons, whereas the spacetime fermions arise from the $(NS_+, R_+), (R_-, NS_+)$ sectors. The mass squared is given by:

$$M^2 = 2(M_L^2 + M_R^2) \tag{2.88}$$

where M_L^2, M_R^2 are the mass-squared operators for the open string theories that are used to build the left and right sectors.

The spectrum of the states contains no tachyons, and the massless states are obtained by combining together the massless states from the various sectors. From the (NS_+, NS_+) sector we get the graviton $g_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton ϕ. From the (R_-, R_+) sector we get a maxwell field A_μ and a three-index antisymmetric gauge field $A_{\mu\nu\rho}$.

- **Type IIB superstrings**

Type IIB superstring theory arises when we pick the subsectors NS_+ and R_- from the left-moving sector, and NS_+, R_- from the right-moving sector. We find four sectors:

$$(NS_+, NS_+),\ (NS_+, R_-),\ (R_-, NS_+),\ (R_-, R_-) \tag{2.89}$$

Again, there are no tachyons. Moreover, the bosons that arise from the (NS_+, NS_+) sector are the same as the ones in IIA theory. There is a difference in the fields obtained from the (R_-, R_-) sector, which include a scalar field A, another Kalb-Ramond field $A_{\mu\nu}$ and finally a totally antisymmetric gauge field with four indices $A_{\mu\nu\rho\sigma}$.

2.3.4 More superstring theories

Earlier, we said that there are actually five superstring theories. In fact, in addition to type IIA and IIB, there are two heterotic superstring theories. In the heterotic string we combine a left-moving open bosonic string with a right-moving open superstring. Out of the 26 left-moving coordinates of the bosonic factor only 10 of them are matched by the right-moving bosonic coordinates of the superstring factor, and the remaining 16 directions give a gauge symmetry. As a result, these theories live in a 10-dimensional spacetime. There are two such theories based on two different gauge groups: these are called $SO(32)$ heterotic[8] and $E_8 \times E_8$ heterotic.
Finally, we have type I superstring theory, which is a supersymmetric theory of open and closed unoriented strings, again with gauge group $SO(32)$. A string theory is said to be unoriented if the states are invariant under an operation that reverses the orientation of the strings[9].
The five theories are not independent: there are many different relationships between them. The limit of type IIA as the string coupling goes to infinity gives a theory that lives in eleven dimensions, called M-theory. Nowadays, it is believed that the five superstring theories and M-theory are different limits of an unique theory. This theory is not a string theory, and it contains exotic objects called 2-branes and 5-branes, which are not D-branes. M-theory is important to us because its low energy limit gives a Supergravity theory.

2.4 Supergravity

The time has come to talk about Supergravity. We are not interested in a full analysis of the topic, which is very vast. We only report the results that are relevant for our later calculations. If the reader is interested, some references are [18] or [20].
Supergravity theories are supersymmetric extensions of General Relativity, and correspond to low energy limits of type IIA and IIB superstring theories. By "low energy" we mean the limit in which the string scale can be ignored and we recover a theory of particles, this corresponds to taking the limit $\alpha \to 0$. In this limit only the massless string states are relevant, so we will

[8] Equivalently to $SO(32)$, the gauge group can also be $\mathrm{Spin}(32)/\mathbb{Z}_2$.
[9] So far, we have implicitly assumed that the strings are oriented.

focus on them. Figure 2.5 at the end of the chapter depicts a simplified scheme involving the five superstring theories, plus the Supergravity theories we are going to discuss, and how they are connected amongst each other.

2.4.1 Eleven-dimensional Supergravity

This theory is the low energy limit of M-theory. Since we are interested in the bosonic content only, we will not discuss the fermionic part of the theory. The bosonic fields are:

- The eleven-dimensional metric G_{MN}, $M, N = 0, ..., 10$, a symmetric tensor.
- The totally antisymmetric 3-form $A_3 = A_{MNR} dx^M \wedge dx^N \wedge dx^R$ with field strength $F_4 = dA_3$

The bosonic part of the action is

$$S = S_{EH} + S_{\text{Kin}} + S_{CS} = \frac{1}{2G_{11}^2} \int \mathcal{R}\sqrt{-G} d^{11}x - \frac{1}{4G_{11}^2} \int |F_4|^2 d^{11}x - \frac{1}{12G_{11}^2} \int A_3 \wedge F_4 \wedge F_4 \quad (2.90)$$

Where G_{11} is the eleven-dimensional Newton constant.Let us now explain what the terms in the action are, one by one.

- S_{EH} is an Einstein-Hilbert term: it is completely analogous to the familiar Einstein-Hilbert term from General Relativity. $\mathcal{R}$ is the Ricci scalar curvature computed from the metric G_{MN}, and G is the determinant of the metric, as usual.
- The term S_{Kin} is a kinetic term for the field A_3. It is completely analogous to the Maxwell action for the electromagnetic field.
- The last term S_{CS} is called the Chern-Simons term. It is a topological term required by supersymmetry.

2.4.2 Compactification

Up to this point, we still have not discussed the topic of dimensional compactification. Our world has three spatial dimensions, whereas superstring theories and Supergravity live in nine or even ten spatial dimensions. In order to deal with these additional dimensions in the effective low-energy theory we apply the procedure of compactification.

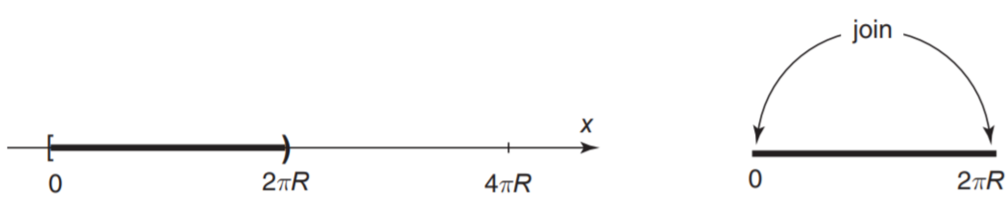

Figure 2.4: The interval $[0, 2\pi R]$ is a fundamental domain for the line with the identification (2.91). The identified space is a circle of radius R. [15]

When we describe an extended dimension, we treat it as a line. For simplicity, let us focus on a world with only one spatial dimension. We call x the coordinate along the line, varying from

$-\infty$ to $+\infty$. Let us now describe the compactification procedure. We introduce a length R, that will be a new length scale in the theory. Then, we select an interval on the line, say for example $[0, 2\pi R]$: this is called a fundamental domain. We now declare that two point whose x coordinates differ by $2\pi R$ are actually the same point

$$x \sim x + 2\pi R \tag{2.91}$$

the two points are now identified.[10] The result of this procedure is that the points $x = 0$ and $x = 2\pi R$ are now, in some sense, the same point, and we find ourselves with a circle. We can compactify multiple dimensions simultaneously. For example, in the next chapter we will see a case in which four extended dimensions are compactified into a four dimensional torus T_4. Topologically, more complicated spaces are also possible, such as the Klein bottle $K3$, but we will not need to consider them here. Compactification has important physical consequences, as we will see later on.

2.4.3 Type IIA Supergravity

Type IIA Supergravity can be obtained in two different ways: we can either take the low energy limit of type IIA superstring theory, or compactify a dimension from eleven-dimensional Supergravity. In both cases, we obtain the same ten-dimensional theory. We now briefly describe how to obtain Type IIA Supergravity by compactification.

Let us choose a spatial coordinate, say $y := x^{10}$, and perform the compactification on a circle of radius R. This radius is now a length scale of the theory, along with the Planck length $l_P \sim G_{10}^{\frac{1}{8}}$. We can rewrite the eleven-dimensional metric as

$$ds_{11}^2 = ds_{10}^2 + e^{2\sigma}(dy + C_\mu dx^\mu)^2 \qquad \mu = 0, ..., 9 \tag{2.92}$$

where we have a line element ds_{10}^2 corresponding to a 10-dimensional metric $g_{\mu\nu}$, a 1 form $C_1 = C_\mu dx^\mu$, and the scalar σ, which is related to the dilaton as $\sigma = \frac{2}{3}\phi$. We can also decompose the field A_3 as follows:

$$A_3 = B_2 \wedge dy + C_3 \tag{2.93}$$

where we introduce the 2-form B_2 and the 3-form C_3. Notice that C_1, C_3 are the same massless fields from type IIA superstring theory that we discussed in section 2.3.3, which we previously named $A_\mu, A_{\mu\nu\rho}$. These fields originally came from the (R_+, R_-) sector. Plus, $g_{\mu\nu}$, B_2 and ϕ are the fields coming from the (NS_+, NS_+) sector. We can find the action for the bosonic part of type IIA Supergravity by substituting (2.92) and (2.93) in (2.90). We introduce the field strengths $F_{p+1} := dC_p$, $H_3 := dB_2$, $\tilde{F}_4 := dC_3 - H_3 \wedge C_1$. The action is

$$\begin{aligned} S_{IIA} = \frac{1}{2G_{10}^2} \int \left(e^\sigma \mathcal{R}_{10} + e^\sigma \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} e^{3\sigma} |F_2|^2\right) \sqrt{-g}\, d^{10}x\, - \\ - \frac{1}{4G_{10}^2} \int \left(e^{-\sigma} |H_3|^2 + e^\sigma |\tilde{F}_4|^2\right) \sqrt{-g}\, d^{10}x - \frac{1}{4G_{10}^2} \int B_2 \wedge F_4 \wedge F_4 \end{aligned} \tag{2.94}$$

where $\mathcal{R}_{10}$ is the Ricci scalar curvature computed from the ten-dimensional metric $g_{\mu\nu}$ and $G_{10} = 2\pi R G_{11}$ is the ten-dimensional Newton constant.

[10](2.91) is an equivalence relation.

We notice that the Einstein-Hilbert term is not written in the canonical form. We can fix this issue by moving to the so-called Einstein frame.

$$(g_E)_{\mu\nu} = e^{\frac{\phi}{6}} g_{\mu\nu} \tag{2.95}$$

this frame is important because we use it to derive physical results, such as the entropy of black holes. Another important frame is the string frame, obtained by performing the substitution

$$(g_s)_{\mu\nu} = e^{\frac{2}{3}\phi} g_{\mu\nu} = e^{\frac{\phi}{2}} (g_E)_{\mu\nu} \tag{2.96}$$

this frame is useful because if one decides to derive type IIA Supergravity by taking the low energy limit of type IIA superstring, they find the action (2.94) written in this frame.
The equations of motion for IIA Supergravity in the string frame read

$$\begin{aligned} &e^{-2\phi}\left(Ric_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{4}H_{MPQ}H_N{}^{PQ}\right) - \frac{1}{2}F_{MP}F_N{}^P - \frac{1}{12}\tilde{F}_{MPQR}\tilde{F}_N{}^{PQR} + \\ &\quad + \frac{1}{4}G_{MN}\left(\frac{1}{2}F_{PQ}F^{PQ} + \frac{1}{24}\tilde{F}_{PQRS}\tilde{F}^{PQRS}\right) = 0 \end{aligned} \tag{2.97a}$$

$$4(d\star d\phi - d\phi\wedge\star d\phi) + \star\mathcal{R} - \frac{1}{2}H_3\wedge\star H_3 = 0 \tag{2.97b}$$

$$d\star(e^{-2\phi}H_3) - F_2\wedge\star\tilde{F}_4 - \frac{1}{2}\tilde{F}_4\wedge\tilde{F}_4 = 0 \tag{2.97c}$$

$$d\star F_2 - H_3\wedge\star\tilde{F}_4 = 0 \tag{2.97d}$$

$$d\star\tilde{F}_4 - H_3\wedge\tilde{F}_4 = 0 \tag{2.97e}$$

where the Hodge star operator $\star$ must be computed with respect to the metric in the string frame.

2.4.4 Type IIB Supergravity and T-duality

What happens if we take the low-energy limit of type IIB superstring theory? We obtain a new Supergravity theory, called type IIB Supergravity. This theory cannot be obtained by compactifying some higher-dimensional theory. However, it turns out that its fields are linked to the ones in type IIA Supergravity by a transformation called T-duality. From a physical point of view, we can wrap a IIA string around a circle of radius R, and a IIB string around a circle of radius $\tilde{R}$. It turns out that if $\tilde{R} = \frac{l_s^2}{R}$, the two theories have the same spectra and they are also equivalent at the interacting level.
When applying a T-duality to the bosonic fields of type IIA Supergravity, we have to rewrite them in the form

$$\begin{cases} ds_{10}^2 = g_{yy}(dy + A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu \\ B_2 = B_{\mu y}dx^\mu \wedge (dy + A_\mu dx^\mu) + \hat{B}_2 \\ C_p = C_{(p-1),y}\wedge(dy + A_\mu dx^\mu) + \hat{C}_p \end{cases} \tag{2.98}$$

after doing so, we have the formula giving the corresponding fields in type IIB Supergravity

$$\begin{cases} ds'^2 = g_{yy}^{-1}(dy + B_{\mu y}dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu \\ e^{2\phi'} = g_{yy}^{-1}e^{2\phi} \\ B_2' = A_\mu dx^\mu \wedge dy + \hat{B}_2 \\ C_p' = \hat{C}_{p-1}\wedge(dy + B_{\mu y}dx^\mu) + C_{p,y} \end{cases} \tag{2.99}$$

This transformation affects both the fields in the (R_+, R_-) sector and the (NS_+, NS_+) sector. The action is left unchanged. The terms in the action (2.94) containing the fields $g_{\mu\nu}$, σ and B_2 are left untouched, so after performing the T-duality the new action becomes

$$\begin{aligned} S_{IIB} = \frac{1}{2G_{10}^2} \int \left(e^{\sigma}\mathcal{R}_{10} + e^{\sigma}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}e^{-\sigma}|H_3|^2\right)\sqrt{-g}d^{10}x - \\ - \frac{1}{4G_{10}^2}\int e^{5\sigma}\left(|F_1|^2 + |\hat{F}_3|^2 + \frac{1}{2}|\hat{F}_5|^2\right)\sqrt{-g}d^{10}x - \frac{1}{4G_{10}^2}\int C_4 \wedge H_3 \wedge F_3 \end{aligned} \tag{2.100}$$

where we introduced the field strengths $\hat{F}_3 \coloneqq F_3 - C_0 \wedge H_3$ and $\hat{F}_5 \coloneqq F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. The (R_-, R_-) sector now contains C_p fields with even p: $p = 0, 2, 4$. In type IIA Supergravity we had a (R_+, R_-) sector containing C_p fields with p odd: $p = 1, 3$.
The equations of motion for IIB Supergravity in the string frame read

$$\begin{aligned} e^{-2\phi}\left(Ric_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{4}H_{MPQ}H_N^{\ PQ}\right) - \frac{1}{2}F_M F^M - \frac{1}{4}\hat{F}_{MPQ}\hat{F}_N^{\ PQ} - \\ - \frac{1}{96}\tilde{F}_{MPQRS}\tilde{F}_M^{\ PQRS} + \frac{1}{4}G_{MN}\left(F_P F^P + \frac{1}{24}\hat{F}_{PQR}\hat{F}^{PQR}\right) = 0 \end{aligned} \tag{2.101a}$$

$$4(d \star d\phi - d\phi \wedge \star d\phi) + \star\mathcal{R} - \frac{1}{2}H_3 \wedge \star H_3 = 0 \tag{2.101b}$$

$$d \star (e^{-2\phi}H_3) - F_2 \wedge \star\hat{F}_4 - \frac{1}{2}\hat{F}_4 \wedge \star\hat{F}_4 = 0 \tag{2.101c}$$

$$d \star F_1 + H_3 \wedge \star\hat{F}_3 = 0 \tag{2.101d}$$

$$d \star \hat{F}_3 + H_3 \wedge \tilde{F}_5 = 0 \tag{2.101e}$$

$$\tilde{F}_5 = \star\tilde{F}_5 \tag{2.101f}$$

where $\tilde{F}_5 \coloneqq F_5 - H_3 \wedge C_2$.
It is important to point out how the T-duality symmetry acts on D-branes. In a few words, we have that

- A T-duality along a direction transverse to the brane maps the Dp-brane to a D(p+1)-brane, increasing its dimension by 1.
- A T-duality along a direction along the brane maps the Dp-brane to a D(p-1)-brane, decreasing its dimension by 1.

This fact is really important, since we will use it in the next chapter when constructing solutions to Supergravity.

2.4.5 S-Duality

T-duality is not the only important transformation applying to Supergravity theories. In fact, the so-called S-duality also exists. This duality relates the weak and strong coupling of IIB theories and is realized by swapping the B_2 and C_2 fields, and changing sign to the dilaton. We have

$$\begin{cases} \phi' = \phi \\ g'_{\mu\nu} = e^{-\phi}g_{\mu\nu} \\ B'_2 = C_2 \\ C'_2 = -B_2 \end{cases} \tag{2.102}$$

The fields C_0 and C_4 are left untouched. This duality is useful mainly for two reasons:

- Starting from a solution of the equations of motion, we can apply an S-duality in order to obtain another solution.
- We can use S-dualities to investigate the strong coupling limit of type IIB Supergravity. In fact, the transformation acts on the coupling constant as $g_s \to \frac{1}{g_s}$, this is clear by looking at what happens to equation (2.76) after changing the sign of the dilaton.

It is clear that S-duality links two type IIB theories: one with weak coupling, and one with strong coupling.

2.4.6 Branes and charges

We saw that Supergravity theories possess many different generalized p-form gauge fields. These objects play an important role in their respective theories, since they couple with branes, thus giving them electric and magnetic charges.
Let us start from a familiar case and then generalize it. In classical electrodynamics we have the gauge field A_1 that gives electric and magnetic charges to point-like particles (which are 0-dimensional objects).[11] The interaction term lagrangian that couples the field A_1 with a particle with spacetime trajectory $\mathcal{P}$ and charge q is

$$\mathcal{L}_{Int} = q \int_{\mathcal{P}} A_1 \tag{2.103}$$

Electric and magnetic charges can be calculated using the field strength tensor $F_2 = dA_1$ and its Hodge dual $\star F_2$: these are integrated over a 2-sphere S_2 obtained by setting t and r constant in polar coordinates.

$$Q_e = \int_{S_2} \star F_2 \tag{2.104}$$

$$Q_m = \int_{S_2} F_2 \tag{2.105}$$

Let us now generalize this discussion. We start by writing the lagrangian (2.103) in an arbitrary number of spacetime dimensions, describing a generic (p-1)-dimensional charged object coupled to a p-form gauge field A_p. We have

$$\mathcal{L}_{Int} = \mu_p \int_{\gamma_p} A_p \tag{2.106}$$

The constant μ_p plays the role of a charge. The spacetime trajectory $\mathcal{P}$ has been substituted with a worldvolume that describes the spacetime "trajectory" spanned by some (p-1)-dimensional objects ((p-1)-branes) that generalize point-like particles (0-branes). At this point we can construct a generalized version of the field strength tensor F_2 by considering the form $F_{p+1} = dA_p$ and its Hodge dual $\tilde{F}_{D-p-1} \coloneqq \star F_{p+1}$. This definition allows us generalize formulas (2.104) and (2.105): a p-form A_p couples electrically to a (p-1)-brane and magnetically to a (D-p-3)-brane, and the charges are given by

$$Q_e = \int_{S_{D-p-1}} \star F_{D-p-1} \tag{2.107}$$

[11] Actually, magnetic monopoles have never been observed.

$$Q_m = \int_{S_{p+1}} F_{p+1} \tag{2.108}$$

Curiously, the fact that in our four-dimensional world both electric and magnetic charges are carried by point-like objects is just a coincidence arising from the fact that $2 + 2 = 4$.
In the following table we briefly summarize the branes that emerge from the gauge fields that appear in the Supergravity theories we discussed.

11D SUGRA	Field	A_3			
	Electric brane Magnetic brane	M2 M5			
Type IIA	Field	B_2	C_1	C_3	
	Electric brane Magnetic brane	$F1$ $NS5$	$D0$ $D6$	$D2$ $D4$	
Type IIB	Field	B_2	C_0	C_2	C_4
	Electric brane Magnetic brane	$F1$ $NS5$	- -	$D1$ $D5$	$D3$ $D3$

What are these objects? The branes labeled with a D turn out to be the familiar D-branes that we have already encountered. The F1 brane is interpreted as the fundamental string, and NS5 is its magnetic dual. Finally, $M2$ and $M5$ are related to the branes contained in M-theory.

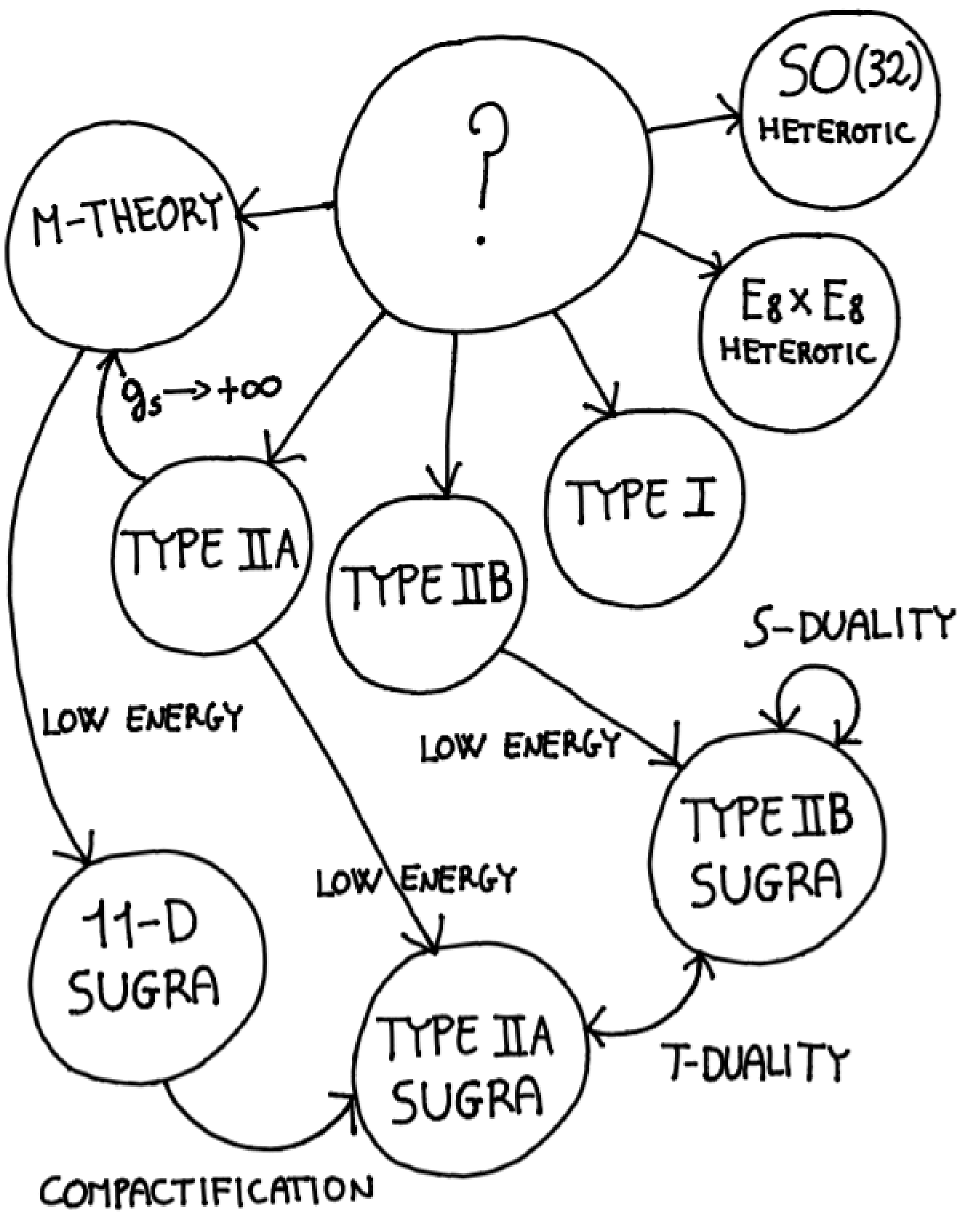

Figure 2.5: A scheme depicting the five superstring theories, along with M-theory and the three discussed Supergravity theories. M-theory and the five superstring theories are manifestations of some still unknown unified theory. M-theory can also be obtained by starting from type IIA superstrings and taking the limit $g_S \to \infty$. The low-energy limit of M-theory gives eleven-dimensional Supergravity, which upon compactification gives type IIA Supergravity. We can obtain type IIA and IIB Supergravity by taking the low energy limit of type IIA and IIB superstrings respectively. Finally, we schematically represented S-duality and T-duality.

Chapter 3

The fuzzball proposal

In this chapter we are going to find solutions to the Supergravity equations of motion. We will be able to interpret them as black holes with thermodynamical properties. We will derive such solutions using the symmetries of Supergravity, starting from simple black hole solutions with no charges associated. The latter are added by making use of the symmetries, as we will see. It is not obvious that the singularities contained in these solutions are allowed in string theory. It is possible that there are no microscopic configurations generating them. We will later analyze this issue focusing on the 2-charge system. The aim of this chapter is to summarize Mathur's work [3], in which this problem was solved. The proposed solution, called fuzzball, describes a microscopic configuration, whose associated entropy reproduces the macroscopic one S_{BH}. Moreover, it contains a description of the microstates needed to obtain S_{BH}. In fact, we will find a family of solutions that all look the same from a large distance from the black hole. However, if we get close, we see that the microstates start differing from each other. This concept is completely analogous to what happens with gases in statistical mechanics. In fact, if we have a box full of gas, we measure an associated macroscopic entropy S, which corresponds to S_{BH}. The entropy is generated by all the possible combinations of microstates, which can be distinguished at the microscopic level. The fact that the solutions look different from each other when we get close to the black hole reminds us of this fact. This whole picture could potentially solve the information paradox, since it would explain where the microstates come from.

3.1 Supergravity solutions

Our task in this section is to find solutions to the Supergravity equations of motion, representing black holes carrying Supergravity charges. We only focus on solutions that represent BPS states. What are BPS states and why are they relevant? Let us see an example.

3.1.1 BPS states

Susskind [21] proposed an interesting approach to studying black holes in string theory. We start by considering a string in a highly excited state, which means $M \gg \frac{1}{\sqrt{\alpha}}$. For now we assume that the coupling constant of the string is really small ($g_s \ll 1$), so we deal with a free string. Equation (2.88) tells us that $N_L, N_R \sim M\sqrt{\alpha} \gg 1$. Since the occupation numbers are really big, we have a large degeneracy of states with mass M, which we denote $\mathcal{N}$. It is possible to count the number of states with mass M and calculate the entropy produced by the string. The result

is $S_{micro} = \log \mathcal{N} \sim \sqrt{\alpha} M$
Let us now consider a strongly coupled string. We have to consider gravity, since we have $G \sim g_s^2$. If the mass M is large enough we obtain a black hole of mass M and we can compute its Bekenstein-Hawking entropy. For a (3+1)-dimensional Schwarzschild black hole we have $A_{hor} = 4\pi r_s^2 = 16\pi M^2$, ultimately giving $S_{Bek} = 4M^2$.
We observe that S_{micro} and S_{BH} grow as different powers of the mass. This is a problem because we expect the two to match. Actually, we made a mistake in our calculations: since the energy levels shift as we change g_s, it is incorrect to compare degeneracies at different values of g_s!
This problem can be avoided if we focus on BPS (Bogomolny-Prasad-Sommerfeld) states, for which $M = Q$ in suitable units. Moreover, their mass could depend on the moduli [1] of the theory. However, charge is quantized, therefore the number of states with fixed charge does not depend on the moduli. Therefore all BPS states with mass M move together as we change g_s. This allows us to compare degeneracies at different values of the coupling g_s. We expect that

$$S_{micro} = S_{BH} \qquad \text{for BPS states} \tag{3.1}$$

If we do not get this agreement, than we would have to give up on string theory as a theory of quantum gravity. On the other hand, if we found such an agreement, we would be very happy, because it would mean that our theory passed a really nontrivial test that allows it to be taken seriously as a candidate for the correct quantum gravity theory. We will see that we get this agreement.
The BPS solutions that we are going to describe are purely bosonic, and are invariant under some of the superymmetries that characterize the theory. There are two methods to find solutions: none of them involves directly solving the equations of motion, since they are extremely complicated. One method consists in exploiting the supersymmetries of the theory, however we will use a more indirect approach. Starting from a neutrally charged solution, we apply S and T dualities and boosts in order to get a solution carrying charges. How is it possible that we get a new solution by applying a boost? The answer is: boosts will always be performed along the compactified directions, so they define coordinate transformations that are not globally defined and relate phisically inequivalent solutions.
In the following sections, we will start from the Schwarzschild solution, which does not carry any Supergravity charge. Then, by applying boosts and dualities, we add the relevant charges. The Schwarzschild solution is obtained by solving equations in General Relativity, and is not determined by any microscopic configuration. Therefore, as anticipated, we are not guaranteed that the solutions we find describe microscopic configurations of strings. In fact, the solutions that we will find represent the macroscopic ensemble of all the states, not the microscopic configuration, which is what we are interested in. Moreover, the 2-charge solution will also yield the wrong entropy. We will face and solve these problems in the following paragraphs. We start by finding the solutions in Supergravity, then we will calculate their associated entropy in two different ways (macroscopic and microscopic) and compare them. Then, we will show that some of them do not correspond to any microscopic configuration of the strings, and finally we will find the anticipated fuzzball solutions, that solve all these problems.

[1] Moduli are continuous parameters characterizing the theory, such as g_s, or the radius R of some compactified dimension.

3.1.2 The 1-charge $\left(\frac{1}{2}\text{-BPS}\right)$ solution

We consider Supergravity with a compactified direction along a circle, that we call y. Let us see how to construct the simplest case: the 1-charge geometry. We consider a solution to the Supergravity equations of motion:

$$\begin{cases} ds^2 = -\big(1-\frac{2M}{r^6}\big)dt^2 + \big(1-\frac{2M}{r^6}\big)^{-1}dx_a dx^a + dy^2 \qquad\qquad a=1,...,8 \\ \phi = 0 \\ B_2 = C_p = 0 \end{cases} \tag{3.2}$$

This solution is a higher-dimensional version of the Schwarzschild's solution along the nine non-compact directions, and is flat along the compactified circle. We do not really worry about specifying the frame (string or Einstein), since the dilaton is zero and they coincide. Finally, notice that this is a solution to both type IIA Supergravity (2.97) and IIB Supergravity (2.101), since the gauge fields vanish. We now perform a boost along the y direction

$$\begin{cases} y = \cosh{(\eta)}y' + \sinh{(\eta)}t' \\ t = \sinh{(\eta)}y' + \cosh{(\eta)}t' \end{cases} \tag{3.3}$$

As anticipated, since y is compact, this boost generates a new solution. After renaming $t', y' \to t, y$ we obtain the new metric

$$\begin{aligned} ds^2 = \big(1+\frac{2M}{r^6}\sinh^2\eta\big)dy^2 + \big(-1+\frac{2M}{r^6}\cosh^2\eta\big)dt^2 + \\ + \frac{4M}{r^6}\sinh\eta\cosh\eta dt dy + \big(1-\frac{2M}{r^6}\big)^{-1}dx_a dx^a \end{aligned} \tag{3.4}$$

Notice that the P charge has become a F1 charge, carried by the B_2 field. This solution represents a wave carrying momentum. The boost had the effect of adding a charge to the solution, which in this case is precisely the momentum P of the wave. Now we perform a T duality along the y direction. We must rewrite the metric (3.4) in the form (2.98) and then we can apply formula (2.99). We get to

$$\begin{cases} ds^2 = \Bigg(1+\frac{2M}{r^6}\sinh^2\eta\Bigg)^{-1}\Bigg[dy^2 + \Bigg(-1+\frac{2M}{r^6}\Bigg)dt^2\Bigg] + \Bigg(1-\frac{2M}{r^6}\Bigg)dx_a dx^a \\ e^{2\phi} = \Bigg(1+\frac{2M}{r^6}\sinh^2\eta\Bigg)^{-1} \\ B_2 = \Bigg(\frac{2M\sinh\eta\cosh\eta}{r^6+2M\sinh^2\eta}\Bigg)dt\wedge dy \\ C_p = 0 \end{cases} \tag{3.5}$$

This is not a BPS solution. In order to get such a solution, we must take the BPS limit, which consists in taking $M \to 0$, $\eta \to \infty$, while keeping the combination $Me^{2\eta}$ fixed to a constant. We set

$$Me^{2\eta} = 2Q_1 \tag{3.6}$$

After taking this limit, the solution becomes

$$\begin{cases} ds^2 = H_1^{-1}\big(-dt^2+dy^2\big) + dx_a dx^a \\ e^{2\phi} = H_1^{-1} \\ B_2 = \big(1-H_1^{-1}\big)dt\wedge dy \\ C_p = 0 \end{cases} \tag{3.7}$$

Where $H_1 \coloneqq \left(1 + \frac{Q_1}{r^6}\right)$. Solution (3.7) describes a static fundamental string F1 wrapped around the direction y.

3.1.3 The 2-charge $\left(\frac{1}{4}\text{-BPS}\right)$ solution

We wish to add a D5-brane to our system. Since a D5-brane is a five-dimensional object, we need 5 compactified dimensions to wrap it around. We already have a circle, so we compactify four more of the non-compact spatial dimensions, say x^5, x^6, x^7, x^8. Different choices can be made: we can compactify them into a four-dimensional torus T_4 or even something more exotic such as a Klein bottle K_3. We choose the torus T_4. Moreover, we introduce spherical coordinates in the four noncompact dimensions x^1, x^2, x^3, x^4:

$$\begin{cases} x^1 = r\sin\theta\cos\phi \\ x^2 = r\sin\theta\sin\phi \\ x^3 = r\cos\theta\cos\psi \\ x^4 = r\cos\theta\sin\psi \end{cases} \tag{3.8}$$

where $\theta \in [0, \frac{\pi}{2}]$ and $\phi, \psi \in [0, 2\pi]$. We start from a Schwarzschild solution analogous to (3.9) reading

$$\begin{cases} ds^2 = -\left(1 - \frac{2M}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3 + dy^2 + dz_a dz^a \qquad a = 1,2,3,4 \\ \phi = 0 \\ B_2 = C_p = 0 \end{cases} \tag{3.9}$$

in which we renamed the torus coordinates $x^5, x^6, x^7, x^8 \to z^1, z^2, z^3, z^4$ for simplicity. An important fact to notice is that in the solution we now have r^2 instead of r^6. This happens because some directions have been compactified to a T_4. From this point on the procedure is similar to the one we just described in the one charge solution: we perform a boost along the y direction with parameter η, which adds the first charge. Then, we perform a T-duality again along y. This yields a solution similar to (3.5). Then we perform a second boost along the y direction with parameter ζ. Its expression is identical to equation (3.3) with $\eta \to \zeta$. The result describes a F1 string along the direction y, denoted $F1_y$, carrying momentum P_y.

$$\begin{cases} ds^2 = \dfrac{F_\eta}{F_\zeta}\left(dy + \dfrac{2M\cosh\eta\sinh\eta}{r^2 + 2M\sinh\zeta}dt\right)^2 + \dfrac{1}{F_\eta F_\zeta}\left(-1 + \dfrac{2M}{r^2}\right)dt^2 + \left(1 - \dfrac{2M}{r^2}\right)^{-1}dr^2 + \\ \qquad + \left(1 - \dfrac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3 + dy^2 + dz_a dz^a \\ e^{2\phi} = F_\eta^{-1} \\ B_2 = \frac{2M}{r^2}F_\eta^{-1}\cosh\eta\sinh\eta\, dt\wedge dy \\ C_p = 0 \end{cases} \tag{3.10}$$

Where $F_\eta \coloneqq \left(1 + \frac{2M}{r^2}\sinh^2\eta\right)$ and F_ζ is F_η with $\eta \to \zeta$. We now perform the BPS limit $M \to 0$, $\zeta, \eta \to \infty$ with

$$Me^{2\eta} = 2\tilde{Q}_1 \qquad\qquad Me^{2\zeta} = 2\tilde{Q}_P \tag{3.11}$$

We obtain

$$\begin{cases} ds^2 = \tilde{Z}_1(r)\big(-dt^2+dy^2+\tilde{K}_P(r)(dt+dy)^2\big)+dr^2+r^2d\Omega_3+dz_adz^a \\ e^{2\phi} = \tilde{Z}_1(r)^{-\frac{1}{2}} \\ B_2 = -\tilde{Z}_1(r)dt\wedge dy \\ C_p = 0 \end{cases} \tag{3.12}$$

where $\tilde{Z}_1(r) \coloneqq 1+\frac{\tilde{Q}_1}{r^2}$ and $\tilde{K}_P(r) \coloneqq \frac{\tilde{Q}_P}{r^2}$.
This is a solution carrying 2 charges: P_y and $F1_y$. We wish to change perspective and see it as a system containing many D1-branes and a D5-brane (D1-D5), instead of a fundamental string carrying momentum. We can move to the D1-D5 system by performing a rather complicated chain of dualities, that we write schematically:

$$\begin{aligned} (F1_yP_y) &\to (D1_yP_y) && \text{S-duality} \\ (D1_yP_y) &\to (D5_{y,T_4}P_y) && \text{Four T-dualities along } z^1,z^2,z^3,z^4 \\ (D5_{y,T_4}P_y) &\to (NS5_{y,T_4}P_y) && \text{S-duality} \\ (NS5_{y,T_4}P_y) &\to (NS5_{y,T_4}F1_y) && \text{T-duality along } y \\ (NS5_{y,T_4}F1_y) &\to (NS5_{y,T_4}F1_y) && \text{T-duality along } z^1 \\ (NS5_{y,T_4}F1_y) &\to (D5_{y,T_4}D1_y) && \text{S-duality} \end{aligned} \tag{3.13}$$

The solution in the final form is

$$\begin{cases} ds^2 = Z_1(r)^{-\frac{1}{2}}Z_5(r)^{-\frac{1}{2}}(dy^2-dt^2)+Z_1(r)^{\frac{1}{2}}Z_5(r)^{\frac{1}{2}}(dr^2+r^2d\Omega_3)+ \\ \qquad +Z_1(r)^{-\frac{1}{2}}Z_5(r)^{\frac{1}{2}}dz_adz^a \\ e^{2\phi} = Z_1(r)^{-1}Z_5(r) \\ C_2 = -Q_5\sin^2\theta d\phi\wedge d\psi+(1-Z_5(r)^{-1})dt\wedge dy \\ B_2 = C_0 = C_4 = 0 \end{cases} \tag{3.14}$$

Where $Z_1(r) \coloneqq 1+\frac{Q_1}{r^2}$ and $Z_5(r) \coloneqq 1+\frac{Q_5}{r^2}$. We notice that the D5-brane has been constructed from the fundamental string F1, thus the charge Q_5 is related to the charge $\tilde{Q}_1$ in equation (3.12) and to the boost parameter η. Analogously, Q_1 is related to $\tilde{Q}_P$.

3.1.4 The 3-charge $\left(\frac{1}{8}\text{-BPS}\right)$ solution

This solution describes the D1-D5-P system: we have a D1-brane and a D5-brane carrying momentum P. For a full computation see for example [22]. We start from (3.10) and apply the chain of transformations (3.13). Then, we perform a third boost along y in order to add the momentum charge. After taking the BPS limit we obtain

$$\begin{cases} ds^2 = Z_1(r)^{-\frac{1}{2}}Z_5(r)^{-\frac{1}{2}}(-dt^2+dy^2+K(r)(dt+dy)^2)+Z_1(r)^{\frac{1}{2}}Z_5(r)^{\frac{1}{2}}(dr^2+r^2d\Omega_3)+ \\ \qquad +Z_1(r)^{-\frac{1}{2}}Z_5(r)^{\frac{1}{2}}dz_adz^a \\ e^{2\phi} = Z_1(r)^{-1}Z_5(r) \\ C_2 = -Q_5\sin^2\theta d\phi\wedge d\psi+(1-Z_5(r)^{-1})dt\wedge dy \\ B_2 = C_0 = C_4 = 0 \end{cases} \tag{3.15}$$

This is called the Strominger-Vafa black hole.

3.2 Computing the entropy

It is now time to compute the entropy for the solutions that we previously found. We do it starting from the gravity side, and then repeat the calculation from a microscopic point of view. Since the solutions are BPS, we expect the entropies to match. This happens in the 1-charge and 3-charge solutions, whereas there is a problem with the 2-charge solution. We will face this issue and solve it in the next section.

3.2.1 Gravity side

Let us suppose we have a 10-dimensional metric. We can compute the entropy in two different ways:

1. We look at the D' noncompact directions. After dimensional reduction, we can find the Einstein metric for such directions. We then calculate the horizon area and write $S_{BH} = \frac{A_{Hor,D'}}{4G_{D'}}$ where $G_{D'}$ is the gravitational constant in D' dimensions.
2. We can compute the 10-dimensional area directly from the 10-D Einstein metric and write the entropy $S_{Bek} = \frac{A_{Hor,10}}{4G_{10}}$.

In the following, we choose the second approach.

- **Entropy for the 1-charge solution**

We rewrite the solution (3.7) after compactifying four directions into a T_4 and moving to spherical coordinates.

$$\begin{cases} ds^2 = H_1^{-1}\big(-dt^2+dy^2\big)+dr^2+r^2d\Omega_3+dz_a dz^a & \qquad a=1,2,3,4 \\ e^{2\phi}=H_1^{-1} \\ B_2=\big(1-H_1^{-1}\big)dt\wedge dy \\ C_p=0 \end{cases} \tag{3.16}$$

Where now we have $H_1 = (1+\frac{Q_1}{r^2})$. We need the line element ds^2 in the Einstein frame. We use equation (2.96) and multiply ds^2 by $e^{-\frac{\phi}{2}} = H_1^{\frac{1}{4}}$ in order to get

$$ds_E^2 = H_1^{-\frac{3}{4}}(-dt^2+dy^2)+H_1^{\frac{1}{4}}(dr^2+r^2d\Omega_3)+H_1^{\frac{1}{4}}dz_a dz^a \tag{3.17}$$

Let us describe in detail how to calculate the area of the horizon. The 1-charge solution is the easiest one, so we use it as a test ground to describe the full procedure in gory detail, that we can apply later to more complicated solutions. We remind that $y\in[0,2\pi R]$, $\theta\in[0,\frac{\pi}{2}]$, $\phi,\psi\in[0,2\pi]$ and choose the radii of the torus associated with the coordinates z_1,z_2,z_3,z_4 such that the total volume of the T_4 is $(2\pi)^4V$. We therefore choose $z_a\in[0,\Xi]$ where $\Xi = 2\pi V^{\frac{1}{4}}$. The formula giving the horizon area is

$$A_{Hor} = \int_0^{2\pi R}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\Xi}\int_0^{\Xi}\int_0^{\Xi}\int_0^{\Xi}\sqrt{g_{8,E}|_{r=0}}\;dy d\theta d\phi d\psi dz_1 dz_2 dz_3 dz_4 \tag{3.18}$$

Where the metric $g_{9,E}$ is the induced spatial metric[2] (Basically (3.17) with $dt=0$). Since this metric has a fairly easy structure (it is diagonal), the determinant is easy to compute. The eight

[2] The metric $g_{9,E}$ is euclidean.

integrals can be factorized into three groups, so we can calculate three volumes separately, and then multiply them by one another at the end. We have

$$A_{Hor} = V_{S_1} V_{S_3} V_{T-4} \tag{3.19}$$

$$\begin{aligned} V_{S_1} &= \int_0^{2\pi R} \sqrt{g_{S_1,E}}\, dy \\ V_{S_3} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{2\pi} \sqrt{g_{S_3,E}}\, d\theta d\phi d\psi \\ V_{T_4} &= \int_0^{\Xi} \int_0^{\Xi} \int_0^{\Xi} \int_0^{\Xi} \sqrt{g_{T_4,E}}\, dz_1 dz_2 dz_3 dz_4 \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} ds^2_{S_1,E} &= H_1^{-\frac{3}{4}} dy^2 \\ ds^2_{S_3,E} &= H_1^{\frac{1}{4}} (r^2 d\Omega_3) \\ ds^2_{T_4,E} &= H_1^{\frac{1}{4}} dz_a dz^a \end{aligned} \tag{3.21}$$

Let us now calculate the three terms. We have $\sqrt{g_{S_1,E}} = H_1^{-\frac{3}{8}}$. In the limit $r \to 0$ we have $H_1^{-\frac{3}{8}} \approx Q_1^{\frac{3}{8}} r^{\frac{3}{4}}$, thus giving $V_{S_1} = Q_1^{-\frac{3}{8}} r^{\frac{3}{4}} (2\pi R)$.
Now let us compute V_{S_3}, corresponding to a term $H_1^{\frac{1}{4}} r^2 d\Omega_3$. In the limit $r \to 0$ this becomes $Q_1^{\frac{1}{4}} r^{\frac{3}{2}}$. We therefore have

$$\begin{aligned} ds^2_{S_3,E} &\to Q_1^{\frac{1}{4}} r^{\frac{3}{2}} (d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\psi^2) \\ \sqrt{g_{S_3,E}} &= Q_1^{\frac{3}{8}} r^{\frac{9}{4}} \sin\theta \cos\theta \end{aligned} \tag{3.22}$$

Applying (3.20) we find $V_{S_3} = (2\pi)^2 Q_1^{\frac{3}{8}} r^{\frac{9}{4}}$. Finally, we consider the torus. The determinant in the limit $r \to 0$ is $g_{T_4,E} = H_1^{\frac{1}{2}} \approx Q_1^{\frac{1}{2}} r^{-1}$. The volume is therefore $V_{T_4} = Q_1^{\frac{1}{2}} r^{-1} \Xi^4 = Q_1^{\frac{1}{2}} r^{-1} (2\pi)^4 V$. Finally, putting everything together we have

$$A_{Hor} = Q_1^{-\frac{3}{8}} r^{\frac{3}{4}} (2\pi R) 2\pi^2 Q_1^{\frac{3}{8}} r^{\frac{9}{4}} Q_1^{\frac{1}{2}} r^{-1} (2\pi)^4 V = 64\pi^7 V R Q_1^{\frac{1}{2}} r^2 \tag{3.23}$$

In the limit $r \to 0$ the horizon area, and therefore the Bekenstein entropy, is just zero.

- **Entropy for the 2-charge solution**

Let us repeat the same calculation, this time starting from equation (3.14). We get the values

$$\begin{aligned} V_{S_1} &= (2\pi R) Q_1^{-\frac{1}{8}} Q_5^{-\frac{3}{8}} r \\ V_{S_3} &= Q_1^{\frac{9}{8}} Q_5^{\frac{3}{8}} 2\pi^2 \\ V_{T_4} &= Q_1^{-\frac{1}{2}} Q_5^{\frac{1}{2}} (2\pi)^4 V \end{aligned} \tag{3.24}$$

$$A_{Hor} = 64\pi^7 V R Q_1^{\frac{1}{2}} Q_5^{\frac{1}{2}} r \to 0 \tag{3.25}$$

Again, the entropy of the black hole is zero. Intuitively this happens because the D1 and D5-branes shrink the S_1 circle, and we need momentum to counterbalance this effect. In the 3-charge solution the entropy is not zero because we add such momentum.

- **Entropy for the 3-charge solution**

In this case we obtain

$$\begin{aligned}
V_{S_1} &= (2\pi R)Q_1^{-\frac{3}{8}}Q_5^{-\frac{1}{8}}Q_P^{\frac{1}{2}} \\
V_{S_3} &= Q_1^{\frac{3}{8}}Q_5^{\frac{9}{8}}2\pi^2 \qquad (3.26)\\
V_{T_4} &= Q_1^{\frac{1}{2}}Q_5^{-\frac{1}{2}}(2\pi)^4V
\end{aligned}$$

$$A_{Hor} = 64\pi^7 VRQ_1^{\frac{1}{2}}Q_5^{\frac{1}{2}}Q_P^{\frac{1}{2}} \qquad (3.27)$$

The horizon area does not go to zero because the r factors cancel each other. Therefore, the entropy does not vanish. We apply the formula $S_{BH} = \frac{A_{Hor}}{4G_{10}}$. There is a relation between the Q_1, Q_5, Q_P in terms of integer charges, that we justify at the end of this section so we do not drift apart from the calculation.

$$\begin{aligned}
Q_1 &= \frac{g_s^2\alpha'^3}{V}N_1 \\
Q_5 &= \alpha' N_5 \qquad (3.28)\\
Q_P &= \frac{g_s^2\alpha'^4}{VR^2}N_P
\end{aligned}$$

Moreover, we have

$$G_{10} = 8\pi^6 g_s^2\alpha^4 \qquad (3.29)$$

Plugging all of this in the entropy formula we get

$$S_{BH} = 2\pi\sqrt{N_1N_5N_P} \qquad (3.30)$$

This is a formula of crucial importance for what will come later. Note that the parameters g_s, V, R have all canceled out. This is crucial to the possibility of reproducing this entropy by some microscopic calculation. In the microscopic description we have a bound state of the charges N_1, N_5, N_P and we will be counting its degeneracy. However, since we are looking at BPS states, the degeneracy will not depend on these continuous parameters. Therefore, the final entropy must not depend on them either.

The relations (3.28) are extremely important, because they relate macroscopic and microscopic charges. They are so important that they deserve at least some justification of how they are derived. We have, for BPS states, $Q_p \propto G_5 M_p$ (we forget about numerical factors). Now, M_p is the mass of the Dp-branes, which is given by

$$M_p = T_pN_pV_p \qquad (3.31)$$

where T_p is the brane tension, N_p is the number of branes and V_p its volume. Let us start from the D1-branes: we have $T_1 =\propto \frac{1}{\alpha'}$ and $V_1 \propto R$. Now, putting everything together and also using (3.29), and $G_5 = \frac{G_{10}}{VR}$ we find

$$Q_1 \propto G_5M_1 \propto \frac{G_{10}}{VR}\frac{1}{\alpha'}RN_1 \propto \frac{g_s^2\alpha'^4}{VR}\frac{1}{\alpha'}RN_1 \propto \frac{g_s^2\alpha'^3}{V}N_1 \qquad (3.32)$$

The numerical factor in the final formula is 1, coming from the cancellation of many different factors that we completely ignored. Again, what matters is a general idea of the derivation, not the details. In a similar way we can derive the remaining formulas for Q_5 and Q_P.

3.2.2 Microscopic side

It is now time to re-compute the entropies we calculated in the previous section. This time we will start from a microscopic point of view. The results will match only in the 1-charge and 3-charge cases. The 2-charge and 3-charge solutions have a singularity which is not allowed in string theory. Moreover, the 2-charge solution has the ulterior problem that generates the wrong entropy.

- **Entropy for the 1-charge solution**

This is the simplest case. The fundamental string F1 is an oscillator in the ground state. Let us consider for example type IIA Supergravity. The bosonic fields are $g_{\mu\nu}$, ϕ, B_2, C_1, C_3. The number of independent on-shell components of these fields corresponds to the number of the base states, which is what we need. The number of on-shell components of such fields is

$$\begin{aligned} &\frac{D}{2}(D-3) &\qquad& \textit{Graviton } g_{\mu\nu} \\ &\binom{D-2}{x} && \textit{Gauge fields } A_x \end{aligned} \tag{3.33}$$

These formulas yield 35 degrees of freedom for the graviton, 28 for B_2, 8 for C_1, 56 for C_3. We add a final degree of freedom from the dilaton field, and our total is 128. Since we have supersymmetry there are also 128 fermionic degress of freedom that need to be accounted for. In total we have a degeneracy of 256. The entropy is therefore

$$S_{Micro} = \log 256 \tag{3.34}$$

Which is not zero as we might expect. However, we have to take the macroscopic limit $N_1 \to \infty$. This entropy does not grow with N_1, therefore, if we were to write a macroscopic entropy, we would write $S = 0$ at the leading order. Thus, in this sense, the two results agree. This means that we failed to make a black hole with nonzero area. The 1-charge solution does not describe a proper black hole. Why is that? Let us see the F1 string from the point of view of M-theory, which means that we have a M2-brane wrapping the directions x^{11} and y. We know that branes have a tension along their worldvolume directions, causing the circles around which they are wrapped to shrink. The length of the x^{11} circle goes to zero at the brane location $r = 0$. This shows up as $\phi \to -\infty$ in the IIA description. Similarly, we get a vanishing of the length of the y circle in the M theory description. This causes the horizon area to vanish.

- **Entropy for the 2-charge solution**

The entropy of the 2-charge system can be computed microscopically in three different ways [3]. We present just one of the three. Let us consider a duality frame in which the solution is generated by the F1-P system, which is a fundamental string wrapped N_1 times, carrying momentum N_P. The string carries right modes, with positive momentum, and left moves, with negative momentum. We count supersymmetric states, which contain right-moving oscillations only[3], otherwise we would not have a supersymmetric string. The momentum can be partitioned among different harmonics of the string: this is where the degeneracy comes from. To each harmonic corresponds a Fourier mode. Since the momentum is carried as multiples of $\frac{1}{R}$, we have $P = \frac{N_P}{R}$. Moreover, the length of the multiwound F1 string is given by $l_s = 2\pi R N_1$. We have

$$P = \frac{2\pi N_1 N_P}{l_s} \tag{3.35}$$

[3]We could also count states with left-moving oscillations only, which are supersymmetric as well.

From this we see that each harmonic contributes to the total momentum with $P_k = \frac{2\pi k}{l_s}$. Let us now focus on a single direction of vibration. If there are m_i units of the Fourier harmonic k_i, then we can write

$$\sum_{i=1}^{\infty} m_i k_i = N_1 N_p \tag{3.36}$$

This equation is really important because it tells us that the degeneracy along one particular direction of vibration is given by counting the partitions of the integer $N_1 N_P$. This is a quite common way to count states in String Theory, that shows up whenever we deal with an equation of the form (3.36). In chapter 5 we are going to deal with a very similar situation involving a 3-charge configuration.
So far we only considered a single direction of vibration. Actually, the momentum is partitioned along 8 bosonic and 8 fermionic modes. A fermionic degree of freedom is worth a half of a bosonic one, therefore the momentum is partitioned amongst 8+4=12 bosonic degrees of freedom. This means that the degeneracy for a single direction is given by the partitions of the integer $\frac{N_1 N_5}{12}$, that we denote with $Part(\frac{N_1 N_5}{12})$. Since we have 12 bosonic degrees of freedom, the total number of states will be $Part(\frac{N_1 N_P}{12})^{12}$. A tool that comes in handy is the Hardy-Ramanujan formula [23] that gives the asymptotic behaviour of the partition function[4]

$$Part(N) \approx \frac{1}{4N\sqrt{3}} e^{2\pi\sqrt{\frac{N}{6}}} \tag{3.37}$$

we can write

$$S_{Micro} = \log\left[Part(\frac{N_1 N_P}{12})^{12}\right] \approx 24\pi\sqrt{\frac{N_1 N_P}{72}} - 12\log\left[\frac{\sqrt{3}}{3} N_1 N_P\right] \tag{3.38}$$

Since we look at the macroscopic limit $N_1, N_5, N_P \to \infty$, the last term is subleading and can be discarded. We obtain

$$S_{Micro} = 2\sqrt{2}\pi\sqrt{N_1 N_P} \tag{3.39}$$

As anticipated, this does not match with the result obtained from the gravity side. This happens because the curvature of the solution gets really big near $r = 0$. Supergravity works well when we deal with small curvatures, since it comes from a low-energy limit. The correct 2-charge entropy can be obtained by adding corrections to the Einstein-Hilbert term of the action, containing higher powers of the Ricci curvature $\mathcal{R}$.

- **Entropy for the 3-charge solution**

The entropy for the F1-NS5-P system is computed in the same way as in the 2-charge case, with two differences. First, one can see that the F1-NS5 form an effective string with length $N_1 N_5$. Second, the string can oscillate only in four transverse directions. This happens because in order to make a F1-NS5 bound state we need the string to be inside the brane, allowing it to vibrate only in the directions parallel to the it, and transverse to the string. There are four such directions, giving $4+2 = 6$ bosonic degrees of freedom. The number of microstates is therefore $Part(\frac{N_1 N_5 N_P}{6})^6$. Using formula (3.37) we obtain

$$S_{Micro} = 2\pi\sqrt{N_1 N_5 N_P} \tag{3.40}$$

which perfectly coincides with (3.30).

[4]The function that counts the number of partitions of an integer is called partition function. The reader must avoid confusion with the partition function in statistical mechanics.

3.3 Constructing the microstates

It is now time to sort out the 2-charge entropy problem. We notice a puzzling fact: suppose we want to keep only two of the charges, setting for example $N_5 = 0$. This yields $S = 0$. However, we found that the entropy for the 2-charge system is not zero, from the string point of view. What is going on? One could think that the 2-charge system we considered is not a good system for describing a black hole, and therefore it should just be disregarded. However, this would be strange, since we found the entropy (3.39) in a similar way as we did with the three-charge case. Actually, the problem resides in equation (3.12) As we said, Supergravity is a good approximation when curvatures are small, which is not the case here: the curvature of the solution (3.12) blows up at $r = 0$. We remind that the F1-P system is made up by a F1 string wrapped around the y direction carrying momentum P, which manifests itself as transverse vibrations of the string. This makes the string bend away from its central axis, therefore it cannot be confined in the $r = 0$ position. This singularity is not allowed in string theory! From this point on we refer to the metric in equation (3.12) as the naive metric for the F1-P system. Our task is now to construct a solution, starting from String Theory, describing a black hole, also reproducing the correct entropy. However, this solution must also describe the microstates responsible for such entropy.
The F1 string is multiwound. We refer to the different parts making up the string as strands[5]. We introduce the variables $u \coloneqq t + y$, $v \coloneqq t - y$. Let us start with a string singly wound string ($N_1 = 1$). This object has only one strand. The latter oscillates with a vibration profile $\vec{F}(v)$[6], and the produced solution is [24] [25]

$$\begin{cases} ds^2 = H(-dudv + Kdv^2 + 2A_i dx^i dv) + dx_i dx^i + dz_a dz^a & i, a = 1,2,3,4 \\ e^{2\phi} = H \\ B_{uv} = -\frac{1}{2}(H-1) \qquad B_{vi} = HA_i \\ C_p = 0 \end{cases} \tag{3.41}$$

where

$$\begin{cases} H^{-1} = 1 + \frac{Q_1}{|\vec{x}-\vec{F}(v)|^2} \\ K = \frac{Q_1|\dot{\vec{F}}(v)|^2}{|\vec{x}-\vec{F}(v)|^2} \\ A_i = -\frac{Q_1\dot{F}_i(v)}{|\vec{x}-\vec{F}(v)|^2} \end{cases} \tag{3.42}$$

where the dot denotes a derivation with respect to v. We are interested in the case $N_1 > 1$, where we have a multiply wound string with many strands. In this case, the solution is obtained by superposing the coefficients H, K, A_i from equation (3.41). Since we are interested in the macroscopic limit $N_1 \to \infty$, we are allowed to turn the sums into integrals over the string. Since the strands all carry the momentum in the same direction along y, this generates a BPS state. The solution takes the same form as (3.41). However, the coefficients are now given by

$$\begin{cases} H^{-1} = 1 + \frac{Q_1}{l_s}\int_0^{l_s} \frac{dv}{|\vec{x}-\vec{F}(v)|^2} \\ K = \frac{Q_1}{l_s}\int_0^{l_s} \frac{|\dot{\vec{F}}(v)|^2 dv}{|\vec{x}-\vec{F}(v)|^2} \\ A_i = -\frac{Q_1}{l_s}\int_0^{l_s} \frac{Q_1\dot{F}_i(v)dv}{|\vec{x}-\vec{F}(v)|^2} \end{cases} \tag{3.43}$$

[5]In the next chapter we will introduce some new objects that are also called strands. However, they are completely different entities, and the reader must avoid confusion.

[6]$\vec{F}$ can depend only on v (not u), otherwise we would not obtain a supersymmetric solution, as we only consider right-moving oscillations.

For the following, it is useful to transform this solution and write it in the D1-D5 frame. We can perform the chain of dualities (3.13) on solutions (3.41) and (3.43). If we label Q_1 the D1 charge and Q_5 the D5 charge, the solution in this frame is

$$\begin{cases} ds^2 = (Z_1Z_2)^{-\frac{1}{2}}\big[-(dt-A_idx^i)^2+(dy+B_idx^i)^2\big]+(Z_1Z_2)^{\frac{1}{2}}dx_idx^i+Z_1^{\frac{1}{2}}Z_2^{-\frac{1}{2}}dz_adz^a \\ Z_1 = 1+\frac{Q_1}{l_s}\int_0^{l_s}\frac{|\dot{\vec{F}}(v)|^2dv}{|\vec{x}-\vec{F}(v)|^2} \\ Z_2 = 1+\frac{Q_5}{l_s}\int_0^{l_s}\frac{dv}{|\vec{x}-\vec{F}(v)|^2} \\ A_i = -\frac{Q_1}{l_s}\int_0^{l_s}\frac{\dot{F}_i(v)dv}{|\vec{x}-\vec{F}(v)|^2} \\ dB = -\star_4 dA \end{cases} \tag{3.44}$$

where the $\star_4$ operator denotes the Hodge dual with respect to the 4D flat euclidean metric, $l_s = \frac{2\pi Q_5}{R}$ and finally $Q_1 = \frac{Q_5}{l_s}\int_0^{l_s}|\dot{F}|^2dv$. Let us now discuss the geometrical features of this solution in different limits and compare it with the naive geometry (3.14).

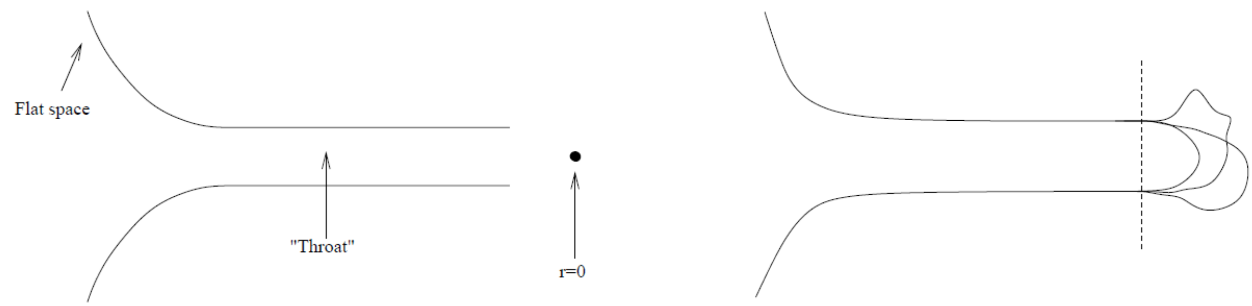

Figure 3.1: On the left: the naive geometry for the extremal D1-D5 black hole. On the right: the actual solutions. The area of the surface denoted with the dotted line reproduces the correct entropy. [3]

- $r^2 \gg Q_1, Q_5$: We have $H_1, H_2 \to 1$ in the naive geometry (3.14) and $Z_1, Z_2 \to 1$, $A_i, B_i \to 0$ in the D1-D5 geometry (3.44). This causes both metrics to be asymptotically flat, and we recover the 10-D Minkowski spacetime.

- $r^2 \approx Q_1, Q_5 \gg |\vec{F}(v)|^2$: $|\vec{F}(v)|$ represents the amplitude of the string's transverse oscillations. In this case from solution (3.44) the coefficients Z_1, Z_2 reduce to $Z_1 \to 1+\frac{Q_1}{r^2}$, $Z_2 \to 1+\frac{Q_5}{r^2}$ and A_i, B_i are again negligible. In this regime the two solutions again coincide.

- $|\vec{F}(v)|^2 \ll r^2 \ll Q_1, Q_5$: This is called the near horizon limit (or also decoupling limit). We neglect the 1's in Z_1, Z_2. We wish to take the limit $r \gg |\vec{F}(v)|$, neglecting the microscopic structure. For both solutions we obtain

$$ds^2 = \frac{r^2}{\sqrt{Q_1Q_5}}(-dt^2+dy^2)+\frac{\sqrt{Q_1Q_5}}{r^2}dr^2+\sqrt{Q_1Q_5}d\Omega_3+\sqrt{\frac{Q_1}{Q_5}}dz_adz^a \tag{3.45}$$

Notice that in this limit, the asymptotically flatness is lost. What is this metric? If we introduce the new coordinate $\bar{r} := \frac{r}{\sqrt{Q_1Q_5}}$ we get

$$ds^2 = \sqrt{Q_1Q_5}\big[\frac{d\bar{r}^2}{\bar{r}^2}+\bar{r}^2(-dt^2+dy^2)\big]+\sqrt{Q_1Q_5}d\Omega_3+\sqrt{\frac{Q_1}{Q_5}}dz_adz^a \tag{3.46}$$

this is $AdS_3 \times S_3 \times T_4$ with $R_{AdS_3} = R_{S_3} = \sqrt{Q_1 Q_5}$, written in Poincaré coordinates. The AdS_3 part will be crucial in order to apply the olographic construction. We notice that the geometries coincide in the limits we have considered so far. This happens because the deviation of the geometry (3.44) from (3.12) is given by the shape of $\vec{F}(v)$, which so far was always discarded.

- $|\vec{x} - \vec{F}(v)| \to 0$: We do not encounter any singularity: the geometry ends in a smooth cap. It looks like that there is a singularity on the profile $\vec{x} = \vec{F}(v)$, but this is not the case: it turns out that this is just a pathology of the coordinates.

Let us talk about the entropy. We found out that there are many microstates, each one parametrized by $\vec{F}(v)$. What are the "typical" microstates? Those for which $|\vec{F}(v)| \sim l_s$, let us see why. Earlier, we argued that the momentum of the string makes it bend away from its central axis, making it an extended object, which is not confined at $r = 0$. If that is the case, we ask ourselves how much space is occupied by it. It turns out [3] that the string fills a spherical region of size $|\vec{x}| \sim \sqrt{\alpha} \sim l_s$. Outside of this ball, the geometry we found becomes the naive geometry. Moreover, we can calculate the area of this surface. The computation is similar to the one we did earlier to calculate the horizon area in the 2-charge case, with $r \to l_s$. This was done more in detail in [3]. We find a remarkable result:

$$\frac{A_{Ball}}{4G_{10}} \sim \sqrt{N_1 N_5} \sim S_{Micro} \tag{3.47}$$

We recover the entropy we found in equation (3.39)! The microstates we constructed do not possess an horizon, but they all differ on the boundary of this region, which we call fuzzball horizon, to avoid confusion with the event horizon. This fact is very interesting because it could potentially lead to a solution of the information paradox. The fuzzball proposal states that, for a given black hole of entropy S, there exist e^S microstates like the one we constructed, that are responsible for it. The microstates do not have an event horizon, so the information paradox cannot arise, as Hawking's calculation assumes that such an horizon exists. If we remove it, we somehow remove the information paradox as well, in fact unlike a classical horizon, the geometry around the fuzzball horizon is not locally equivalent to Rindler, therefore the state of the emitted pairs is not maximally entangled. This idea resembles the concept of coarse-graining from statistical mechanics: if we keep only the part of the geometry outside the fuzzball horizon, then we have kept that part of each state which is common to all microstates, and discarded the part where they differ. In usual statistical systems entropy appears when we describe a system by a few macroscopic parameters (which all the microstates share) while ignoring the details that are different through the microstates.

As we have shown above, we now have a full understanding of the microstates responsible for the 2-charge extremal black hole's entropy. The same cannot be said for the 3-charge case, for which we know the naive solution , but we know only a limited class of solutions describing the individual microstates, e.g. [26] or [27]. It is possible that not all of them are smooth and horizonless. Moreover, the microstates for the 3-charge case might not be well described in Supergravity. To properly describe the 3-charge solutions it is very well possible that we need String Theory. Moreover, non-extremal black holes are still not well understood either.

In order to enforce the validity of the fuzzball conjecture, we must have a deeper understanding of the 3-charge solutions. In particular, we wish to provide a full gravitational description of the microstates. In Chapter 5, we investigate two known solutions and find some new ones, that interpolate between the two. To fully understand, however, we first need to introduce Conformal

Field Theories and the AdS/CFT correspondence, which are important theoretical tools that will be extremely important in our discussion.

Chapter 4

The D1-D5 CFT and Holography

The AdS/CFT correspondence is a powerful tool that creates a bridge between Supergravity and Conformal Field Theories. So far, we introduced only one side of the correspondence, involving gravity. We now introduce Conformal Field Theory (CFT). In this chapter we refer to David Tong's notes [28]. Another reference for the topic of CFT is the textbook [29]. The second part of the chapter is devolved to introducing the D1-D5 CFT, which is a particular conformal field theory that we will work with, which is dual via AdS/CFT to the gravity description that we discussed so far. For that, we refer to the works [30] [31].
The correspondence is useful for studying black hole microstates, because it relates geometries to states in the CFT. In the following chapter we will consider a vacuum state in the D1-D5 CFT, corresponding to $\mathrm{AdS}_3 \times S_3 \times T_4$, and apply operators on it. Acting with operators means applying coordinate transformations on the gravity side. This will allow us to construct new geometries. Note that not all geometries can be obtained by acting with operators on the vacuum. We will come back on this point later.

4.1 Introducing Conformal Field Theory

The main goal of this section is to introduce some of the most fundamental results in Conformal Field Theory, that we need later. We focus on two-dimensional theories, which we need in this work. However, keep in mind that there are higher dimensional theories that are of great interest in the field of the AdS/CFT correspondence. We begin by introducing conformal transformations. A conformal transformation is a change of coordinates under which the metric of the worldsheet transforms as

$$g'_{\mu\nu}(\tau,\sigma) = \Omega^2(\tau,\sigma) g_{\mu\nu}(\tau,\sigma) \tag{4.1}$$

A conformal field theory (CFT) is a theory which is invariant under conformal transformations. From a physical point of view, this means that the theory looks the same at all length scales: it does not care about distances, but it does care about angles. We are interested in the case in which the metric $g_{\mu\nu}$, also called background, is fixed, and not dynamical. In this situation, equation (4.1) is interpreted as a global symmetry and we can find the corresponding conserved currents. Even though Conformal Field Theories are a subset of general quantum field theories, we use a special language to describe them.
We are ultimately interested in worldsheets with Minkowski signature, however it is easier to work with euclidean worldsheets. Our results can then be transposed to Minkowski worldsheets with little effort. The Euclidean worldsheet coordinates are reached by performing a Wick rotation.

That is, we define $(\sigma^1, \sigma^2) := (\sigma, i\tau)$. We immediately move to the Euclidean analogue of the light-cone coordinates, defined as

$$\begin{aligned} z &:= \sigma^1 + i\sigma^2 \\ \bar{z} &:= \sigma^1 - i\sigma^2 \end{aligned} \tag{4.2}$$

The holomorphic derivatives are

$$\begin{aligned} \partial_z &:= \partial = \frac{1}{2}(\partial_1 - i\partial_2) \\ \partial_{\bar{z}} &:= \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \end{aligned} \tag{4.3}$$

these obey $\partial z = \bar{\partial}\bar{z} = 1$, $\partial\bar{z} = \bar{\partial}z = 0$.
If we work with a flat Euclidean metric $ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dzd\bar{z}$ we have $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$. In the coordinates z and $\bar{z}$, conformal transformations of flat space are particularly simple: they are holomorphic changes of coordinates

$$\begin{aligned} z &\to z' = f(z) \\ \bar{z} &\to \bar{z}' = \bar{f}(\bar{z}) \end{aligned} \tag{4.4}$$

Note that under this transformation we have $ds^2 = dzd\bar{z} \to |\frac{df}{dz}|^2 dzd\bar{z}$, which is precisely of the form (4.1). Note that we have an infinite number of such transformations, one for each function $f(z)$. This is a peculiarity of CFTs in two dimensions. For theories defined on $\mathbb{R}^{p,q}$, the conformal group is $SO(p+1, q+1)$ with $p + q > 2$. For many purposes, it is easier to treat z and $\bar{z}$ as independent variables. This corresponds to extending the worldsheet from $\mathbb{R}^2$ to $\mathbb{C}^2$, allowing us to apply various theorems from complex calculus. If we wish to recover the $\mathbb{R}^2$ worldsheet we simply set $\bar{z} = z^*$. A couple of particularly simple, yet important transformations in two dimensions are

- $z \to z + a$: translations.
- $z \to \alpha z$: rotations if $|\alpha| = 1$ and scalings for $\alpha \in \mathbb{R}$, $\alpha \neq 1$.

4.1.1 The stress-energy tensor

An object of crucial importance is the stress-energy tensor. It is defined as the matrix containing the conserved currents which arise from translational invariance

$$\delta\sigma^\mu = \epsilon^\mu \tag{4.5}$$

after some calculations we can obtain the following expression [28]

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}} \frac{\partial S}{\partial g^{\mu\nu}} \tag{4.6}$$

If we have a flat worldsheet we can choose $g_{\mu\nu} = \delta_{\mu\nu}$ and the corresponding stress-energy tensor is divergenceless $\partial^\mu T_{\mu\nu} = 0$. If the worldsheet is curved, then the stress-energy tensor is covariantly conserved $\nabla^\mu T_{\mu\nu} = 0$. In conformal theories, $T_{\mu\nu}$ has an important property: it is traceless. To prove this, let us vary the action with respect to a scale transformation

$$\delta g_{\mu\nu} = \epsilon(\sigma) g_{\mu\nu} \tag{4.7}$$

then, using (4.6) and (4.7) we can write

$$\delta S = \int \frac{\partial S}{\partial g_{\mu\nu}} \delta g_{\mu\nu} \ d\sigma^1 d\sigma^2 = \frac{1}{4\pi} \int \epsilon(\sigma) \sqrt{g} T_\mu{}^\mu \ d\sigma^1 d\sigma^2 \tag{4.8}$$

which is zero for any $\epsilon(\sigma)$ because scaling transformations are a symmetry in CFT. This tells us that

$$T_\mu{}^\mu = 0 \tag{4.9}$$

The vanishing of the trace of the stress-energy tensor is a peculiarity of classical Conformal Field Theories, such as Maxwell theory or Yang-Mills theory in four-dimensions. However, at the quantum level, this feature often disappears. This is not the case for CFTs: T is traceless also at the quantum level.
In complex coordinates the tracelessness condition (4.9) becomes

$$T_{z\bar{z}} = 0 \tag{4.10}$$

the conservation equation $\partial_\mu T^{\mu\nu} = 0$ becomes $\partial T^{zz} = \bar{\partial} T^{\bar{z}\bar{z}} = 0$, or equivalently, by lowering the indices, $\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$. These conditions tell us that $T_{zz}(z)$ is a holomorphic function, whereas $T_{\bar{z}\bar{z}}(\bar{z})$ is antiholomorphic. From this point on we use the simplified notation

$$T(z) \coloneqq T_{zz}(z) \qquad \qquad \bar{T}(\bar{z}) \coloneqq T_{\bar{z}\bar{z}}(\bar{z}) \tag{4.11}$$

It is possible to compute the currents associated with an infinitesimal conformal transformation

$$z' = z + \epsilon(z) \qquad \qquad \bar{z}' = \bar{z} + \epsilon(\bar{z}) \tag{4.12}$$

For constant ϵ this is a translation. For $\epsilon \sim z$, we get a dilatation or a rotation. The corresponding currents are

$$\begin{aligned} J^z &= 0 \qquad & J^{\bar{z}} &= T(z)\epsilon(z) \\ \bar{J}^{\bar{z}} &= 0 \qquad & \bar{J}^z &= \bar{T}(\bar{Z})\bar{\epsilon}(\bar{z}) \end{aligned} \tag{4.13}$$

The currents $J, \bar{J}$ are respectively holomorphic and antiholomorphic and they satisfy the conditions $\partial_{\bar{z}} J^{\bar{z}} = 0$ and $\bar{\partial}_z \bar{J}^z = 0$, which means they are conserved.

4.1.2 Operators

In QFT, fields are promoted to operators. In CFT, operators are states. This happens because an operator acting on the vacuum defines a state, and thanks to conformal symmetry this does not depend on the coordinates. It is important to keep this fact in mind in order to avoid confusion between operators and states.
We now define the operator product expansion (OPE) between two local operators (or fields). This is a statement about what happens as they approach each other. The idea is that if we insert two local operators at nearby points, we can closely approximate them with a string of operators at one of the two points. We denote with $\mathcal{O}_i$ the local operators of the CFT[1] and define

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \sum_k C^k_{ij}(z-w,\bar{z}-\bar{w})\mathcal{O}_k(w,\bar{w}) \tag{4.14}$$

[1] with i, j, k running over the set of all operators

where $C_{ij}^k(z-w,\bar{z}-\bar{w})$ is a set of functions that depend only by the separation between the two operators. We understand equation (4.14) as a statement which holds as operator insertions inside time-ordered correlation functions:

$$\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})...\rangle = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\langle \mathcal{O}_k(w,\bar{w})...\rangle \tag{4.15}$$

where the ... can be any chosen operator insertion. It is tedious to write $\langle ... \rangle$ all the time, so we stick with the notation in equation (4.14). Note that in this definition we always assume the operators to be time-ordered. It is important to note that the OPEs may have a singularity as $z = w$. This singularity is important because it contains the same information contained in the commutation relations, plus, it bears information about how the operators transform under symmetries. In many equations we will simply write the singular terms in the OPE and denote the non-singular terms as +...

A result of crucial importance are the Ward identities. A full derivation is found in [29]. Let us consider a conformal transformation. If we treat z and $\bar{z}$ as independent variables, we have two Ward identities: one for the transformation $\delta z = \epsilon(z)$, another for $\delta\bar{z} = \epsilon(\bar{z})$. The two read

$$\begin{aligned} \delta\mathcal{O}_i(\sigma_i) &= -Res[J_z(z)\mathcal{O}_i(\sigma_i)] = -Res[\epsilon(z)T(z)\mathcal{O}_i(\sigma_i)] \\ \delta\mathcal{O}_i(\sigma_i) &= -Res[\bar{J}_{\bar{z}}(\bar{z})\mathcal{O}_i(\sigma_i)] = -Res[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\mathcal{O}_i(\sigma_i)] \end{aligned} \tag{4.16}$$

This is an important result: if we know the OPE between some operator and the stress-energy tensor, then we immediately know how the operator transforms under conformal symmetry.

Another important concept in QFT is the conformal weight. An operator $\mathcal{O}$ is said to have weight $(h,\bar{h})$ if, under $\delta z = \epsilon z$ and $\delta\bar{z} = \bar{\epsilon}\bar{z}$, it transforms as

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O} + z\partial\mathcal{O}) - \bar{\epsilon}(\bar{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}) \tag{4.17}$$

There are a couple of important observations to make:

- h and $\bar{h}$ are real numbers. If the CFT is unitary, all operators have $h,\bar{h} \geq 0$.
- h and $\bar{h}$ are related to two familiar concepts: the spin S and the scaling dimension Δ. We have that $S = h - \bar{h}$ is the eigenvalue of the operator under rotations along a fixed axis. Moreover, $\Delta = h + \bar{h}$ is the the familiar dimension that we associate to the field when performing dimensional analysis. As an example, we have $\Delta[\partial] = +1$
- Why are S and Δ defined like this? We have to remember how rotations and scalings act on the z and $\bar{z}$ coordinates, that is $L = -i(\sigma^1\partial_2 - \sigma^2\partial_1) = z\partial - \bar{z}\bar{\partial}$ for rotations and $D = \sigma^\mu\partial_\mu = z\partial + \bar{z}\bar{\partial}$ for scale transformations. The link between the concepts is quite clear from these expressions.

An operator is said to be primary if its OPE with T and $\bar{T}$ truncates at order $\frac{1}{(z-w)^2}$ and $\frac{1}{(\bar{z}-\bar{w})^2}$ respectively.

$$\begin{aligned} T(z)\mathcal{O}(w,\bar{w}) &= h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + ... \\ \bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) &= \bar{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + ... \end{aligned} \tag{4.18}$$

The OPEs above determine how primary operators transform under conformal transformations, together with the Ward identities. Primary operators are important because they have particularly simple transformation rules. Let us focus on $\delta z = \epsilon(z)$. We have, using (4.18) and (4.16)

$$\delta\mathcal{O}(w,\bar{w}) \overset{4.16}{=} -Res[\epsilon(z)T(z)\mathcal{O}(w,\bar{w})] \overset{4.18}{=} -Res\left[\epsilon(z)\left(h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + ...\right)\right] \tag{4.19}$$

since we look at smooth transformations, we require $\epsilon(z)$ to be regular at $z = w$. This allows us to Taylor expand it near that point

$$\epsilon(z) = \epsilon(w) + \epsilon'(w)(z-w) + o(z-w)^2 \tag{4.20}$$

We plug this in (4.19) and we are done. We can follow an analogous reasoning to find the transformation law for an antiholomorphic primary operator. Our results are

$$\begin{aligned} \delta\mathcal{O}(w,\bar{w}) &= -h\epsilon'(w)\mathcal{O}(w,\bar{w}) - \epsilon(w)\partial\mathcal{O}(w,\bar{w}) \\ \delta\mathcal{O}(w,\bar{w}) &= -\bar{h}\bar{\epsilon}'(\bar{w})\mathcal{O}(w,\bar{w}) - \bar{\epsilon}(\bar{w})\partial\mathcal{O}(w,\bar{w}) \end{aligned} \tag{4.21}$$

These relations can be easily integrated in order to obtain the general transformation of a primary operator under a finite conformal transformation

$$z \to \hat{z}(z) \qquad\qquad \bar{z} \to \hat{\bar{z}}(\bar{z}) \tag{4.22}$$

the transformation reads

$$\mathcal{O}(z,\bar{z}) \to \left(\frac{\partial\hat{z}}{\partial z}\right)^{-h}\left(\frac{\partial\hat{\bar{z}}}{\partial\bar{z}}\right)^{-\bar{h}}\mathcal{O}(z,\bar{z}) \tag{4.23}$$

One of the main objects of interest in CFT is the spectrum of weights $(h, \bar{h})$ of primary fields. This is equivalent to computing the particle mass spectrum in quantum field theory. In statistical mechanics, the weights of primary operators are critical exponents.

4.1.3 The central charge

In any CFT, the stress-energy tensor is not a primary operator. T has weight $(2, 0)$.[2] The reason is quite simple: its associated spin S has to be 2, since it is a symmetric 2-tensor. Moreover, since we obtain the energy by integrating T over space, Δ has to be 2. This information yields the correct weight, which is obtained by solving $h + \bar{h} = 2$ and $h - \bar{h} = 2$. The TT OPE takes the form

$$T(z)T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + ... \tag{4.24}$$

We may ask ourselves how many singular terms are there in this OPE. Each addend has $\Delta = 4$, so the terms on the right side are of the form

$$\frac{\mathcal{O}_n}{(z-w)^n} \tag{4.25}$$

where $\mathcal{O}_n$ has $\Delta = 4-n$. Since in any unitary CFT there are no operators with $h, \bar{h} < 0$, we have to truncate the expansion at $(z-w)^{-4}$. We note, however, that a term like $(z-w)^{-3}$ is not acceptable

[2] $\bar{T}$ has weight $(0, 2)$

because the OPE must be time-ordered, and such a term changes sign under $z \leftrightarrow w$. The $(z-w)^{-1}$ may seem unacceptable as well, however, there is a trick to save it, which involves Taylor expanding $T(z)$ and its derivative, and using the expansion to show that $T(z)T(w) = T(w)T(z)$. See [28]. We write the expansions

$$\begin{aligned} T(z)T(w) &= \frac{\frac{C}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \ldots \\ \bar{T}(\bar{z})\bar{T}(\bar{w}) &= \frac{\frac{\bar{C}}{2}}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{(\bar{z}-\bar{w})} + \ldots \end{aligned} \tag{4.26}$$

The constants C and $\bar{C}$ are called the central charges. (Or also left-moving and right-moving charges). T is not primary, unless $C = 0$. Plus, it is possible to show that all CFTs have $C, \bar{C} > 0$. Note that C is not always an integer.

The most important fact about the central charges is their relation to the number of degrees of freedom of the theory. This link is made precise in a theorem by Zamolodchikov, called c-theorem, which states the following.

Conformal field theories are fixed points of the renormalization group, looking the same at all length scales. We can perturb a CFT by introducing an extra term in the action:

$$S \to S + \gamma \int \mathcal{O}(\sigma) d^2\sigma \tag{4.27}$$

where $\mathcal{O}$ is some local operator in the theory and γ is a constant. The possible perturbations fall in three different categories, depending on the conformal dimension Δ of the operator $\mathcal{O}$:

- $\Delta < 2$: γ has positive dimension: $[\gamma] = 2$. Such deformations are important in the infrared and are called relevant. The renormalization group flow takes us away from the original CFT.

- $\Delta = 2$: γ is dimensionless. The deformed action defines a new CFT. We call these deformations marginal.

- $\Delta > 2$: γ has negative dimension. The infrared physics is still described by the unperturbed CFT, whereas the ultraviolet physics is altered. These are called irrelevant deformations.

We expect a loss of information as we flow from an ultraviolet theory to an infrared theory. This is indeed the case, in fact the theorem states that there exists a function C on the space of all theories which monotonically decreases along RG flows. At the fixed points, C is a constant and coincides with the central charge of the CFT.

4.2 The AdS/CFT conjecture

The AdS/CFT correspondence is an idea originally formulated in 1997 by Juan Martin Maldacena. The conjecture relates (d+1)-dimensional theories on AdS spacetime and d-dimensional conformal field theories. When one of the two is strongly coupled, the other one is weakly coupled. In the following section we will justify and explain this statement in more detail.

4.2.1 Justifying the conjecture

Throughout the years, many different clues pointing in the direction of the correspondence have been found. We still do not have a precise proof, that is why we call it a conjecture. However, there is strong evidence that the correspondence is true. Maldacena, followed by Witten, originally proposed the conjecture [32] [33] by studying D-branes and black holes in String Theory. However, different approaches can be followed. In this section we describe two of them.
We first propose the line of reasoning that led Susskind to the formulation of the holographic principle [34]. Let us suppose that we have some region of spacetime Γ containing a thermodynamic system with entropy $S > \frac{A_{\partial\Gamma}}{4G}$. Now we throw in some mass, in such a way to form a black hole that perfectly fills Γ, which means that its horizon's area is $A_{\partial\Gamma}$. The result of this process is that the entropy of the thermodynamic system has decreased, since now we have precisely $S = \frac{A_{\partial\Gamma}}{4G}$. The entropy of the region outside Γ has decreased as well. This is a violation of the generalized second principle (1.9). So, if we assume that (1.9) is true, this statement means that the entropy of a thermodynamic system is bounded by $S = \frac{A}{4G}$. This means that in the presence of gravity the number of degrees of freedom of the system generating the entropy does not grow with its volume, like in quantum field theory, but it scales as the area of its boundary. This led Susskind to think that in quantum gravity theories the physics happening inside some spacetime region can be described by means of some nongravitational theory defined on the boundary of the system.
Another interesting line of thought can be followed, and involves seeing the AdS/CFT correspondence as a realization of the open-closed string duality. In the previous chapters we have introduced D-branes. These objects can be seen from an open or closed string point of view: from the open string point of view, they are the objects in which the string endpoint lie, whereas in the closed string description they are charged objects with respect to Ramond-Ramond fields. Plus, since they are massive objects with $M \propto \frac{1}{g_s}$, they curve the spacetime in which the strings propagate. We will now consider D-branes from both point of views.

- **Open string description**

After quantizing the strings on the D-branes, we obtain an open string spectrum that can be identified with fluctuations of the brane [15]. For a single D-brane, the massless spectrum contains scalar fields ϕ_i which describe fluctuations of the brane in the transverse direction, and a $U(1)$ gauge field A_μ that lives on the brane. Something interesting happens when we consider N parallel D-branes. Strings starting and ending on the same brane give raise to a $U(1)$ gauge field, so we have an overall gauge group $U(1)^N$. However, strings can start and end on different branes: if the separation between the branes is r, then the corresponding fields have mass $m = \frac{r}{2\pi\alpha}$. We denote the gauge fields with $(A_\mu)^a_b$ where a labels the brane containing the starting point of the string, and b contains the ending point. If the branes are all on top of each other, we have $m = 0$, therefore we obtain $N(N-1)$ massless fields and the resulting theory is a non-abelian gauge theory with gauge group $U(N)$. In a similar fashion, one finds that the massless scalars become NxN matrices $(\phi_i)^a_b$ that transform in the adjoint representation of the gauge group.
Let us make an explicit example: we consider N D3-branes from type IIB superstring theory. In the low energy limit, we can write an effective action for the massless modes

$$S = S_{bulk} + S_{branes} + S_{bulk-branes} \tag{4.28}$$

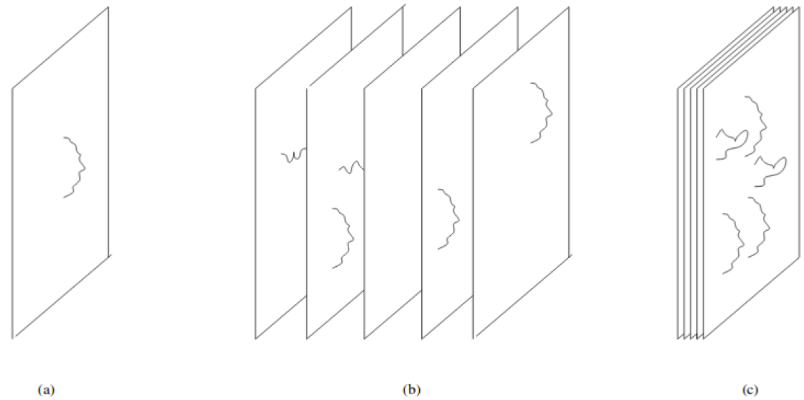

Figure 4.1: Open strings stretching between (a) a single D-brane (b) separated D-branes (c) coincident D-branes. [31]

where S_{bulk} is the 10-dimensional Supergravity action (2.100) (plus higher derivative corrections), S_{branes} is the action for the branes and finally $S_{bulk-branes}$ describes the interactions between the bulk modes and the brane modes.
Let us focus on S_{branes}. In the low energy limit the theory contains the fields $(A_\mu)^a_b$ $(\mu = 0, 1, 2, 3)$ and $(\phi_i)^a_b$ $(i = 1, 2, 3, 4, 5, 6)$. They all transform in the adjoint representation of $U(N)$. It turns out that S_{branes} describes a N=4 Super Yang-Mills theory with gauge group $U(N)$ in (3+1)-dimensions. The bosonic lagrangian of this theory is [35]

$$\mathcal{L} = -\frac{1}{g_{YM}} Tr\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi^i D^\mu\phi^i + \sum_i\sum_j[\phi^i, \phi^j]\right) \tag{4.29}$$

Where we have a relation between the Yang-Mills coupling g_{YM} and the string coupling:

$$g^2_{YM} = 4\pi g_S \tag{4.30}$$

Thanks to supersymmetry, the β function vanishes and this theory possesses conformal invariance. The $U(N)$ gauge group is equivalent to $U(1) \times SU(N)$. We can forget about the $U(1)$ symmetry because it describes the rigid motion of the brane system's center of mass, which is trivial.
The action (4.28) also contains S_{bulk}, describing closed string modes interacting among themselves, and $S_{bulk-branes}$, which describes interactions between closed string and open string modes. All these interactions are always proportional to some power of $\sqrt{G_N} \sim g_S\alpha^2$. If we take the low energy limit, which means $l_s \to 0$ with g_S and N fixed, we have $\sqrt{G_N} \to 0$ and all these interactions vanish. This makes sense because gravity is free at large distances. We end up with two theories: free Supergravity in the bulk and a 4-dimensional gauge theory on the branes.

- **Closed string description**

D3-branes are massive objects that act as sources for Supergravity fields. If we denote with x_i the three directions along which the brane is extended, we can write the D3-brane Supergravity metric [36]:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \\ H &= 1 + \frac{R^4}{r^4} \qquad R^4 = 4\pi g_S \alpha'^2 N \end{aligned} \tag{4.31}$$

Let us study some limits: for $r \gg R$ we obtain Minkowski spacetime, so the metric is asymptotically flat. In the near horizon limit $r \ll R$ we have $H \approx \frac{R^4}{r^4}$ and we obtain the $AdS_5 \times S_5$ geometry. In the low energy limit the interacting sector reduces to closed string in $AdS_5 \times S_5$ with AdS radius R [31]. This happens because of the red-shift factor $H^{-\frac{1}{4}}$: the energy measured by an observer at spatial infinity E_∞ is related to the energy E_r measured by an observer at a constant position r by the formula

$$E_\infty = H^{-\frac{1}{4}} E_r = \frac{r}{R} E_r \tag{4.32}$$

This means that the closer we get to $r = 0$, the lower the energy measured at infinity. In QFT, the energy gets bigger for small r, which is the opposite that happens here. In fact, this phenomenon is an effect given by the presence of gravity.

- **Open-Closed string duality**

As we have previously seen, superstring theories are reparametrization-invariant. This means that we can exchange the τ and σ coordinates on the worldsheet. This has the consequence that we can see tree level closed string processes as open string loop diagrams, as shown in figure 4.2.

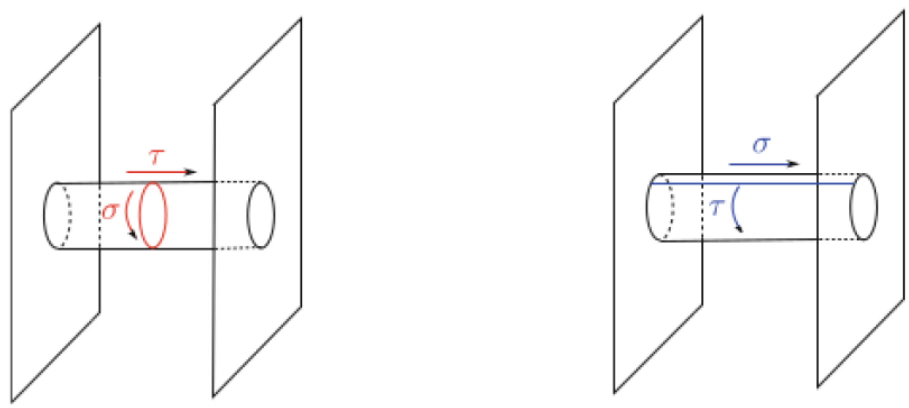

Figure 4.2: We can interpret the process on the left as an exchange of a closed string between two D-branes. If we swap τ and σ we obtain the process on the right, which can be viewed an open string loop diagram. [31]

4.2.2 The correspondence

We have seen that the open and closed string description are somewhat dual to one another. In the low energy limit we found a 4D Yang-Mills theory in the open string description, and a closed string theory on AdS$_5$ in the closed string description. This duality leads to the conjecture that

$$\big[\text{N=4 SU(N) super-YM in (3+1)-D}\big] \leftrightarrow \big[\text{IIB superstring on } AdS_5 \times S_5\big] \tag{4.33}$$

We can find a relation between the parameters of the two theories by means of equations (4.30), (4.31) and (2.10)

$$\frac{R^4}{l_s^4} = g_{YM}^2 N \tag{4.34}$$

This is an important relation, because it tells us that when a theory is strongly coupled, the other one is weakly coupled, in fact

- If $g_{YM}^2 N \ll 1$, then $R \ll l_s$. This is a regime where the Yang-Mills theory is weakly coupled, and the perturbative expansion is reliable. However, the radius R characterizing the gravitational effects becomes extremely small, and the closed string description becomes intractable since one faces highly stringy behaviours.
- If If $g_{YM}^2 N \gg 1$, then $R \gg l_s$. The geometry becomes weakly curved, making the Supergravity description reliable. However, the Yang-Mills description becomes highly coupled and the loop expansion becomes really difficult to deal with.

It is now clear why the AdS/CFT conjecture is such a powerful tool: if we have to deal with a strongly coupled theory, we can instead work with its dual, which often makes calculations much easier. The fact that the two theories behave so differently makes the conjecture really difficult to prove. However, there are several clues that make us think it is true. [37]

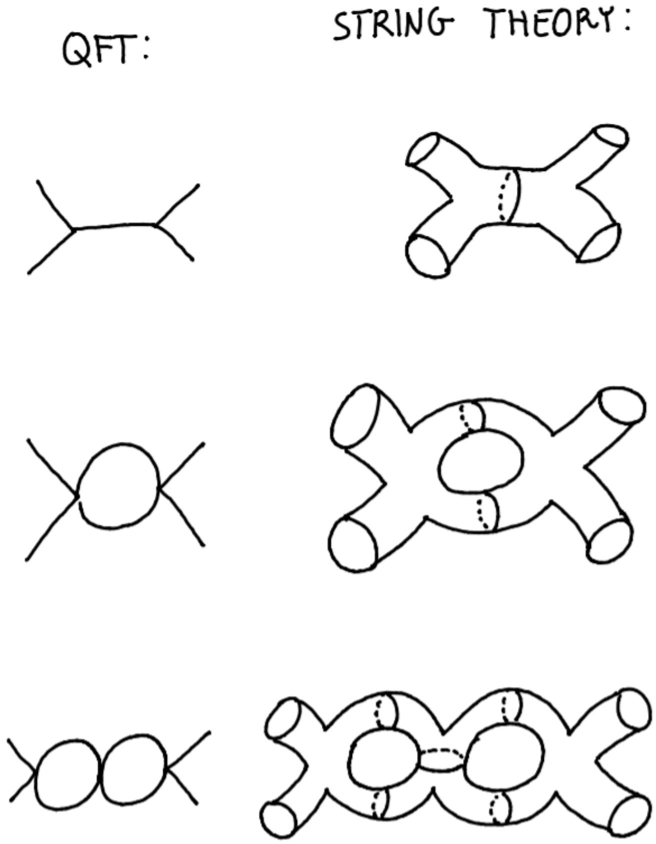

Figure 4.3: A comparison between Feynman diagrams in QFT vs Feynman diagrams in String Theory: the lines describing particles are now replaced with surfaces, describing strings. When $N \to \infty$ we are discarding diagrams like the second and the third one, which contain loops. Notice that the loops in String Theory are characterized by the genus of the surfaces that represent them. At the tree level, we have surfaces with genus 0, which are spheres.

Finally, notice that we have implicitly assumed that $N \to \infty$. The two theories are dual to each other in this limit, which the most interesting case for us because it corresponds to the thermodynamical limit for macroscopic black holes[3]. How can we assume that $N \to \infty$ and what does it mean? In string theory we have two possible expansions: a curvature expansion, controlled by l_s (4.34). The second one is a loop expansion, controlled by g_s. The loop expansion is analogous to the loop expansion in QFT, in which if $\hbar \to 0$ we can discard the loop and consider only processes at the tree-level. In String Theory the procedure is similar: if $g_s \to 0$ we can discard the loops, and only keep diagrams at the tree level, corresponding to spheres, as we can see in figure 4.3. Now, let us fix the quantity $g_{YM}^2 N$, which is called 't Hooft coupling. If $N \to \infty$, then we must have $g_{YM}^2 \to 0$. Because of (4.30) this means $g_s \to 0$, meaning that we can ignore the loop expansion, making our calculations easier.

4.3 The D1-D5 CFT

The Strominger-Vafa black hole that we introduced in Chapter 3 is not made by a stack of D3-branes, but by binding D1 and D5-branes. We therefore introduce and discuss the D1-D5 system. We have seen that the geometries constructed in section 3.3 become $\mathrm{AdS}_3 \times \mathrm{S}_3 \times \mathrm{T}_4$ in the near horizon limit. The so-called D1-D5 CFT is the dual description, in the low energy limit, of such geometry.

	0	1	2	3	4	5	6	7	8	9
D5	–	•	•	•	•	–	–	–	–	–
D1	–	•	•	•	•	–	•	•	•	•

Table 4.1: Diagram showing the D1-D5 brane configuration. The dots indicate that the object is "point-like" in the corresponding direction and the lines indicate that the object is extended in the corresponding direction. The 0 direction is time; the 5 direction is the S_1, the 1-4 directions are the noncompact spatial part of $\mathbb{R}^{4,1}$, and the 6-9 directions are the T_4. [30]

The D1-D5 system is obtained by compactifiying IIB string theory on $T_4 \times S_1$ with a bound state of D5-branes wrapping the whole compact space and D1-branes wrapping the circle S_1. From this point on we choose the T_4 to be really tiny compared to the S_1. In fact, we choose the radii of the T_4 such that $V_{T_4}^{\frac{1}{4}} \sim l_s$, and $R_{S_1} \gg l_s$. With this distinction, we can tell the S_1 direction apart from the T_4 directions. The D1-D5 system breaks $\frac{1}{4}$ supersymmetries, so we expect the dual theory to be a two-dimensional CFT with 8 supercharges.There are two different ways to construct this CFT. We briefly outline them. For an exhaustive treatment see [30] or [38].

- **First method**

This first method is a bit convoluted and not exactly easy to treat, however we include it for completeness. As we pointed out earlier, Dp-branes are objects in which open string endpoints lie. In a similar fashion to the example from the previous section, the strings can either start and end on D1-branes and D5-branes. We can split them in three groups:

- 5-5 strings: They have both endpoints on D5-branes and give raise to a $U(N_5)$ gauge theory in (5+1)-dimensions with 16 supercharges. Here, N_5 is the number of parallel D5-branes.

[3] There exist a strong form of the conjecture, stating that the two theories are equivalent for every value of N and g_s.

- 1-1 strings: They have both endpoints on D1-branes and give raise to a $U(N_1)$ gauge theory in (1+1)-dimensions with 16 supercharges. Here, N_1 is the number of parallel D1-branes.
- 5-1 strings: They originate from D5-branes and end on D1-branes and transform under the fundamental representation of $U(N_5)$, and under the antifundamental representation of $U(N_1)$. They break the number of supersymmetries of the theory down to 8. [4]

Since we have taken the size of the torus to be $\sim l_s^4$, we can dimensionally reduce the theory to a (1+1)-dimensional theory parametrized by the time and the S_1 coordinate y. Moreover, since we want to take the low energy limit, we are interested in the supersymmetric minima of the theory. It turns out that there are two classes of minima, which correspond to two different regions of the moduli space of the theory.

- The Coulomb branch: The states of the 1-1 and 5-5 strings that parametrize the displacement of the branes along the transverse directions[5] acquire a VeV. This causes the branes to separate from each other, breaking the gauge group down.
- The Higgs branch: The states of the 1-5 and 5-1 strings that parametrize the displacement of the D1-branes inside of the D5-branes acquire a VeV. In this case, the branes do not separate from each other and a bound state is described, with the D1-branes moving inside of the D5-branes. The D1-branes can move along four directions, which means that we have four different VeVs, each one representing a displacement along one direction.

Notice that the process we just described is precisely the Higgs mechanism in string theory. In fact, when we have strings stretched along parallel D-branes with separation d, it can be shown [15] that the mass squared operator is

$$M^2 = \left(\frac{d}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1) \tag{4.35}$$

where N is some number operator. Massless states have N=1, and the separation between the D-branes is $d = 0$. If we have a massless state and we separate the D-branes, then it is impossible to keep the mass to zero, because the term containing the separation appears squared. Therefore, if we split the D-branes, some fields become massive and some symmetries are broken.
We are interested in bound states of D1-branes and D5-branes, so we would like to work in the Higgs branch. Making progress along this route is quite complicated, therefore we turn our attention to an alternative method to describe the D1-D5 CFT.

- **Second method**

An alternative approach to describe the Higgs branch is to consider the D1-branes as instantonic solutions of the 6-dimensional $U(N_5)$ gauge theory on the D5-branes: these are strings that wrap S_1 but are localized on T_4. We are therefore interested in N_1 instantons in the D5 theory: these form a family of solutions whose parameters form the instanton moduli space. In fact, it can be shown that we can treat N_1 as an instantonic number, and the D1-branes behave just like instantons moving in a 4D space. We can describe the Higgs branch with a (1+1)-dimensional

[4] Of course 1-5 strings also exist and behave just like 5-1 strings.

[5] If one explicitly writes the states they can see that they contain the separation between the branes in their expression. [15]

sigma model whose target space is the moduli space of the N_1 instantons on T_4. This space is very complicated in general, however, for some particular values of the closed string moduli, it reduces to the so-called orbifold point:

$$SP(T_4)^{N_1N_5} := \frac{(T_4)^{N_1N_5}}{S_{N_1N_5}} \tag{4.36}$$

Here, S_n is the symmetric group of order n. Basically, this space is constructed by taking N_1N_5 copies of the T_4 and then taking the symmetrized cartesian product. From this point on we will always work at the orbifold point, where all interactions vanish and the theory contains free bosons and fermions only. This description allows us to visualize the CFT as a collection of N_1N_5 strings wrapped around the S_1 circle, with target space T_4. Notice that we need the symmetrization because if we exchange two or more strings, there is no physical distinction between the configurations.

One important thing to point out is that the strings can wrap the circle more than once. There are two opposite situations: one in which we have N_1N_5 strings wrapped around the circle just once. This is called the untwisted sector of the theory. Then, we can for example join two strings together and create a string which is wrapped two times around the circle, or we can go on and join w strings together in order to get a single string wrapping the circle w times. The extreme situation is when we have a single string wrapped N_1N_5 times. We will refer to these multi-wound strings as strands, but it is very important not to confuse them with the strands of Chapter 3: in that chapter we were describing an oscillating F1 (or D1) string, whereas here we are describing an effective string made by binding D1-branes and D5-branes. Our theory therefore consists in m_i strands, each one wound w_i times along S_1, such that

$$\sum_i m_i w_i = N_1N_5 \tag{4.37}$$

In the two extreme configurations we have N_1N_5 singly wound strands, one single strand wound N_1N_5 times respectively. For a visual representation of this formula see figures 4.4 and 4.5 at the end of the chapter.

Finally, we would like to briefly list the bosonic symmetries of the two theories (Supergravity and the D1-D5 CFT), and see that thay match. First, on the gravity side, we have $\mathrm{AdS}_3\times\mathrm{S}_3\times\mathrm{T}_4$. This space has a $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ isometry group on AdS_3. Moreover, it has an $SO(4)_E$ isometry group of S_3 and an $SO(4)_I$ on the torus, which is broken by compactification. On the other hand, the CFT is characterized by the conformal group in two dimensions, which is infinite-dimensional and is generated by the Virasoro generators $L_n, \bar{L}_n$, $(n \in \mathbb{Z})$. However, only the subalgebra generated by the operators with $n = 0, \pm 1$ is globally defined [36]. This subalgebra generates a $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ group of transformations, which we can identify with the isometries on AdS_3. Moreover, the CFT has two $SO(4)$ symmetries that can be identified with $SO(4)_E$ and $SO(4)_I$.

4.3.1 Field content

In the following, we briefly describe the fields contained in the left sector of the theory[6]. For more information see [30] or [39]. The symmetries of the D1-D5 CFT are generated by the $\mathcal{N} = (4,4)$ superconformal algebra, which is spanned at every point in the complex plane by the following local operators:

- The stress-energy tensor $T(z)$

[6] An identical discussion can be made for the right sector.

- Four supersymmetry currents $G^{\alpha A}(z)$
- The $SO(4)_E \sim SU(2)_L \times SU(2)_R$ R-symmetry current $J^a(z)$
- There also is a global $SO(4)_I \sim SU(2)_1 \times SU(2)_2$ symmetry on the torus

We introduce the following conventions for the fundamental representation indices of the various symmetry groups:

$$\begin{array}{llll} \alpha, \beta : & SU(2)_L & \dot{\alpha}, \dot{\beta} : & SU(2)_R \\ A, B : & SU(2)_1 & \dot{A}, \dot{B} : & SU(2)_2 \end{array} \tag{4.38}$$

All these indices can take the values 1,2 or +,-. For the vectorial representation of $SO(4)_I$ we use the indices i,j=1,2,3,4. Finally, we denote the adjoint indices of $SU(2)_{L/R}$ with a, b, c that can take the values 1,2,3 or +,-,3.

In the free field description, the theory contains bosons and fermions

$$X_r^{A\dot{A}}(z) \qquad \text{(bosons)} \tag{4.39}$$

$$\psi_r^{\alpha\dot{A}}(z) \qquad \text{(fermions)} \tag{4.40}$$

The index r is the copy index, and keeps track of which one of the $N_1 N_5$ copies of the sigma model we are in. Since in a single copy we have the central charges $c = \bar{c} = 6 = 4 + 2$ (4 from the bosons, 2 from the fermions), the complete theory has central charges

$$c = \bar{c} = 6 N_1 N_5 \tag{4.41}$$

At this point, various quantities of interest can be calculated, that we report here. First, in the $SU(2)_{L/R}$ basis with $\alpha, \beta = +, -$ we define the invariant tensor $\epsilon_{\alpha\beta}$ such that

$$\epsilon_{12} = -\epsilon_{21} = \epsilon^{21} = -\epsilon^{12} = 1 \tag{4.42}$$

and therefore

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma \tag{4.43}$$

The OPEs between the fields on a single strand are

$$X_r^{A\dot{A}}(z) X_s^{B\dot{B}}(w) \sim \frac{\epsilon^{\dot{A}\dot{B}}\epsilon^{AB}}{(z-w)^2}\delta_{rs} \tag{4.44}$$

$$\psi_r^{\alpha\dot{A}}(z) \psi_s^{\beta\dot{B}}(w) \sim \frac{\epsilon^{\alpha\beta}\epsilon^{AB}}{(z-w)^2}\delta_{rs} \tag{4.45}$$

Now, we can express the generators of the theory in terms of free fields as

$$T(z) = \frac{1}{2}\sum_{r=1}^{N} \epsilon_{\dot{A}\dot{B}}\epsilon_{AB}\partial X_r^{A\dot{A}}\partial X_r^{B\dot{B}} + \frac{1}{2}\sum_{r=1}^{N}\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}\psi_r^{\alpha\dot{A}}\psi_r^{\beta\dot{B}} \tag{4.46}$$

$$G^{\alpha A}(z) = \sum_{r=1}^{N} \psi_r^{\alpha\dot{A}}\partial X_r^{\dot{B}A}\epsilon_{\dot{A}\dot{B}} \tag{4.47}$$

$$J^a(z) = \frac{1}{4}\sum_{r=1}^{N} \epsilon_{\dot{A}\dot{B}}\psi_r^{\alpha\dot{A}}\epsilon_{\alpha\beta}(\sigma^{*a})^\beta_\gamma\psi_r^{\gamma\dot{B}} \tag{4.48}$$

where σ^a are the Pauli matrices. From these expansions we can figure out the OPEs of the currents.

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{4.49}$$

$$G^{\alpha A}(z)G^{\beta B}(w) \sim -\frac{c}{3}\frac{\epsilon^{\alpha\beta}\epsilon^{AB}}{(z-w)^3} + \epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^\alpha_\gamma\big[\frac{2J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}\big] - \epsilon^{\alpha\beta}\epsilon^{AB}\frac{T(w)}{z-w} \tag{4.50}$$

$$J^a(z)J^b(w) \sim \frac{c}{12}\frac{\delta^{ab}}{(z-w)^2} + i\epsilon^{ab}_c\frac{J^c(w)}{z-w} \tag{4.51}$$

$$T(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \tag{4.52}$$

$$T(z)G^{\alpha A}(w) \sim \frac{3}{2}\frac{G^{\alpha A}(w)}{(z-w)^2} + \frac{\partial G^{\alpha A}(w)}{z-w} \tag{4.53}$$

$$J^a(z)G^{\alpha A}(w) \sim \frac{1}{2}(\sigma^{*a})^\alpha_\beta\frac{G^{\beta A}(w)}{z-w} \tag{4.54}$$

Finally, after labeling the modes of the currents T, $G^{\alpha A}$ and J^a with L_n, $G^{\alpha A}_n$ and J^a_n, we can find the commutators that generate the infinite dimension affine algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \tag{4.55}$$

$$[J^a_m, J^b_n] = \frac{c}{12}m\delta^{ab}_{m+n,0} + i\epsilon^{abc}J^c_{m+n} \tag{4.56}$$

$$\{G^{\alpha A}_m, G^{\beta B}_n\} = -\frac{c}{6}\big(m^2 - \frac{1}{4}\big)\epsilon^{\alpha\beta}\epsilon^{AB}\delta_{m+n,0} + (m-n)\epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^\alpha_\gamma J^a_{m+n} - \epsilon^{AB}\epsilon^{\alpha\beta}L_{m+n} \tag{4.57}$$

$$[J^a_m, G^{\alpha A}_n] = \frac{1}{2}(\sigma^{*a})^\alpha_\beta G^{\beta A}_{m+n} \tag{4.58}$$

$$[L_m, J^a_n] = -nJ^a_{m+n} \tag{4.59}$$

$$[L_m, G^{\alpha A}_n] = -\big(\frac{m}{2} - n\big)G^{\alpha A}_{m+n} \tag{4.60}$$

Equation (4.55) is familiar: it is the Virasoro algebra that we have already introduced back in equation (2.48). The subalgebra of the affine algebra spanned by the set $\{L_0, L_{\pm 1}, J^a_0, G^{\alpha A}_{\pm\frac{1}{2}}\}$ is the global part of the algebra, which does not contain the central charge. This is fairly easy to check, since we just have to substitute L_m with $L_{0,\pm 1}$ in (4.55), J^a_m with J^a_0 in (4.56) and $G^{\alpha A}_m$ with $G^{\alpha A}_{\pm\frac{1}{2}}$ in (4.57).

4.3.2 The spectrum in the untwisted sector

In this work we are mainly interested in the untwisted sector of the theory, which means that, as said before, we deal with strands that are singly wound along S_1. By the way, it is possible to define twist operators [30] [39] that bring to the twisted sector.

We can construct the irreducible representations of an affine algebra by selecting some highest weight state $|\psi\rangle$, called a primary, which is annihilated by all the positive modes of the generators, and acting on it with the negative modes, thus building a set of descendant states. There are different types of primary fields, depending on the generator we act upon them with, such as:

$$\begin{aligned} &L_n\,|\psi\rangle = 0 \; n>0 && \text{Virasoro primary} \\ &J^a_n\,|\psi\rangle = 0 \; n>0 && \text{Affine primary} \\ &G^{+A}_{-\frac{1}{2}}\,|\psi\rangle = 0 && \text{Chiral primary} \end{aligned} \tag{4.61}$$

Generally, the states are labelled by eigenvalues of the Cartan subalgebra spanned by $\{L_0, J^3_0\}$. This allows us to label the states in the left sector[7] with their spin[8] m and conformal dimension h. After choosing a basis, we identify the primaries and then generate the descendands, until we fill the spectrum.
In a way similar to superstring theories, we can impose different boundary conditions to the fermions, which generate two different sectors of the D1-D5 CFT. The conditions are

$$\begin{aligned} &\psi^{\alpha\dot{A}}_r(e^{2\pi i}z) = -\psi^{\alpha\dot{A}}_r(z) && \text{(Ramond)} \\ &\psi^{\alpha\dot{A}}_r(e^{2\pi i}z) = \psi^{\alpha\dot{A}}_r(z) && \text{(Neveu-Schwarz)} \end{aligned} \tag{4.62}$$

We begin analyzing the two sectors in the untwisted theory. In the NS sector, the boundary conditions for bosons and fermions read

$$\begin{aligned} \partial X^{A\dot{A}}_r(e^{2\pi i}z) &= \partial X^{A\dot{A}}_r(z) \\ \psi^{\alpha\dot{A}}_r(e^{2\pi i}z) &= \psi^{\alpha\dot{A}}_r(z) \end{aligned} \tag{4.63}$$

and the mode expansions are

$$\begin{aligned} \partial X^{A\dot{A}}_r(z) &= \sum_{n=-\infty}^{+\infty} \alpha^{A\dot{A}}_{r,n} z^{-n-1} \\ \psi^{\alpha\dot{A}}_r(z) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{\alpha\dot{A}}_{r,n} z^{-n-\frac{1}{2}} \end{aligned} \tag{4.64}$$

We notice that, in this sector, bosons have integer mode expansions, while fermions have semi-integer mode expansions. We have a primary field corresponding to the identity operator, which is the vacuum state characterized by $(h,m) = (\bar{h},\bar{m}) = (0,0)$. In each copy we can define a bosonic and a fermionic vacuum. Each of them is the tensor product of a vacuum from the holomorphic sector and the antiholomorphic sector. The bosonic vacuum on one copy is

$$\alpha^{A\dot{A}}_{r,n}\,|0\rangle_r = 0, \qquad n\geq 0,\; A,\dot{A}=1,2 \tag{4.65}$$

And the fermionic vacuum is defined by

$$\psi^{\alpha\dot{A}}_{r,n}\,|0\rangle_r = 0, \qquad n\geq 0,\; A,\dot{A}=1,2,\; \alpha,\dot{\alpha}=+,- \tag{4.66}$$

Let us now turn our attention to chiral primaries. These states are the analog of the highest weight state for $SU(2)$ [40]. If we make use of the anticommutator (4.57) we can derive a constraint for the states of the theory. [30]

$$h \geq |m| \tag{4.67}$$

[7] One also has the right-moving quantities $\bar{m}$ and $\bar{h}$

[8] Previously, the spin S denoted the spin in the 2D sense. The spin here has nothing to do with that spin.

Chiral primaries are annihilated by definition by $G^{+A}_{-\frac{1}{2}}$. This implies that they are annihilated also by J^+_0 (see for example (4.57)). Therefore, they are also SU(2) highest weight states, and we have

$$h = m \qquad \text{for chiral primaries} \tag{4.68}$$

Supergravity particles can be identified as the global subalgebra descendants of chiral primaries, which are chiral primary operators which are acted upon only by $L_{-1}, J^-_0, G^{-A}_{-\frac{1}{2}}$.
In the untwisted sector there are four chiral primaries

$$|0\rangle_{NS}\,, \quad \psi^{+\dot{A}}_{-\frac{1}{2}}|0\rangle_{NS}\,, \quad \psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}|0\rangle_{NS} \tag{4.69}$$

The first with $h = m = 0$, the second ones with $h = m = \frac{1}{2}$ and the last one with $h = m = 1$.
Why are chiral primaries so important? In the moduli space of the theory we have a really important point, which is the orbifold point: the one we are working in. When we want to describe the system from a gravitational point of view, we must move away from the orbifold point and reach the so-called gravity point. When doing so, the majority of the supersymmetric states cease to be supersymmetric. Moreover, bad things can happen to h. In general, h is a function of the moduli of the theory. When leaving the orbifold point, h can change or even become infinity! That happens because $\alpha' \to 0$ at the gravity point, and $M \propto \frac{1}{\alpha'}$. Chiral primaries are also called protected states, because this does not happen. In fact, m is quantized, so it cannot depend on the moduli of the theory, and $h = m$, therefore, h cannot depend on the moduli either. The descendant states are also protected because the operators we use to define them are symmetries, and exist at every point of the moduli space.
Let us now turn to the Ramond sector. The bosons satisfy the same boundary conditions of the NS sector (4.63), whereas the boundary conditions for the fermions are

$$\psi^{\alpha\dot{A}}_r(e^{2\pi i}z) = -\psi^{\alpha\dot{A}}_r(z) \tag{4.70}$$

bosons satisfy the same expansions (4.64). Fermions satisfy

$$\psi^{\alpha\dot{A}}_r(z) = \sum_{n=-\infty}^{+\infty} \psi^{\alpha\dot{A}}_{r,n} z^{-n-\frac{1}{2}} \tag{4.71}$$

Now, the bosonic vacuum is defined in the same way as the NS sector. There is a difference in the fermionic vacuum, which is actually made of sixteen degenerate states, coming from the nontrivial action of the zero modes of the fermions $\psi^{\alpha A}_{r,0}$ on the ground state[9]. The states in the left sector all have $h = \bar{h} = \frac{1}{4}$ and are the following:

$$|+\rangle_r\,, \quad \psi^{-\dot{A}}_{r,0}|+\rangle_r\,, \quad \psi^{-1}_{r,0}\psi^{-2}_{r,0}|+\rangle_r \tag{4.72}$$

Notice that there are also four states in the right sector analogous to these. Putting them together we make 4×4=16 states. Among these states, there is one which is especially important to us, which is the fermionic highest weight vacuum state with $\alpha = +$, namely

$$|+\rangle_r \qquad (h.m) = (\frac{1}{4}, \frac{1}{2}) \tag{4.73}$$

[9] These resembles the R ground states we discussed in chapter 2.

defined by the condition

$$\psi^{+\dot{A}}_{r,0} \left|+\right\rangle_r = 0, \qquad \forall \dot{A} \tag{4.74}$$

In the complete orbifold theory, the R vacuum is obtained by tensoring (4.73) repeatedly.

$$\bigotimes_{r=1}^{N_1N_5} \left|+\right\rangle_r = \left|+\right\rangle^{N_1N_5}, \qquad\qquad (h,m) = (\frac{N_1N_5}{4}, \frac{N_1N_5}{2}) \tag{4.75}$$

4.3.3 Spectral flow

Spectral Flow is a map between states in the R and NS sector of the CFT. In the left sector, the weight and spin change as

$$h_R = h_{NS} + \alpha m_{NS} + \frac{c\alpha^2}{24}, \qquad m_R = m_{NS} + \frac{c\alpha}{12} \tag{4.76}$$

where α is a integer parameter. For the right sector we can apply an independent transformation with parameter $\bar{\alpha}$. It is important to note that, in the untwisted sector, spectral flow with odd α exchanges NS and R boundary conditions of the left fermions, whereas they are preserved by spectral flow with odd α. We can therefore make use of the spectral flow to switch from the R sector to the NS sector and viceversa. In particular, spectral flow with $\alpha = -1$ maps chiral primaries into R ground states. For example, the NS vacuum $\left|0\right\rangle_{NS}$ is mapped to the maximally spinning R vacuum (4.73):

$$\left|0\right\rangle_{NS} \to \left|+\right\rangle_r \tag{4.77}$$

Spectral flow can also be viewed from the gravity side, thanks to the AdS/CFT correspondence. Strictly speaking, black hole microstates are dual to states in the R sector, because only the geometries dual to these states can be asymptotically flat. Since we work in the near horizon limit, where the geometries are asymptotically $\mathrm{AdS}_3 \times S_3 \times T_4$, we can freely switch between the R and the NS sector. However, if we wish to re-extend the geometries at infinity, we must switch to the R sector.

What we outlined so far should be enough to understand and follow the next chapter. For a more exhaustive treatment see [36], [40], [41], [42].

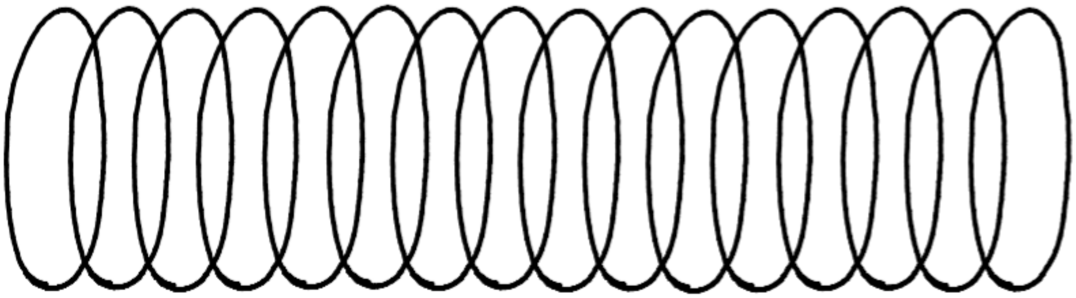

Figure 4.4: A visual representation of the strands. Here we have a system of $N_1N_5 = 16$ strands wrapped around a circle once. In terms of equation (4.37) the only nonzero terms correspond to $i = 1$ and we have $w_1 = 1$ and $m_1 = 16 = N_1N_5$.

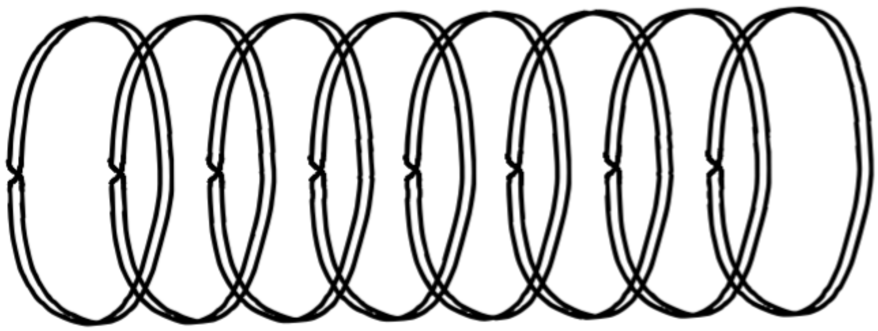

Figure 4.5: A visual representation of the strands. Here we have a system of $N_1N_5 = 16$ strands wrapped around a circle, with winding number 2. In terms of equation (4.37) the only nonzero terms correspond to $i = 2$ and we have $w_2 = 2$ and $m_2 = 8 = \frac{N_1N_5}{w_2} = \frac{N_1N_5}{2}$.

Chapter 5

A class of microstates

In this chapter reside all the original calculations in this work. We construct a new class of 3-charge solutions that correspond to microstates, by applying large diffeomorphisms to known 2-charge solutions. We start from a system of N_1N_5 singly wound strands in the state (4.73) and excite them with some of the operators we introduced in the previous chapter. First, we apply Virasoro operators on the strands and see that, in the regime when $N_P \gg N$, the generated entropy scales like the Bekenstein entropy. We show that the correct entropy is obtained in every regime only if we consider strands with winding $w \approx N$. After that, we begin to construct the corresponding Supergravity solution, describing microstates. We construct a class of microstates that captures only a small part of the total entropy: in fact we start from the state (4.73) and apply only the operators $J^{\pm}_{\mp 1}$ on it. This is a very preliminary calculation that will hopefully lead to the development of an entirely new class of microstates, obtained by applying different combinations of the $J^{\pm}_{n}$ operators. We will introduce a continuous parameter χ that will assume values between 0 and $\frac{\pi}{2}$. The solutions corresponding to the extremal values of χ are known. What is new are the solutions interpolating between the two values, for generic χ. After finding these solutions we repeat the same calculations in order to find microstates corresponding to the operators $J^{\pm}_{\mp n}$, with $n > 1$, generalizing the previous ones. At the end of the chapter we compute the relevant charges that correspond to the states obtained by acting with $J^{\pm}_{\mp 1}$ and briefly comment on them. Every single step, even the ones that are algebrically trivial, will be described in detail.

5.1 Computing the entropy

Our task is to find states that reproduce at least the trend of the Bekenstein entropy (3.30). Such states capture a finite fraction of the entropy, and therefore are typical states of the statistical ensemble, in the limit of macroscopic charges. Let us start by introducing the system of strings we will work with. We saw in the previous chapter that the D1-D5 CFT can be visualized as a collection of N_1N_5 strands wrapped around the circle S_1. We consider $N := N_1N_5$ strands singly wound around the circle, so we have $w = 1$ (see figure 4.4). We put all these strands in the Ramond ground state (4.73). We slightly change notation and label the state with an additional index w, which for now is 1, indicating that the strands are singly wound. Concretely, we consider N strands, each one in the state

$$|0\rangle_{NS,r,1} \tag{5.1}$$

so that the total system is in the state

$$\frac{1}{N!}\sum_{\sigma\in S(N)}\bigotimes_{r=1}^{N}|0\rangle_{NS,\sigma(r),1}=|0\rangle_{NS,1}^{N} \tag{5.2}$$

Where σ is a permutation living in the symmetric group $S(N)$. The symmetrization is required because we are working at the orbifold point (4.36). This state describes a 2-charge solution on the Supergravity side. We must add the third charge, which is momentum. This is accomplished by acting upon the strands with right Virasoro operators L_{-n}. Note that we must act with right or left operators only, because otherwise we would obtain a solution that is not supersymmetric. We have

$$L_{-n}\,|{+}{+}\rangle_{r,1}=0 \qquad n\leq 1 \tag{5.3}$$

so the Virasoro operators excite the strands only for $n>1$. By acting with these operators, we can build a huge number of states. The most general possible state can be written as

$$\frac{1}{N!}\sum_{\sigma\in S(N)}\bigotimes_{r=1}^{N}\prod_{n=2}^{\infty}(L_{-n})^{N_{n,\sigma(r)}}\,|0\rangle_{NS,\sigma(r),1} \tag{5.4}$$

where the integers $N_{n,l}$ indicate how many times some operator L_{-n} has been applied to the l-th strand. There are two additional constraints that we have to consider: the first one comes from the fact that the winding numbers of the strands must add up to N. This is exactly condition (4.37) that we rewrite here for convenience

$$\sum_i m_i w_i = N \tag{5.5}$$

Notice that this constraint is trivially satisfied by states of the form (5.4) because all strands have winding number 1. Moreover, every Virasoro operator L_{-n} contributes n unities to the total momentum, therefore we have the condition

$$\sum_{r=1}^{N}\sum_{n=2}^{\infty} nN_{n,r}=N_P \tag{5.6}$$

It is helpful to see how this last formula works in two examples, in order to fix the ideas

- **Example 1:**

Let us consider the following state:

$$(L_{-7})^3(L_{-3})^6\,|{+}{+}\rangle_{1,1}\otimes(L_{-2})^5(L_{-8})^6(L_{-4})^4\,|{+}{+}\rangle_{2,1} \tag{5.7}$$

In this state we have $N=2$ and the nonzero occupation numbers are $N_{7,1}=3$, $N_{3,1}=6$, $N_{2,2}=5$, $N_{8,2}=6$, $N_{4,2}=4$. N_P is given by

$$N_P = 7\times 3+3\times 6+2\times 5+8\times 6+4\times 4=218$$

Notice that this state also trivially satisfies the constraint (5.5)

- **Example 2:**

Let us say we have $N = 3$ and $N_P = 4$. How many states are there possessing these charges? Actually, only three. In fact, because of the symmetrization (4.36), many states are actually the same state. The first state is

$$\begin{aligned}
&(L_{-4})\,|{+}{+}\rangle_{1,1} \otimes |{+}{+}\rangle_{2,1} \otimes |{+}{+}\rangle_{3,1} \;\text{ or}\\
&|{+}{+}\rangle_{1,1} \otimes (L_{-4})\,|{+}{+}\rangle_{2,1} \otimes |{+}{+}\rangle_{3,1} \;\text{ or}\\
&|{+}{+}\rangle_{1,1} \otimes |{+}{+}\rangle_{2,1} \otimes (L_{-4})\,|{+}{+}\rangle_{3,1}
\end{aligned}$$

the second one is

$$\begin{aligned}
&(L_{-2})^2\,|{+}{+}\rangle_{1,1} \otimes |{+}{+}\rangle_{2,1} \otimes |{+}{+}\rangle_{3,1} \;\text{ or}\\
&|{+}{+}\rangle_{1,1} \otimes (L_{-2})^2\,|{+}{+}\rangle_{2,1} \otimes |{+}{+}\rangle_{3,1} \;\text{ or}\\
&|{+}{+}\rangle_{1,1} \otimes |{+}{+}\rangle_{2,1}\,(L_{-2})^2 \otimes |{+}{+}\rangle_{3,1}
\end{aligned}$$

and the last one is

$$\begin{aligned}
&(L_{-2})\,|{+}{+}\rangle_{1,1} \otimes (L_{-2})\,|{+}{+}\rangle_{2,1} \otimes |{+}{+}\rangle_{3,1} \;\text{ or}\\
&(L_{-2})\,|{+}{+}\rangle_{1,1} \otimes |{+}{+}\rangle_{2,1} \otimes (L_{-2})\,|{+}{+}\rangle_{3,1} \;\text{ or}\\
&|{+}{+}\rangle_{1,1} \otimes (L_{-2})\,|{+}{+}\rangle_{2,1} \otimes (L_{-2})\,|{+}{+}\rangle_{3,1}
\end{aligned}$$

The notation is now clear. Now, there is an important point that needs to be discussed, which are the different regimes corresponding to different values of the ratio $\frac{N_P}{N}$.

5.1.1 Regimes

Let us go back for a moment to the computation of the 3-charge black hole entropy we faced in section 3.2.1. The horizon is made up by a 3-sphere, a circle and a torus. The latter is way smaller than the other objects ($V_{T_4} \sim l_s^4$), therefore we can forget about it. Let us then compute the ratio of the radii of the sphere and the circle. We go back to equation (3.26). We have

$$\begin{aligned}
2\pi R_{S_1} &= V_{S_1} = (2\pi R) Q_1^{-\frac{3}{8}} Q_5^{-\frac{1}{8}} Q_P^{\frac{1}{2}} && \Rightarrow && R_{S_1} = R Q_1^{-\frac{3}{8}} Q_5^{-\frac{1}{8}} Q_P^{\frac{1}{2}}\\
\frac{4}{3}\pi R_{S_3}^3 &= V_{S_3} = Q_1^{\frac{3}{8}} Q_5^{\frac{9}{8}} 2\pi^2 && \Rightarrow && R_{S_3} = \left(\frac{3\pi}{2}\right)^{\frac{1}{3}} Q_1^{\frac{1}{8}} Q_5^{\frac{3}{8}}
\end{aligned}$$

Which gives

$$\left(\frac{R_{S_1}}{R_{S_3}}\right)^2 = \left(\frac{2}{3\pi}\right)^{\frac{2}{3}} R^2 \frac{Q_P}{Q_1 Q_5}$$

Finally, substituting with the integer charges (3.28) we get

$$\left(\frac{R_{S_1}}{R_{S_3}}\right)^2 = \left(\frac{2}{3\pi}\right)^{\frac{2}{3}} \frac{N_p}{N} \tag{5.8}$$

Equation (5.8) can be interpreted as follows:

- When $N \gg N_P$ the S_3 is much bigger in size than the S_1. The full horizon resembles a spherical horizon, since the S_1 is somewhat hidden. Therefore, we call this the black hole regime.
- When $N_P \gg N$ the S_1 plays a prominent role and the horizon resembles a long and thin ring: this is called the black string regime.
- Finally, when $N \approx N_P$ we find ourselves in the intermediate regime.

Going back to our D1-D5-CFT, we want to take the macroscopic limit, namely $N, N_P \to \infty$. Our goal is to find some configuration that produces the correct Bekenstein entropy (3.30) in the black hole regime. We anticipate a result: the state (5.4) has winding 1. For this winding we find the correct entropy only in the black string regime. Instead, if we consider states with $w \sim N$, we find the correct entropy in every regime. We will justify all these statements in the following two sections.

5.1.2 Entropy for $w = 1$

Let us calculate the entropy arising from the state (5.4). For this computation, we follow two similar calculations that have been performed in [43] and [44]. The first thing we must do is writing down some partition function. Then we compute its logarithm, and finally use it to compute the entropy in the thermodynamic limit, which is given by a well known formula.
We introduce the fugacities

$$p := e^{-\alpha}, \qquad q := e^{-\beta}, \qquad \alpha, \beta > 0 \tag{5.9}$$

The grand-canonical partition function is defined as

$$Z = \mathrm{tr}\big(p^N q^{N_P}\big) \tag{5.10}$$

where the trace is computed over the $\frac{1}{8}-$BPS states of the form (5.4). We can expand the partition function as

$$Z(p, q) = \sum_N \sum_{N_P} C(N, N_P) p^N q^{N_P} \tag{5.11}$$

Our goal is to find the coefficients $C(N, N_P)$ from the knowledge of $Z(p, q)$, which we now find. We could do that by computing (5.10) directly. However, there is a faster route: a single strand where L_{-k} is applied n_k times contributes

$$\prod_{n_k=0}^{\infty} pq^{\sum_{k=2}^{\infty} kn_k} \tag{5.12}$$

to Z. Summing over all possible numbers of such strands produces a geometric series whose resummation gives

$$Z(p, q) = \prod_{\{n_i\}=0}^{\infty} \frac{1}{1 - pq^{\sum_{h=2}^{\infty} hn_h}}, \qquad i \geq 2 \tag{5.13}$$

Or, in a more intelligible way,

$$Z(p, q) = \prod_{n_2=0}^{\infty} \prod_{n_3=0}^{\infty} \prod_{n_4=0}^{\infty} \prod_{n_5=0}^{\infty} \dots \left(\frac{1}{1 - pq^{2n_2+3n_3+4n_4+5n_5+\dots}} \right) \tag{5.14}$$

This partition function is a generalization of the one written by Mayerson an Shigemori in [44] in equation 4.10: in fact a single Virasoro operator (L_{-k}) produces an infinite product in the partition function. Since we can potentially apply infinite Virasoro operators ($L_{-2}, L_{-3}, ...$), our partition function will contain an infinite product for each operator, which means a countable infinity of infinite products. We now take the logarithm: this has the effect of turning the products into sums

$$\begin{aligned}\log\left(Z(p,q)\right) &= \log\left[\prod_{n_2=0}^{\infty}\prod_{n_3=0}^{\infty}\prod_{n_4=0}^{\infty}\prod_{n_5=0}^{\infty}\cdots\left(\frac{1}{1-pq^{2n_2+3n_3+4n_4+5n_5+\ldots}}\right)\right] = \\ &= \sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\cdots\left[\log\left(\frac{1}{1-pq^{2n_2+3n_3+4n_4+5n_5+\ldots}}\right)\right] = \\ &= \sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\cdots\left[-\log\left(1-pq^{2n_2+3n_3+4n_4+5n_5+\ldots}\right)\right]\end{aligned} \tag{5.15}$$

Now we wish to write the log as a power series

$$\log\left(1-x\right) = -\sum_{r=1}^{\infty}\frac{x^r}{r}, \qquad x\in[-1,1) \tag{5.16}$$

Since $p, q < 1$ the expansion is well defined, because the argument of the log falls in the right interval. After expanding in power series, we exchange the order of the sums, bringing the r sum out. Then, we perform some algebra to find:

$$\begin{aligned}\log\left(Z(p,q)\right) &= \sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\cdots\left[-\log\left(1-pq^{2n_2+3n_3+4n_4+5n_5+\ldots}\right)\right] = \\ &= \sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\cdots\sum_{r=1}^{\infty}\frac{1}{r}\left(pq^{2n_2+3n_3+4n_4+5n_5+\ldots}\right)^r = \\ &= \sum_{r=1}^{\infty}\frac{p^r}{r}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\cdots\left(q^{2n_2+3n_3+4n_4+5n_5+\ldots}\right)^r = \\ &= \sum_{r=1}^{\infty}\frac{p^r}{r}\sum_{n_2=0}^{\infty}\left[(q^{2r})^{n_2})\right]\sum_{n_3=0}^{\infty}\left[(q^{3r})^{n_3})\right]\sum_{n_4=0}^{\infty}\left[(q^{4r})^{n_4})\right]\sum_{n_5=0}^{\infty}\left[(q^{5r})^{n_5})\right]\ldots\end{aligned} \tag{5.17}$$

Now we evaluate the sums, which are all geometric series, in order to get

$$\begin{aligned}\log\left(Z(p,q)\right) &= \sum_{r=1}^{\infty}\frac{p^r}{r}\frac{1}{1-q^{2r}}\frac{1}{1-q^{3r}}\frac{1}{1-q^{4r}}\frac{1}{1-q^{5r}}\ldots = \\ &= \sum_{r=1}^{\infty}\left[\frac{p^r}{r}\prod_{i=2}^{\infty}\frac{1}{1-q^{ir}}\right]\end{aligned} \tag{5.18}$$

So, in conclusion we have

$$\log\left(Z(p,q)\right) = \sum_{r=1}^{\infty}\left[\frac{p^r}{r}\prod_{i=2}^{\infty}\frac{1}{1-q^{ir}}\right] \tag{5.19}$$

It is now time to introduce some approximations. We consider the black string regime $N_P \gg N$. In terms of α, β this means, as we will confirm later, $\alpha > 1$, $\beta \ll 1$ with $0 < \beta \ll \alpha$. First, we rewrite equation (5.19) as

$$\log\left(Z(p,q)\right) = \sum_{r=1}^{\infty}\left[\frac{p^r}{r}\prod_{i=2}^{\infty}\frac{1}{1-q^{ir}}\right] = \sum_{r=1}^{\infty}\left[\frac{p^r}{r}(1-q^r)\prod_{i=1}^{\infty}\frac{1}{1-q^{ir}}\right] \tag{5.20}$$

Now, by Taylor expanding, we have $(1-q^r)\approx\beta r$, which gives

$$\log\left(Z(p,q)\right) \approx \beta\sum_{r=1}^{\infty}\left[rp^r\prod_{i=1}^{\infty}\frac{1}{1-q^{ir}}\right] \tag{5.21}$$

Now we have to handle the product, that we call $\tilde{Z}$. We start by taking the log:

$$\log\tilde{Z} = \log\left(\prod_{i=1}^{\infty}\frac{1}{1-q^{ir}}\right) = -\sum_{i=1}^{\infty}\log\left(1-q^{ir}\right) \tag{5.22}$$

Now, since $\beta\ll 1$, the sum at the right-hand side can be approximated with an integral

$$-\sum_{i=1}^{\infty}\log\left(1-q^{ir}\right) \approx -\int_1^{\infty}\log\left(1-e^{-\beta xr}\right)dx \tag{5.23}$$

We now change variables as $y\coloneqq\beta x$. The new differential is $dy=\beta dx$, so we have

$$-\int_1^{\infty}\log\left(1-e^{-\beta xr}\right)dx = -\frac{1}{\beta}\int_{\beta}^{\infty}\log\left(1-e^{-ry}\right)dy \tag{5.24}$$

Now, since $\beta\ll 1$, we can take the lower integration limit to be 0. Plus, we re-use the power series expansion (5.16) with $x=e^{-ry}$ to write

$$-\frac{1}{\beta}\int_{\beta}^{\infty}\log\left(1-e^{-ry}\right)dy = \frac{1}{\beta}\int_0^{\infty}\sum_{h=1}^{\infty}\frac{e^{-rhy}}{h}dy \tag{5.25}$$

We can check that we are allowed to swap the integral and the series. Then, we evaluate the integral and get

$$\frac{1}{\beta}\int_0^{\infty}\sum_{h=1}^{\infty}\frac{e^{-rhy}}{h}dy = \frac{1}{\beta}\sum_{h=1}^{\infty}\frac{1}{h}\int_0^{\infty}e^{-rhy}dy = \frac{1}{\beta}\sum_{h=1}^{\infty}\frac{1}{h}\frac{1}{rh} \tag{5.26}$$

The series is $\zeta(-2)$, which is equal to $\frac{\pi^2}{6}$. In the end we are left with

$$\log\tilde{Z} \approx \frac{1}{\beta r}\sum_{h=1}^{\infty}\frac{1}{h^2} = \frac{\pi^2}{6}\frac{1}{\beta r} \tag{5.27}$$

Which means

$$\tilde{Z}\approx e^{\frac{\pi^2}{6}\frac{1}{\beta r}} \tag{5.28}$$

We substitute this all the way back in (5.21)

$$\log\left(Z(p,q)\right) \approx \beta\sum_{r=1}^{\infty}\left[rp^r e^{\frac{\pi^2}{6}\frac{1}{\beta r}}\right] = \beta\sum_{r=1}^{\infty}re^{-\alpha r+\frac{\pi^2}{6}\frac{1}{\beta r}} \tag{5.29}$$

Now, for every $\alpha > 0$ we notice that the first term in the sum is much bigger than the others, so we only keep that term to find

$$\log\left(Z(p,q)\right) \approx \beta p e^{\frac{\pi^2}{6\beta}} \tag{5.30}$$

In the large charge limit ($N, N_P \to \infty$) we can use the thermodynamic limit, meaning that we can apply the formulas

$$\begin{aligned} N &= -\partial_\alpha \log Z \\ N_P &= -\partial_\beta \log Z \end{aligned} \tag{5.31}$$

Finally, the entropy is given by

$$S(N, N_P) = \log\left[C(N, N_P)\right] = \log Z + \alpha N + \beta N_P \tag{5.32}$$

From now on the computation is fairly easy, because we just need to compute partial derivatives of the function (5.30) and finally substitute them in (5.32). We have

$$N = -\partial_\alpha\left[\beta p e^{\frac{\pi^2}{6\beta}}\right] = -\frac{dp}{d\alpha}\partial_p\left[\beta p e^{\frac{\pi^2}{6\beta}}\right] = -(-p)\left[\beta e^{\frac{\pi^2}{6\beta}}\right] = \beta p e^{\frac{\pi^2}{6\beta}} \tag{5.33}$$

$$N_P = -p e^{\frac{\pi^2}{6\beta}}\left[1 - \frac{\pi^2}{6\beta}\right] \approx \frac{p}{\beta}\frac{\pi^2}{6} e^{\frac{\pi^2}{6\beta}} \tag{5.34}$$

Where we dropped the 1 in equation (5.34) since the other term is much bigger in the limit $\beta \ll 1$. Now we get p from equation (5.34) and plug it in equation (5.33) to obtain β. We find

$$p = \beta N_P \frac{6}{\pi^2} e^{-\frac{\pi^2}{6\beta}} \tag{5.35}$$

$$\beta = \sqrt{\frac{\pi^2}{6}\frac{N}{N_P}} \tag{5.36}$$

Substituting β back into equation (5.35) we obtain

$$p = \sqrt{\frac{\pi^2}{6}\frac{N}{N_P}}\frac{6}{\pi^2} N_P e^{-\frac{\pi^2}{6\beta}} \tag{5.37}$$

$$\alpha = -\log p = -\log\sqrt{\frac{6}{\pi^2}} - \log\sqrt{N N_P} + \sqrt{\frac{\pi^2}{6}\frac{N_P}{N}} \tag{5.38}$$

Finally, we can compute the entropy by substituting (5.38) and (5.36) into formula (5.32), this yields

$$S(N, N_P) = N + 2\pi\sqrt{\frac{N N_P}{6}} - N\log\sqrt{\frac{6}{\pi^2}} - N\log\sqrt{N N_P} \tag{5.39}$$

In the limit $N_P \gg N$ we can drop all terms but the second one, since they are subleading in the macroscopic limit $N, N_P \to \infty$. Therefore, the entropy in the black string regime is

$$S = 2\pi\sqrt{\frac{N N_P}{6}} = 2\pi\sqrt{\frac{N_1 N_5 N_P}{6}} \tag{5.40}$$

This is almost exactly the Bekenstein entropy we wanted. The prefactors are different by a factor $\sqrt{6}$. This is not a surprise, since we have considered only a subset of all possible states. As previously stated, we want to reproduce the correct scaling of the entropy in the large charge limit. This has been achieved, in the black string regime.
In the other regimes, these approximations cannot be applied anymore. Therefore, we have to try something else. However, one can notice that the entropy does not depend on N. This happens because when $N \gg N_P$ it does not matter how many strands we have, because the momentum can always be partitioned in the same number of ways, since the strands are indistinguishable. Let us make an example to clarify this. Let us say we have a situation with $N_P = 4$ and $N = 3$. As we saw earlier in an example, there are three possible states. What happens if we instead have $N = 4$? Nothing: we still have three possible states because the new strand makes no difference, since we cannot tell it apart from the others. What about $N = 100$? $N = 1000$? $N = 10^{100}$? Nothing changes. Therefore, the entropy cannot depend on N in this regime, and we cannot have the correct scaling $S \propto \sqrt{NN_P}$.

5.1.3 Entropy for $w > 1$

If we allow multiply wound strings into our configuration, the situation gets more complicated, since fractional Virasoro operators come into play. If a string is wound w times, then we can apply the operators $L_{-\frac{2}{w}}$, $L_{-\frac{3}{w}}$, $L_{-\frac{4}{w}}$ and so on. We could compute the entropy in a similar way as we did in the previous section by generalizing the constraint (5.6) in order to include multiply wound strings. Then we would obtain a more complicated partition function, generalizing (5.13), which leads to some heavy calculations. Fortunately, we can avoid this: due to the fact that the modes are now fractional, the calculations reduce to the previous ones with an effective momentum wN_P: when $w \sim N \to \infty$ and $N_P \to \infty$ we are always in the regime $wN_P \gg N$ and thus we can rely on our result in equation (5.40), since the condition $N_p \gg N$ is automatically satisfied. This argument allows us to avoid repeating any calculations. We now confirm this approximation by checking the extremal situation of a single strand with winding N, which is really simple to study. The state (5.4) is now different. We only have one string, so we can drop the tensor product, since it contains only the $r = 1$ term. Plus, keeping into account the fractionary modes, we have

$$\prod_{n=2}^{\infty} (L_{-\frac{n}{N}})^{N_{\frac{n}{N},1}} \left|{++}\right\rangle_{1,N} \tag{5.41}$$

After fixing the momentum N_P, we can apply the operators $L_{-\frac{2}{N}}$, $L_{-\frac{3}{N}}$, $L_{-\frac{4}{N}}$ and so on, until we reach $L_{max} = L_{-\frac{NN_P}{N}} = L_{-N_P}$, which adds exactly N_P momentum charges on the strand, which is the maximum value we can have. This single strand contains therefore NN_P fractionary modes.
We now recall the computation performed in chapter 3 when we calculated the entropy for the 2-charge solution. In particular, we focus on equation (3.36). In this situation we can write a modified version of it, featuring NN_P modes instead of N_1N_P.

$$\sum_{i=1}^{\infty} m_i k_i = NN_P \tag{5.42}$$

For winding $w = N$ the sum contains just one term, and we have $m = 1$ and $k = NN_P$. In the same way as we did previously, we compute the entropy by counting the partitions of the integer

NN_P. This immediately gives the correct result.

$$S = \log\big[Part(NN_P)\big] \approx 2\pi\sqrt{\frac{NN_P}{6}} = 2\pi\sqrt{\frac{N_1N_5N_P}{6}} \tag{5.43}$$

Again, we are off by a factor $\sqrt{6}$, which is not a big deal. The important fact that we must point out is that this computation is valid in all three regimes, in fact we only needed to assume that $NN_P \to \infty$ when applying the Hardy-Ramanujan formula in equation (5.43). In the $w = 1$ case we were also forced to assume $N_P \gg N$, which defines the black string regime. We therefore proved what we anticipated earlier: the correct entropy is obtained in all three regimes only for $w \sim N$.

5.2 General solution

We consider type IIB Supegravity compactified on $S_1 \times T_4$. We are looking for microstate solutions describing the D1-D5-P black hole, therefore we seek solutions with the same number of supersymmetries as preserved charges by the D1-D5-P system. Three-charge D1-D5-P microstates preserve $\frac{1}{8}$ supersymmetries. The general solution that possesses this property was found in [45] under the assumption that the geometry is invariant under rotations of the torus T_4. It is given by

$$ds^2_{10} = -\frac{2\alpha}{\sqrt{Z_1Z_2}}(dv+\beta)\big[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\big] + \sqrt{Z_1Z_2}ds^2_4 + \sqrt{\frac{Z_1}{Z_2}}d\hat{s}^2_4 \tag{5.44a}$$

$$e^{2\Phi} = \alpha\frac{Z_1}{Z_2} \tag{5.44b}$$

$$B_2 = -\frac{\alpha Z_4}{Z_1Z_2}(du+\omega)\wedge(dv+\beta) + a_4\wedge(dv+\beta) + \delta_2 \tag{5.44c}$$

$$C_0 = \frac{Z_4}{Z_1} \tag{5.44d}$$

$$C_2 = -\frac{\alpha}{Z_1}(du+\omega)\wedge(dv+\beta) + a_1\wedge(dv+\beta) + \gamma_2 \tag{5.44e}$$

$$C_4 = \frac{Z_4}{Z_2}\hat{vol}_4 - \frac{\alpha Z_4}{Z_1Z_2}\gamma_2\wedge(du+\omega)\wedge(dv+\beta) + x_3\wedge(dv+\beta) \tag{5.44f}$$

$$C_6 = \hat{vol}_4\wedge\big[-\frac{\alpha}{Z_2}(du+\omega)\wedge(dv+\beta) + a_2\wedge(dv+\beta) + \gamma_1\big] \tag{5.44g}$$

Where

$$\alpha = \frac{Z_1Z_2}{Z_1Z_2 - Z_4^2} \tag{5.45}$$

There is a lot going on in these formulas:

- $d\hat{s}^2_4$ is a flat metric on the compact torus T_4.
- $\hat{vol}_4$ is the volume form associated with $d\hat{s}^2_4$.
- ds^2_4 is a nontrivial euclidean metric along the four non-compact directions, which are diffeomorphic to $\mathbb{R}^4$.
- u, v are obtained by the coordinates along the time and the S_1 directions after moving to light-cone coordinates:

$$u = \frac{t-y}{\sqrt{2}} \qquad v = \frac{t+y}{\sqrt{2}} \tag{5.46}$$

Plus, we have many different p-forms defining the ansatz, which are

- The 0-forms on $\mathbb{R}^4$: Z_1, Z_2, Z_4[1] and $\mathcal{F}$.
- The 1-forms β, ω, a_1, a_4. If we call x^i the coordinates on the non-compact space, then we have $\beta = \beta_i dx_i$ and so on with the other forms.
- The 2-forms γ_2 and δ_2. Again, we can write $\gamma_2 = (\gamma_2)_{ij} dx_i \wedge dx_j$.
- A 3-form x_3 such that $x_3 = (x_3)_{ijk} dx_i \wedge dx_j \wedge dx_k$
- Finally, we can introduce a 1-form a_2 and a 2-form γ_1 that appear in C_6, which is a 6-form dual to C_2. We include C_6 in the full solution even though it obviously carries the same information as C_2, in order to make clear where a_2 and γ_1 appear.

All these objects, ds_4^2 included, depend in general on the coordinates x^i and v. If we know all of them, it means that we know the full solution. In order to preserve supersymmetry and satisfy the equations of motion, these objects must satisfy some constraints, which were found and discussed in detail in [45] and [46], that we report. We define the operator

$$\mathcal{D} := d - \beta \wedge \frac{d}{dv} \tag{5.47}$$

Plus, note that the external derivative d and the Hodge dual $\star_4$ must be performed over the non-compact space $\mathbb{R}^4$. Plus, the dot denotes a derivative with respect to v. There are two more objects that we can define, in order to compactify the notation:

$$\psi := \frac{1}{8} \epsilon^{ABC} (J_A)^{ij} (\dot{J}_B)_{ij} J_C \tag{5.48}$$

$$L := \dot{\omega} + \frac{\mathcal{F}}{2} \dot{\beta} - \frac{1}{2} \mathcal{D} \mathcal{F} \tag{5.49}$$

We now list the equations. Notice that the first two are the most complicated ones to check, since one has to find the 2-forms J_A.

- Equations for ds_4^2, β: we can find three 2-forms J_A such that

$$dJ_A = \frac{d}{dv}(\beta \wedge J_A), \qquad J_A = -\star_4 J_A, \qquad J_A \wedge J_B = -2\delta_{AB} vol_4 \tag{5.50}$$

- Equations for Z_1, a_2, γ_1:

$$\star_4 (\mathcal{D} Z_1 + \dot{\beta} Z_1) = \mathcal{D}\gamma_1 - a_2 \wedge \mathcal{D}\beta \tag{5.51}$$

$$\Theta_2 - Z_1 \psi = \star_4 (\Theta_2 - Z_1 \psi) \qquad \text{with } \Theta_2 := \mathcal{D} a_2 - \dot{\beta} \wedge a_2 + \dot{\gamma}_1 \tag{5.52}$$

- Equations for Z_2, a_1, γ_2:

$$\star_4 (\mathcal{D} Z_2 + \dot{\beta} Z_2) = \mathcal{D}\gamma_2 - a_1 \wedge \mathcal{D}\beta \tag{5.53}$$

$$\Theta_1 - Z_2 \psi = \star_4 (\Theta_1 - Z_2 \psi) \qquad \text{with } \Theta_1 := \mathcal{D} a_1 - \dot{\beta} \wedge a_1 + \dot{\gamma}_2 \tag{5.54}$$

[1] These objects are analogous to the Z_i characterizing the solutions in chapter 3.

- Equations for Z_4, a_4, δ_2:

$$\star_4(\mathcal{D}Z_4 + \dot{\beta}Z_4) = \mathcal{D}\delta_2 - a_4 \wedge \mathcal{D}\beta \tag{5.55}$$

$$\Theta_4 - Z_4\psi = \star_4(\Theta_4 - Z_4\psi) \qquad \text{with } \Theta_4 := \mathcal{D}a_4 - \dot{\beta} \wedge a_4 + \dot{\delta}_2 \tag{5.56}$$

- Equations for ω, $\mathcal{F}$:

$$\mathcal{D}\omega + \star_4\mathcal{D}\omega + \mathcal{F}\mathcal{D}\beta = Z_1 \star_4 \Theta_1 + Z_2\Theta_2 - Z_4(\Theta_4 + \star_4\Theta_4) \tag{5.57}$$

$$\begin{aligned} &\star_4 \mathcal{D} \star_4 L + 2\dot{\beta}_i L^i + \frac{1}{4}\frac{Z_1Z_2}{\alpha}\dot{g}^{ij}\dot{g}_{ij} - \frac{1}{2}\frac{d}{dv}\left[\frac{Z_1Z_2}{\alpha}g^{ij}\dot{g}_{ij}\right] - \dot{Z}_1\dot{Z}_2 - Z_1\ddot{Z}_2 - \ddot{Z}_1Z_2 + (\dot{Z}_4)^2 + \\ &+ 2Z_4\ddot{Z}_4 + \frac{1}{2}\star_4\Big[(\Theta_1 - Z_2\psi)\wedge(\Theta_2 - Z_1\psi) - (\Theta_4 - Z_4\psi)\wedge(\Theta_4 - Z_4\psi) + \\ &+ \frac{Z_1Z_2}{\alpha}\psi\wedge\psi - 2\psi\wedge\mathcal{D}\omega\Big] = 0 \end{aligned} \tag{5.58}$$

- Equation for x_3:

$$\mathcal{D}x_3 - \dot{\beta}\wedge x_3 - \Theta_4 \wedge \gamma_2 + a_1 \wedge (\mathcal{D}\delta_2 - a_4 \wedge \mathcal{D}\beta) = Z_2^2 \star_4 \frac{d}{dv}\left(\frac{Z_4}{Z_2}\right) \tag{5.59}$$

The forms $\gamma_1, \gamma_2, \delta_2, a_1, a_2, a_4$ are gauge dependent. However, the quantities $\Theta_1, \Theta_2, \Theta_4$ are not. It is then useful to write down the gauge-invariant form of equations (5.51) through (5.56)

- Equations for Z_1, Θ_1:

$$\mathcal{D}\star_4(\mathcal{D}Z_1 + \dot{\beta}Z_1) = -\Theta_1 \wedge \mathcal{D}\beta \tag{5.60}$$

$$\mathcal{D}\Theta_1 - \dot{\beta}\wedge\Theta_1 = \frac{d}{dv}\star_4(\mathcal{D}Z_1 + \dot{\beta}Z_1) \tag{5.61}$$

$$(1 - \star_4)\Theta_1 = 2Z_1\psi \tag{5.62}$$

- Equations for Z_2, Θ_2:

$$\mathcal{D}\star_4(\mathcal{D}Z_2 + \dot{\beta}Z_2) = -\Theta_2 \wedge \mathcal{D}\beta \tag{5.63}$$

$$\mathcal{D}\Theta_2 - \dot{\beta}\wedge\Theta_2 = \frac{d}{dv}\star_4(\mathcal{D}Z_2 + \dot{\beta}Z_2) \tag{5.64}$$

$$(1 - \star_4)\Theta_2 = 2Z_2\psi \tag{5.65}$$

- Equations for Z_4, Θ_4:

$$\mathcal{D}\star_4(\mathcal{D}Z_4 + \dot{\beta}Z_4) = -\Theta_4 \wedge \mathcal{D}\beta \tag{5.66}$$

$$\mathcal{D}\Theta_4 - \dot{\beta}\wedge\Theta_4 = \frac{d}{dv}\star_4(\mathcal{D}Z_4 + \dot{\beta}Z_4) \tag{5.67}$$

$$(1 - \star_4)\Theta_4 = 2Z_4\psi \tag{5.68}$$

5.3 Constructing the microstates

It is now time to construct the microstates corresponding to the states we analyzed. Actually, we will work with a system that is slightly different from the one just discussed. In fact, we apply J_n^a operators instead of Virasoro operators. The two systems produce the same entropy, but it is easier to write a partition function involving states such as (5.4) and (5.41), because the L_{-k} operators can be applied an indefinite number of times. The J^-_n operators can only be applied $n-1$ times on the NS vacuum, then we have

$$(J^-_n)^n \left|0\right\rangle_{NS,r,1} = 0 \tag{5.69}$$

This is why we computed the entropy produced by the L_{-k}. On the gravity side, however, it is easier to work with the J^a_n operators. We now introduce the states that we are using to build the microstates on the gravity side.
Again, we start from N singly wound strings. We put each string in the NS ground state $\left|0\right\rangle_{NS,1}$[2]. Therefore, we start from the state

$$\left|\psi_0\right\rangle := \frac{1}{N!}\sum_{\sigma\in S(N)}\bigotimes_{r=1}^{N}\left|0\right\rangle_{NS,\sigma(r),1} \tag{5.70}$$

Let us now apply some operators on this state. We start from J^+_{-1}. Let us look at (4.69): we have that $J^+_{-1}\left|0\right\rangle_{NS,r,1} \propto \psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}\left|0\right\rangle_{NS,r,1}$. Now, if we were to apply this operator twice, we would get zero, in fact $(J^+_{-1})^2\left|0\right\rangle_{NS,r,1} \propto \psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}\psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}\left|0\right\rangle_{NS,r,1} = 0$. We can apply the operator J^+_{-1} on a strand in the NS ground state just once.
This being said, we can build a more general state by summing the states:

$$\begin{aligned}
&\left|\psi_0\right\rangle\\
&(J^+_{-1})\left|\psi_0\right\rangle = (J^+_{-1})\left|0\right\rangle_{NS,1,1}\otimes\left|0\right\rangle_{NS,2,1}\otimes\left|0\right\rangle_{NS,3,1}\otimes\left|0\right\rangle_{NS,4,1}\otimes\ldots\\
&(J^+_{-1})^2\left|\psi_0\right\rangle = (J^+_{-1})\left|0\right\rangle_{NS,1,1}\otimes(J^+_{-1})\left|0\right\rangle_{NS,2,1}\otimes\left|0\right\rangle_{NS,3,1}\otimes\left|0\right\rangle_{NS,4,1}\otimes\ldots\\
&(J^+_{-1})^3\left|\psi_0\right\rangle = (J^+_{-1})\left|0\right\rangle_{NS,1,1}\otimes(J^+_{-1})\left|0\right\rangle_{NS,2,1}\otimes(J^+_{-1})\left|0\right\rangle_{NS,3,1}\otimes\left|0\right\rangle_{NS,4,1}\otimes\ldots
\end{aligned} \tag{5.71}$$

And so on. How do we accomplish this? The answer is: by applying the operator $e^{\chi J^+_{-1}}$, $\chi\in\mathbb{R}$. We have

$$e^{\chi J^+_{-1}}\left|\psi_0\right\rangle = \sum_{n=0}^{\infty}\frac{\chi^n}{n!}(J^+_{-1})^n\left|\psi_0\right\rangle \tag{5.72}$$

This series expansion captures every single state in equation (5.71)! Such states are called superdescendants [47]. (5.72) is analogous to a coherent state and admits a good gravity description. The parameter χ is crucial because it measures how many strands are excited. If $\chi = 0$ in (5.72) we recover the identity operator, and the state $\left|\psi_0\right\rangle$ is mapped into itself. When $\chi = \frac{\pi}{2}$ we obtain a state with every single strand excited. This last statement is not trivial at all and must be justified. We will do so later.
Here comes another important point: the operations we perform on the CFT states correspond via AdS/CFT to changes of coordinates performed on some metric on the gravity side. For example, the state $\left|\psi_0\right\rangle$ corresponds to the $\mathrm{AdS}_3\times\mathrm{S}_3\times$ T_4 metric.

$$\left|\psi_0\right\rangle \leftrightarrow \mathrm{AdS}_3\times\mathrm{S}_3\times T_4 \tag{5.73}$$

Why is that? Intuitively, the NS vacuum is the only state which is invariant under the conformal transformations generated by $L_0, L_{\pm1}, \bar{L}_0, \bar{L}_{\pm1}$, corresponding to a $SL(2,\mathbb{C})$ symmetry. Moreover,

[2]Again, 1 is the winding number

the NS vacuum possesses a $SO(4)$ symmetry generated by $J_0^a, \bar{J}_0^a$. From the gravitational point of view, the $AdS_3 \times S_3$ geometry is the only one that has an invariance under the geometrical version of the transformations that we mentioned. Therefore, it has an $SL(2,\mathbb{C})$ invariance and a $SO(4)$ invariance.
For similar reasons, the fact that $J_0^\pm |\psi_0\rangle = 0$ corresponds to performing a rotation on the S_3, which has no effect on the metric. On the other hand, since the operators J_{-1}^+ excite the strands, we expect to find some new solution on the gravity side. J_{-1}^+ generates some coordinate transformation on $\mathrm{AdS}_3 \times \mathrm{S}_3 \times \mathrm{T}_4$ that, after being applied, will yield a new metric that still solves the Supergravity equations of motion. However, since the transformation corresponding to the operator J_{-1}^+ is complex, we apply the following operator instead:

$$e^{\chi(J_{-1}^+ - J_1^-)} \qquad \chi \in \mathbb{R} \tag{5.74}$$

An important question that might arise is: why do we get new solutions just by changing coordinates? The reason is because the transformations that we perform do not vanish on the boundary of $_3$. This is similar to what happened with the boosts in chapter 3.
Let us now outline what we are going to do from this point on. First, we start from a simple case, that is, we apply operator (5.74) at first order in χ and see what we find on the gravity side. Then, we move on and repeat the same calculation for every χ. At that point we will show that the values $\chi = 0$ and $\chi = \frac{\pi}{2}$ correspond to two already well-known solutions: the one with $\chi = 0$ is trivial because it is still going to be $\mathrm{AdS}_3 \times \mathrm{S}_3 \times \mathrm{T}_4$, however we will use it as a check. The other solution is not trivial, and will be briefly discussed later. We will also outline all the inbetween solutions, interpolating between the two extremal cases. We expect the corresponding charges to interpolate between the charges of the two known solutions as well. Finally, we will repeat the same discussion applying the more general operator

$$e^{\chi(J_{-n}^+ - J_n^-)} \qquad \chi \in \mathbb{R} \tag{5.75}$$

and see what we find.

5.4 Case $n = 1$

It is now time to describe how construct we the microstates, starting from the action of some operator on the CFT state $|\psi_0\rangle$. We describe only the general procedure here, the details will be discussed case by case in the next sections. We report here the full solution, dual to the state $|\psi_0\rangle$, which is a 2-charge solution. The third charge is added when the strands are excited and we add momentum. As we said, the geometry dual to the vacuum state $|\psi_0\rangle$ is $\mathrm{AdS}_3 \times S_3$. However, we can also obtain it [47] as a particular case of the general construction of Lunin and Mathur (3.44) by using a circular profile in the plane $x_1 - x_2$. Reviewing this construction also helps to embed the vacuum solution into the general BPS ansatz reviewed in Chapter 3. The profile is

$$g_1(v) = a\cos\big(\frac{2\pi v}{L}\big) \qquad g_2(v) = a\sin\big(\frac{2\pi v}{L}\big) \qquad g_3 = g_4 = 0 \tag{5.76}$$

L is the length of the multiply wound fundamental string that is dual to the D1-D5 system and is given by $L = 2\pi\frac{Q_5}{R}$, where R is the radius of the compact circle S_1. The solution, in the near-horizon limit is of the form (5.44), with the following coefficients:

$$ds_4^2 = (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2 \tag{5.77}$$

$$\mathcal{F} = 0 \tag{5.78}$$

$$Z_1 = \frac{Q_1}{r^2 + a^2\cos^2\theta} \tag{5.79}$$

$$Z_2 = \frac{Q_5}{r^2 + a^2\cos^2\theta} \tag{5.80}$$

$$\beta = \frac{Ra^2}{\sqrt{2}(r^2 + a^2\cos^2\theta)}(\sin^2\theta d\phi - \cos^2\theta d\psi) \tag{5.81}$$

$$\omega = \frac{Ra^2}{\sqrt{2}(r^2 + a^2\cos^2\theta)}(\sin^2\theta d\phi + \cos^2\theta d\psi) \tag{5.82}$$

$$a_1 = a_4 = Z_4 = \delta_2 = 0 \tag{5.83}$$

$$\gamma_2 = -Q_5\frac{(r^2+a^2)\cos^2\theta}{r^2 + a^2\cos^2\theta}d\phi \wedge d\psi \tag{5.84}$$

$$\Theta_1 = 0 \tag{5.85}$$

Now, let us see what to do with this solution.

1. First, we find the coordinate transformation associated to the operator we wish to apply. The transformation can be deduced from the known isometries of S_3. This may sound vague, but we will see some examples when we tackle the specific cases.

2. As we pointed out in (5.73), the ground state $|\psi_0\rangle$ corresponds to the $\mathrm{AdS}_3 \times \mathrm{S}_3 \times \mathrm{T}_4$ metric ds^2_{10}, which is obtained by substituting the coefficients (5.77) to (5.85) in the general metric (5.44a).

$$\begin{aligned} ds^2_{10} &= \sqrt{Q_1Q_5}\left[\frac{1}{r^2+a^2}dr^2 + \frac{r^2}{Q_1Q_5}dy^2 - \frac{r^2+a^2}{Q_1Q_5}dt^2 + d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2\right] + d\hat{s}^2_4 = \\ &= ds^2_{AdS_3} + ds^2_{S_3} + d\hat{s}^2_4 \end{aligned} \tag{5.86}$$

   Where
$$a = \frac{\sqrt{Q_1Q_5}}{R} \tag{5.87}$$

   We apply the change of coordinates that we found in step one, which only acts on the angular coordinates of the S_3, and we obtain a new metric, that we call ds'^2_{10}.

3. After that, we apply the spectral flow and switch to the Ramond sector. This has to be done, since we argued that asymptotic flatness can be achieved only in this sector. We obtain a third metric ds''^2_{10}.

4. We move to light-cone coordinates and obtain a fourth metric ds'''^2_{10}. This last metric can be written in the form (5.44) with $Z_4 = 0$.

5. Now we can extract the coefficients characterizing the solutions by comparison. Notice that none of the coordinate transformations we wish apply touches the T_4 part of the metric. Therefore, we can omit it and only work on the $\mathrm{AdS}_3 \times \mathrm{S}_3$ sector because at this stage, the T_4 parts would just cancel out.

6. We repeat steps 2 through 5 on the 2-form C_2 and use it to extract a_1, γ_2, or equivalently Θ_1.

Note that as we said not all states admit a dual describable in Supergravity. This fact could pose a problem. However, it is not our case, because when we apply the coordinate transformations we are always able to find some legitimate solution within Supergravity.

5.4.1 Case $n = 1$, linear order

Let us start from a simple case. Our goal is to apply the operator (5.74) at first order in the parameter χ. This corresponds to a state in which only a small fraction of the total strands are excited. We have, at first order in χ:

$$e^{\chi(J^+_{-1} - J^-_1)} \approx \mathbb{I} + \chi(J^+_{-1} - J^-_1) \tag{5.88}$$

The first thing we have to do is compute the (real) coordinate transformation corresponding to this operator at the linearized order. To do so, we need to apply equations (3.19) and (3.21) of [47]. We find

$$\begin{aligned} J^+_{-1} &\overset{3.21}{=} e^{-i\frac{t+y}{R}} J^+_0 \overset{3.19}{=} \frac{1}{2} e^{-i\frac{t+y}{R}} e^{i(\phi+\psi)} (\partial_\theta + i\cot\theta\partial_\phi - i\tan\theta\partial_\psi) \\ J^-_1 &\overset{3.21}{=} e^{i\frac{t+y}{R}} J^-_0 \overset{3.19}{=} \frac{1}{2} e^{i\frac{t+y}{R}} e^{-i(\phi+\psi)} (-\partial_\theta + i\cot\theta\partial_\phi - i\tan\theta\partial_\psi) \end{aligned} \tag{5.89}$$

Now we perform some algebra in order to find

$$\begin{aligned} &J^+_{-1} - J^-_1 = \\ &= \frac{1}{2} e^{-i\frac{t+y}{R}} e^{i(\phi+\psi)} (\partial_\theta + i\cot\theta\partial_\phi - i\tan\theta\partial_\psi) - \frac{1}{2} e^{i\frac{t+y}{R}} e^{-i(\phi+\psi)} (-\partial_\theta + i\cot\theta\partial_\phi - i\tan\theta\partial_\psi) = \\ &= \frac{1}{2}\big[e^{-i(\frac{t}{R}+\frac{y}{R}-\phi-\psi)} + e^{i(\frac{t}{R}+\frac{y}{R}-\phi-\psi)}\big]\partial_\theta + \frac{i}{2}\big[e^{-i(\frac{t}{R}+\frac{y}{R}-\phi-\psi)} - e^{i(\frac{t}{R}+\frac{y}{R}-\phi-\psi)}\big]\big(\cot\theta\partial_\phi - \tan\theta\partial_\psi\big) = \\ &= \cos\big(\frac{t}{R}+\frac{y}{R}-\phi-\psi\big)\partial_\theta + \cot\theta\sin\big(\frac{t}{R}+\frac{y}{R}-\phi-\psi\big)\partial_\phi - \tan\theta\sin\big(\frac{t}{R}+\frac{y}{R}-\phi-\psi\big)\partial_\psi \end{aligned}$$

From this we can read the vector field associated with the operator $J^+_{-1} - J^-_1$. However, we need the one corresponding to the operator $\mathbb{I} + \chi(J^+_{-1} - J^-_1)$, which is

$$\begin{cases} \theta = \theta' + \chi\big[\cos\big(\frac{t'}{R} + \frac{y'}{R} - \phi' - \psi'\big)\big] \\ \phi = \phi' + \chi\big[\cot\theta'\sin\big(\frac{t}{R} + \frac{y}{R} - \phi - \psi\big)\big] \\ \psi = \psi' + \chi\big[\tan\theta'\sin\big(\frac{t}{R} + \frac{y}{R} - \phi - \psi\big)\big] \end{cases} \tag{5.90}$$

Where the first terms in the right hand sides come from the contribution of the identity operator $\mathbb{I}$, and the other terms come from $\chi(J^+_{-1} - J^-_1)$. This is the coordinate transformation we apply to the AdS$_3$ $\times$ S$_3$$\times$ T$_4$ metric. As anticipated, we omit the T_4 part.

$$ds^2_{10} = \sqrt{Q_1Q_5}\left[\frac{1}{r^2+a^2}dr^2 + \frac{r^2}{Q_1Q_5}dy^2 - \frac{r^2+a^2}{Q_1Q_5}dt^2 + d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2\right] \tag{5.91}$$

From this point on, we performed the majority of the calculations using Wolfram Mathematica. The intermediate steps, in the majority of cases, are extremely lengthy and complicated, therefore there is no point in reporting them here, because they would take up pages and pages. From here we take ds^2_{10} and we can proceed in two ways that give the same result:

- We put (5.91) in matrix form and apply the metric's transformation law

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} \tag{5.92}$$

- We can directly substitute (5.90) and the differentials $d\theta, d\phi, d\psi, dt, dy$ into (5.91) and find a new metric. We follow this path because it is easier to implement in Mathematica.

After this substitution, we end up with a new metric that depends on the primed coordinates, named ds'^2_{10}. This completes step 1 and 2 that we cited in the previous section. We now tackle step 3 and apply the spectral flow, which from the NS sector to the R sector is given by

$$\begin{cases} \phi'' = \phi' + \frac{t'}{R} \\ \psi'' = \psi' + \frac{y'}{R} \end{cases} \tag{5.93}$$

The result is a metric that depends on the doubly primed coordinates ds''^2_{10}. Now we move to light-cone coordinates:

$$\begin{cases} t'' = \frac{v+u}{\sqrt{2}} \\ y'' = \frac{v-u}{\sqrt{2}} \end{cases} \tag{5.94}$$

We get a metric ds'''^2_{10} that depends on $\theta'', \phi''\psi'', r, u, v$. We rename $\theta'' \to \theta$,$\phi'' \to \phi$,$\psi'' \to \psi$ for simplicity. Now, as anticipated, the metric ds'''^2_{10} can be written in the form (5.44a). Therefore, we can now extract the coefficients from it. Let us proceed one step at a time. First, we denote g_{AB} the coefficients of ds'''^2_{10}:

$$ds'''^2_{10} = g_{AB} dX_A dX_B \tag{5.95}$$

where $A, B = r, u, v, \theta, \phi, \psi$. Plus, we define two quantities that will appear quite frequently in our calculations, which are

$$\hat{v}_1 \coloneqq \frac{2\sqrt{2}av}{\sqrt{Q_1 Q_5}} - \phi - \psi \tag{5.96}$$

$$\Sigma_1 \coloneqq r^2 + a^2 \cos^2\theta \tag{5.97}$$

Now, let us compare the metric we found with (5.44a) and get the coefficients.

- Z:

If we look at (5.44a) we notice that the only thing multiplying $dudv$ is $-\frac{2}{Z}$. Therefore, we have $g_{uv} = -\frac{2}{Z}$, from which

$$Z = -\frac{2}{g_{uv}} \tag{5.98}$$

Then, we find

$$Z = \frac{\sqrt{Q_1 Q_5}}{\Sigma_1} - \chi \frac{a^2\sqrt{Q_1 Q_5}\cos\hat{v}_1 sin 2\theta}{\Sigma_1^2} \tag{5.99}$$

- Z_1 and Z_2:

We can use Z to determine Z_1 and Z_2. From the solution corresponding to $|\psi_0\rangle$ we have the expressions

$$\hat{Z}_1 = \frac{Q_1}{\Sigma_1} \qquad \hat{Z}_2 = \frac{Q_5}{\Sigma_1} \tag{5.100}$$

We wish to know how these transform under the three changes of coordinates. We can make use of the fact that the dilaton is left untouched, which means

$$e^{2\Phi'} = e^{2\Phi} \implies \frac{\hat{Z}_1}{\hat{Z}_2} = \frac{Z_1}{Z_2} \tag{5.101}$$

Plus, we know that $\hat{Z}_1$ and $\hat{Z}_2$ transform at the linear order as

$$\begin{aligned} Z_1 &= \hat{Z}_1 + x_1 \\ Z_2 &= \hat{Z}_2 + x_2 \end{aligned} \tag{5.102}$$

Finally, the last bit of information that we need is the transformation rule for Z, that we just found in (5.99). We remind that $Z = \sqrt{Z_1 Z_2}$ [47], therefore we have a system of four equations in the unknowns x_1, x_2, Z_1, Z_2

$$\begin{cases} Z_1 = \hat{Z}_1 + x_1 \\ Z_2 = \hat{Z}_2 + x_2 \\ \frac{\hat{Z}_1}{\hat{Z}_2} = \frac{Z_1}{Z_2} \\ \sqrt{Z_1 Z_2} = \sqrt{\hat{Z}_1 \hat{Z}_2} - \chi \frac{a^2\sqrt{Q_1 Q_5}\cos\hat{v}_1 sin 2\theta}{\Sigma_1^2} \end{cases} \tag{5.103}$$

This can be solved by hand. The result is

$$Z_1 = \frac{Q_1}{\Sigma_1} - \chi \frac{Q_1 a^2 \cos\hat{v}_1 \sin 2\theta}{\Sigma_1^2} \tag{5.104}$$

$$Z_2 = \frac{Q_5}{\Sigma_1} - \chi \frac{Q_5 a^2 \cos\hat{v}_1 \sin 2\theta}{\Sigma_1^2} \tag{5.105}$$

We can check that when computing $\sqrt{Z_1 Z_2}$ at first order in χ using these expressions we obtain precisely (5.99).

- $\mathcal{F}$:

The coefficient $\mathcal{F}$ can be read from the coefficient multiplying dv^2, we have $g_{vv} = -\frac{\mathcal{F}}{Z}$ giving

$$\mathcal{F} = -Z g_{vv} \tag{5.106}$$

At first order in χ we have

$$\mathcal{F} = 0 \tag{5.107}$$

- β:

The 1-form β appears in the coefficients multiplying $du dx_i$, where we remind that $x_i = \{r, \theta, \phi, \psi\}$. We find

$$\beta_i = -\frac{Z}{2} g_{ui} \tag{5.108}$$

This gives

$$\beta = \left[\frac{\sqrt{2}}{2}\frac{a\sqrt{Q_1 Q_5}}{\Sigma} - \chi\frac{\sqrt{2}}{2}\frac{a^3\sqrt{Q_1 Q_5}\cos\hat{v}_1 \sin 2\theta}{\Sigma^2}\right](\sin^2\theta d\phi - \cos^2\theta d\psi) \tag{5.109}$$

- ω:

The 1-form ω appears, together with β_i and $\mathcal{F}$, in the coefficients multiplying $dvdx_i$. We now know $\mathcal{F}$ and β, so we use them to extract ω_i

$$\begin{aligned}\omega = &\frac{\sqrt{2}}{2}\frac{a\sqrt{Q_1Q_5}}{\Sigma}(\sin^2\theta d\phi + \cos^2\theta d\psi) + \chi\sqrt{2}\frac{a\sqrt{Q_1Q_5}}{\Sigma}\sin\hat{v}_1 d\theta - \\ &- \chi\frac{\sqrt{2}}{2}\frac{a\sqrt{Q_1Q_5}}{\Sigma^2}\cos\hat{v}_1\sin 2\theta\big[(r^2+a^2)d\phi + r^2 d\psi\big]\end{aligned} \tag{5.110}$$

- ds_4^2:

In order to compute this metric, which is along the x_i coordinates, we take our $ds_{10}^{'''2}$ and subtract some terms from (5.44a), where we substitute the coefficients $Z, \mathcal{F}, \beta, \omega, Z_1, Z_2$ we just found. We have

$$ds_4^2 = \frac{1}{Z}\left[ds_{10}^{'''2} + \frac{2}{Z}(dv+\beta)\big[du + \omega + \frac{\mathcal{F}}{2}(dv+\beta)\big]\right] \tag{5.111}$$

Now we extract the metric, which reads

$$\begin{aligned}ds_4^2 = &\left[\frac{\Sigma}{r^2+a^2}dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2\right] + \\ &\chi\left[\frac{a^2}{r^2+a^2}\cos\hat{v}_1\sin 2\theta dr^2 + a^2\cos\hat{v}_1\sin 2\theta d\theta^2 + a^2\sin\hat{v}_1\sin^2\theta d\theta d\phi - \right. \\ &\left. - a^2\sin\hat{v}_1\cos^2\theta d\theta d\psi + a^2\cos\hat{v}_1\sin\theta\cos\theta d\theta d\psi\right]\end{aligned} \tag{5.112}$$

- Θ_1:

For our last step, we have to re-apply the same coordinate transformations on the 2-form C_2. The form corresponding to the state $|\psi_0\rangle$ is obtained by substituting the coefficients (5.77) through (5.85) in the general solution (5.44e). We have

$$C_2 = -Q_5\big[\frac{r^2}{Q_1Q_5}dt\wedge dy + \cos^2\theta d\phi\wedge d\psi\big] \tag{5.113}$$

We can get a_1 and γ_2 from this quantity. However, they are gauge-dependent quantities, so we turn our attention to F_3, from which we can read the 2-form Θ_1, which bears the same information. We compute $F_3 = dC_2$, where the exterior derivative acts on all coordinates. We get

$$F_3 = 2Q_5(-\hat{vol}_{AdS_3} + \hat{vol}_{S_3}) \tag{5.114}$$

where

$$\begin{aligned}\hat{vol}_{AdS_3} &= \frac{r}{Q_1Q_5}dr\wedge dt\wedge dy \\ \hat{vol}_{S_3} &= \big(\sin\theta\cos\theta\big)d\theta\wedge d\phi\wedge d\psi\end{aligned} \tag{5.115}$$

Now we apply the three coordinate transformations again and get a new 3-form, which we call $F_3^{'''}$. Note that we could apply the transformations on (5.113) as well, and after getting $dC_2^{'''}$ we could compute $F_3^{'''}$ by taking the exterior derivative. The two methods yield the same result. Now, we can write our F_3''' in the form of equation (3.51b) of [45], with a different notation and some coefficients set to zero, That is

$$\begin{aligned} F_3 = &-(du+\omega)\wedge(dv+\beta)\wedge\left[\mathcal{D}\left(\frac{1}{Z_1}\right)-\frac{\dot\beta}{Z_1}\right]+(dv+\beta)\wedge\left(\Theta_1-\frac{\mathcal{D}\omega}{Z_1}\right)+\\ &+\frac{1}{Z_1}(du+\omega)\wedge\mathcal{D}\beta+\star_4(\mathcal{D}Z_2+Z_2\dot\beta)\end{aligned} \tag{5.116}$$

Using the computed coefficients, we calculate the quantity

$$E_3 = F_3''' + (du+\omega)\wedge(dv+\beta)\wedge\left[D\left(\frac{1}{Z_1}\right)-\frac{\dot\beta}{Z_1}\right]+(dv+\beta)\wedge\frac{D\omega}{Z_1} \tag{5.117}$$

From (5.116) we deduce that the 3-form E_3 must be of the form

$$E_3 = (\Theta_1)_{ij}dv\wedge dx_i\wedge dx_j + \Xi_{ijk}dx_i\wedge dx_j\wedge dx_k \tag{5.118}$$

Allowing us to read off the components of the 2-form Θ_1, which must be totally antisymmetric, because its symmetric part mutiplied by the antysimmetric form $dv\wedge dx_i\wedge dx_j$, is zero. We have

$$\begin{aligned}\Theta_1 = &\chi 2\sqrt{2}\sqrt{\frac{Q_5}{Q_1}}\frac{a(r^2+a^2)}{\Sigma}\cos\hat v_1(d\theta\wedge d\phi-\cot^2\theta d\theta\wedge d\psi)+\\ &+\chi 4\sqrt{2}\frac{a(r^2+a^2)}{\Sigma^2}\cot\theta\sin\hat v_1(r^2+a^2\cos^2\theta\cos 2\theta)d\phi\wedge d\psi\end{aligned} \tag{5.119}$$

Now that our solution is complete, we can check it by verifying that it satisfies the equations of motion we previously listed in section 5.2. This was done for some of the easiest ones to check, and everything is consistent. This concludes our discussion regarding the linear order. It is now time to move to the finite order, which is the most interesting case.

5.4.2 Case $n = 1$, finite order

The procedure to find the coefficients is almost identical to the infinitesimal case. Therefore, we will not write everything in detail again. What is different is how we find the coordinate transformation on the S_3. We have to find a way to generalize equation (5.90). The other tiny difference, that we point out, is how we find Z_1 and Z_2 starting from Z. In this case it is even easier, because instead of (5.103), we solve the system

$$\begin{cases}\frac{\hat Z_1}{\hat Z_2}=\frac{Z_1}{Z_2}\\ \sqrt{Z_1Z_2}=Z\end{cases} \tag{5.120}$$

Since we know $Z,\hat Z_1,\hat Z_2$, we find Z_1 and Z_2 easily. This said, let us focus our attention on generalizing the first-order change of coordinates.
Our goal is to apply the operator

$$e^{\chi(J^+_{-1}-J^-_1)} \tag{5.121}$$

on the state $|\psi_0\rangle$. We start with some basic algebra. We use equation (3.21) of [47], together with $J_0^{\pm} = J_0^1 \pm iJ_0^2$ and the fact that $J_0^i = \frac{\sigma_i}{2}$ are the $SU(2)$ generators to rewrite

$$\begin{aligned} J_{-1}^{+} - J_{1}^{-} &= e^{-i\frac{t+y}{R}} J_0^{+} - e^{i\frac{t+y}{R}} J_0^{-} = \\ &= e^{-i\frac{t+y}{R}} (J_0^1 + iJ_0^2) - e^{i\frac{t+y}{R}} (J_0^1 - iJ_0^2) = \\ &= \big[e^{-i\frac{t+y}{R}} - e^{i\frac{t+y}{R}}\big] J_0^1 + i\big[e^{-i\frac{t+y}{R}} - e^{i\frac{t+y}{R}}\big] J_0^2 = \\ &= -2i \sin\big(\frac{t+y}{R}\big) J_0^1 + 2i\cos\big(\frac{t+y}{R}\big) J_0^2 = \\ &= -i\sin\big(\frac{t+y}{R}\big)\sigma_1 + i\cos\big(\frac{t+y}{R}\big)\sigma_2 \end{aligned} \tag{5.122}$$

Now we proceed by analogy with equation (3.15) of [47]. Our transformation is

$$\begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} = \begin{bmatrix} z_1' & -\bar{z}_2' \\ z_2' & \bar{z}_1' \end{bmatrix} e^{i\chi\cos(\frac{t'+y'}{R})\sigma_2 - i\chi\sin(\frac{t'+y'}{R})\sigma_1} \tag{5.123}$$

where

$$\begin{aligned} z_1 &:= \sin\theta e^{i\phi} \qquad & z_2 &:= \cos\theta e^{i\psi} \\ z_1' &:= \sin\theta' e^{i\phi'} \qquad & z_2' &:= \cos\theta' e^{i\psi'} \end{aligned} \tag{5.124}$$

and the bars represent complex conjugations. We now have to evaluate the matrix exponential on the right side of (5.123), and then explicitly calculate the product. The exponential can be computed by using the standard techniques because the matrix

$$i\chi\cos\big(\frac{t'+y'}{R}\big)\sigma_2 - i\chi\sin\big(\frac{t'+y'}{R}\big)\sigma_1 \tag{5.125}$$

is diagonalizable. If we compute the exponential and multiply it by the matrix in (5.123) we obtain something of the form

$$\begin{bmatrix} \sin\theta e^{i\phi} & -\cos\theta e^{-i\psi} \\ \cos\theta e^{i\psi} & \sin\theta e^{-i\phi} \end{bmatrix} = \begin{bmatrix} A(t',y',\theta',\phi',\psi';\chi) & B(t',y',\theta',\phi',\psi';\chi) \\ C(t',y',\theta',\phi',\psi';\chi) & D(t',y',\theta',\phi',\psi';\chi) \end{bmatrix} \tag{5.126}$$

This yields four equations, one for each entry of the matrices. Writing explicitly the functions A, B, C, D, the four equations are

$$\sin\theta e^{i\phi} = e^{-i\chi} e^{-i\psi'} \Big[e^{i\frac{t'+y'}{R}} e^{i\chi} \cos\theta' \sin\chi + e^{i(\phi'+\psi')} e^{i\chi} \sin\theta' \cos\chi \Big] \tag{5.127a}$$

$$-\cos\theta e^{-i\psi} = \frac{1}{2} e^{-i\chi} e^{-i\psi'} \Big[-\cos\theta'(e^{2i\chi} - 1) + 2e^{i(\phi'+\psi')} e^{-i\frac{t'+y'}{R}} e^{i\chi} \sin\theta' \sin\chi \Big] \tag{5.127b}$$

$$\cos\theta e^{i\psi} = e^{-i\chi} e^{-i\phi'} \Big[-e^{i\frac{t'+y'}{R}} e^{i\chi} \sin\theta' \sin\chi + e^{i(\phi'+\psi')} e^{i\chi} \cos\theta' \cos\chi \Big] \tag{5.127c}$$

$$\sin\theta e^{-i\phi} = \frac{1}{2} e^{-i\chi} e^{-i\phi'} \Big[\sin\theta'(e^{2i\chi} + 1) + 2e^{i(\phi'+\psi')} e^{-i\frac{t'+y'}{R}} e^{i\chi} \cos\theta' \sin\chi \Big] \tag{5.127d}$$

These transformations look really complicated and difficult to handle. However, there is a clever way to sum them that gets rid of the complex numbers. We obtain another set of four equations

$$\text{(5.127a)+(5.127d):}\quad \sin\theta\cos\phi = \sin\theta'\cos\phi'\cos\chi + \cos\theta'\cos\big(\frac{t'+y'}{R} - \psi'\big)\sin\chi \tag{5.128a}$$

$$\text{(5.127b)+(5.127c):}\quad \cos\theta\sin\psi = \cos\theta'\sin\psi'\cos\chi - \sin\theta'\sin\big(\frac{t'+y'}{R} - \phi'\big)\sin\chi \tag{5.128b}$$

$$\text{(5.127a)-(5.127d):}\quad \sin\theta\sin\phi = \sin\theta'\sin\phi'\cos\chi + \cos\theta'\sin\big(\frac{t'+y'}{R} - \psi'\big)\sin\chi \tag{5.128c}$$

$$\text{(5.127b)-(5.127c):}\quad \cos\theta\cos\psi = \cos\theta'\cos\psi'\cos\chi - \sin\theta'\cos\big(\frac{t'+y'}{R} - \phi'\big)\sin\chi \tag{5.128d}$$

We are almost done. We calculate the ratio $\frac{(5.128c)}{(5.128a)}$, which gets us $\tan\phi$. Moreover, $\tan\psi$ is obtained by the ratio $\frac{(5.128b)}{(5.128d)}$. Finally, we get $\sin^2\theta$ by calling R' the right-hand side of equation (5.128a) and applying trigonometric identities. We have

$$\sin\theta = \frac{R'}{\cos\phi} \tag{5.129}$$

We square both sides and apply the identity $\cos^{-2}\phi = 1 + \tan^2\phi$ to get

$$\sin^2\theta = R'^2(1 + \tan^2\phi) \tag{5.130}$$

since we know R' and $\tan\phi$, we are done. The finite version of transformation (5.90) is[3] therefore

$$\begin{cases} \sin^2\theta = \big[\sin\theta'\sin\phi'\cos\chi + \cos\theta'\sin\big(\frac{t'+y'}{R} - \psi'\big)\sin\chi\big]^2 + \\ \qquad + \big[\sin\theta'\cos\phi'\cos\chi + \cos\theta'\cos\big(\frac{t'+y'}{R} - \psi'\big)\sin\chi\big]^2 \\ \tan\phi = \frac{\sin\theta'\sin\phi'\cos\chi + \cos\theta'\sin\left(\frac{t'+y'}{R} - \psi'\right)\sin\chi}{\sin\theta'\cos\phi'\cos\chi + \cos\theta'\cos\left(\frac{t'+y'}{R} - \psi'\right)\sin\chi} \\ \tan\psi = \frac{\cos\theta'\sin\psi'\cos\chi - \sin\theta'\sin\left(\frac{t'+y'}{R} - \psi'\right)\sin\chi}{\cos\theta'\cos\psi'\cos\chi - \sin\theta'\cos\left(\frac{t'+y'}{R} - \psi'\right)\sin\chi} \end{cases} \tag{5.131}$$

This looks rather complicated because we have to apply square roots and inverse trigonometric functions. However, it is the correct transformation. It is immediate to check that for $\chi = 0$ it reduces to the identity. Moreover, we can expand it to first order in χ to recover the infinitesimal transformation (5.90). Now we follow the same steps as before, in order to fine the relevant coefficients, that we directly report. We define

$$\bar{\Sigma}_1 \coloneqq r^2 + a^2\cos^2\theta + a^2\cos 2\theta\sin^2\chi + a^2\cos\hat{v}_1\sin 2\theta\sin\chi\cos\chi \tag{5.132}$$

Notice that $\bar{\Sigma}_1 \to \Sigma$ when $\chi \to 0$. We have

$$Z = \frac{\sqrt{Q_1 Q_5}}{\bar{\Sigma}_1} \tag{5.133}$$

$$Z_1 = \frac{Q_1}{\bar{\Sigma}_1} \tag{5.134}$$

[3]An alternative way of achieving this is by using equation (3.22) of [47] and conjugating with spectral flow.

$$Z_2 = \frac{Q_5}{\bar{\Sigma}_1} \tag{5.135}$$

$$\mathcal{F} = -\frac{4a^2 \sin^2 \chi}{\bar{\Sigma}_1} \tag{5.136}$$

$$\beta = \frac{\sqrt{2}}{2}\frac{a\sqrt{Q_1 Q_5}}{\bar{\Sigma}_1}(\sin^2\theta d\phi - \cos^2\theta d\psi) \tag{5.137}$$

$$\begin{aligned}
\omega = &\left[\frac{1}{2\bar{\Sigma}_1}\sqrt{2}a\sqrt{Q_1Q_5}\sin\hat{v}_1\sin(2\chi)\right]d\theta+\\
&+\left[-\frac{1}{2\bar{\Sigma}_1}\sqrt{2}a\sqrt{Q_1Q_5}\sin\theta\big[\cos\hat{v}_1\cos\theta\sin 2\chi+\sin\theta(\cos(2\chi)-2)\big]+\right.\\
&\left.+\frac{1}{\bar{\Sigma}_1^2}2\sqrt{2}\sqrt{Q_1Q_5}a^3\sin^2\theta\sin^2\chi\right]d\phi+\\
&+\left[-\frac{1}{2\bar{\Sigma}_1}\sqrt{2}a\sqrt{Q_1Q_5}\cos\theta\big[-\cos\hat{v}_1\sin\theta\sin 2\chi+\cos\theta(\cos(2\chi)-2)\big]-\right.\\
&\left.-\frac{1}{\bar{\Sigma}_1^2}2\sqrt{2}\sqrt{Q_1Q_5}a^3\cos^2\theta\sin^2\chi\right]d\psi
\end{aligned} \tag{5.138}$$

$$\begin{aligned}
ds_4^2 = &\left(\frac{\bar{\Sigma}_1}{r^2+a^2}\right)dr^2+\bar{\Sigma}_1 d\theta^2+\left[(r^2+a^2)\sin^2\theta+a^2\sin^2\theta\sin^2\chi+\right.\\
&\left.+\frac{2}{\bar{\Sigma}_1}a^4\sin^4\theta\sin^2\chi\right]d\phi^2+\left[r^2\cos^2\theta-a^2\cos^2\theta\sin^2\chi+\frac{2}{\bar{\Sigma}_1}a^4\cos^4\theta\sin^2\chi\right]d\psi^2+\\
&+a^2\sin^2\theta\sin\hat{v}_1\sin\chi\cos\chi d\theta d\phi-a^2\cos^2\theta\sin\hat{v}_1\sin\chi\cos\chi d\theta d\psi+\\
&+\left[\frac{1}{2}a^2\cos\hat{v}_1\sin 2\theta\sin\chi\cos\chi-\frac{2}{\bar{\Sigma}_1}a^4\sin^2\theta\cos^2\theta\sin^2\chi\right]d\phi d\psi
\end{aligned} \tag{5.139}$$

$$\begin{aligned}
\Theta_1 = &\sqrt{\frac{Q_5}{Q_1}}\frac{4}{\bar{\Sigma}_1^2}\sqrt{2}a^3 r\sin^2\chi\big[(-\sin^2\theta)dr\wedge d\phi+(\cos^2\theta)dr\wedge d\psi\big]+\\
&+\sqrt{2}\sqrt{\frac{Q_5}{Q_1}}\frac{a(r^2+a^2)}{\bar{\Sigma}_1}\cos\hat{v}_1\sin 2\chi(d\theta\wedge d\phi-\cot^2\theta d\theta\wedge d\psi)+\\
&+2\sqrt{2}\frac{a(r^2+a^2)}{\bar{\Sigma}_1^2}\cot\theta\sin\hat{v}_1\sin 2\chi(r^2+a^2\cos^2\theta\cos 2\theta)d\phi\wedge d\psi
\end{aligned} \tag{5.140}$$

As a check, we can expand these objects to first order in χ and see that they match the ones we found in the previous section. As anticipated, we recover known solutions for $\chi = 0$ and $\chi = \frac{\pi}{2}$.

5.4.3 Special cases

We start with $\chi = 0$. This corresponds to applying the identity operator, which means that we do nothing on the state $|\psi_0\rangle$, so we are left with the trivial solution that we already know. This case is not so interesting after all, but we can still substitute $\chi = 0$ in expressions (5.133) through (5.171) for a check. If this is done, we find that the coefficients perfectly match equations (5.77) through (5.85).

Let us now turn to the more interesting case $\chi = \frac{\pi}{2}$. Let us first write the coefficients obtained by substituting in equations (5.133) to (5.171). We have

$$\hat{\Sigma}_1 = \Sigma\big(\chi = \frac{\pi}{2}\big) = r^2 + a^2(2\cos\theta^2 - \sin^2\theta) \tag{5.141}$$

and the coefficients

$$Z = \frac{\sqrt{Q_1 Q_5}}{\hat{\Sigma}_1} \tag{5.142}$$

$$Z_1 = \frac{Q_1}{\hat{\Sigma}_1} \tag{5.143}$$

$$Z_2 = \frac{Q_5}{\hat{\Sigma}_1} \tag{5.144}$$

$$\mathcal{F} = -\frac{4a^2}{\hat{\Sigma}_1} \tag{5.145}$$

$$\beta = \frac{\sqrt{2}}{2}\frac{a\sqrt{Q_1 Q_5}}{\hat{\Sigma}_1}(\sin^2\theta d\phi - \cos^2\theta d\psi) \tag{5.146}$$

$$\begin{aligned}\omega = &\left[\frac{3}{2}\frac{\sqrt{2}}{\hat{\Sigma}_1}\sqrt{Q_1 Q_5} + 2\frac{\sqrt{2}}{\hat{\Sigma}_1^2}a^3\sqrt{Q_1 Q_5}\right]\sin^2\theta d\phi + \\ &+\left[\frac{3}{2}\frac{\sqrt{2}}{\hat{\Sigma}_1}\sqrt{Q_1 Q_5} - 2\frac{\sqrt{2}}{\hat{\Sigma}_1^2}a^3\sqrt{Q_1 Q_5}\right]\cos^2\theta d\psi\end{aligned} \tag{5.147}$$

$$\begin{aligned}ds_4^2 = &\left(\frac{\hat{\Sigma}_1}{r^2+a^2}\right)dr^2 + \hat{\Sigma}_1 d\theta^2 + \left[(2a^2+r^2)\sin^2\theta + \frac{2}{\hat{\Sigma}_1}a^4\sin^4\theta\right]d\phi^2 + \\ &+\left[(r^2-a^2)\cos^2\theta + \frac{2}{\hat{\Sigma}_1}a^4\cos^4\theta\right]d\psi^2 - \left(\frac{2}{\hat{\Sigma}_1}a^4\sin^2\theta\cos^2\theta\right)d\phi d\psi\end{aligned} \tag{5.148}$$

$$\Theta_1 = \sqrt{\frac{Q_5}{Q_1}}\frac{4}{\hat{\Sigma}_1^2}\sqrt{2}a^3 r\left[(-\sin^2\theta)dr\wedge d\phi + (\cos^2\theta)dr\wedge d\psi\right] \tag{5.149}$$

This solution is already known. In fact, this is the three-charge solution that has been discussed in [26], which corresponds to a state with every strand excited. The objects in equations (5.141) to (5.149) exactly coincide with equations (2.13) and (2.14) of [26] with $\psi \to -\psi$. This is not immediate, as the notations used are different. We have a match with if we set $n = k = \eta = 1$ in [26]. This yields $f \to \hat{\Sigma}_1$, $Q \to \sqrt{Q_1 Q_5}$, $h \to Z$, $\eta \to 1$, $\gamma_1 \to -a$, $\gamma_2 \to 2a$, $Q_P \to 2a^2$. The metric ds_4^2 can be put in the Gibbons-Hawking form [26].

5.5 Case $n > 1$

An interesting situation to look at involves $J_n^{\pm}$ operators, generalizing what we did so far with $J_1^{\pm}$. Again, we start from the linearized order and then proceed to the finite case.

5.5.1 Case $n > 1$, linear order

We apply on $|\psi_0\rangle$ the operator

$$\mathbb{I} + \chi(J_{-n}^{+} - J_n^{-}) \tag{5.150}$$

Finding the coordinate transformation is really easy, because we just have to repeat the same calculations we did in section 5.4.1 with a tiny difference, that is, now we have the formula

$$J^{\pm}_{\mp n} = e^{\mp in\frac{t+y}{R}} J^{\pm}_0 \tag{5.151}$$

which is obtained by repeatedly applying equation (3.21) of [47]. Therefore, we can redo all the algebra in equation (5.89) with $e^{\mp i\frac{t+y}{R}} \to e^{\mp in\frac{t+y}{R}}$. The resulting transformation generalizes (5.90):

$$\begin{cases} \theta = \theta' + \chi\big[\cos\big(n\frac{t'+y'}{R} - \phi' - \psi'\big)\big] \\ \phi = \phi' + \chi\big[\cot\theta' \sin\big(n\frac{t'+y'}{R} - \phi - \psi\big)\big] \\ \psi = \psi' + \chi\big[\tan\theta' \sin\big(n\frac{t'+y'}{R} - \phi - \psi\big)\big] \end{cases} \tag{5.152}$$

After changing coordinates we define the quantity

$$\hat{v}_n := \frac{(n+1)\sqrt{2}av}{\sqrt{Q_1 Q_5}} - \phi - \psi \tag{5.153}$$

and the coefficients are

$$Z = \frac{\sqrt{Q_1Q_5}}{\Sigma_1} - \chi\frac{na^2\sqrt{Q_1Q_5}\cos\hat{v}_n sin2\theta}{\Sigma_1^2} \tag{5.154}$$

$$Z_1 = \frac{Q_1}{\Sigma_1} - \chi\frac{Q_1 na^2\cos\hat{v}_n\sin 2\theta}{\Sigma_1^2} \tag{5.155}$$

$$Z_2 = \frac{Q_5}{\Sigma_1} - \chi\frac{Q_5 na^2\cos\hat{v}_n\sin 2\theta}{\Sigma_1^2} \tag{5.156}$$

$$\mathcal{F} = 0 \tag{5.157}$$

$$\beta = \left[\frac{\sqrt{2}}{2}\frac{a\sqrt{Q_1Q_5}}{\Sigma} - \chi\frac{\sqrt{2}}{2}\frac{na^3\sqrt{Q_1Q_5}\cos\hat{v}_n\sin 2\theta}{\Sigma^2}\right](\sin^2\theta d\phi - \cos^2\theta d\psi) \tag{5.158}$$

$$\begin{aligned} \omega = &\frac{\sqrt{2}}{2}\frac{a\sqrt{Q_1Q_5}}{\Sigma}(\sin^2\theta d\phi + \cos^2\theta d\psi) + \chi\sqrt{2}\frac{na\sqrt{Q_1Q_5}}{\Sigma}\sin\hat{v}_n d\theta - \\ &- \chi\frac{\sqrt{2}}{2}\frac{na\sqrt{Q_1Q_5}}{\Sigma^2}\cos\hat{v}_n\sin 2\theta\big[(r^2+a^2)d\phi + r^2 d\psi\big] \end{aligned} \tag{5.159}$$

$$\begin{aligned} ds_4^2 = &\left[\frac{\Sigma}{r^2+a^2}dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2\right] + \\ &\chi\left[\frac{na^2}{r^2+a^2}\cos\hat{v}_n\sin 2\theta dr^2 + na^2\cos\hat{v}_n\sin 2\theta d\theta^2 + na^2\sin\hat{v}_n\sin^2\theta d\theta d\phi - \right.\\ &\left. - na^2\sin\hat{v}_n\cos^2\theta d\theta d\psi + na^2\cos\hat{v}_n\sin\theta\cos\theta d\theta d\psi\right] \end{aligned} \tag{5.160}$$

$$\begin{aligned} \Theta_1 = &\chi(n+1)\sqrt{2}\sqrt{\frac{Q_5}{Q_1}}\frac{a(r^2+na^2)}{\Sigma}\cos\hat{v}_n(d\theta\wedge d\phi - \cot^2\theta d\theta\wedge d\psi) + \\ &+ \chi 2(n+1)\sqrt{2}\frac{a\cot\theta}{\Sigma^2}\sin\hat{v}_n\Big(r^4 + a^4\cos^2\theta(n\cos^2\theta - \sin^2\theta) + \\ &+ r^2a^2(2\cos^2\theta - n\sin^2\theta\cos 2\theta)\Big)d\phi\wedge d\psi \end{aligned} \tag{5.161}$$

We can check that for $n = 1$ we re-obtain the solution we previously found, at first order in χ.

5.5.2 Case $n > 1$, finite order

In this case the coordinate transformation generalizing (5.131) is

$$\begin{cases} \sin^2\theta = \left[\sin\theta'\sin\phi'\cos\chi + \cos\theta'\sin\left(n\frac{t'+y'}{R} - \psi'\right)\sin\chi\right]^2 + \\ \qquad + \left[\sin\theta'\cos\phi'\cos\chi + \cos\theta'\cos\left(n\frac{t'+y'}{R} - \psi'\right)\sin\chi\right]^2 \\ \tan\phi = \frac{\sin\theta'\sin\phi'\cos\chi+\cos\theta'\sin\left(n\frac{t'+y'}{R}-\psi'\right)\sin\chi}{\sin\theta'\cos\phi'\cos\chi+\cos\theta'\cos\left(n\frac{t'+y'}{R}-\psi'\right)\sin\chi} \\ \tan\psi = \frac{\cos\theta'\sin\psi'\cos\chi-\sin\theta'\sin\left(n\frac{t'+y'}{R}-\psi'\right)\sin\chi}{\cos\theta'\cos\psi'\cos\chi-\sin\theta'\cos\left(n\frac{t'+y'}{R}-\psi'\right)\sin\chi} \end{cases} \tag{5.162}$$

We define

$$\Sigma_n := r^2 + a^2\cos^2\theta + na^2\cos 2\theta\sin^2\chi + na^2\cos\hat{v}_n\sin 2\theta\sin\chi\cos\chi \tag{5.163}$$

Notice that $\Sigma_n(n=1) = \bar{\Sigma}_1$. The coefficients are

$$Z = \frac{\sqrt{Q_1 Q_5}}{\Sigma_n} \tag{5.164}$$

$$Z_1 = \frac{Q_1}{\Sigma_n} \tag{5.165}$$

$$Z_2 = \frac{Q_5}{\Sigma_n} \tag{5.166}$$

$$\mathcal{F} = -\frac{2n(n+1)a^2}{\Sigma_n}\sin^2\chi \tag{5.167}$$

$$\beta = \frac{\sqrt{2}}{2}\frac{a\sqrt{Q_1Q_5}}{\Sigma_n}(\sin^2\theta d\phi - \cos^2\theta d\psi) \tag{5.168}$$

$$\begin{aligned} \omega = &\left[\frac{1}{2\Sigma_n}\sqrt{2}na\sqrt{Q_1Q_5}\sin\hat{v}_n\sin(2\chi)\right]d\theta - \\ &-\left[\frac{\sqrt{2}}{2}\frac{a}{\Sigma_n}\sqrt{Q_1Q_5}\sin\theta\big[n\cos\hat{v}_n\cos\theta\sin 2\chi - \sin\theta(1+2n\sin^2\chi)\big] - \right. \\ &\left. -\frac{a^3}{\Sigma_n^2}n(n+1)\sqrt{2}\sqrt{Q_1Q_5}\sin^2\theta\sin^2\chi\right]d\phi + \\ &+\left[\frac{\sqrt{2}}{2}\frac{a}{\Sigma_n}\sqrt{Q_1Q_5}\cos\theta\big[n\cos\hat{v}_n\sin\theta\sin 2\chi + \cos\theta(1+2n\sin^2\chi)\big] - \right. \\ &\left. -\frac{a^3}{\Sigma_n^2}n(n+1)\sqrt{2}\sqrt{Q_1Q_5}\cos^2\theta\sin^2\chi\right]d\psi \end{aligned} \tag{5.169}$$

$$\begin{aligned} ds_4^2 = &\left(\frac{\Sigma_n}{r^2+a^2}\right)dr^2 + \Sigma_n d\theta^2 + \left[(r^2+a^2)\sin^2\theta + na^2\sin^2\theta\sin^2\chi + \right. \\ &\left. + \frac{n(n+1)}{\Sigma_n}a^4\sin^4\theta\sin^2\chi\right]d\phi^2 + \left[r^2\cos^2\theta - na^2\cos^2\theta\sin^2\chi + \right. \end{aligned} \tag{5.170}$$

$$
\begin{aligned}
&+ \frac{n(n+1)}{\Sigma_n} a^4 \cos^4\theta \sin^2\chi \Big] d\psi^2 + na^2 \sin^2\theta \sin\hat{v}_n \sin\chi \cos\chi d\theta d\phi - \\
&- na^2 \cos^2\theta \sin\hat{v}_n \sin\chi\cos\chi d\theta d\psi + \Big[\frac{1}{2} na^2 \cos\hat{v}_1 \sin 2\theta \sin\chi \cos\chi - \\
&- \frac{n(n+1)}{\Sigma_n} a^4 \sin^2\theta \cos^2\theta \sin^2\chi \Big] d\phi d\psi
\end{aligned}
$$

$$
\begin{aligned}
\Theta_1 = &\sqrt{\frac{Q_5}{Q_1}} \frac{2}{\Sigma_n^2} n(n+1)\sqrt{2} a^3 r \sin^2\chi \big[(-\sin^2\theta) dr\wedge d\phi + (\cos^2\theta) dr \wedge d\psi\big] + \\
&+ \frac{\sqrt{2}}{2}(n+1)\sqrt{\frac{Q_5}{Q_1}} \frac{a(r^2+na^2)}{\Sigma_n} \cos\hat{v}_1 \sin 2\chi (d\theta\wedge d\phi - \cot^2\theta d\theta \wedge d\psi) + \\
&+ (n+1)\sqrt{2} \frac{a\cot\theta}{\Sigma^2} \sin\hat{v}_n \sin 2\chi \Big(r^4 + a^4\cos^2\theta(n\cos^2\theta - \sin^2\theta) + \\
&+ r^2 a^2 (2\cos^2\theta - n \sin^2\theta \cos 2\theta)\Big) d\phi\wedge d\psi
\end{aligned}
\tag{5.171}
$$

We can check that for $n = 1$ we re-obtain the solution we previously found, for finite χ.

5.6 Charges

It is now time to compute the asymptotic charges associated with the solution with finite χ. We start from the gravity side, then repeat the computation from the CFT side. We eventually compare the results. This gives us a first nontrivial check of the validity of our calculations, because CFT and Supergravity live in two separate points of the moduli space. The charges do not depend on the moduli, so they must coincide.

5.6.1 $n = 1$: Gravity side

We are interested in the charges Q_1, Q_5, the momentum Q_P, and finally the angular momenta J and $\tilde{J}$. Notice that we are working in four spatial dimensions: in three dimensions there is only one angular momentum, associated to rotations on a plane. In four dimensions, there are two independent 2-planes, therefore there are 2 commuting angular momenta, which are simultaneously diagonalizable. The general procedure to extract these charges is not straight-forward, because it involves extending the solution in the asymptotically flat region. This means that we must perform the opposite operation of taking the near-horizon limit, i.e. restoring the 1's in Z_1 and Z_2. However, if we do this, then the equations of motion are not satisfied anymore, therefore we need to make some adjustments. The general procedure is described in [48]. Our situation is much simpler, since we can avoid this extension. In fact, we would end up with terms proportional to v_1, which, when integrated over the compact directions, give null contribution. We follow the computation done in [27].

First of all, we can extract the D1 and D5 Supergravity charges from the large-r behaviour of the factors Z_1, Z_2. In this case, these are left untouched by the transformation and do not depend on χ. Q_1 and Q_5 are just given by equation (3.28).

The momentum charge Q_P is encoded in $\mathcal{F}$, which controls the $dtdy$ term in the full metric, which is the one that is generally linked to momentum. Q_P can be extracted from the large-r expansion of the coefficient $\mathcal{F}$:

$$-\frac{\mathcal{F}}{2} \approx \frac{Q_P}{r^2} + \mathcal{O}(r^{-3}) \tag{5.172}$$

Comparing with equation (5.136) we extract

$$Q_P = 2a^2 \sin^2\chi = 2\frac{Q_1 Q_5}{R^2}\sin^2\chi \tag{5.173}$$

By substituting Q_1, Q_5, Q_P using the formulas in equation (3.28) we finally have

$$N_P = 2N_1 N_5 \sin^2\chi = 2N\sin^2\chi \tag{5.174}$$

In a similar fashion as Q_P, the angular momenta are encoded in β and ω, which control the $dtd\phi$ and $dtd\psi$ terms respectively. In order to compute J and $\bar{J}$ we first compute the quantities β_0 and μ:

$$\begin{aligned} \beta_0 &= \frac{\beta_\phi + \beta_\psi}{2} \\ \mu &= \frac{\omega_\phi + \omega_\psi}{2} \end{aligned} \tag{5.175}$$

where the terms in the right hand sides are computed at large r. Then, we apply the formula

$$\frac{\beta_0 + \mu}{\sqrt{2}} = \frac{J - \tilde{J}\cos 2\theta}{2r^2} + \mathcal{O}(r^{-3}) \tag{5.176}$$

From which we read off the charges J, $\tilde{J}$. After performing some algebra we have

$$\begin{aligned} \beta_0 &= -\frac{\sqrt{2}}{4}a\frac{\sqrt{Q_1 Q_5}}{r^2}\cos 2\theta \\ \mu &= \frac{\sqrt{2}}{4}a\frac{\sqrt{Q_1 Q_5}}{r^2}(1 + 2\sin^2\chi) \\ \frac{\beta_0 + \mu}{\sqrt{2}} &= \frac{1}{2r^2}\left[\frac{a\sqrt{Q_1 Q_5}}{2}(1 + 2\sin^2\chi) - \frac{a\sqrt{Q_1 Q_5}}{2}\cos 2\theta\right] \end{aligned} \tag{5.177}$$

By comparison the angular momenta are

$$\begin{aligned} J &= \frac{a}{2}\sqrt{Q_1 Q_5}(1 + 2\sin^2\chi) \\ \tilde{J} &= \frac{a}{2}\sqrt{Q_1 Q_5} \end{aligned} \tag{5.178}$$

The quantized angular momenta are given by

$$\begin{aligned} j &= \frac{VR}{g_s \alpha^4} J \\ \tilde{j} &= \frac{VR}{g_s \alpha^4} \tilde{J} \end{aligned} \tag{5.179}$$

Using these two equations, together with (5.178) and (3.28) again we obtain

$$j = \frac{1}{2}N_1 N_5 (1 + 2\sin^2\chi) = \frac{N}{2}(1 + 2\sin^2\chi) \tag{5.180}$$

$$\tilde{j}=\frac{N_1N_5}{2}=\frac{N}{2} \tag{5.181}$$

We note that from (5.174), (5.180), (5.181),we have

$$j-\tilde{j}=\frac{N_P}{2} \tag{5.182}$$

We now turn to the CFT side of the problem.

5.6.2 $n=1$: CFT side

We now recalculate the charges computed in the previous section, starting from the D1-D5 CFT. We anticipate that the charges will perfectly match.
What we wish to do is calculate the average values of the operators L_0, $\bar{L}_0$ J_0^3, $\bar{J}_0^3$ on the state

$$|\psi_\chi\rangle=e^{\chi(J^+_{-1}-J^-_1)}|\psi_0\rangle \tag{5.183}$$

We start by calculating the average values of $\bar{L}_0$ and $\bar{J}_0^3$, which are really easy. We compute them in the R sector, in which the state $|\psi_\chi\rangle$ is expressed as[4] [47]:

$$|\psi_\chi\rangle_R=e^{\chi(J^+_{-2}-J^-_2)}|\psi_0\rangle_R \tag{5.184}$$

Where $|\psi_0\rangle_R$ is a state in which all the strands are in the state $|{+}{+}\rangle_1$. Since the left operators commute with the right operators, $\bar{L}_0|\psi_0\rangle_R=\frac{N}{4}|\psi_0\rangle_R$ and $\bar{J}_0^3|\psi_0\rangle_R=\frac{N}{2}|\psi_0\rangle_R$ we have

$$\begin{aligned}\langle\psi_\chi|_R\,\bar{L}_0\,|\psi_\chi\rangle_R&=\langle\psi_0|_R\left[e^{\chi(J^+_{-2}-J^-_2)}\right]^\dagger\bar{L}_0e^{\chi(J^+_{-2}-J^-_2)}|\psi_0\rangle_R=\\&=\langle\psi_0|_R\left[e^{\chi(J^+_{-2}-J^-_2)}\right]^\dagger e^{\chi(J^+_{-2}-J^-_2)}\bar{L}_0|\psi_0\rangle_R=\\&=\frac{N}{4}\langle\psi_0|_R\left[e^{\chi(J^+_{-2}-J^-_2)}\right]^\dagger e^{\chi(J^+_{-2}-J^-_2)}|\psi_0\rangle_R=\frac{N}{4}\end{aligned} \tag{5.185}$$

In the same exact way we can find

$$\langle\psi_\chi|_R\,\bar{J}_0^3\,|\psi_\chi\rangle_R=\frac{N}{2} \tag{5.186}$$

This coincides with the result in equation (5.181). We now turn to the more complicated computations, which involve J_0^3 and L_0. We perform the calculations in the NS sector, and then, after we find the results, we transpose them to the R sector. We start by reporting two identities that will prove extremely useful [47]:

$$e^{\chi(J^+_{-1}-J^-_1)}=e^{\tan\chi J^+_{-1}}\big(\cos\chi\big)^{\hat{N}-2J_0^3}e^{-\tan\chi J^-_1} \tag{5.187}$$

$$e^{\gamma J^-_1}e^{\alpha J^+_{-1}}=e^{\frac{\alpha}{1+\alpha\gamma}J^+_{-1}}e^{2\log(1+\alpha\gamma)\left(\frac{\hat{N}}{2}-J_0^3\right)}e^{\frac{\gamma}{1+\alpha\gamma}J^-_1} \tag{5.188}$$

We start by rewriting the state $|\psi_\chi\rangle$. We have

$$\begin{aligned}|\psi_\chi\rangle&=e^{\chi(J^+_{-1}-J^-_1)}|\psi_0\rangle\overset{(5.187)}{=}e^{\tan\chi J^+_{-1}}\big(\cos\chi\big)^{\hat{N}-2J_0^3}e^{-\tan\chi J^-_1}|\psi_0\rangle=\\&=e^{\tan\chi J^+_{-1}}\big(\cos\chi\big)^{\hat{N}-2J_0^3}|\psi_0\rangle=e^{\tan\chi J^+_{-1}}\big(\cos\chi\big)^{N}|\psi_0\rangle=\\&=\big(\cos\chi\big)^{N}e^{\tan\chi J^+_{-1}}|\psi_0\rangle\end{aligned} \tag{5.189}$$

[4] For convenience we write an R subscript to label the states and operators in the R sector, while the objects with no subscript are in the NS sector.

Where we used $J_1^- |\psi_0\rangle = 0$ from the first to the second line, and $J_0^3 |\psi_0\rangle = 0$ in the following passage. Now, the NS vacuum, as previously said, has $h = m = 0$, therefore

$$L_0 |\psi_0\rangle = 0 \tag{5.190}$$
$$J_0^3 |\psi_0\rangle = 0 \tag{5.191}$$

We now have all the elements to compute the mean values of L_0 and J_0^3 on the state $|\psi_\chi\rangle$. We start from L_0, following a completely analogous procedure to that done in [47]:

$$\begin{aligned}
\langle\psi_\chi| L_0 |\psi_\chi\rangle &\overset{(5.189)}{=} \big(\cos\chi\big)^{2N} \langle\psi_0| e^{\tan\chi J_1^-} L_0 e^{\tan\chi J_{-1}^+} |\psi_0\rangle = \\
&= \big(\cos\chi\big)^{2N}\Big[\langle\psi_0| e^{\tan\chi J_1^-}\big[L_0, e^{\tan\chi J_{-1}^+}\big]|\psi_0\rangle + \langle\psi_0| e^{\tan\chi J_1^-} e^{\tan\chi J_{-1}^+} L_0 |\psi_0\rangle\Big] \overset{(5.190)}{=} \\
&= \big(\cos\chi\big)^{2N} \langle\psi_0| e^{\tan\chi J_1^-}\big[L_0, e^{\tan\chi J_{-1}^+}\big]|\psi_0\rangle
\end{aligned}$$

We can now realize the operator J_{-1}^+ as a multiplicative operator $J_{-1}^+ \to \xi$ and L_0 as a differential operator $L_0 \to \xi\partial_\xi$. This choice is consistent with the commutator $[L_0, J_{-1}^+] = J_{-1}^+$: it is in fact fairly easy to check that $[\xi\partial_\xi, \xi] = \xi$. This allows us to rewrite the commutator $\big[L_0, e^{\tan\chi J_{-1}^+}\big]$ as

$$\big[L_0, e^{\tan\chi J_{-1}^+}\big] \to \alpha\partial_\alpha e^{\alpha J_{-1}^+} \tag{5.192}$$

where $\alpha = \tan\chi$. In fact, if $f(\xi)$ is a test function we have

$$\begin{aligned}
\big[\xi\partial_\xi, e^{\alpha\xi}\big] f(\xi) &= \xi\partial_\xi(e^{\alpha\xi} f(\xi)) - e^{\alpha\xi}\xi\partial_\xi(f(\xi)) = \\
&= \xi\alpha e^{\alpha\xi} f(\xi) + e^{\alpha\xi}\xi\partial_\xi(f(\xi)) = \\
&= \xi\alpha e^{\alpha\xi} f(\xi) = \alpha\partial_\alpha(e^{\alpha\xi}) f(\xi)
\end{aligned}$$

Moving on, we have

$$\begin{aligned}
\langle\psi_\chi| L_0 |\psi_\chi\rangle &= \big(\cos\chi\big)^{2N} \langle\psi_0| e^{\tan\chi J_1^-}\big[L_0, e^{\tan\chi J_{-1}^+}\big]|\psi_0\rangle = \\
&= \big(\cos\chi\big)^{2N} \langle\psi_0| e^{\gamma J_1^-}\alpha\partial_\alpha(e^{\alpha J_{-1}^+})|\psi_0\rangle\big|_{\alpha,\gamma=\tan\chi} = \\
&= \big(\cos\chi\big)^{2N}\alpha\partial_\alpha \langle\psi_0| e^{\gamma J_1^-} e^{\alpha J_{-1}^+}|\psi_0\rangle\big|_{\alpha,\gamma=\tan\chi} \overset{(5.188)}{=} \\
&= \big(\cos\chi\big)^{2N}\alpha\partial_\alpha \langle\psi_0| e^{\frac{\alpha}{1+\alpha\gamma}J_{-1}^+} e^{2\log(1+\alpha\gamma)\left(\frac{\hat N}{2}-J_0^3\right)} e^{\frac{\gamma}{1+\alpha\gamma}J_1^-}|\psi_0\rangle\big|_{\alpha,\gamma=\tan\chi} = \\
&= \big(\cos\chi\big)^{2N}\alpha\partial_\alpha \langle\psi_0| e^{2\log(1+\alpha\gamma)\left(\frac{\hat N}{2}-J_0^3\right)}|\psi_0\rangle\big|_{\alpha,\gamma=\tan\chi} = \\
&= \big(\cos\chi\big)^{2N}\alpha\partial_\alpha\big[e^{N\log(1+\alpha\gamma)}\big]\langle\psi_0|\psi_0\rangle\big|_{\alpha,\gamma=\tan\chi} = \\
&= \big(\cos\chi\big)^{2N}\alpha e^{N\log(1+\alpha\gamma)} N\frac{\gamma}{1+\alpha\gamma}\big|_{\alpha,\gamma=\tan\chi} = \\
&= \big(\cos\chi\big)^{2N}\big(1+\tan^2\chi\big)^N\frac{\tan^2\chi}{1+\tan^2\chi}N = \\
&= N\sin^2\chi
\end{aligned}$$

We can repeat the exact same calculation for the operator J_0^3. This works because $J_0^3|\psi_0\rangle = 0$ and when we realize it as a multiplicative operator we have $J_0^3 \to \xi$, exactly like L_0, because of the commutator $[J_0^3, J_{-1}^+] = J_{-1}^+$. To sum up, in the NS sector, we have

$$\langle\psi_\chi| L_0 |\psi_\chi\rangle = N\sin^2\chi \tag{5.193}$$

$$\langle\psi_\chi| J_0^3 |\psi_\chi\rangle = N \sin^2\chi \tag{5.194}$$

We now move to the R sector. We report the formulas that link the operators in the different sectors, originally found in [47]

$$\begin{aligned}(J_R^\pm)_n &= (J^\pm)_{n\pm 1}\\ (J_R^3)_n &= J_n^3 + \frac{c}{12}\delta_{n,0}\\ (L_R)_n &= L_n + J_n^3 + \frac{c}{24}\delta_{n,0}\end{aligned} \tag{5.195}$$

Keeping in mind that $c = 6N$ we have

$$\langle\psi_\chi|_R\, (J_R)_0^3 \,|\psi_\chi\rangle_R = \langle\psi_\chi| J_0^3 |\psi_\chi\rangle + \frac{N}{2} \overset{(5.194)}{=} N\left(\sin^2\chi + \frac{1}{2}\right) = \frac{N}{2}\big(1 + 2\sin^2\chi\big) \tag{5.196}$$

perfectly matching equation (5.180). Finally, the last equation in (5.195) yields

$$\langle\psi_\chi|_R\, (L_R)_0 \,|\psi_\chi\rangle_R = \langle\psi_\chi| L_0 |\psi_\chi\rangle + \langle\psi_\chi| J_0^3 |\psi_\chi\rangle + \frac{N}{4} = 2N\sin^2\chi + \frac{N}{4} \tag{5.197}$$

We have $(N_P)_R = (L_R)_0 - (\bar{L}_R)_0$. Using equations (5.197) and (5.185) we finally have

$$\begin{aligned}\langle\psi_\chi|_R\, (N_P)_R \,|\psi_\chi\rangle_R &= \langle\psi_\chi|_R\, (L_R)_0 \,|\psi_\chi\rangle_R - \langle\psi_\chi|_R\, (\bar{L}_R)_0 \,|\psi_\chi\rangle_R = \\ &= 2N\sin^2\chi + \frac{N}{4} - \frac{N}{4} = 2N\sin^2\chi\end{aligned} \tag{5.198}$$

Again, this perfectly matches the result found from the gravity side.

5.6.3 $n > 1$: Gravity side

We repeat the calculation of the charges in the case $n > 1$. On the gravity side, the computation of N_P, J and $\bar{J}$ is the same as in the case $n = 1$.
Q_P can be read by $\mathcal{F}$, given in equation (5.167). By applying again formula (5.172) and substituting with the quantized charges we obtain

$$N_P = n(n+1)N\sin^2\chi \tag{5.199}$$

Going on, we have the coefficients β_0 and μ that are computed from (5.168) and (5.169) respectively. We have:

$$\begin{aligned}\beta_0 &= \frac{\sqrt{2}}{4}\frac{a\sqrt{Q_1Q_5}}{r^2}(\sin^2\theta - \cos^2\theta)\\ \mu &= \frac{\sqrt{2}}{4}\frac{a\sqrt{Q_1Q_5}}{r^2}(1 + 2n\sin^2\chi)\\ \frac{\beta_0+\mu}{\sqrt{2}} &= \frac{1}{2r^2}\left[\frac{\sqrt{2}}{2}a\sqrt{Q_1Q_5}(1+2n\sin^2\chi) - \frac{a}{2}\sqrt{Q_1Q_5}\cos 2\theta\right]\end{aligned} \tag{5.200}$$

From this we can read off J and $\bar{J}$. After substituting with the quantized the charges we find

$$j = \frac{N}{2}(1 + 2n\sin^2\chi) \tag{5.201}$$

$$\bar{j} = \frac{N}{2} \tag{5.202}$$

again, we have the relation

$$j - \bar{j} = \frac{N_P}{n+1} \tag{5.203}$$

5.6.4 $n > 1$: CFT side

The computation from the CFT side is again really similar to the one for $n = 1$. However, we need to generalize some results.
Let us take a look at the identities (5.187) and (5.188). We need to extend them to the case $n > 1$. We try to guess them and then verify that our result is indeed correct. First, we rewrite them by using the commutator

$$[J_1^-, J_{-1}^+] = N - 2J_0^3 \tag{5.204}$$

which can be obtained by (4.56) substituting $J^{\pm}_{\mp 1}$ with $J^1_{\mp 1} \pm iJ^2_{\mp 1}$. We have

$$e^{\chi(J_{-1}^+ - J_1^-)} = e^{\tan\chi J_{-1}^+}\big(\cos\chi\big)^{[J_1^-, J_{-1}^+]} e^{-\tan\chi J_1^-} \tag{5.205}$$

$$e^{\gamma J_1^-} e^{\alpha J_{-1}^+} = e^{\frac{\alpha}{1+\alpha\gamma} J_{-1}^+} e^{\log(1+\alpha\gamma)[J_1^-, J_{-1}^+]} e^{\frac{\gamma}{1+\alpha\gamma} J_1^-} \tag{5.206}$$

In this form, we can attempt a generalization by guessing

$$e^{\chi(J_{-n}^+ - J_n^-)} = e^{\tan\chi J_{-n}^+}\big(\cos\chi\big)^{[J_n^-, J_{-n}^+]} e^{-\tan\chi J_n^-} \tag{5.207}$$

$$e^{\gamma J_n^-} e^{\alpha J_{-n}^+} = e^{\frac{\alpha}{1+\alpha\gamma} J_{-n}^+} e^{\log(1+\alpha\gamma)[J_n^-, J_{-n}^+]} e^{\frac{\gamma}{1+\alpha\gamma} J_n^-} \tag{5.208}$$

We have to evaluate the commutator, but this is easy, in fact we apply (4.56) again to get

$$\begin{aligned}
[J_n^-, J_{-n}^+] &= [J_n^1 - iJ_n^2, J_{-n}^1 + iJ_{-n}^2] = \\
&= [J_n^1, J_{-n}^1] + i[J_n^1, J_{-n}^2] - i[J_n^2, J_{-n}^1] + [J_n^2, J_{-n}^2] = \\
&= n\frac{N}{2} + i(i\epsilon^{12c} J_0^c) - i(i\epsilon^{21c} J_0^c) + n\frac{N}{2} = \\
&= nN - \epsilon^{123} J_0^3 + \epsilon^{213} J_0^3 = \\
&= nN - 2J_0^3
\end{aligned} \tag{5.209}$$

Then, our identities in the case $n > 1$ are

$$e^{\chi(J_{-n}^+ - J_n^-)} = e^{\tan\chi J_{-n}^+}\big(\cos\chi\big)^{nN - 2J_3^0} e^{-\tan\chi J_n^-} \tag{5.210}$$

$$e^{\gamma J_n^-} e^{\alpha J_{-n}^+} = e^{\frac{\alpha}{1+\alpha\gamma} J_{-n}^+} e^{\log(1+\alpha\gamma)(nN - 2J_3^0)} e^{\frac{\gamma}{1+\alpha\gamma} J_n^-} \tag{5.211}$$

These are correct because they will yield us the correct charges that perfectly match the ones found from the gravity side. Another way to check these would be, for example, by expanding them in χ, order by order. The most correct way to deduce these identities, however, is by complexifying the SU(2) algebra [47].
Additionally, there is a difference in the realization of the commutators. In the $n > 1$ case we have

$$[L_0, J_{-n}^+] = nJ_{-n}^+ \tag{5.212}$$

$$[J_0^3, J_{-n}^+] = J_{-n}^+ \tag{5.213}$$

These can be checked by using (4.56) and (4.59). The first commutator is realized by choosing $L_0 \to \xi\partial_\xi$ and $J_{-n}^+ \to \xi^n$, the second one is the same as in the $n = 1$ case: $J_0^3 \to \xi\partial_\xi$ and $J_{-n}^+ \to \xi$. This will allow us to write

$$[L_0, e^{\tan\chi J_{-n}^+}] \to n\alpha\partial_\alpha e^{\alpha J_{-n}^+} \tag{5.214}$$

$$[J_0^3, e^{\tan\chi J_{-n}^+}] \to \alpha\partial_\alpha e^{\alpha J_{-n}^+} \tag{5.215}$$

With $\alpha = \tan\chi$.
Now that this has been taken care of, we can repeat the same calculations we did in the previous section. First of all, thanks to (5.210), we have, on the NS sector,

$$|\psi_\chi\rangle = \big(\cos\chi\big)^{nN} e^{\tan\chi J_{-n}^+} |\psi_0\rangle \tag{5.216}$$

Then, keeping in mind that $J_n^- |\psi_0\rangle = 0$ for every n (this can be proven by using for example (4.59)), we have

$$\begin{aligned}
\langle\psi_\chi| L_0 |\psi_\chi\rangle &= \big(\cos\chi\big)^{2nN} \langle\psi_0| e^{\tan\chi J_n^-} \big[L_0, e^{\tan\chi J_{-n}^+}\big] |\psi_0\rangle \overset{(5.214)}{=} \\
&= n\big(\cos\chi\big)^{2nN} \langle\psi_0| e^{\gamma J_n^-} \alpha\partial_\alpha (e^{\alpha J_{-n}^+}) |\psi_0\rangle \big|_{\alpha,\gamma=\tan\chi} = \\
&= n\big(\cos\chi\big)^{2nN} \alpha\partial_\alpha \langle\psi_0| e^{\gamma J_n^-} e^{\alpha J_{-n}^+} |\psi_0\rangle \big|_{\alpha,\gamma=\tan\chi} \overset{(5.211)}{=} \\
&= n\big(\cos\chi\big)^{2nN} \alpha\partial_\alpha \big[e^{nN\log(1+\alpha\gamma)}\big] \langle\psi_0|\psi_0\rangle \big|_{\alpha,\gamma=\tan\chi} = \\
&= n\big(\cos\chi\big)^{2nN} \big(1+\tan^2\chi\big)^N \frac{\tan^2\chi}{1+\tan^2\chi} nN = \\
&= n^2 N \sin^2\chi
\end{aligned}$$

In a similar way we find, by using (5.215) instead of (5.214)

$$\langle\psi_\chi| J_0^3 |\psi_\chi\rangle = nN\sin^2\chi \tag{5.217}$$

Moreover, in the R sector we have

$$\langle\psi_\chi|_R (\bar{J}_0^3)_R |\psi_\chi\rangle_R = \frac{N}{2} \tag{5.218}$$

$$\langle\psi_\chi|_R (\bar{L}_0)_R |\psi_\chi\rangle_R = \frac{N}{4} \tag{5.219}$$

These two results are again trivial because the operators in the left and right sector commute. The final step is to transpose the L_0 and J_0^3 charges to the R sector. Therefore, we re-apply formulas (5.195) and obtain

$$\langle\psi_\chi|_R (J_R)_0^3 |\psi_\chi\rangle_R = \langle\psi_\chi| J_0^3 |\psi_\chi\rangle + \frac{N}{2} = nN\sin^2\chi + \frac{N}{2} = \frac{N}{2}(1+2n\sin^2\chi) \tag{5.220}$$

$$\begin{aligned}
\langle\psi_\chi|_R (N_P)_R |\psi_\chi\rangle_R &= \langle\psi_\chi|_R (L_0)_R - (\bar{L}_0)_R |\psi_\chi\rangle_R = \langle\psi_\chi|_R (L_0)_R |\psi_\chi\rangle_R - \frac{N}{4} = \\
&= \langle\psi_\chi| L_0 |\psi_\chi\rangle + \langle\psi_\chi| J_0^3 |\psi_\chi\rangle + \frac{N}{4} - \frac{N}{4} = n^2 N\sin^2\chi + nN\sin^2\chi = \\
&= n(n+1)N\sin^2\chi
\end{aligned} \tag{5.221}$$

As expected, a perfect match with (5.199) and (5.201).
Finally, we point out that the n dependence of N_P and j has a simple explanation. The operator $J_{-(n+1)}^+$ can act n times on $|\psi_0\rangle$, and raises N_P by a factor $n+1$, and j by a factor $+1$. That is the reason why N_P depends on $n(n+1)$ and j depends on n.

Conclusions

This work has involved many connected topics, let us briefly summarize them. We started from reviewing the laws of black hole thermodynamics, and saw how they eventually lead to the information paradox, when we take the Hawking radiation into account. We saw how we need some quantum gravity theory to face the problem, and hopefully solving it. After that, we introduced the basic aspects of String Theory and saw how the theory is quantized. We briefly outlined how to add fermions to the theory to obtain superstrings, and then described the five superstring theories known, together with M-theory. Then, we saw how the Supergravity theories are obtained by them, and introduced the S-duality and T-duality, describing the important role they play in this framework.

After introducing Supergravity, we briefly reviewed the fuzzball conjecture, explaining how it could solve the information paradox, and describe the microstates of a 2-charge black hole at the same time. A general solution was found, that represents such microstates. Nowadays we know only a limited class of microstates that represent the so-called $\frac{1}{8}$-BPS black hole, which possesses three charges. The aim of the thesis was to look for some potentially new microstates. To do that, we needed to introduce some more concepts.

We started from introducing Conformal Field Theories (CFT), which are a very powerful tool that constitutes a microscopic guide to construct the dual geometries, thanks to the AdS/CFT correspondence, that was briefly introduced and partially justified as well. After that, we introduced the D1-D5 CFT, which is the dual description of the 3-charge black hole. We outlined the untwisted sector of the theory, listing the relevant operators in it and their OPEs, and introduced the primary fields and saw why they are so important.

At this point we had introduced everything we needed and could start our calculations. First of all, we defined our starting state, which is made of $N_1 N_5$ strands all in the NS vacuum. The goal was to find a family of typical states, which capture a finite fraction of the entropy. We applied L_{-n} operators on the strands and showed that for singly wound strands, we get the correct entropy only in the black string regime. However, when the winding is of order $N_1 N_5$, we get the correct entropy in every regime. Then, we moved to the construction of the corresponding microstates in Supergravity, this time applying J^a_n instead of L_{-n}, because they are easier to perform calculations with. We applied the operator

$$e^{\chi(J^+_{-1}-J^-_1)} \tag{5.222}$$

and built the corresponding microstates. We showed that for $\chi = 0$ and $\chi = \frac{\pi}{2}$ we obtain known solutions, and the microstates corresponding to generic χ interpolate between them. We extracted the relevant charges from the generic solution from the gravity side and the CFT side and see that they coincide. We also applied the more generic operator

$$e^{\chi(J^+_{-n}-J^-_n)} \tag{5.223}$$

generalizing the solutions we obtained in the previous case.
The final aim of this work was to make a preliminary step that could allow us to find a larger class of microstates. In fact, the future aim is to carefully study the solutions we outlined, in order to obtain clues on how to generalize them. Our hope is to find more general solutions, which are not obtainable by applying coordinate transformations on simple states. The existence of such solutions is crucial to reproduce the whole entropy, as we have shown. One way to do that would be by rewriting the 4D metrics we have constructed, and in particular the 2-forms appearing in the equations (5.50), in the general form predicted by supersymmetry, to highlight structures that can be generalized.
In conclusion, the fuzzball proposal is a powerful idea that could potentially solve the information paradox, and also allow us to understand black holes at a much deeper level. However, substantial work still needs to be done to ensure its validity. For example, it would be important to understand how to coarse-grain geometries in order to reproduce the macroscopic properties of the black hole. Also, it would be crucial to find the whole set of microstate solutions. As we have seen in this work, we are still far from that.

Appendices

Appendix A

General Relativity Tools

In this Appendix, we briefly summarize some basic concepts in General Relativity. The goal is to give an intuitive understanding of what is going on, in order to follow the thesis, nothing more. For this reason, we will be deliberately vague on many topics that would need entire books to be explained in a full fashion.

A.1 Covariant derivative

The covariant derivative is a map that replaces the ordinary partial derivative. We need this replacement because the map provided by the partial derivative depends on the coordinate system used. The covariant derivative operator ∇ performs the function of the partial derivative, but in a way independent of coordinates. The covariant derivative can be defined on general tensors, but the definition is rather complicated. In this work we will only act with the covariant derivative on vectors V^μ and 1-forms ω_μ:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{A.1}$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \tag{A.2}$$

The matrices $\Gamma^\rho_{\mu\nu}$ are called Christoffel symbols and describe the covariant derivative. An important fact to point out is that the Christoffel symbols are not tensors.

A.2 The metric and the Einstein equations

The metric tensor $g_{\mu\nu}$, is one of the most important objects of the whole theory, because it contains all the information about the curvature of the spacetime. $g_{\mu\nu}$ is a symmetric tensor, from which we can calculate various quantities characterizing the curvature of the spacetime we work in. For a fixed $g_{\mu\nu}$, we can calculate the Christoffel symbols:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}) \tag{A.3}$$

the Riemann tensor:

$$R^\sigma_{\mu\rho\nu} = \partial_\rho \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\rho\lambda} - \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda} \tag{A.4}$$

the Ricci tensor:

$$Ric_{\mu\nu} = R^\sigma_{\mu\sigma\nu} \tag{A.5}$$

the scalar curvature:

$$\mathcal{R} = g^{\mu\nu} Ric_{\mu\nu} \tag{A.6}$$

and finally, the Einstein tensor:

$$G_{\mu\nu} = Ric_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} \tag{A.7}$$

The most important equations in General Relativity are the Einstein's equations, which can be obtained by varying the following action with respect to the metric $g^{\mu\nu}(x)$

$$S = \frac{c^4}{16\pi G} S_{EH} + S_M = \frac{c^4}{16\pi G}\int \mathcal{R}\sqrt{-g}\, d^4x + S_M \tag{A.8}$$

where the first term containing the scalar curvature is called Einstein-Hilbert action, and g is the determinant of the metric tensor. The Einstein equations are

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{A.9}$$

where the object at the right-hand side is the stress-energy tensor, defined by

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{A.10}$$

The Einstein's equations are a system of second-order differential equations, where the unknowns are the components of the metric tensor $g_{\mu\nu}$.

A.3 Geodesics

A geodesic is a curve that represents in some sense the shortest path between two points in the spacetime. The general equation defining the geodesic $x^\mu(t)$ can be written as

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{dt}\frac{dx^\sigma}{dt} = f(t)\frac{dx^\mu}{dt} \tag{A.11}$$

A parameter t is said to be affine if it is related to the proper time τ in the way

$$\tau \to t = a\tau + b \tag{A.12}$$

for some constants a and b. If t is affine, then the geodesic equation simplifies to

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{dt}\frac{dx^\sigma}{dt} = 0 \tag{A.13}$$

An important result in General Relativity is that particles in free fall move along geodesics.

A.4 The Schwarzschild solution

We say that a spacetime is stationary when the metric coefficients do not depend on time. A spacetime is static if it is stationary, and there are no off-diagonal terms in the metric. The

Schwarzschild solution is the unique static solution with spherical symmetry to the Einstein's equations in the vacuum ($T_{\mu\nu} = 0$):

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad \text{(A.14)}$$

This is one of the most important solutions to the Einstein's equations because it describes a black hole. We notice that the metric has a pathology in $r = 0$ and $r = 2GM$. $r = 2GM$ defines a spherical surface, called the event horizon of the black hole. The point at $r = 0$ is called a singularity. The important difference between the horizon and the singularity is that the horizon is a pathology of the chosen coordinate system. The singularity is not. In General Relativity, singularities are rather complicated objects to define. In a very intuitive way, we can see them as points or regions of the spacetime where the geodesics end abruptly.

A.5 The Kerr solution

This solution to the Einstein's equations in the vacuum describes a rotating black hole, carrying angular momentum per mass unit $a = \frac{J}{M}$. The metric, which is stationary but not static, is

$$\begin{aligned} ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \\ + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2 \end{aligned} \qquad \text{(A.15)}$$

where

$$\Delta(r) = r^2 - 2GMr + a^2 \qquad \text{(A.16)}$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta \qquad \text{(A.17)}$$

This black hole has a ring singularity defined by $\rho^2 = 0$. Moreover, there are two event horizons at $r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$.

A.6 Killing vectors

A Killing vector is a vector K^μ satisfying the equation

$$\nabla_\mu K_\nu - \nabla_\nu K_\mu = 0 \qquad \text{(A.18)}$$

Killing vectors are extremely important because they give clues about conserved quantities in the spacetime. If the metric is independent of some coordinate x^{σ_*}, the vector ∂_{σ_*} will satisfy the Killing equation.
In the Schwarzschild spacetime there are four Killing vectors: three corresponding to the conservation of angular momentum, and the fourth one corresponding to the conservation of the energy.
In the Kerr spacetime there are only two Killing vectors: one expresses the axial symmetry of the solution, the second one, again, is linked to energy conservation.
A Killing horizon is a null hypersurface defined by the vanishing of the norm of a Killing vector.

$$K^\mu K_\mu = 0 \qquad \text{(A.19)}$$

In the Kerr spacetime, there are two Killing horizons. The region between the two is called ergosphere. It can be shown that inside the ergosphere, no stationary observer is possible. Physically, this means that if someone ends up inside the ergosphere, they must move in the direction of rotation of the black hole.

Bibliography

[1] Jacob D. Bekenstein. Black holes and entropy. *Phys. Rev. D*, 7:2333–2346, Apr 1973.

[2] S. W. Hawking. Particle Creation by Black Holes. *Commun. Math. Phys.*, 43:199–220, 1975. [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[3] S.D. Mathur. The fuzzball proposal for black holes: an elementary review. *Fortschritte der Physik*, 53(7-8):793–827, Jul 2005.

[4] Paul Townsend. Black holes, 1997, https://arxiv.org/abs/gr-qc/9707012v1.

[5] Mathias Pielahn, Gabor Kunstatter, and Alex B. Nielsen. Dynamical surface gravity in spherically symmetric black hole formation. *Physical Review D*, 84(10), Nov 2011.

[6] Jacob D. Bekenstein. Generalized second law of thermodynamics in black hole physics. *Phys. Rev. D*, 9:3292–3300, 1974.

[7] Sean M. Carroll. *Spacetime and Geometry: an Introduction to General Relativity*. Pearson, 2004.

[8] Don N. Page. Particle emission rates from a black hole: Massless particles from an uncharged, nonrotating hole. *Physical Review. D, Particles Fields*, 13(2), 1 1976.

[9] Don N. Page. Information in black hole radiation. *Physical Review Letters*, 71(23):3743–3746, Dec 1993.

[10] Don N Page. Time dependence of hawking radiation entropy. *Journal of Cosmology and Astroparticle Physics*, 2013(09):028–028, Sep 2013.

[11] Hong Zhe Chen, Robert C. Myers, Dominik Neuenfeld, Ignacio A. Reyes, and Joshua Sandor. Quantum extremal islands made easy, part i: Entanglement on the brane, 2020.

[12] Hong Zhe Chen, Robert C. Myers, Dominik Neuenfeld, Ignacio A. Reyes, and Joshua Sandor. Quantum extremal islands made easy, part ii: Black holes on the brane, 2020.

[13] Juan Hernandez, Robert C. Myers, and Shan-Ming Ruan. Quantum extremal islands made easy. part iii. complexity on the brane. *Journal of High Energy Physics*, 2021(2), Feb 2021.

[14] P. Chen, Y.C. Ong, and D.-h. Yeom. Black hole remnants and the information loss paradox. *Physics Reports*, 603:1–45, Nov 2015.

[15] Barton Zwiebach. *Introduction to String Theory*. Cambridge University Press, 2009.

[16] Joseph Polchinski. *String Theory, Volume 1: An introduction to the bosonic string*. Cambridge University Press, 1998.

[17] Joseph Polchinski. *String Theory, Volume 2: Superstring theory and beyond*. Cambridge University Press, 1998.

[18] Antoine Van Proeyen Daniel Z. Freedman. *Supergravity*. Cambridge University Press, May 2012.

[19] F. Gliozzi, Joel Scherk, and David I. Olive. Supersymmetry, Supergravity Theories and the Dual Spinor Model. *Nucl. Phys. B*, 122:253–290, 1977.

[20] Horatiu Nastase. Introduction to supergravity, 2012.

[21] Leonard Susskind. Some speculations about black hole entropy in string theory. 10 1993.

[22] Luca Mattiello. Black hole microstates in string theory, 2014, http://tesi.cab.unipd.it/46492/1/Mattiello_Luca.pdf.

[23] G. H. Hardy and S. Ramanujan. Asymptotic Formulaæ in Combinatory Analysis. *Proceedings of the London Mathematical Society*, s2-17(1):75–115, 01 1918.

[24] Atish Dabholkar, Jerome P. Gauntlett, Jeffrey A. Harvey, and Daniel Waldram. Strings as solitons and black holes as strings. *Nuclear Physics B*, 474(1):85–121, Aug 1996.

[25] Curtis G. Callan, Juan M. Maldacena, and Amanda W. Peet. Extremal black holes as fundamental strings. *Nuclear Physics B*, 475(3):645–676, Sep 1996.

[26] Stefano Giusto and Samir D. Mathur. Geometry of d1–d5–p bound states. *Nuclear Physics B*, 729(1-2):203–220, Nov 2005.

[27] Iosif Bena, Stefano Giusto, Rodolfo Russo, Masaki Shigemori, and Nicholas P. Warner. Habemus superstratum! a constructive proof of the existence of superstrata. *Journal of High Energy Physics*, 2015(5), May 2015.

[28] David Tong. String theory, 2014, https://www.damtp.cam.ac.uk/user/tong/string.html.

[29] David Sénéchal Philippe di Francesco, Pierre Mathieu. *Conformal Field Theory*. Springer-Verlag New York, 1997.

[30] Steven G. Avery. Using the d1-d5 cft to understand black holes, 2010, https://arxiv.org/abs/1012.0072.

[31] Sami Rawash. A holographic description of black hole microstates, 2018, http://tesi.cab.unipd.it/59745/.

[32] Juan Maldacena. *International Journal of Theoretical Physics*, 38(4):1113–1133, 1997.

[33] Edward Witten. Anti de sitter space and holography, 1998, https://arxiv.org/abs/hep-th/9802150.

[34] Leonard Susskind. The world as a hologram. *Journal of Mathematical Physics*, 36(11):6377–6396, Nov 1995.

[35] Eric D'Hoker and Daniel Z. Freedman. Supersymmetric gauge theories and the AdS / CFT correspondence. In *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and EXTRA Dimensions*, 1 2002.

[36] Ofer Aharony, Steven S. Gubser, Juan Maldacena, Hirosi Ooguri, and Yaron Oz. Large n field theories, string theory and gravity. *Physics Reports*, 323(3-4):183–386, Jan 2000.

[37] Elias Kiritsis. *String Theory in a Nutshell*. Princeton University Press, 2007.

[38] Justin R. David, Gautam Mandal, and Spenta R. Wadia. Microscopic formulation of black holes in string theory. *Physics Reports*, 369(6):549–686, Nov 2002.

[39] Andrea Galliani. Holography for correlators in black hole microstates, 2018, http://paduaresearch.cab.unipd.it/11494/.

[40] Wolfgang Lerche, Cumrun Vafa, and Nicholas P. Warner. Chiral Rings in N=2 Superconformal Theories. *Nucl. Phys. B*, 324:427–474, 1989.

[41] Jan de Boer, Jan Manschot, Kyriakos Papadodimas, and Erik Verlinde. The chiral ring of ads3/cft2and the attractor mechanism. *Journal of High Energy Physics*, 2009(03):030–030, Mar 2009.

[42] M. Gunaydin, G. Sierra, and P. K. Townsend. The Unitary Supermultiplets of $d = 3$ Anti-de Sitter and $d = 2$ Conformal Superalgebras. *Nucl. Phys. B*, 274:429–447, 1986.

[43] Masaki Shigemori. Counting superstrata. *Journal of High Energy Physics*, 2019(10), Oct 2019.

[44] Daniel Mayerson and Masaki Shigemori. Counting d1-d5-p microstates in supergravity. *SciPost Physics*, 10(1), Jan 2021.

[45] Stefano Giusto, Luca Martucci, Michela Petrini, and Rodolfo Russo. 6d microstate geometries from 10d structures. *Nuclear Physics B*, 876(2):509–555, Nov 2013.

[46] Iosif Bena, Stefano Giusto, Masaki Shigemori, and Nicholas P. Warner. Supersymmetric solutions in six dimensions: a linear structure. *Journal of High Energy Physics*, 2012(3), Mar 2012.

[47] Stefano Giusto and Rodolfo Russo. Superdescendants of the d1d5 cft and their dual 3-charge geometries. *Journal of High Energy Physics*, 2014(3), Mar 2014.

[48] Robert C. Myers and M. J. Perry. Black Holes in Higher Dimensional Space-Times. *Annals Phys.*, 172:304, 1986.