• Fields

A field $\phi = \phi(x^\mu)$ is a function of space-time

Not in our case

The system is described by the L function, in general : $L = L(\phi, \partial_\mu \phi, \dots, \partial_\mu^{(n)} \phi, x^\mu)$

The eq. of motion come from the action :

$$S = \int L\, dt = \int dt \int d^3x\, \mathcal{L} = \int \underbrace{d^4x}_{\star}\, \mathcal{L}$$, $\mathcal{L}$ is the Lagrangian density

$\star$: The integration volume is invariant because $d^4x' = |\det \Lambda| d^4x = d^4x$

$\Rightarrow$ If S is invariant, $\mathcal{L}$ is invariant, $\partial_\mu \phi \partial^\mu \phi$ is the kinetic energy

We consider $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ because otherwise a Poinc. transf. would add a term $\Rightarrow$ in order to be Poinc. inv., $\mathcal{L}$ has to depend from x^μ only from the fields

Conjugated momenta to ϕ_i : $\boxed{\pi_i(x^\mu) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}}$

Hamiltonian density : $\boxed{\mathcal{H} = \mathcal{H}(\phi_i, \pi_i) = \pi_i \dot{\phi}_i - \mathcal{L}}$

Variation of the field: $\begin{cases} \delta\phi(\underline{x}, t) = 0 & \forall\, \underline{x}, t \in \Sigma \\ \delta\phi(\underline{x}, t_1) = \delta\phi(\underline{x}, t_2) = 0 & \forall\, \underline{x} \end{cases}$

Σ : surface of the volume where we integrate $S = \int d^4x\, \mathcal{L}$

$\phi_{i,\mu} = \partial_\mu \phi_i$

$$0 = \delta S = \int d^4x \left\{ \mathcal{L}(\phi', \partial_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) \right\} = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \delta\phi_{i,\mu} \right\} =$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \partial_\mu \delta\phi_i \right] = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \delta\phi_i \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \right) \delta\phi_i \right] =$$

$$= \int d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \right] + \underbrace{\cancel{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \delta\phi_i \right)}}_{\int = 0 \text{ because } \delta\phi_i = 0 \text{ on the surface}} \right\}$$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} = 0}$$ Euler-Lagrange equation

OBS : Motion eq.s. are not affected with a Lagrangian $\mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu$

• Nöther's theorem

Let $L = L(\phi_i, \partial_\mu \phi_i, x^\mu)$ and let's suppose that the system undergoes a certain continuous transformation

Transformation F : $\begin{cases} x^\mu \to x'^\mu = x^\mu + \delta x^\mu \\ \phi_i \to \phi_i'(x') \\ \mathcal{L} \to \mathcal{L}'(\phi_i'(x'), \phi'_{i,\mu}(x'), x') \end{cases}$, F is totally generic

$\Rightarrow S$ transforms : $S(V) \to S'(V') = \int d^4x' \, \mathcal{L}'(\phi', \partial_\mu \phi', x'^\mu)$

A symmetry is a transformation F such as $S'(V') = S(V)$

For the field $\phi_i(x)$ we have two possibilities:

- Geometric simmetry: affects points and functional form, $\phi_i(x) \to \tilde{\phi}_i(x')$
- Internal simmetry: affects only functional form

Example: Lorentz transformations

If ϕ is a scalar, $\phi'(x') = \phi(x)$,

but the field can have a structure $\Rightarrow \phi(x) \to \phi'(x') = \underset{\text{Linear operator}}{S(\Lambda)}\phi(x)$

$A^\mu(x^\mu)$ is a 4-vector and transforms as a 4-vector

Nöther theorem: if the system is invariant under a symmetry, for every param. of the symm. there is a quantity conserved

Variations of fields:

we remove it neglecting the II order

$$\Delta\phi_i := \phi_i'(x') - \phi_i(x) = \underbrace{\phi_i'(x') - \phi_i'(x)}_{\partial_\mu \cancel{\phi_i'}\,\delta x^\mu} + \underbrace{\phi_i'(x) - \phi(x)}_{\delta\phi_i} = \partial_\mu \phi_i \, \delta x^\mu + \delta\phi_i$$

$$\mathcal{L}'(\phi', \phi'_{,\mu}) - \mathcal{L}(\phi, \phi_{,\mu}, \dots) \simeq \delta\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\delta\phi_{i,\mu} + \partial_\mu\mathcal{L}\,\delta x^\mu \qquad \text{(in general)}$$

Now we impose $0 = \delta S = \int d^4x \, \mathcal{L}' - \int d^4x \, \mathcal{L}$, we have to take into account the jacobian of $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$: $\boxed{J = \frac{\partial x'^\mu}{\partial x^\mu} = 1 + \partial_\mu \delta x^\mu}$

$$0 = \int d^4x \left[\left(1+\partial_\mu \delta x^\mu\right)\mathcal{L}' - \mathcal{L}\right] = \int d^4x \left\{ \delta\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial \phi_{i,\mu}} + \partial_\mu \mathcal{L}\,\delta x^\mu + (\partial_\mu \delta x^\mu)\mathcal{L}\right\} =$$

$$= \int d^4x \left\{ \delta\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\delta\phi_i\right] - \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\right)\delta\phi_i + \partial_\mu\mathcal{L}\,\delta x^\mu + \partial_\mu\delta x^\mu \mathcal{L}\right\} =$$

$$= \int d^4x \left\{ \delta\mathcal{L} + \underbrace{\cancel{\left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\right]}}_{=0}\delta\phi_i + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\delta\phi_i + \mathcal{L}\,\delta x^\mu\right]\right\}$$

because of eq. of motion

$$= \int_V d^4x \left\{ \delta\mathcal{L} + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\delta\phi_i + \mathcal{L}\,\delta x^\mu\right]\right\}$$, d^4X is arbitrary $\Rightarrow$

$$\Rightarrow \boxed{\delta\mathcal{L} + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\delta\phi_i + \mathcal{L}\,\delta x^\mu\right] = 0}$$

The transf. is a simmetry $\Rightarrow$ the eq. of motion must not change

$\Rightarrow \delta\mathcal{L}$ cannot be more than $\boxed{\delta\mathcal{L} = \partial_\mu \delta\Omega^\mu}$, with $\delta\Omega^\mu$ which vanishes on the frontier of the integration domain

$\Rightarrow \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\delta\phi + \mathcal{L}\,\delta x^\mu + \delta\Omega^\mu\right] = 0 \Rightarrow \exists\, J^\mu$ such that $\boxed{\partial_\mu J^\mu = 0}^{\$}$,

where $J^\mu = \frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\delta\phi + \mathcal{L}\,\delta x^\mu + \delta\Omega^\mu$

The conservation of the current J^μ implies the conservation of the charge

We define $Q = \int_V d^3x\, J^0 \Rightarrow$

$$\Rightarrow \frac{dQ}{dt} = \int_V d^3x\, \partial_0 J^0 \overset{\$}{=} \int_V d^3x\, \partial_i J^i \overset{\text{Dir. th.}}{=} \int_{\partial V} d\Sigma\, \underline{J}\cdot\underline{n} = 0 \Rightarrow Q = \text{const}$$

If the Nöther's symmetries form a group, the group's algebra is induced on the charges

• Invariance under Lorentz transformations

Let's consider the case $\delta\Omega^\mu = 0$ ($\mathcal{L}$ = const) and the infinitesim. transformation

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^{\mu\nu} x_\nu$$

with $\varepsilon^{\mu\nu}$ antisymmetric $\Rightarrow \frac{4\cdot(4-1)}{2} = 6$ independent parameters

$\Rightarrow$ we expect 6 conserved quantities.

Let's consider a field $\phi^i(x)$, with i a Lorentz-index:

$$\boxed{\phi^i(x) \to S(\Lambda)^i_j \phi^j(x) \simeq \left(\delta^i_j - \frac{1}{2}\Sigma_{\nu\rho}\varepsilon^{\nu\rho}\right)^i_j \phi^j(x)}$$

- $\Sigma_{\nu\rho}$ are the **generators of the Lorentz group**, (or, better, a repr. of the generators in the base of the fields)
- $\varepsilon^{\nu\rho}$ represents the "angles" of rotation.

Overall, the transf. must satisfy

$$\boxed{\begin{aligned} \delta x^\mu &= \varepsilon^{\mu\nu} x_\nu \\ \Delta\phi^i &= -\frac{1}{2}\left(\Sigma_{\nu\rho}\varepsilon^{\nu\rho}\right)^i_j \phi^j \end{aligned}} \quad (\star)$$

The current is given by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\delta\phi^i + \mathcal{L}\delta x^\mu \overset{\odot,\text{ pag } 21}{=\!=} \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\left(\Delta\phi^i - \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\partial_\nu\phi^i\delta x^\nu\right) + \mathcal{L}\,\delta x^\mu$$

$$\overset{(\star)}{=} -\frac{1}{2}\left(\Sigma_{\nu\rho}\varepsilon^{\nu\rho}\right)^i_j \phi^j \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}} - \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\partial_\nu\phi^i\varepsilon^{\nu\rho}x_\rho + g^\mu_\nu\varepsilon^{\nu\rho}x_\rho\mathcal{L} =$$

$$= -\frac{1}{2}\left(\Sigma_{\nu\rho}\varepsilon^{\nu\rho}\right)^i_j \phi^j \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}} - \varepsilon^{\nu\rho}x_\rho\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\phi^i_{,\nu} - g^\mu_\nu\mathcal{L}\right) =$$

$$= -\frac{1}{2}\left(\Sigma_{\nu\rho}\varepsilon^{\nu\rho}\right)^i_j \phi^j \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}} - \varepsilon^{\nu\rho}x_\rho T^\mu_\nu$$

where we put $\boxed{T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\phi^i_{,\nu} - g^\mu_\nu\mathcal{L}}$, T^μ_ν is the **energy-momentum tensor**

$\varepsilon^{\nu e}$ is antisymmetric $\Rightarrow$ the only non-zero contribution to $\varepsilon^{\nu e} x_e T^\mu_\nu$ is given by the antisymmetric part of $x_e T^\mu_\nu$ (in ν and e)

$$\Rightarrow \quad J^\mu = -\frac{1}{2}(\Sigma_{\nu e}\,\varepsilon^{\nu e})^i_j\,\phi^j \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} + \frac{1}{2}\varepsilon^{\nu e}\left[x_e T^\mu_\nu - x_\nu T^\mu_e\right]$$

$$= \frac{1}{2}\varepsilon^{\nu e}\left\{-(\Sigma_{\nu e})^i_j\,\phi^j \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} + (x_e T^\mu_\nu - x_\nu T^\mu_e)\right\}$$

$$\Rightarrow \quad \boxed{J^\mu = \frac{1}{2}\varepsilon^{\nu e} M^\mu_{\nu e}}$$

Where $\boxed{M^\mu_{\nu e} = -(\Sigma_{\nu e})^i_j\,\phi^j \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} + (x_e T^\mu_\nu - x_\nu T^\mu_e)}$ is the generalized energy-momentum tensor

which owns 24 (4 in μ and 6 in $e\nu$) independent components

$M^\mu_{\nu e}$ is a generalization of the angular momentum, it is made up of

- an "orbital" momentum $x_e T^\mu_\nu - x_\nu T^\mu_e$ which comes from the action of the Lorentz group on space-time coordinates
- an intrinsic (spin) momentum $-\frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}}(\Sigma_{e\nu})^i_j\,\phi^j$ which comes from the action of the Lorentz group on spinorial components of the field

$$\partial_\mu J^\mu = 0 \overset{\varepsilon^{e\nu} = \text{const}}{\Longrightarrow} \partial_\mu M^\mu_{\nu e} = 0$$

$\Rightarrow$ 6 indep. charges $M_{\nu e} := \int d^3x\, M^0_{\nu e}$; $\frac{dM_{\nu e}}{dt} = 0$

25

Scalar field and 4 - momentum and angular momentum conservation

Let's consider a scalar field : $\phi'(x') = \phi(x) \Rightarrow \Delta\phi = 0$ and a translation by a const. 4-vector a^μ $\Rightarrow \begin{cases} \Delta\phi = 0 \\ \delta x^\mu = a^\mu \end{cases}$

$$\Rightarrow \delta\phi(x) = -\partial_\mu \phi(x) \delta x^\mu = -\partial_\mu \phi(x) a^\mu$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \delta\phi_i + \mathcal{L}\delta x^\mu = -\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \partial_\nu \phi_i a^\nu + \mathcal{L} a^\mu =$$

$$= -\left[\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \partial_\nu \phi_i - g^\mu_\nu \mathcal{L}\right] a^\nu = -T^\mu_\nu a^\nu$$

$$\partial_\mu J^\mu = 0 \xlongequal{a^\nu = \text{const}} \partial_\mu T^\mu_\nu = 0 \Rightarrow \begin{cases} \text{4 local conservation laws} \\ P_\nu = \int d^3x \, T^0_\nu \, , \ \boxed{\dot{P}_\nu = 0} \, * \end{cases}$$

In fact, the 4 eq. $\begin{cases} \partial_\mu T^\mu_0 = 0 \\ \vdots \\ \partial_\mu T^\mu_3 = 0 \end{cases}$ imply $\begin{cases} \partial_0 T^0_0 = \partial_i T^i_0 \\ \vdots \\ \partial_0 T^0_3 = \partial_i T^i_3 \end{cases}$ Integrating in d^3x $*$ is obtained

$$T^0_0 = \frac{\partial \mathcal{L}}{\partial \dot\phi_i} \dot\phi_i - \mathcal{L} = \mathcal{H} \ ; \ T^0_i = \frac{\partial \mathcal{L}}{\partial \dot\phi_k} \partial_i \phi_k = P_i$$

Instead of translations, we can consider proper Lor. transf:

$$J^\mu = \frac{1}{2} \varepsilon^{\ell\nu} \left[x_\ell T^\mu_\nu - x_\nu T^\mu_\nu\right] = \frac{1}{2} \varepsilon^{\ell\nu} M^\mu_{\ell\nu} \ ; \quad \partial_\mu J^\mu = 0 \Rightarrow \partial_\mu M^\mu_{\ell\nu} = 0 \ ;$$

Cons. quantities:

$$M^0_{ij} = \int d^3x \left[x_i T^0_j - x_j T^0_i\right] = \int d^3x \, (x_i \underbrace{p_j}_{\text{momentum density}} - x_j p_i) = \varepsilon_{ijk} L_k = \begin{pmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & L_1 & 0 \end{pmatrix}$$

$$M^0_{12} = x_1 P_2 - x_2 P_1 = L_3$$

• Lorentz and Poincaré symmetries in QFT

• ~~Groups~~ Lor and Poinc symm in QFT

Group := collection of objects where

i) $a, b \in G \Rightarrow ab \in G$
ii) $a(bc) = (ab)c$
iii) $\exists\, e \in G$ such that $ae = ea = a \quad \forall a \in G$
iv) $\exists\, a^{-1} \in G$ s.t. $aa^{-1} = a^{-1}a = \mathbb{1}$

If $ab = ba \Rightarrow$ G is abelian

Homomorphism := $\phi : G_1 \to G_2$ such tha $\forall g_1, g_2 \in G_1 \Rightarrow \phi(g_1)\phi(g_2) = \phi(g_1 g_2)$

• Group representation

Let V be a vectorial space, the general linear group of V, $GL(V)$ is the group of all invertible linear transformations $V \to V$,

A representation of a group G on the space V is a group homomorphism $D_R : G \to GL(V)$,
$g \in G \mapsto D_R(g) \in GL(V)$

such that $\begin{cases} D_R(g_1)\, D_R(g_2) = D_R(g_1 g_2) & \forall g_1, g_2 \in G \\ D_R(e) = \mathbb{1} \end{cases}$

A typical exampl of a representation is a matrix representation.

Two repr. D_1, D_2 are equivalent if there exists $T : T D_1(g) T^{-1} = D_2(g)$

S subspace of V is invariant $D_R(g)$ if $\forall x \in S$ and $\forall g \in G$ we have $D_R(g)(x) \in S$

- **Lie groups**

A Lie group is a group whose elements g depend in a continuous and differentiable way on a set of real parameters ϑ^a, $a=1,\dots,N$
$\Rightarrow g = g(\vartheta^a)$.

We choose the coordinates ϑ^a such that $g(0)=e$

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Two repr. D_1, D_2 are equivalent if there exists $T : T D_1(g) T^{-1} = D_2(g)$, when we change the representation, in general, the explicit form of $D_R(g)$ will change, but there exists a property of the Lie group that is indipendent from the representation, the Lie algebra.

Given the assumption of smoothness, we can consider ϑ^a infinitesimal id est, in the neighbourhood of the identity element, we have

$$\boxed{D_R(\vartheta) \simeq 1 + i \vartheta_a T_R^a}$$ (at the first order, because $D_R(0) = \mathbb{1}$)

Where $T_R^a = -i \left.\dfrac{\partial D_R}{\partial \vartheta_a}\right|_{\vartheta=0}$ which are called the generators of the group in the representation R

I can write every element of the group in term of the generators

Now let be the transformation that follows:

$$D_R(\vartheta + d\vartheta) = D_R(\vartheta) D_R(d\vartheta) \simeq D_R(\vartheta)(1 + i\, d\vartheta T_R) = D(\vartheta) + i\, d\vartheta T_R D(0)$$

- $D(\vartheta + d\vartheta) - D(\vartheta) \simeq i T_R\, d\vartheta$
- $D(\vartheta + d\vartheta) - D(\vartheta) \simeq \dfrac{dD}{d\vartheta}\, d\vartheta$

$$\Rightarrow \frac{dD}{d\vartheta} = i T_R \vartheta D(\vartheta) \Rightarrow D_R(\vartheta) = e^{i T_R \vartheta}$$

$\Rightarrow \boxed{D_R(g) = e^{iT_R g}} \Rightarrow R$ is a unitary representation.

How the generation should be satisfying group conditions?

D has to satisfy $D(g_1)D(g_2) = D(g_1 g_2)$,

and, for some $\delta_a(\alpha,\beta)$, $\boxed{e^{i\alpha_a T_R^a}\, e^{i\beta_a T_R^a} = e^{i\delta_a(\alpha,\beta) T_R^a}}$ $\star$

in general, $\delta_a \neq \alpha_a + \beta_a$

$$i\delta_c T_R^c \overset{\star}{=} \ln\Big[\Big(1 + i\alpha_a T^a + \tfrac{1}{2}(i\alpha_a T^a)^2 + \dots\Big)\Big(1 + i\beta_a T^a + \tfrac{1}{2}(i\beta_a T^a)^2 + \dots\Big)\Big]$$

infinitesimal transformations

$$= \ln\Big[1 + i(\alpha_a + \beta_a)T^a - \tfrac{1}{2}(\alpha_a T^a)^2 - \tfrac{1}{2}(\beta_a T^a)^2 - \alpha_a\beta_b T^a T^b + \dots\Big) \simeq$$

$$\simeq i(\alpha_a + \beta_a)T^a - \alpha_a\beta_b T^a T^b - \tfrac{1}{2}(\alpha_a T^a)^2 - \tfrac{1}{2}(\beta_a T^a)^2 + \tfrac{1}{2}(\alpha_a T^a + \beta_b T^b)^2$$

$$= i(\alpha_c + \beta_c)T^c - \tfrac{1}{2}\alpha_a\beta_b(T^a T^b - T^b T^a)$$

$$= i(\alpha_c + \beta_c)T^c - \tfrac{1}{2}\alpha_a\beta_b[T^a, T^b]$$

$$\Rightarrow \alpha_a\beta_b[T^a, T^b] = 2i(\alpha_c + \beta_c - \delta_c)T^c = i\gamma_c T^c$$

$\delta_c \propto a + b + \mathcal{O}((\alpha+\beta)^2)$

$\hookrightarrow \propto \alpha\beta$

Where $\gamma_c = f_c^{ab}\alpha_a\beta_b$ because $\uparrow$, then we can divide for $\alpha_a\beta_b$

$\Rightarrow \boxed{[T^a, T^b] = i f_c^{ab} T^c}$, the **Lie algebra of the group.**

$\downarrow$

Lie bracket

$\boxed{\text{OBS}}$ Even if the expression of T^a depends on the representation used, the structure constants f^{ab}_c are independent of the representation, in fact, if f^{ab}_c were to depend on R, γ^a and therefore δ^a would also depend on R, so it would be of the form $\delta^a_R(\alpha,\beta)$, then, from $*$, $g_1 g_2$ would depend on R which is not possible because $g_1 g_2$ is a property of the group.

$\boxed{\text{OBS}}$ If the structure constants f^{ab}_c are $\emptyset$, the generators commute and the group is abelian, in an abelian group, $\delta_a = \alpha_a + \beta_a$

Any d-dimensional abelian Lie algebra is isomorphic to the direct sum of d 1-dimensional abelian Lie algebras

$\Rightarrow$ All irreducible repr. of ab. groups are one-dimentional

In the study of the representations, an important role is played by the Casimir operators. These are operators constructed from the T_a that commute with all the T_a . In each irreducible representation, the Casimir operators are proportional to the identity matrix, and the proportionality constant labels the representation.

For example, the angular momentum algebra is $[J^i, J^j] = i\,\varepsilon^{ijk} J^k$ and the Casimir operator is $\underline{J}^2$.

On an irreducible representation, $J^2 = j(j+1)\mathbb{1}$, $j = 0, \frac{1}{2}, 1, \dots$

Example :

$R(\phi)$, $0 \leq \phi \leq 2\pi$; $|x'|^2 = x' R^+ R x = |x|^2$ $(R^+R = RR^+ = \mathbb{1})$
↳ Rotation $\det R = \pm 1$

$R(\phi_1) R(\phi_2) = R(\varphi_1 + \varphi_2)$

$R(d\varphi) \simeq 1 - iJd\varphi \Rightarrow R(\varphi) = e^{-i\varphi J}$

Representation ;

$D(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ what J is ?

$D(d\phi) = \begin{pmatrix} 1 & d\phi \\ -d\phi & 1 \end{pmatrix} = 1 - iJd\varphi \rightarrow J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$R(\varphi) = e^{-i\varphi J} \simeq 1 - iJ\phi - \frac{1}{2}\phi^2 J^2 = 1\cos\phi - iJ\sin\phi$

$R(\phi) = R(\phi \pm 2\pi)$

J has to be hermitian ,

Other represent.: $U(\phi) = f(J) \Rightarrow [U, J] = 0 \Rightarrow$ common basis

$J|\alpha\rangle = \alpha|\alpha\rangle$

$U|\alpha\rangle = e^{-i\phi\alpha}|\alpha\rangle \Rightarrow U(\phi) = U(\phi \pm 2\pi) \Rightarrow \underset{m}{\overset{\shortparallel}{\alpha}} \in \mathbb{Z}$

$J|m\rangle = m|m\rangle$ $\qquad m=0 \quad U^0(\phi) = 1$
$U|m\rangle = e^{-i\phi m}|m\rangle$ $\qquad m=1 \quad U^1(\phi) = e^{-i\phi}$
$\qquad m=-1 \quad U^{-1}(\phi) = e^{i\phi}$
$\qquad m = \pm 2$

$J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $J\hat{e}_\pm = \pm\hat{e}_\pm$; $R(\phi)\hat{e}_\pm = e^{\mp i\phi}\hat{e}_\pm$

08/10/2020

- Poincaré algebra

The element of Poincaré group is $T(\Lambda, a)$

$$\boxed{\begin{aligned} &T(\Lambda', a')\,T(\Lambda, a) = T(\Lambda'\Lambda, \Lambda' a + a') \\ &T^{-1}(\Lambda', a') = T(\Lambda'^{-1}, -\Lambda'^{-1} a') \end{aligned}}$$

Infinitesimal transformation : $\Lambda^\mu_{\ \nu} = \delta^\mu_\nu + \delta\omega^\mu_{\ \nu}$; $a^\mu \simeq \delta a^\mu$

$\Rightarrow T(\Lambda, a) \simeq T(\delta^\mu_\nu + \delta\omega^\mu_{\ \nu}, \delta a^\mu)$

- $T(\mathbb{1}, \delta a^\mu) \simeq \mathbb{1} - \delta a_\mu P^\mu$
  has to be

- $T(\delta^\mu_\nu + \delta\omega^\mu_{\ \nu}, 0) \simeq 1 - \frac{i}{2}\delta\omega_{\mu\nu} J^{\mu\nu}$

$\begin{cases} P^{\mu\dagger} = P^\mu \\ J^{\mu\nu\dagger} = J^{\mu\nu} \end{cases}$ because under T the physics doesn't change

Now we study the relation the generators have to satisfy

Let's see what happens if I take

$\left[T(\Lambda, b)\, T(\mathbb{1} + \delta\omega, \delta a)\, T^{-1}(\Lambda, b)\right]^{\star} = T(\Lambda(1+\delta\omega), \Lambda\delta a + b)\, T^{-1}(\Lambda, b) =$

$= T(\Lambda(1+\delta\omega), \Lambda\delta a + b)\, T(\Lambda^{-1}, -\Lambda^{-1} b) =$

$= T(\Lambda(1+\delta\omega)\Lambda^{-1}, -\Lambda(1+\delta\omega)\Lambda^{-1} b + \Lambda\delta a + b) =$

$= \underline{T(\Lambda(1+\delta\omega)\Lambda^{-1}, \Lambda\delta a - \Lambda\delta\omega\Lambda^{-1} b)}$

$T(\mathbb{1} + \delta\omega, \delta a) \simeq 1 - \frac{i}{2}\delta\omega_{\mu\nu} J^{\mu\nu} - i\delta a_\mu P^\mu$, we insert in $\star$

Expanding

$$\Rightarrow \star = \boxed{T(\Lambda, b)\left[1 - \frac{i}{2}\delta\omega_{\mu\nu} J^{\mu\nu} - i\delta a_\mu P^\mu\right] T^{-1}(\Lambda, b)}$$

$$\parallel$$

$$\boxed{1 - \frac{i}{2}(\Lambda\delta\omega\Lambda^{-1})_{\mu\nu} J^{\mu\nu} - i(\Lambda\delta a - \Lambda\delta\omega\Lambda^{-1} b)_\mu P^\mu}$$

$$\Rightarrow \begin{cases} \overbrace{T(\Lambda,b)\, J^{\mu\nu}\, T^{-1}(\Lambda,b)}^{\#} = \underbrace{\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\left(J^{\rho\sigma} - b^{\rho}p^{\sigma} + b^{\sigma}p^{\rho}\right)}_{\#} & (1) \\ T(\Lambda,b)\, P^{\mu}\, T^{-1}(\Lambda,b) = \Lambda^{\mu}{}_{\nu}\, P^{\nu} & (2) \end{cases}$$

Now let be $b^{\rho} = \delta a^{\rho}$ $

- From eq. (1)

$$\begin{cases} \# \overset{\$}{=} \left(1 - \frac{i}{2}\delta\omega_{\mu'\nu'}J^{\mu'\nu'} - i\,\delta a_{\mu'}P^{\mu'}\right) J^{\mu\nu} \left(1 - \frac{i}{2} \;---\; \right) \\ \qquad\qquad \| \\ \# \overset{\$}{=} \left(\delta^{\mu}_{\rho} + \delta\omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}_{\sigma} + \delta\omega^{\nu}{}_{\sigma}\right)\left(J^{\rho\sigma} - \delta a^{\rho\sigma} + \delta a^{\sigma} p^{\rho}\right) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \cancel{J^{\mu\nu}} - \frac{i}{2}\delta\omega_{\mu'\nu'}J^{\mu'\nu'}J^{\mu\nu} + \frac{i}{2}\delta\omega_{\mu'\nu'}J^{\mu\nu}J^{\mu'\nu'} - i\,\delta a_{\mu'}P^{\mu'}J^{\mu\nu} + i\,\delta a_{\mu'}J^{\mu\nu}P^{\mu'} \\ \qquad\qquad \| \\ \cancel{J^{\mu\nu}} + \delta a^{\nu}P^{\mu} - \delta a^{\mu}P^{\nu} + \delta\omega^{\nu}{}_{\sigma}J^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}J^{\rho\nu} \end{cases}$$

We adjust the equation:

$$\boxed{\begin{array}{c} \frac{i}{2}\delta\omega_{\mu'\nu'}\left[J^{\mu\nu}, J^{\mu'\nu'}\right] + i\,\delta a_{\mu'}\left[J^{\mu\nu}, P^{\mu'}\right] \\ \| \\ \delta\omega^{\nu}{}_{\sigma}J^{\mu\sigma} + \delta\omega^{\mu}{}_{\rho}J^{\rho\nu} + \delta a^{\nu}P^{\mu} - \delta a^{\mu}P^{\nu} \end{array}}$$

- From eq. (2)

$$\cancel{P^{\mu}} + i\,\delta a_{\mu'}\left[P^{\mu}, P^{\mu'}\right] + \frac{i}{2}\delta\omega_{\mu'\nu'}\left[P^{\mu}, J^{\mu'\nu'}\right] = \cancel{P^{\mu}} + \delta\omega^{\mu}{}_{\rho}P^{\rho}$$

$$\boxed{i\,\delta a_{\mu'}\left[P^{\mu}, P^{\mu'}\right] + \frac{i}{2}\delta\omega_{\mu'\nu'}\left[P^{\mu}, J^{\mu'\nu'}\right] = \delta\omega^{\mu}{}_{\rho}P^{\rho}}$$

33

Now we take corresponding terms

II $\Rightarrow$ $[P^\mu, P^\nu] = 0$

II, I $\Rightarrow$ $[P^\mu, J^{\lambda\sigma}] = i\,(P^\lambda \eta^{\mu\sigma} - P^\sigma \eta^{\mu\lambda})$

I $\Rightarrow$ $[J^{\mu\nu}, J^{\rho\sigma}] = i\,[J^{\nu\sigma}\eta^{\mu\rho} + J^{\rho\nu}\eta^{\sigma\mu} - J^{\mu\sigma}\eta^{\nu\rho} - J^{\rho\mu}\eta^{\sigma\nu}]$

This is the Lie algebra of the Poincaré group

p^0 is the generator of the translation in t-component

$p^0 = H$

$\underline{p} = (p^1, p^2, p^3)$

$\underline{J} = (\underbrace{J^{23}}_{=J^1}, \underbrace{J^{31}}_{=J^2}, \underbrace{J^{12}}_{=J^3})$

They commute with $p^0 = H$

$\underline{K} = (\underbrace{J^{10}}_{=K^1}, \underbrace{J^{20}}_{=K^2}, \underbrace{J^{30}}_{=K^3})$

$[J_i, J_j] = i\,\varepsilon_{ijk} J_k$, this is the algebra of the ang. mom. (closed algebra)

$[J_i, K_j] = i\,\varepsilon_{ijk} K_k$

$[\underline{K}_i, \underline{K}_j] = -i\,\varepsilon_{ijk} J_k$

$[J_i, p_j] = i\,\varepsilon_{ijk} p_k$

$[K_i, p_j] = i\,H\,\delta_{ij}$

$[J_i, H] = [p_i, H] = 0$

$[K_i, H] = i\,P_i$

The displacement is given by

$$U(\mathbb{1}, a) = e^{-i a_\mu P^\mu}$$

The rotation is given by

$$U(R, 0) = e^{-i\,\underline{J}\cdot\underline{\theta}}$$

For Riccardo:
Remember that
$U(\ ) = e^{-i\ldots}$
is simply the same
of the definition
at page 2,7

Can we recast the relations to understand the structure?

General element: $S(\Lambda) = e^{-\frac{i}{2}\omega_{\rho\sigma} J^{\rho\sigma}}$

we riparametrize: $\begin{cases} \eta_i = \frac{i}{2}\,\varepsilon_{ijm}\,\omega^{lm} \\ \xi_i = \omega_{0i} \end{cases}$ $\Rightarrow S(\Lambda) = e^{-i\eta_k J^k - i\,\xi_k K^k}$

The operators which commute with all the generators (the Casimir) are

$$J^2 - K^2 = \frac{1}{2} J_{\mu\nu} J^{\mu\nu}$$
$$\underline{J} \cdot \underline{K} = -\frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} J_{\mu\nu} J_{\rho\sigma}$$

We can take the complex combinations
$$\begin{cases} N^i = \frac{1}{2}(J^i + iK^i) \\ M^i = \frac{1}{2}(J^i - iK^i) \end{cases}$$

and we obtain the algebra
$$\begin{cases} [N^i, N^j] = i\,\varepsilon^{ijk} N^k \\ [M^i, M^j] = i\,\varepsilon^{ijk} M^k \\ [N^i, M^j] = 0 \end{cases}$$

In this space we decoupled J and K,

the casimir of the new representation are N^2 and M^2

Also we can have a representation as follows:

$\overset{n}{SU(2)} \otimes \overset{m}{SU(2)}$ $\quad n, m = 0, \frac{1}{2}, 1, \dots$, the basis is $\{|k,l\rangle = |n,k\rangle \otimes |m,l\rangle$

$(2n+1)(2m+1)$

$$J^3 |k,l\rangle = (N^3 + M^3)|k,l\rangle = (l+k)|k,l\rangle$$

$$K^3 |k,l\rangle = i(N^3 - M^3)|k,l\rangle$$

Let's consider $\phi(x): M^4 \to \mathbb{C}^n \quad (\phi_1(x), \dots, \phi_n(x))$,

how does it transform?

$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$

representation in this structure: $\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$

$\phi(x) \to \phi'(x') = S(\Lambda)\phi(x)$ I transform the vector nature of ϕ

Transformation on the structure of the field

- Trivial representation: $S(\Lambda) = \mathbb{1} \Rightarrow \phi(x) \to \phi'(x') = \phi(x)$, scalar field

Let be a vector $V^\mu = (V^0(x), \underline{V}(x))$, the vector representation transforms $V^\mu = (V^0(x), \underline{V}(x))$, the vector representation transforms with Λ^μ_ν.

$V^\mu(x) \to V'^\mu(x') = \Lambda^\mu_\nu V^\mu(x)$,

In this case, $S(\Lambda) = \Lambda^\mu_\nu$, $\quad (\Lambda^\mu_\nu)_x = \begin{pmatrix} \cosh\phi_x & \sinh\phi_x & 0 & 0 \\ \sinh\phi_x & \cosh\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Generators of rotation: come from $\quad R_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\vartheta_x & \sin\vartheta_x & 0 \\ 0 & -\sin\vartheta_x & \cos\vartheta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Generators: $t^a = -i \frac{d}{d\vartheta_a} S(\Lambda) \Big|_{\vartheta_a = 0}$

Gen. of boosts: $K_x = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; $K_y = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; $K_z = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Gen. of rotations: $J_x = -i \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$

- Spinorial representations

Tensor representations do not exhaust all interesting finite-dimensional repr. of the Lorentz group.

Let's consider the spatial rotations, i.e. the SO(3) subgroup of the Lorentz group, the tensor repres. of SO(3) are constructed with scalars ϕ, spatial vectors v^i, tensors T^{ij}, $i=1,2,3$.
In SO(3), a 2π rotation is equivalent to the identity.

$$R_{\underline{n}}(\vartheta) = e^{i\underline{J}\cdot\underline{\vartheta}} \quad ; \quad [J_i, J_j] = i\,\varepsilon_{ijk} J_k$$

SU(2) $\quad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{C}$; $\quad U^\dagger U = U U^\dagger = \mathbb{1} \qquad \det U = \underline{1}$

U can be param. in this way: $\quad U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

Spinor: 2-component objects, $\quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$; $\xi' = U\xi$

The generators of SU(2) are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \ \sigma_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \boxed{\left[\tfrac{1}{2}\sigma_i, \tfrac{1}{2}\sigma_j\right] = i\,\varepsilon_{ijk}\tfrac{1}{2}\sigma_k}$$

$$h = \underline{\sigma}\cdot\underline{x} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} , \quad \mathrm{Tr}\, h' = \mathrm{Tr}\,(U h U^\dagger) = \mathrm{Tr}\, h \,, \ -\det h = x^2+y^2+z^2$$

$$\det h = \det h' = -(x'^2 + y'^2 + z'^2)$$

If you rotate by 2π, in SO(2) you obtain $\mathbb{1}$, in SU(2) you obtain $-\mathbb{1}$, there is correspondence 1 to 1 between SO(3) and SU(2), e.v. are

$(2j+1)$; $j = +\frac{1}{2}, \dots$

$J_i = \frac{1}{2}\sigma_i$;

$[k_i, k_j] = -i\,\varepsilon_{ijk} J_k = -i\,\varepsilon_{ijk}\frac{1}{2}\sigma_k$; 2 different repr. of boosts : $K_i = \pm\frac{i}{2}\sigma_i$

$[J_i, k_j] = i\,\varepsilon_{ijk} K_k$

Right spinor : $\phi_R \rightarrow \phi_R' = e^{\frac{1}{2}\underline{\sigma}(\overset{\text{rotat.}}{\underline{\theta}} - i\overset{\text{boost}}{\underline{\phi}})}\phi_R$

Left spinor : $\phi_L \rightarrow \phi_L' = e^{\frac{1}{2}\underline{\sigma}(\underline{\theta} + i\underline{\phi})}\phi_L$

This 2 kind of spinors transf. in a peculiar way under parity,

$P\phi_R = \phi_L$, $P\phi_L = \phi_R$

If you have a theory which doesn't preserve parity, you need two representations

The **Dirac spinor** is a 4d vector which puts together R and L spinors in this way : $\psi = \begin{pmatrix}\phi_R \\ \phi_L\end{pmatrix}$

$$\psi \rightarrow \psi' = \begin{pmatrix} e^{\frac{1}{2}\underline{\sigma}(\underline{\theta} - i\underline{\phi})} & 0 \\ 0 & e^{\frac{1}{2}\underline{\sigma}(\underline{\theta} + i\underline{\phi})} \end{pmatrix}\psi$$

• Representation on one-particle states

The repr. of the Poinc. group on fields allows us to write Poincaré invariant lagrangians, as we will do next.

At the quantum level, we will want to understand how the concept of particle emerges from field quantization.

It is useful to see how the Poinc. group can be represented using as a basis the Hilbert space of a free particle.

Let's take a state as e.s. of operator P, $P^\mu \phi_{\rho,\sigma} = p^\mu \phi_{\rho,\sigma}$

Wigner theorem: on the Hilbert space any symmetry can be repr. by a unitary operator

$\Rightarrow$ In this base space, a Poinc. tr. is represented by a unitary matrix, and the generators J^i, K^i, P^i, H by hermitian operators.

The representations are labeled by the Casimir operators

- The first Casimir is $P^\mu P_\mu$, on a one-part. state $P^\mu P_\mu = m^2$
- The Pauli-Nubanski 4-vector is $W^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$, the second Casimir is given by $W^\mu W_\mu$ which commutes with P^μ and $J^{\mu\nu}$

$W^\mu W_\mu$ is inv. $\Rightarrow$ if $m \neq 0$, we can compute it in the system where the particle is at rest

$$\Rightarrow \begin{cases} P^\mu = (m, \underline{0}) \\ W^\mu = -\frac{1}{2} m \,\varepsilon^{\mu\nu\rho 0} J_{\nu\rho} \end{cases} ; \quad \begin{cases} W^0 = 0 \\ W^i = m J^i \end{cases} \Rightarrow W^\mu W_\mu = m^2 J^2 \quad \text{which is invariant}$$

$\Rightarrow$ We can label the particle with m and spin

$p^\mu = (m, \underline{0})$ is invariant under rotation, the rot. in this case is called transf. of little group.

If $m = 0$, we can move where $P^\mu = (E, 0, 0, E)$, not inv. under rot.

it is invariant under rotation on y axis, for example, J_3 is the comp of the spin in the direction of motion.

Another characteristic is the elicity, it combines spin and direction of motion.