# Electromagnetic interaction of fields

## Quantum Electrodynamics

We want to study the QED interaction between e.m. field and 1/2 spin field

In classical theory, the inter. between field and particle is described by

the minimal substitution $\boxed{P^\mu \to P^\mu - eA^\mu}$, in quantum mech.: $\boxed{i\partial^\mu \to i\partial^\mu - eA^\mu}$

Eqs of motion : $\underline{F} = e(\underline{E} + \frac{1}{c}\underline{v} \wedge \underline{H})$ ; $\frac{d}{dt} m\gamma\underline{v} = e\underline{E} + \frac{e}{c}\underline{v}\wedge\underline{H}$

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\phi - \frac{e}{c}\underline{v}\cdot\underline{A}$$

Moving to $\mathcal{H}$: $p = \frac{\partial\mathcal{L}}{\partial \underline{v}} = m\gamma\underline{v} + \frac{e}{c}\underline{A} \Rightarrow \mathcal{H} = p\cdot\underline{v} - \mathcal{L} = m\gamma c^2 + e\phi$

Free particle: $\left(\frac{H}{c}\right)^2 + |p|^2 = m^2c^2$,

we move to interacting particle : $\left(\frac{H-e\phi}{c}\right)^2 + \left(p - \frac{e}{c}\underline{A}\right)^2 = m^2c^2$

- Lagrangian density

The free theory which describes photons and electrons is obtained by combining the Maxwell Lagrangian with the Dirac Lagrangian

$$\mathcal{L}_0 = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{*} + \bar{\Psi}(\not{\partial} - m)\Psi$$

We want to move from free part. Lagrangian to obtain interacting $\mathcal{L}$, to do so, we use the quantum form minimal substitution : $i\partial^\mu \to i\partial^\mu - eA^\mu$

$$\Rightarrow \mathcal{L} = \bar{\Psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \bar{\Psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \underbrace{- eA_\mu \underbrace{\bar{\Psi}\gamma^\mu\psi}_{\text{current}}}_{\text{interaction}}$$

$$\Rightarrow \boxed{\mathcal{L} = \mathcal{L}_0 - eA_\mu\bar{\Psi}\gamma^\mu\psi}$$

## Gauge transformation and gauge principle

Let's see what happens with the gauge $A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda(x)$

invariant

$$\Rightarrow \mathcal{L}' = \mathcal{L}_0 - e\bar{\psi}\gamma'_\mu \psi A^\mu + e\bar{\psi}\gamma_\mu \psi\, \partial^\mu \Lambda(x) = \mathcal{L} + e\bar{\psi}\gamma_\mu\psi\,\partial^\mu\Lambda(x)$$

$\Rightarrow \mathcal{L}$ not inv under gauge, but there is still a symmetry.

We can try to transform the field by a phase dependent on the point:

$$\mathcal{L} = \psi(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\psi$$

$$\boxed{\psi(x) \to e^{-ie\Lambda(x)}\psi(x)}$$

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \bar{\psi}e^{ie\Lambda(x)}(i\not{\partial} - m)\, e^{-ie\Lambda(x)}\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma_\mu\psi A^\mu$$

$$= \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\not{A}\psi + \bar{\psi}e^{ie\Lambda}(i\not{\partial}e^{-ie\Lambda})\psi$$

$\Rightarrow \mathcal{L} \to \mathcal{L} - e\bar{\psi}\gamma_\mu\psi\partial^\mu\Lambda(x)$, it is exactly $-$ the term of gauge tr

$\Rightarrow$ If you use the same $\Lambda$ of gauge, $\mathcal{L}$ inv!

$$\Rightarrow \mathcal{L} \text{ invariant under local gauge: } \begin{cases} \psi(x) \to \psi'(x) = e^{-ie\Lambda(x)}\psi(x) \\ \bar{\psi}'(x) = \bar{\psi}e^{ie\Lambda(x)} \\ A^\mu \to A'^\mu = A^\mu + \partial^\mu\Lambda(x) \end{cases}$$

**Covariant derivative**: $D^\mu := \partial^\mu + ieA^\mu \Rightarrow$

$$\Rightarrow D^\mu\psi \to D'^\mu\psi' = (\partial^\mu + ieA^\mu + ie\partial^\mu\Lambda)e^{-ie\Lambda(x)}\psi = e^{-ie\Lambda(x)}D^\mu\psi \Rightarrow$$

$\Rightarrow D$ transforms like the field.

$$\Rightarrow \boxed{\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ free-matter field, invariant $\xrightarrow{\text{local phase}}$ $\mathcal{L}$ not anymore invariant

$A^\mu$ is a connection that makes $\mathcal{L}$ invariant under a local phase

$$U = e^{-ie\Lambda(x)} \simeq 1 - i\underset{\text{generator}}{e}\,\underset{\text{parameter}}{\Lambda} \Rightarrow U(1) \text{ local invariance}$$

In order to have an inv $\mathcal{L}$, we can look to more complicated groups

- <u>Quantization of the interaction lagrangian</u>

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$ , $\mathcal{L}_{int} = -e\bar{\psi}\gamma_\mu\psi A^\mu$

$$\begin{cases} \pi_\psi = \dfrac{\partial\mathcal{L}}{\partial\dot{\psi}} = \dfrac{\partial}{\partial\dot{\psi}}(\mathcal{L}_0 + \mathcal{L}_{int}) = \dfrac{\partial\mathcal{L}_0}{\partial\dot{\psi}} = i\psi^\dagger \\ \pi^\mu = \dfrac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = F^{\mu 0} - g^\mu_0(\partial_\lambda A^\lambda) \end{cases} ; \quad H = \overbrace{\pi_i\dot{\phi}^i - \mathcal{L}_0}^{H_0} \underbrace{-\mathcal{L}_{int}}_{+H_{int}} , \quad H_{int} = -\mathcal{L}_{int}$$

Minimal prescription, complications: (non derivative interactions?)

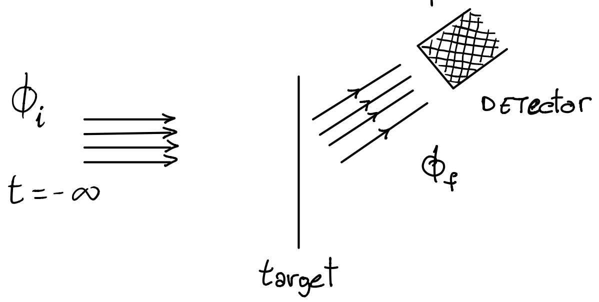
$$\mathcal{L} = \left[(\partial_\mu + ieA_\mu)\phi\right]^\dagger\left[(\partial^\mu + ieA^\mu)\phi\right] - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= \mathcal{L}_0 - ie\left(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right)A + e^2A^2\phi^\dagger\phi$$

$$\pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^\dagger - ie\phi^\dagger A^0 \Rightarrow \text{not same comm. relations}$$

# • Time evolution of quantum systems

Let's consider a set of particle e.g. of momentum, (ideally all with the same momentum)




Schrodinger picture: the evolution of the state is given by $\boxed{\frac{\partial}{\partial t}\phi_s(t) = H_s\phi_s(t)}$ $\rightarrow H_s^\circ + H_s^{int}$

## • The S-matrix, interaction representation

The state $|i\rangle$ evolves as $|\phi(t)\rangle = e^{-iHt}|i\rangle$ and at the final time it has evolved into $e^{-iHT_f}|i\rangle$

The amplitude for the process of $|i\rangle$ that evolves into $|f\rangle$ is given by $\langle f|e^{-iHt}|i\rangle$

We can define the limit $\quad S_{fi} = \lim_{t\to\infty} \langle f|\phi(t)\rangle = \langle f|S|i\rangle$

S is an operator that maps an initial state to a final state, $|i\rangle \to S|i\rangle$

In quantum field theory the Heisenberg representation is often more useful than the Schrödinger representation.
The reason is that in QFT the operators are just the fields, so in the Heisenberg representation the quantum fields depend both on x and t while in the Schrödinger representation they depend only on x .
The Heisenberg representation is therefore more natural from the point of view of Lorentz covariance.

In many physically interesting cases, $H_s = H_s^\circ + H_s^{int}$, where $H_s^\circ$ is exactly diagonalizable, $H_s^{int}$ is a "small" modification to $H_s^\circ$

We want to describe the scattering in such a way that if we switch off the interaction, we obtain the free field description

$$\phi(t) = \phi_{Heis}(t) := e^{iH^0_{Shr} t}\, \phi_{Shr}(t)$$
$$O_H(t) := e^{iH^0_s t}\, O_s\, e^{-iH^0_s t}$$
$\Rightarrow$ the physics remains the same: $\langle \phi' | O | \phi \rangle = \langle \phi_s' | O_s' | \phi_s \rangle$

$$i \frac{\partial}{\partial t} \phi(t) = i \frac{\partial}{\partial t}\left(e^{iH^s_0 t}\phi_s(t)\right) = -H^s_0 e^{iH^s_0 t}\phi_s + e^{iH^s_0 t}\, i\frac{\partial}{\partial t}\phi_s = \left[\odot \text{ pag } 108\right] =$$
$$= \cancel{-H^s_0 e^{iH^s_0 t}}\phi_s + e^{iH^s_0 t}(\cancel{H^s_0} + H^s_{int})\phi_s = e^{iH^s_0 t} H^s_{int}\phi_s =$$
$$= e^{iH^s_0 t} H^{int}_s e^{-iH^s_0 t}\phi = \underbrace{H^{int}}_{H_I}\phi \qquad \text{(in interaction picture)}$$

$\Rightarrow$ If $H^{int} = 0 \Rightarrow \frac{\partial}{\partial t}\phi = 0$, free case, as we wanted to obtain.

03/11/2020

- ## Time-dependent perturbation theory and Dyson's formula

In interaction representation, $\boxed{i\frac{\partial}{\partial t}\phi(t) = H_I\phi(t)}$ (i), this eq. defines a translation operator in time between $t=-\infty$ and $t$: $|\phi(t)\rangle = U_I(t)|\phi(-\infty)\rangle$

Let's integrate equation (i):

$$\phi(t) = \phi(-\infty) - i\int_{-\infty}^{t} H_I(t_1)\phi(t_1)\,dt_1 = \phi(-\infty) - i\int_{-\infty}^{t} H_I(t_1)\phi(-\infty)\,dt + o(H_I^2)$$

But $\phi(t_1) = \phi(-\infty) - i\int_{-\infty}^{t} H_I(t_2)\phi(t_2)\,dt_2$

$$\Rightarrow \phi(t) = \phi(-\infty) - i\int_{-\infty}^{t} H_I(t_1)\left(\phi(-\infty) - i\int_{-\infty}^{t} H_I(t_2)\phi(t_2)\,dt_2\right)dt_1 =$$

$$= \phi(-\infty)\left(1 + (-i)\int_{-\infty}^{t} H_I(t_1)\,dt_1 + (-i)^2\int_{-\infty}^{t}\int_{-\infty}^{t} H_I(t_1)H_I(t_2)\,dt_1 dt_2\right) + o(H_I^3)$$

You can go on: $\phi(t) = \phi(-\infty) + \sum_{n=1}^{N} (-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n \, H_I(t_1) \dots H_I(t_n) \phi(t_n)$

You can send $N \to \infty$: $\phi(t) = \left(1 + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{t} dt_1 \dots \int_{-\infty}^{t_{n-1}} dt_n \, H(t_1) \dots H(t_n)\right) \phi(-\infty)$

$\Rightarrow S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} \int_{-\infty}^{t_1} \dots \int_{-\infty}^{t_{n-1}} H(t_1) \dots H(t_n) \, dt_1 \dots dt_n$ , $t_1 > t_2 > \dots > t_n$

We can use the T product $\Rightarrow$ $\boxed{S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} T(H(t_1) \dots H(t_n))}$

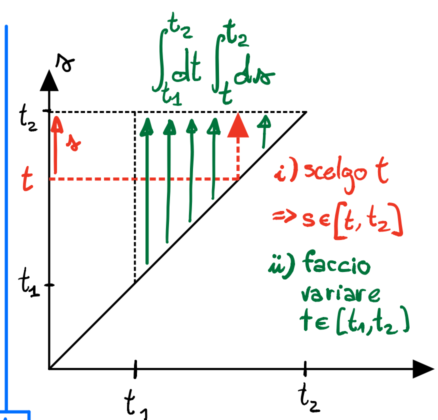
Check: II-order term:

$A_2 = \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} ds \, T(H(t)H(s)) = \int_{t_1}^{t_2} dt \left( \int_{t_1}^{t} H(t)H(s) + \int_{t}^{t_2} H(s)H(t) \right) =$

$= \int_{t_1}^{t_2} dt \int_{t_1}^{t} ds \, H(t)H(s) + \int_{t_1}^{t_2} dt \int_{t}^{t_2} ds \, H(s)H(t) = \boxed{\star} =$

$= \int_{t_1}^{t_2} dt \int_{t_1}^{t} ds \, H(t)H(s) + \int_{t_1}^{t_2} ds \int_{t_1}^{s} dt \, H(s)H(t) = 2 \int_{t_1}^{t_2} \int_{t_1}^{t} H(t)H(s)$

$2!$

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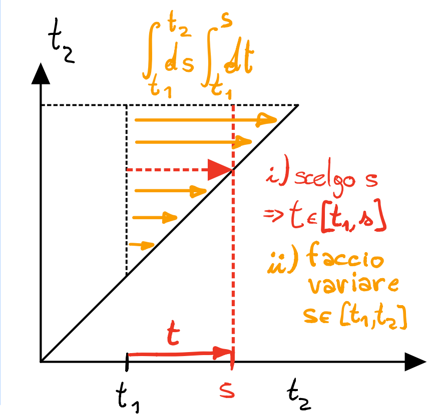


Let $\boxed{T\left(\int_a^b O(t)\, dt\right)^n := \int_a^b dt_1 \dots \int_a^b dt_n \, T(O(t_1) \dots O(t_n))}$

$\Rightarrow$ We obtain the Dyson's formula:

$$\boxed{S = T\left(e^{-i\int_{-\infty}^{\infty} dt \, H_I(t)}\right)}$$

## • Properties of the time-ordered exponential

$$T\left(e^{\int_{t_1}^{t_3} O(t)dt}\right) = T\left(e^{\int_{t_2}^{t_3} O(t)dt}\right) T\left(e^{\int_{t_1}^{t_2} O(t)dt}\right) \quad , \quad t_1 \leq t_2 \leq t_3$$

Proof :

$$T\left(\int_{t_1}^{t_3} O(t)dt\right)^n = \int_{t_1}^{t_3} ds_1 \dots \int_{t_1}^{t_3} ds_n T_t\left(O(s_1)\dots O(s_n)\right) = \left(\int_{t_2}^{t_3} + \int_{t_1}^{t_2}\right) ds_1 \dots \left(\int_{t_2}^{t_3} + \int_{t_1}^{t_2}\right) ds_n T(\dots) =$$

$$= \left[\text{It works, I cheked}\right] = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} \int_{t_2}^{t_3} ds_1 \dots \int_{t_2}^{t_3} ds_{n-k} \int_{t_1}^{t_2} dz_1 \dots \int_{t_1}^{t_2} dz_k \; T(\dots) =$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} \int_{t_2}^{t_3} ds_1 \dots \int_{t_2}^{t_3} ds_{n-k} \, T(O(s_1)\dots O(s_{n-k})) \int_{t_1}^{t_2} dz_1 \dots \int_{t_1}^{t_2} dz_k \, T(O(z_1)\dots O(z_k)) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} T\left(\int_{t_2}^{t_3} dt\, O(t)\right)^{n-k} \cdot T\left(\int_{t_1}^{t_2} dt\, O(t)\right)^{k} = (T_{2,3} + T_{1,2})^n$$

$$\Rightarrow T(e^{\int}) = \sum T \frac{(\int)^n}{n!} = \sum \frac{1}{n!}(T_{23} + T_{12})^n = T(e^{\int + \int}) = T(e^{\int}) T(e^{\int})$$

■

## • Unitarity and Lorentz invariance of the scattering matrix

$$U(t_f, t_i) = T\left(e^{-i\int_{t_i}^{t_f} O(t)dt}\right) = \left[\begin{matrix}\text{we divide} \\ N\Delta t = t_f - t_i\end{matrix}\right] = T\left(e^{-i\int_{t_{N-1}}^{t_N} O(t)\,dt}\right) \dots T\left(e^{-i\int_{t_1}^{t_2} O(t)dt}\right) =$$

$$= \lim_{N\to\infty} e^{-iO(t_N)\Delta t} e^{-iO(t_{N-1})\Delta t} \dots e^{-iO(t_2)\Delta t}$$

$\Rightarrow \quad U^\dagger U = 1 \quad \Rightarrow \quad$ Sending $t_f - t_i = \infty$ , $S^\dagger S = \mathbb{1}$ , S unitary !

We know that $S = T\left(e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)}\right)$ , but $H_I(t) = -L_I = -\int d^3x\, \mathcal{L}_I$

$$\Rightarrow \boxed{S = T\left(e^{i\int \mathcal{L}_I(t) d^4x}\right)} \quad \Rightarrow S \text{ is Lorentz-invariant}$$

# • Wick's theorem

Now we need to evaluate the probability density of the transition $|\langle f|S|i\rangle|^2$, to do so, we demonstrate the Wick's theorem.

$$\boxed{T\left(e^{-i\int d^4x\, j(x)\phi(x)}\right) = :e^{-i\int d^4x\, j(x)\phi(x)}:\, e^{-\frac{1}{2}\iint dx\,dy\, j(x)j(y)\langle 0|T(\phi(x)\phi(y))|0\rangle}}$$

Where $j(x)$ is a function, $\phi(x)$ is a field.

This gives order-by-order what we need to calculate $|S_{if}|^2$

1) $O(t) = \int d^3x\, j(x)\phi(x)$ ; 2) $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ ; 3) $[\quad] = [[A,B],B] = 0$

$$\blacktriangleright\; T\left(e^{-i\int_{t_i}^{t_f} O(t)dt}\right) = \lim_{N\to\infty} e^{-iO(t_N)\Delta t} \cdots e^{-iO(t_1)\Delta t} = \left[e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}\right] =$$

$$= \lim_{N\to\infty} e^{-iO(t_N)\Delta t} \cdots e^{-i\Delta t(O(t_2)+O(t_1))} e^{-\frac{1}{2}\Delta t^2 [O(t_2),O(t_1)]} =$$

$$= \lim_{N\to\infty} e^{-i\Delta t\left(\sum_{i=1}^{N} O(t_i)\right)} e^{-\frac{1}{2}\Delta t^2 \sum_{1\le i<j<N} [O(t_i),O(t_j)]} =$$

$$= e^{-i\int_{t_i}^{t_f} O(t)dt}\, e^{-\frac{1}{2}\int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2\, \theta(t_1-t_2)[O(t_1),O(t_2)]}$$

$$\Rightarrow T\left(e^{-i\int d^4x\, j(x)\phi(x)}\right) = \underbrace{e^{-i\int d^4x\, j(x)\phi(x)}}_{\#}\, e^{-\frac{1}{2}\int d^4x\, d^4y\, \theta(x^0-y^0) j(x)j(y)[\phi(x),\phi(y)]}$$

► Let's rewrite # in terms of normal ordering:

$$e^{-i\int d^4x\, j(x)\phi(x)} = e^{-i\int d^4x\, j(x)(\phi^{(-)}(x)+\phi^{(+)}(x))} = \left[e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}\right] =$$

$$= e^{-i\int d^4x\, j(x)\phi^{(-)}} e^{-i\int d^4x\, j(x)\phi^{(+)}} e^{+\frac{i}{2}\int d^4x\, d^4y\, j(x)j(y)[\phi^{(-)}(x);\phi^{(+)}(y)]}$$

$$= :e^{-i\int d^4x\, j(x)\phi(x)}:\, e^{+\frac{i}{2}\int d^4x\, d^4y\, j(x)j(y)[\phi^{(-)}(x);\phi^{(+)}(y)]}$$

$$\Rightarrow T\left(e^{-i\int d^4x\, j(x)\phi(x)}\right) = :e^{-i\int d^4x\, j(x)\phi(x)}:\, e^{-\frac{i}{2}\int d^4x\, d^4y\, j(x)j(y)\{\theta(x^0-y^0)[\phi(x),\phi(y)] - [\phi^{(-)}(x),\phi^{(+)}(y)]\}}$$

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▶ $\theta(x^0-y^0)[\phi(x),\phi(y)] - [\phi^{(-)}(x),\phi^{(+)}(y)]$ is a c-number, c-number: $\langle 0|c|0\rangle = c\langle 0|0\rangle$

$$\langle 0|\theta(x^0-y^0)[\phi(x),\phi(y)] - [\phi^{(-)}(x),\phi^{(+)}(y)]|0\rangle = \left[\cancel{-\phi^{(-)}\phi^{(+)}} + \phi^{(+)}\phi^{(-)}|0\rangle \text{ , annihilation}|0\rangle = 0\right] =$$

$$= \langle 0|\theta(x^0-y^0)\,\phi(x)\phi(y) - \theta(x^0-y^0)\phi(y)\phi(x) + \phi^+\phi^-|0\rangle =$$

$$\left[\begin{array}{l} \langle 0|\phi(y)\phi(x)|0\rangle = \langle 0|\,\cancel{\phi^{(+)}(y)\phi^{(+)}(x)} + \phi^{(+)}(y)\phi^{(-)}(x) + \cancel{\phi^{(-)}(y)\phi^{(+)}(x)} + \cancel{\phi^{(-)}(y)\phi^{(-)}(x)}\,|0\rangle \\ \Rightarrow \quad \langle 0|\phi^+(y)\phi^-(x)|0\rangle = \langle 0|\phi(y)\phi(x)|0\rangle \end{array}\right]$$

$$= \langle 0|\theta(x^0-y^0)\,\phi(x)\phi(y) - \theta(x^0-y^0)\phi(y)\phi(x) + \phi(y)\phi(x)|0\rangle =$$

$$= \langle 0|\theta(x^0-y^0)\,\phi(x)\phi(y) - \cancel{\theta(x^0-y^0)\phi(y)\phi(x)} + (\cancel{\theta(x^0-y^0)} + \theta(y^0-x^0))\phi(y)\phi(x)|0\rangle =$$

$$= \langle 0|\theta(x^0-y^0)\,\phi(x)\phi(y) + \theta(y^0-x^0)\phi(y)\phi(x)|0\rangle$$

$$= \langle 0|T(\phi(x)\phi(y))|0\rangle$$

■

Let's define

$$\boxed{\begin{cases} \langle j\phi\rangle := \int d^4x\, j(x)\phi(x) \\ \langle\langle j^2\phi^2\rangle\rangle := \int d^4x\, d^4y\; j(x)j(y)\langle 0|T(\phi(x)\phi(y))|0\rangle \end{cases}}$$

Let's consider the expansion of the theorem:

$$T\Big(\underbrace{1}_{S^{(0)}} + \underbrace{(-i)\langle j\phi\rangle}_{S^{(1)}} + \underbrace{\frac{(-i)^2}{2}\langle j\phi\rangle^2}_{S^{(2)}} + \dots\Big) = :\Big(1 - i\langle j\phi\rangle + \frac{(-i)^2}{2}\langle j\phi\rangle^2 + \dots\Big): \Big(1 - \frac{1}{2}\langle\langle j^2\phi^2\rangle\rangle + \frac{1}{8}\langle\langle\dots\rangle\rangle^2 + \dots\Big)$$

Let's take, for istance, the $2^{nd}$ order: $\quad T(\langle j\phi\rangle^2) = :\langle j\phi\rangle^2: + \langle\langle j^2\phi^2\rangle\rangle$

$$\Rightarrow \int d^4x\, d^4y\; j(x)j(y)\,T(\phi(x),\phi(y)) = \int d^4x\, d^4y\; j(x)j(y)\; :\phi(x)\phi(y): +$$

$$- \int d^4x\, d^4y\; j(x)j(y)\,\langle 0|T(\phi(x)\phi(y))|0\rangle$$

this is called the contraction

$$\Rightarrow \boxed{T(\phi(x)\phi(y)) = :\phi(x)\phi(y): + \langle 0|T(\phi(x)\phi(y))|0\rangle}$$

Since $T$ is symmetric, on the RHS you need something symmetric, at all orders!

$$T(\phi_a(x)\phi_b(x)) = \phi_a(x)\phi_b(x)$$

$$T(\phi_a(x)\phi_b(x)) = :\phi_a(x)\phi_b(x): + \langle 0|T(\phi_a(x)\phi_b(x))|0\rangle$$

$$\Rightarrow :\phi_a(x)\phi_b(x): = \phi_a(x)\phi_b(x) - \langle 0|T(\phi_a(x)\phi_b(x))|0\rangle$$

Wick's theorem expansion to 2nd order:

- Dirac : $iS_F(x_1-x_2) = \langle 0|T(\psi_\alpha(x_1)\bar\psi_\beta(x_2))|0\rangle = T(\psi_\alpha(x_1)\bar\psi_\beta(x_2)) - :\psi_\alpha(x_1)\bar\psi_\beta(x_2):$

  $$\overbrace{\psi_\alpha(x_1)\bar\psi_\beta(x_2)} := T(\psi_\alpha(x_1)\bar\psi_\beta(x_2)) - :\psi_\alpha(x_1)\bar\psi_\beta(x_2):$$

- K-G : $iD_F(x_1-x_2) = \langle 0|T(\phi(x_1)\phi^\dagger(x_2))|0\rangle = T(\phi(x_1)\phi^\dagger(x_2)) - :\phi(x_1)\phi^\dagger(x_2):$

  $$\overbrace{\psi(x_1)\psi(x_2)} := T(\phi(x_1)\phi^\dagger(x_2)) - :\phi(x_1)\phi^\dagger(x_2):$$

- E-M : $ig^{\mu\nu}D_F(x_1-x_2) = \langle 0|T(A^\mu(x_1)A^\nu(x_2))|0\rangle = T(A^\mu(x_1)A^\nu(x_2)) - :A^\mu(x_1)A^\nu(x_2):$

  $$\overbrace{A^\mu(x_1)A^\nu(x_2)} := T(A^\mu(x_1)A^\nu(x_2)) - :A^\mu(x_1)A^\nu(x_2):$$

## • Time ordered product of n fields

$$D(x_i-x_j) = \overbrace{\phi_i\phi_j}$$

$$\boxed{T(\phi_1\dots\phi_m) = :\phi_1\dots\phi_n: + \sum_{\text{combinations}}\Big(\underbrace{:\phi_1\dots\phi_m:}_{m-2} D_{ij} + D_{ij}D_{kn}\dots D_{mn}\Big)}$$

$$\langle 0|T(\phi_1\dots\phi_m)|0\rangle = \cancel{\langle 0|:\phi_1\dots\phi_m:|0\rangle}^{\,=0,\ \text{ann}|0\rangle=0} + \langle 0|\sum(\ )|0\rangle = \boxed{\begin{array}{l}\text{ahn}|0\rangle = 0 \Rightarrow \\ \Rightarrow \text{survive only} \\ \text{the term with all} \\ \text{contractions}\end{array}} = \sum_{\text{combinations}} \underbrace{D_{ab}D_{cd}\dots D_{mn}}_{\text{Product of } \frac{m}{2} \text{ terms}}$$

$$\#\text{ of combin} = \prod_{k=0}^{\frac{m}{2}-1}\binom{m-2k}{2}$$

It is the product of the number of ways to choose two fields between $m, m-2, m-4 \dots$

Examples

- $m=2 \Rightarrow T(\phi_1\phi_2) = :\phi_1\phi_2: + \overbrace{\phi_1\phi_2}^{=D_{12}}$ , $\langle 0|T(\phi_1\phi_2)|0\rangle = D_{12}$
- $m=4 \Rightarrow T(\phi_1\phi_2\phi_3\phi_4) = :\phi_1\phi_2\phi_3\phi_4: + :\phi_1\phi_2:D_{34} + :\phi_1\phi_3:D_{24} + D_{14}:\phi_3\phi_2: + \dots$

$$\dots + D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$$

$$\langle 0|T(\phi_1\phi_2\phi_3\phi_4)|0\rangle = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$$

# • Time ordered product of normal ordered fields

Problem: usually we compute fields in the same point and interaction lagrangians are given in terms of normal ordering.

For example, in QED, $\mathcal{L}_{int} \propto :\bar{\Psi}(x)\not{A}(x)\Psi(x):$, they are already time-ordered!

What happens when I have to compute $T(:\phi_a(x)\phi_b(x):\phi_c(x_1)\phi_d(x_2))$?

► Since $\phi_a$ and $\phi_b$ are evaluated at the same $x$: $T(\phi_a(x)\phi_b(x)) = \phi_a(x)\,\phi_b(x)$

► We know from Wick's theorem that

$$T(\phi_a(x)\phi_b(y)) = :\phi_a(x)\phi_b(y): + \langle 0|T(\phi_a(x)\phi_b(y))|0\rangle$$

Putting them together, we get

$$\boxed{:\phi_a(x)\phi_b(x): = \phi_a(x)\phi_b(x) - \langle 0|\overset{?}{T}(\phi_a(x)\phi_b(x))|0\rangle}$$

$$\Rightarrow :\phi_a(x)\phi_b(x):\phi_c(x_1)\phi_d(x_2) = \phi_a(x)\phi_b(x)\phi_c(x_1)\phi_d(x_2) - \langle 0|T(\phi_a(x)\phi_b(x))|0\rangle\phi_c(x_1)\phi_d(x_2)$$

$$\Rightarrow \boxed{\begin{gathered} T(:\phi_a(x)\phi_b(x):\phi_c(x_1)\phi_d(x_2)) \\ \parallel \\ T(\phi_a(x)\phi_b(x)\phi_c(x_1)\phi_d(x_2)) - T(\phi_c(x_1)\phi_d(x_2))\langle 0|T(\phi_a(x)\phi_b(x))|0\rangle \end{gathered}} \quad (✡)$$

It is equivalent to calculating $T(\text{all fields})$, but when I consider all the contractions, I don't consider the one of $\phi_a, \phi_b$, because it's removed (✡)

In general: $T(:A(x)B(x):C_1(x_1)\ldots C_n(x_n)) = T(ABC_1\ldots C_n) - \underbrace{AB}C_1\ldots C_n =$

$= :ABC_1\ldots C_n: + \sum :A\underbrace{B\ldots C_k}\ldots C_n + \sum :\underbrace{A\;\underbrace{B\ldots\ldots}}\ldots + \ldots \quad - \underbrace{AB}C_1\ldots C_n$

⟵ all the contractions ⟶

# • Application: QED

## • Evaluation of the scattering matrix at the first order in QED

We consider the scattering of an electron in a static external field

We know that $S = T\left(e^{i\int d^4x \mathcal{L}_{int}}\right)$, in this case, $\mathcal{L}_{int} = -e\bar{\psi}\gamma_\mu A^\mu \psi = -e\bar{\psi}\not{A}\psi$

$$S^{(1)} = T(\langle j\phi\rangle) = :\langle j\phi\rangle: = -ie\int d^4x :\bar{\psi}(x)\not{A}\psi(x): = -ie\int d^4x (\bar{\psi}^+ + \bar{\psi}^-)(\not{A}^+ + \not{A}^-)(\psi^+ + \psi^-)$$

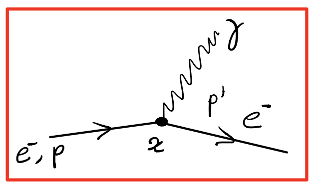
We can remove ::, the proof is to take $T(\ )$ pag 115 assuming $T(\text{constant}) = 0$

$\psi^{(+)}$ ann. $e^-$, $\psi^{(-)}$ creates $e^+$ ; $\bar{\psi}^{(+)}$ creates $e^+$, $\bar{\psi}^{(-)}$ annihil $e^-$ ; $A^{(+)\mu}$ ann $\gamma$, $A^{(-)\mu}$ creates $\gamma$

Let's consider the operator $\underbrace{\bar{\psi}^-}_{\text{creates } e^-}\underbrace{\not{A}^-}_{\text{creates }\gamma}\underbrace{\psi^+}_{\text{ann. } e^-}$ $\Rightarrow$ $\bar{\psi}^-\not{A}^-\psi^+|e^-\rangle = |\gamma e^-\rangle$ final state

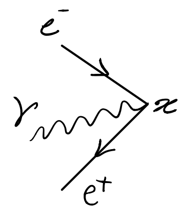
We can calculate $\langle e^-\gamma|\underline{\bar{\psi}^-(x)\not{A}^-\psi^+(x)}|e^-\rangle$

which is the prob amplitude of $e^- \to e^-\gamma$




Other non-0 prob. amplitudes :

$\langle 0|\bar{\psi}^+\not{A}^+\psi^+|e^-\gamma e^+\rangle$ :




This is not physical, we want to sandwitch the matrix between two physical states, for example, 4-momentum has to be conserved at each vertex

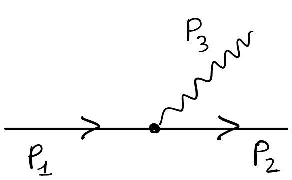
$\langle\gamma|A^-\bar{\psi}^+\gamma_\mu\psi^+|e^-e^+\rangle$, pair "annihilation"

NO PROPAGATOR, NOTHING HAPPENS

In the radiation of $\gamma$ from $e^-$ : $e^-(p_1) \to \gamma(p_3) + e^-(p_2)$, let's square

supposing all are on mass-shell: $\cancel{m^2} = 0 + 2\underline{p_2}\cdot\underline{p_3} + \cancel{m^2} \Rightarrow \underline{p_2}\cdot\underline{p_3} = 0$.

If $e^-$ is at rest, $\underline{p}_1 = \underline{0} \Rightarrow \underline{p}_2 + \underline{p}_3 = 0 \Rightarrow (\underline{p}_2 + \underline{p}_3)\cdot(\underline{p}_2 + \underline{p}_3) = 0 \Rightarrow \underline{p}_2^2 + \underline{p}_3^2 + 2\underbrace{\cancel{\underline{p}_2\underline{p}_3}}_{=0} = 0$




$\Rightarrow \underline{p}_2^2 + \underline{p}_3^2 = 0 \Rightarrow \underline{p}_2 = \underline{p}_3 = 0$

If $\underline{p}_3 \neq 0 \Rightarrow$ the process is not physical

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# • Evaluation of the scattering matrix at the second order in QED

$$S^{(2)} = T\left(\int \bar{\Psi} \cancel{A} \Psi\right)^2 = \frac{(-ie)^2}{2} \int d^4x_1 d^4x_2 \; T\left(\bar{\Psi}(x_1)\cancel{A}(x_1)\psi(x_1)\; \bar{\Psi}(x_2)\cancel{A}(x_2)\psi(x_2)\right)$$

► $\langle 0|\, T\,(\psi_\alpha(x_1)\bar{\psi}_\beta(x_2))\,|0\rangle = i\, S_F(x_1-x_2)_{\alpha\beta}$

► $\langle 0|\, T\,(A_\mu(x_1), A_\nu(x_2))\,|0\rangle = i\, g_{\mu\nu}\, D(x_1-x_2)$

$\psi_\alpha, \psi_\beta$ aren't anymore free fields, they are interacting

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## One-contraction contributions

Let's consider the case of a contraction of the Dirac field $\psi$

$$S^{(2)} = \frac{(-ie)^2}{2} \int d^4x_1 d^4x_2 \; : \bar{\psi}(x_1)\, \cancel{A}(x_1) \langle 0|\, T\,(\psi(x_1)\bar{\psi}(x_2))\, |0\rangle\, \cancel{A}(x_2)\psi(x_2) : +$$

$$+ \frac{(-ie)^2}{2} \int d^4x_1 d^4x_2 \; : \bar{\psi}(x_2)\, \cancel{A}(x_2) \langle 0|T(\psi(x_2)\, \bar{\psi}(x_1))\, |0\rangle \cancel{A}(x_1)\, \psi(x_1) :$$

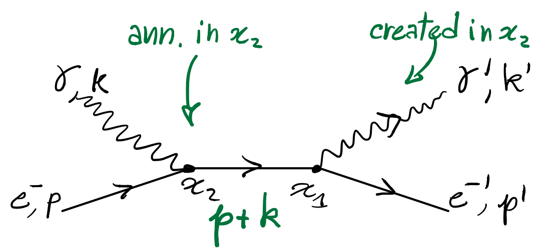
$$= (-ie)^2 \int dx_1 dx_2 \; : \bar{\psi}(x_1)\, \cancel{A}(x_1)\; i\, S_F(x_1-x_2)\, \cancel{A}(x_2)\, \psi(x_2) :$$

$(\bar{\psi}^+ + \bar{\psi}^-)(\cancel{A}^+ + \cancel{A}^-)$ : processes with photon and fermion

■ Compton Scattering : $\gamma + e^- \to \gamma + e^-$

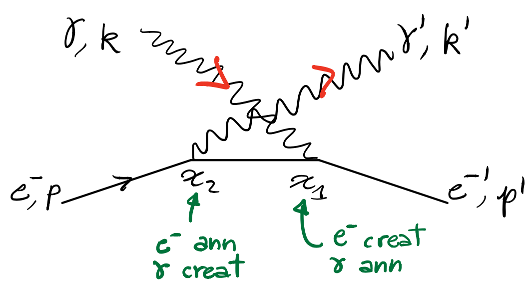
$\begin{cases} |i\rangle = |\gamma e^-\rangle \\ |f\rangle = |\gamma e^-\rangle \end{cases}$ You have to select the normal ord. operator which ann. $e^-$ and $\gamma$, then propagation to $x_1$, then creation of $e^-$ and $\gamma$

$$S_a = (-ie)^2 \int d^4x_1 d^4x_2 \; \bar{\Psi}_1^- \cancel{A}_1^-\; i\, S_F(x_1-x_2)\, \cancel{A}_2^+\, \Psi_2^+$$




Other possibility with same $|i\rangle, |f\rangle$ :

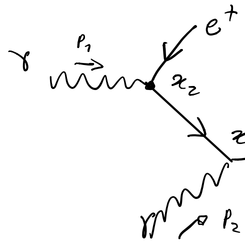
$$S_b = (-ie)^2 \int d^4x_1 d^4x_2 \; \bar{\Psi}_1^- \cancel{A}_1^+\; i\, S_F(x_1-x_2)\, \cancel{A}_2^-\, \Psi_2^+$$

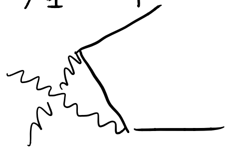


- Positron Compton scattering : $\gamma + e^+ \to \gamma + e^+$

ann $\gamma$ creat $e^+$

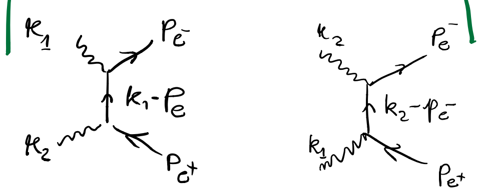
$$S_a = (-ie)^2 \int d^2x_1 d^2x_2 \, \bar{\Psi}_1^- \cancel{A}_1^- \, iS_F(x_1-x_2) \cancel{A}_2^+ \psi_2^-$$




or we can have 

- Pair creation and pair annihilation

PAIR CREATION




$\gamma + \gamma \to e^+ + e^-$

$e^+ + e^- \to \gamma + \gamma$

We contract $\underbrace{\cancel{A}_1 \cancel{A}_2}$ :

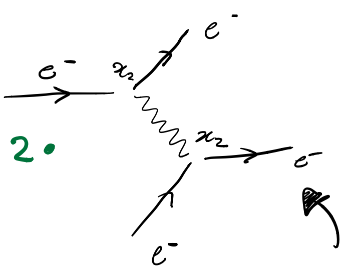
$$S^{(2)} = \frac{(-ie)^2}{2} \int d^4x_1 d^4x_2 : \bar{\Psi}_1 \gamma_\mu \psi_1 \left[ i g^{\mu\nu} D(x_1 - x_2) \right] \bar{\Psi}_2 \gamma_\nu \psi_2 :$$

$$\bar{\psi}_1 \cancel{A}_1 \underbrace{\psi \bar{\Psi}_2 \cancel{A}_2} \psi_2$$

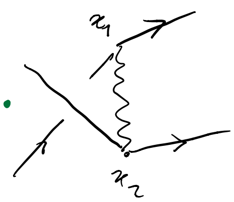
- Möller scattering: $e^- + e^- \to e^- + e^-$

$$S_a = \frac{(-ie)^2}{2} \int dx_1 dx_2 \, \bar{\Psi}_1^- \gamma_\mu \psi_1^+ \, i g^{\mu\nu} D(x_1 - x_2) \bar{\Psi}_2^- \gamma_\nu \psi_2^+$$

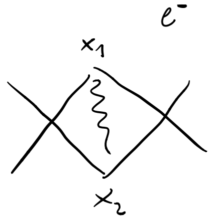
2.




(−) 2.
sign

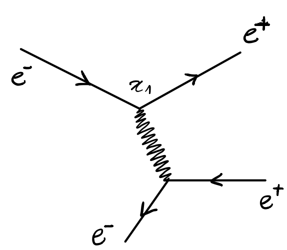


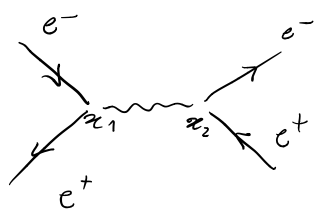

In total, sum of 4 objects

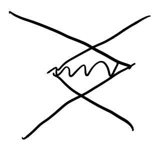



If i consider $x_1 \leftrightarrow x_2$
I have the same

- Bhabha scattering: $e^- + e^+ \to e^- + e^+$



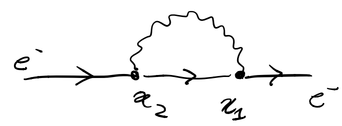
## Two-contractions contributions

You can contr a ferm. field and photon field:

- Electron and positron self-energy

ann. of $e^-$ in $x_2$

$$S^{(2)} = (-ie)^2 \int d^4x_1 d^4x_2 \; \bar{\psi}_1^- \gamma_\mu \underbrace{\left[ i S_F(x_1-x_2)\, i g^{\mu\nu} D(x_1-x_2) \right]}_{\text{2 virtual states propagated in } x_1} \gamma_\nu \psi_2^+$$



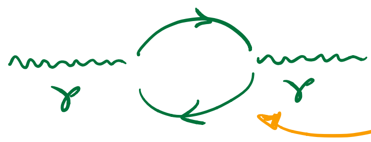

$e^-$ emits and absorbs a photon, self enegy

- Vacuum polarization of the photon

$$S^{(2)} = \frac{(-ie^2)}{2} \int d^4x_1 d^4x_2 \; \bar{\psi}_\alpha(x_1)\, \gamma^\mu_{\alpha\beta}\, \underbrace{\psi_\beta(x_1) \bar{\psi}_\gamma(x_2)}_{\text{contraction}}\, \gamma^\nu_{\gamma\delta}\, \psi_\alpha(x_2)\, \underbrace{A_\mu(x_1) A_\nu(x_2)}_{\text{external fields}}$$

contraction

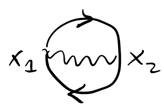
$$= \frac{(-ie)^2}{2} \int dx_1 dx_2 \left( -i S_F(x_2-x_1)_{\delta\alpha}\, \gamma^\mu_{\alpha\beta} \; i S_F(x_1-x_2)_{\beta\gamma}\, \gamma^\nu_{\gamma\delta}\, A_\mu(x_1) A_\nu(x_2) \right)$$




fermionic propagator

## Three contraction term

If we contract three fields ⇒




$$\Rightarrow \quad S^{(2)}_{all} = \int d^4x_1 d^4x_2 \left( -i S_F(x_2-x_1) \gamma^\mu_{\alpha\beta} \; i S_F(x_1-x_2) \gamma^\nu_{\gamma\delta} \right) i\, g_{\mu\nu} D(x_2-x_1)$$

quantum connection? of a vacuum (term/state?)

- S-matrix in momentum space

$$S := T\left(e^{i\int d^4x\, \mathcal{L}_I(x)}\right) \longrightarrow :\bar{\psi}\not{A}\psi: \;, \quad S^{(2)} = \frac{(-ie)^2}{2}\int d^4x_1 d^4x_2\, T\left(\bar{\psi}\not{A}\psi_{x_1}\bar{\psi}\not{A}\psi_{x_2}\right)$$

we want to evaluate it

$$i\,S_F(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, i\,S_F(p) \quad \Rightarrow \quad \boxed{S_F(p) = \frac{\not{p}+m}{p^2-m^2+i\varepsilon}}$$

$$i\,g_{\mu\nu}D(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, i\,g_{\mu\nu}D(p) \quad \Rightarrow \quad \boxed{D(p) = -\frac{1}{p^2+i\varepsilon}}$$

$$\psi(x) = \sum_{\pm n}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\left(\underbrace{b(p,m)\,u(p,n)\,e^{-ipx}}_{\psi^{(+)}} + d^\dagger(p,n)\,v(p,n)\,e^{ipx}\right)$$

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{2\omega_p}\sum_{\lambda=0}^{3}\varepsilon_\mu^\lambda(p)\left[a_\lambda(p)\,e^{-ipx} + a_\lambda^\dagger(p)\,e^{ipx}\right]$$

► $$\psi^{(+)}(x)\,|e^-(p',n')\rangle = \sum_{\pm n}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\,u(p,n)\,\underline{b(p,n)\,b^\dagger(p',n')}|0\rangle\, e^{-ipx}$$

$\hookrightarrow$ $bb^\dagger|0\rangle = (\delta(p-p')\delta_{nn'} - b^\dagger b)|0\rangle$

$$= \sum_{\pm n}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\,u(p,n)\,\delta_{nn'}\,\delta(p-p')\,e^{-ipx} = \frac{1}{2\pi}\sqrt{\frac{m}{E_{p'}}}\,u(p',n')\,e^{-ip'x}|0\rangle$$

► $$A_\mu^{(+)}(x)\,|\gamma(k,\lambda)\rangle = \frac{1}{(2\pi)^{3/2}\sqrt{2E_k}}\,\varepsilon_\mu^\lambda(k)\,e^{-ikx}|0\rangle$$

If we have a finite volume $V$ :

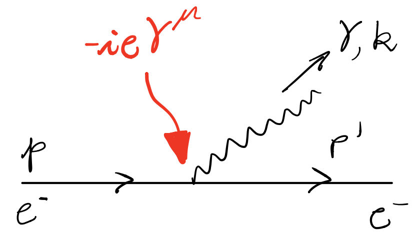
$$\psi^{(+)}|e^-(p',n')\rangle = \sqrt{\frac{m}{VE_{p'}}}\,u(p',n')\,e^{-ip'x}|0\rangle$$

$$A_\mu^{(+)}|\gamma(k,\lambda)\rangle = \frac{1}{\sqrt{2VE_k}}\,\varepsilon_\mu^\lambda(k)\,e^{-ikx}|0\rangle$$

Let's consider different processes and then derive the rules

## The interaction vertex for particles

$|i\rangle = |e^-(p)\rangle \quad , \quad |f\rangle = |e^-(p'), \gamma(k)\rangle$




Not a physical vertex because the static source which generates the field, breaks translational invariance

$$\langle f|S^{(1)}|i\rangle = -ie\int d^4x \langle e^-\gamma|\bar{\psi}^-(x)\cancel{A}^-(x)\psi^+(x)|e^-\rangle = [\Delta] =$$

$$= -ie\int d^4x \langle e^-\gamma|\bar{\psi}^-(x) A^{(-)}_\mu(x)\gamma^\mu \sqrt{\frac{m}{VE_p}}\, e^{-ipx} u(p,n)|0\rangle$$

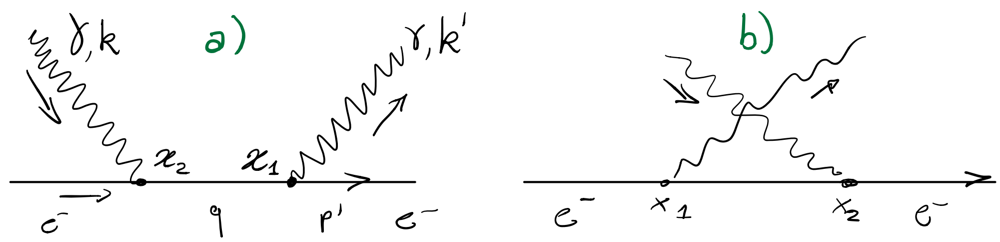
$$\mathcal{M}_{if}$$

$$= \int d^4x \sqrt{\frac{m}{VE_p}}\sqrt{\frac{m}{VE_{p'}}}\frac{1}{\sqrt{2E_kV}}\, e^{i(p'+k-p)x}\left(\bar{u}(p')(-ie\gamma^\mu)u(p)\,\varepsilon^\lambda_\mu(k)\right)$$

follows the fermionic arrow inverted

$$= (2\pi)^4\delta(p'+k-p)\sqrt{\frac{m}{VE_p}}\sqrt{\frac{m}{VE_{p'}}}\frac{1}{\sqrt{2VE_k}}\mathcal{M}_{if} \;; \quad \text{Where } \mathcal{M} = \bar{u}(p')\underline{(-ie\gamma^\mu)}u(p)\,\varepsilon^\lambda_\mu(k)$$

## Second-order process:




$$\langle f|S_a|i\rangle =$$

$$= (-ie)^2\int d^4x_1 d^4x_2 \langle e^-(p')\gamma(k')|\bar{\psi}^-(x_1)\cancel{A}^{(-)}(x_1)\, iS_F(x_1-x_2)\cancel{A}^{(+)}(x_2)\psi^+(x_2)|e^-(p)\gamma(k)\rangle =$$

$$= (-ie)^2\int d^4x_1 d^4x_2 \sqrt{\frac{m}{VE_p}}\sqrt{\frac{m}{VE_{p'}}}\frac{1}{\sqrt{2VE_k}}\frac{1}{\sqrt{2VE_{k'}}}\,\bar{u}(p',n')\cdot\left(e^{ip'x_1}\cancel{\varepsilon}^\lambda(k')e^{ik'x_1}\right)\cdot$$

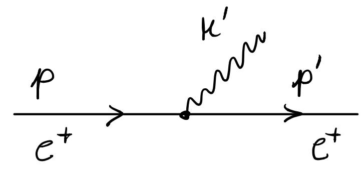
$$\cdot\frac{1}{(2\pi)^4}\int d^4q\, e^{-iq(x_1-x_2)}S_F(q)\left(\cancel{\varepsilon}^\lambda(k)e^{-ikx_2}u(p,n)e^{-ipx_2}\right) =$$

$$\propto\int\frac{d^4q}{(2\pi)^4}(2\pi)^4\delta(p'+k'-q)(2\pi)^4\delta(q-p-k)\propto(2\pi)^4\delta(p'+k'-p-k) =$$

$$= \prod_f\sqrt{\frac{m}{VE_f}}\prod_b\frac{1}{\sqrt{2VE_b}}(2\pi)^4\delta(p_f-p_i)\,\mathcal{M}_a$$

Where $\mathcal{M}_a = \bar{u}(p')(-ie\gamma^\mu)\,\varepsilon^{\lambda'}_\mu(k')\dfrac{1}{(\cancel{p}+\cancel{k})-m+i\varepsilon}(-ie\gamma^\nu)\,\varepsilon^\lambda_\nu(k)\,u(p)$

## Interaction vertex for antiparticle processes




$$\bar{\psi}^{(+)}|e^+(p)\rangle = \sqrt{\frac{m}{VE_p}}\, e^{-iPx}\, \bar{v}(p,n)|0\rangle$$

$|i\rangle = |e^+(p)\rangle$ , $|f\rangle = |e^+(p')\gamma(k)\rangle$

$$\langle f|S|i\rangle = -ie\int d^4x\, \langle e^+(p')\gamma(k)|\, \psi^- \cancel{A}^- \bar{\psi}^+\, |e^+(p)\rangle$$

$\uparrow$ = exactly $:\bar{\psi}^+ \cancel{A}\, \psi^-:$

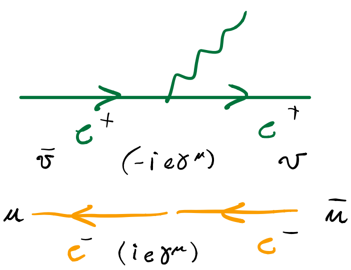
$$\langle e^+(p')|\bar{v}\, d\, \gamma^\mu\, d^+\, v\, |e^+(p)\rangle$$

$$= -ie\int d^4x\, \overbrace{\bar{v}(p,n)}^{\text{incoming } e^+}\, \varepsilon^\lambda(k')\, v(p',n')\, e^{-i(P-k-P')x} =$$

$$= (2\pi)^4\, \delta(P-k-P')\, \bar{v}(p,n)(-ie\gamma^\mu)\, v(p',n')\, \varepsilon^\lambda_\mu(k)$$

$\longleftarrow \quad \mathcal{M} \quad \longrightarrow$

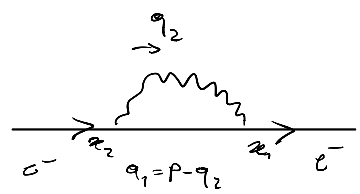
the reading order of $\mathcal{M}$ follows the fermionic arrow order

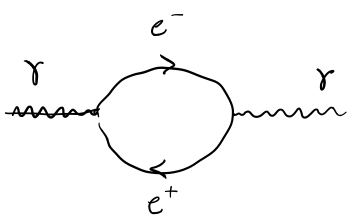



→ against the arrow

## Loop, self energy

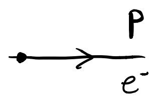
2 integrals — — — — ?



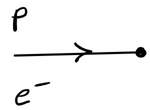



# • Feynman rules

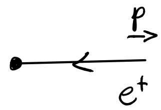
By calculations we obtained the **Feyman rules** we can summarize :
We can summarize the rules :

► Outgoing particle 

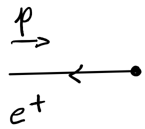
: $\sqrt{\frac{m}{VE_p}}\ \bar{u}(p)$

► Incoming particle 

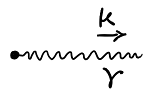
: $\sqrt{\frac{m}{VE_p}}\ u(p)$

► Outgoing antipart. 

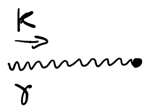
: $\sqrt{\frac{m}{VE_p}}\ v(p)$

► Incoming antipart. 

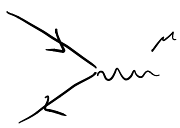
: $\sqrt{\frac{m}{VE_p}}\ \bar{v}(p)$

► Outgoing photon 

: $\frac{1}{\sqrt{2VE_k}}\ \varepsilon^{\lambda}_{\mu}(k)$

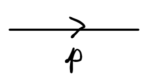
► Incoming photon 

: $\frac{1}{\sqrt{2VE_k}}\ \varepsilon^{\lambda}_{\mu}(k)$

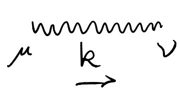
► Interaction vertex 

: $-ie\gamma^{\mu}$

► Propagator



: $\frac{1}{\not{p}-m+i\varepsilon}$ $\left(\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}\right)$



: $\frac{-i\,g_{\mu\nu}}{k^2+i\varepsilon}(2\pi)^4\delta(P_f-P_i)$

► Loops 

: $\int\frac{d^4k}{(2\pi)^4}$ ; $k$ is the momentum which loops

# • The production cross-section

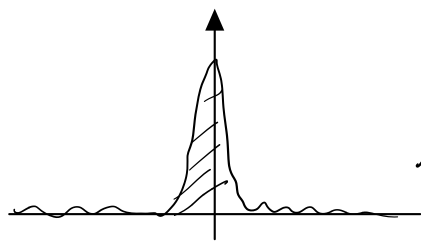
$$|S_{fi}|^2 = \left|(2\pi)^4 \, \delta^4\left(\sum_i P_i^{ext}\right) \prod_f \sqrt{\frac{m}{VE_p}} \prod_b \frac{1}{\sqrt{2VE_p}} \mathcal{M}\right|^2$$

f : fermions
b : bosons

i) $(2\pi)^4 \delta^4(P_f - P_i) = \lim_{\substack{T\to\infty \\ V\to\infty}} \int_V d^3x \int_{-T/2}^{T/2} dt \; e^{i(P_f - P_i)x}$

ii) $(2\pi)\,\delta(E_f - E_i) = \lim_{T\to\infty} \int_{-T/2}^{T/2} dt \; e^{i\Delta E t} = \lim_{T\to\infty} \dfrac{2\sin\left(\frac{\Delta E}{2}T\right)}{\Delta E}$

iii) $\lim_{T\to\infty} \dfrac{4\sin^2\left(\frac{\Delta E}{2}T\right)}{\Delta E^2} \sim 2\pi T\,\delta(E_f - E_i)$



$A = 2\pi T$

i) $\Rightarrow \left|(2\pi)^4 \delta^4(P_f - P_i)\right|^2 = \lim_{\substack{V\to\infty \\ T\to\infty}} (2\pi)^4 V T\, \delta^4(P_f - P_i)$

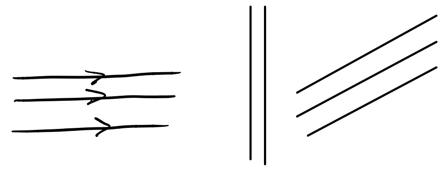
$\prod_{bos} \dfrac{1}{2VE_b} \prod_{ferm} \dfrac{m}{VE_f}$

We define $\omega_{fi} = \dfrac{|S_{fi}|^2}{T} = V(2\pi)^4 \delta^4(P_f - P_i) \prod_{ext} \dfrac{1}{2VE_p} \prod_{ferm} (2m)\,|\mathcal{M}|^2$

all part. that go in or out

Number of states : $d^3 n = \dfrac{V}{(2\pi)^3} d^3 P_f \;\rightarrow\; \omega_{fi} \cdot \left(\prod_f \dfrac{L^3}{(2\pi)^3} d^3 P_f\right)$

# Scattering cross-section



target

$n$ : # of inc. part per unit time and $S$

$N_i$ : # of inc. part in $\Delta t$ $\Rightarrow$ $\boxed{N_i = nS\Delta t}$

$N$ : # of scattered particles per unit time and unit target particle

$N_d$ : # of scattered particles in $\Delta t$ per unit target $\Rightarrow$ $N_d = N\Delta t$

$$\Rightarrow \quad \sigma := \frac{N}{n} = \frac{N_d}{\Delta t}\frac{S\Delta t}{N_i} = \frac{N_d}{N_i}S \quad , \quad \boxed{\sigma = \frac{N_d}{N_i}S}$$

$$n = \frac{N_i}{S\Delta t}\frac{L}{L} = \frac{NL}{V\Delta t} \quad \Rightarrow \quad \boxed{n = \rho\cdot|v_{rel}|}$$

$$\Rightarrow \quad d\sigma = \frac{1}{\rho\,|v_{rel}|}\,\omega_{fi}\prod_f \frac{V}{(2\pi)^3}\,d^3p_f$$

$$\rho = \frac{1}{V} \quad \Rightarrow \quad d\sigma = \frac{V^2}{|V_{rel}|}(2\pi)^4\delta^4(P_f - P_i)\prod_{ext}\frac{1}{2VE}\prod_{ferm}(2m)|M|^2\prod_f\frac{V}{(2\pi)^3}\,d^3p_f$$

$$\text{Case} \quad \underset{\text{incom}}{2} \longrightarrow \underset{\text{scatt}}{n}$$

$$d\sigma = (2\pi)^4\delta^4(P_f - P_i)\frac{1}{4E_1E_2|v_{rel}|}\prod_{ferm}(2m)\,|M|^2\prod_f\frac{d^3p_f}{(2\pi)^3\,2E_f}$$

$$v_{rel} = v_1 - v_2 = \frac{p_1}{E_1} - \frac{p_2}{E_2}$$

$$\Rightarrow \quad E_1E_2|v_{rel}| = E_1E_2\left|\frac{p_1}{E_1} - \frac{p_2}{E_2}\right|$$ can be rewritten in covariant form

If $\underline{p_2 = 0}$ $\Rightarrow$ $E_1E_2|v_{rel}| = \cancel{E_1}\,m_2\frac{|p_1|}{\cancel{E_1}} = m_2\sqrt{E_1^2 - m_1^2} = \sqrt{m_2^2E_1^2 - m_1^2m_2^2} = \underbrace{\sqrt{(P_1P_2)^2 - m_1^2m_2^2}}_{\text{COVARIANT}}$

$$\Rightarrow \quad \boxed{d\sigma = (2\pi)^4\delta^4(P_f - P_i)\frac{1}{4\sqrt{(P_1P_2)^2 - m_1^2m_2^2}}\prod_{ferm}(2m)\,|M|^2\prod_f\frac{d^3p_f}{(2\pi)^3\,2E_f}} \quad (\star)$$

# • Calculation of the cross section for the scattering process $e^+ + e^- \rightarrow \mu^+ + \mu^-$

Let's consider the process $e^+ e^- \rightarrow \mu^+ \mu^-$

I) We take the formula for $d\sigma$ from pag 125 (*)

$$d\sigma_{e^+e^-\rightarrow\mu^+\mu^-} = (2\pi)^4 \delta^4 (P_1+P_2-P_3-P_4) \frac{1}{4\sqrt{(P_1 P_2)^2 - m_1^2 m_2^2}} (2m_e)^2 (2m_\mu)^2 \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} |\mathcal{M}|^2$$

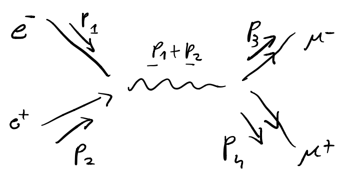
$\mathcal{L}_{int} = -e :\left[\bar{\Psi}_e \not{A} \Psi_e + \bar{\Psi}_\mu \not{A} \Psi_\mu\right]:$ , we want to select $\langle \mu^+\mu^- | \quad | e^+ e^- \rangle$

We take the 2nd order contribution (the first with physical sense)

$$\frac{(-ie)^2}{2} T\left( :(\bar{\Psi}_e \not{A} \Psi_e + \bar{\Psi}_\mu \not{A} \Psi_\mu)_{x_1}: \; :(\bar{\Psi}_e \not{A} \Psi_e + \bar{\Psi}_\mu \not{A} \Psi_\mu)_{x_2}: \right)$$

$$T\left( :(\bar{\Psi}_e \underbrace{\not{A} \Psi_e)_{x_1} : \, :(\bar{\Psi}_\mu \not{A}}_{\text{only 1 possible contraction}} \Psi)_{x_2} : \right) + T\left( :(\bar{\Psi}_\mu \not{A} \Psi_\mu)_{x_1}: \, :(\bar{\Psi}_\mu \not{A} \Psi_\mu)_{x_2}: \right)$$

II) We write the $\mathcal{M}$ elements from the Feynman rules

The system 


$$\mathcal{M} = \bar{u}_i(P_3, n_3)(-ie\gamma_\nu)_{ij} v_j(P_4, n_4) \frac{-i g^{\mu\nu}}{(P_1+P_2)^2} \bar{v}_k(P_2, n_2)(-ie\gamma_\mu)_{kl} u_l(P_1, n_1)$$

$$= ie^2 \bar{u}_{3,i} (\gamma_\nu)_{ij} v_{4,j} \frac{1}{(P_1+P_2)^2} \bar{v}_{2,k} (\gamma^\nu)_{kl} u_{1,l}$$

$$\mathcal{M}^* = -ie^2 \bar{v}_{4j} (\gamma_\nu)_{ji} u_{3i} \frac{1}{(P_1+P_2)^2} \bar{u}_{1l} (\gamma^\nu)_{lk} v_{2k}$$

$$\Rightarrow |\mathcal{M}|^2 = \frac{e^4}{(P_1+P_2)^4} \left( \bar{u}_3 (\gamma_\nu)_{ij} v_{4j} \, \bar{v}_{4j} (\gamma_\nu)_{ji} u_{3i} \right) \left( \bar{v}_{2k} (\gamma^\nu)_{kl} u_{1l} \, \bar{u}_{1l} (\gamma^\nu)_{lk} v_{2k} \right)$$

# III) Polarizations and traces

Usually in lit we have a polar. beam

Unpolarized beam:

$$\frac{1}{4}\sum_{s,s'}|\mathcal{M}|^2=\frac{1}{4}\frac{e^4}{(P_1+P_2)^4}\sum\overbrace{\left[\left(\bar{u}_3(\gamma_\nu)_{ij}v_{4j}\,\bar{v}_{4j}(\gamma_\nu)_{ji}u_{3i}\right)\left(\bar{v}_{2k}(\gamma^\nu)_{kl}u_{1l}\,\bar{u}_{1l}(\gamma^\nu)_{lk}v_{2k}\right)\right]}^{\text{trace}}$$

$$\Rightarrow \frac{1}{4}\sum_{s,s'}|\mathcal{M}|^2=\frac{1}{4}\frac{e^4}{(P_1+P_2)^4}\,tr\left(\sum u_3\,\bar{u}_3\,\gamma_\nu\,v_4\,\bar{v}_4\,\gamma_{\nu'}\right)tr\left(\sum v_2\,\bar{v}_2\,\gamma^\nu\sum u_1\bar{u}_1\gamma^{\nu'}\right)$$

$$\sum_{\pm h} u_3\bar{u}_3=\frac{\not{P}_3+m_\mu}{2m_\mu}\qquad \sum_{\pm h} v_4\bar{v}_4=\frac{\not{P}_4-m_\mu}{2m_\mu}$$

$$\Rightarrow \frac{1}{4}\sum_{s,s'}|\mathcal{M}|^2=\frac{e^4}{64(P_1+P_2)^4}\frac{1}{m_\mu^2m_e^2}\underbrace{tr\left((\not{P}_3+m_\mu)\gamma_\nu(\not{P}_4-m_\mu)\gamma_{\nu'}\right)}_{tr1}\underbrace{tr\left((\not{P}_1+m_e)\gamma^{\nu'}(\not{P}_2-m_e)\gamma^{\nu}\right)}_{tr2}$$

read: traces of $\gamma$ matrices, week 4

- $tr1=tr\left(\not{P}_3\gamma_\nu\not{P}_4\gamma_{\nu'}-\cancel{m_\mu\not{P}_3\gamma_\nu\gamma_{\nu'}}+\cancel{m_\mu\gamma_\nu\not{P}_4\gamma_{\nu'}}-m_\mu^2\gamma_\nu\gamma_{\nu'}\right)=$
  $=tr\left(\not{P}_3\gamma_\nu\not{P}_4\gamma_{\nu'}-m_\mu^2\gamma_\nu\gamma_{\nu'}\right)$

$tr(\gamma_\nu\gamma_{\nu'})=g_{\nu\nu'}$ ; $tr(\not{P}_3\gamma_\nu\not{P}_4\gamma_{\nu'})=4(P_{3\nu}P_{4\nu'})+4(P_{3\nu'}P_{4\nu})-4(P_3P_4)g_{\nu\nu'}$

$$\Rightarrow \boxed{tr1=4(P_{3\nu}P_{4\nu'})+4(P_{3\nu'}P_{4\nu})-4(P_3P_4)g_{\nu\nu'}-4m_\mu^2g_{\nu\nu'}}$$

- $\boxed{tr2=4(P_{1\nu}P_{2\nu'})+4(P_{1\nu'}P_{2\nu})-4(P_1P_2)g_{\nu\nu'}-4m_e^2g_{\nu\nu'}}$

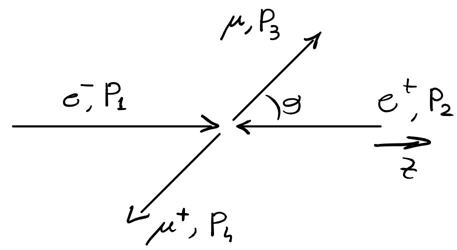
$$\Rightarrow tr1\,tr2=32(P_1P_3)(P_2P_4)+32(P_1P_4)(P_2P_3)+32m_e^2(P_3P_4)+32m_\mu^2(P_1P_2)+64m_e^2m_\mu^2$$

$$\Rightarrow \frac{1}{4}\sum|\mathcal{M}|^2=\frac{e^4}{2(P_1+P_2)^4}\frac{1}{m_e^2m_\mu^2}\left[(P_1P_3)(P_2P_4)+(P_1P_4)(P_2P_3)+m_e^2(P_3P_4)+m_\mu^2(P_1P_2)+2m_e^2m_\mu^2\right]\quad \textcircled{A}$$

## IV) We consider the case $E_1 = E_2$, $\vec{p}_1 = -\vec{p}_2$

$$p_1^\mu = (E_1, \vec{p}_1) \qquad p_2^\mu = (E_2, -\vec{p}_1)$$

$$P_1^2 = m_e^2 = P_2^2 \Rightarrow E_1 = E_2 = E$$




$$\Rightarrow P_1^\mu = (E, \vec{p}) \; ; \; P_2^\mu = (E, -\vec{p}) \Rightarrow (P_1 + P_2)^2 = 4E^2$$

cylindrical symmetry around $z$ direction

$$\begin{cases} P_3^\mu = (E_3, \vec{p}_3) \\ P_4^\mu = (E_4, -\vec{p}_3) \end{cases} \Rightarrow E_3 = E_4 \Rightarrow \begin{cases} P_3^\mu = (E, \vec{p}\,') ; \\ P_4^\mu = (E, -\vec{p}\,') \end{cases}$$

cons. of energy

$$\Rightarrow \begin{cases} (P_1 P_2) = E^2 + p^2 \;,\; (P_3 P_4) = E^2 + p'^2 \\ (P_1 P_3) = (P_2 P_4) = E^2 - \vec{p}\cdot\vec{p}\,' = E^2 - pp'\cos\vartheta \\ (P_1 P_4) = (P_2 P_3) = E^2 + pp'\cos\vartheta \end{cases}$$

## V) We put the products $P_a P_b$ into (4)

$$\Rightarrow \frac{1}{4}\sum |\mathcal{M}|^2 = \frac{e^4}{2 m_\mu^2 m_e^2 16 E^4}\left[(E^2 - pp'\cos\vartheta)^2 + (E^2 + pp'\cos\vartheta)^2 + m_e^2(E^2 + p'^2) + m_\mu^2(E^2 + p^2) + 2 m_e^2 m_\mu^2\right]$$

$$d\sigma = (2\pi)^4 \delta^4(P_f - P_i)\frac{1}{\left[4\sqrt{(P_1 P_2)^2 - m_1^2 m_2^2}\right]}\frac{d^3 p_3}{(2\pi)^3 2E_3}\frac{d^3 p_4}{(2\pi)^3 2E_4}\left[(2m_e)^2(2m_\mu)^2 \frac{1}{4}\sum_{r,s,r',s'} |\mathcal{M}|^2\right]$$

## VI) We approximate $m_e \ll m_\mu$

$$\odot \left[16\, m_e^2 m_\mu^2 \frac{1}{4}\sum |\mathcal{M}|^2\right] \underset{m_e \ll m_\mu}{\simeq} \frac{e^4}{2E^4}\left(2E^4 + 2\underbrace{p^2}_{\substack{\| m_e = 0 \\ E^2}} p'^2\cos^2\vartheta + m_\mu^2 E^2 + m_\mu^2 \underbrace{p^2}_{\substack{/\!/ m_e = 0 \\ E^2}}\right)$$

$$\simeq e^4 (E^2 + p'^2\cos^2\vartheta + m_\mu^2)\frac{1}{E^2}$$

$$\odot \left[4\sqrt{(P_1 \cdot P_2)^2 + \cancel{m_e^2 m_\mu^2}}\right] \overset{\substack{\text{in our case} \\ p \simeq E}}{\simeq} 8E^2$$

(129)

$$\Rightarrow d\sigma = (2\pi)^4 \delta^4(P_f - P_i) \frac{e^4}{8E^4}(E^2 + p'^2\cos^2\vartheta + m_\mu^2) \frac{d^3p_3}{(2\pi)^3 2E} \frac{d^3p_4}{(2\pi)^3 2E} =$$

$$= \underbrace{\delta(2E - 2E')}_{=\frac{1}{2}(E-E')} \frac{e^4}{128\pi^2 E^6}(E^2 + p'^2\cos^2\vartheta + m_\mu^2)\, d^3p'$$

$$d^3p' = |p'|^2 dp'\, d\Omega = p'^2 dp'\, d(\cos\vartheta)\, d\varphi$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{e^4}{128\pi^2 E^6} \frac{1}{2} \int \delta(E-E')(E^2 + p'^2\cos^2\vartheta + m_\mu^2) p'^2 dp'$$

We change variable $\int dp' \to \int dE'$, $p'^2 = E'^2 - m_\mu^2 \Rightarrow \frac{dp'}{dE'} = \frac{E'}{p'} = \frac{E'}{\sqrt{E'^2 - m_\mu^2}}$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{e^4}{128\pi^2 E^6} \frac{1}{2} \int \delta(E-E')\left(E^2 + (E'^2 - m_\mu^2)\cos^2\vartheta + m_\mu^2\right)\sqrt{E'^2 - m_\mu^2}\, E'\, dE' =$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{e^4}{128\pi^2 E^6} \frac{1}{2} \int (E^2 + E^2\cos^2\vartheta - m_\mu^2\cos^2\vartheta + m_\mu^2)\sqrt{E^2 - m_\mu^2}\, E\, dE =$$

VII) We consider the high energy limit : $E \gg m_\mu$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{e^4}{128\pi^2 E^6} \frac{1}{2} E^4 (1 + \cos^2\vartheta)$$

Let $\alpha = \frac{e^2}{4\pi} \Rightarrow \alpha^2 = \frac{e^4}{16\pi^2}$

$$\Rightarrow \frac{d\sigma}{d\cos\vartheta\, d\varphi} = \frac{\alpha^2}{16E^2}(1 + \cos^2\vartheta)$$ , next we integrate $\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta\,(1+\cos^2\vartheta) = \frac{16\pi}{3}$

$$\Rightarrow \boxed{\sigma = \frac{\alpha^2\pi}{3E^2} \simeq 5.6\times 10^{-5}\, \frac{1}{E^2}}$$