• Anti-commutation relations and positive definite energy density.

We would like to identify $b, b^\dagger, d, d^\dagger$ with annihilation and creation operators, also we would like to avoid negative En. for 1-particle states

If we ignored the statystic and imposed comm. rel, H would gain a "−1" sign
$\Rightarrow$ ~~states~~ with $E<0$

In order to get rid of $E<0$, we can impose anti-commutation relations obtaining:

$$\boxed{\begin{aligned} &[b(p,n), b^\dagger(p',n')]_+ = [d(p,n), d^\dagger(p',n')]_+ = \delta_{nn'}\,\delta(p-p') \\ &[d(p,n), d(p',n')]_+ = \quad \dots \quad = 0 \end{aligned}}$$

$$\Rightarrow \boxed{\begin{aligned} :H: &= \sum \int d^3p\, E\,[b^\dagger b + d^\dagger d] \\ :\underline{P}: &= \sum \int d^3p\, \underline{p}\,[b^\dagger b + d^\dagger d] \end{aligned}}$$

We subtracted Energy of the vacuum

From Nöther's theorem, we obtain conserved quantities from $j^\mu = \bar\psi \gamma^\mu \psi$, $\partial_\mu j^\mu = 0$

$$\frac{d}{dt} Q = \frac{d}{dt} \int d^3x\, j^0 = 0 \quad ; \quad Q = \int d^3x\, \psi^\dagger \psi = \sum_{\pm n} \int d^3p\, [b^\dagger(p,n) b(p,n) + d(p,n) d^\dagger(p,n)]$$

"+" sign because there is no derivative as in $H = i\int \psi \frac{\partial}{\partial t} \psi$

If, instead, $\mathcal{L} = \bar\psi (i\partial\!\!\!/ - eA\!\!\!/ - m)\psi$

couples ψ field and e.m. field

$\hookrightarrow e\bar\psi\gamma^\mu\psi A_\mu$, a term with electric-charge and the conserved current

$\Rightarrow$ We write $j^\mu = e\bar{\psi}\gamma^\mu\psi$; $Q = e\int d^3x\, \psi^+\psi$ I interpret this as the electric charge

$\Rightarrow :Q: = \int d^3p\, e\left[b^+b - d^+d\right]$ makes us capable to distinguish e.s. of the charge with $E>0$ and $e>0$ and es. of the charge with $E>0$ and $e<0$

• Fock space, Fermi statistic

The energy is made of a b-energy and a d-energy

We see that the Hilbert space of the theory is that of a set of Fermi oscillators. Explicitly, it consists of:

- $|0\rangle$, the vacuum, such that $b_n(p)|0\rangle = d_{n'}(p')|0\rangle = 0$ for each n, n', p, p'
- The states with given occupation numbers:

$$|i_1, i_2, \dots; j_1, j_2, \dots\rangle = [b^+_{n_1}(p_1)]^{i_1}[b^+_{n_2}(p_2)]^{i_2}\dots[d^+_{n'_1}(p'_1)]^{j_1}[d^+_{n'_2}(p'_2)]^{j_2}\dots|0\rangle$$

As can be seen from the expression for the momentum, the a and b operators destroy relativistic particles of mass m and spin 1/2 . The quantum states are those of a perfect gas made up of fermions of two different types, with equal mechanical properties.

1-particle states : $\begin{cases} b^+|0\rangle = |p^{(n)}\rangle \\ d^+|0\rangle \end{cases}$ Fock space : $b^+|0\rangle = |p_b\rangle$, $d^+|0\rangle = |p_d\rangle$

If we wanted to write the wavef. of a 2-part. state : $\int dp_1 dp_2\, f(p_1,p_2)\, b^+_{p_1} b^+_{p_2}|0\rangle$

It has to satysfy Fermi-statystic, otherwise it's $\emptyset$ $\Rightarrow$ ψ antisymm

• Locality and causality

MAIANI pag 125 many interesting infos

Let's impose anti-comm rel. for the fields, we will be interested in what happens to the observables :

$$[\psi(\underline{x},t),\psi^\dagger(\underline{y},t)]_+ = \sum_{\substack{\pm n\\ \pm n'}} \int \frac{d^3p}{(2\pi)^3}\, d^3p' \sqrt{\frac{m}{E}}\sqrt{\frac{m}{E'}} \Big\{ \left[b u e^{-ipx} + d^\dagger v e^{ipx}\right]\left[d v^\dagger e^{-ipy} + b^\dagger u^\dagger e^{ipx}\right] + [\ ][\ ]\Big\} =$$

will be useful *

We found $[b,d]=0$, $[b^\dagger,d^\dagger]=0$, let's impose it:

$$= \sum_{\substack{\pm n\\ \pm n'}} \int \frac{d^3p\, d^3p'}{(2\pi)^3} \frac{m}{\sqrt{EE'}} \delta_{n,n'}\, \delta(\underline{p}-\underline{p}')\Big\{$$

$e^{i\underline{p}(\underline{x}-\underline{y})}$ ⟵ because $x^0=y^0$ * ⟶ $e^{-i\underline{p}(\underline{x}-\underline{y})}$

$$u(p,n)\,\bar{u}(p',n')\,\gamma^0\gamma^0\, e^{ip(x-y)} + v(p,n)\,\bar{v}(p',n')\,\gamma^0\gamma^0\, e^{-ip(x-y)}\Big\} =$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{m}{E}\left\{\frac{\not p+m}{2m} e^{i\underline{p}\cdot(\underline{x}-\underline{y})} + \frac{(\not p-m)}{2m} e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}\right\}\gamma^0 =$$

$\tilde{\not p}$, $-\underline{p}$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{m}{E} e^{i\underline{p}\cdot(\underline{x}-\underline{y})}\left\{\frac{(\not p+m)}{2m} + \frac{(\tilde{\not p}-m)}{2m}\right\}\gamma^0 \quad ; \quad \not p + \tilde{\not p} = 2E\gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{\cancel{m}}{\cancel{E}} e^{i\underline{p}(\underline{x}-\underline{y})} \frac{\cancel{2E}\gamma^0}{\cancel{2m}}\gamma^0 = \delta^3(\underline{x}-\underline{y})$$

$$\Rightarrow [\psi(\underline{x},t),\psi^\dagger(\underline{y},t)]_+ = \delta^3(\underline{x}-\underline{y})$$

$$\Rightarrow [\psi(\underline{x},t),\psi(\underline{y},t)]_+ = 0$$

$$\Rightarrow [\psi^\dagger(\underline{x},t),\psi^\dagger(\underline{y},t)]_+ = 0$$

} Same calculations

If ψ was an observable, $[\psi(\underline{x},t), \psi(\underline{y},t)]_- = 0$, but it is not $\emptyset$

We would like to preserve casuality for the observables, the observables are bilinear in the field

We can prove that the bilinears in Dirac field satisfy comm-relations

A, B anti-comm with c

$$[AB, C] = A[B, C] + [A, C]B =$$

$$= A(BC - CB) + (AC - CA)B = ABC - ACB + ACB - CAB =$$

$$= A\underbrace{[B,C]_+}_{=0} - \underbrace{[A,C]_+}_{=0} B \Rightarrow \quad [AB, C] = 0$$

Let's consider $j^\mu = \bar{\psi}\gamma^\mu\psi$; $[j^\mu(x), j^\nu(y)]_{(x-y)^2<0} = 0$

$$\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(x)\gamma^\nu\psi(y) =$$

$$= \sum_{\alpha\beta}\sum_{\varrho\sigma} \bar{\psi}_\alpha(x)\gamma^\mu_{\alpha\beta}\psi_\beta(x)\bar{\psi}_\varrho(y)\gamma^\nu_{\varrho\sigma}\psi_\sigma(y) =$$

$$= \sum\sum \gamma^\mu_{\alpha\beta}\gamma^\nu_{\varrho\sigma}\bar{\psi}_\alpha(x)\psi_\beta(x)\bar{\psi}_\varrho(y)\psi_\sigma(y)$$

$$- \quad \varrho \curvearrowright \beta$$

$$= -\sum\sum\gamma^\mu_{\alpha\beta}\gamma^\nu_{\varrho\sigma}\bar{\psi}_\varrho(y)\bar{\psi}_\alpha(x)\psi_\sigma(y)\psi_\beta(x)$$

$$= \sum\sum\gamma^\mu\gamma^\nu\bar{\psi}_\varrho(y)\psi_\sigma(y)\bar{\psi}_\alpha(x)\psi_\beta(x) = \bar{\psi}(y)\gamma^\nu\psi(y)\bar{\psi}(x)\gamma^\mu\psi(x)$$

$$\Rightarrow [\quad] = 0$$

• Massless Dirac field, the Weyl neutrino

More details MAIANI p. 197

We consider the Dirac equation in the limit of zero mass:

$$\boxed{i\not{\partial}\psi(x)=0} \; *$$

The Lor group can be recoupled, two different irreducible repres. of Lor. group.

In a new representation

$$\gamma^0=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix};\quad \gamma^i=\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix};\quad \gamma_5=\begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix},\ \gamma_5 \text{ is diagonal}$$

⇒ We can separate the solutions into two invariant subspaces by means of the projection operators:

$$P_R=\frac{1+\gamma_5}{2};\quad P_L=\frac{1-\gamma_5}{2}\qquad P_R\psi=\frac{1+\gamma_5}{2}\psi=\begin{pmatrix}\psi_R\\0\end{pmatrix}$$

We can select $P_R\psi=\frac{1+\gamma_5}{2}\psi=\begin{pmatrix}\psi_R\\0\end{pmatrix}$; $P_L\psi=\frac{1-\gamma_5}{2}\psi=\begin{pmatrix}0\\\psi_L\end{pmatrix}$

⇒ * becomes 2 eq.s : $\begin{cases}\partial^0\psi_R-i\underline{\sigma}\cdot\underline{\nabla}\psi_R=0\\ \partial^0\psi_L+i\underline{\sigma}\cdot\underline{\nabla}\psi_L=0\end{cases}\quad\begin{cases}i\partial^0\psi_R+\underline{\sigma}\cdot\underline{p}\,\psi_R=0\\ i\partial^0\psi_L-\underline{\sigma}\cdot\underline{p}\,\psi_L=0\end{cases}$

Elicity: $$\boxed{h:=\frac{\underline{\sigma}\cdot\underline{p}}{2|\underline{p}|}}$$

$p^2=0;\quad E=|p|$

$\psi_R=\psi_R^0\,e^{\mp ip_\mu x^\mu}\Rightarrow(\pm p^0+\underline{\sigma}\cdot\underline{p})\psi_R=0\Rightarrow h\psi_R=\mp\frac{1}{2}$

$-$: neutrinos $h=-\frac{1}{2}$

$+$: antineutrinos $h=+\frac{1}{2}$

Both ψ_R and ψ_L are legitimated, but we find L-neutrinos and R-antineutrinos, we would select one of the two R or L with $P_{R,L}$ projectors

• Electromagnetic field

• Maxwell's equations and gauge invariance

$$\begin{cases} \underline{\nabla}\cdot\underline{E} = \rho & \mathrm{I} \\ \underline{\nabla}\cdot\underline{H} = 0 & \mathrm{II} \end{cases} \quad ; \quad \begin{matrix} \underline{\nabla}\wedge\underline{E} + \frac{1}{c}\frac{\partial}{\partial t}\underline{H} = 0 & \mathrm{III} \\ (\underline{\nabla}\wedge\underline{H}) - \frac{1}{c}\frac{\partial}{\partial t}\underline{E} = \frac{1}{c}\,\underline{j} & \mathrm{IV} \end{matrix}$$

$\underset{+\mathrm{I}}{\underline{\nabla}\cdot}\ \mathrm{IV} \Rightarrow \frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\underline{j} = 0$ (Continuity eq.), let be $\underline{H} = \underline{\nabla}\wedge\underline{A}$

$$\mathrm{III}:\ \underline{\nabla}\wedge\underbrace{\left(\underline{E} + \frac{1}{c}\frac{\partial}{\partial t}\underline{A}\right)}_{-\underline{\nabla}\phi} = 0 \qquad \Rightarrow \underline{E} = -\underline{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\underline{A} \quad \star$$

$$\mathrm{I}:\ \underline{\nabla}\cdot\underline{E} = \rho \ \overset{\star}{\Rightarrow} \qquad \boxed{\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\underline{\nabla}\cdot\underline{A} = \rho} \ ;$$

$$\mathrm{IV}:\qquad \left(-\underline{\nabla}(\underline{\nabla}\cdot\underline{A}) + \nabla^2\underline{A}\right) + \frac{1}{c}\frac{\partial}{\partial t}\underline{\nabla}\phi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{A} = \frac{1}{c}\,\underline{j}$$

$$\Rightarrow \boxed{\nabla^2\underline{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{A} = -\frac{1}{c}\,\underline{j} - \underline{\nabla}\left(\underline{\nabla}\cdot\underline{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi\right)}$$

Gauge transformations:

eq.s are invariant under the transformations
$$\begin{cases} \underline{A} \to \underline{A} + \underline{\nabla}\chi & \text{, gives the same } \underline{H} \\ \phi \to \phi - \frac{1}{c}\frac{\partial\chi}{\partial t} & \text{, gives the same } \underline{E} \end{cases}$$

Due to gauge, we can choose α such that
$$\begin{cases} \left(\underline{\nabla}\cdot\underline{A}' + \frac{1}{c}\frac{\partial}{\partial t}\phi = 0\right) \\ \nabla^2\underline{A}' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{A}' = -\frac{1}{c}\,\underline{j} \end{cases}$$

$$\nabla^2\chi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\chi = -\underline{\nabla}\cdot\underline{A} - \frac{1}{c}\frac{\partial\phi}{\partial t} \quad \Rightarrow \quad \boxed{\begin{cases} \nabla^2\phi' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi' = -\rho \\ \nabla^2\underline{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{A} = -\frac{1}{c}\,\underline{j} \end{cases}} \ ✡$$

We can impose up to 2 constraints $\Rightarrow$ we can impose $\boxed{\nabla^2\chi' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\chi' = 0}$ too

• Covariant form of Maxwell's equations, Lorentz gauge

Let be $\begin{cases} j^\mu = (c\rho, \underline{j}) \\ A^\mu = (\phi, A^\mu) \end{cases}$, we know $\partial_\mu j^\mu = 0$ (j^μ external current)

Covariant form of D'Alambertian: $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \partial_\mu \partial^\mu = \partial^2$

$$\Rightarrow \boxed{\partial^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu}$$

$A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi \quad \Rightarrow$ Lorentz gauge: $\boxed{\partial_\nu A^\nu = 0}$

Electromagnetic tensor: $\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -F^{\nu\mu}}$

$$\underline{E} = -\underline{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\underline{A} \Rightarrow E^i = \partial^i A^0 - \partial^0 A^i$$

$$\underline{H} = \underline{\nabla} \wedge \underline{A} \qquad \Rightarrow H^j = \varepsilon_{ijk}\partial^j A^k$$

$\Rightarrow F^{\mu\nu}$ can be written as a 4×4 matrix $F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -H^3 & +H^2 \\ E^2 & H^3 & 0 & -H^1 \\ E^3 & -H^2 & H^1 & 0 \end{pmatrix}$

$\Rightarrow$ The homogeneous Maxwell's eq.s can be written as $\boxed{\partial^\mu F^{\nu\sigma} + \partial^\sigma F^{\mu\nu} + \partial^\nu F^{\sigma\mu} = 0}$ ⊙ .

It becomes more symmetric if we define the dual of F:

$\boxed{\mathcal{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}}$ (Exchanged E, H components)

$\Rightarrow$ ⊙ becomes $\boxed{\partial_\mu \mathcal{F}^{\mu\nu} = 0}$

The dual tensor does not contain sources, unlike F.
Because the conversion from F to $\mathcal{F}$ requires the interchange of electric and magnetic fields, $\partial_\mu \mathcal{F}^{\mu\nu} = 0$ implies the absence of magnetic monopoles.

• Lagrangian density

We have the two objects A^μ, $F^{\mu\nu}$, we look for a scalar, so we have to contract indices.

▪ From Lagrangian to eq.s of motion

A choice : $\boxed{\mathcal{L} = a\, F_{\mu\nu} F^{\mu\nu}}$, not the only because with $F^{\mu\nu}$ we can compose more complex scalars, such as $F_{\mu\lambda} F^{\lambda}{}_{\sigma} F^{\sigma\mu}$

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, let's see that it gives the correct motion equations

$\Rightarrow$ We want to find $\boxed{\dfrac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \dfrac{\partial \mathcal{L}}{\partial (A_{\mu,\nu})} = 0}$

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) =$$

$$= -\frac{1}{4}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu\right]$$

they are the same

they are the same

$$= -\frac{1}{2}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right]$$

the Maxw. eq we know

$$\Rightarrow \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = 0 \Rightarrow \partial^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$$

▪ From eq.s of motion to Lagrangian

Let's impose $0 = \delta S = \int d^4x \left(\partial^2 A_\nu - \partial_\nu(\partial^\mu A_\mu)\right)\delta A^\nu$

We want to integrate back by parts to move the δ in top of the integral

$$0 = \int d^4x \left(\partial_\mu \partial^\mu A_\nu \delta A^\nu - \partial_\nu(\partial^\mu A_\mu)\delta A^\nu\right) =$$

total derivative, surface term

$$= \int d^4x \left\{ \partial^\mu \cancel{\left[\partial_\mu A_\nu \delta A^\nu\right]} - \partial^\mu A_\nu \delta(\partial_\mu A^\nu) - \partial^\mu \cancel{\left[\partial_\nu A_\mu \delta A^\nu\right]} + \partial_\nu A_\mu (\partial^\mu A^\nu) \right\}$$

$$= \int d^4x \left\{ - \partial_\mu A_\nu \delta(\partial^\mu A^\nu) + \partial_\nu A_\mu \delta(\partial^\mu A^\nu) \right\}$$

$$= \int d^4x \left\{ -\frac{1}{2}\delta(\partial^\mu A^\nu)\partial_\mu A_\nu - \frac{1}{2}\delta(\partial^\nu A^\mu)\partial_\nu A_\mu + \frac{1}{2}\delta(\partial^\mu A^\nu)\partial_\nu A_\mu + \frac{1}{2}\delta(\partial^\nu A^\mu)\partial_\mu A_\nu \right\}$$

$$= \int d^4x \left(-\frac{1}{2}\delta(F_{\mu\nu})F^{\mu\nu}\right) = \delta \int d^4x \left(\underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}}\right)$$

$\mathcal{L}$ is gauge-independent and a scalar

- Counting of the actual degrees of freedom in covariant form

The components of A^μ are not all independent

Let's consider the transform of the field: $A^\mu(x) = \int d^4p \, e^{ip^\mu x_\mu} A^\mu(p)$

The eq.s of motion are $\quad \partial^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$

$$\Rightarrow 0 = \int d^4p \left[-p^2 A^\mu(p) + p^\mu(p_\nu A^\nu)\right] \Rightarrow \boxed{-p^2 A^\mu(p) + p^\mu(p_\nu A^\nu) = 0} \; *$$

Let's take a basis for $p^\mu = (E, \mathbf{p})$, $\tilde{p}^\mu = (E, -\mathbf{p})$

Let be $\varepsilon^\lambda_\mu(p)$, $\lambda = 1,2$ such that $\varepsilon^\lambda_\mu p^\mu = \varepsilon^\lambda_\mu \tilde{p}^\mu = 0$ #

$\lambda = 1,2 \Rightarrow \varepsilon^1_\mu(p), \varepsilon^2_\mu(p)$ are two 4-vectors

Let's suppose $A^\mu(p) = a_\lambda(p)\varepsilon^{\lambda\mu} + b\, p^\mu + c\,\tilde{p}^\mu$

$*$
$$\Rightarrow 0 = -P^2(a_\lambda \varepsilon^{\lambda\mu} + bP^\mu + c\tilde{P}^\mu) + P^\mu(a_\lambda P_\nu \varepsilon^{\lambda\nu} + bP^2 + cP_\nu \tilde{P}^\nu)$$

$$\Rightarrow 0 = -P^2 a_\lambda \varepsilon^{\lambda\mu} - \cancel{P^2 b P^\mu} - P^2 c\tilde{P}^\mu + \cancel{a_\lambda \underbrace{P^\mu P_\nu \varepsilon^{\lambda\nu}}_{=0\ \#}} + \cancel{P^2 b P^\mu} + cP^\mu P_\nu \tilde{P}^\nu$$

$$\Rightarrow 0 = -P^2 a_\lambda \varepsilon^{\lambda\mu} - P^2 c\tilde{P}^\mu + cP_\nu \tilde{P}^\nu P^\mu$$

$P^2 = 0$ because $m = 0$ $\Rightarrow cP_\nu \tilde{P} P^\mu = 0$ $\Rightarrow \begin{cases} c = 0 \\ a_\lambda \text{ can be } \neq 0 \\ b \text{ is not constrained} \end{cases}$

In a certain gauge, $b' = 0$ $\Rightarrow$ remain only the 2 transverse components which are physical

Proof:

$$A_\mu(x) \longrightarrow A'_\mu(x) + \partial_\mu \chi \quad \Rightarrow A_\mu(p) \longrightarrow A_\mu(p) + iP_\mu \chi(p)$$

$$A_\mu(p) = a_\lambda \varepsilon^\lambda_\mu + bP_\mu + c\tilde{P}_\mu \rightarrow a_\lambda \varepsilon^\lambda_\mu + (b + i\chi)P_\mu + c\tilde{P}_\mu$$

$\Rightarrow$ we can choose χ such a way that $b + i\chi = 0$

$\Rightarrow$ pysics is only associated to a_λ component, the transverse component

In Coulomb gauge : $\boxed{A^0 = \underline{\nabla} \cdot \underline{A} = 0}$

• Canonical Quantization of the field A^μ, problems!

Problem: we have an object with apparently 4 components, but 2 independent, we will see that we end up with a nonsense

We introduce the conjugated momenta $\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu}$

Next we impose $\left[A_\mu(\underline{x},t), \pi_\nu(\underline{y},t)\right] = i\eta_{\mu\nu}\delta^3(\underline{x}-\underline{y})$

Equal time, different points

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\left[A_{\mu,\nu}A^{\mu,\nu} - A_{\mu,\nu}A^{\nu,\mu}\right]$$

$$\Rightarrow \pi^\mu = -A^{\mu 0} + A^{0\mu} = F^{0\mu}, \text{ but } \pi_0 \overset{= F^{00}}{\equiv} 0 \Rightarrow \left[A_0, \pi_0\right] \equiv 0, \text{ problem!}$$

• Gupta-Bleuler quantization

28/10/2020

We encountered a problem, but we can impose the Lorentz gauge obtaining $\mathcal{L} = \underbrace{\underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{gauge-invariant}} - \frac{1}{2}(\partial_\mu A^\mu)^2}_{\text{not anymore gauge invariant}}$

Proof: Because of Lor. gauge, we want to recover $\partial^2 A^\mu = 0$

$\partial^2 A^\mu = 0$ comes from a lagrangian like $\mathcal{L} = -\frac{1}{2}\partial_\nu A_\mu \partial^\nu A^\mu$

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{GF}$$

gauge fixing

$$\Rightarrow \mathcal{L}_{GF} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \left(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu\right) =$$

$$= \cancel{-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu} + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu + \cancel{\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu} =$$

$$= \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu \overset{\text{PARTS}}{=} \partial_\nu\left(\frac{1}{2}(\partial_\mu A_\nu)A^\mu\right) - \frac{1}{2}(\partial^\nu\partial_\mu A_\nu)A^\mu$$

$$= \cancel{\partial_\nu\left(\tfrac{1}{2}(\partial_\mu A_\nu)A^\mu\right)} + \cancel{\partial_\mu\left(-\tfrac{1}{2}(\partial^\nu A_\nu)A^\mu\right)} + \tfrac{1}{2}(\partial_\mu A^\mu)^2$$

They are two 4-divergences, don't affect eq. of motion

$$\Rightarrow \boxed{\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)^2}$$ which doesn't affect eq.s of motion

• Plane wave solutions, physical states

28/10/2020

Now we can calculate $\pi^\mu = \dfrac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = F^{\mu 0} - \eta^{\mu 0}(\partial_\nu A^\nu)$

If $\mu=0$, $\pi^0 = 0 - 1(\partial_\nu A^\nu) \neq 0$, good

We can enlarge the space were the operators A, π act, we don't want $\pi^0 = 0$ at the operator level, but will be $\varnothing$ in the physics:

$$\langle phy | -\partial_\nu A^\nu | phy \rangle = 0$$

In the Fock space we included non-physical states where $\langle |\pi| \rangle \neq 0$ and this preserves the commutation relations

$$[A_\mu(\underline{x},t), \pi_\nu(\underline{y},t)] = i\,\eta_{\mu\nu}\,\delta(\underline{x}-\underline{y}) \;;\quad \begin{cases} \pi^0 = -\dot{A}^0 - \partial_i A^i \\ \pi^i = -\dot{A}^i - \partial_i A^0 \end{cases}$$

$$\Rightarrow [A_0(\underline{x},t), -\dot{A}_0(\underline{y},t) - \partial_i A^i(\underline{y},t)] = [A_0(\underline{x},t), -\dot{A}_0(\underline{y},t)] = i\,\delta^3(\underline{x}-\underline{y})$$

In fact $[A_0(x), -\partial_i A^i(x)] = \partial_i(-A_0(x)A^i(y) + A_0(x)A^i(y)) = 0$

because ∂_i acts on y, not on x

$$[A^i(\underline{x},t), -\dot{A}^i - \partial^i A^0] = [A^i, -\dot{A}^i] = -i\,\delta^3(\underline{x}-\underline{y})$$

$$[A_\mu(\underline{x},t), \dot{A}_\nu(\underline{y},t)] = -i\eta_{\mu\nu}\,\delta^3(\underline{x}-\underline{y})$$

FROM HERE MAGGIORE-BENHAR

93

We can choose the frame where $P^\mu = (p,0,0,p)$

Let be $\varepsilon^{(0)}_\mu = (1,0,0,0) = n^\mu$; $n^2 = 1$; $\varepsilon^{(1)}_\mu = (0,1,0,0)$

$\varepsilon^{(2)}_\mu = (0,0,1,0)$; $\varepsilon^{(3)}_\mu = (0,0,0,1)$

The two vectors $\varepsilon^{(1)}_\mu, \varepsilon^{(2)}_\mu$ satisfy $\boxed{\varepsilon^{(1,2)}_\mu n^\mu = 0 = \varepsilon^{(1,2)}_\mu P^\mu}$

$\varepsilon^\lambda_\mu \varepsilon^{\lambda'\mu} = -\delta^{\lambda\lambda'}$ (space-like vectors)

$$\begin{cases} \varepsilon^3_\mu n^\mu = \varepsilon^3_\mu \varepsilon^{(\lambda)\mu} = 0 \\ \varepsilon^3_\mu \varepsilon^{3\mu} = -1 \end{cases} ; \quad \varepsilon^{(3)}_\mu = \overbrace{\frac{P_\mu - (n\cdot P)n_\mu}{(n \cdot P)}}^{(0,0,0,1)}$$

All these vectors are orthonormal: $\begin{cases} \varepsilon^\lambda_\mu \varepsilon^{\lambda'\mu} = \eta^{\lambda\lambda'} \\ \varepsilon^\lambda_\mu \varepsilon^{\lambda'}_\nu = \eta_{\lambda\lambda'} = \eta_{\mu\nu} \end{cases}$ In this r.f. $\varepsilon^{(\lambda)}_\mu = \delta^\lambda_\mu$

We want to write the fields in the system and then look to what happens with a boost

with a boost

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E}} \sum_{\lambda=0}^{3} \varepsilon^\lambda_\mu(p) \left[a_\lambda(p) e^{-iPx} + a^\dagger_\lambda(p) e^{iPx} \right]$$

From this and $[A_\mu(\underline{x},t), \pi_\nu(\underline{y},t)] = i\eta_{\mu\nu}\delta(\underline{x}-\underline{y})$ it follows that

$$[a_\lambda(p), a^\dagger_{\lambda'}(p')] = -\eta_{\lambda\lambda'}\delta^3(\underline{p}-\underline{p}')$$

$$[a,a] = [a^\dagger, a^\dagger] = 0$$

do not have the same behaviour for every polarization

Problem! If we consider $|1,\lambda\rangle = \int d^3p\, f(p)\, a^\dagger |0\rangle$ and we interpret like a 1-particle state:

$$\langle 1,\lambda' | 1,\lambda\rangle = \int d^3p\, d^3p'\, f^* f \langle 0 | [a_\lambda, a^\dagger_{\lambda'}] | 0\rangle = -\eta_{\lambda\lambda'} \int d^3p\, |f|^2$$

$\Rightarrow$ for $\lambda = 0$, the norm (that for 1-part. states is the prob) is < 0 !

We can solve this observing that the states that create the problem have no counterpart in the canonical quantization of the electromagnetic field; we have seen in fact (pag 90 of these notes) that the physical states are only those associated with transverse polarization vectors. The states created by a†p,0 and by a†p,3 (in the frame where p is along the third axis) are unphysical.

The theory that we have quantized so far is not electrodynamics, because of the extra term (∂μAμ)^2 in the Lagrangian:

The basic idea of the covariant quantization of the electromagnetic field, or Gupta–Bleuler quantization, is to start from an apparently different theory and to recover a quantum theory of the electromagnetic field imposing a restriction on the Fock space: we define the subspace of physical states requiring that for any two physical states |phys⟩, |phys'⟩

$$\langle phys | \partial_\mu A^\mu | phys \rangle = 0 \quad \star$$

It is true that $\partial_\mu A^\mu(x) = (\partial_\mu A^\mu)^+ + (\partial_\mu A^\mu)^-$, with

$$\boxed{\partial^\mu A_\mu^{(+)} = -i \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E}}\, e^{-ipx}\, p^\mu \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(p)\, a_\lambda(p)} \quad \text{and} \quad \partial_\mu A_\mu^{(-)} = (\partial_\mu A^{(+)}_\mu)^\dagger \ \#$$

\# A sufficient condition to satisfy $\star$ is to require that

$$\Rightarrow \quad \begin{cases} \partial_\mu A^{(+)\mu} |phys\rangle = 0 \\ \langle phys| \partial_\mu A^{(+)\mu\dagger} = 0 \end{cases}$$

in this way, $\langle phys | \underbrace{\partial_\mu A^{(+)\mu} + \partial_\mu A^{(-)\mu}}_{\partial_\mu A^\mu} | phys \rangle = 0 \quad \Rightarrow \star$ satisfied

$\Rightarrow$ A physical state is such that $\boxed{(\partial_\mu A^\mu)^\dagger |phys\rangle = 0}$

Now we can use $\begin{cases} p^\mu \varepsilon_\mu^{(1)} = p^\mu \varepsilon_\mu^{(2)} = 0 \\ p^\mu \varepsilon_\mu^{(3)} = -(n\,p) = -p^\mu \varepsilon_\mu^{(0)} \end{cases}$

$$\Rightarrow \partial^\mu A_\mu^{(+)} = -i \int \frac{d^3p}{(\dots)}\, e^{-ipx} \big(a_0(p) - a_3(p)\big)(n \cdot p)$$

$\partial^\mu A_\mu^{(+)} |phys\rangle = 0 \quad \Rightarrow \quad \boxed{(a_0(p) - a_3(p))|phys\rangle = 0}$ ph

this is the condition for physical states we use now

We see that the two transverse photons a†p,1|0⟩ and a†p,2|0⟩, and any linear combination of them, **|ψ_T**⟩ are not physical states.
This is good news, since we know from (pag 90) that these are the true degrees of freedom of the photon.

Consider now the subspace generated by a†p,0|0⟩ and a†p,3|0⟩.
We see that neither a†p,0|0⟩ nor a†p,3|0⟩ are physical states, this also good news, since the state a†p,0|0⟩ is just the negative norm state, and a†p,3|0⟩, even if it has a positive norm, does not correspond to a physical polarization state.

However, $|n_0, n_3\rangle = \dfrac{\left[a_0^\dagger(p) - a_3^\dagger(p)\right]^m}{m}|0\rangle$ satisfies the condition $\boxed{ph}$, in fact $\left[a_0 - a_3, a_0^\dagger - a_3^\dagger\right] = -\delta(p-p') + \delta(p-p') = 0$

transverse, without constraints

$|\phi\rangle$

$\Rightarrow$ The most general physical state can be written as $\boxed{|phys\rangle = \underbrace{|\psi_T\rangle}_{\text{transverse, without constraints}} + c\underbrace{|n_0, n_3\rangle}_{=|\phi\rangle}}$

The question now is: what shall we do with |ϕ⟩, which has no counterpart in the canonical quantization? First of all, observe that |ϕ⟩ has zero norm:

$$\langle\phi|\phi\rangle = \langle 0|(a_0 - a_3)(a_0^\dagger - a_3^\dagger)|0\rangle = \langle 0|\left[a_0, a_0^\dagger\right] + \left[a_3, a_3^\dagger\right]|0\rangle = \langle 0|\emptyset|0\rangle = \emptyset$$

$\langle\psi_T|\phi\rangle = 0 \Rightarrow \langle phys|\phi\rangle = 0 \Rightarrow |\phi\rangle$ is ortogonal to all physical states

Let us next look at the contribution of |ϕ⟩ to an observable, like the energy

$$:H: = \int d^3x \, :(\pi^\mu \dot{A}_\mu - \mathcal{L}):$$

$$= \int d^3x \, :\Big(F^{\mu 0}\dot{A}_\mu - \eta^{\mu 0}(\partial_\lambda A^\lambda)\dot{A}_\mu + \overbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(\partial_\lambda A^\lambda)^2}^{-\mathcal{L}}\Big):$$

$$= \int d^3p \; E\left\{-a_0^\dagger a_0 + \sum_{\lambda=1}^{3} a_\lambda^\dagger a_\lambda\right\}$$

$$\langle ph|:H:|ph\rangle = \langle ph| \underbrace{\cancel{-(\text{"0"}-\text{"3"})}}_{\text{gives } \emptyset \text{ because of } \boxed{ph} \text{ pag } 94} + a_1^\dagger a_1 + a_2^\dagger a_2|ph\rangle$$

Contrybute only from $\lambda=1, \lambda=2$, for the momentum too

$\Rightarrow$ the physics is governed only by the independent component

Free field propagators

Non homogeneous equations and Green's functions

Let's consider the Klein-Gordon field with current:

$$(\partial^2+m^2)\phi(x)=j(x) \quad ①$$

In order to solve the equation, we want to find the solution of

$$\boxed{(\partial^2+m^2)G(x,x')=\delta^4(x-x')} \quad ②,$$

in this way we obtain $\phi=\underbrace{\phi_0}_{\text{homog. part}}+\int d^4x'\, G(x,x')\, j(x')$

In fact $\underbrace{(\partial^2+m^2)\phi}_{\overset{①}{=}\, j(x)} = (\partial^2+m^2)\left(\phi_0+\int d^4x'\, G(x,x')\, j(x')\right)$

$$j(x) = \underbrace{\cancel{(\partial^2+m^2)\phi_0}}_{=0,\ \text{homog.}} + \int d^4x'\,(\partial^2+m^2)\,G(x,x')\,j(x')$$

$$\Rightarrow \quad j(x) = \int d^4x'\ \delta^4(x-x') = j(x) \quad \checkmark$$

$$G(x,x')=\frac{1}{(2\pi)^4}\int d^4p\ e^{-iP(x-x')}\ \tilde{G}(P)$$

$$\overset{②}{\Rightarrow} (-P^2+m^2)\,\tilde{G}(p)=1$$

$$\delta^4(x-x') = \frac{1}{(2\pi)^4}\int d^4p\ e^{-iP(x-x')}$$

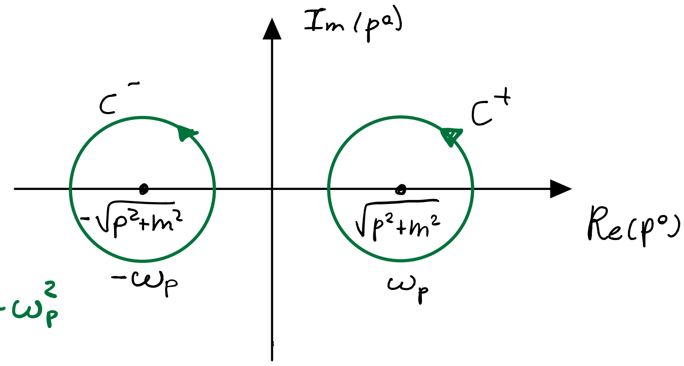
$\Rightarrow$ The Green function is $\boxed{\tilde{G}(p) = -\dfrac{1}{P^2-m}}$

$$\Rightarrow G(x,x') = -\frac{1}{(2\pi)^4}\int d^4p\ \frac{e^{-iP(x-x')}}{P^2-m^2}$$

We must take account of the fact that the denominator in the integral is singular at the points which correspond to propagation of free waves, p^0 = ±ω_p. To do this, it is convenient to work in the complex plane of the variable p^0 . The singularity of G(p) is found on the real axis, for p^0 = ±ω_p , and each particular solution is found by assigning a path in the complex plane to carry out the integral in p^0.

$$p^0=\pm\sqrt{p^2+m^2}=\pm\omega_p$$

Closed paths



$$\Delta^+(x) := -i \int_{C^+} \frac{d^4p}{(2\pi)^4} \frac{e^{-iPx}}{P^2-m^2}$$

$p^{0\,2} - (p^2+m^2) = p^{0\,2} - \omega_p^2$

$$= -i \int \frac{d^3p}{(2\pi)^3} \int_{C^+} \frac{dp^0}{(2\pi)} \frac{e^{-iPx}}{(p^0-\omega_p)(p^0+\omega_p)} \Rightarrow \text{[Res. th.]} \Rightarrow$$

$$\Rightarrow \boxed{\Delta^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-i(\omega_p t - \mathbf{p}\cdot\mathbf{x})}}$$

Analogously, we define

$$\Delta^-(x) := -i \int_{C^-} \frac{d^4p}{(2\pi)^4} \frac{e^{-iPx}}{P^2-m^2} \Rightarrow \text{[Again residual theorem]} \Rightarrow$$

$$\Rightarrow \boxed{\Delta^-(x) = -\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{i(\omega t + \mathbf{p}\cdot\mathbf{x})}}$$

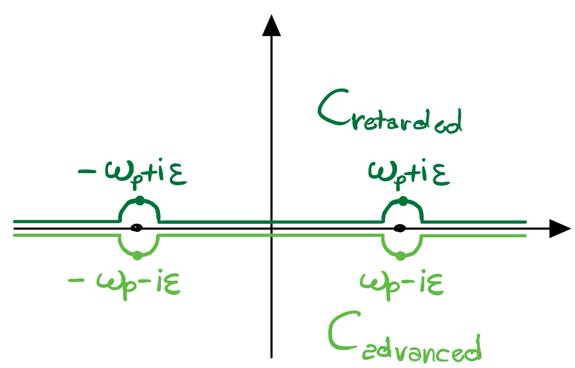
Let's evaluate $(\partial^2+m^2)\Delta^\pm(x) = -i \int \frac{d^3p}{(2\pi)^4} \frac{(\partial^2+m^2)e^{-iPx}}{P^2-m^2} =$

$$= \int_{C^\pm} \frac{d^3p}{(2\pi)^4} e^{-iPx} = \text{[Integral on closed path of analytical function]} = 0$$

$\Rightarrow$ These integrals give solutions to the homogeneous equation.

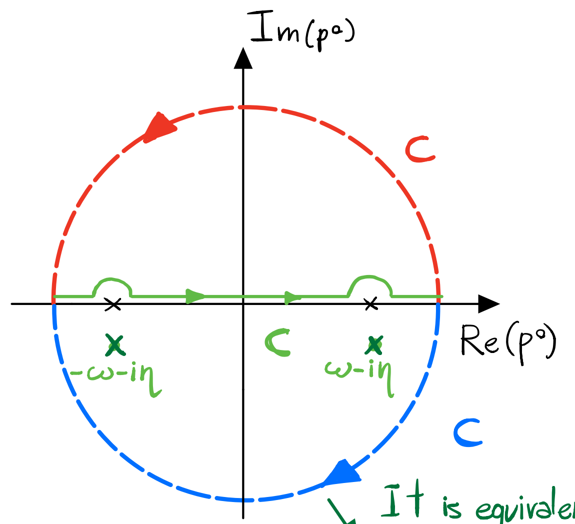
Closed paths

In general, these paths give a solution to the inhomogeneous equation.
Two different paths give the same result if we can continuously deform one into the other without encountering any singular point of G(p), otherwise they differ by combinations of the integrals around the singularity, i.e. by solutions of the inhomogeneous equation.



$$\hat{G}(p) = \frac{e^{-iP(x-x')}}{(p_0+i\eta)^2 - p^2 - m^2} = \frac{e^{-iP(x-x')}}{(p_0-\omega+i\eta)(p_0+\omega+i\eta)}$$

Retarded solution



The **retarded Green's function**, G_ret, corresponds to the condition that G(x) = 0 for t-t' < 0, which is that the result should be different from zero only after switching on the source at the coordinate origin (causality condition).
In this case, the integration path must be completely above the singularity.

It is equivalent to integrate over C+C or to shift Re axis by iη

With this condition:

- If $t-t'<0 \Rightarrow$ I integrate over C + C

no singularities, 0 Jordan's lemma

$$\Rightarrow \overset{\text{no singularities}}{0} = \int_C + \int_C = G + 0 = G \quad \Rightarrow \quad G = 0$$

- If $t-t'>0 \Rightarrow 2\pi i \sum \mathrm{Res} = -\int_C - \int_C = G_{ret}(x-x')$

$\Rightarrow$ We put $\boxed{p^0 \to p^0 + i\eta}$

$$\Rightarrow G_{ret}(x-x') = -\frac{\theta(x^0-x^{0\prime})}{(2\pi)^4}\int d^4p \, \frac{e^{-iP(x-x')}}{(p^0+i\eta)^2 - p^2 + m^2} =$$

$$= -\frac{\theta(x^0-x^{0\prime})}{(2\pi)^4}\int d^3p \, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\int dp^0 \frac{e^{-ip_0(x^0-x^{0\prime})}}{(p^0-\omega+i\eta)(p^0+\omega+i\eta)} = \left(\text{we do partial fractioning}\right)$$

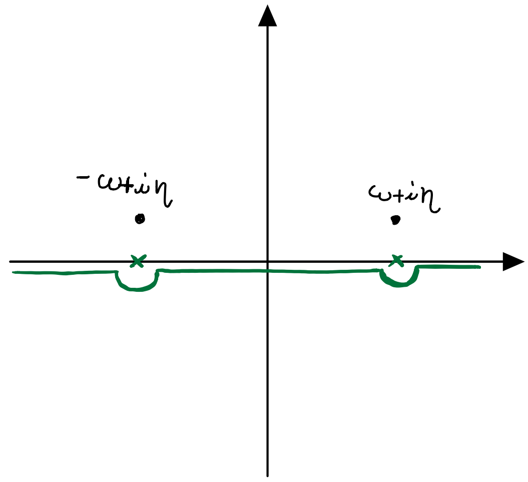
$$= -\frac{\theta(x^0-x^{0\prime})}{(2\pi)^4}\int d^3p \, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\int \frac{dp^0}{2\omega_p} e^{-ip_0(x^0-x^{0\prime})}\left(\frac{1}{p^0-\omega+i\eta} - \frac{1}{p^0+\omega+i\eta}\right) =$$

Res. th.

$$\overset{\text{Res. th.}}{=} i\frac{\theta(x^0-x^{0\prime})}{(2\pi)^4}\int \frac{d^3p}{2\omega_p}\left(e^{-i\omega(x^0-x^{0\prime})+i\mathbf{p}(\mathbf{x}-\mathbf{x}')} - e^{i\omega(x^0-x^{0\prime})+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\right)$$

$$\Rightarrow \boxed{G_{ret}(x-x') = \theta(x^0-x^{0\prime})\left(i\Delta^+(x-x') + i\Delta^-(x-x')\right)}$$

Advanced solution



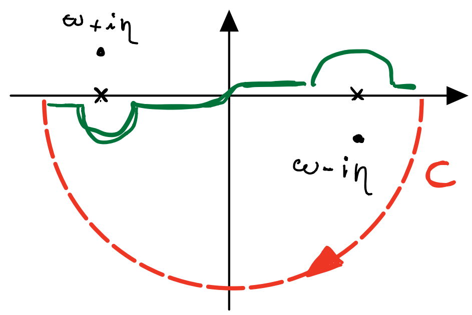
The symmetric condition, that G should vanish for t > 0, takes us to the **advanced Green's function**

$$\tilde{G}(p) = \frac{e^{-iP(x-x')}}{(p_0-i\eta)^2-p^2-m^2} = \frac{e^{-iP(x-x')}}{(p_0-\omega-i\eta)(p_0+\omega-i\eta)}$$

The only solution (residues) is in the upper plane

$$\Rightarrow \boxed{G_{adv}(x-x') = -\theta(x^{0'}-x^0)\left(i\Delta^+(x-x') + i\Delta^-(x-x')\right)}$$

Feynman propagator



In order to select the path that gives positive energy solutions and propagation forward in time, we use the **Feynman Propagator**

The Feynman propagator is obtained from the condition that it should coincide with iΔ(+)(x) for t-t' > 0 and with iΔ(−)(x) for t-t' < 0.

This condition determines an integration path, C_F , that originates from the negative real axis from k0 passing below the singularity in p_0 < 0, and above the one at p_0 > 0.

The same result is obtained, obviously, integrating along the real axis, after having moved the poles in the complex planes by an infinitesimal amount, ε > 0, in the following way:

$$\begin{cases} p^0 = -\omega(p) \rightarrow p^0 = -\omega + i\varepsilon \\ p^0 = +\omega(p) \rightarrow p^0 = \omega - i\varepsilon \end{cases}$$

$$D_F(x-x') = -\int_{C_F} \frac{d^4p}{(2\pi)^4} \frac{e^{-iP(x-x')}}{p^2-m^2} =$$

$$= -\theta(x^0-x^{0'}) \int \frac{d^3p}{(2\pi)^3} e^{ip(x-x')} \int \frac{dp^0}{2\omega_p} e^{-ip^0(x^0-x^{0'})} \left[\frac{1}{p^0-\omega_p+i\eta} - \frac{1}{p^0+\omega_p+i\eta}\right]$$

$$-\theta(x^{0'}-x^0) \int \frac{d^3p}{(2\pi)^4} e^{ip(x-x')} \int \frac{dp^0}{2\omega_p} e^{-ip^0(x^0-x^{0'})} \left[\frac{1}{p^0-\omega_p+i\eta} - \frac{1}{p^0+\omega_p+i\eta}\right]$$

$$\Rightarrow D_F(x-x') = \theta(x^0-x^{0'})\, i\Delta^+(x-x') + \theta(x^{0'}-x^0)\, i\Delta^-(x-x')$$

$$\Rightarrow D_F(x-x') = i\int \frac{d^3p}{(2\pi)^3 2\omega_p} \left[\theta(x^0-x^{0'})\, e^{-iP(x-x')} + \theta(x^{0'}-x^0)\, e^{iP(x-x')}\right]$$

• Propagator as the vacuum expectation value of T-ordered product of two fields.

Feynman propagator has an interpretation in terms of particle and antiparticle

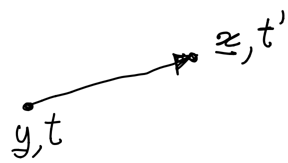
What is the probability amplitude that a state is annihilated ?

Let's take $|\psi(y,t)\rangle = \phi^\dagger(y)|0\rangle$, ϕ creates a particle and ann. an antiparticle

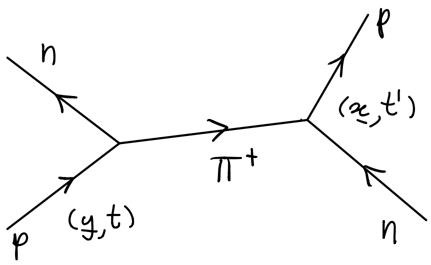
$$\phi^\dagger(y)|0\rangle = \int d^3p\left[\underbrace{f_p^{+} b(p)|0\rangle}_{=0} + f_p^{(+)\dagger} a^\dagger(p)|0\rangle\right] = \int \frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_p}} e^{iPx} a^\dagger(p)|0\rangle$$

$$\Rightarrow \quad \theta(t'-t)\langle\psi(x,t')|\psi(y,t)\rangle = \theta(t'-t)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle$$

At a certain point y you create a paticle, it travels up to x where it is annihilated

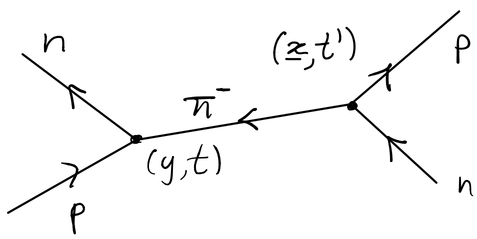


For example :



, they exchange a π^+ created in (y,t) and annihilated in (x,t')

Reversed process



π^- created in (x,t') and annihilated in (y,t)

These processes give the same initial and final state

Time-ordered product:

$$\langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle := \theta(x^0-y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle$$

$$\langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle = \langle 0|\theta(x^0-y^0)\phi(x)\phi^\dagger(y) + \theta(y^0-x^0)\phi^\dagger(y)\phi(x)|0\rangle = \langle 0|T(\phi^\dagger\phi)|0\rangle$$

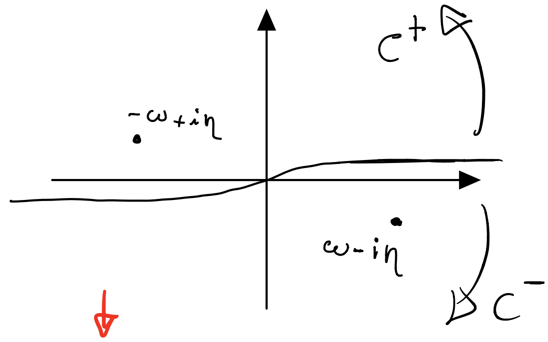
i ?

$$\Rightarrow \langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle = \int\frac{d^3p}{(2\pi)^3\, 2\omega}\left(\theta(x^0-y^0)e^{-iP(x-y)} + \theta(y^0-x^0)\,e^{iP(x-y)}\right) = D_F(x-y)$$

30/10/19

101

$$D_F(x-x') = -\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-iP(x-x')}}{P^2-m^2+i\varepsilon}$$



$$= -\frac{\theta(x^0-x^{0\prime})}{(2\pi)^4}\int d^3p\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\int_{C^-} dp^0\, e^{-ip_0(x^0-x^0)}\left(\frac{1}{p_0-\omega+i\varepsilon}-\frac{1}{p_0+\omega-i\varepsilon}\right)+$$

$$-\frac{\theta(x^{0\prime}-x^0)}{(2\pi)^4}\int \quad // \quad \int_{C^+} dp^0\, e^{-ip_0(x^0-x^{0\prime})}\left(\; // \;-\; // \;\right) =$$

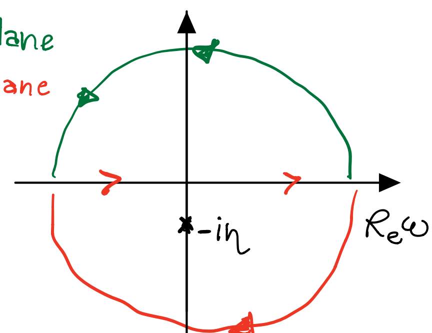
$$= i\,\theta(x^0-x^{0\prime})\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\,e^{-i\omega_p(x^0-x^{0\prime})+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}+$$

$$+i\,\theta(x^{0\prime}-x^0)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\,e^{i\omega_p(x^0-x^{0\prime})+i\underset{\mathbf{p}\to-\mathbf{p}}{\mathbf{p}}\cdot(\mathbf{x}-\mathbf{x}')}$$

$$\Rightarrow D_F(x-x') = i\,\theta(x^0-x^{0\prime})\int\frac{d^3p}{(2\pi)^3\,2\omega_p}\,e^{-iP(x-x')} + i\,\theta(x^{0\prime}-x^0)\int\frac{d^3p}{(2\pi)^3\,2\omega_p}\,e^{iP(x-x')}$$

$$\theta(t) = \lim_{\eta\to 0}\frac{i}{2\pi}\int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega t}}{\omega+i\eta}$$

$t<0$: upper plane
$t>0$: lower plane



- upper plane : no poles , $\int = 0$
- lower plane: $\lim_{\eta\to 0}\left(-2\pi i\,\mathrm{Res}(f,-i\eta)\right) = 1$

$$\Rightarrow -i D_F(x-y) = i\int\frac{d^3p}{(2\pi)^4}\left\{\int\frac{d\omega}{2\omega_p}\left[\frac{e^{-i\omega(x^0-y^0)}\,e^{-iP(x-y)}}{\omega+i\eta}+\frac{e^{i\omega(x^0-y^0)}\,e^{iP(x-y)}}{\omega+i\eta}\right]\right\}$$

$$\omega = P^0-\omega_p$$

$$= i\int\frac{d^4p}{(2\pi)^4}\frac{1}{2\omega_p}\left\{\frac{e^{-iP^0(x^0-y^0)+\cancel{i\omega_p(x^0-y^0)}-\cancel{i\omega_p(x^0-y^0)}-iP_i(x-y)^i}}{p^0-\omega_p+i\eta}+\frac{e^{i\overset{\to -P_\mu}{P_\mu}(x-y)^\mu}}{\underbrace{P^0-\omega_p+i\eta}_{-(p^0+\omega_p-i\eta)}}\right\}$$

102

$$= i\int \frac{d^4p}{(2\pi)^4}\frac{1}{2\omega_p} e^{-iP(x-y)}\left[\frac{1}{p^0-\omega_p+i\eta} - \frac{1}{p^0+\omega_p-i\eta}\right]$$

► $(\partial^2+m^2)\, D_F(x-y) = (\partial^2+m^2)\left[-\int \frac{d^4p}{(2\pi)^4}\frac{e^{-iP(x-y)}}{P^2-m^2+i\eta}\right] =$

$$= -\int \frac{d^4p}{(2\pi)^4}\frac{(-P^2+m^2)}{P^2-m^2} e^{-iP(x-y)} = \delta^4(x-y)$$

► $(\partial^2+m^2)\langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle = \partial_0^2\langle 0|T(\ \ )|0\rangle + \langle 0|T(-\nabla^2+m^2)\,\phi(x)\phi^\dagger(y)|0\rangle =$

$= \underset{\substack{\uparrow \\ \text{acts only on } x}}{\partial_0^2}\left\{\theta(x^0-y^0)\langle\phi_x\phi_y^\dagger\rangle + \theta(y^0-x^0)\langle\phi_y^\dagger\phi_x\rangle\right\} + \overset{\text{SPACE}}{\dots}$

$$\partial_0\theta(x^0-y^0) = \delta(x^0-y^0)$$

$$\partial_0\theta(-(x^0-y^0)) = -\delta(x^0-y^0)$$

$= \partial_0\{\langle 0|\delta(x^0-y^0)\,\phi(x)\phi^\dagger(y)|0\rangle - \langle 0|\delta(x^0-y^0)\,\phi^\dagger(y)\phi(x)|0\rangle$

$+ \langle 0|\theta(x^0-y^0)\,\dot\phi(x)\phi^\dagger(y)|0\rangle + \langle 0|\theta(y^0-x^0)\,\phi^\dagger(y)\dot\phi(x)|0\rangle\} + \overset{\text{SPACE}}{\dots}$

$= \partial_0\left\{\underline{\langle 0|\delta(x^0-y^0)[\phi(x),\phi^\dagger(y)]|0\rangle}\right\} + \partial_0\underline{\langle 0|T(\dot\phi(x)\phi^\dagger(y))|0\rangle} + \overset{\text{space}}{\dots}$

$$\partial_0\left[\delta(x^0-y^0)F(x)\right] = \dot F(x)\Big|_{x^0=y^0} + \delta(x^0-y^0)\dot F(x)$$

$= \cancel{-\langle 0|[\dot\phi(x)\phi^\dagger(y)]_{x^0=y^0}|0\rangle} + \langle 0|\delta(x^0-y^0)[\dot\phi(x),\phi^\dagger(y)]|0\rangle +$ linked to $\pi(x)$

$+ \cancel{\langle 0|\delta(x^0-y^0)[\dot\phi(x),\phi^\dagger(y)]|0\rangle} + \langle 0|T(\ddot\phi(x)\phi^\dagger(y))|0\rangle$

$+ \langle 0|T((-\nabla^2+m^2)\phi(x)\phi^\dagger(y))|0\rangle = -i\delta^4(x-y) + \langle 0|T(\underbrace{(\partial^2+m^2)}_{=0}\phi(x)\phi^\dagger(y))|0\rangle =$

$= -i\delta^4(x-y)$

$$\Rightarrow \boxed{D_F(x-y) = i\langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle}$$

• Feynman's propagator for the Dirac field

Now we want to find the propagator for Dirac field: by analogy to the case of the scalar field, the Feynman propagator is defined as

$$\boxed{S_F(x-y)_{\alpha\beta} = -i\langle 0|T(\psi_\alpha(x)\bar{\psi}_\beta(y))|0\rangle}$$

$$T(\psi_\alpha(x)\bar{\psi}_\beta(y)) := \theta(x^0-y^0)\,\psi\bar{\psi} - \theta(y^0-x^0)\,\bar{\psi}\psi = -T(\bar{\psi}\psi), \text{ antisymm.}$$

T has to be antisymmetric because ψ must fulfill anticommutation relations

$$(i\not{\partial}-m)_{\alpha\beta}\langle 0|T(\psi_\beta(x)\bar{\psi}_\delta(y))|0\rangle = \qquad \left[(i\gamma^\mu\partial_\mu - m)\psi = 0\right]$$

$$= \langle 0|i\gamma^0_{\alpha\beta}\,\delta(x^0-y^0)\left[\psi_\beta(x),\bar{\psi}_\delta(y)\right]_+|0\rangle +$$

$$+\langle 0|T(i\gamma^0_{\alpha\beta}\partial_0\psi_\beta(x)\bar{\psi}_\delta(y))|0\rangle + \langle 0|T((i\gamma^i_{\alpha\beta}\partial_i - m)\psi_\beta(x)\bar{\psi}_\delta(y))|0\rangle =$$

$$= \langle 0|i\gamma^0_{\alpha\beta}\delta(x^0-y^0)\left[\psi_\beta(x),\bar{\psi}_\delta(y)\right]_+|0\rangle + \langle 0|(i\not{\partial}-m)\psi_\beta(x)\bar{\psi}_\delta(y))|0\rangle$$

$$= \langle 0|i\gamma^0_{\alpha\beta}\delta(x^0-y^0)\left[\psi_\beta(x),\bar{\psi}_\delta(y)\right]_+|0\rangle = \left(\left[\psi_\beta(x),\psi^\dagger_\delta(y)\right]_+ = \delta_{\beta\delta}\,\delta(\underline{x}-\underline{y})\right) =$$

$$= \langle 0|i\gamma^0_{\alpha\beta}\,\delta_{\beta\sigma}\,\delta^4(x-y)\,\gamma^0_{\gamma\delta}|0\rangle = i\,\delta_{\alpha\delta}\,\delta^4(x-y)$$

We want to show $S_F(x-y) = -(i\not{\partial}+m)D_F(x-y)$:

$$(i\not{\partial}-m)S_F(x-y) = -(i\not{\partial}-m)(i\not{\partial}-m)D_F(x-y)$$

$$= -(\partial^2+m^2)D_F(x-y) = \delta^4(x-y) \quad \text{, correct!}$$

$$\Rightarrow S_F(x-y) = +(i\not{\partial}+m)\int\frac{d^4p}{(2\pi)^4}\frac{e^{-iP(x-y)}}{p^2-m^2+i\varepsilon} = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-iP(x-y)}}{p^2-m^2+i\varepsilon}(\not{p}+m)$$

- Feynman' s propagator for the Electromagnetic field

$$\langle 0|T(A_\mu(x)\,A_\nu(y))|0\rangle = i\eta_{\mu\nu}\, D_F(x-y)$$

$$\hookrightarrow -\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip(x-y)}}{p^2+i\varepsilon}$$ (massless field)

$$m=0 \Rightarrow (\partial^2+m^2) = \partial^2$$

$$\Rightarrow \quad \partial^2\langle 0|T(A_\mu(x)\,A_\nu(y))|0\rangle = i\eta_{\mu\nu}\,\delta^4(x-y)$$

$$\tilde{G}(p) = \frac{\eta_{\mu\nu}}{p^2+i\varepsilon}$$

Parameter

Feynman gauge

More in general: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\lambda A^\lambda)^2$, we took $\lambda=1$

$$\partial^2 A_\mu + \lambda\,\partial_\mu(\partial_\lambda A^\lambda) = 0$$

but physics doesn't change, $\lambda \neq 1$ is useful in the propagator