## • Unitarity in γ_0 metric



$S(\Lambda)$ leaves $(i\not\partial - m)\psi' = 0$

$S(\Lambda)$ is not unitary because the repr. is finite

$$\sigma_{\mu\nu}^{\dagger} = \left[\frac{i}{2}[\gamma_\mu, \gamma_\nu]\right]^{\dagger} = \frac{i}{2}\left(\gamma_\mu^\dagger \gamma_\nu^\dagger - \gamma_\nu^\dagger \gamma_\mu^\dagger\right) = \frac{i}{2}\left[\gamma_\mu^\dagger, \gamma_\nu^\dagger\right] \neq \sigma_{\mu\nu}$$

$\Rightarrow$ When we do $S^\dagger(\Lambda) = e^{\frac{1}{4}\sigma_{\mu\nu}^\dagger \varepsilon^{\mu\nu}} \neq S^{-1}(\Lambda)$

proof compon.-by-component

$$\gamma^0 \sigma_{\mu\nu}^\dagger \gamma_0 = \sigma_{\mu\nu}$$

$$\gamma^0 \frac{i}{2}\left(\gamma_\mu^\dagger \gamma_\nu^\dagger - \gamma_\nu^\dagger \gamma_\mu\right)\gamma_0 = \begin{cases} 0,0 & = 0 = \sigma_{00} \\ 0,i & \frac{i}{2}\gamma^0\left(\gamma_0(-\gamma_i) - (-\gamma_i)\gamma_0\right)\gamma_0 = \sigma_{0i} \end{cases}$$

$$\boxed{\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)}$$

We have to verify that group structure is preserved:

$S(\Lambda_1)S(\Lambda_2) = S(\Lambda_1\Lambda_2)$

- $S^{-1}(\Lambda_1)\gamma^\mu S(\Lambda_1) = \Lambda_{1\,\nu}^{\mu}\gamma^\nu \rightarrow (\Lambda_1^{-1})^{\rho}{}_{\mu} S^{-1}(\Lambda_1)\gamma^\mu S(\Lambda_1) = \gamma^\rho$

  $(\Lambda_1^{-1})^{\rho}{}_{\mu}$

- $S^{-1}(\Lambda_2)\gamma^\rho S(\Lambda_2) = \Lambda_{2\,\alpha}^{\rho}\gamma^\alpha$

$$\Rightarrow (\Lambda_1^{-1})^{\rho}{}_{\mu} S^{-1}(\Lambda_2) S^{-1}(\Lambda_1)\gamma^\mu S(\Lambda_1) S(\Lambda_2) = \Lambda_{2\,\alpha}^{\rho}\gamma^\alpha$$

$$\Rightarrow S^{-1}(\Lambda_2) S^{-1}(\Lambda_1)\gamma^\sigma S(\Lambda_1)S(\Lambda_2) = \Lambda_{1\,\rho}^{\sigma}\Lambda_{2\,\alpha}^{\rho}\gamma^\alpha$$
$$= (\Lambda_1\Lambda_2)^{\sigma}_{\alpha}\gamma^\alpha$$

$\Rightarrow S(\Lambda)$ is a repr. of the Lor. group that behaves correctly

# • Dirac conjugated

We note that $\psi^\dagger\psi$ is not invariant, since $S(\Lambda)$ is not unitary:

$$\psi' = S\psi \Rightarrow \psi'^\dagger\psi' = \psi^\dagger S^\dagger S\psi \neq \psi^\dagger\psi$$

it is not surprising because it has to be the temporal component of a 4-vector

We can construct a scalar:

Let be $\bar{\psi} = \psi^\dagger\gamma^0 \Rightarrow$

$$\Rightarrow \bar{\psi}'\psi' = \psi^{\dagger\prime}\gamma^0\psi' = (S\psi)^\dagger\gamma^0 S\psi = \psi^\dagger S^\dagger \gamma^0 S\psi \overset{\mathbb{1} = \gamma^{0\,2}}{=}$$

$$= \psi^\dagger\gamma^0\gamma^0 S^\dagger\gamma^0 S\psi = \psi^\dagger\gamma^0 S^{-1}S\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi$$

# • Current and probability density

$\psi^\dagger\psi$ is positive definite, we can construct a continuity equation obtained from D.E.:

$$\psi^\dagger\left[i\frac{\partial}{\partial t}\psi = (-i\underline{\alpha}\cdot\underline{\nabla} + \beta m)\psi\right] - \left[-i\frac{\partial}{\partial t}\psi^\dagger = \psi^\dagger(i\overleftarrow{\underline{\nabla}}\cdot\underset{=\alpha^\dagger}{\underline{\alpha}} + \beta m)\right]\psi$$

$$\Rightarrow i\psi^\dagger\frac{\partial}{\partial t}\psi + i\left(\frac{\partial}{\partial t}\psi^\dagger\right)\psi = \psi^\dagger(-i\underline{\alpha}\cdot\vec{\underline{\nabla}} + \cancel{\beta m})\psi - (i\underline{\nabla}\psi^\dagger\cdot\underline{\alpha} + \cancel{m\psi^\dagger\beta})\psi$$

$$\Rightarrow i\frac{\partial}{\partial t}(\psi^\dagger\psi) = -i\psi^\dagger\underline{\alpha}(\underline{\nabla}\psi) - i(\underline{\nabla}\psi^\dagger)\underline{\alpha}\psi = -i\underline{\nabla}(\psi^\dagger\underline{\alpha}\psi)$$

$\boxed{j^\mu := (\psi^\dagger\psi, \psi^\dagger\underline{\alpha}\psi)}$, $j^\mu$ is such that $\boxed{\partial_\mu j^\mu = 0}$

$$\Rightarrow \frac{d}{dt}\int d^3x\, j^0 = \frac{d}{dt}\int d^3x\, \psi^\dagger\psi = 0,$$

$\psi^\dagger\psi$ is proportional to the prob. density

We prove that $j^\mu$ is a 4-vector:

$$j^\mu = \bar\psi \gamma^\mu \psi \ ;$$

$$j'^\mu(x') = \bar\psi' \gamma^\mu \psi' = \psi'^\dagger \gamma^0 \gamma^\mu \psi' = (S\psi)^\dagger \gamma^0 \gamma^\mu (S\psi) =$$

$$= \psi^\dagger S^\dagger \gamma^0 \gamma^\mu S \psi = \psi^\dagger \gamma^0 S^{-1} \gamma^\mu S\psi = \bar\psi S^{-1} \gamma^\mu S \psi =$$

$$= \bar\psi \Lambda^\mu_\nu \gamma^\nu \psi = \Lambda^\mu_\nu \bar\psi \gamma^\nu \psi = \Lambda^\mu_\nu j^\nu$$

- Lagrangian density and conjugated momenta

$$(i\partial\!\!\!/ - m)\psi = 0 \xRightarrow{\dagger} (i\partial\!\!\!/\psi)^\dagger - m\psi^\dagger = 0 \Rightarrow (-i\,\partial_\mu \psi^\dagger \gamma^{\mu\dagger} - m\psi^\dagger = 0)\,\gamma^0$$

$$\Rightarrow -i\,\partial_\mu \psi^\dagger \gamma^0\gamma^0 \gamma^{\mu\dagger}\gamma^0 - m\psi^\dagger\gamma^0 = 0 \Rightarrow -i\partial_\mu \bar\psi \gamma^0 \gamma^{\mu\dagger}\gamma^0 - m\bar\psi = 0$$

$$\gamma^0 \gamma^{\mu\dagger}\gamma^0 = \gamma^\mu \Rightarrow -i\partial_\mu \bar\psi \gamma^\mu - m\bar\psi = 0 \Rightarrow -i\partial\!\!\!/\bar\psi - m\bar\psi = 0$$

$$\Rightarrow \begin{cases} (i\partial\!\!\!/ - m)\psi = 0 \\ \bar\psi(i\overleftarrow{\partial\!\!\!/} + m) = 0 \end{cases} \ ; \text{ two independent fields}$$

The Lagr density is $\boxed{\mathcal{L} = \bar\psi(i\overrightarrow{\partial\!\!\!/} - m)\psi}$ , we don't have derivative on $\bar\psi$

$$\delta S = \delta \int \bar\psi (i\partial\!\!\!/ - m)\psi$$

$$\psi: \ \frac{\partial \mathcal{L}}{\partial \bar\psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \bar\psi_{,\mu}} = 0 \Rightarrow (i\partial\!\!\!/ - m)\psi = 0 \longrightarrow$$ Correct different. equations

$$\bar\psi: \ \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} = 0 \Rightarrow -m\bar\psi - \partial_\mu(\bar\psi i \gamma^\mu) = -m\bar\psi - (i\partial^\mu \bar\psi \gamma_\mu) = 0$$

Conjugated momenta aren't both present because Dir. eq. is 1st order

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = i\psi^\dagger \ ; \ \pi_{\psi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot\psi^\dagger} = 0 \ ,$$

$\dot\psi^\dagger$ is not present in $\mathcal{L}$

$\Rightarrow \mathcal{H}$ cannot be found as usual

# • Energy density, Nöther's theorem

The hamiltonian cannot be found because $\pi_{\psi^+}=0$, the derivative are not invertible, to solve the problem we use Nöther's theorem, we start calculating the energy density.

$$\mathcal{H} = \pi_\psi \dot{\psi} - \mathcal{L} = i\psi^+\dot\psi - \bar{\psi}(i\cancel{\partial}-m)\psi = i\psi^+\partial_0\psi - \bar{\psi}(i\gamma^0\partial_0\psi + i\partial_i\gamma^i\psi - m\psi)$$

$$= -i\bar{\psi}\partial_i\gamma^i\psi + m\bar{\psi}\psi = -i\bar{\psi}\cancel{\partial}\psi + m\bar{\psi}\psi + i\bar{\psi}\partial_0\gamma^0\psi =$$

$$= -\bar{\psi}(i\cancel{\partial}-m)\psi + i\bar{\psi}\partial_0\gamma^0\psi \overset{(i\cancel{\partial}-m)\psi=0}{=\!=\!=} i\bar{\psi}\gamma^0\partial_0\psi = i\psi^+\partial_0\psi$$

$$\Rightarrow \boxed{\mathcal{H} = i\psi^+\partial_0\psi}$$

From. N.th: $$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\phi^i_{,\nu} - \eta^\mu_\nu\mathcal{L} \quad , \quad \phi^i = \psi, \bar{\psi}$$

$$\Rightarrow T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial\psi_{,\mu}}\psi_{,\nu} = i\bar{\psi}\gamma^\mu\psi_{,\nu}$$

$$\partial_\mu T^\mu_\nu = i\partial_\mu(\bar{\psi}\gamma^\mu\psi_{,\nu}) = i(\partial_\mu\bar{\psi})\gamma^\mu\psi_{,\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu\psi_{,\nu}) =$$

$$= -im\bar{\psi}\psi_{,\nu} + im\bar{\psi}\psi_{,\nu} = 0$$

$\Rightarrow$ we have a continuity equation

$\Rightarrow P_\nu$ are conserved, $P_\nu = \int d^3x\, T^0_\nu$ ; $\frac{dP_\nu}{dt} = 0$

$$\epsilon = \int d^3x\, i\psi^+\partial_0\psi$$
$$\underline{P} = \int d^3x\, i\psi^+\underline{\nabla}\psi$$

$$P_\nu = \int d^3x\, T^0_\nu = i\int d^3x\, \psi^+\partial_\nu\psi \quad ; \quad \epsilon = i\int d^3x\, \psi^+\partial_0\psi$$

Imposing the invariance under homgeneous Lor transf., we find other conserved quantities, the 6 charges of the angular momentum.

# Angular momentum and spin

$$M^{\mu}_{\rho\nu} = x_\rho T^{\mu}_{\nu} - x_\nu T^{\mu}_{\rho} - (\Sigma_{\rho\nu})^i_j \, \phi^j \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}}$$

$-\frac{1}{2}\Sigma_{\rho\nu}\varepsilon^{\rho\nu} \rightarrow -\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}$

Generators of Lor tr.

$$\partial_\mu M^{\mu}_{\rho\nu} = 0 \,; \quad M_{\rho\nu} := \int d^3x \, M^{0}_{\rho\nu} \,, \qquad T^{\mu}_{\nu} = i\bar{\psi}\gamma^\mu \psi_{,\nu}$$

$$\boxed{\underline{J} := (M_{23}, M_{31}, M_{12}) = \int d^3x \, \psi^+ \left[ -i\,\underline{x}\wedge\underline{\nabla} + \frac{1}{2}\begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} \right] \psi}$$

Spin

$\Rightarrow$ The ang. mom. is written as an orbital part and a spin-part.

20/10/2020

# Other symmetries of the Lagrangian

## U(1) transformations

A symmetry for the Lagr. is to redefine the fields with a charge
$$\begin{cases} \psi(x) \rightarrow e^{-i\vartheta}\psi(x) = \psi'(x) \\ \bar{\psi}(x) \rightarrow e^{i\vartheta}\bar{\psi}(x) = \bar{\psi}'(x) \end{cases}$$

$$\Rightarrow \mathcal{L}' = \cancel{e^{i\vartheta}}\bar{\psi}(x)(i\cancel{\partial} - m)\cancel{e^{-i\vartheta}}\psi = \mathcal{L}$$

It is a global int. symmetry $\Rightarrow J^\mu = \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\delta\phi^i \;;\; \delta\phi^i = -i\vartheta\psi$

$$\Rightarrow J^\mu = \bar{\psi}\, i\gamma^\mu(-i\vartheta\psi) = \vartheta\bar{\psi}\gamma^\mu\psi \;;\quad \partial_\mu J^\mu = 0 \Rightarrow \frac{d}{dt}\int d^3x\, \psi^+\psi = 0$$

## Parity

$\Lambda_{p}{}^{\mu}{}_{\nu}$ is not connected to $\mathbb{1}$, $\Lambda_{p}{}^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ Action on the vectors, it changes the sign of the space component, parity is unitary, time rev. is anti-unitary,

$S(\Lambda_p) = ?$ $\qquad S^{-1}(\Lambda_p)\gamma^\mu S(\Lambda_p) = \Lambda_{p}{}^{\mu}{}_{\nu}\gamma^\nu \quad \forall\mu$

$$\gamma^\mu S = S\Lambda_{p}{}^{\mu}{}_{\nu}\gamma^\nu \Rightarrow \begin{cases} \gamma^0 S = S\gamma^0 \\ \gamma^i S = -S\gamma^i \end{cases} \Rightarrow S \propto \gamma^0 \;;\quad S = \eta_p\gamma^0$$

intrinsic parity of the particle

$\eta_p \rightarrow \pm 1$

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- Time reversal

$$\Lambda_T{}^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} = -\eta_{\mu\nu} \quad ; \quad S^{-1}(\Lambda_T)\gamma^{\mu}S(\Lambda_T) = \Lambda_T{}^{\mu}{}_{\nu}\gamma^{\nu}$$

$\Rightarrow S(\Lambda_T) = \eta_T \gamma^1\gamma^2\gamma^3$ , imposing that it is idempotent, we obtain $\eta_T = i$

$\Rightarrow S(\Lambda_T) = i\gamma^1\gamma^2\gamma^3$

$\boxed{[\gamma^{\mu}, \gamma^{\nu}] = 2\eta^{\mu\nu}}$

# • Matrix γ_5

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad ; \quad \gamma_5^{\dagger} = (i\gamma^0\gamma^1\gamma^2\gamma^3)^{\dagger} = -i\,\overset{-}{\gamma^{3\dagger}}\,\overset{-}{\gamma^{2\dagger}}\,\overset{-}{\gamma^{1\dagger}}\,\overset{+}{\gamma^{0\dagger}} = +i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

↳ not a Lorentz index

$$[\gamma_5, \gamma^0]_+ = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3 = 0$$

$$[\gamma_5, \gamma^i]_+ = 0 \qquad \gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

We can rewrite it in a more covariant form, using $i\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}$

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{even } P(0123) \text{ (even permutations)} \\ -1 & \text{odd } P(0123) \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_{\mu\nu\rho\sigma}\, i\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} = 24\gamma^5 \quad \Rightarrow \gamma^5 = \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{5}$$

How $\gamma^5$ transforms: $S^{-1}(\Lambda)\gamma^5 S(\Lambda)$

$$\varepsilon_{\alpha\beta\gamma\delta}\det\Lambda = \varepsilon_{\mu\nu\rho\sigma}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\delta}\Lambda^{\sigma}{}_{\gamma}$$

$$\Rightarrow S^{-1}(\Lambda)\gamma^5 S(\Lambda) = \frac{i}{24}S^{-1}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}S =$$

$$= \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\,\underbrace{S^{-1}\gamma^{\mu}S}\,\underbrace{S^{-1}\gamma^{\nu}S}\,\underbrace{S^{-1}\gamma^{\rho}S}\,\underbrace{S^{-1}\gamma^{\sigma}S} =$$

$$= \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\,\Lambda^{\mu}{}_{\alpha}\gamma^{\alpha}\,\Lambda^{\nu}{}_{\beta}\gamma^{\beta}\,\Lambda^{\rho}{}_{\delta}\gamma^{\delta}\,\Lambda^{\sigma}{}_{\gamma}\gamma^{\gamma} =$$

$$= \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\delta}\Lambda^{\sigma}{}_{\gamma}\gamma^{\alpha}\gamma^{\beta}\gamma^{\delta}\gamma^{\gamma} =$$

$$= \frac{i}{24}\varepsilon_{\alpha\beta\delta\gamma}\det\Lambda = (\det\Lambda)\gamma^5 = \begin{cases} \gamma^5 & \text{for proper orthocronous transf} \\ -\gamma^5 & \text{otherwise} \end{cases}$$

# • Bilinear covariants and their behaviour under Lor. transform

It is possible to construct a basis of 16 objects: $\{\overset{1}{\mathbb{1}}, \overset{4}{\gamma^\mu}, \overset{1}{\gamma_5}, \overset{6}{\sigma^{\mu\nu}}, \overset{4}{\gamma^\mu\gamma^5}\} = \Gamma^\alpha$

The object $\bar{\psi}\Gamma^\alpha\psi$ can be connected to an observable:

⊙ $\bar{\psi}\psi \longrightarrow \bar{\psi}'\psi' = \bar{\psi}\psi$ is an invariant

⊙ $\bar{\psi}\gamma^\mu\psi$ is a 4-vector $\rightarrow \bar{\psi}'\gamma^\mu\psi' = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$

⊙ $\bar{\psi}\gamma_5\psi \rightarrow \bar{\psi}'\gamma_5\psi = \psi'^\dagger\gamma^0\gamma_5 S\psi = \psi^\dagger S^\dagger\gamma^0\gamma^5 S\psi = \bar{\psi}S^{-1}\gamma^5 S\psi =$

$$= (\det\Lambda)\,\bar{\psi}\gamma^5\psi$$

$\Rightarrow$ is a scalar for prop. orthocr.

⊙ $\bar{\psi}\gamma^\mu\gamma^\nu\psi \rightarrow \bar{\psi}'\gamma^\mu\gamma^\nu\psi' = \Lambda^\mu_\rho\Lambda^\nu_\sigma\,\bar{\psi}\gamma^\rho\gamma^\sigma\psi$,

$\Rightarrow$ transforms as a rank 2 contravariant tensor

⊙ $\bar{\psi}'\gamma^\mu\gamma_5\psi' = (\det\Lambda)\Lambda^\mu_\nu\,\bar{\psi}\gamma^\nu\gamma_5\psi$

## • Algebra and traces of $\gamma$ matrices

$$[\gamma_\mu,\gamma_\nu]=2\eta_{\mu\nu}$$

$$\begin{cases} i)\ \gamma^\mu\gamma_\mu = 4\mathbb{1} \\ ii)\ \gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu(-\gamma^\mu\gamma^\nu+2\eta^{\mu\nu}) = -\gamma_\mu\gamma^\mu\gamma^\nu + 2\gamma_\mu\eta^{\mu\nu} = -4\mathbb{1}+2\gamma^\nu = -2\gamma^\nu \\ iii)\ \gamma_\mu\gamma^\lambda\gamma^\nu\gamma^\mu = 4\eta^{\lambda\nu} \end{cases}$$

$$\not{a} = \gamma_\mu a^\mu$$

$$\begin{cases} i)\ \not{a}\not{a} = \gamma_\mu a^\mu\gamma_\nu a^\nu = \gamma_\mu\gamma_\nu a^\mu a^\nu = a^\mu a^\nu(-\gamma_\nu\gamma_\mu+2\eta_{\mu\nu}) = a_\mu a^\mu = a^2 \\ ii)\ \gamma_\mu\not{a}\gamma^\mu = -2\not{a} \end{cases}$$

1) $\mathrm{tr}\,\gamma^\mu=0$ ; $\mathrm{tr}\,\gamma^5=0$ ;

A,B not matrices

2) $\mathrm{tr}(\not{A}\not{B}) = \mathrm{tr}(\gamma^\mu\gamma^\nu A_\mu B_\nu) = A_\mu B_\nu\,\mathrm{tr}(\gamma^\mu\gamma^\nu) = A_\mu B_\nu\,\mathrm{tr}(\gamma^\nu\gamma^\mu) =$

$= \frac{1}{2}A_\mu B_\nu\,\mathrm{tr}(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu) = \frac{1}{2}A_\mu B_\nu\,\mathrm{tr}([\gamma^\mu,\gamma^\nu]_+) =$

$= \frac{1}{\cancel{2}}A_\mu B_\nu\,\mathrm{tr}(\cancel{2}\eta^{\mu\nu}\mathbb{1}) = 4\,A\cdot B$ scalar product

3) $\mathrm{tr}(\not{A}\not{B}\not{C})=0$

4) $\mathrm{tr}(\not{A}\not{B}\not{C}\not{D}) = 4(A\cdot B)(C\cdot D)+4(A\cdot D)(B\cdot C)-4(A\cdot C)(B\cdot D)$

$\mathrm{tr}(\gamma_5\not{A}\not{B})=0$ ; $\mathrm{tr}(\gamma_5\not{A}\not{B}\not{C})=0$ ; $\mathrm{tr}(\gamma_5\not{A}\not{B}\not{C}\not{D}) = 4i\,\varepsilon_{ijkl}A^iB^jC^kD^l$

# • Plane wave solutions of Dirac equation

The Dirac eq. has solutions in the form of relativistic plane waves:

$\psi(x) = u(p)\, e^{-ip_\mu x^\mu}$ , $p^\mu = (E, \vec{p})$ , $u(p)$ is a 4-component spinor, contains the structure

$$\Rightarrow (i\gamma^\mu \partial_\mu - m)\, u(p)\, e^{-ip_\mu x^\mu} = (i\gamma^\mu(-ip_\mu) - m)\, u(p)\, e^{-ip_\mu x^\mu} = (\not{p} - m)\, u(p) \cancel{e^{-ip_\mu x^\mu}} = 0$$

$$\Rightarrow \quad \boxed{(\not{p}^\mu - m)\, u(p) = 0}$$

$$\begin{pmatrix} p^0 - m & -\underline{\sigma}\cdot\underline{p} \\ \underline{\sigma}\cdot\underline{p} & -p^0 - m \end{pmatrix} \overset{\text{spinor}}{\begin{pmatrix} \varphi \\ \chi \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 0 = \det\begin{pmatrix} & \\ & \end{pmatrix} = m^2 - p^{0\,2} + (\underline{\sigma}\cdot\underline{p})^2$$

$$(\underline{\sigma}\cdot\underline{p})^2 = \sigma_i \sigma_j p^i p^j \; ; \; \left.\begin{aligned} [\sigma_i, \sigma_j] &= 2\varepsilon_{ijk}\sigma_k = \sigma_i\sigma_j - \sigma_j\sigma_i \\ [\sigma_i, \sigma_j]_+ &= 2\delta_{ij} = \sigma_i\sigma_j + \sigma_j\sigma_i \end{aligned}\right\} \Rightarrow 2\sigma_i\sigma_j = 2\delta_{ij} + 2\varepsilon_{ijk}\sigma_k$$

$$\Rightarrow (\underline{\sigma}\cdot\underline{p})^2 = \sigma_i\sigma_j p^i p^j = \underline{p}^2 \Rightarrow m^2 - p^{0\,2} + p^2 = 0 \Rightarrow p^0 = \pm\sqrt{p^2 + m^2} = \pm\omega_p \quad \text{(again)}$$

$$\Rightarrow \begin{aligned} \psi^{(+)} &= u(p)\, e^{-ip_\mu x^\mu} \\ \psi^{(-)} &= v(p)\, e^{ip_\mu x^\mu} \end{aligned} \; ; \; \begin{aligned} 0 &= (i\not{\partial} - m)\, \psi^{(+)} = (\not{p}^\mu - m)\, u(p)\, e^{-ip_\mu x^\mu} \\ 0 &= (i\not{\partial} - m)\, \psi^{(-)} = (-\not{p}^\mu - m)\, v(p)\, e^{-ip_\mu x^\mu} \end{aligned} \Rightarrow \begin{cases} (\not{p}^\mu - m)\, u(p) = 0 \\ (\not{p}^\mu + m)\, v(p) = 0 \end{cases}$$

$$\text{If } \; \underline{p}^\mu = (m, 0) \Rightarrow \begin{cases} (\gamma^0 - 1)\, u(m, \underline{0}) = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \Rightarrow u_3 = u_4 = 0 \\ (\gamma^0 + 1)\, u(m, \underline{0}) = 0 \Rightarrow \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \Rightarrow v_1 = v_2 = 0 \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} u(m, \underline{0}) &= \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \alpha\, u^{(1)}(m, \underline{0}) + \beta\, u^{(2)}(m, \underline{0}) \\ v(m, \underline{0}) &= \alpha_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \alpha_1\, v^{(1)}(m, \underline{0}) + \beta_1\, v^{(2)}(m, \underline{0}) \end{aligned}}$$

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Now we know $u(m,\underline{0})$ and $v(m,\underline{0})$, now we can find $u^\alpha(m,p)$, $v^\alpha(m,p)$

$$(\not{p}-m)(\not{p}+m) = \not{p}\not{p}-m^2 = \boxed{\not{p}\not{p}=p^{\mu 2}} = p^{\mu 2}-m^2 = 0 \quad \star$$

We know that $\begin{cases}(\not{p}-m)u(p)=0 \\ (\not{p}+m)v(p)=0\end{cases}$ $\star$ $\Rightarrow$ $\boxed{\begin{cases}u^{(\alpha)}(p) = C_\alpha(\not{p}+m)\,u^{(\alpha)}(m,\underline{0}) \\ v^{(\alpha)}(p) = d_\alpha(-\not{p}+m)\,v^{(\alpha)}(m,0)\end{cases}}$ \$

In the ref. syst. where $p=0$ : $\begin{cases}\bar{u}^{(\alpha)}(m,\underline{0})\,u^{(\beta)}(m,\underline{0}) = \delta^{\alpha\beta} \\ \bar{v}^{(\alpha)}(m,\underline{0})\,v^{(\beta)}(m,\underline{0}) = -\delta^{\alpha\beta} \\ \bar{u}^{(\alpha)}(m,\underline{0})\,v^{(\beta)}(m,\underline{0}) = 0 \quad \text{(they are orthogonal)}\end{cases}$

$\begin{cases}\bar{u}^1u^1 \quad (1\,0\,0\,0)\gamma^0\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 1 \\ \bar{u}^2u^2 = (0\,1\,0\,0)\gamma^0\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 1\end{cases}$ ok! $\quad \begin{cases}\bar{v}^1v^1 = (0\,0\,1\,0)\gamma^0\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = -1 \\ \bar{v}^2v^2 = (0\,0\,0\,1)\gamma^0\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = -1\end{cases}$ OK!

$\Rightarrow$ we have found $\bar{u}u$, $\bar{v}v$ which are scalar objects, the relations stands in all ref. frames.

$$\Rightarrow \delta^{\alpha\beta} = \bar{u}^{(\alpha)}(p)\,u^{(\beta)}(p) =$$

$$= C_\alpha^* \underbrace{u^{(\alpha)\dagger}(m,\underline{0})}\underbrace{(\not{p}+m)^\dagger}\gamma^0 C_\beta(\not{p}+m)u^{(\beta)}(m,\underline{0}) \quad ; \qquad \gamma_0(\not{p}+m)^\dagger\gamma_0 = (\not{p}+m)$$

$\gamma^0\gamma^0$

$$= C_\alpha^* C_\beta\, \bar{u}^{(\alpha)}(m,0)(\not{p}+m)^2u^{(\beta)}(m,0) = C_\alpha^* C_\beta\, \bar{u}^{(\alpha)}[\underbrace{\not{p}\not{p}}_{p_\mu p^\mu} + m^2 + 2m\not{p}]u^{(\beta)} =$$

$$= C_\alpha^* C_\beta\, \bar{u}^\alpha(m,0)(2m\not{p}+2m^2)u^{(\beta)}(m,0) = C_\alpha^* C_\beta\, \bar{u}^{(\alpha)}(m,0)\,2m\not{p}\,u^{(\beta)}(m,0) + \ldots =$$

$$= C_\alpha^* C_\beta\, 2m\, P_\mu\, \bar{u}^{(\alpha)}(m,0)\gamma^\mu u^{(\beta)}(m,0) + 2m^2 C_\alpha^* C_\beta\, \bar{u}^\alpha(m,0)u^{(\beta)}(m,0) =$$

$$\boxed{\gamma^i u^{(\beta)}(m,0) = 0\,,\ \gamma^0 u^{(\beta)}(m,0) \neq 0}$$

$$= C_\alpha^* C_\beta\, 2m\, P_0\, \bar{u}^{(\alpha)}(m,0)\gamma^0 u^{(\beta)}(m,0) + 2m^2 C_\alpha^* C_\beta\,\delta^{\alpha\beta} =$$

$$= C_\alpha^* C_\beta\, 2mE\, \bar{u}^{(\alpha)}(m,0)\, u^{(\beta)}(m,0) + 2m^2 C_\alpha^* C_\beta\,\delta^{\alpha\beta} =$$

$$= C_\alpha^* C_\beta\, 2mE\,\delta^{\alpha\beta} + 2m^2 C_\alpha^* C_\beta\,\delta^{\alpha\beta} = C_\alpha^* C_\beta\, 2m(E+m)\,\delta^{\alpha\beta}$$

$$\Rightarrow |C_\alpha|^2\, 2m(E+m) = 1 \quad \Rightarrow \quad \boxed{C_\alpha = \frac{1}{\sqrt{2m(E+m)}}} \quad \text{up to a phase}$$

For $v$ components:

$$-\delta^{\alpha\beta} = \bar{v}^{(\alpha)}(p)\, v^{(\beta)}(p) = d_\alpha^* d_\beta\, \bar{v}^{\alpha} (-\not{p}+m)^2 v^{\beta} \Rightarrow |d_\alpha|^2\, 2m(E+m) = -1$$

$$\Rightarrow \boxed{d_\alpha = \frac{1}{\sqrt{2m(E+m)}}} \quad \text{up to a phase}$$

$$\Rightarrow \begin{cases} u^{(\alpha)}(p) = \dfrac{\not{p}+m}{\sqrt{2m(E+m)}}\, u^{(\alpha)}(m,\underline{0}) \\ v^{(\alpha)}(p) = \dfrac{\not{p}+m}{\sqrt{2m(E+m)}}\, v^{(\alpha)}(m,\underline{0}) \end{cases} \quad \Bigg\| \quad u^{(\alpha)}(m,0) = \begin{pmatrix} \phi^{(\alpha)}(m,0) \\ 0 \end{pmatrix} ; \quad v^{(\alpha)}(m,0) = \begin{pmatrix} 0 \\ \chi^{(\alpha)}(m,0) \end{pmatrix}$$

$$\not{p} = p_\mu \gamma^\mu = \begin{pmatrix} E & -\underline{p}\cdot\underline{\sigma} \\ \underline{p}\cdot\underline{\sigma} & -E \end{pmatrix} \Rightarrow \left[ \$ \text{ pag } 70 \right]$$

$$\Rightarrow u^{(\alpha)}(p) = \begin{pmatrix} E+m & -\underline{\sigma}\cdot\underline{p} \\ \underline{\sigma}\cdot\underline{p} & -E+m \end{pmatrix} \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} \phi^{(\alpha)} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{E+m}{2m}}\, \phi^{(\alpha)}(m,\underline{0}) \\ \dfrac{\underline{p}\cdot\underline{\sigma}}{\sqrt{2m(E+m)}}\, \phi^{(\alpha)}(m,\underline{0}) \end{pmatrix}$$

$$u^{(\alpha)\dagger}(p)\, u^{(\alpha)}(p) = \left( \sqrt{\frac{E+m}{2m}}\, \phi^\dagger ,\ \frac{\underline{p}\cdot\underline{\sigma}}{\sqrt{2m(E+m)}}\, \phi^\dagger \right) \begin{pmatrix} \sqrt{\dfrac{E+m}{2m}}\, \phi^{(\alpha)}(m,\underline{0}) \\ \dfrac{\underline{p}\cdot\underline{\sigma}}{\sqrt{2m(E+m)}}\, \phi^{(\alpha)}(m,\underline{0}) \end{pmatrix} =$$

$$= \frac{E+m}{2m} \underbrace{\phi^\dagger \phi}_{=1} + \frac{(\underline{p}\cdot\underline{\sigma})^2}{2m(E+m)} \underbrace{\phi^\dagger \phi}_{=1} = \frac{E^2+m^2+2Em+p^2}{2m(E+m)} = \frac{2E^2+2Em}{2m(E+m)} = \frac{E}{m}$$

with $(\underline{p}\cdot\underline{\sigma})^2 = p^2$.

$$v^{(\alpha)\dagger}(p)\, v^{(\alpha)}(p) = \ldots = \frac{E}{m}$$

We require that $\psi$ is normalized

$$\Rightarrow \begin{cases} \hat{u}^{(\alpha)}(p) = \sqrt{\dfrac{E}{m}}\, u^{(\alpha)}(p) \\ \hat{v}^{(\alpha)}(p) = \sqrt{\dfrac{E}{m}}\, v^{(\alpha)}(p) \end{cases} \Rightarrow \begin{cases} \psi^{(\alpha)(+)} = \sqrt{\dfrac{m}{E}}\, u^{(\alpha)}(p)\, e^{-ip_\mu x^\mu} \\ \psi^{(\alpha)(-)} = \sqrt{\dfrac{m}{E}}\, v^{(\alpha)}(p)\, e^{ip_\mu x^\mu} \end{cases}$$

$$\boxed{(\psi_1, \psi_2) = \int d^3x\, \psi_1^\dagger \psi_2}$$

$$(\psi_\alpha^{(+)}, \psi_\beta^{(+)}) = N^2 \frac{m}{E}\int d^3x \, u^{(\alpha)\dagger}(p)\, u^{(\beta)}(q)\, e^{i(P-Q)_\mu x^\mu} = N^2 \frac{\cancel{m}}{\cancel{E}}\frac{\cancel{E}}{\cancel{m}}\, \delta^{\alpha\beta}\,\delta(p-q)(2\pi)^3$$

$$\Rightarrow \quad \boxed{N = \frac{1}{(2\pi)^{3/2}}}$$

$$\Rightarrow \quad \boxed{\psi(x) = \sum_{\alpha=1}^{2}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E}}\left(b_{(\alpha)}(p)\, u_{(\alpha)}(p)\, e^{-iP_\mu x^\mu} + d^*_{(\alpha)}(p)\, v_{(\alpha)}(p)\, e^{iP_\mu x^\mu}\right)}$$

The struct. is the same as the scalar field's

## • Energy projectors

Orthogonality relations : $\begin{cases} (\not{P}+m)(\not{P}+m) = 2m(\not{P}+m) \\ (\not{P}+m)(-\not{P}+m) = 0 \end{cases}$

Let's define $\Lambda_\pm := \dfrac{\pm\not{P}+m}{2m}$ , how do they operate on the field?

$$\Lambda_+\psi = \alpha\Lambda_+ u(p) + \beta\Lambda_+ v(p) = \alpha\Lambda_+ \frac{\not{P}+m}{\sqrt{2m(E+m)}}\, u(m,0) + \cancel{\beta\Lambda_+ \frac{-\not{P}+m}{\sqrt{2m(E+m)}}\, v(m,0)}$$

$$= \alpha \frac{\cancel{2m}}{\cancel{2m}}\frac{\not{P}+m}{\sqrt{2m(E+m)}}\, u(m,0) = \alpha\, u(p) \quad \leftarrow \text{bel}$$

$\Lambda_\pm$ are projectors :

- ► $\Lambda_\pm^2 = \frac{1}{4m^2}(\pm\not{P}+m)(\pm\not{P}+m) = \frac{\cancel{2m}}{2\cancel{4}m^{\cancel{2}}}(\pm\not{P}+m) = \Lambda_\pm$ (idempotent)
- ► $\Lambda_+\Lambda_- = 0$ (orthogonal)
- ► $\Lambda_+ + \Lambda_- = \frac{1}{\cancel{2m}}(\cancel{\not{P}}+\cancel{m}-\cancel{\not{P}}+\cancel{m}) = \mathbb{1}$ (sum to $\mathbb{1}$)

Thanks to energy projectors, we can select observables

# Energy projectors in terms of spinors

► $\Lambda_+ = \sum_{\alpha=1}^{2} u_\alpha(p)\bar{u}_\alpha(p)$

$$\sum_{\alpha=1}^{2} u_\alpha(p)\bar{u}_\alpha(p) = \sum u_\alpha u_\alpha^\dagger \gamma^0 = \frac{\sum_\alpha}{2m(E+m)} (\not{p}+m) u_\alpha(m,\underline{0})\, u_\alpha^\dagger(m,\underline{0}) \overset{\gamma^0\gamma^0}{\downarrow} (\not{p}+m)^\dagger \gamma^0 =$$

: matrix

$$= \frac{\sum_\alpha}{2m(E+m)} (\not{p}+m) u_\alpha(m,\underline{0})\, \bar{u}_\alpha(m,\underline{0})(\not{p}+m)$$

$$\sum_\alpha u_\alpha(m,\underline{0})\bar{u}_\alpha(m,\underline{0}) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}(0\,1\,0\,0) + \begin{pmatrix}1\\0\\0\\0\end{pmatrix}(1\,0\,0\,0) = \frac{1+\gamma^0}{2}$$

$$\Rightarrow \sum_{\alpha=1}^{2} u_\alpha(p)\bar{u}_\alpha(p) = \frac{1}{2m(E+m)}(\not{p}+m)\frac{(1+\gamma^0)}{2}(\not{p}+m) = \frac{1}{4m(E+m)}\left[(\not{p}+m)^2 + (\not{p}+m)\gamma^0(\not{p}+m)\right] =$$

$$(\not{p}+m)\gamma^0(\not{p}+m) = (\not{p}+m)(p_\nu\gamma^0\gamma^\nu + \gamma^0 m) = \left[[\gamma_\mu,\gamma_\nu]_+ = 2\eta_{\mu\nu}\right] =$$

$$= (\not{p}+m)\left(p_\nu(2\eta^{0\nu} - \gamma^\nu\gamma^0) + \gamma^0 m\right) =$$

$$= (\not{p}+m)(2\underbrace{p_0}_{=E} - \not{p}\gamma^0 + \gamma^0 m) = (\not{p}+m)(2E + (\not{p}-m)\gamma^0) =$$

$$= 2E(\not{p}+m) + \underbrace{(\not{p}+m)(\not{p}-m)}_{0}\gamma^0 = 2E(\not{p}+m)$$

$$\Rightarrow \sum_{\alpha=1}^{2} u_\alpha(p)\bar{u}_\alpha(p) = \frac{1}{4m(E+m)}\left[2m^2 + 2m\not{p} + 2E(\not{p}+m)\right] = \frac{1}{4m(E+m)}\left[2m(\not{p}+m) + 2E(\not{p}+m)\right] =$$

$$= \frac{\cancel{2}}{\underset{2}{\cancel{4}}m\cancel{(E+m)}}(\not{p}+m)\cancel{(E+m)} = \frac{(\not{p}+m)}{2m} = \Lambda_+$$

■

► In the same way $\sum_{\alpha=1}^{2} v_\alpha(p)\bar{v}_\alpha(p) = -\Lambda_- = \frac{\not{p}-m}{2m}$

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- <u>Spin projectors</u>

$$\sigma_{12} = \frac{i}{2}[\gamma_1, \gamma_2] = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad \text{e.v.} = 1, \ \text{e.v.} = -1$$

Let be $\Sigma(\hat{n}) = \frac{1+\sigma_{12}\hat{n}^3}{2}$, it is connected to $\sigma_3$

$$\sigma_{12} = \frac{i}{2}[\gamma_1, \gamma_2] = i\gamma^1\gamma^2 = -(\gamma^0 \underbrace{i\gamma^0\gamma^1\gamma^2\gamma^3}_{\gamma^5}\overset{=-1}{\gamma^3\gamma^3}) = -\gamma^0\gamma^5\gamma^3 = \gamma_5\gamma_3\gamma_0$$

$$= \frac{1+\gamma_5\gamma_3\gamma_0\hat{n}^3}{2} = \frac{1}{2}\begin{pmatrix} 1+\sigma_3 & 0 \\ 0 & -1+\sigma_3 \end{pmatrix} = \frac{1+\gamma_5\hat{\not{n}}\gamma_0}{2}$$

We can also write $\Sigma(-\hat{n})$, it selects other spin components

$$\Sigma(\pm\hat{n}) = \frac{1 \pm \gamma_5\hat{\not{n}}\gamma_0}{2} \; ; \quad \Sigma(+\hat{n})\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\alpha\\ 0\\ \gamma\\ \delta\end{pmatrix}, \quad \Sigma(-\hat{n})\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}0\\ \beta\\ 0\\ \delta\end{pmatrix}$$

$\to 0?$

removing $\gamma_0$

New projector: $\Sigma(\hat{n}) = \frac{1+\gamma_5\hat{\not{n}}}{2}$ : $\Sigma(+\hat{n})\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\alpha\\ 0\\ 0\\ \delta\end{pmatrix}$

I boost $\Rightarrow \Sigma(\pm n) = \frac{1+\gamma_5\hat{\not{n}}}{2}$ ; $n$ has to be spacelike: $n^2 = 1$

not anymore on z

$$\Sigma^2(\pm n) = \frac{1}{4}\left[1 + (\gamma_5\not{n})^2 \pm 2\gamma_5\not{n}\right] = \frac{1}{4}\left[2 \pm 2\gamma_5\not{n}\right] = \Sigma(\pm n)$$

$$\Sigma(+n) + \Sigma(-n) = \mathbb{1}$$

$$\Sigma(+n)\Sigma(-n) = 0$$

# • Non-relativistic limit and recovering of Schroedinger's equation

We consider an electron in a known electromagnetic field $A^\beta \equiv (\phi, A)$.
The Dirac equation in the presence of the field is obtained with the **minimal substitution**

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$$\partial^\mu \to \partial^\mu + ieA^\mu$$

where $e$ is the electronic charge $\quad \hookrightarrow$ interaction with the charge

$$\Rightarrow (i\partial\!\!\!/ - m)\psi = 0 \longrightarrow [i\gamma_\mu(\partial^\mu + ieA^\mu) - m]\psi = 0 \Rightarrow (i\partial\!\!\!/ - eA\!\!\!/ - m)\psi$$

or in equivalent form $\boxed{(i\gamma_0\partial^0 - e\gamma_0 A^0 - m)\psi(x) + \gamma_i(i\partial^i - eA^i)\psi = 0}$ #

We want to describe the physics of a particle with $v \ll c$

$$E = \sqrt{p^2 + m^2} \simeq m + \frac{p^2}{2m}$$

In these conditions it is helpful to isolate the rapidly varying phase factor which corresponds to the rest energy, and rewrite the solution to (#) in the form

$$\boxed{\psi = \psi_0 \exp\{-imt\}}$$

Where $\psi_0$ oscillates much more slowly than $e^{-imt}$

$$\Rightarrow \gamma_0\partial_0(\psi_0 e^{-imt}) = \gamma_0(\partial_0\psi_0)e^{-imt} - im\gamma_0\psi_0 e^{-imt}$$

Negative energy solut. are of difficult interpretation, we can show that in our limit we can neglect negative energy solutions

Let's put $\boxed{\phantom{xxx}}$ in Dirac eq. #, $\psi(x) = \begin{pmatrix}\phi(x)\\ \chi(x)\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}(i\partial^0 - eA^0 + m)\begin{pmatrix}\phi\\ \chi\end{pmatrix} - m\begin{pmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} + \begin{pmatrix}0 & -\sigma^i\\ \sigma^i & 0\end{pmatrix}(i\partial^i - eA^i)\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0$$

$$\Rightarrow \begin{cases}(i\partial^0 - eA^0)\phi - \sigma_i(i\partial^i - eA^i)\chi = 0\\ -(i\partial^0 - eA^0 + 2m)\chi + \sigma_i(i\partial^i - eA^i)\phi = 0\end{cases}$$

$$\chi = \frac{\underline{\sigma}\cdot(\underline{p} - e\underline{A})}{2m}\phi \Rightarrow (i\partial^0 - eA^0)\phi = \frac{[\underline{\sigma}\cdot(\underline{p} - e\underline{A})]^2}{2m}\phi$$

we recovered Schrodinger equation

# • The gyromagnetic factor of electron

$$(\underline{\sigma}\cdot(\underline{p}-e\underline{A}))^2 = \sigma_i\sigma_j(p^i - eA^i)(p^j - eA^j) \;;$$

$$\sigma_i\sigma_j = \frac{1}{2}[\sigma_i,\sigma_j]_+ + \frac{1}{2}[\sigma_i,\sigma_j] = \delta_{ij} + \varepsilon_{ijk}\sigma^k$$

$$\Rightarrow (\underline{\sigma}\cdot(\underline{p}-e\underline{A}))^2 = (\underline{p}-e\underline{A})^2 + \varepsilon_{ijk}\sigma_k\left[\cancel{p^i p^j} - e(p^iA^j + A^ip^j) + \cancel{e^2A^iA^j}\right]$$

because they are totally symmetric and $\varepsilon_{ijk}$ is antisymm

$$= (\underline{p}-e\underline{A})^2 - e\underline{\sigma}\cdot\underbrace{(\underline{\nabla}\wedge\underline{A})}_{\text{magnetic field } B}$$

$$\Rightarrow (i\partial^0 - eA^0)\phi = \left(\frac{(\underline{p}-e\underline{A})^2}{2m} - \frac{e}{2m}\underline{\sigma}\cdot\underline{B}\right)\phi$$

; we reproduced the hamilt. of interac. between $e^-$ and B field

We can write $\underline{\mu} = -\frac{e}{m}\hat{s} = -g\frac{e}{2m}\hat{s}$ ,

The coefficient ***g*** is known as the ***gyromagnetic ratio*** and expresses the relationship between the magnetic moment, given in units of Bohr magnetons, and the corresponding angular momentum.

23/10/2020

Dirac imagined a sea of negative solutions filled, vacuum state, of infinite energy

A negative energy-electron can jump to a positive energy state

$\Rightarrow E_0 = -NE \longrightarrow -(N-1)E = -NE + E$ , $Q_0 = N|e| \longrightarrow (N-1)e$

The photon creates a pair, an $e^-$, $E>0$ and a positron $\Rightarrow e^+$ was predicted.

- # Charge conjugation

$\psi$ under charge conjugation

charge of the particle

Dir. eq. with interaction: $(i\not\partial - e\not A - m)\psi(x) = 0 \xrightarrow{C} (i\not\partial + e\not A - m)\psi^c(x) = 0$

I want to find $C$ s.t. $C(C\psi) = e^{i\varphi}\psi$ ★

irrelevant

$\bar\psi^t = \gamma^{0t}\psi^*$

Dirac eq. for $\bar\psi$: $-i\bar\psi\overleftarrow{\not\partial} - e\bar\psi\not A - m\bar\psi = 0 \xrightarrow{\text{transpose}} [\gamma^{\mu t}(-i\partial_\mu - eA_\mu) - m]\bar\psi^t = 0$

$$\Rightarrow \quad \mathbb{C}\left[\gamma^{\mu t}(-i\partial_\mu - eA_\mu) - m\right]\mathbb{C}^{-1}\mathbb{C}\,\bar\psi^t = 0$$

$\mathcal{L}$ should be invariant under $C$ $\Rightarrow$ we must have $\bar\psi\psi = \bar\psi_c\psi_c \xRightarrow[\text{176/177}]{\text{MAIANI PAG}} \mathbb{C}\gamma^{\mu t}\mathbb{C}^{-1} = -\gamma^{\mu t}$

It will be valid $\psi^c(x) = \eta_c\,\mathbb{C}\,\bar\psi^t(x)$ ; $\gamma^{0t} = \gamma^0$; $\gamma^{1t} = -\gamma^1$ ; $\gamma^{2t} = \gamma^2$ ; $\gamma^{3t} = -\gamma^3$

$$\Rightarrow \quad \mathbb{C} = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \quad ; \ C^t = -C \ ; \ C^t = C^{-1}$$

Let's take a peculiar solution for Dirac field: $\psi'(x) = \underbrace{\left(\frac{\pm\not p + m}{2m}\right)}_{\text{DEFINITE ENERGY}}\underbrace{\left(\frac{1 \pm \gamma_5\not n}{2}\right)}_{\text{DEFINITE SPIN}}\psi(x)$

Let's see what happens applying $\mathbb{C}$:

$$\psi'^c(x) = \eta_c\,\mathbb{C}\,\bar\psi'^t(x) = \eta_c\,\mathbb{C}\,\gamma^0\left(\frac{\pm\not p^* + m}{2m}\right)\left(\frac{1 \pm \gamma_5\not n^*}{2}\right)\psi^*(x)$$

► $\gamma^0\gamma^{\mu *} = \gamma^{\mu t}\gamma_0$ (\$) ► $\gamma_0\gamma^{\mu t}\gamma_0 = \gamma^\mu \Rightarrow (\gamma_0\gamma^{\mu t}\gamma_0)^t = \gamma^{\mu t} \Rightarrow \gamma_0\gamma^{\mu *}\gamma_0 = \gamma^{\mu t}$

► $[\mathbb{C}, \gamma_5] = 0$ ► $\gamma^{\mu t} = \begin{cases} \gamma^0 \\ -\gamma^1 \\ \gamma^2 \\ -\gamma^3 \end{cases} \Rightarrow \mathbb{C}\gamma^\mu = \begin{cases} -\gamma^\mu\mathbb{C} & , \mu = 0,2 \\ \gamma^\mu\mathbb{C} & , \mu = 1,3 \end{cases}$

$$\Rightarrow \psi'^c(x) \overset{\$}{=} \eta_c\,\mathbb{C}\left(\frac{\pm\not p^t + m}{2m}\right)\left(\frac{1 \mp \gamma_5\not n^t}{2}\right)\bar\psi^t(x) =$$

$$= \left[\gamma_0\psi^*(x) = \bar\psi^t(x)\right] = \eta_c\left(\frac{\mp\not p + m}{2m}\right)\left(\frac{1 \mp \gamma_5\not n}{2}\right)\overbrace{\mathbb{C}\,\bar\psi^t(x)}^{\psi^c(x)}$$

Let's take a particular spin: $\psi(x) = e^{imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

we moved from $E<0$, spin $\downarrow$ to $E>0$, spin $\uparrow$

$$\psi^c(x) = \eta_c\,\mathbb{C}\,\bar\psi^t = \eta_c\,\overbrace{i\gamma^2\gamma^0}^{\mathbb{C}}\overbrace{\gamma^0\psi^*}^{\bar\psi^t} = \eta_c\, i\gamma^2\psi^* = \eta_c e^{-imt}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \eta_c e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

# • Dirac field quantization

$$\begin{cases} \mathcal{L} = \bar\psi (i\not\partial - m)\psi , \\ \pi_\psi = i\psi^\dagger ; \\ \pi_{\bar\psi} = 0 \end{cases}$$

In principle I cannot do Leg. transf., but from N. th I know $E$, $P^\mu$ are conserved

$$\underset{\uparrow \\ \text{Energy}}{\mathcal{H}} = i\psi^\dagger \frac{\partial\psi}{\partial t} \quad ; \quad \underset{\uparrow \\ \text{trimomentum}}{\underline{P}} = i\psi^\dagger \underline{\nabla}\psi$$

Additional conserved quantity: charge, it comes from the invariance of $\mathcal{L}$ under a global phase

$$Q = \int d^3x\, \bar\psi \gamma^0 \psi ; \qquad \frac{d}{dt} Q = 0$$

Now we promote the field as an operator giving some rules

$$\psi(x) = \sum_{\pm n} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left( b(p,n)\, u(p,n)\, e^{-iP_\mu x^\mu} + d^\dagger(p,n)\, v(p,n)\, e^{iP_\mu x^\mu} \right)$$

OPERATORS

$$\psi^\dagger(x) = \sum_{\pm n} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left( d(p,n)\, v^\dagger(p,n)\, e^{-iP_\mu x^\mu} + b^\dagger(p,n)\, u^\dagger(p,n)\, e^{iP_\mu x^\mu} \right)$$

With $(\psi, \psi^\dagger) = 1$ .

Now I want to express energy density

- ► $\bar u(p,n)\, u(p,n') = -\bar v(p,n)\, v(p,n') = \delta_{nn'}$
- ► $u^\dagger(p,n)\, u(p,n') = v^\dagger(p,n)\, v(p,n') = \frac{E}{m}\, \delta_{nn'}$
- ► $\bar v(p,n)\, u(p,n') = 0 = v^\dagger(p,n)\, u(\tilde p,n') = u^\dagger(p,n)\, v(\tilde p,n')$ ; where $p^\mu = (E, \underline{p})$ , $\tilde p^\mu = (E, -\underline{p})$
- ► $u^\dagger(p,n)\, v(\tilde p,n') = u^\dagger(m,\underline{0}) \underbrace{\overset{\gamma_0\gamma_0}{}(\not p+m)^{\dagger\,\gamma_0\gamma_0}(-\tilde{\not p}+m)}_{2m(E+m)} v(m,\underline{0}) =$

$$= \bar u(m,0) \underbrace{\gamma_0 (\not p+m)^\dagger \gamma_0}_{(\not p+m)} \underbrace{\gamma_0(-\tilde{\not p}+m)}_{(-\not p+m)} v(m,\underline{0}) = 0$$

$$(\not p+m) \cdot (-\not p+m) = 0$$

# • Energy density and momentum in terms of creation and annihilation operators

hamiltonian

$$H = \int d^3x\, \mathcal{H} = i\int d^3x\, \psi^\dagger \frac{\partial \psi}{\partial t} =$$

$$= \underset{+1}{\underline{i}} \int d^3x \sum_{\substack{\pm n \\ \pm n'}} \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E}} \int \frac{d^3p'}{(2\pi)^{3/2}}\sqrt{\frac{m}{E'}} \left[ d(p,n)\, v^\dagger(p,n)\, e^{-ip_\mu x^\mu} + b^\dagger(p,n)\, u^\dagger(p,n)\, e^{ip_\mu x^\mu} \right] \cdot$$

$$\cdot\, (-\underline{i}E') \left[ b(p',n')\, u(p',n')\, e^{-ip'_\mu x^\mu} - d(p',n')\, v(p',n')\, e^{ip'_\mu x^\mu} \right] =$$

$$= \int \otimes \frac{mE'}{\sqrt{EE'}} \Big\{ \underbrace{\cancel{d(p,n)\, v^\dagger(p,n)\, b(p',n')\, u(p',n')\, e^{-i(P+P')_\mu x^\mu}}}_{=0} +$$

$$- d(p,n)\, v^\dagger(p,n)\, d^\dagger(p',n')\, v(p',n')\, e^{-i(p-p')_\mu x^\mu} +$$

$$+ b^\dagger(p,n)\, u^\dagger(p,n)\, b(p',n')\, u(p',n')\, e^{-i(p'-p)_\mu x^\mu} +$$

$$\underbrace{\cancel{- b^\dagger \; u^\dagger \; d^\dagger \; v \; e^{i(P+P')_\mu x^\mu}}}_{=0} \Big\} = \left[ \text{integrate first in } x \right] =$$

because of $\delta(p+p')$ that appears

$$\Rightarrow \quad \boxed{H = \int d^3x\, \psi^\dagger \frac{\partial \psi}{\partial t} = \sum_{\pm n} \int d^3p\, E \left[ b^\dagger(p,n)\, b(p,n) - d(p,n)\, d^\dagger(p,n) \right]}$$

Calculating P, the integrals are the same

$$\boxed{\underline{P} = \int d^3x\, \psi^\dagger(-i\nabla)\psi = \sum_{\pm n} \int d^3p\, \underline{p} \left[ b^\dagger(n)\, b(n) - d(n)\, d^\dagger(n) \right]}$$

To have a consistent theory, in order to recover a − sign in the charge in normal order, we impose anti comm. relations: in this way, wavefun. will be invariant under exchange of particles (good)