• Natural units

In our formulas are always present the constants c, $\hbar$, we use $c=\hbar=1$, which gives natural units

- $\ell = v \cdot t \Rightarrow [\ell] = [t]$
  $\hookrightarrow c = 1$
- $E = mc^2 \Rightarrow [E] = [m]$
- Action: $Et = \hbar = 1 \Rightarrow [E] = [t^{-1}] = [\ell^{-1}]$
- Momentum: $p = \frac{E}{c} \Rightarrow [E] = [p]$

$c = 3 \times 10^8 \, m\,s^{-1}$; $\hbar = 1.05 \cdot 10^{-34} \, J\,s^{-1}$

$1 = c = 3 \times 10^8 \, m\,s^{-1} = 3 \times 10^{23} \, fm\,s^{-1} \Rightarrow 1\,fm = 3.3 \times 10^{-24}\,s$

$1 = \hbar = 1.05 \cdot 10^{-34} J\,s^{-1} = 0.7 \times 10^{-15} eV\,s^{-1} = 7 \cdot 10^{-22}\,MeV\,s$

$\Rightarrow 1\,MeV^{-1} = 7 \times 10^{-22}\,s$

$\lambda_{COMPTON} = \frac{\hbar}{m_e c} = 400\,fm \Rightarrow m_e \sim 0.5\,MeV$

$1\,barn = 10^{-24}\,cm^2 \Rightarrow 1\,GeV^{-2} = 0.389\,mbarns$

$\sigma_H \simeq 48.58\,pb = \ldots = 0.125\,TeV^{-2}$

$\hookrightarrow$ Cross section of X-bosons production

• Klein-Gordon equation

Let's consider as a first trivial representation of Poinc. group a scalar field $\phi(x)$

In Poincaré group, $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$; $\phi(x) \rightarrow \phi'(x') = \phi(x)$

• Schrödinger eq. and non relativistic Q.M. point of view

Schrödinger point of view : $\boxed{i\hbar \frac{\partial}{\partial t}\psi = H\psi = -\hbar^2 \frac{\nabla^2}{2m}\psi(x,t)}$

wavefunction   free particle

This eq. can be obtained quantizing $\epsilon = \frac{p^2}{2m}$: $\epsilon \rightarrow i\hbar \frac{\partial}{\partial t}$; $\underline{p} \rightarrow -i\hbar \underline{\nabla}$

The problem of this equation is that special relativity is excluded

Probability density: $|\psi|^2$, it should be conserved : $\int d^3x |\psi|^2 = 1$

and $\frac{d}{dt}\int d^3x |\psi|^2 = 0$ (Indipendent L2 scalar product)

complex conjugated equation

You can obtain ψ from Schr. eq : $-i\hbar \frac{\partial}{\partial t}\psi^* = H\psi^* = \hbar^2 \frac{\nabla^2}{2m}\psi^*$

You subtract equations multiplied by ψ and ψ^* :

$$i\hbar \frac{\partial}{\partial t}(\psi^*\psi) = \frac{\hbar^2}{2m}(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi) =$$

$$= \frac{\hbar^2}{2m}\underline{\nabla}\cdot(\psi\underline{\nabla}\psi^* - \psi^*\underline{\nabla}\psi)$$

$$\Rightarrow \frac{\partial}{\partial t}\rho = -\underline{\nabla}\cdot\underline{J} \Rightarrow \boxed{\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0}$$ (Continuity equation)

Integrating on $\mathbb{R}^3$, $\frac{\partial}{\partial t}\int d^3x |\psi|^2 = 0$, it is interpreted as the probability.

$|\psi|^2$ is a real positive quantity

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- ## Klein-Gordon equation

We can use $\boxed{E^2 = p^2 + m^2}$ #

The first problem is that $E = \pm\sqrt{p^2+m^2}$, for every momentum, we have two solutions for E and one of them is negative

Quantising #, we obtain a second-order equation with the problem of negative equation.

Taking the square root : $i\frac{\partial}{\partial t}\phi = \sqrt{-i\nabla^2 + m^2}\,\phi$, not good because

The Hilbert space should be linear and we want to preserve locality which implies casuality

Now we have $\boxed{-\frac{\partial^2}{\partial t^2}\phi = \left(-\nabla^2 + m^2\right)\phi}$ which can be put in covariant form

$\frac{\partial}{\partial x^\mu}$, $\frac{\partial}{\partial x_\mu}$: covariant and contravariant gradient

$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \partial_\mu\partial^\mu = \partial_t^2 - \partial_i^2$, it is an invariant $\Rightarrow \partial_{\mu'}\partial^{\mu'} = \partial_\mu\partial^\mu$

$\Rightarrow \left(\partial_\mu\partial^\mu + m^2\right)\phi = 0$, $\left(\Box + m^2\right)\phi = 0$ (Klein-Gordon equation)

However:

- we have a 2nd-order derivative in time equation $\Rightarrow$ we have to impose conditions on wavefunction and its derivative which are connected to the position and the momentum, which is not possible because of Heisenberg principle
- $|\psi|^2$ is not positive definite

K-G eq. is covariant, which is good :

ϕ is a scalar

$$\left(\partial'_\mu\partial'^\mu + m^2\right)\phi' = 0$$
$$\phi' = \phi$$
$$\left(\partial_\mu\partial^\mu + m^2\right)\phi = 0$$

K-G eq. is real in parameters, if ψ is solution $\Rightarrow \psi^*$ is solution:

$$\phi^*(\partial_\mu\partial^\mu + m^2)\phi = 0$$
$$\phi(\partial_\mu\partial^\mu + m^2)\phi^* = 0 \Rightarrow \phi^*(\partial_\mu\partial^\mu + \cancel{m^2})\phi - \phi(\partial_\mu\partial^\mu + \cancel{m^2})\phi^* = 0$$

$$\Rightarrow \phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^* = 0 \Rightarrow \partial_\mu \underbrace{i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)}_{:= J^\mu} = 0$$

4-vector

$\partial_\mu J^\mu = 0$; $\frac{d}{dt}\int d^3x\, J^0 = 0$, good point!

$$\Rightarrow \|\phi\|^2 := \int d^3x \; i\,\phi^* \overleftrightarrow{\partial_0} \phi$$

(Correct time - independent scalar product for K-G field)

We had to assume complex ϕ, another problem

⇒ The equation was discarted because it was complicated to connect it to concepts of Schrödinger's point of view

Solution

$(\Box + m^2)\phi = 0$ is interpreted as the classical eq. for the field, in fact it is equal to Maxw. eq., then it is possible to quantize the field and find the particle linked to the field.

This procedure is called second quantization

• Lagrangian and hamiltonian density of the Klein-Gordon field

We start from the equation to obtain the Lagr. density:

$$\phi(x),\quad 0=\int d^4x\,\underbrace{(\partial_\mu\partial^\mu\phi+m^2\phi)}_{\text{Diff. eq., K-G. eq}}\,\underbrace{\delta\phi}\;=\;\longrightarrow \text{Variation of the field}$$

$$=\int d^4x\;\underbrace{\cancel{\partial_\mu\left[\partial^\mu\phi\,\delta\phi\right]}}_{\text{Total derivative, div. theorem}}-\partial^\mu\phi\,\underbrace{\partial_\mu\delta}_{\text{they commute}}\phi+m^2\phi\,\delta\phi=$$

Otherwise $\mathcal{H}$ would be negative definite

$$=\int d^4x\left\{-\partial^\mu\phi\,\delta(\partial_\mu\phi)+\frac{m^2}{2}\delta(\phi^2)\right\}=-\delta\int d^4x\;\underbrace{\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi-m^2\phi^2\right)}_{\mathcal{L}(\phi,\phi_{,\mu}),\ \text{lagr. density}}$$

$\Rightarrow$ Lag. dens. of scalar fields: $\boxed{\mathcal{L}=\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi-m^2\phi^2\right)}$

Now we find conjugated momentum of ϕ: $\boxed{\pi_\phi=\dfrac{\partial\mathcal{L}}{\partial\dot\phi}=\dot\phi}$

Hamiltonian density

$$\mathcal{H}=\pi_\phi\,\partial_0\phi(\phi,\pi)-\mathcal{L}(\phi,\pi)=\boxed{\partial_\mu\phi\,\partial^\mu\phi=\dot\phi^2-(\nabla\phi)^2}=\pi_\phi^2-\frac{1}{2}\left(\pi_\phi^2-(\nabla\phi)^2-m^2\phi^2\right)$$

$$\Rightarrow\quad\boxed{\mathcal{H}=\frac{1}{2}\left[\pi^2+(\nabla\phi)^2-m^2\phi^2\right]}$$

(Sum of positive quantities)

From Nöther's theorem point of view we find the conserved quantities $\boxed{T^\mu_{\ \nu}=\dfrac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\phi_{,\nu}-\eta^\mu_{\ \nu}\mathcal{L}}$

$$T^0_{\ 0}=\frac{\partial\mathcal{L}}{\partial\phi_{,0}}\phi_{,0}-\mathcal{L}\ \overset{\frac{\partial\mathcal{L}}{\partial\phi_{,0}}=\phi_{,0}}{=\!=\!=}\ (\partial_0\phi)^2-\mathcal{L}=\frac{1}{2}\left((\partial_0\phi)^2+(\nabla\phi)^2+m^2\phi^2\right)$$

$$T^0_{\ i}=\frac{\partial\mathcal{L}}{\partial\phi_{,0}}\phi_{,i}-\cancel{\underset{\eta^0_{\ i}=0}{\eta^0_{\ i}}\,\mathcal{L}}=\pi\,\phi_{,i}$$

$$M^0_{ij}=\int\left(x_i T^0_{\ j}-x_j T^0_{\ i}\right)d^3x\qquad\text{(Angular momentum)}$$

• Charged scalar field and global U(1) symmetry

We can move from a real field to a complex field, which means that the field is charged

We consider the lagrangian of two real fields ϕ_1, ϕ_2 :

$$\begin{cases} \phi = \dfrac{\phi_1 + i\phi_2}{\sqrt{2}} \; ; \; \phi^* = \dfrac{\phi_1 - i\phi_2}{\sqrt{2}} \\ \left[\phi_1 = \dfrac{\phi + \phi^*}{\sqrt{2}} \; ; \; \phi_2 = \dfrac{\phi - \phi^*}{i\sqrt{2}} \right] \star \end{cases}$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^2 + \partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^2\right) =$$

$\star$

$\Longrightarrow$ $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$, this bring us back to Nöth. th.

$\mathcal{L}$ has an internal symmetry, in fact ϕ can be rescaled by a global phase

$$\begin{aligned} &\phi(x) \to \phi'(x) = e^{i\vartheta}\phi(x) \\ &\phi^*(x) \to \phi^{*\prime}(x) = e^{-i\vartheta}\phi^*(x) \end{aligned} \quad , \; \vartheta \in \mathbb{R} \;, \vartheta \text{ constant}$$

It is a global internal symmetry!

→ because our tr. doesn't include x

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \delta\phi_i + \cancel{\mathcal{L}\, \delta x^\mu}$$

i is a non-Lor. index, we have 2 independent fields

$$= \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi^*_{,\mu}} \delta\phi^* \quad ; \quad \begin{aligned} &\delta\phi = e^{i\vartheta}\phi - \phi \simeq i\vartheta\phi \\ &\delta\phi^* = e^{-i\vartheta}\phi - \phi \simeq -i\vartheta\phi^* \end{aligned}$$

$\Rightarrow$ $$J^\mu = i\vartheta\,\phi\,\partial^\mu\phi^* - i\vartheta\,\phi^*\partial^\mu\phi$$

$\Rightarrow$ $$0 = \partial_\mu J^\mu = \vartheta\, \partial_\mu\left[i\,\phi\partial^\mu\phi^* - i\,\phi^*\partial^\mu\phi\right]$$

$\Rightarrow$ $$\boxed{\frac{d}{dt}\int d^3x \; \underbrace{i\,\phi\overleftrightarrow{\partial_0}\phi^*}_{\text{Correct scalar product for K-G fields}} = 0}$$

• Plane wave solutions of the KG equation

We want to derive a general solution for the scalar field of the form

$\phi(x) = A e^{-iP_\mu x^\mu}$, solution of $\quad (\partial_\mu \partial^\mu + m^2)\phi(x) = 0$

As we knew

$\Rightarrow \quad (-P_\mu P^\mu + m^2) A e^{-iP_\mu x^\mu} = 0 \quad \Rightarrow P_\mu P^\mu = m^2 \quad \Rightarrow E^2 = p^2 + m^2$

Two frequences : $\begin{cases} E = +\sqrt{p^2+m^2} = \omega_p \\ E = -\sqrt{p^2+m^2} = -\omega_p \end{cases}$

$\Rightarrow$ two kind of solutions $\begin{cases} \phi^+ = A e^{-i\omega_p t + i\underline{p}\cdot\underline{x}} \\ \phi^- = A e^{+i\omega_p t + i\underline{p}\cdot\underline{x}} \end{cases}$ (Positive and negative energy solution)

$\Rightarrow$ The general solution is the superposition of solutions over all momentum space $\quad \boxed{\phi(x) = \int d^3p \left(a e^{-i\omega_p t + i\underline{p}\cdot\underline{x}} + b e^{i\omega_p t + i\underline{p}\cdot\underline{x}} \right)}$

Normalization : we can normalize to the δ with the scalar product we have found

$$(\phi_1, \phi_2) = i \int d^3x \, \phi_1^* \overleftrightarrow{\partial_0} \phi_2 = i \int d^3x \left[\phi_1^* \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1^* \right]$$

$$(f_p^+, f_{p'}^+) = i\,|A|^2 \int d^3x \left[e^{iP_\mu x^\mu} \partial_0 e^{-iP'_\mu x^\mu} - e^{-iP'_\mu x^\mu} \partial_0 e^{iP_\mu x^\mu} \right] =$$

$$= i\,|A|^2 \int d^3x \left[(-i\omega_p') e^{i(P-P')_\mu x^\mu} - i\omega_p e^{i(P-P')_\mu x^\mu} \right] =$$

$$= |A|^2 \int d^3x \, (\omega_{p'} + \omega_p) e^{i(P-P')_\mu x^\mu} =$$

$$= |A|^2 \int d^3x \, (\omega_p + \omega_{p'}) \, e^{i(\omega_p - \omega_{p'})t} \, e^{-i(\underline{p}-\underline{p}')\cdot x} =$$

$$= |A|^2 (\omega_p + \omega_{p'}) \cancel{e^{i(\omega_p - \omega_{p'})t}} (2\pi)^3 \underline{\delta(\underline{p}-\underline{p}')} \propto \delta(\underline{p}-\underline{p}')$$

$$= |A|^2 (2\pi)^3 \, 2\omega_p \, \delta(\underline{p}-\underline{p}')$$

$$\Rightarrow A = \frac{1}{\sqrt{2\omega_p}}\,\frac{1}{(2\pi)^{3/2}} \Rightarrow \boxed{f_p^+ = \frac{e^{-ip_\mu x^\mu}}{\sqrt{2\omega_p}\,(2\pi)^{3/2}}}$$

$$(f_p^-, f_{p'}^-) = i\,|A|^2 \int d^3x \Big[e^{-i\omega_p t - i\underline{x}\cdot\underline{p}}\, \partial_0\, e^{i\omega_{p'} t + i\underline{x}\cdot\underline{p}'} - \\ + e^{i\omega_{p'} t + i\underline{x}\cdot\underline{p}}\, \partial_0\, e^{-i\omega_p t - i\underline{x}\cdot\underline{p}} \Big] = \dots$$

$$= -|A|^2 (\omega_{p'} + \omega_p)\, e^{-i(\omega_p - \omega_{p'})} (2\pi)^3\, \delta(\underline{p} - \underline{p}')$$

$$= -|A|^2\, 2\omega_p (2\pi)^3\, \delta(\underline{p} - \underline{p}') \qquad \Rightarrow \boxed{f_p^- = \frac{e^{ip_\mu x^\mu}}{\sqrt{2\omega_p}\,(2\pi)^{3/2}}}$$

f^- and f^+ are orthogonal:

$$(f_p^-, f_{p'}^+) = i|A|^2 \int d^3x \Big[e^{-i\omega_p t - i\underline{p}\cdot\underline{x}}\, \partial_0\, e^{-i\omega_{p'} t + i\underline{p}'\cdot\underline{x}} + \\ + e^{-i\omega_{p'} t + i\underline{p}'\cdot\underline{x}}\, \partial_0\, e^{-i\omega_p t - i\underline{p}\cdot\underline{x}} \Big] =$$

$$= |A|^2 (\omega_{p'} - \omega_p)\, e^{-i(\omega_p + \omega_{p'})t}\, \delta(\underline{p} - \underline{p}') = 0$$

The classical solution can be written in terms of f^+ and f^-

$p \to -p$ because I integrate over d^3p

$$\phi(x) = \int d^3p \left(\alpha(p) f_p^+ + \underline{\beta(p) f_p^-}\right) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \Big[\alpha(p)\, e^{-ip_\mu x^\mu} + \overset{\beta(p)}{\overset{\shortparallel}{\tilde{\beta}(-p)}}\, \underbrace{e^{ip_\mu x^\mu}}_{f_p^{+*}} \Big]$$

If we impose $\phi(x) \in \mathbb{R} \Rightarrow \phi(x) = \phi^*(x)$

$\Rightarrow \beta(p) = \alpha^*(p)$

$\Rightarrow$ for real fields, $\phi(x) = \int d^3p \left(\alpha(p) f_p^+ + \alpha^*(p) f_p^-\right)$ #

$$\psi(x,t) = \int A(\underline{p})\, \frac{e^{i\underline{p}\cdot\underline{x} - i\omega t}}{(2\pi)^{3/2}}\, d^3\underline{p}$$

impossible, negative E, the eq. cannot be a quantum mechanics eq.

Now we have to find an operator of the Hilbert space, we will do the same done with x and p in QM, but with fields

f^+, f^- is a base, they can't be promoted as operators, but $\alpha(p), \alpha^*(p)$ can be promoted as operators, it is correct because $\alpha(p), \alpha^*(p)$ will be found in the hamiltonian as in the harmonic oscillator

depends on $t \Rightarrow$ we will have operators dependent on t
$\Rightarrow$ The correct point of view is Heisemberg's instead of Schrödinger's.

- Recap on states and operators

Schrödinger's picture, $\psi(x,t)$: $i\hbar \frac{\partial}{\partial t}\psi = H\psi$, $\langle \psi_1 | \underbrace{O_s}_{\text{indep. on time}} | \psi_2 \rangle$

The physics should be the same: $\langle O \rangle_s = \langle O \rangle_{Heisemb.}$

Unitary transformations: $\langle \psi' | \phi' \rangle = \langle \psi | \underbrace{U^\dagger U}_{= \mathbb{1}} | \phi \rangle = \langle \psi | \phi \rangle$

- Conjugated momenta

$\phi = \phi(\underline{x}, t)$ now plays the role of $\hat{q}(t)$

$\pi_\phi = \pi_\phi(\underline{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ now plays the role of $\hat{p}(t)$

Now we have to impos commutation relations between the fields,
$[\hat{q}, \hat{p}] = i\hbar \mathbb{1}$, if there are more d.o.f. $\Rightarrow [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$

$\Rightarrow$ We impose $\boxed{[\phi(\underline{x}, t), \pi(\underline{y}, t)] = i\hbar\, \delta(\underline{x} - \underline{y})}$ Equal-time commutation relation

- Hamiltonian density in normal modes

What does it mean? How the quantization takes place?

Let us consider the K-G hamiltonian:

$$H = \int d^3x\, \mathcal{H} = \int d^3x \frac{1}{2}\left(\dot\phi^2 + (\underline{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$\blacktriangleright\ \phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}\left(a(p)e^{-ip_\mu x^\mu} + a^*(p)e^{ip_\mu x^\mu}\right)$$

$$\blacktriangleright\ \dot\phi(x) = -i\int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}\,\omega_p\left(a(p)e^{-ip_\mu x^\mu} - a^*(p)e^{ip_\mu x^\mu}\right)$$

$$\blacktriangleright\ \underline{\nabla}\phi(x) = i\int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}\,\underline{p}\left(a(p)e^{-ip_\mu x^\mu} - a^*(p)e^{ip_\mu x^\mu}\right)$$

$$\Rightarrow H = \frac{1}{2}\int \frac{d^3p\, d^3p'}{\sqrt{4\omega_p\omega_{p'}}}\int \frac{d^3x}{(2\pi)^3}\Big\{$$

$$\omega_p\omega_{p'}\left[a_p e^{-ip_\mu x^\mu} - a_p^* e^{ip_\mu x^\mu}\right]\cdot\left[a_{p'}e^{-ip'_\mu x^\mu} - a^*_{p'}e^{ip'_\mu x^\mu}\right]$$

$$-\underline{p}\cdot\underline{p}'\left[a_p e^{-ip_\mu x^\mu} - a_p^* e^{ip_\mu x^\mu}\right]\cdot\left[a_{p'}e^{-ip'_\mu x^\mu} - a^*_{p'}e^{ip'_\mu x^\mu}\right]$$

$$+m^2\left[a_p e^{-ip_\mu x^\mu} + a_p^* e^{ip_\mu x^\mu}\right]\cdot\left[a_{p'}e^{-ip'_\mu x^\mu} + a^*_{p'}e^{ip'_\mu x^\mu}\right]\Big\} =$$

$$= \frac{1}{2}\int \frac{d^3p\, d^3p'}{\sqrt{4\omega_p\omega_{p'}}}\int \frac{d^3x}{(2\pi)^3}\Big\{$$

$$a_p a_{p'}\, e^{-i(\underline{p}+\underline{p}')\cdot\underline{x}}\left(-\omega_p\omega_{p'} - \underline{p}\cdot\underline{p}' + m^2\right) +$$

$$+ a_p^* a_{p'}^*\, e^{i(\underline{p}+\underline{p}')\cdot\underline{x}}\left(-\omega_p\omega_{p'} - \underline{p}\cdot\underline{p}' + m^2\right) +$$

$$+ a_p a_{p'}^*\, e^{-i(\underline{p}-\underline{p}')\cdot\underline{x}}\left(\omega_p\omega_{p'} + \underline{p}\cdot\underline{p}' + m^2\right) +$$

$$+ a_p^* a_{p'}\, e^{i(\underline{p}-\underline{p}')\cdot\underline{x}}\left(\omega_p\omega_{p'} + \underline{p}\cdot\underline{p}' + m^2\right)\Big\} = \left[\begin{array}{l}\text{Integrating in } x \\ \text{several } \delta() \text{ appear}\end{array}\right] =$$

$$= \frac{1}{2}\int \frac{d^3p\, d^3p'}{\sqrt{4\omega_p\omega_{p'}}}\Big\{$$

$$(a_p a_{p'} + a_p^* a_{p'}^*)\,\delta(\mathbf{p}+\mathbf{p}')(-\omega_p\omega_{p'} - \mathbf{p}\cdot\mathbf{p}' + m^2)\, e^{i(\omega_p+\omega_{p'})t}$$

$$+ (a_p a_{p'}^* + a_p^* a_{p'})\,\delta(\mathbf{p}-\mathbf{p}')(\omega_p\omega_{p'} + \mathbf{p}\cdot\mathbf{p}' + m^2)\, e^{i(\omega_p-\omega_{p'})t}\Big\}$$

I take $p' \to -p'$ in the first integral

$$= \frac{1}{\cancel{2}}\int \frac{d^3p\, d^3p'}{\sqrt{4\omega_p\omega_{p'}}}\Big\{$$

for $p=p'$, $-\omega^2+p^2+m^2=0$

$$(a_p a_{p'} + a_p a_{p'}^*)\,\delta(\mathbf{p}-\mathbf{p}')\,\overbrace{\cancel{(-\omega_p\omega_{p'} + \mathbf{p}\cdot\mathbf{p}' + m^2)}}^{=0}\, e^{i(\omega_p+\omega_{p'})t}$$

$$+ (a_p a_{p'}^* + a_p^* a_{p'})\,\delta(\mathbf{p}-\mathbf{p}')\,\underbrace{(\omega_p\omega_{p'} + \mathbf{p}\cdot\mathbf{p}' + m^2)}_{=(\cancel{2}\omega_p^2)}\, \underbrace{e^{i(\omega_p-\omega_{p'})t}}_{=1 \text{ for } \mathbf{p}=\mathbf{p}'}\Big\}$$

$$= \frac{1}{2}\int \frac{d^3p}{\cancel{\omega_p}}\, \omega_p^{\cancel{2}}\,(a_p a_p^* + a_p^* a_p)$$

$$\Rightarrow \boxed{H = \frac{1}{2}\int d^3p\, \omega_p\, (a_p a_p^* + a_p^* a_p)},$$

the hamiltonian of a harmonic oscillator

• Field in terms of creation and annihilation operators

We started with $\phi(x)$ solution of $(\partial_\mu \partial^\mu + m^2)\phi(x) = 0$, $\phi(x)$ is the classical (real) field with $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$, we can also move to the hamiltonian point of view: $\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$;

$$\mathcal{H} = \Pi_\phi \dot{\phi} - \mathcal{L} = \frac{1}{2}\left[\Pi_\phi^2 + (\underline{\nabla}\phi)^2 + m^2\phi^2\right]$$

Now we want to quantize the classical field:

- We write the field in plane waves:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \left(a(p) e^{-iP_\mu x^\mu} + a^*(p) e^{iP_\mu x^\mu} \right) \quad \#$$

- We find the hamiltonian

$\boxed{\text{trick : we ignored that } a, a^* \text{ are numbers}}$

$$H = \frac{1}{2}\int d^3p\, \omega_p \left[a(p) a^*(p) + a^*(p) a(p) \right]$$

It is a sum of single harmonic oscillators

- $\phi(x)$ is also a function of the time $\Rightarrow$ the d.o.f. in the index of $q_i \longrightarrow \hat{q}_i(t)$ now are into the field

$\uparrow$

$\phi(x,t) \to$ heisemb. operator

- In # the only part we can cast to operator is $a(p)$ and $a^*(p)$

$$\Rightarrow \hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \left(\hat{a}(p) e^{-iP_\mu x^\mu} + \hat{a}^\dagger(p) e^{iP_\mu x^\mu} \right) \quad \#$$

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• Harmonic oscillator: a recap

$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$; $\hat{p}, \hat{q}, \hat{H}$ are operators ; $[\hat{q}, \hat{p}] = i\hbar$

we are interested in $H|E\rangle = E|E\rangle$.

We can move to $\hat{Q} = \sqrt{\frac{m\omega}{\hbar}}\,\hat{q}$; $\hat{P} = \frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}$; $H = \frac{1}{2}\hbar\omega(\hat{P}^2 + \hat{Q}^2)$

$a = \frac{1}{\sqrt{2}}(Q + iP)$; $a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) \Rightarrow [a, a^\dagger] = \mathbb{1}$

$$\Rightarrow \boxed{H = \frac{1}{2}\hbar\omega(P^2 + Q^2) = \frac{1}{2}\hbar\omega(aa^\dagger + a^\dagger a)}$$

we diagonalized the hamiltonian

$$= \frac{1}{2}\hbar\omega(2a^\dagger a + \mathbb{1}) = \hbar\omega\left(a^\dagger a + \frac{\mathbb{1}}{2}\right) = \hbar\omega\left(\hat{N} + \frac{\mathbb{1}}{2}\right)$$

$\hat{N}|\nu\rangle = \nu|\nu\rangle$, $\nu \in \mathbb{N}$

$\hat{H}|\nu\rangle = \hbar\omega\left(\nu + \frac{1}{2}\right)|\nu\rangle \Rightarrow E_\nu = \hbar\omega\left(\nu + \frac{1}{2}\right)$

Now we can construct our Hilbert space

$a^\dagger|\nu\rangle = \sqrt{\nu+1}\,|\nu+1\rangle$

$a|\nu\rangle = \sqrt{\nu}\,|\nu-1\rangle$

$a^\dagger|0\rangle = |1\rangle$ .... $\Rightarrow |\nu\rangle = \frac{(a^\dagger)^\nu}{\sqrt{\nu!}}|0\rangle$

ground state

Now we can apply what we know for the field:

$$\hat{\phi}(x) = \int d^3p\,\left(\hat{a}\, f_p^{(+)} + \hat{a}^\dagger f_p^{(-)}\right) \; ; \; \hat{\pi}_\phi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

and then we impose

the commutation relations

• Canonical quantization and commutation relations on the fields

Canonical quantization:

$$\begin{cases} \left[\hat{\phi}(\underline{x},t),\hat{\phi}(\underline{y},t)\right]=0 \\ \left[\hat{\pi}(\underline{x},t),\hat{\pi}(\underline{y},t)\right]=0 \\ \left[\hat{\phi}(\underline{x},t),\hat{\pi}(\underline{y},t)\right]=i\hbar\mathbb{1}\delta(\underline{x}-\underline{y}) \end{cases} \quad \star$$

First we can extract $a, a^\dagger$, we do it by the scalar product:

$$(f^+,f^+)=\delta(\underline{p}-\underline{p}') \qquad \boxed{a(p)=(f^+,\phi)=i\int d^3x\, f^+\overleftrightarrow{\partial_0}\phi}$$

$$(f^-,f^-)=\delta(\underline{p}-\underline{p}') \qquad \boxed{a^\dagger(p)=-(f^-,\phi)}$$

$$(f^+,f^-)=0$$

$$\Rightarrow a(p)=i\int\frac{d^3x}{(2\pi)^{3/2}}\int\frac{d^3p'}{\sqrt{2\omega_p}}\Big[e^{iP_\mu x^\mu}\left(-i\omega_{p'}a(p')e^{-iP'_\mu x^\mu}+i\omega_{p'}a^\dagger(p')e^{iP'_\mu x^\mu}\right)-$$
$$+\left(a(p')e^{-iP'_\mu x^\mu}+a^\dagger(p')e^{iP'_\mu x^\mu}\right)(i\omega_p)e^{iP_\mu x^\mu}\Big]=$$

$$=\int d^3p\int d^3x\Big[\omega_{p'}a_{p'}\left(e^{-i(P'-P)_\mu x^\mu}\right)-\omega_{p'}a^\dagger_{p'}e^{i(P+P')_\mu x^\mu}+$$
$$+\omega_p a_{p'}e^{-i(P'-P)_\mu x^\mu}+\omega_p a^\dagger_{p'}e^{i(P'+P)_\mu x^\mu}\Big]=$$

$$=\int d^3p'\int d^3x\left[(\omega_p+\omega_{p'})a(p')e^{-i(P'-P)_\mu x^\mu}+\cancel{(\omega_p-\omega_{p'})a^\dagger(p')e^{i(P'+P)_\mu x^\mu}}\right]$$

We send $p\to p$ and then we obtain zero

$$=\int\frac{d^3p'}{\cancel{\sqrt{4\omega_p\omega_{p'}}}}\delta(\underline{p}-\underline{p}')\cancel{2\omega_p}\,a(p')=a(p)$$, the scalar product works

The same can be done for $a^\dagger$.

Now we see what relations satisfy $a, a^\dagger$ thanks to $\star$.

In our case, π_ϕ is $\dot{\phi}$

$$\blacktriangleright\ a_p = (f^+, \phi) = \int \frac{d^3x}{(2\pi)^{3/2}\sqrt{2\omega_p}} (\omega_p \phi + i\dot{\phi}) e^{iP_\mu x^\mu}$$

$$\blacktriangleright\ a_p^\dagger = -(f^-, \phi) = \int \frac{d^3x}{(2\pi)^{3/2}\sqrt{2\omega_p}} (\omega_p \phi - i\dot{\phi}) e^{-iP_\mu x^\mu}$$

$$a_p a_{p'}^\dagger = \int \frac{d^3x\, d^3y}{(2\pi)^3\sqrt{4\omega_p\omega_{p'}}} (\omega_p \phi_x + i\dot{\phi}_x) e^{iP_\mu x^\mu} (\omega_{p'}\phi_y - i\dot{\phi}_y) e^{-iP'_\mu y^\mu} =$$

$$= \int \frac{d^3x\, d^3y}{(2\pi)^3\sqrt{4\omega_p\omega_{p'}}} (\underline{\omega_p\omega_{p'}\phi_x\phi_y} - i\omega_p\phi_x\dot{\phi}_y + i\omega_{p'}\dot{\phi}_x\phi_y + \underline{\dot{\phi}_x\dot{\phi}_y}) e^{-i(P'_\mu y^\mu - P_\mu x^\mu)}$$

$$a_{p'}^\dagger a_p = \int \frac{d^3x\, d^3y}{(2\pi)^3\sqrt{4\omega_p\omega_{p'}}} [\underline{\omega_p\omega_{p'}\phi_x\phi_y} + i\omega_p\phi_x\dot{\phi}_y - i\omega_{p'}\dot{\phi}_x\phi_y + \underline{\dot{\phi}_x\dot{\phi}_y}) e^{-i(P'_\mu y^\mu - P_\mu x^\mu)}$$

$\star \Rightarrow \boxed{[a_p, a_{p'}^\dagger] = \delta(\underline{p} - \underline{p}')}$ $\star$ as it should if we want creation and annihilation operators

$\Rightarrow [a_{p'}, a_{p'}] = [a_{p'}^\dagger, a_{p'}^\dagger] = 0$

$$\Rightarrow H = \frac{1}{2}\int d^3p\, \omega_p (a a^\dagger + a^\dagger a) \overset{\star}{=} \frac{1}{2}\int d^3p\, \omega_p\, a^\dagger a + \boxed{\frac{1}{2}\left(\int d^3p\, \omega_p\right)\delta(0)}\ \$$$

\$ is a divergent integral ⇒ infinite energy, but we can remove it imagining it is the energy of the ground state

⇒ We use just $H = \frac{1}{2}\int d^3p\, \omega_p\, a^\dagger a$, it works: $H|0\rangle = 0$

⇒ We imposed that the energy of the vacuum is zero

The notation $H = \frac{1}{2}\int d^3p\, \omega_p\, a^\dagger a$ is called $:H:$ (Normal ordering)

• Fock space

The Fock space is the space were $\hat{\phi}(x)$ acts, we construct it using a and $a^\dagger$ operators : $a|0\rangle = 0$, $|1\rangle = a^\dagger|0\rangle$

▪ One-particle states

We want to evaluate $:H:\left(a_p^\dagger|0\rangle\right) = \int d^3p' \, \omega_{p'} \, a_{p'}^\dagger \underline{a_{p'} a_p^\dagger} |0\rangle =$

$$= \begin{bmatrix}\text{comm.}\\ \text{relat.}\end{bmatrix} = \int d^3p' \, \omega_{p'} \, a_{p'}^\dagger \left(\underbrace{a_p^\dagger a_{p'}}_{=0} + \delta(\underline{p}-\underline{p}') \right)|0\rangle =$$

$$= \int d^3p' \, \omega_{p'} \, a_{p'}^\dagger|0\rangle \delta(\underline{p}-\underline{p}') = \omega_p \left(a_p^\dagger|0\rangle\right) \Rightarrow$$

$\Rightarrow a_p^\dagger|0\rangle$ is e.s. of $:H:$ with e.v. ω_p

Trimomentum:

Nöther's theorem

$$P^i \overset{\uparrow}{=} \int d^3x \, T^{0i} = \int d^3x \, \partial_0\phi \, \partial^i\phi = \ldots = \frac{1}{2}\int d^3p (a_p a_p^\dagger + a_p^\dagger a_p) p^i$$

now we adopt the normal ordering

$$:\underline{P}: = \int d^3p \, \underline{p} \, (a_p^\dagger a_p) \quad \Rightarrow \quad :\underline{P}:(a_p^\dagger|0\rangle) = \ldots = \underline{p}(a_p^\dagger|0\rangle)$$

► $|p\rangle = a_p^\dagger|0\rangle$

► $|f\rangle = \int d^3p \, f(p) \, a_p^\dagger|0\rangle$

wavefunction :

$$\langle p'|f\rangle = \underset{\langle 0|a_{p'}}{\underset{\parallel}{\langle p'|}} \int d^3p \, f(p) a_p^\dagger|0\rangle = \int d^3p \, f(p) a_{p'} a_p^\dagger|0\rangle = f(p')$$

$\langle f, f\rangle = \int d^3p \, |f(p)|^2 = 1$, normalized to the δ

- <u>Two particle states</u>

15/10/2020

I can construct a 2-particle state as follows:

$$:H: \int a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle = \int d^3p' \, \omega_{p'} \, a^\dagger_{p'} a_{p'} a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle = \overset{\text{comm. relat.s}}{\dots} = (\omega_{p_1} + \omega_{p_2}) \left(a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle \right)$$

2-particle state

also this is an e.s. of $:H:$ with energy the sum of the energies

$$|f(p_1, p_2)\rangle = \int d^3p_1 \, d^3p_2 \, f(p_1, p_2) \, a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle$$

- <u>Bose symmetry</u>

$$\left[a^\dagger_{p_1}, a^\dagger_{p_2} \right] = 0 \Rightarrow a^\dagger_{p_1} a^\dagger_{p_2} = a^\dagger_{p_2} a^\dagger_{p_1}$$

$$\Rightarrow a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle = |p_1, p_2\rangle = |p_2, p_2\rangle$$

symmetric

$\Rightarrow$ In the integral will survive only the symmetric part, the scalar field is related to bosons

- Complex field

We now consider a free complex scalar field $\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$; $\phi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$ ← different fields with same mass

$$\boxed{\phi_1(x) = \int d^3p \left(a_p^1 f_p^+ + a_p^{1\dagger} f_p^{+*} \right)} \; ; \; \boxed{\phi_2(x) = \int d^3p \left(a_p^2 f_p^+ + a_p^{2\dagger} f_p^{+*} \right)}$$

$$\boxed{\phi(x)} = \int d^3p \left[\frac{a_p^1 + i a_p^2}{\sqrt{2}} f_p^+ + \frac{a_p^{1\dagger} + i a_p^{2\dagger}}{\sqrt{2}} f_p^{+*} \right] = \boxed{\int d^3p \left[a_p f_p^+ + b_p^\dagger f_p^{+*} \right]}$$

$$a_p = \frac{a_p^1 + i a_p^2}{\sqrt{2}} \; ; \; b_p^\dagger = \frac{a_p^{1\dagger} + i a_p^{2\dagger}}{\sqrt{2}}$$

$$\boxed{\phi^\dagger(x)} = \int d^3p \left(\frac{(a_p^1 - i a_p^2)}{\sqrt{2}} f_p^+ + \frac{a_p^{1\dagger} - i a_p^{2\dagger}}{\sqrt{2}} f_p^{+*} \right) = \boxed{\int d^3p \left(b_p f^+ + a^\dagger f^{+*} \right)}$$

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi^\dagger \; ; \; \Pi_{\phi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot\phi^\dagger} = \dot\phi$$

We impose commutation relations:

$$[\phi(\underline{x},t), \phi^\dagger(\underline{y},t)] = i\delta^3(\underline{x}-\underline{y}) \; ; \; [\phi^\dagger(\underline{x},t), \dot\phi(\underline{y},t)] = i\delta^3(\underline{x}-\underline{y})$$

$$\Rightarrow [a_p, a_{p'}^\dagger] = \delta(\underline{p}-\underline{p}') = [b_p, b_{p'}^\dagger]$$

- Energy: $H = \int d^3x \left(\dot\phi^\dagger \dot\phi + \underline{\nabla}\phi^\dagger \cdot \underline{\nabla}\phi + m^2 \phi^\dagger \phi \right)$

$$:H: = \int d^3p \, \omega_p \left[b_p^\dagger b_p + a_p^\dagger a_p \right]$$ totally symm. expression

- Momentum: $:P: = \int d^3p \, \underline{p} \left[b_p^\dagger b_p + a_p^\dagger a_p \right]$ totally symm. expression

- Fock space :

$$a_p|0\rangle = 0,\ b_p|0\rangle = 0 \quad ; \quad a^+|0\rangle = |p\rangle\ ,\ b^+_p|0\rangle = |p\rangle$$

$$a^+_p a_p (a^+_{p'}|0\rangle) = \delta(\not{p}-\not{p}')\, a^+_p|0\rangle \quad ; \quad b^+_p b_p b^+_{p'}|0\rangle = \delta(\not{p}'-\not{p})\, b^+_p|0\rangle$$

$$b^+_p b_p a^+_{p'}|0\rangle = b^+_p a^+_{p'} \underbrace{b_p|0\rangle}_{=0} = 0 \ ; \quad a^+_p a_p b^+_{p'}|0\rangle = 0$$

$$:H: a^+_p|0\rangle = \omega_p a^+_p|0\rangle \quad ; \quad :H: b^+_p|0\rangle = \omega_p b^+_p|0\rangle$$

- Charge of the scalar (U(1)) complex field:

$$\boxed{Q_{U(1)} := i\int d^3x\, \phi^+ \overleftrightarrow{\partial}_0 \phi}$$

$$\Rightarrow Q_{U(1)} = i\int d^3x \left(\phi^+\partial_0\phi - \phi\,\partial_0\phi^+\right) =$$

$$= i\int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{4\omega_p\omega_q}} \Big\{$$

$$\left(a^+_p e^{iP_\mu x^\mu} + b_p e^{-iP_\mu x^\mu}\right) \cancel{\partial_0}\left(-i\omega_q\, a_q e^{-iQ_\mu x^\mu} + i\omega_q\, b^+_q e^{iQ_\mu x^\mu}\right) -$$

$$+ \left(a_q e^{-iQ_\mu x^\mu} + b^+_q e^{iQ_\mu x^\mu}\right) \cancel{\partial_0}\left(-i\omega_p\, a^+_p e^{iP_\mu x^\mu} + -i\omega_p\, b_q e^{-iP_\mu x^\mu}\right)\Big\}$$

$$\Rightarrow \boxed{:Q: = \int d^3p\, (a^+_p a_p - b^+_p b_p)}$$

the "–" sign permits to distinguish a and b particles

$$:Q: a^+_p|0\rangle = (+1)\, a^+_p|0\rangle$$

$$:Q: b^+_p|0\rangle = (-1)\, b^+_p|0\rangle$$

- Properties of the commutator

We constructed a theory imposing comm. relat. at the same time, we should demonstrate that the commutator is Lorentz-invariant.

Considered an event, only the events inside the light cone are casually connected, otherwise it is needed to send a signal to connect the events.

Let's take $[\phi(x), \phi(y)]_{x^0=y^0} = 0$, if it is Lor-invariant, if we move to another ref. frame, it has to be $\emptyset$

$$[\phi(x), \phi(y)] = \int \frac{d^3p_1\, d^3p_2}{(2\pi)^3 \sqrt{4\omega_{p_1}\omega_{p_2}}} \cdot$$

$$\cdot \left[\left(e^{-iP_{1\mu}x^\mu} a_{p_1} + e^{iP_{1\mu}x^\mu} a^\dagger_{p_1} \right), \left(e^{-iP_{2\mu}y^\mu} a_{p_2} + e^{iP_{2\mu}y^\mu} a^\dagger_{p_2} \right) \right]$$

$$[a_{p_1}, a_{p_2}] = [a^\dagger_{p_1}, a^\dagger_{p_2}] = 0$$

$$\Rightarrow [\phi(x), \phi(y)] = \int (\quad) \overset{\delta}{[a_{p_1}, a^\dagger_{p_2}]}\, e^{-iP_{1\mu}x^\mu + iP_{2\mu}y^\mu} + \overset{-\delta}{[a^\dagger_{p_1}, a_{p_2}]}\, e^{iP_{1\mu}x^\mu - iP_{2\mu}y^\mu} =$$

$$= \int \frac{d^3p_1}{(2\pi)^{3/2}\, 2\omega_{p_1}} \left(e^{-iP_{1\mu}(x-y)^\mu} - e^{iP_{1\mu}(x-y)^\mu} \right) = \Delta(x-y)$$

$\Delta(x-y)\big|_{x^0=y^0} = 0$, It is invariant if both integral measure and integrand are invariant.

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- Integral measure

We can start demonstrating that

$$\int \frac{d^3p}{(2\pi)^3 2\omega_p} f(p) = \int \frac{d^4p}{(2\pi)^4} \delta(P^2-m^2)\, \underset{\text{Heaviside}}{\vartheta}(p_0)\, f(p)$$

Proof:

$$P^2-m^2 = P^{0\,2} - (p^2+m^2) = (p^0 - \sqrt{p^2+m^2})(p^0+\sqrt{p^2+m^2})$$

$$\Rightarrow \delta(P^2-m^2) = \left(\frac{\delta(p^0+\omega_p)}{|p^0-\omega_p|} + \frac{\delta(p^0-\omega_p)}{|p^0+\omega_p|}\right)\vartheta(p_0) = \frac{\delta(p^0-\omega_p)}{2\omega_p}$$

Applying the δ we obtain the LHS

~ ϑ_0 is invariant because it is the sign of the temporal part, which remains the same under Lor. transf.

~ P^2-m^2 is the sum of invariants

$\Rightarrow$ The integral measure is invariant

- The integrand is inv. because there are scalar products

$\Rightarrow \Delta(X-Y)$ is invariant

• Dirac equation

Dirac looked for a 1st order eq. in time and space:

$$\boxed{i\frac{\partial}{\partial t}\psi(\underline{x},t)=\left(-i\,\underline{\alpha}\cdot\underline{\nabla}+\beta m\right)\psi(\underline{x},t)}$$

vector of functions

(1927)

i) It has to satisfy $E^2=p^2+m^2 \Rightarrow \frac{\partial^2}{\partial t^2}\psi=(\nabla^2-m^2)\psi$ ✭

ii) It has to be relativistically covariant

iii) $|\psi|^2$ has to be positive-definite

• Properties of α and β matrices

We can square Dirac equation

$$-\frac{\partial^2}{\partial t^2}\psi=(-i\underline{\alpha}\cdot\underline{\nabla}+\beta m)^2\psi=\left(-\underline{\alpha^i\alpha^j\partial_i\partial_j}\,\underline{-im(\alpha^i\beta+\beta\alpha^i)\partial_i}+\underline{\beta^2}m^2\right)\psi$$

► $\alpha^i\alpha^j\partial_i\partial_j=$ [is symm. in i and j] $=\frac{1}{2}(\alpha^i\alpha^j+\alpha^j\alpha^i)\partial_i\partial_j \overset{✭}{\Rightarrow}$

$\Rightarrow \frac{1}{2}(\alpha^i\alpha^j+\alpha^j\alpha^i)=\delta^{ij}$

► ✭ $\Rightarrow$ $\alpha^i\beta+\beta\alpha^i=0$;

► ✭ $\Rightarrow$ $\beta^2=\mathbb{1}$

$$\Rightarrow \boxed{\begin{cases}\beta^2=\mathbb{1}\\ [\alpha^i,\beta]_+=0 \;\#\\ [\alpha^i,\alpha^j]_+=2\delta^{ij} \;\$\end{cases}}$$

It is needet that $-i\underline{\alpha}\cdot\underline{\nabla}+\beta m$ is Hermitian $\Rightarrow \begin{cases}\alpha^{i\dagger}=\alpha^i\\ \beta^\dagger=\beta\end{cases}$

$\overset{\$}{\Rightarrow}$ $\alpha^{i\,2}=\mathbb{1}$ ⇒ The e.v. of α^i are only ± 1

$Tr(\beta)=0$, $Tr(\alpha^i)=0$

$Tr(\alpha^i)=Tr(\overset{\mathbb{1}}{\beta^2}\alpha^i)\overset{\#}{=}Tr(-\beta\alpha^i\beta)=-Tr(\alpha^i\beta^2)=-Tr(\alpha^i)$

$\Rightarrow Tr(\alpha^i)=-Tr(\alpha^i)\Rightarrow Tr(\alpha^i)=0$

$\Rightarrow \alpha^i$ will have even dimentions but cannot be 2x2 because σ_i form a base with the identity $\Rightarrow \alpha_i$ cannot anticommute with σ_i

$\Rightarrow$ At least α^i will be 4x4 which is good because Dirac spinors have 4 parameters

The usual representation is $\boxed{\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}}$; $\boxed{\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}}$

Pauli

- <u>Covariance of Dirac's equation</u>

We can now define $\begin{cases} \gamma^0 = \beta \\ \gamma^i = \beta \alpha^i \end{cases} \Rightarrow \boxed{\gamma^\mu = (\gamma^0, \gamma^i)}$

$$\beta\left(i \frac{\partial}{\partial t} \psi(\underline{x}, t) = (-i \underline{\alpha} \cdot \underline{\nabla} + \beta m)\psi(\underline{x}, t)\right)$$

$$\beta i \frac{\partial}{\partial t}\psi = (-i\beta \underline{\alpha} \cdot \underline{\nabla} + \underbrace{\beta\beta}_{=\mathbb{1}} m)\psi$$

$$\Rightarrow i\gamma^0 \partial_0 \psi = (-i\gamma^i \partial_i + m)\psi$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0$$

It is manefestely covariant, we will demonstrate it

$$\Rightarrow \boxed{(i\not{\partial} - m)\psi = 0}$$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} ; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \Rightarrow \boxed{[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}}$$

Clifford Algebra

$$\gamma^{0\,2} = \mathbb{1}_4, \quad \gamma^{i\,2} = -\mathbb{1}_4 ; \quad \gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i$$

The repres. of γ^μ matrices we used is not unique:
given a non-singular matrix S, $\tilde{\gamma}^\mu = S^{-1}\gamma^\mu S$ satisfies the same Clifford Algebra.

We would like to show that, if $\psi(x)$ satisfies the Dirac equation in a given frame of reference O, the wave function determined by an observer in another system O′ satisfies the Dirac equation in O′.

We consider the homogeneous Lorentz tr $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Correspondingly, the components of ψ should transform linearly, to respect the superposition principle, with a matrix which depends on the transformation Λ :

$$\psi'(x') = S(\Lambda)\,\psi(x)$$

Note that we do not know a priori the form of $S(\Lambda)$. The relativistic invariance of the Dirac equation requires that it should be possible to determine $S(\Lambda)$ so that:

- the transformations are in accord with the combination rule $S(\Lambda 1 \Lambda 2)=S(\Lambda 1)S(\Lambda 2)$;
- they lead to a ψ' which satisfies the Dirac equation in O′ if ψ satisfies it in O.

$$\Rightarrow \quad \boxed{(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \longrightarrow (i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0} \quad (\star)$$

The matrix γ^μ remains the same because $\tilde{\gamma}^\mu$ obeys to the same Clifford algebra

Let's impose $(\star)$

$$(i\gamma^\mu\partial_\mu - m)S^{-1}(\Lambda)\psi'(x) = 0\,; \qquad \partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu\,\partial'_\nu$$

$$S(\Lambda)\,(i\gamma^\mu\Lambda^\nu{}_\mu\,\partial'_\nu - m)S^{-1}(\Lambda)\psi'(x') = 0$$

$$\Rightarrow \left[i\,S(\Lambda)\left(\gamma^\mu\Lambda^\nu{}_\mu\,\partial'_\nu\right)S^{-1}(\Lambda) - m\right]\psi'(x') = 0$$

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It has to be equal to $(i\gamma^{\mu}\partial'_{\mu}-m)\psi'(x')=0$

$$\Rightarrow S(\Lambda)\gamma^{\mu}\Lambda^{\nu}_{\mu}S^{-1}(\Lambda)=\gamma^{\nu} \quad \Rightarrow \quad \boxed{\gamma^{\mu}\Lambda^{\nu}_{\mu}=S^{-1}(\Lambda)\gamma^{\nu}S(\Lambda)} \quad \odot$$

Can I find such a transf.? YES

Let's consider Infinitesimal transformations:

$$\begin{cases} \Lambda^{\mu}_{\nu}\simeq\delta^{\mu}_{\nu}+\varepsilon^{\mu}_{\nu} \\ S(\Lambda)\simeq 1-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\nu}, \; S^{-1}(\Lambda)\simeq 1+\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\nu} \end{cases}$$

We put into $\odot$

$$\Rightarrow\left(1+\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\right)\gamma_{\rho}\left(1-\frac{i}{4}\sigma_{\alpha\beta}\varepsilon^{\alpha\beta}\right)=\gamma_{\rho}+\underline{\varepsilon_{\rho\alpha}\gamma^{\alpha}}$$

|| $\varepsilon_{\rho\alpha}$ is antisimmetric

$$\frac{1}{2}\varepsilon^{\alpha\beta}(\eta_{\alpha\rho}\gamma_{\beta}-\eta_{\beta\rho}\gamma_{\alpha})$$

$$\Rightarrow \frac{i}{4}\varepsilon^{\mu\nu}[\sigma^{\mu\nu},\gamma_{\rho}]=\frac{1}{2}\varepsilon^{\alpha\beta}(\eta_{\alpha\rho}\gamma_{\beta}-\eta_{\beta\rho}\gamma_{\alpha})$$

$$\Rightarrow [\sigma^{\mu\nu},\gamma_{\rho}]=-2i(\eta_{\alpha\rho}\gamma_{\beta}-\eta_{\beta\rho}\gamma_{\alpha})$$

A solution is $\sigma_{\mu\nu}=\frac{i}{2}[\gamma_{\mu},\gamma_{\nu}]$

It is possible to find it with theory group

$$\Rightarrow \boxed{S(\Lambda)=e^{\frac{1}{8}[\gamma_{\mu},\gamma_{\nu}]\varepsilon^{\mu\nu}}=e^{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}}},$$

$\sigma_{\mu\nu}=\frac{i}{2}[\gamma_{\mu},\gamma_{\nu}]$, 6 generators

$\sigma_{00}=\sigma_{ii}=0$;

$\sigma_{0i}=-i\begin{pmatrix}0 & \sigma^{i}\\ \sigma^{i} & 0\end{pmatrix}$; $\sigma_{ij}=\varepsilon_{ijk}\begin{pmatrix}\sigma^{k} & 0\\ 0 & \sigma^{k}\end{pmatrix}$