28/09/2020

Relativistic quantum mechanics

Introduction

Goal: to be able to calculate a cross-section which is measurable

To a particle is connected a wavefunction → density of probability

Special relat. is not included in Schrodinger equation, transformations will be Lorentz transformation, speed of particle will be close to speed of light.

A first attempt made by Schrodinger for relat. covar. equation gave the Klein-Gordon equation

Schr. starts from $E = \frac{p^2}{2m} + V$, it is not correct in relativity

$E \to \hbar \frac{\partial}{\partial t}$, $p^2 \to (-i\hbar\nabla)^2$

K-G starts from the correct $\frac{E^2}{c^2} = p^2 + m^2c^2$; $\hbar = c = 1$

$E \to \hbar \frac{\partial}{\partial t}$, $p \to -i\hbar\nabla$

Substituting derivatives : $\boxed{\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0}$

It is intended to be a wavefunction

Problem : prob. density is negative in some cases, a nonsense

Problem*: the eq. is quadratic ⇒ 2 solutions for E, $E = \pm\sqrt{p^2+m^2}$, a positive and a negative energy

From a classical point of view the gap from $-E$ and $+E$ is not a problem, but E is not bounded from below and a particle continues to fall, we don't see this collapse

1927: Dirac proposed a new equation, in order to solve the problem of prob. density, proposed a first-order equation in time and space

$$\boxed{i\frac{\partial}{\partial t}\psi(x,t)=(-i\,\underline{\alpha}\cdot\underline{\nabla}+\beta m)\,\psi(x,t)}$$ *

This structure also gives the spin-orbit interaction

→ gives $|\psi(x,t)|^2$ positive definite

Dirac brought to the discovery of e^+, you have unbounded negative solutions which are filled by fermions, a sea of electrons. One of these electrons can jump to $E>0$, it leaves a hole which is filled by a positive charge with same mass and positive energy.
The hole is a paticle with mass of e^- but positive charge and $E>0$

Problem: connected to e.m. field; e.m. is a classic field, is not quantised

There were experiments that showed the wave nature of e^-,

Solution: change in the point of view, e.m. needs to be quantised, max. equations are not quantised, we can quantise interaction part and matter part

* is the classical eq. that the field has to fulfill and then we move to a quantised version for the field, the SECOND quantization

For the sec. quantisation we have to find the classical diff. eq. the field has to obey, one has to introduce transformations, canonical variables

We want to build a theory invariant under Lorentz transformations

Idea: we find a class of reference frames were the physics has to be the same

In newtonian physics velocity has to be constant:
$$\begin{cases} \underline{x}'=\underline{x}-\underset{\text{constant}}{\underline{v}}\,t \\ t'=t \end{cases}$$

Electromagnetism is described by max. eq.:
$$\begin{cases} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\varphi=\rho \\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\underline{A}=\underline{J} \end{cases}$$

Are not invariant under in fact velocity are added, not in Maxw. eq.s.

Three possibilities:
$$\begin{cases} \text{etherium} \\ \text{new transformations} \\ \text{both} \end{cases}$$

③ 29/09/2020

• Special relativity

• Einstein postulates of relativity

Einstein postulates: $\begin{cases} \text{Laws of physics are the same in every r.f.} \\ \text{Speed of light is the same in every r.f.} \end{cases}$

Time and space mix up, this affects the composition of velocity

Event: $(ct, \underline{x})$

Suppose to emit a light ray in (x_1, y_1, z_1, t_1) to (x_2, y_2, z_2, t_2)

$$\Rightarrow \begin{cases} -[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2] + c^2(t_1-t_2)^2 = 0 \\ -[(x_1'-x_2')^2 + (y_1'-y_2')^2 + (z_1'-z_2')^2] + c^2(t_1'-t_2')^2 = 0 \end{cases}$$

$(\Delta \ell = c\Delta t$, viaggia la luce$)$

$\leftarrow$ in another r.f.

• Invariant interval

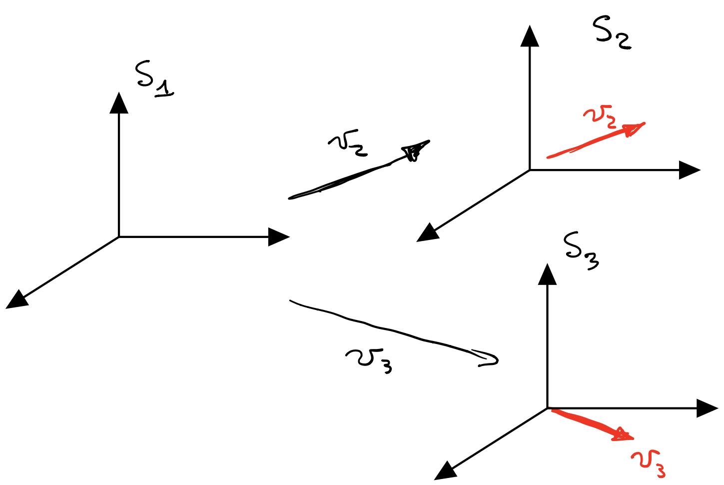
$\Delta s^2 := [c^2(t_1-t_2)^2 - (x_1-x_2)^2 - (y_1-y_2)^2 - (z_1-z_2)^2]$, it is the distance

We pass to the differential: $\begin{cases} ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ ds'^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 \end{cases}$

It must be $ds^2 = a\, ds'^2$ because they go to $\varnothing$ at the same time

"a" cannot depend from direction, because the space is homogeneus, only from the module of velocity

Now let's consider three systems as follows:



$$\begin{cases} ds_1^2 = a(v_2)\, ds_2^2 \\ ds_1^2 = a(v_3)\, ds_3^2 \\ ds_2^2 = a(v_{23})\, ds_3^2 \end{cases} \Rightarrow \frac{a(v_3)}{a(v_2)} = a(v_{23})$$

$\Rightarrow ds^2 = ds'^2$

$\Rightarrow s^2 = c^2 t^2 - \ell^2$, it is not positive definite

• Type of intervals, light cone

Let's see if there is a r.f. in which the events happen in the same x *

$\Rightarrow s^2 = c^2t^2 - \ell^2 = s' = c^2t'^2 - \cancel{\ell'^2} = c^2t'^2 > 0$ $\ell'^2 \overset{*}{=} 0$

► If $s^2 > 0 \Rightarrow$ time-like interval

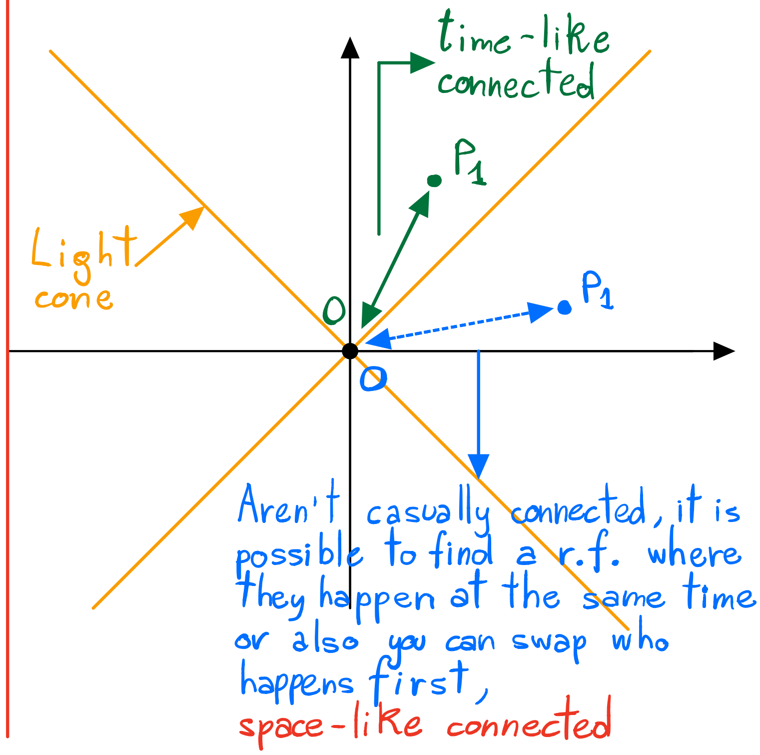
In similar way, we request the events happen at the same time

$\Rightarrow s^2 = c^2t^2 - \ell^2 = s'^2 = 0 - \ell^2 < 0$

► If $s^2 < 0 \Rightarrow$ space-like interval

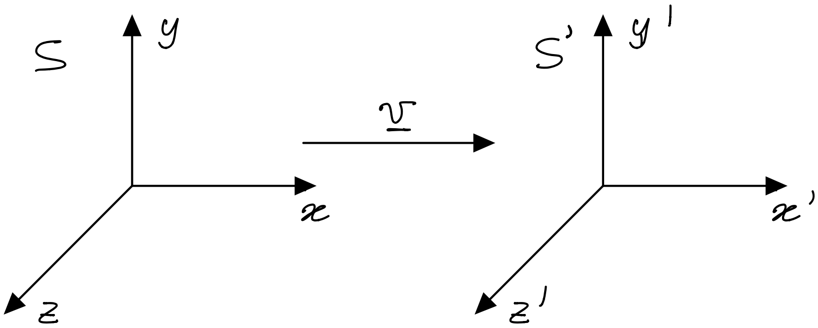
► If $s^2 = 0 \Rightarrow$ light-like interval

Light cone :



• Lorentz transformations

Let's consider a system S and another S' which moves with respect to S at speed $\underline{v}$



$$\Rightarrow \begin{cases} t' = \dfrac{t - \frac{v}{c^2}x}{\sqrt{1-\beta^2}} \\[2ex] x' = \dfrac{x - vt}{\sqrt{1-\beta^2}} \\[2ex] y' = y \\ z' = z \end{cases}$$

$$\Rightarrow \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = A \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} ; \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} ;$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}}, & -\beta\frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ -\beta\frac{1}{\sqrt{1-\beta^2}}, & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we obtain the transformation (boost on x) :

Because of homogeneity of spacetime, we can have only a linear transformation

$$\Rightarrow \begin{cases} t' = a_{00}t + a_{01}x + a_{02}y + a_{03}z \\ x' = a_{10}t + a_{11}x + a_{12}y + a_{13}z \\ y' = \dots \\ z' = \dots \end{cases}$$

$$\text{If } t=t'=0 \Rightarrow O \equiv O' \Rightarrow \begin{cases} \text{If } x=0 \Rightarrow x'=0 \\ \text{If } y=0 \Rightarrow y'=0 \Rightarrow \\ \text{If } z=0 \Rightarrow z'=0 \end{cases} \begin{cases} a_{13}=0, a_{31}=0 \\ a_{23}=0, a_{32}=0 \end{cases} \Rightarrow \begin{cases} y' = a_{20}t + a_{22}y \\ z' = a_{30}t + a_{33}z \end{cases}$$

We see from S what happens in S'

$\Delta x = v\Delta t$, $\Delta y = 0$, $\Delta z = 0$ $\Rightarrow$ $a_{10} + a_{11}v = 0$

$\Delta x' = 0 = \Delta y = \Delta z$ $\Rightarrow$ $a_{20} = a_{30} = 0$ because are hortogonal to the movement

In similar way, $a_{00} = a_{11}$

$$\Rightarrow \begin{cases} t' = a_{00}t + a_{01}x \\ x' = a_{00}(x - vt) \\ y' = a_{22}y \\ z' = a_{33}z \end{cases}$$

If we put a bar in S its lenght l is $a_{22}l'$ also if we put in S' its lenght l' is $a_{22}l$ $\Rightarrow$ $a_{22}=1$, $a_{33}=1$

Now we impose the invariance of ds^2:

$ds^2 = c^2dt^2 - dl^2 = ds'^2 = c^2dt'^2 - dl'^2$

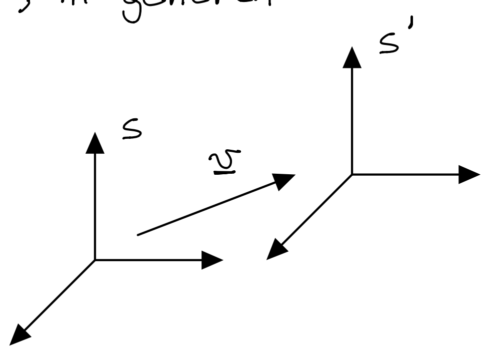
$$c^2dt^2 - dx^2 = c^2dt'^2 - dx'^2$$

$$\Rightarrow \quad c^2dt^2 - dx^2 = c^2(a_{00}dt + a_{01}dx)^2 - a_{00}^2(dx - v\,dt)^2$$

$$\Rightarrow a_{00} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \; ; \; a_{10} = \frac{v^2}{c^2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

We found the transformation for a boost on x, in general

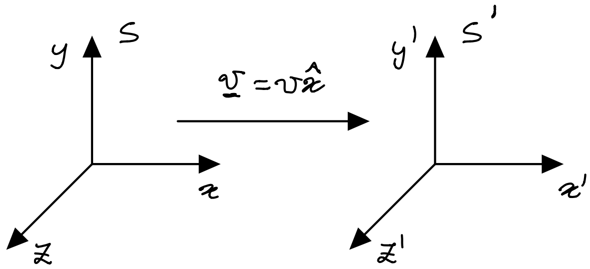
$$\begin{cases} x^{0'} = \frac{1}{\sqrt{1-\beta^2}}(x^0 - \beta \underline{x}_{\parallel} \cdot \hat{v}) \\ \underline{x}'_{\parallel} = \frac{1}{\sqrt{1-\beta^2}}(\underline{x}_{\parallel} - \underline{v}t) = \frac{1}{\sqrt{1-\beta^2}}(\underline{x} - \beta x^0) \\ \underline{x}'_{\perp} = \underline{x}_{\perp} \end{cases}$$



$$\begin{cases} \underline{x}_{\parallel} = \dfrac{\underline{x}\cdot\underline{v}}{\|\underline{v}\|}\,\dfrac{\underline{v}}{\|\underline{v}\|} \\ \underline{x}_{\perp} = \underline{x} - (\underline{x}\cdot\hat{v})\hat{v} \end{cases} ; \quad \hat{v} = \frac{\underline{v}}{|\underline{v}|}$$

- Composition of velocity

Let's consider a boost on x :



$$\begin{cases} t' = \dfrac{t - \frac{v}{c^2}x}{\sqrt{\ }} \\ x' = \dfrac{x - vt}{\sqrt{\ }} \\ y' = y \\ z' = z \end{cases} \qquad \begin{cases} t = \dfrac{t' + \frac{v}{c^2}x}{\sqrt{\cdots}} \\ x = \dfrac{x' + vt}{\sqrt{\cdots}} \\ \cdots \\ \cdots \end{cases}$$

$$u_x = \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \frac{(u_x' - v)}{\sqrt{\cdots}}\frac{\left(1 - \frac{v}{c^2}u_x\right)}{\sqrt{\cdots}} = \frac{u_x' - \frac{v}{c^2}u_x u_x' - v + \beta^2 u_x}{\sqrt{\cdots}}$$

$$\Rightarrow u_x = \frac{u_x' + v}{1 + \frac{v}{c^2}u_x'} \quad ; \quad u_y = \frac{dy}{dt} = \frac{dy}{dt'}\frac{dt'}{dt} = \frac{u_y'\sqrt{1-\beta^2}}{1 + \frac{v}{c^2}u_x'}$$

$u_x' \leq c$; $v \leq c$, they don't add up :

$(c - u_x') \geq 0$; $(c - v) \geq 0$

$(c - u_x')(c - v) = c^2 - cv - u_x'c + u_x'v \geq 0$

$$1 + \frac{u_x' v}{c^2} \geq \frac{u_x' + v}{c} \quad ; \quad u_x = \frac{u_x' + v}{1 + \frac{v}{c^2}u_x'} \Rightarrow \frac{u_x' + v}{c} = \left(1 + \frac{v}{c^2}u_x'\right)\frac{u_x}{c}$$

$$\cancel{\left(1 + \frac{u_x' v}{c^2}\right)} \geq \frac{u_x' + v}{c} = \cancel{\left(1 + \frac{v}{c^2}u_x'\right)}\frac{u_x}{c} \Rightarrow \boxed{\frac{u_x}{c} < 1}$$

If $u_x' \to c$
$\Rightarrow u_x \to c$

1) Consider a bar of length l_0 in S', show that from S we see l_0 contracted
2) Show that $\Delta t'$ between two events in S' and Δt in S $> \Delta t'$
3) Bruno rossi experiment : $\mu \to e^- + \bar{\nu}_e + \nu_\mu$

$N(t) = N_0 e^{-t/\tau}$ $\quad \tau = 2.16\,\mu s$ in the rest frame of μ

We see 560 μ /hour
after 2000m what we see ? $\quad v = 0.995\,c$

30/09/20

Vectors and tensors

After the introduction of Lorentz transformations, we now want to study how mathematical objects, that will be used to describe our Physics, transform under Lorentz transformations (LT). This is the subject of Tensor Analysis.
Let us start introducing a more general definition of vectors in a Euclidean space.

Vectors and contravariant components

In Special Relativity (SR) we have to deal with different kind of vectors. The fact that in Newtonian mechanics, for instance, we do need just the usual Euclidean definition is simply due to the fact that usually we use an ortonormal system of basis vectors for my vectorial space. In this sytuation the metric tensor reduces to a Kronecker delta function and it becomes impossible to appreciate the difference between different definitions of vectors.

Let us consider a vector space $\mathcal{V}$ on $\mathbb{R}$, let $\{\underline{e}_i\}$ be a basis for $\mathcal{V}$

$\Rightarrow$ If $\underline{v} \in \mathcal{V} \Rightarrow \underline{v} = v^i \underline{e}_i$, contravariant vector

Of course you can move from one basis to another: $\boxed{\underline{e}'_i = \Lambda^j_{\ i}\, \underline{e}_j}$, $(\det \Lambda \neq 0)$;

$$\Rightarrow \underline{v} = v^j \underline{e}_j = v'^i \underline{e}'_i = \underbrace{v'^i \Lambda^j_{\ i}}\, \underline{e}_j \Rightarrow v^j = \Lambda^j_{\ i} v'^i$$

$(\Lambda^{-1})^i_{\ j}$

$$\Rightarrow \boxed{v'^i = (\Lambda^{-1})^i_{\ j}\, v^j}$$

Contravariant, they transform with the inverse

In matrix notation: : $\underline{v} = \Lambda^T \underline{v}'$; $\underline{v}' = (\Lambda^T)^{-1} \underline{v} = (\Lambda^{-1})^T \underline{v}$

- Dual vectors and covariant components

For every V there is a dual space V^*, which is the vect. sp. of linear funtionals on V: $\sigma : V \longrightarrow \mathbb{R}$, $\underline{v} \longrightarrow \sigma(\underline{v})$

V^* is a vectorial space $\Rightarrow$ exist a basis $\{\underline{k}^i\}$ $\Rightarrow$ $\sigma = \sigma_i \underline{k}^i$

$$\Rightarrow \sigma(\underline{v}) = \sigma_i \underline{k}^i(\underset{\uparrow}{\underline{v}}) = \sigma_i \underline{k}^i(v^j \underline{e}_j) = \sigma_i v^j \underline{k}^i(\underline{e}_j)$$

Contravariant

We say that the two chosen basis are dual if $\underline{k}^i(\underline{e}_j) = \delta^i_j$

$\Rightarrow \sigma_i(\underline{v}) = \sigma_i v^i$ it is not a scalar product! $\sigma, \underline{v}$ belong to different spaces

Let be two dual bases $\Rightarrow$ $\underline{k}^i(\underline{v}) = \underline{k}^i(v^j e_j) = v^j \underline{k}^i(\underline{e}_j) = v^j \delta^i_j = v^i$

$\Rightarrow$ The dual base selects the i-component, on the other hand:

$\sigma(\underline{e}_j) = \sigma_i \underline{k}^i(\underline{e}_j) = \sigma_j$

We know that $\underline{e}'_i = \Lambda^j_i \underline{e}_j$, how are $\underline{k}$ affected by the change?

$\{\underline{k}^i\} \rightarrow \{\underline{k}'^i\}$ $\quad \sigma = \sigma'_i \underline{k}'^i$

$$\sigma'_i = \sigma(\underline{e}'_i) = \sigma_j \underline{k}^j(\underline{e}'_i) = \sigma_j \underline{k}^j(\Lambda^l_i \underline{e}_l) = \sigma_j \Lambda^l_i \underline{k}^j(\underline{e}_l) = \sigma_j \Lambda^l_i \delta^j_l = \sigma_j \Lambda^j_i$$

$$\Rightarrow \boxed{\sigma'_i = \Lambda^j_i \sigma_j}$$

Covariant, transforms with Λ

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- ## Scalar product and Metric Tensor

The scalar product between two vectors of $\mathcal{V}$ is an application of $\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ bilinear, symmetric and non degenerate: $\underline{v},\underline{w}\in\mathcal{V}\longrightarrow(\underline{v},\underline{w})\in\mathbb{R}$

The functional $f_{\underline{v}} := (\underline{v},\cdot)$ is such that $\underline{w}\in\mathcal{V}\to f_{\underline{v}}(\underline{w}) = (\underline{v},\underline{w})\in\mathbb{R}$

Let $\underline{v} = v^i\underline{e}_i$, $v_\mu = (\underline{v},\underline{e}_\mu) = (v^\nu\underline{e}_\nu,\underline{e}_\mu) = v^\nu(\underline{e}_\nu,\underline{e}_\mu) = v^\nu g_{\mu\nu}$,

$g_{\mu\nu}$ is the metric tensor.

$$\boxed{v_\mu = g_{\mu\nu}v^\nu}$$, relation between cov and contr. indices

If g_{ij} is pos. def., there exists a basis where $g_{ij} = \delta_{ij}$, not the case of special relativity, the MinKowski space has got a not pos-def. distance.

$$v'_\mu = (\underline{v},\underline{e}'_\mu) = (v^\ell\underline{e}_\ell,\Lambda^\nu_\mu\underline{e}_\nu) = v^\ell\Lambda^\nu_\mu(\underline{e}_\ell,\underline{e}_\nu) = v^\ell\Lambda^\nu_\mu g_{\ell\nu} = \Lambda^\nu_\mu v_\nu$$

$$g'_{ij} = (\underline{e}'_i,\underline{e}'_j) = (\Lambda^k_i\underline{e}_k,\Lambda^\ell_j\underline{e}_\ell) = \Lambda^k_i\Lambda^\ell_j(\underline{e}_k,\underline{e}_\ell) = \Lambda^k_i\Lambda^\ell_j g_{k\ell}$$

$$\Rightarrow \boxed{g'_{ij} = \Lambda^k_i\Lambda^\ell_j g_{k\ell}}$$

Every component transforms in covariant way

We can also define the inverse of the metric tensor such that

$$\boxed{g^{\mu\nu}g_{\nu\ell} = \delta^\mu_\ell}$$

$$(\underline{v},\underline{w}) = v^i w^j(\underline{e}_i,\underline{e}_j) = v^i w^j g_{ij}$$

$$(\underline{u},\underline{v}) = u_\nu v_\mu g^{\mu\nu}$$

$$\boxed{g'^{\ell m} = (\Lambda^{-1})^\ell_\gamma(\Lambda^{-1})^m_\delta g^{\gamma\delta}}$$

- Tensor of rank (p,q)

It is a collection of "n"$^{p+q}$ numbers, usually expressed as

$$T^{i_1 \dots i_p}_{j_1 \dots j_q} = (\Lambda^{-1})^{l_1}_{j_1} \dots (\Lambda^{-1})^{l_q}_{j_q} \Lambda^{i_1}_{k_1} \dots \Lambda^{i_p}_{k_p} T'^{k_1 \dots k_p}_{l_1 \dots l_q}$$

$$T'^{k_1 \dots k_p}_{l_1 \dots l_q} = \Lambda^{j_1}_{l_1} \dots \Lambda^{j_q}_{l_q} (\Lambda^{-1})^{k_1}_{i_1} \dots (\Lambda^{-1})^{k_p}_{i_p} T^{i_1 \dots i_p}_{j_1 \dots j_q}$$

$A^{i_1, \dots, i_p}_{j_1, \dots, j_q} = B^{i_1 \dots i_p}_{j_1, \dots, j_q}$ means that is valid for every component and it is valid under Lorentz-transformation.

- We define a sum:

$$\underset{(p,q)}{A^{i_1, \dots, i_p}_{j_1, \dots, j_q} + B^{i_1, \dots, i_p}_{j_1, \dots, j_q}} = \underset{(p,q)}{C^{i_1, \dots, i_p}_{j_1, \dots, j_q}}$$

It is demonstrated showing that LHS transforms as a (p,q) tensor

- We def. a product:

$A \in (p,q)$; $B \in (k,l)$ $\Rightarrow$ $C^{i_1, \dots i_p \dots i_{p+k}}_{j_1, \dots, j_q \dots j_{q+l}} = A^{i_1 \dots i_p}_{j_1 \dots j_q} B^{i_1 \dots i_k}_{j_1 \dots j_l}$

- We can contract indices

$A^\mu B_\nu = C^\mu_\nu$ is the product

$A^\mu B_\mu = g^{\mu\nu} A_\mu B_\nu$ is the contraction

- Relation between tensors

If $\underbrace{T^{\alpha}_{\beta} = U^{\alpha}_{\beta}}$, we transform $\Rightarrow$ $T'^{\alpha}_{\beta} = \Lambda^\alpha_\gamma \Lambda^\delta_\beta T^\gamma_\delta = \Lambda^\alpha_\gamma \Lambda^\delta_\beta U^\gamma_\delta = U'^{\alpha}_{\beta}$

The relation between tensors is covariant

(11)

- ## M^4 space, $\eta_{\mu\nu}$ tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu}$$, the space is called M^4

Scalar product: $x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu = \eta^{\mu\nu} x_\mu x_\nu = x^{0\,2} - x^{1\,2} - x^{2\,2} - x^{3\,2}$ → Not positive definite

$v^\mu = (v^0, \underline{v})$ is contravariant, from the def. we obtain the covariant as $v_\mu = \eta_{\mu\nu} v^\nu \Rightarrow v_\mu = (v^0, -\underline{v})$

($\underline{v}$: spatial comp. $\in \mathbb{R}^3$)

Boost: $\Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\qquad x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$

$$\eta_{\sigma\varrho} x^\sigma x^\varrho = x^2 = x'^2 = \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu_{\ \sigma} x^\sigma \Lambda^\nu_{\ \varrho} x^\varrho = \eta_{\mu\nu} \Lambda^\mu_{\ \sigma} \Lambda^\nu_{\ \varrho} x^\sigma x^\varrho$$

$$\Rightarrow \boxed{\eta_{\mu\nu} \Lambda^\mu_{\ \sigma} \Lambda^\nu_{\ \varrho} = \eta_{\sigma\varrho}} \quad \star$$

The boosts $\Lambda^\mu_{\ \nu}$ form a group:

- If $v = 0 \Rightarrow \Lambda^\mu_{\ \nu}(v=0) = \mathbb{1} \Rightarrow \mathbb{1} \in \{\Lambda^\mu_{\ \nu}\}$
- $x''^\mu = \Lambda^\mu_{\ \nu} x'^\nu = [\Lambda^\mu_{\ \nu} \Lambda^\nu_{\ \sigma}] x^\sigma$, $\Lambda^\mu_{\ \nu} \Lambda^\nu_{\ \sigma}$ is a boost again:

$$\# = \underline{\eta_{\mu\nu} (\Lambda^\mu_{\ \varrho} \Lambda^\varrho_{\ \sigma})(\Lambda^\nu_{\ \omega} \Lambda^\omega_{\ \delta})} = \underbrace{\eta_{\mu\nu} \Lambda^\mu_{\ \varrho} \Lambda^\nu_{\ \omega}}_{\overset{\star}{=}\ \eta_{\varrho\omega}} \Lambda^\varrho_{\ \sigma} \Lambda^\omega_{\ \delta} =$$

$$= \eta_{\varrho\omega} \Lambda^\varrho_{\ \sigma} \Lambda^\omega_{\ \delta} \overset{\star}{=} \eta_{\sigma\delta} \Rightarrow \Lambda^\mu_{\ \varrho} \Lambda^\varrho_{\ \sigma} ; \Lambda^\nu_{\ \omega} \Lambda^\omega_{\ \delta}$$ are boosts because # behaves as $\star$

- There exist Λ^{-1}: $\Lambda^{-1} = \Lambda(-v)$, because I move in the opposite direction: $\Lambda^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ■

• Dynamics of a classical free particle

Velocity in newt. mechanics: $\underline{v} = \frac{dx}{dt} \longrightarrow \frac{dx^\mu}{dt}$, not a good choice
not invariant

$ds^2 = c^2dt^2 - dl^2 = ds'^2$, we can move to the r.f. where the system is at rest,

$\Rightarrow dl^2 = 0 \Rightarrow$ Proper time $\boxed{d\tau = dt\sqrt{1-\beta^2}}$

In that r.f, $ds^2 = c^2 d\tau^2 \Rightarrow d\tau$ is invariant

$$\Rightarrow u^\mu = (u^0, \underline{u}) := \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} \quad ; \quad u^0 = \frac{dx^0}{dt\sqrt{\dots}} = \gamma c \quad ; \quad \underline{u} = \frac{d\underline{x}}{dt\sqrt{\dots}} = \gamma \underline{v}$$

It is a time-like vector: $u^\mu u_\mu = \frac{c^2}{1-\beta^2} - \frac{v^2}{1-\beta^2} = c^2 > 0$

$$a^\mu := \frac{d^2x^\mu}{d\tau^2} = \gamma \frac{du^\mu}{dt} \quad ; \quad a^0 = \frac{\underline{v}\cdot\underline{a}}{c(1-\beta^2)} \quad ; \quad \underline{a} = \frac{\underline{a}}{(1-\beta^2)} + \frac{1}{c^2}\frac{\underline{v}\cdot\underline{a}}{(1-\beta^2)}\underline{v}$$

$a^\mu u_\mu = 0$, they are orthogonal

$\underline{p} = m\underline{v} \Rightarrow$ we define $\boxed{P^\mu := m u^\mu = (m\gamma c, m\gamma \underline{v})}$

Also a time-like vector: $P^2 = P^\mu P_\mu = P^{0\,2} - |\underline{p}|^2 = \frac{m^2c^2}{1-\beta^2} - \frac{m^2v^2}{1-\beta^2} = m^2c^2 > 0$

Lagrangian: $L = -mc^2\sqrt{1-\frac{v^2}{c^2}} \qquad \underline{p} = \frac{\partial L}{\partial \underline{v}} = \frac{m\underline{v}}{\sqrt{1-\beta^2}}$

$E = \underline{p}\cdot\underline{v} - L = m\gamma \underline{v}^2 + \frac{mc^2}{\gamma} = m\gamma c^2$; if $\beta \to 0$, $E \simeq mc^2 + \frac{mv^2}{2}$

$$P^2 \begin{array}{l} = m^2 \\ = \epsilon^2 - p^2 \end{array} \Rightarrow \epsilon^2 = p^2 + m^2$$

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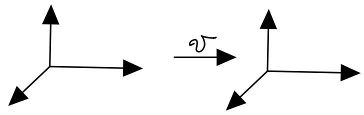
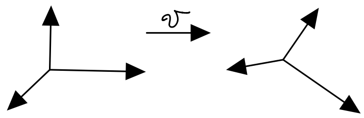
• Lorentz transformations form a group

$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$,

simple boost :



boost and rotation :



You can connect them with a rotation and a boost

Boost: $\Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; Rotation: $\Lambda^\mu_{\ \nu} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$, where R is a 3D rotation $RR^\dagger = \mathbb{1}$

$\boxed{\text{Lorentz transformation are such that } x'^2 = x^2}$

$$x'^2 = x^2 = \eta_{\mu\nu} x^\mu x^\nu \Rightarrow \eta_{\mu\nu} \Lambda^\mu_{\ \sigma} \Lambda^\nu_{\ \varrho} = \eta_{\sigma\varrho}$$

$$\Rightarrow \boxed{\Lambda^\mu_{\ \sigma} \eta_{\mu\nu} \Lambda^\nu_{\ \varrho} = \eta_{\sigma\varrho}} \quad \star$$

Lorentz tr. form a group:

- $\mathbb{1}$ is a Lorentz transf. : $\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu}$ satisfies $\star$

- $x^\mu \xrightarrow{\Lambda_1} x'^\mu \xrightarrow{\Lambda_2} x''^\mu$; $x^\mu \xrightarrow[\Lambda_1\Lambda_2]{} x''^\mu$; $\Lambda_1\Lambda_2$ is a Lor. tr., proof is the same as in page (11)

- Λ^{-1} is a Lorentz transformation:

$$\eta_{\mu\nu} \Lambda^\mu_{\ \sigma} \Lambda^\nu_{\ \varrho} = \eta_{\sigma\varrho} \Rightarrow \eta^{\sigma'\varrho} \eta_{\mu\nu} \Lambda^\mu_{\ \sigma} \Lambda^\nu_{\ \varrho} = \eta^{\sigma'\varrho} \eta_{\sigma\varrho} = \delta^{\sigma'}_{\ \sigma}$$

DOVREBBE VENIRE UNA COSA EQUIVALENTE A

$$\Rightarrow \boxed{\eta^{\sigma'\varrho} \eta_{\mu\nu} \Lambda^\nu_{\ \varrho} = (\Lambda^{-1})^\mu_{\ \sigma}} \qquad \eta^{\mu\varrho} \eta_{\nu\sigma} \Lambda^\sigma_{\ \varrho} = (\Lambda^{-1})^\mu_{\ \sigma}$$

$$\Rightarrow \overbrace{\eta_{\mu\nu} (\Lambda^{-1})^\mu_{\ \sigma} (\Lambda^{-1})^\nu_{\ \varrho}}^{\$} = \eta_{\mu\nu} (\eta^{\mu\xi} \eta_{\sigma\omega} \Lambda^\omega_{\ \xi})(\eta^{\nu\xi'} \eta_{\varrho\omega'} \Lambda^{\omega'}_{\ \xi'}) =$$

$$= \underbrace{\overbrace{\eta_{\mu\nu} \eta^{\mu\xi}}^{\delta^\xi_\nu} \eta^{\nu\xi'}}_{\eta^{\xi\xi'}} \eta_{\sigma\omega} \eta_{\varrho\omega'} \Lambda^\omega_{\ \xi} \Lambda^{\omega'}_{\ \xi'} = \eta^{\xi\xi'} \underline{\eta_{\sigma\omega}} \eta_{\varrho\omega'} \underline{\Lambda^\omega_{\ \xi}} \Lambda^{\omega'}_{\ \xi'} =$$

$$= \left[\eta_{\sigma\omega} \Lambda^{\omega}_{\ \xi} = \underbrace{\eta_{\sigma'\omega} \Lambda^{\omega}_{\ \xi} \Lambda^{\sigma'}_{\ \ell}}_{\star\ \eta_{\xi\ell}} (\Lambda^{-1})^{\ell}_{\ \sigma} = \eta_{\xi\ell} (\Lambda^{-1})^{\ell}_{\ \sigma} \right]$$

$$= \underbrace{\eta^{\xi\xi'} \eta_{\xi\ell}}_{\delta^{\xi'}_{\ell}} (\Lambda^{-1})^{\ell}_{\ \sigma} \eta_{\ell\omega'} \Lambda^{\omega'}_{\ \xi'} = (\Lambda^{-1})^{\ell}_{\ \sigma} \Lambda^{\omega'}_{\ \ell} \eta_{\ell\omega'} = \eta_{\ell\sigma}$$

$\Rightarrow$ \$ behaves as $\star$ $\Rightarrow$ Λ^{-1} is a Lor. tr. $\blacksquare$

$$\begin{cases} \det \Lambda = \pm 1 & \text{because } \det(\eta \Lambda) = \det(\eta) \Rightarrow (\det \Lambda)^2 = 1 \\ \det(\delta^{\mu}_{\ \nu}) = +1 \end{cases}$$

$\Lambda^{\mu}_{\ 0}$ is a 4-vector

$$(\Lambda^0_{\ 0})^2 - (\Lambda^i_{\ 0})^2 = \eta_{\mu\nu} \Lambda^{\mu}_{\ 0} \Lambda^{\nu}_{\ 0} = \eta_{00} = 1 \Rightarrow (\Lambda^0_{\ 0})^2 \geq 1$$

There are two kinds of transformations : $\begin{cases} \Lambda^0_{\ 0} \geq 1 \\ \Lambda^0_{\ 0} \leq -1 \end{cases}$

Proper ortochronous Lorentz group $\rightarrow$ $L_+^{\uparrow}$

	$\Lambda^0_{\ 0}$	$\det \Lambda$
$L_+^{\uparrow}$	≥ 1	$+1$
$L_+^{\downarrow}$	≤ -1	$+1$
$L_-^{\uparrow}$	≥ 1	-1
$L_-^{\downarrow}$	≤ -1	-1

- Examples
  - P such that $x^{\mu} = (x^0, \underline{x}) \longrightarrow x'^{\mu} = (x^0, -\underline{x})$

    $P \in L_-^{\uparrow}$, space reversal $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
  - $T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in L_-^{\downarrow}$, time reversal

$$\begin{tikzcd}
L_+^{\uparrow} \arrow[rr, leftrightarrow, "PT"] \arrow[dd, leftrightarrow, "P"'] \arrow[ddrr, "T"] & & L_+^{\downarrow} \arrow[dd, leftrightarrow, "P"] \arrow[ddll, "T"'] \\
& & \\
L_-^{\uparrow} \arrow[rr, leftrightarrow, "PT"'] & & L_-^{\downarrow}
\end{tikzcd}$$

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Poincaré group

Translation + Lor. tr.

Let's consider a transform. $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu + a^\mu$; $\Delta x'^\mu = \Lambda^\mu_{\ \nu} \Delta x^\nu$

$T(\Lambda, a)$ is a group:

- $T(\Lambda', a') T(\Lambda, a) = T(\Lambda'\Lambda, \overbrace{\Lambda' a + a'}^{\text{Constant vector}})$

  In fact

  $x''^\mu = \Lambda'^\mu_{\ \nu} x'^\nu + a'^\mu = \Lambda'^\mu_{\ \nu}(\Lambda^\nu_{\ \rho} x^\rho + a^\nu) + a'^\mu = \Lambda'^\mu_{\ \nu} \Lambda^\nu_{\ \rho} x^\rho + \Lambda'^\mu_{\ \nu} a^\nu + a'^\mu$

  $T(\Lambda'\Lambda, \Lambda' a + a') \in T(\Lambda, a)$ because $\Lambda\Lambda'$ is a L.t. and $\Lambda' a + a'$ is a constant vector

- Identity : $\mathbb{1} = T(\delta^\mu_{\ \nu}, 0) \in T(\Lambda, a)$
- Inverse : $T^{-1}(\Lambda, a) = T(\Lambda^{-1}, -\Lambda^{-1} a) \in T(\Lambda, a)$ because Λ^{-1} is a L.t. and $\Lambda^{-1} a$ is a const. vector.

02/10/2020

Infinitesimal transformation

We can consider an infinitesimal transformation

$$\Rightarrow \begin{cases} \Lambda^\mu_{\ \nu} \to \delta^\mu_{\ \nu} + \delta\omega^\mu_{\ \nu} & , \delta\omega^\mu_{\ \nu} \text{ is a small deformation, we call } \delta\omega^\mu_{\ \nu} = \varepsilon^\mu_{\ \nu} \\ a^\mu \to \delta a^\mu & , \delta a^\mu \text{ is a small displacement, we call } \delta a^\mu = \varepsilon^\mu \end{cases}$$

$$x'^\mu = (\delta^\mu_{\ \nu} + \varepsilon^\mu_{\ \nu}) x^\nu + \varepsilon^\mu \simeq x^\mu + \varepsilon^\mu_{\ \nu} x^\nu + \varepsilon^\mu$$

$$\Rightarrow \boxed{\cancel{\eta_{\sigma\rho}}} = (\delta^\mu_{\ \sigma} + \varepsilon^\mu_{\ \sigma}) \eta_{\mu\nu} (\delta^\nu_{\ \rho} + \varepsilon^\nu_{\ \rho}) = (\delta^\mu_{\ \sigma} \eta_{\mu\nu} + \varepsilon^\mu_{\ \sigma} \eta_{\mu\nu})(\delta^\nu_{\ \rho} + \varepsilon^\nu_{\ \rho})$$

$$= \delta^\mu_{\ \sigma} \eta_{\mu\nu} \delta^\nu_{\ \rho} + \varepsilon^\mu_{\ \sigma} \eta_{\mu\nu} \delta^\nu_{\ \rho} + \delta^\mu_{\ \sigma} \eta_{\mu\sigma} \varepsilon^\nu_{\ \rho} =$$

$$= \eta_{\sigma\rho} + \varepsilon^\mu_{\ \sigma} \eta_{\mu\nu} \delta^\nu_{\ \rho} + \delta^\mu_{\ \sigma} \eta_{\mu\nu} \varepsilon^\nu_{\ \rho} = \boxed{\cancel{\eta_{\sigma\rho}} + \varepsilon_{\sigma\rho} + \varepsilon_{\rho\sigma}} \Rightarrow \boxed{\varepsilon_{\sigma\rho} = -\varepsilon_{\rho\sigma}}$$

Lagrangian description of the mechanics

Lagrangian and Hamiltonian mechanics

$$L = L(q, \dot{q}, t) \;;\; \text{action}: \quad S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

Hamilton principle: $\delta S = 0$ (Lagrangian motion equation)

Lagrangian and relativity

We need that our equation are invariant, so S must be composed by Lorentz invariants, dt is not an invariant, but $d\tau = \frac{dt}{\gamma}$ is

$$\Rightarrow S := \int_{\tau_1}^{\tau_2} L(x^\mu, \dot{x}^\mu, \tau)\, \underbrace{d\tau}_{\text{invariant}} \Rightarrow (L \text{ must be LT invariant})?$$

If L is LT invariant, then it satisfies $L'(x^{\mu\prime}, \dot{x}^{\mu\prime}, \tau) = L(x^\mu, \dot{x}^\mu, \tau)$

We can vary anything in the path

Let's do a path transformation, it can be any transf., not only LT :

$$\begin{cases} \tau \rightarrow \tau' = \tau + \delta\tau(\tau) \\ x^\mu \rightarrow x'^\mu(\tau') = x^\mu(\tau) + \delta x^\mu(\tau) \\ \dot{x}^\mu \rightarrow \dot{x}'^\mu(\tau') = \dot{x}^\mu(\tau) + \delta\dot{x}^\mu(\tau) \end{cases}$$

$$\delta S = \int L'(x'^\mu, \dot{x}'^\mu, \tau')\, d\tau' - \int L(x^\mu, \dot{x}^\mu, \tau)\, d\tau$$

- $L(x'^\mu(\tau'), \dot{x}'^\mu(\tau'), \tau') \simeq L(x^\mu(\tau), \dot{x}^\mu(\tau), \tau) + \frac{\partial L}{\partial \tau}\delta\tau + \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta\dot{x}^\mu =$

$$= \left[\frac{dL}{d\tau} = \frac{\partial L}{\partial \tau} + \frac{\partial L}{\partial x^\mu}\frac{\partial x^\mu}{\partial t} + \frac{\partial L}{\partial \dot{x}^\mu}\frac{\partial \dot{x}^\mu}{d\tau} \Rightarrow \frac{\partial L}{\partial \tau} = \frac{dL}{d\tau} - \frac{\partial L}{\partial x^\mu}\dot{x}^\mu - \frac{\partial L}{\partial \dot{x}^\mu}\ddot{x}^\mu \right] =$$

$$= L(\tau) + \frac{dL}{d\tau}\delta\tau + \left[\frac{\partial L}{\partial x^\mu}(\delta x^\mu - \dot{x}^\mu \delta\tau)\right] + \left[\frac{\partial L}{\partial \dot{x}^\mu}(\delta\dot{x}^\mu - \ddot{x}^\mu\delta\tau)\right]$$

- $d\tau' = d\tau + \frac{d}{d\tau}\delta\tau\, d\tau = d\tau\left(1 + \frac{d}{dt}\delta\tau\right)$

(17)

$$\Rightarrow \delta S = \int d\tau \left(1+\frac{d}{d\tau}\delta\tau\right)\Big[$$

$$L + \frac{dL}{d\tau}\delta\tau + \frac{\partial L}{\partial x^\mu}(\delta x^\mu - \dot{x}^\mu \delta\tau) + \frac{\partial L}{\partial \dot{x}^\mu}(\delta \dot{x}^\mu - \ddot{x}^\mu \delta\tau)\Big] - \int d\tau L$$

$$= \int \left[\frac{d}{d\tau}(L d\tau) + \frac{\partial L}{\partial x^\mu}(\delta x^\mu - \dot{x}^\mu\delta\tau) + \frac{\partial L}{\partial \dot{x}^\mu}(\delta\dot{x}^\mu - \ddot{x}^\mu \delta\tau)\right] d\tau$$

Let's define the local variation $\delta_o x^\mu = x'^\mu(\tau) - x^\mu(\tau)$

$$\delta x^\mu(\tau) = x'^\mu(\tau') - x^\mu(\tau) = \underbrace{x'^\mu(\tau') - x'^\mu(\tau)}_{\simeq \text{ derivative}} + \overbrace{x'^\mu(\tau) - x^\mu(\tau)}^{=\delta_o x^\mu} \simeq$$

can be canceled at the first order

$$= \delta_o x^\mu + \dot{\cancel{x}}'^\mu(\tau)\delta\tau \Rightarrow$$

$\Rightarrow$ The local variation is $\boxed{\delta_o x^\mu = \delta x^\mu(\tau) - \dot{x}^\mu(\tau)\delta\tau}$

$$\delta\dot{x}^\mu(\tau) = \dot{x}'^\mu(x') - x^\mu(\tau) = \underbrace{\dot{x}'^\mu(\tau') - \dot{x}'^\mu(\tau)}_{\simeq \text{ derivative}} + \overbrace{\dot{x}'^\mu(\tau) - \dot{x}^\mu(\tau)}^{=\delta_o \dot{x}^\mu} \simeq$$

$$\simeq \delta_o \dot{x}^\mu + \ddot{x}^\mu(\tau)\delta\tau = \frac{d}{d\tau}\delta_o x^\mu + \ddot{x}^\mu(\tau)\delta\tau$$

$$\Rightarrow \boxed{\delta\dot{x}^\mu - \ddot{x}^\mu\delta\tau = \frac{d}{d\tau}\delta_o x^\mu}$$

$$\Rightarrow \delta S = \int d\tau\left[\frac{d}{d\tau}(L\delta\tau) + \frac{\partial L}{\partial x^\mu}\delta_o x^\mu + \underline{\frac{\partial L}{\partial \dot{x}^\mu}\frac{d}{d\tau}(\delta_o x^\mu)}\right]$$

we integrate by parts:

$$\delta S = \int d\tau\left[\frac{d}{d\tau}(L\delta\tau) + \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\delta_o x^\mu\right) - \left(\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu}\right)\delta_o x^\mu\right] =$$

$$= \int d\tau\left[-\left(\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu}\right)\delta_o x^\mu + \underbrace{\frac{d}{d\tau}\left[\left(L - \frac{\partial}{\partial \dot{x}^\mu}\dot{x}^\mu\right)\delta\tau\right]}_{*}\right] +$$

$$+ \underbrace{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\delta x^\mu\right)}_{\star}\Big]$$

Let $\delta x^\mu(\tau_1) = \delta x^\mu(\tau_2) = 0 \Rightarrow \int \star = \int \star = 0$

$$\Rightarrow \delta S = \int d\tau \left\{ -\left(\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu}\right)\delta_0 x^\mu \right\} \qquad \forall\ \delta_0 x^\mu$$

$$\Rightarrow \delta S = 0 \iff \boxed{\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0}\ ,$$ Lagrange equation

- Relativistic free particle

L has to be written as $L = \frac{\partial L}{\partial \dot{x}^\mu}\dot{x}^\mu$ (omogeneous function)

Let $L = \alpha\sqrt{\dot{x}^\mu \dot{x}_\mu} \Rightarrow$

$$\Rightarrow S = \int \alpha \sqrt{\dot{x}^\mu \dot{x}_\mu}\, d\tau = \alpha \int \sqrt{dx_\mu dx^\mu} = \alpha \int \sqrt{ds^2} = \alpha \int ds = \alpha c \int_{\tau_1}^{\tau_2} d\tau =$$

$$= \alpha c \int \sqrt{1 - \frac{v^2}{c^2}}\, dt = \alpha c \int \sqrt{1-\beta^2}\, dt$$

$$\Rightarrow L = \frac{\alpha c}{\gamma} \simeq \alpha c\left(1 - \frac{\beta^2}{2}\right) = \alpha c - \alpha \frac{v^2}{2c};$$

The classic limit has to be $L = T = \frac{1}{2}mv^2 \Rightarrow \alpha = -mc$

$$\Rightarrow L = -mc^2 + \frac{mv^2}{2} + \dots \Rightarrow L = -mc^2\sqrt{1-\beta^2} = -\frac{mc^2}{\gamma}$$

$$\boxed{L = -mc\sqrt{\dot{x}^\mu \dot{x}_\mu}}$$

If $\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}^\mu} = \text{const}$, a conservation law

$$\frac{\partial L}{\partial \dot{x}^\mu} = -\left(\frac{mc}{\sqrt{1-\beta^2}}, -\frac{m\vec{v}}{\sqrt{1-\beta^2}}\right) = -P_\mu$$

(19)

- Lagrangian invariant under Poinc. tr

$$L(\dot{x}'^{\mu}, x'^{\mu}, \tau) \simeq L(\dot{x}^{\mu}, x^{\mu}, \tau) + \frac{\partial L}{\partial x^{\mu}} \delta x^{\mu} + \frac{\partial L}{\partial \dot{x}^{\mu}} \delta \dot{x}^{\mu} =$$

$$= L(\dot{x}^{\mu}, x^{\mu}, \tau) + \frac{\partial L}{\partial x^{\mu}} \delta x^{\mu} + \frac{\partial L}{\partial \dot{x}^{\mu}} \frac{d}{d\tau} \delta x^{\mu} =$$

$$= L(\dot{x}^{\mu}, x^{\mu}, \tau) + \cancel{\frac{\partial L}{\partial x^{\mu}} \delta x^{\mu}} + \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\mu}} \delta x^{\mu}\right) - \cancel{\left(\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^{\mu}}\right)} \delta x^{\mu}$$

BECAUSE OF MOTION EQUATION

$$= L(\dot{x}^{\mu}, x^{\mu}, \tau) + \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\mu}} \delta x^{\mu}\right)$$

If L is LT invariant $\Rightarrow$ it is needed $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\mu}} \delta x^{\mu}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}^{\mu}} \delta x^{\mu} = \text{const.}$ *

$$\left(\Rightarrow P_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}} = \text{cost}\right)$$

Let's consider a Poinc. transf:

$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} + a^{\mu} \to (\delta^{\mu}_{\nu} + \varepsilon^{\mu}_{\ \nu}) x^{\nu} + \varepsilon^{\mu}$ (infinitesimal)

$$P_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}} \overset{*}{\Rightarrow} \text{const} = P_{\mu}(\varepsilon^{\mu}_{\ \nu} x^{\nu} + \varepsilon^{\mu}) = P_{\mu} \underbrace{\varepsilon^{\mu}_{\ \nu} x^{\nu}}_{= \varepsilon^{\mu\nu} x_{\nu}} + P_{\mu} \varepsilon^{\mu}$$

$$P_{\mu} x_{\nu} \varepsilon^{\mu\nu} = \left[\cancel{\tfrac{1}{2}(P_{\mu} x_{\nu} + P_{\nu} x_{\mu})} + \tfrac{1}{2}(P_{\mu} x_{\nu} - P_{\nu} x_{\mu})\right] \varepsilon^{\mu\nu}$$

Symm

Anti-symm

$$\Rightarrow \underbrace{\tfrac{1}{2} \varepsilon^{\mu\nu} (P_{\mu} x_{\nu} - P_{\nu} x_{\mu})}_{\text{Angular momentum-like}} + \overbrace{\varepsilon^{\mu} P_{\mu}}^{= \text{const}} = \text{const} \Rightarrow \tfrac{1}{2} \varepsilon^{\mu\nu} (P_{\mu} x_{\nu} - P_{\nu} x_{\mu}) = \text{const}$$

$$x_{\mu} P_{\nu} = M_{\mu\nu} = \text{const}$$

$\Rightarrow M$ and P are constants for the imposition of Lorentz invariation.