BU5526 - Portfolio Analysis
Lecture Notes

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Contents

Chapter 1

Introduction

1.1 Asset Pricing

1.1.1 Asset

Something valuable that an entity owns, benefits from, or has use of, in generating income.

1.1.2 Asset Pricing

Needed to buy/sell at a fair price.

- Obtained by discovery process:
  - Demand and supply forces.
- The price of an asset is simply the current value of its cash-flows.

$$PV = \frac{FV_1}{(1+r)^1} + \frac{FV_2}{(1+r)^2} + ... + \frac{FV_n}{(1+r)^n}$$

- What we need:
  - Prediction of cash flows;
  - Discount rate.

1.1.3 How to price a stock:

- Dividend Discount Model;
- FCF Model;
- Multipliers and Comparable approach.

Being able to estimate the required **rate of return** is everything you need to calculate the **value of any asset** - once you have predicted **future cash-flows**, including:

- Estimating the value of a stock;
- Estimating the value of a firm.

1.2 Portfolio Management

Prudent administration of **investable** (liquid) assets, aimed at achieving an **optimum risk-reward ratio**.

1.3 Mathematical Concepts

1.3.1 Mean

An average of different observations.

- Useful to describe a population;
- **Arithmetical**: sum of observations divided by the number of observations (if with equal weights).

$$\overline{X} = \frac{X_1 + X_2 + ... + X_n}{n}$$

  or

$$\overline{X} = W_1 \times X_1 + W_2 \times X_2 + ... + W_n \times X_n$$

- **Geometrical**: n root of the product of observations.

$$\overline{X}_G = \sqrt[n]{X_1 \times X_2 \times ... \times X_n}$$

  or

$$\overline{X}_G = \sqrt[n]{X_1^{W_1} \times X_2^{W_2} \times ... \times X_n^{W_n}}$$

There is a reason to use arithmetical or geometrical mean.

- **Equally weighted**: when all the observations have the same importance.
- **Unequally weighted**: different importance for different observations.

1.3.2 Variance

Seen as an extension of the mean.

- Dispersion to the mean (higher/lower):
  - Average of differences from the mean.

$$\sigma_x^2 = \frac{(X_1 - \mu_X)^2 + (X_2 - \mu_X)^2 + ... + (X_n - \mu_X)^2}{n}$$

1.3.3 Important Statistical Concept

- The formula is correct if we possess all the data on the population;
- If we only have a sample, we need to reflect "impreciseness" by removing "one degree of freedom".

$$\sigma_x^2 = \frac{(X_1 - \mu_X)^2 + (X_2 - \mu_X)^2 + ... + (X_n - \mu_X)^2}{n-1}$$

This will **always** be the case in finance.

1.3.4 Standard Deviation

$$\sigma_X = \sqrt[2]{\frac{(X_1 - \mu_n)^2 + (X_2 - \mu_n)^2 + ... + (X_n - \mu_n)^2}{(n-1)}}$$

1.3.5 Skewness

Brings back the sign. A positive skewness means more positive-value and reversely.

$$\sigma_X^3 = \frac{(X_1 - \mu_n)^3 + (X_2 - \mu_n)^3 + ... + (X_n - \mu_n)^3}{n}$$

1.3.6 Kurtosis

Outweighs extremes - dropping the sign.

$$\sigma_X^4 = \frac{(X_1 - \mu_n)^4 + (X_2 - \mu_n)^4 + ... + (X_n - \mu_n)^4}{n}$$

A large Kurtosis means a lot of extreme values.

- To get a **meaningful estimate: excess** kurtosis needs to be provided.
- Kurtosis of a normal distribution is 3.

Note: Unbiased equations of these two indicators are slightly more complex, but computer packages provide them automatically.

1.4 Assumptions of Mean-Variance Analysis

- Allows describing different assets simply.
- Assumes returns are normally distributed.

1.4.1 Normal distribution

- Its mean and median are equal;
- It's defined by two parameters, mean and variance;
- It's defined around its mean with:
  - 68% of observations within $\pm 1\sigma$ of the mean.
  - 95% of observations within $\pm 2\sigma$ of the mean.
  - 99% of observations within $\pm 3\sigma$ of the mean
- Returns are not normally distributed.
  - Skewed: not symmetric around the mean.
  - Characterized by high probability of extreme event.

1.5 Returns on Financial Assets

1.5.1 Holding period return

Return from holding an asset for a specific period of time.

$$R = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}}$$

Capital gain + Dividend yield
Holding period returns = compound returns

$$R = [(1 + r_1) \times (1 + r_2) \times (1 + r_3)] - 1$$

1.5.2 Geometric Mean Return

$$\overline{R} = \sqrt[T]{(1 + r_{i_1}) \times (1 + r_{i_2}) \times ... \times (1 + r_{i_T})} - 1$$

1.5.3 Annualized Return

These returns should be capitalized.

- Such as:

  $$r_{annual} = (1 + r_{period})^C - 1$$

  with C being the number of periods in a year.

1.5.4 Portfolio Return

- When several assets are combined into a portfolio, we can compute the portfolio return.
- Weighted average of the returns of individual assets

$$R_p = W_1 \times R_1 + W_2 \times R_2$$

1.5.5 Historical and Expected Returns

- Computed from historical data.
- What the investor expects to earn.
- Historical returns are different from expected returns.

1.5.6 Ways of Calculating Expected Returns

- Gut feeling;
- Modelling - calculated from a formula.

1.5.7 Standard deviation - Volatility

- Historical standard deviation;
- Defined as **risk** of equity return.

1.5.8 Portfolio Variance

- Cannot simply add up two variances.
- Covariance:

$$cov(X, Y) = E(XY) - E(X) \times E(Y)$$

  - The more the two assets move in the same way, the higher the covariance.
  - A negative covariance means the assets move in opposite directions.

- Variance:

$$\sigma_v^2 = wCw^T$$

1.5.9 Correlation and Portfolio Risk

The correlation among assets determine the portfolio's risk.

- Is a measure of tendency for N investments to act similarly;
- Can range from -1 and $+1$.

$$\rho_{ij} = \frac{cov(R_i, R_j)}{\sigma_i \sigma_j}$$

Chapter 2

The Optimal Portfolio

2.1 Portfolio

2.1.1 The Portfolio Perspective on Investing

One of the biggest challenges faced by individuals and institutions is to decide on how to invest for future needs. Should they invest in individual securities, or should they take a portfolio approach?
A portfolio and evaluating individual securities in relation to their contribution to the investment characteristics is important to achieve a "good" and "safe" return.

2.1.2 Diversification: Avoiding Disaster

Portfolio diversification helps investors avoiding disasters. It is of our utmost priority to balance investment weights as a way of having a good risk-return ratio.
On the other hand, it does not mean that, by reducing risk the portfolio will have reduced profits.

- The portfolio approach provides investors with a way to reduce the risk associated with their investment.
- Investors can choose the risk-return trade-off they prefer.
- A lower risk does not necessarily mean lower profits.

2.1.3 Reducing Risk

A Portfolio generally offers equivalent expected returns with lower overall volatility.
How to calculate its standard deviation?

- An interesting feature of combining assets is that:

- - While the combined return is the weighted average of individual returns, the combined variance is impacted by the covariance between securities.

- Covariance is how securities move together.

Variance of a Portfolio

$$\sigma_p{}^2 = wCw^T$$

or

$$\mu_v = w_1\mu_1 + w_2\mu_2$$

Standard Deviation

$$\sigma_p = \sqrt{\sigma_p{}^2}$$

or

$$\sigma_v{}^2 = w_1{}^2\sigma_1{}^2 + w_2{}^2\sigma_2{}^2 + 2w_1w_2c_{12}$$

Remember that using correlation instead of variance, we can reformulate the variance equation:

$$\sigma_v{}^2 = w_1{}^2\sigma_1{}^2 + w_2{}^2\sigma_2{}^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}$$

This equation shows that as long as $\rho_{12} is < 1$ adding more assets will reduce the overall variance of the portfolio.

2.1.4 Correlation and Portfolio Risk

- A major reason that portfolios can effectively reduce risk is that combining securities whose returns do not move together provides diversification.

- The less correlated the assets, the better the risk-return trade-off obtained within a portfolio.

2.1.5 Return and Risk of a Portfolio

- This is the fundamental idea of **diversification**;

- Because assets do not move perfectly together (correlation = 1), combining several assets will reduce the overall variance;

- Assets don't need to be negatively correlated, just not perfectly correlated (correlation < 1)

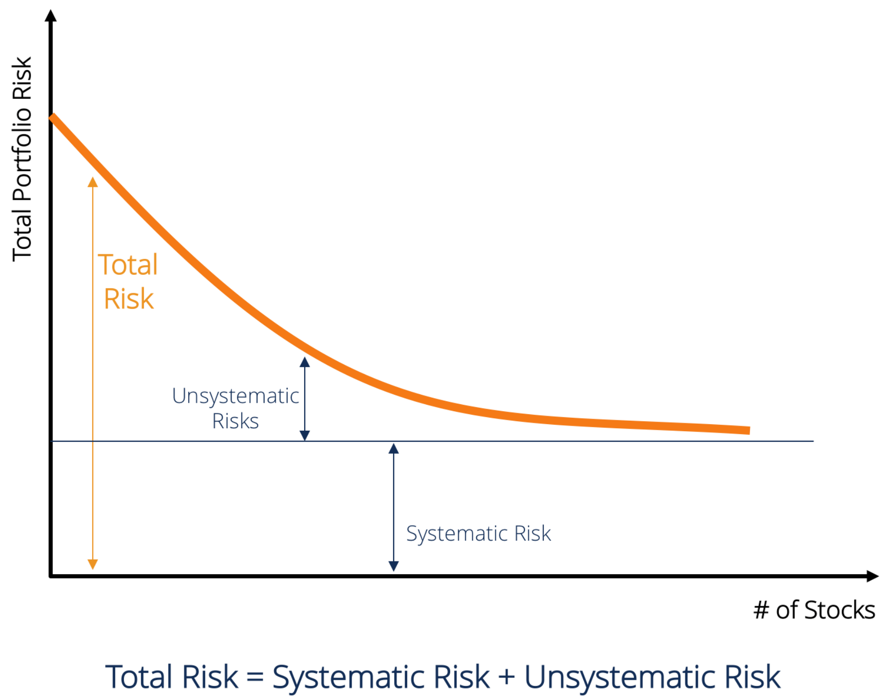
- The more uncorrelated the assets, the greater the risk-return trade-off.

2.1.6 Avenues for Diversification

- Diversify with asset classes;
- Diversify with index funds;
- Diversify among countries;
- Buy insurance;
- Buy put options;
- Evaluate the assets.

2.1.7 Diversification: Not necessarily Downside Protection

- It does not mean that diversification eliminates the entire risk.
- It removes only unsystematic (idiosyncratic) risk.
  Total Risk = Systematic Risk + Unsystematic Risk



Unsystematic Risk Is the risk specific to a company or industry. It is also known as diversifiable risk. It can be reduced through diversification.

Systematic Risk Is the risk specific to the entire market. Also known as undiversifiable risk, affecting the overall market.

- The benefit of diversification varies over time.
  - Correlation among assets might change;
  - Diversification does not protect against a generalized decrease in returns and increase in risk.
- When all assets move together, like during a financial crisis, diversification benefits shrink.
  - Hedging against risk becomes more difficult.

2.1.8 The Emergence of Modern Portfolio Theory

- The concept and intuition of the benefit of diversification has been around for a long time.
- However, its modern concept is theorized more recently.
- The main conclusion is that: **investors should not only hold portfolios but also focus on the correlation among the securities included.**

2.2 Efficient Portfolio Frontier

2.2.1 Investment Opportunity Set

Include as many assets as possible, from different sectors.

2.2.2 Minimum Variance Frontier

- We create a portfolio of assets;
- We are interested in pushing the frontier onto the north-west:
  - Minimizing the variance for a given return.
- We calculate the combinations with different weights that minimize the variance for a given return.
- If we combine all the available assets and select the weights giving the lower variance for a given return, we end up with the **minimum variance frontier**.
- All portfolios on this frontier display a lower variance and a higher return than any individual asset.

Note: The frontier uses all the available assets. An efficient portfolio is *any* portfolio on the minimum variance frontier.

2.3 The Efficient Portfolio

2.3.1 Risk-free Asset

- To select the most efficient portfolio on the frontier, we need to add a risk-free asset.
- A risk-free asset will allow us to:
  - Find the most efficient portfolio;
  - Draw the capital allocation line (CAL);
  - Create any risk-return combination for investors.
- The efficient **investor portfolio** will always be a **combination** of:
  - The risk-free asset;
  - The efficient portfolio.

2.4 Capital Allocation Line

- A capital allocation line is a line that allocates capital between two assets;
- We will allocate between a risk-free and a risky asset;
- We will still think in terms of risk and return trade-off;
- We need two elements: an **intercept**, and a **slope**.

2.4.1 A Risk-free Asset

In our case, assume a portfolio of two assets, a risk-free asset and a risky asset. Expected return can be determined as:

$$E(R_p) = W_1 R_f + (1 - W_1)E(R_i)$$

Because the risk-free asset has 0 risk, its variance is equal to zero, hence, the variance of this portfolio can be calculated as:

$${\sigma_p}^2 = (1 - W_1)^2 {\sigma_i}^2$$

And volatility:

$$\sigma_p = \sqrt{(1 - W_1)^2 {\sigma_i}^2} = (1 - W_1)\sigma_i$$

If we combine the portfolio return and standard deviation formula, we can rewrite the expected return in terms of risk.
The expected return of a portfolio that mixes a risk-free asset and the optimal portfolio is based on two elements:

$$E[R_p] = R_f + \frac{E(R_i) - R_f}{\sigma_i}\sigma_i$$

- The risk-free rate: R_f
  - It is the Y-intercept;
  - You cannot earn less than that.
- The market price of risk: $\frac{E(R_i)-R_f}{\sigma_i}\sigma_p$
  - It is the slope of the capital allocation line (CAL);
  - The highest, the best: plus it is high, plus the risk is profitable.

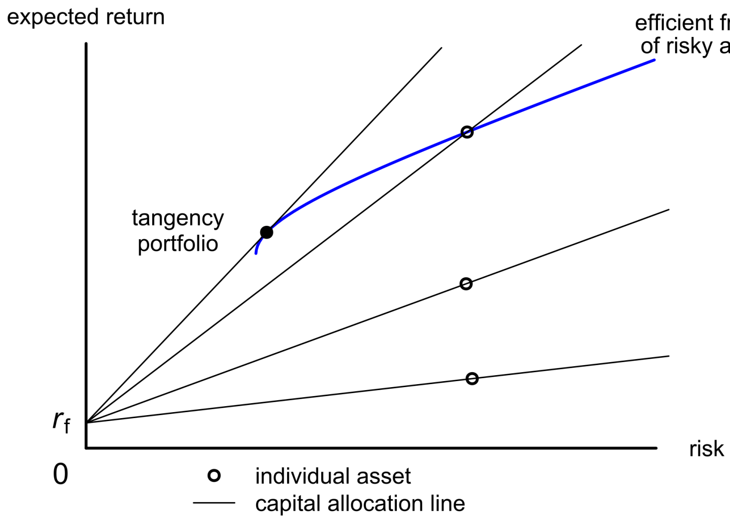
2.4.2 The Capital Allocation Line (CAL)

- The combination between a risky asset and a risk-free asset is called the **capital allocation line**;
- The risk-free rate does not change;
- Maximizing this angle maximizes your benefits for the risk you are taking.
- You do not control the market price of risk;
  - It is priced by the market.
- At an investor level, you can vary your risk, by changing your weights in the portfolio.

2.4.3 CAL and Optimal Portfolio

- Using the CAL, we will now find **the** optimal portfolio;
- It is one of the portfolios on the efficient frontier;
- By using the risk-free asset, you maximize your risk-return ration.
- Mathematically, it is the point that is tangent between the efficient frontier and the CAL.
- You cannot achieve a point better than one on the CAL;
  - This would only be by changing the set of assets.

- By holding a risk-free asset, an investor can achieve a point of higher return than the Markowitz Efficient Frontier with the same risk-level.
  - Negative weight on the risk-free asset (Y).
- The CAL and the optimum portfolio evolve over time.
  - If the risk-free asset return changes, it will affect the CAL Y-intercept;
  - Hence, it will modify the CAL, and which portfolio is optimum.



2.5 Optimal Investor Portfolio

All investors will choose a **combination** of the risk-free and optimum portfolio. They will just change their weightings. But how do we know which portfolio will be chosen by a **specific investor**?

2.5.1 The concept of Risk Aversion

The choice of portfolio will differ across individuals because each individual has a different risk aversion.

- Risk-seeking: utility increase with uncertainty.

- The individual will preferred a gamble with an expected value of £45 than a certain £50;
- Lottery and casinos.

- Risk-neutral
  - The individual is indifferent between the gamble or a £50 guaranteed income;
  - A billionaire may be indifferent in this case.

- Risk-adverse
  - The investor will prefer a certain value of £50 than an expected value of £45;
  - The risk-return trade-off is an indicator of risk-aversion;
  - Historical data supports risk-aversion: higher returns, come with higher risks.

2.5.2 Utility Theory

- The **utility** he derives from the guaranteed income of £50 greater than the **utility** he derives from the alternative.
- Individuals are different in their preferences.
  - All risk-averse investors will not rank their investments in the same manner;
  - With a guaranteed outcome of £40, some may find it inadequate.
- We can calculate the utility of an investor.
- This is not its return: it combines return and risk.
  - Utility is a function of return for each individual;
  - Individuals prefer higher utility.
- We usually assume that investors are adverse to risk.
- There are plenty of ways to **modelize** utility.

The formula

$$U = E(r) - \frac{1}{2}A\sigma^2$$

- U = Utility of an investment;
- E(r) = Expected Return;

- A = Measure of risk tolerance or risk aversion - can be either negative or positive;
  - Risk adverse investor: $A > 0$
  - Risk neutral investor: $A = 0$
  - Risk seeking investor: $A < 0$
- σ^2 = Variance or risk.
- Higher returns = higher utility;
- Higher variance reduce/increase the utility, depending on A;
- A risk-free asset generates the same utility for all individuals.

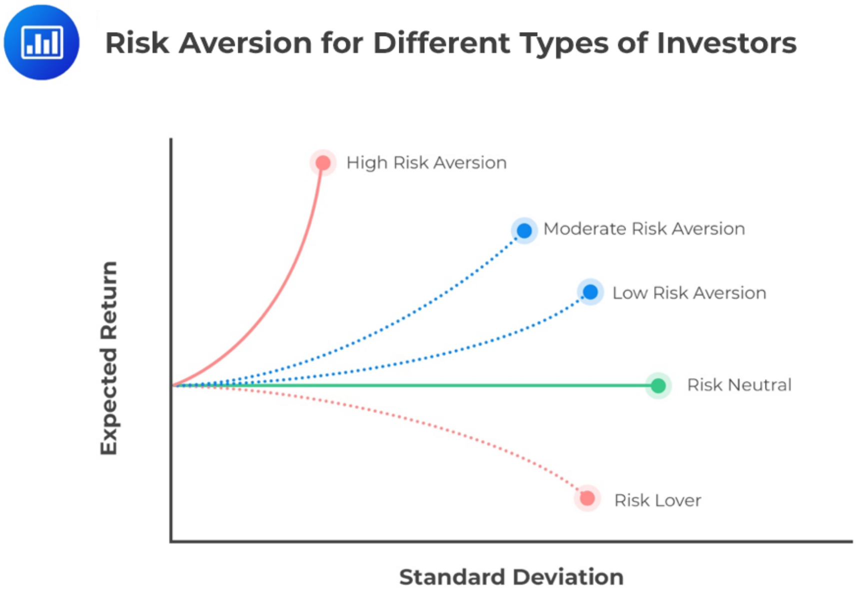
2.6 Indifference Curve

An indifference curve plots the combination of risk-return pairs that an investor would accept to maintain a given level of utility. For each investor there is an infinity of indifference curves, but each indifference curve for each investor never cross over each other.

- Represent the trade-off between risk and returns, with the *same utility.*
- All risk-adverse investors will prefer curves on the north-west.
- Risk-adverse curves are convex because as risk increases, an investor needs even higher returns to compensate.
- The **slope** of curves is the **same for one investor** and different among investors.

2.7 Optimal Investor Portfolio

- To know which combination is optimal for a given investor, we will use its indifference curve.
- The optimal investor's portfolio is the one which is tangent with the indifference curve.
  - It gives them the highest **achievable** return.
  - Lower or higher curbs, respectively, give:
    * A lower utility;
    * An unachievable utility.



- There is a different optimal portfolio for each investor - depending on its risk-aversion.
  - All the points on CAL(P) are achievable;
  - The selection will depend on the investor's preference.

2.8 The Two-Fund Separation Theorem

- Efficient Frontier;
- The Capital Allocation Line;
- Indifference Curve;

These concepts can be synthesized as a key theorem of modern portfolio theory.

- **The two-fund separation theorem:** we can divide an investor's problem into two distinct steps.
  - The investment decision;
  - The financing decision.
- For the investment decision: all investors, regardless of taste, risk preferences and initial wealth, will hold a combination of two funds:

- A risk-free asset;
- The optimal portfolio of risky assets.

- **Note:** All investors should lie on CAL(P).
- For the financing decision: each investor chooses the appropriate weight of risk-free and risky portfolio (P).
- The utility indifference curve of each investor will determine the investor's allocation to risky assets.
  - Portfolios before the optimal risky portfolio are obtained by lending at the risk-free rate.
  - Portfolios beyond the optimal risky portfolio are obtained by borrowing at the risk-free rate.

2.9 Summary of the Optimal Portfolio Choice

- Build the minimum variance frontier using all the assets available;
- Calculate and draw the capital allocation line (P) that is tangent to the minimum variance frontier.
- Use the investor's utility curves to obtain the portfolio that lies on the utility curve and on the capital allocation line (P).