patch test

Riccardo Petrucci

July 2024

1 Equation

Solve

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) + \mathbf{S}(\mathbf{U}) = \mathbf{0} \tag{1}$$

where $\mathbf{U} = [A,\ Q]^T, \mathbf{F}(\mathbf{U}) = [Q,\ \alpha Q^2/A + C_1]^T, \mathbf{S}(\mathbf{U}) = [0,\ K_r Q/A]^T$
$C_1 = \int_{A_0}^{A} \frac{A}{\rho_f}\frac{\partial \psi}{\partial A} dA, \psi = \beta\frac{\sqrt{A}-\sqrt{A_0}}{A_0}$.

We want to perform a patch test, so we introduce a forcing term

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) + \mathbf{S}(\mathbf{U}) = \mathbf{f} \tag{2}$$

to retrieve the exact solution $A(t,z) = 1 + zt, Q(t,z) = zt$.
Setting $A_0 = \beta = K_r = \rho_f = 1$, we obtain

$$\mathbf{f} = \begin{bmatrix} t+x \\ -\frac{t^3x^2}{(1+tx)^3} + \frac{2t^2x}{1+tx} + \frac{tx}{1+tx} + 0.5(1+tx)^{0.5} + x \end{bmatrix} \tag{3}$$

$$\frac{\partial \mathbf{f}}{\partial t} = \begin{bmatrix} 1 \\ +\frac{2t^3x^3}{(1+tx)^3} - \frac{5t^2x^2}{(1+tx)^2} + \frac{4tx}{1+tx} + \frac{tx}{4\sqrt{1+tx}} + 0.5(1+tx)^{0.5} \end{bmatrix} \tag{4}$$

2 Numerical discretization

We will use a second order Taylor-Galerkin scheme:

for $n \geq 0$, find $\mathbf{U}_h^{n+1} \in V_h$ which satisfies the following equations $\forall i = 1, 2, \ldots, N-1$

$$\begin{gathered}
(\mathbf{U}_h^{n+1}, \Phi_i) = (\mathbf{U}_h^n, \Phi_i) + \Delta t \left(\mathbf{F}^n - \frac{\Delta t}{2}\mathbf{H}^n\mathbf{S}^n, \frac{\partial \Phi_i}{\partial z} \right) + \frac{\Delta t^2}{2}\left(\frac{\partial \mathbf{S}^n}{\partial \mathbf{U}_n}\frac{\partial \mathbf{F}^n}{\partial z}, \Phi_i \right) \\
-\Delta t \left(\mathbf{S}^n - \frac{\Delta t}{2}\frac{\partial \mathbf{S}^n}{\partial \mathbf{U}_n}\mathbf{S}^n, \Phi_i \right) - \frac{\Delta t^2}{2}\left(\mathbf{H}^n \frac{\partial \mathbf{F}^n}{\partial z}, \frac{\partial \Phi_i}{\partial z} \right) \\
+\Delta t \left(\mathbf{f}^n, \Phi_i \right) + \frac{\Delta t^2}{2}\left(-\frac{\partial \mathbf{S}^n}{\partial \mathbf{U}_n}\mathbf{f}^n + \frac{\partial \mathbf{f}^n}{\partial t}, \Phi_i \right) + \frac{\Delta t^2}{2}\left(\mathbf{H}^n\mathbf{f}^n, \frac{\partial \Phi_i}{\partial z} \right)
\end{gathered}$$

In the L^2 norm i will expect a second order convergence, both in space and time

3 Boundary condition

- Inlet: Set $A = A_{inlet}$, **Dirichlet BC**
- Outlet: **Non reflecting BC** that is

$$\mathbf{l}_2^T \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{S} - \mathbf{f} \right) = 0 \quad \text{at } z = 1 \tag{5}$$

- **Compatibility condition**

$$\mathbf{l}_2^T \left(\frac{\partial \mathbf{U}}{\partial t} + +\mathbf{H}\frac{\partial \mathbf{U}}{\partial z} + \mathbf{B} - \mathbf{f} \right) = 0 \quad \text{at } z = 0 \tag{6}$$

$$\mathbf{l}_1^T \left(\frac{\partial \mathbf{U}}{\partial t} + +\mathbf{H}\frac{\partial \mathbf{U}}{\partial z} + \mathbf{B} - \mathbf{f} \right) = 0 \quad \text{at } z = L \tag{7}$$

At the discrete level they can be written as:

$$\mathbf{l}_2^T \mathbf{U}^{n+1} = \mathbf{l}_2^T \mathbf{CC} \tag{8}$$

$$\mathbf{l}_1^T \mathbf{U}^{n+1} = \mathbf{l}_1^T \mathbf{CC} \tag{9}$$

where $\mathbf{CC} = \mathbf{U}^n - \Delta t \mathbf{H}\frac{\partial \mathbf{U}}{\partial z} - \Delta t \mathbf{B} + \Delta t \mathbf{f}$. This only a first order approximation of the time derivative, and moreover we are considering a fully explicit method. This could spoil the global convergence rate.

Putting together eq. (5) and eq. (9) at the outlet we have:

$$\begin{bmatrix} A^{n+1} \\ Q^{n+1} \end{bmatrix} = \frac{1}{2c_\alpha} \begin{bmatrix} 1 & -1 \\ c_\alpha + \alpha Q_n / A_n & c_\alpha - \alpha Q_n / A_n \end{bmatrix} \begin{bmatrix} \mathbf{l}_1^T \mathbf{CC} \\ \mathbf{l}_2^T (\mathbf{U}^n - \Delta t \mathbf{S}) \end{bmatrix}. \tag{10}$$

where

$$l_1 = \begin{bmatrix} c_\alpha - \alpha \frac{Q}{A} \\ 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} -c_\alpha - \alpha \frac{Q}{A} \\ 1 \end{bmatrix}, \tag{11}$$

$$c_\alpha = \sqrt{\frac{\beta}{2\rho f A_0}\sqrt{A} + \left(\frac{Q}{A}\right)^2 \alpha(\alpha - 1)}. \tag{12}$$