EXERCISE 1. *Suppose the people who own page 3 in the web of Figure 1 are infuriated by the fact that its importance score, computed using formula (2.1), is lower than the score of page 1. In an attempt to boost page 3's score, they create a page 5 that links to page 3; page 3 also links to page 5. Does this boost page 3's score above that of page 1?*

EXERCISE 2. *Construct a web consisting of three or more subwebs and verify that* $\dim(V_1(\mathbf{A}))$ *equals (or exceeds) the number of the components in the web.*

EXERCISE 3. *Add a link from page 5 to page 1 in the web of Figure 2. The resulting web, considered as an undirected graph, is connected. What is the dimension of* $V_1(\mathbf{A})$*?*

EXERCISE 4. *In the web of Figure 2.1, remove the link from page 3 to page 1. In the resulting web page 3 is now a dangling node. Set up the corresponding substochastic matrix and find its largest positive (Perron) eigenvalue. Find a non-negative Perron eigenvector for this eigenvalue, and scale the vector so that components sum to one. Does the resulting ranking seem reasonable?*

EXERCISE 5. *Prove that in any web the importance score of a page with no backlinks is zero.*

EXERCISE 6. *Implicit in our analysis up to this point is the assertion that the manner in which the pages of a web* W *are indexed has no effect on the importance score assigned to any given page. Prove this, as follows: Let* W *contains* n *pages, each page assigned an index* 1 *through* n*, and let* $\mathbf{A}$ *be the resulting link matrix. Suppose we then transpose the indices of pages* i *and* j *(so page* i *is now page* j *and vice-versa). Let* $\tilde{\mathbf{A}}$ *be the link matrix for the relabelled web.*

- *Argue that* $\tilde{\mathbf{A}} = \mathbf{PAP}$*, where* $\mathbf{P}$ *is the elementary matrix obtained by transposing rows* i *and* j *of the* $n \times n$ *identity matrix. Note that the operation* $\mathbf{A} \to \mathbf{PA}$ *has the effect of swapping rows* i *and* j *of* $\mathbf{A}$*, while* $\mathbf{A} \to \mathbf{AP}$ *swaps columns* i *and* j*. Also,* $\mathbf{P}^2 = \mathbf{I}$*, the identity matrix.*
- *Suppose that* $\mathbf{x}$ *is an eigenvector for* $\mathbf{A}$*, so* $\mathbf{Ax} = \lambda\mathbf{x}$ *for some* λ*. Show that* $\mathbf{y} = \mathbf{Px}$ *is an eigenvector for* $\tilde{\mathbf{A}}$ *with eigenvalue* λ*.*
- *Explain why this shows that transposing the indices of any two pages leaves the importance scores unchanged, and use this result to argue that any permutation of the page indices leaves the importance scores unchanged.*

EXERCISE 7. *Prove that if* $\mathbf{A}$ *is an* $n \times n$ *column-stochastic matrix and* $0 \leq m \leq 1$*, then* $\mathbf{M} = (1-m)\mathbf{A} + m\mathbf{S}$ *is also a column-stochastic matrix.*

EXERCISE 8. *Show that the product of two column-stochastic matrices is also column-stochastic.*

EXERCISE 9. *Show that a page with no backlinks is given importance score* $\frac{m}{n}$ *by formula (3.2)*

EXERCISE 10. *Suppose that* $\mathbf{A}$ *is the link matrix for a strongly connected web of* n *pages (any page can be reached from any other page by following a finite number of links). Show that* $\dim(V_1(\mathbf{A})) = 1$ *as follows. Let* $(\mathbf{A}^k)_{ij}$ *denote the* (i,j)*-entry of* $\mathbf{A}^k$*.*

- *Note that page* i *can be reached from page* j *in one step if and only if* $A_{ij} > 0$ *(since* $A_{ij} > 0$ *means there's a link from* j *to* i*!) Show that* $(\mathbf{A}^2)_{ij} > 0$ *if and only if page* i *can be reached from page* j *in exactly two steps. Hint:* $(\mathbf{A}^2)_{ij} = \sum_k A_{ik}A_{kj}$*; all* A_{ij} *are non-negative, so* $(\mathbf{A}^2)_{ij} > 0$ *implies that for some* k *both* A_{ik} *and* A_{kj} *are positive.*
- *Show more generally that* $(\mathbf{A}^p)_{ij} > 0$ *if and only if page* i *can be reached from page* j *in EXACTLY* p *steps.*
- *Argue that* $(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^p)_{ij} > 0$ *if and only if page* i *can be reached from page* j *in* p *or fewer steps (note* $p = 0$ *is a legitimate choice—any page can be reached from itself in zero steps!)*
- *Explain why* $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$ *is a positive matrix if the web is strongly connected.*
- *Use the last part (and Exercise 8) so show that* $\mathbf{B} = \frac{1}{n}(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1})$ *is positive and column-stochastic (and hence by Lemma 3.2,* $\dim(V_1(\mathbf{B})) = 1$*).*
- *Show that if* $\mathbf{x} \in V_1(\mathbf{A})$ *then* $\mathbf{x} \in V_1(\mathbf{B})$*. Why does this imply that* $\dim(V_1(\mathbf{A})) = 1$*?*

EXERCISE 11. *Consider again the web in Figure 2.1, with the addition of a page 5 that links to page 3, where page 3 also links to page 5. Calculate the new ranking by finding the eigenvector of* $\mathbf{M}$ *(corresponding to* $\lambda = 1$*) that has positive components summing to one. Use* $m = 0.15$*.*

EXERCISE 12. *Add a sixth page that links to every page of the web in the previous exercise, but to which no other page links. Rank the pages using* $\mathbf{A}$*, then using* $\mathbf{M}$ *with* $m = 0.15$*, and compare the results.*

EXERCISE 13. *Construct a web consisting of two or more subwebs and determine the ranking given by formula (3.1).*

EXERCISE 14. *For the web in Exercise 11, compute the values of* $\|\mathbf{M}^k\mathbf{x}_0 - \mathbf{q}\|_1$ *and* $\frac{\|\mathbf{M}^k\mathbf{x}_0-\mathbf{q}\|_1}{\|\mathbf{M}^{k-1}\mathbf{x}_0-\mathbf{q}\|_1}$ *for* $k = 1, 5, 10, 50$, *using an initial guess* $\mathbf{x}_0$ *not too close to the actual eigenvector* $\mathbf{q}$ *(so that you can watch the convergence). Determine* $c = \max_{1\le j\le n} |1 - 2\min_{1\le i\le n} M_{ij}|$ *and the absolute value of the second largest eigenvalue of* $\mathbf{M}$.

EXERCISE 15. *To see why the second largest eigenvalue plays a role in bounding* $\frac{\|\mathbf{M}^k\mathbf{x}_0-\mathbf{q}\|_1}{\|\mathbf{M}^{k-1}\mathbf{x}_0-\mathbf{q}\|_1}$, *consider an* $n \times n$ *positive column-stochastic matrix* $\mathbf{M}$ *that is diagonalizable. Let* $\mathbf{x}_0$ *be any vector with non-negative components that sum to one. Since* $\mathbf{M}$ *is diagonalizable, we can create a basis of eigenvectors* $\{\mathbf{q}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$, *where* $\mathbf{q}$ *is the steady state vector, and then write* $\mathbf{x}_0 = a\mathbf{q} + \sum_{k=1}^{n-1} b_k\mathbf{v}_k$. *Determine* $\mathbf{M}^k\mathbf{x}_0$, *and then show that* $a = 1$ *and the sum of the components of each* $\mathbf{v}_k$ *must equal 0. Next apply Proposition 4 to prove that, except for the non-repeated eigenvalue* $\lambda = 1$, *the other eigenvalues are all strictly less than one in absolute value. Use this to evaluate* $\lim_{k\to\infty} \frac{\|\mathbf{M}^k\mathbf{x}_0-\mathbf{q}\|_1}{\|\mathbf{M}^{k-1}\mathbf{x}_0-\mathbf{q}\|_1}$.

EXERCISE 16. *Consider the link matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{bmatrix}.$$

Show that $\mathbf{M} = (1-m)\mathbf{A} + m\mathbf{S}$ *(all* $S_{ij} = 1/3$*) is not diagonalizable for* $0 \le m < 1$.

EXERCISE 17. *How should the value of* m *be chosen? How does this choice affect the rankings and the computation time?*