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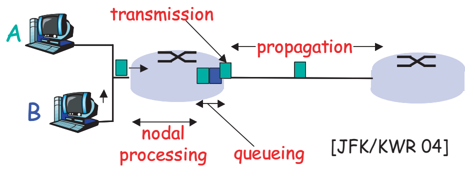
A few Slides on Foundations of Performance Analysis of Data Networks



The intention here is to consolidate some useful results (not to teach the underlying material).

Outline

1. Overview of random variables
2. Some useful discrete r.v. distributions
3. Some useful continuous r.v. distributions
4. Little's Theorem
5. The M/M/1 Queue

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1. Overview of random variables

1. RVs arise in 3 possible ways:
   - direct observation of a random experiment
   - mapping from a sample space to real numbers (e.g., HHT $\to 2$)
   - functions of other random variables
2. Examples:
   - $X =$ number of packets queued in a router for transmission
     - S_X denotes the range of the r.v. X
     - $S_X = \{0, 1, \ldots\}$
   - $Y =$ time delay seen by a packet while waiting in a queue
     - $S_Y \in [0, \infty]$
   - $Y = (X - \mu_X)^2$, here Y is derived from X
     (in general, we write $Y = g(X)$, for some function $g()$)
3. We describe a discrete r.v. X by a *probability mass function* (PMF) $P_X(x)$,

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defined as:

$$P_X(x) = Prob[X = x]$$

e.g.,

$$P_X(x) = \begin{cases} 1/4 & x = 0 \\ 3/4 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

4. $P_X(.)$ satisfies
   - $\sum_{x \in S_X} P_X(x) = 1$
   - For every event $B \subseteq S_X$: $Prob[B] = \sum_{x \in B} P_X(x)$
5. X can also be described by the *Cumulative Distribution Function* (CDF) $F_X(x)$, defined as:

   $$F_X(x) = Prob[X \leq x]$$
6. A continuous r.v. X can be described by its CDF $F_X(x)$, or its *probability density function* (PDF), denoted $f_X(x)$:

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- $$Prob[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(u)du$$
- $$F_X(x) = \int_{-\infty}^{x} f_X(u)du$$

7. Mean (denoted $E[X]$ or μ_X):
   - discrete: $E[X] = \sum_{x \in S_X} x P_X(x)$
   - continuous: $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
8. Variance: $Var[X] = E[(X - \mu_X)^2] = E[X^2] - \mu_X^2$
   - standard deviation: $\sigma_X = \sqrt{Var[X]}$

2. Some Useful Discrete Distributions

2.1 Discrete Uniform $[k, k+1, \ldots, l]$

- X is equally likely to take one the possible values

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- PMF:

$$P_X(x) = \begin{cases} \frac{1}{l-k+1} & x = k, k+1, k+2, \ldots, l \\ 0 & \text{otherwise} \end{cases}$$

- $E[X] = \frac{k+l}{2}$
- $Var[X] = \frac{(l-k)(l-k+2)}{12}$

2.2 Bernoulli

- X is binary (success/fail); so $S_X = \{0, 1\}$

$$P_X(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

- $E[X] = 1.p + 0.(1-p) = p$
- $Var[X] = p(1-p)$

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2.3 Geometric

- Perform a sequence of independent Bernoulli (success/fail) trials until we get the first success.
- $X =$ the number of independent trials performed until we get the first success. (So, $S_X = \{1, 2, 3, \ldots\}$)
- PMF:

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- $E[X] = \frac{1}{p}$ (intuition?)
- $Var[X] = \frac{(1-p)}{p^2}$

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2.4 Binomial

- Perform a sequence of n, $n \geq 1$, independent Bernoulli (success/fail) trials, let
  $X =$ the number of successes in n trials
- PMF:

$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

- $E[X] = np$
- $Var[X] = np(1-p)$
- Application: justifying statistical multiplexing

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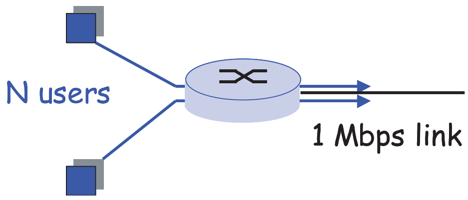

- Assume: 1 Mbps link, $N = 35$ users, each user sends at 100 kbps when active, each user is active 10% of time
- Circuit-switching: can accommodate at most 10 users
- Packet-switching: What is the prob. that the aggregate traffic demand exceeds the link capacity?
  - ∗ Model user activity during a short period as a Bernoulli trial with Prob[active]$= p = 0.1$ ($q = 1 - p = 0.9$)
  - ∗ So, Prob [# of active users during a short period ≥ 11]

$$= \sum_{i=11}^{N=35} \binom{N}{i} p^i q^{N-i}$$
$$= .0004$$

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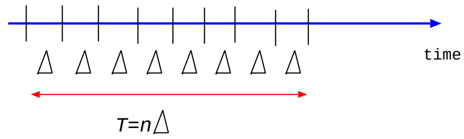
2.5 Poisson (with parameter α, where $\alpha = \lambda T$)

- First, let's introduce the Poisson process :
  - events: arrivals, departures, births, deaths, hits to a WEB server, etc.
  - event rate: λ events/sec
  - view time as slotted: each slot is Δ sec. (Δ is infinitesimal)



  - $$\text{Prob[exactly } n \text{ events in one } \Delta] = \begin{cases} \lambda\Delta & n = 1 \\ 1 - \lambda\Delta & n = 0 \\ 0 & \text{otherwise} \end{cases}$$
  - So, the number of possible events in one Δ is a Bernoulli r.v.
  - Number of events in non-overlapping intervals are independent r.v.s
- Now, suppose we take an interval of T sec, and let

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$X =$ number of events in T secs

- PMF: let $\alpha = \lambda T$ (average number of events in T)

$$P_X(x) = \begin{cases} \frac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- $E[X] = \alpha$
- $Var[X] = \alpha$
- Example. Suppose the number of hits to a WEB server in a 10-sec interval is a Poisson r.v. K, with $\alpha = 5$ hits.

1. Find Prob[no hit in 10 sec].
2. Find Prob [at least 2 hits occur in 2 seconds].

Solution.

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1. $P_K(k) = \begin{cases} \frac{5^k e^{-5}}{k!} & k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$

   Prob[no hit in 10 sec]= Prob[$K = 0$]= $\frac{e^{-5}}{0!} = .0067$

2. By definition, $\alpha = \lambda T$, so $\lambda = \frac{5}{10} = 0.5$ hits/sec.

   For $T = 2$ sec, $\alpha = \lambda T = 0.5 \times 2 = 1.0$ hit.

   Let N be the number of hits in a 2-sec interval then

$$P_N(n) = \begin{cases} \frac{e^{-1}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

   Prob[at least 2 hits in 2 secs] $= 1 - P_N(0) - P_N(1) = 0.264$

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3. Some Useful Continuous Distributions

3.1 Uniform in the interval $[a, b]$

- PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x < b \\ 0 & \text{otherwise} \end{cases}$$

- CDF

$$F_X(x) = Prob[X \leq x] = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x > b \end{cases}$$

- $E[X] = (b + a)/2$
- $Var[X] = (b - a)^2/12$

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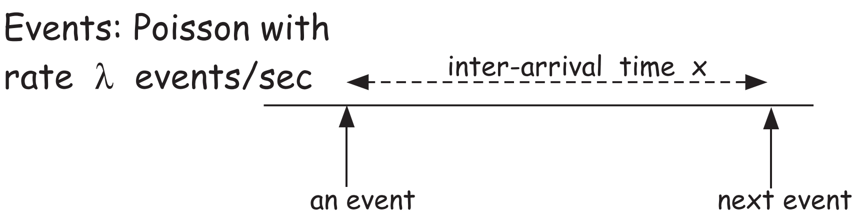
3.2 Exponential

- Consider a Poisson process with rate λ events/sec, let $X =$ the time between two consecutive events (inter-arrival time):



Q: What is X's distribution?

- A: X is *exponentially* distributed. CDF:

$$F_X(x) = Prob[X \leq x] = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

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- Why?

$$\begin{aligned} Prob[X \leq x] &= Prob\,[\text{at least one event in } x] \\ &= 1 - Prob\,[\text{zero event in } x] \\ &= 1 - \left[\frac{(\lambda x)^n e^{-\lambda x}}{n!}\right]_{n=0} \\ &= 1 - e^{-\lambda x}. \end{aligned}$$

- $E[X] = 1/\lambda$
- $Var[X] = 1/\lambda^2$
- Example. The call duration T in some cellular network is exponentially distributed with an average of 3 minutes per call. Find the probability that a call lasts between 2 and 4 minutes.

Solution.

1. $\lambda = 1/3$ minutes/call
2. $P[2 \leq T \leq 4] = F_T(4) - F_T(2) = e^{-2/3} - e^{-4/3} = 0.25.$

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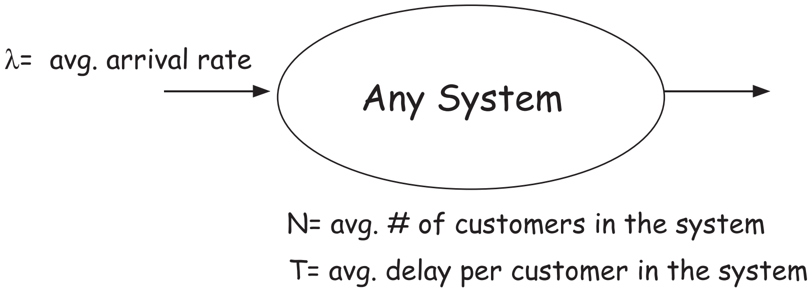
4. J. Little's Theorem [1961]



- Theorem: If the limits N, λ, and T exist, and the system is in equilibrium (avg. arrival rate $=$ avg. departure rate) then $\boxed{N = \lambda T}$.
- Example (The Burger King Spy).
  Spying on McDonald's, the spy recorded:
  - Arrivals: $\lambda = 32$ customer's/hour
  - Each customer exits after 12 minutes on the average

  Q: What is N (the avg. # of customers at McDonald's)?

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Solution.

1. $\lambda = 32$ customers/hour $= \frac{32}{60} \approx 0.53$ customers/minute
2. Assuming equilibrium, by Little's Theorem:

$$N = \lambda T = 0.53 \times 12 \approx 6.4 \text{ customers.}$$

5. The M/M/1 (Infinite Buffer) Queue

- Assumptions:
  - arrival process: Poisson with rate λ pkt/sec
  - service time: exponentially distributed with average $\frac{1}{\mu}$ sec. per pkt (that is, μ pkts/sec can be viewed as the departure rate)

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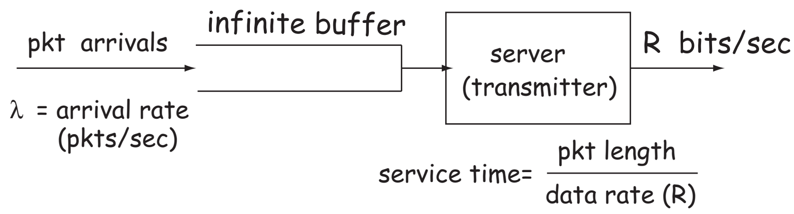

- A steady state solution exists only if $\lambda < \mu$.
- Then

$$\boxed{\text{avg. \# of pkts (queued + in-service)} = \frac{\lambda}{\mu - \lambda} \text{ pkts.}}$$

- By Little's Theorem:

$$\boxed{\text{avg. pkt delay in system} = \frac{1}{\mu - \lambda}.}$$