FE-610 Stochastic Calculus for Financial Engineers

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Assignment #5

6.1 , 6.2 , 6.3 , 11.1 , 11.2

Canvas Problems

1. Let a stock price $S(t)$ be a geometric Brownian motion under the risk-neutral measure.

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)$$

Where r and $\sigma > 0$ are constant and $\widetilde{W}$ is a brownian motion under the risk-neutral measure. Consider a derivative security that pays $S^2(t)$ at time T. Construct a portfolio that trades in the stock and a money market account with constant rate of interest r so that the final value of the portfolio $X(T)$ is $S^2(T)$ almost surely. In particular, specify what $X(0)$ should be, and how many shares of stock $\Delta(t)$ the portfolio should hold at each time t?

2. We have a stochastic interest rate given by the process $R(t) = r + \sigma W(t)$ where r and σ are positive constants. Using this interest rate, determine the value of a zero-coupon bond at time t. (Assume an affine-yield model)

6.1 Consider

$$dX(u) = \big(a(u) + b(u)X(u)\big)\,du + \big(\gamma(u) + \sigma(u)X(u)\big)\,dW(u)$$

W is Brownian motion and a, b, γ, σ are adapted to $\mathcal{F}(u)$, $u \geq 0$. Let

$$Z(u) = \exp\left\{\int_t^u \sigma(v)\,dW(v) + \int_t^u \left(b(v) - \tfrac{1}{2}\sigma^2(v)\right)dv\right\}$$

$$Y(u) = X + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)}\,dv + \int_t^u \frac{\gamma(v)}{Z(v)}\,dW(v)$$

(i) Show that $Z(t) = 1$ and

$$dZ(u) = b(u)Z(u)\,du + \sigma(u)Z(u)\,dW(u)$$

Let $Z(u) = e^{R(u)}$, where $R(u) = \int_t^u \sigma(v)\,dW(v) + \int_t^u \left(b(v) - \tfrac{1}{2}\sigma^2(v)\,dv\right)$

Find $\underline{dZ(u)}$

$f(a,b) = e^b$, so $f_a = 0$

$f_b = e^b$

$f_{bb} = e^b$

$$dZ(u) = 0 + e^{R(u)}\, dR(u) + e^{R(u)} (dR(u))^2$$

$$dR(u) = \sigma(u)\, dW(u) + \left(b(u) - \tfrac{1}{2}\sigma^2(u)\right) du$$

$(dR(u))^2 = 0$ (dt term makes this 0)

$$dZ(u) = e^{R(u)}\, dR(u)$$

(ii)

$$dY(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)}\, du + \frac{\gamma(u)}{Z(u)}\, dW(u), \quad u \geq t$$

Show $X(u) = Y(u) Z(u)$ Solves the SDE $dX(u)$

What is $X(u)$?

1. $dX(u)$
2. Integrate

$$dX(u) = d\big(Y(u)Z(u)\big) \Rightarrow \quad \text{(product Rule)}$$

$$dY(u)Z(u) + dZ(u)Y(u) + dZ(u)\,dY(u)$$

$$(dZ)(dY) = \big(\sigma Z(t)\,dW(t)\big) \cdot \frac{\gamma(u)}{\cancel{Z(t)}}\,dW(t)$$

$$= \sigma(t)\gamma(t)\,dt$$

$$dY(u)Z(u) = \frac{\big(a(u) - \sigma(u)\gamma(u)\big)du + \gamma(u)\,dW(u)}{\cancel{Z(u)}} \cdot \cancel{Z(u)}$$

$$dY(u)Z(u) = \big(a(u) - \sigma(u)\gamma(u)\big)du + \gamma(u)\,dW(u)$$

$$dZ(u)Y(u) = \big(b(u)Z(u)\,du + \sigma(u)Z(u)\,dW(u)\big) \cdot \left(X + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)\,du}{Z(v)} + \int_t^u \frac{\gamma(v)}{Z(v)}\,dW(v)\right)$$

$$= X\,b(u)Z(u)\,du + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)\,du}{Z(v)} \cdot b(u)Z(u)\,du$$

$$+ X\sigma(u)Z(u)\,dW(u) + \underbrace{\int_t^u \frac{\gamma(v)}{\cancel{Z}}\,dW(v) \cdot \sigma(u)\cancel{Z(u)}\,dW(u)}_{\int_t^u \gamma(v)\sigma(v)\,dv}$$

$$dX(u) = (a(u) - \sigma(u)\gamma(u))du + \gamma(u)dW(u)$$

$$+ \underbrace{Xb(u)Z(u)du + X\sigma(u)Z(u)dW(u)}_{X(dZ(u))} + \int_t^u \gamma(v)\sigma(v)dv$$

$$dX(u) = (a(u) - \sigma(u)\gamma(u))du + \gamma(u)dW(u)$$

$$+ X \circ dZ(u) + \int_t^u \gamma(v)\sigma(v)dv \Rightarrow$$

Integrate

$$\int dX(u) = \int_0^T (a(u) - \sigma(u)\gamma(u))du + \int_0^T \gamma(u)dW(u)$$

$$+ \int_0^T X\,dZ(u) + \int_0^T \int_t^u \gamma(v)\sigma(v)dv$$

6.2 Suppose interest rate is :

$$dR(t) = \alpha(t, R(t))\,dt + \gamma(t, R(t))\,dW(t)$$

Find

$$d(D(t)X(t)) = dD(t)\,X(t) + dX(t)\,D(t) + dD(t)\,dX(t)$$

$$D(t) = e^{-r(T-t)}$$

$$dD(t) = re^{-r(T-t)}\,dt$$

$$f(a,b) = e^{-rT+rt} = e^{-rT}e^{ra}$$

$$f_a = e^{-rT}re^{ra}$$

$$f_b = 0 = f_{bb}$$

$$X(t) = X$$

$$dD(t)\,X(t) + dX(t)\,D(t) + dD(t)\,dX(t) \Rightarrow$$

$$re^{-r(T-t)}\,dt\,X(t) + dX(t)\,e^{-r(T-t)} + re^{-r(T-t)}\,dt\,dX(t)$$

(ii) Let

$$\text{Sign}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

and

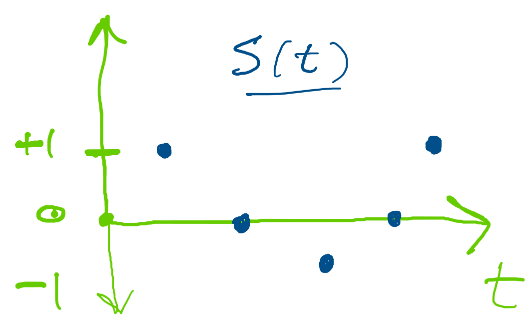
$$S(t) = \text{Sign}\left\{ \left[B(t, R(t), T_2) - B(t, R(t), T_1) \right] f_r(t, R(t), T_1)\, f_r(t, R(t), T_2) \right\}$$

Show There is arbitrge unless

$$B(t, R(t), T_1) = B(t, R(t), T_2).$$

$$B(t, r, T) = -\frac{1}{f_r(t, r, T)} \left[-r f(t, r, T) + f_t(t, r, T) + \frac{1}{2} \gamma^2(t, r) f_{rr} \right]$$

$S(t) = +1$, 0, or -1

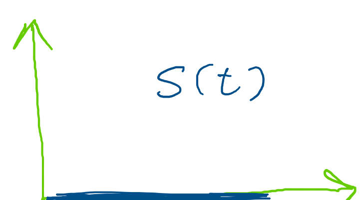


If $B_1 = B_2$, then

$$\underbrace{\left[B(t, R(t), T_2) - B(t, R(t), T_1) \right]}_{=0}$$

Which means $\text{Sign}(x) = 0 \Rightarrow S(t) = 0$.

If the stock price is always 0, then there is no arbitrage.



6.2 (iii) Let a Maturity $T > 0$ be given + consider a portfolio $\Delta(t)$ that invests only in the bond of maturity T, financing this by investing or borrowing at interest rate $R(t)$. Show that the value of the portfolio satisfies

$$d(D(t)X(t)) = \Delta(t)D(t)\Big[-R(t)f(t,R(t),T) + f_t(t,R(t),T) + \alpha(t,R(t))f_r(t,R(t),T) + \tfrac{1}{2}\gamma^2(t,R(t))f_{rr}(t,R(t),T)\Big]dt + D(t)\Delta(t)\gamma(t,R(t))f_r(t,R(t),T)\,dW(t) \qquad (6.9.5)$$

Show if $f_r(t,r,T)=0$, then there is arbitrage.

$$f_t(t,r,T) + \tfrac{1}{2}\gamma^2(t,r)f_{rr}(t,r,T) = rf(t,r,T) \qquad (6.\ \ .6)$$

If $f_r(t,r,T)=0$, then (6.9.3) must hold no matter how we choose $B(t,r,T)$.

Assuming $f_r(t,r,T)=0$, then

$$d(D(t)X(t)) = \Delta(t)D(t)\Big[-R(t)f(t,R(t),T) + f_t(t,R(t),T) + \alpha(t,R(t))\cdot 0 + \tfrac{1}{2}\gamma^2(t,R(t))\cdot 0\Big]dt + D(t)\Delta(t)\gamma(t,R(t))\cdot 0\cdot dW(t) \implies$$

$$d(D(t)X(t)) = \Delta(t)D(t)\Big[-R(t)f(t,R(t),T) + f_t(t,R(t),T)\Big]dt$$

Lemma 5.4.5 says the discounted portfolio $(D(t)X(t))$ must be a martingale under risk-neutral measure $\widetilde{\mathbb{P}}$.

In the equation above, we have a dt term, so it's not a martingale, which violates Lemma 5.4.5. Arbitrage exists.

6.3 Solve the ordinary differential equations (6.5.8) + (6.5.9) to produce solutions $C(t,T)$ and $A(t,T)$ given in (6.5.10) + (6.5.11).

(6.5.8) $$C'(t,T) = b(t)\,C(t,T) - 1$$

(6.5.9) $$A'(t,T) = -a(t)\,C(t,T) + \frac{1}{2}\sigma^2(t)\,C^2(t,T)$$

(i) Show $$\frac{d}{ds}\left[e^{-\int_0^s b(v)\,dv} C(s,T)\right] = -e^{-\int_0^s b(v)\,dv}$$

$$\frac{d}{ds}\left(e^{-\int_0^s b(v)\,dv} C(s,T)\right)$$ (product Rule)

$$\underbrace{\left(e^{-\int_0^s b(v)\,dv}\right)'}_{} \cdot C(s,T) + C(s,T)'\, e^{-\int_0^s b(v)\,dv}$$

$$(e^u)' = u'e^u = -b(s)\,e^{-\int_0^s b(v)\,dv}$$

$$-b(s)\,e^{-\int_0^s b(v)\,dv} C(s,T) + \big(b(s)\,C(s,T) - 1\big) \cdot e^{-\int_0^s b(v)\,dv}$$

$$\underline{-b(s)\,e^{-\int_0^s b(v)\,dv} C(s,T)} + \underline{b(s)\,C(s,T)\,e^{-\int_0^s b(v)\,dv}} - e^{-\int_0^s b(v)\,dv}$$

$$\Longrightarrow -e^{-\int_0^s b(v)\,dv}$$

Thus,

$$\frac{d}{ds}\left[e^{-\int_0^s b(v)\,dv} C(s,T)\right] = -e^{-\int_0^s b(v)\,dv}$$

6.3 (ii) Integrate eq. (i) to obtain (6.5.10).

(6.5.10) $$c(t,T) = \int_t^T e^{-\int_t^s b(v)\,dv}\,ds$$

$$\int_t^T \frac{d}{ds}\left[e^{-\int_0^s b(v)\,dv}\,c(s,T)\right] = \int_t^T -e^{-\int_0^s b(v)\,dv}$$

$$e^{-\int_0^s b(v)\,dv}\underbrace{c(T,T)}_{0} - e^{-\int_0^t b(v)\,dv}\,c(t,T) = \int_t^T -e^{-\int_0^s b(v)\,dv}$$

Terminal condition $(c(T,T) = A(T,T) = 0)$

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$$-e^{-\int_0^t b(v)\,dv}\,c(t,T) = \int_t^T -e^{-\int_0^s b(v)\,dv} \Rightarrow$$

$$c(t,T) = \int_t^T \frac{-e^{-\int_0^s b(v)\,dv}}{-e^{-\int_0^t b(v)\,dv}}\,ds$$

$$\frac{e^{-x}}{e^{+x}} = e^{-x-x}$$

$$c(t,T) = \int_t^T e^{-\int_0^s b(v)\,dv - \int_0^t b(v)\,dv}\,ds \Rightarrow$$

$$c(t,T) = \int_t^T e^{-\int_t^s b(v)\,dv}\,ds$$

6.3 (iii) Replace t by s in (6.5.9), integrate the resulting equation from $s=t$ to $s=T$, use $A(T,T)=0$, and obtain (6.5.11)

(6.5.9) $$\frac{d}{ds}A(s,T) = -a(s)C(s,T) + \frac{1}{2}\sigma^2(s)C^2(s,T)$$

(6.5.11) $$A(t,T) = \int_t^T \left(a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T)\right)ds$$

$$\Rightarrow \int_t^T A(s,T) = \int_t^T -a(s)C(s,T) + \frac{1}{2}\sigma^2(s)C^2(s,T)$$

$$\underbrace{A(T,T)}_{0} - A(t,T) = \int_t^T -\left(a(s)C(s,T) + \frac{1}{2}\sigma^2(s)C^2(s,T)\right) ds$$

$$-A(t,T) = \int_t^T -\left(a(s)C(s,T) + \frac{1}{2}\sigma^2(s)C^2(s,T)\right) ds$$

$$\Rightarrow A(t,T) = \int_t^T \left(a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T)\right) ds$$

11.1 Let $M(t)$ be the compensated Poisson process Thm. 11.2.4

(i) Show $M^2(t)$ is a sub martingale.

We must show $E\left(M^2(t) \mid \mathcal{F}(s)\right) \geq M^2(s)$

Note: $M(t) = N(t) - \lambda t$

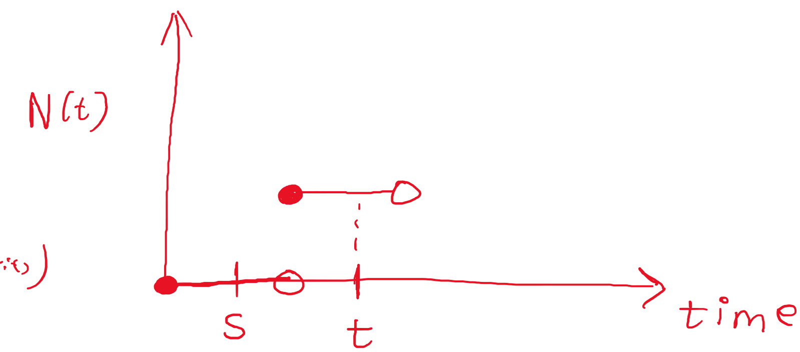
$$N(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Proof

$$E\left(M^2(t) \mid \mathcal{F}(s)\right) = E\left((N(t) - \lambda t)^2 \mid \mathcal{F}(s)\right) \quad \boxed{\geq}$$

if $t \geq s$

then $N(t) \geq N(s)$ (Monotonicity)



$$E\Big(\underbrace{(N(s) - \lambda s)}_{\text{This is } M(s)}{}^2 \,\Big|\, \mathcal{F}(s)\Big) = E\Big(\underbrace{M^2(s)}_{} \,\Big|\, \mathcal{F}(s)\Big) = M^2(s)$$

This is known with information up to time s.

Sub Martingale since $E\left(M^2(t) \mid \mathcal{F}(s)\right) \geq M^2(s)$

11.1 (ii) Show that $M^2(t) - \lambda t$ is martingale.

We must show $E\left(M^2(t) - \lambda t \mid \mathcal{F}(s)\right) = M^2(s) - \lambda s$

$$E\left(\underbrace{M^2(t) - \lambda t}_{\text{Apply Linearity of expected Value}} \mid \mathcal{F}(s)\right) \Rightarrow$$

$$E\left(M^2(t) \mid \mathcal{F}(s)\right) - E\left(\underbrace{\lambda t \mid \mathcal{F}(s)}_{\substack{\lambda = \text{constant} \\ t = \text{apply the filtration.}}}\right) \Rightarrow$$

$$E\left(\underbrace{M^2(t) \mid \mathcal{F}(s)}\right) - \lambda s$$

$$E\left(N^2(t) - 2N(t)\lambda t + \lambda^2 t^2 \mid \mathcal{F}(s)\right) - \lambda s \Rightarrow$$

$$N^2(s) - 2N(s)\lambda s + \lambda^2 s^2 - \lambda s$$

$$\underbrace{(N(s) - \lambda s)^2}_{M^2(s)} - \lambda s$$

$$M^2(s) - \lambda s$$

Martingale since

$$E\left(M^2(t) - \lambda t \mid \mathcal{F}(s)\right) = M^2(s) - \lambda s$$

11.2 Prove $\mathbb{P}\{N(s+t)=k \mid N(s)=k\} = 1-\lambda t$

$$\mathbb{P}\{N(s+t)=k \mid N(s)=k\} = \frac{\mathbb{P}(N(s+t)=k)}{\mathbb{P}(N(s)=k)} \Rightarrow$$

$$\frac{e^{-\lambda(s+t)}}{e^{-\lambda(s)}} = \frac{e^{-\lambda s-\lambda t}}{e^{-\lambda s}} = \frac{e^{-\lambda s}\cdot e^{-\lambda t}}{e^{-\lambda s}} =$$

$$\frac{\cancel{e^{-\lambda s}}}{\cancel{e^{-\lambda s}}} \cdot \frac{e^{-\lambda t}}{1} = e^{-\lambda t} \Rightarrow \ln\left(e^{-\lambda t}\right) \Rightarrow$$

$$-\lambda t \cancel{\ln(e)} \Rightarrow 1-\lambda t$$

Canvas problem 1

Let $V(t) = \tilde{E}\left(D(t)V(T) \mid \mathcal{F}(t)\right)$

where $D(t) = e^{-r(T-t)}$

$V(T) = S^2(T)$, so

$$V(t) = \tilde{E}\left(e^{-r(T-t)} S^2(T) \mid \mathcal{F}(t)\right) \Rightarrow$$

$$e^{-r(T-t)} \tilde{E}\left(S^2(T) \mid \mathcal{F}(t)\right) = e^{-r(T-t)} \tilde{E}\left(S^2(T) \cdot \frac{S^2(t)}{S^2(t)} \middle| \mathcal{F}\right)$$

$$\Rightarrow e^{-r(T-t)} \tilde{E}\left(\frac{\cancel{S^2(0)}\, e^{2\sigma W(T) + 2(r - \frac{1}{2}\sigma^2)T}}{\cancel{S^2(0)}\, e^{2\sigma W(t) + 2(r-\frac{1}{2}\sigma^2)t}} \cdot S^2(t) \middle| \mathcal{F}(t)\right) \Rightarrow$$

$$e^{-r(T-t)} \tilde{E}\left(e^{2\sigma W(T) + 2(r - \frac{1}{2}\sigma^2)T - 2\sigma W(t) - 2(r - \frac{1}{2}\sigma^2)t} \cdot S^2(t) \mid \mathcal{F}(t)\right)$$

$$e^{-r(T-t)} \tilde{E}\left(e^{\underbrace{2\sigma(W(T) - W(t))}_{\text{ind.}} + \underbrace{2(r - \frac{1}{2}\sigma^2)(T-t)}_{\text{measurable}}} \cdot S^2(t) \mid \cancel{\mathcal{F}(t)}\right)$$

$$e^{-r(T-t)} e^{2(r - \frac{1}{2}\sigma^2)(T-t)} \cdot S^2(t) \cdot \tilde{E}\left[\underbrace{e^{\sigma(W(T) - W(t))}}_{\text{MGF}}\right]$$

$$e^{\frac{1}{2}\sigma^2 2^2 (T-t)}$$

$$V(t) = e^{-r(T-t) + (r-\frac{1}{2}\sigma^2)(T-t) + \frac{1}{2}\sigma^2(T-t)} \cdot S^2(t)$$

$$V(t) = e^{-rT + rt + 2rT - 2rt - \sigma^2 T + \sigma^2 t + 2\sigma^2 T - 2\sigma^2 t}$$

$$V(t) = e^{rT - rt - \sigma^2 T + \sigma^2 t + 2\sigma^2 T - 2\sigma^2 t} S^2(t)$$

$$V(t) = e^{r(T-t) + \sigma^2 T - \sigma^2 t} S^2(t)$$

$$V(t) = e^{r(T-t) + \sigma^2(T-t)} S^2(t)$$

Find $\Delta(t)$

Two methods.

(1) $\dfrac{\partial V(t, S(t))}{\partial S(t)}$

(2) $d\left(e^{-rt} V(t)\right)$

use method (1)

(1) $$\frac{\partial}{\partial S}\left(e^{r(T-t) + \sigma^2(T-t)} S^2(t)\right) =$$

$$e^{r(T-t) + \sigma^2(T-t)} \frac{\partial}{\partial S} S^2(t) =$$

$$\boxed{\Delta(t) = e^{r(T-t) + \sigma^2(T-t)} 2 \cdot S(t)}$$

Canvas problem 2

Let $R(t) = r + \sigma \tilde{W}(t)$ determine value of ZCB, $V(T) = 1$

$$V(t) = \tilde{E}\left(e^{-\int_t^T R(u)\,du} V(T) \,\middle|\, \mathcal{F}(t)\right) =$$

$$\tilde{E}\left(e^{-\int_t^T r + \sigma \tilde{W}(u)\,du} \,\middle|\, \mathcal{F}(t)\right) =$$

$$\tilde{E}\left(e^{-\int_t^T r\,du + \int_t^T \sigma \tilde{W}(u)\,du} \,\middle|\, \mathcal{F}(t)\right) =$$

$$\tilde{E}\left(\underbrace{e^{-r(T-t)}}_{\text{ind}} {}^{+ \sigma \int_t^T \tilde{W}(u)\,du} \,\middle|\, \mathcal{F}(t)\right) =$$

$$e^{-r(T-t)} \tilde{E}\left(e^{\sigma \int_t^T \tilde{W}(u)\,du} \,\middle|\, \mathcal{F}(t)\right)$$

Apply filtration.

$$e^{-r(T-t)} \tilde{E}\left(e^{\sigma \underbrace{\int_t^t \tilde{W}(u)\,du}_{0}}\right) = \boxed{e^{-r(T-t)} = V(t)}$$