Week 5 – Practice Problem (Strictly for Fun)

5) An Ornstein-Uhlenbeck process follows the form:

$$dX(t) = -\theta X(t)\,dt + \sigma\,dW(t)$$

Find the closed form solution for $X^2(t)$.

hint for $X(t)$ was $d(e^{\theta t} X(t))$

Take $d(e^{2\theta t} X^2(t))$

$$d\left(e^{2\theta t} X^2(t)\right)$$

$$f(t, X(t)) = e^{2\theta t} X$$

$$f(a, b) = e^{2\theta a} b^2$$

$$f_a(a, b) = 2\theta e^{2\theta a} b^2$$

$$f_b = e^{2\theta a}\, 2b$$

$$f_{bb} = e^{2\theta a}\, 2$$

$$df(t, X(t)) = f_a\,dt + f_b\,(dX(t)) + \frac{1}{2} f_{bb}\,(dX(t))^2$$

$$d\left(e^{2\theta t} X^2(t)\right) = 2\theta e^{2\theta t} X^2(t)\,dt + e^{2\theta t}\, 2X(t) + \frac{1}{\cancel{2}} e^{2\theta t} \cdot \cancel{2}\,(dX(t))^2$$

$$d\left(e^{2\theta t}X^2(t)\right) = 2\theta e^{2\theta t}X^2(t)\,dt +$$

$$e^{2\theta t}\,2X(t)\,dX(t) \; +$$

we know $dX(t)$!

$$e^{2\theta t}\,(dX(t))^2$$

$\sigma^2 dt$

$$\Rightarrow 2\theta e^{2\theta t}X^2(t)\,dt$$

$$+\, e^{2\theta t}\,2X(t)\left[-\theta X(t)\,dt + \sigma\,dW(t)\right]$$

$$+\, e^{2\theta t}\sigma^2 dt \quad \Rightarrow$$

$$\underline{2\theta e^{2\theta t}X^2(t)\,dt}$$

$$\underline{-2\theta e^{2\theta t}X^2(t)\,dt} + e^{2\theta t}\,2X(t)\,\sigma\,dW(t)$$

$$+\, e^{2\theta t}\sigma^2 dt \quad \Rightarrow \qquad \text{so,}$$

$$d\left(e^{2\theta t}X^2(t)\right) = e^{2\theta t}\,2X(t)\,\sigma\,dW(t) + e^{2\theta t}\sigma^2 dt$$

Integrate

$$\int_0^t d\left(e^{2\theta u} X^2(u)\right) = \int_0^t e^{2\theta u}\, 2X(u)\sigma\, dW(u)$$

$$+ \int_0^t e^{2\theta u} \sigma^2\, du \quad \Longrightarrow$$

$$e^{2\theta t} X^2(t) - \underbrace{e^{2\theta 0}}_{1} X^2(0) = \int_0^t e^{2\theta u}\, 2X(u)\sigma\, dW(u)$$

$$+ \int_0^t e^{2\theta u} \sigma^2\, du \quad \Longrightarrow$$

$$X^2(t) = e^{-2\theta t}\Big(X^2(0) + \underbrace{\int_0^t e^{2\theta u}\, 2X(u)\sigma\, dW(u)}_{\substack{\text{Itô} \\ \text{Integral} \\ [\text{Martingale, } E(I(t))=0]}} + \underbrace{\int_0^t e^{2\theta u} \sigma^2\, du}_{\substack{\text{Reimann} \\ \text{Integral}}} \Big)$$