FE-610 Stochastic Calculus for Financial Engineers

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Assignment #4

5.1, 5.2, 5.6, 5.7, 5.10

5.1

a.) Let $f(x) = S(0)\, e^{x}$ and

$$X(t) = \int_0^t \sigma(s)\, dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right) ds$$

Find $df(X(t))$

$$f(t, X(t)) = S(0)\, e^{X(t)}$$

$$f(a, b) = S(0)\, e^{b}$$

$$f_a = 0$$

$$f_b = S(0)\, e^{b} = f_{bb}$$

$$df(t, X(t)) = S(0)\, e^{X(t)}\, dX(t) + \frac{1}{2} S(0)\, e^{X(t)} (dX(t))^2$$

$$df(t, X(t)) = \underbrace{S(0)\, e^{X(t)}}_{f(X(t))} \underbrace{\left[dX(t) + \frac{1}{2} dX(t)^2 \right]}_{*}$$

What is $dX(t)$ + $(dX(t))$?

$$X(t) = \int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right) ds$$

$$dX(t) = \sigma(t)\,dW(t) + \left(\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t)\right) dt$$

$$(dX(t))^2 = \left(\boxed{\sim}\,dW(t) + \boxed{\sim}\,dt\right)\left(\boxed{\sim}\,dW(t) + \boxed{\sim}\,dt\right) \quad \text{FOIL}$$

$$= \sigma^2(t)\,dt$$

$$\circledast \left(dX(t) + \frac{1}{2}\,dX(t)^2\right)$$

$$\sigma(t)\,dW(t) + \alpha(t)\,dt - R(t)\,dt - \underline{\frac{1}{2}\sigma^2(t)}\,dt$$

$$(dX(t))^2 \longrightarrow + \underline{\frac{1}{2}\sigma^2(t)\,dt} \Rightarrow$$

Cancels out

$$\sigma(t)\,dW(t) + \left(\alpha(t) - R(t)\right) dt \qquad \text{(put it together)}$$

$$df(X(t)) = f(X(t)) \cdot \left[\sigma(t)\,dW(t) + \left(\alpha(\quad - R(t)\right) dt\right]$$

5.1 (ii) Use Itô product Rule + Simplify

$$d(D(t)S(t)) = S(t)\,dD(t) + D(t)\,dS(t) + dD(t)\,dS(t)$$

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t)$$

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \tfrac{1}{2}\sigma^2(s)\right)ds\right\}$$

$$dD(t) = -R(t)D(t)\,dt$$

$$D(t) = e^{-\int_0^t R(s)\,ds}$$

$$S(t)\,dD(t) + D(t)\,dS(t) + \underbrace{dD(t)\,dS(t)}_{0} \Rightarrow$$

$$d(D(t)S(t)) = -S(t)R(t)D(t)\,dt + D(t)\alpha(t)S(t)\,dt + D(t)\sigma(t)S(t)\,dW(t)$$

$$d(D(t)S(t)) = S(t)D(t)\left[(-R(t) + \alpha(t))\,dt + \sigma\,dW(t)\right]$$

5.2 Show eq. (5.2.30) can be written as

$$D(t)Z(t)V(t) = E\left[D(T)Z(T)V(T) \mid \mathcal{F}(t)\right]$$

(5.2.30) = $D(t)V(t) = \tilde{E}\left[D(T)V(T) \mid \mathcal{F}(t)\right]$

(5.2.11) = $Z(t) = \exp\left\{-\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du\right\}$

(5.2.21) = $\Theta(t) = \dfrac{\alpha(t) - R(t)}{\sigma(t)}$

(5.2.30)

$$D(t)V(t) = \tilde{E}\left[D(T)V(T) \mid \mathcal{F}(t)\right] \Rightarrow$$

$$D(t)V(t) = \tilde{E}\left[D(T)V(T)\frac{Z(T)}{Z(T)} \mid \mathcal{F}(t)\right] \Rightarrow$$

$$D(t)V(t) = \frac{1}{Z(t)}\tilde{E}\left[D(T)V(T)Z(T) \mid \mathcal{F}(t)\right] \Rightarrow$$

$$D(t)V(t)Z(t) = \tilde{E}\left[D(T)V(T)Z(T) \mid \mathcal{F}(t)\right]$$

5.6 use 2-D Levy (4.6.5)

to prove 2-D Girsanov Thm. (5.4.1)

$$Z(t) = \exp\left\{ -\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \underbrace{\|\Theta\|^2}_{\text{Euclidean norm}}\,du \right\}$$

Euclidean norm

$$\sum_{j=1}^{d} \Theta_j^2(u)$$

Also,

$\Theta(t) = (\Theta_1(t), \ldots, \Theta_d(t))$ is a d dim. adapted process.

Thm. (5.2.1a) $\tilde{W}(t) = W(t) + \int_0^t \Theta(u)\,du$

$$Z(t) = \exp\left\{ -\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du \right\}$$

$*$ Set $Z = Z(t)$. Show $W(t)$ is BM under $\tilde{\mathbb{P}}$, then $\hat{W}(t)$ is also Brownian motion

5.6 cont.

Use Lévy

① $\tilde{W}(0)=0$ $\boxed{\checkmark}$

$$\tilde{W}(t) = W(t) + \int_0^t \theta(u)\,du$$

set $t=0$

$$\tilde{W}(0) = W(0) + \underbrace{\int_0^0 \theta(u)\,du}_{0} = 0$$

② $\tilde{W}(t)$ depends on $W(t)$ which is cont., so $\tilde{W}(t)$ must also be cont.

③ $$[\tilde{W}_i, \tilde{W}_i](t) = \int_0^t (d\tilde{W}_i(t))^2$$

$$\tilde{W}(t) = W(t) + \int_0^t \theta(u)\,du \Rightarrow$$

$$d\tilde{W}(t) = dW(t) + \theta(t)\,dt$$ (what is $d\tilde{W}(t)^2$?)

$$\underbrace{(dW(t)+\theta(t)dt)(dW(t)+\theta(t)dt)}_{\underline{\underline{dt}}}$$

$$\int_0^t du = t - 0 = t$$

The QV $= t$ $\boxed{\checkmark}$

④ $\tilde{W}_1$ is independent from $\tilde{W}_2$
if $[\tilde{W}_1, \tilde{W}_2](t) = 0$

$$d\tilde{W}_1(t)\, dW_2(t) = 0$$

⑤ Show $\tilde{W}(t)$ is a martingale.

Lemma 5.2.2

$$\tilde{E}[Y \mid \mathcal{F}(s)] = \frac{1}{Z(s)} E[Y Z(t) \mid \mathcal{F}(s)]$$

$$\circledast \quad \tilde{E}[\tilde{W}(t) \mid \mathcal{F}(s)] = \frac{1}{Z(s)} E[\hat{W}(t) Z(t) \mid \mathcal{F}(s)]$$

$$= \tilde{W}(s)$$

$$\frac{1}{Z(s)} \tilde{E}[\tilde{W}(t) Z(t) \mid \mathcal{F}(s)] = \frac{1}{Z(s)} \cdot \tilde{E}[\tilde{W}(t) \mid \mathcal{F}(s)] \cdot$$

$$\underbrace{\tilde{E}[Z(t) \mid \mathcal{F}(s)]}_{Z(s)}$$

$$\Rightarrow \frac{Z(s)}{Z(s)} \cdot \tilde{E}[\tilde{W}(t) \mid \mathcal{F}(s)] = \tilde{W}(s)$$

By the Girsanov Thm. $\tilde{W}(t)$ is a Brownian Motion.

5.7 (i)

(arbitrage Thm. 5.4.23)

$\mathbb{P}(X(T) \geq 0) = 1$, $\mathbb{P}(X(T) > 0) > 0$

+ $X(0) = 0$

Show if $X_2(0)$ is positive $(X_2(0) > 0)$

then

$$\mathbb{P}\left\{X_2(T) \geq \frac{X_2(0)}{D(T)}\right\} = 1, \quad \mathbb{P}\left\{X_2(T) > \frac{X_2(0)}{D(T)}\right\} > 0$$

Let $D(T) = e^{-r(T-t)}$

$$\mathbb{P}\left\{X_2(T) \geq \underbrace{X_2(0)}_{\text{This term is positive}} \cdot \underbrace{e^{+r(T-t)}}_{\text{This term is positive + Constant.}}\right\}$$

(i.e) $\$5 \cdot e^{.02(1 \text{ year})} = \$5.10$

$X_2(T)$ is $\geq X_2(0)\, e^{r(T-t)}$ So $\mathbb{P} = 1$

5.7 (ii)

Show that if a multidimensional Market model has a portfolio value process $X_2(T)$ s.t. $X_2(0)$ is positive and (5.4.24) holds, then the model has portfolio value process $X_1(0)$ s.t. $X_1(0)=0$ and (5.4.23) holds.

(5.4.24)

$$\mathbb{P}\left\{X(T) \geq \frac{X(0)}{D(T)}\right\} = 1, \quad \mathbb{P}\left\{X(T) > \frac{X(0)}{D(T)}\right\} > 0$$

(5.4.23)

$$\mathbb{P}(X(T) \geq 0) = 1, \quad \mathbb{P}(X(T) > 0) > 0$$

If Thm. (5.4.24) holds. then

$$\mathbb{P}\left\{X(T) \geq \frac{X(0)}{D(T)}\right\} = 1$$

Since $X_2(0)$ is positive then $\frac{X_2(0)}{D(t)} > 0$

So $\mathbb{P}\{X(T) \geq 0\} = 1$, which is exactly Thm. 5.2.23 !

5.10 (Chooser option)

(i) Show @ t_0 the value of chooser option is:

$c(t_0) + (-F(t_0))^+$

Let the value $= \max\{C_0, P_0\}$,

option value $= \max\{C_0, \underbrace{C_0 - F_0}\}$

$C_t - P = F$,

so $P = C - F$

Option Value $= C(t_0) + (-F_0)^+$

(ii) Show the value @ $t_0 \Rightarrow$

$\text{Value}_{t_0} = C(T) + P(t_0)$

$K = e^{-r(T-t_0)} K$

If $T = t_0$, then

$K = e^{-r(t_0 - t_0)} K = e^0 K = K$

$\Rightarrow$ $\text{Value}(t_0) = C(T) + P(t_0)$