FE – 621 Computational Method in Finance (Assignment 3)

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Introduction

The purpose of this analysis is to analyze different hedging strategies. The first section begins with a common strategy called Delta Hedging. The additional sections of this paper alter this strategy and outline the results.

1. Delta Hedge

The purpose of this exercise is to implement a delta hedging strategy. The hedge begins by selling a call option with a fixed strike price K = $50, and initial stock price $S_0 = \$50$. Additionally, there are other inputs to our Black-Scholes closed form solution, such as the risk-free rate, r = 0%, dividend rate d = 0%, volatility $\sigma = 30\%$, and time to maturity, T = .25 years. Based upon these inputs the price of the call is $2.99.

Next, the future stock prices are simulated using Geometric Brownian Motion. The equation used to simulate a path is listed below which comes from the textbook by Hull on page 471 (equation 21.16).

$$S(t + \Delta t) = S(t)e^{\left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}}$$

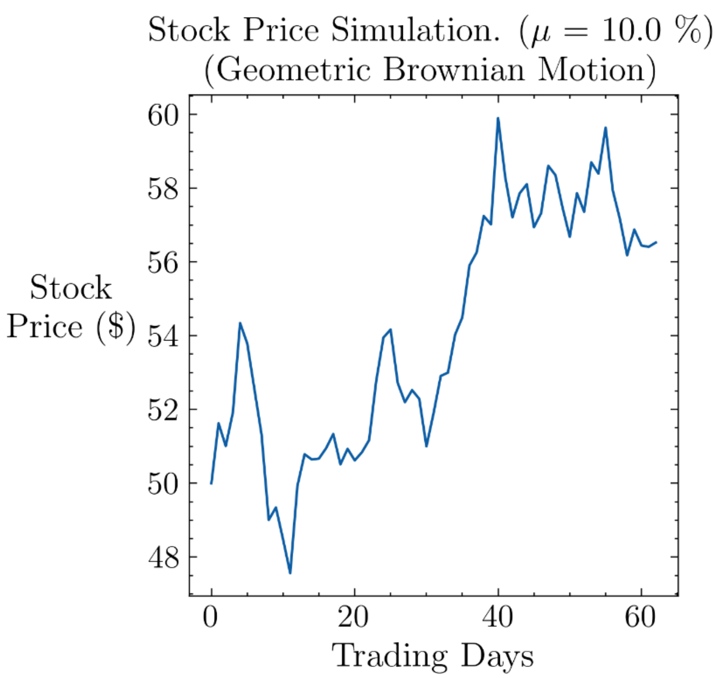
This equation calcaulates the future stock price, $S(t + \Delta t)$, based upon the current price, S(t), and other factors. Namely, the drift rate, $\hat{\mu}$, and time interval, Δt. ε is a random sample from a standard normal distribution $\sim N(0,1)$.

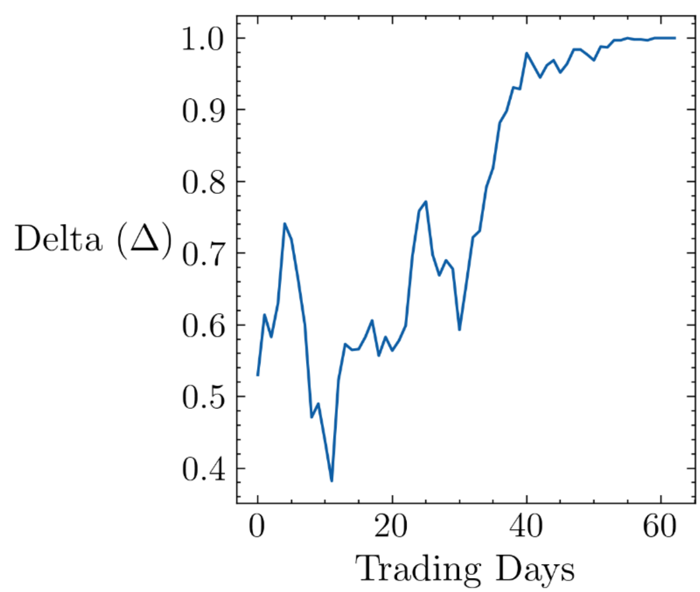
Once future stock prices are calculated, the option greek delta (Δ) is calculated at every time step. This determines the total number of shares to purchase in the first time step. In future time steps the number of shares to purchase is $(\Delta_{n+1} - \Delta_n)$. This process is repeated untill the maturity date. In order to purchase these shares, money is borrowed at a interest rate of r = 5%.

The table below is a sample result of a simulated hedge.

Trading Days	Stock Price	Delta	Shares to Purchase	Cumulative Shares	Borrowed Value
1	$ 50.00	0.53	.53	.53	$ (26.50)
2	$ 51.63	0.614	.08	.64	$ (33.04)
⋮	⋮	⋮	⋮	⋮	⋮
63	$56.53	1	0	1	$ (33)

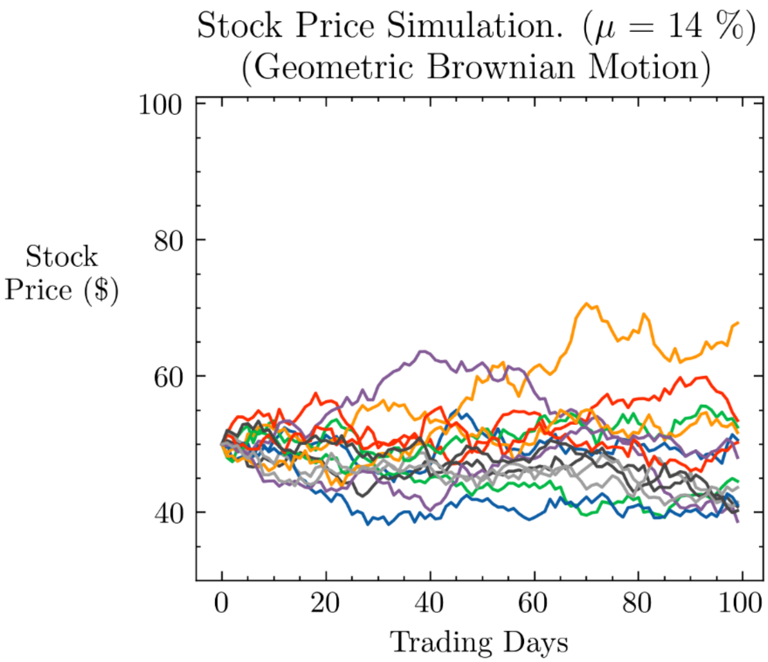
Below are two plots. The first shows the simulated stock price for 63 days. The second plot shows the Δ for every time step. There is a positive relationship between stock price and Δ. It's interesting how Δ approaches 1.0 at the end of maturity.

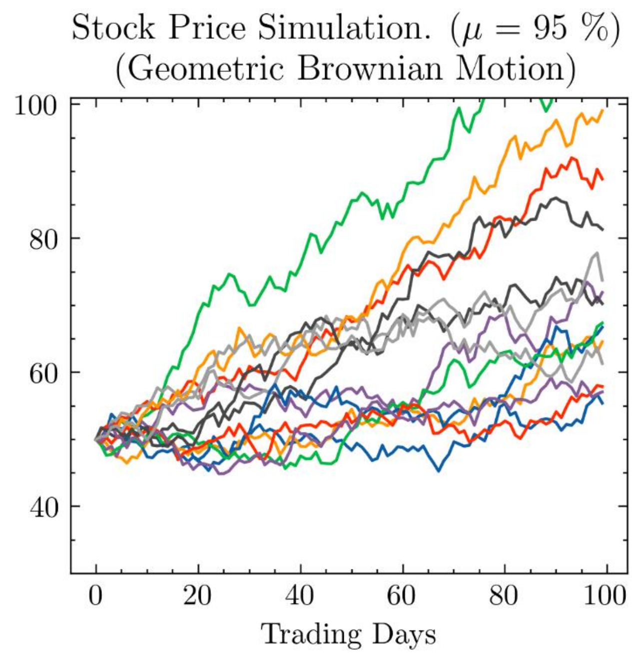




At the end of maturity, t = T we calcualte the total profit and loss (PnL). The position is exited by selling all shares and paying the client if nessecary (-max($S_t - K, 0$)).

What if we simulate many paths? Below is an example with 15 different paths. The difference between the two is μ, which is the "drift" term. As drift increases, so does the possibility the stock price will rise. This relation is logical due to the equation on page one, since there is a positive relationship between μ and $S(t + \Delta t)$.





Below is a histogram of PnL by simulating 100,000 paths. Each path is assocaited with a specific hedging scenario and one final PnL, so the sample size for each histogram is 100,000. The results show daily re-heding has less possibility of losing money.

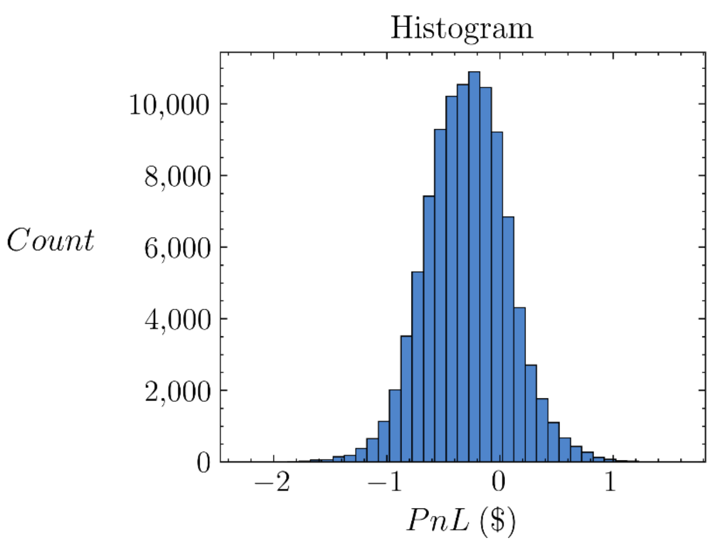


Figure: *Results based upon* ***daily*** *re-hedging and* $\mu =$ 10%

Mean ≈ -$.29 , Median ≈ -$.29 , Standard Deviation ≈ $.36

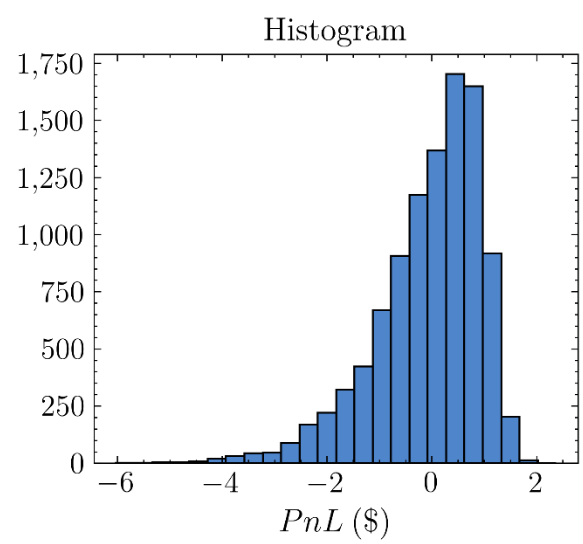


Figure: *Results based upon* ***weekly*** *re-hedging and* $\mu =$ 10%

Mean ≈ -$0.08, Median ≈ -$0.14, Standard Deviation ≈ $1

Special note, if the interest rate changes to 0%, then the distribution becomes centered around $0.

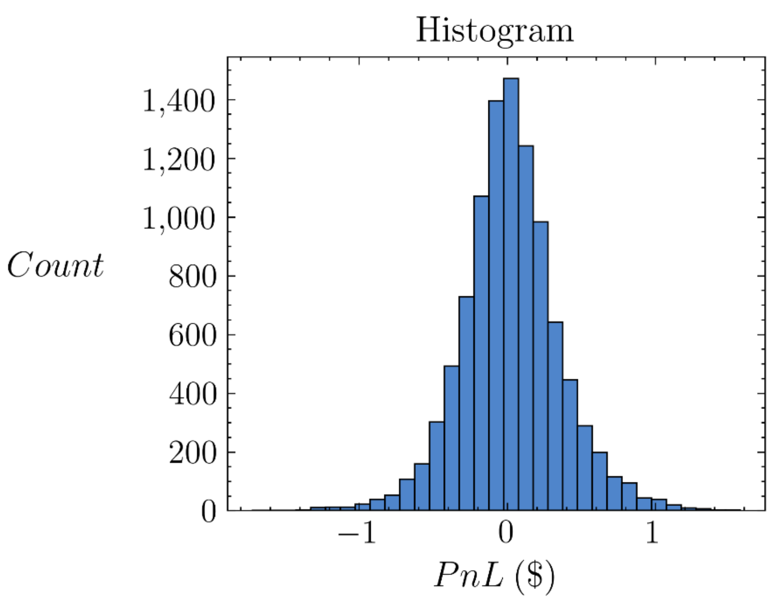


Figure: *Results based upon **daily** re-hedging and* $\mu =$ 10%

Mean ≈ $.25 , Median ≈ $.015 , Standard Deviation ≈ $.33

Skewness = .12, sample size (n) = 10,000

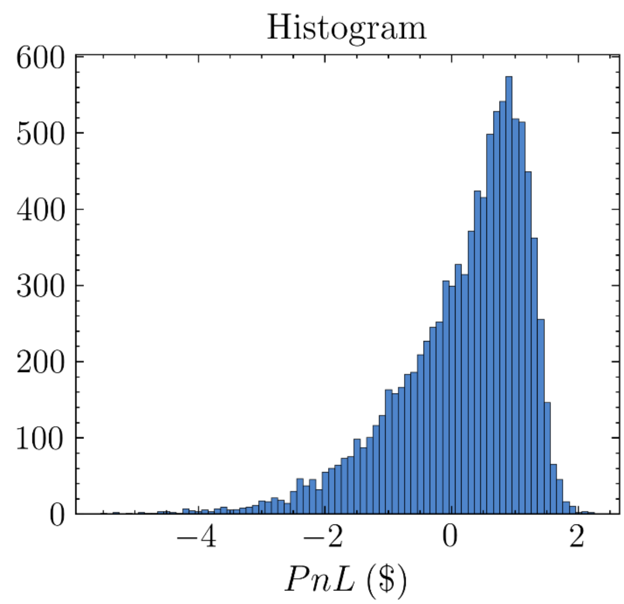
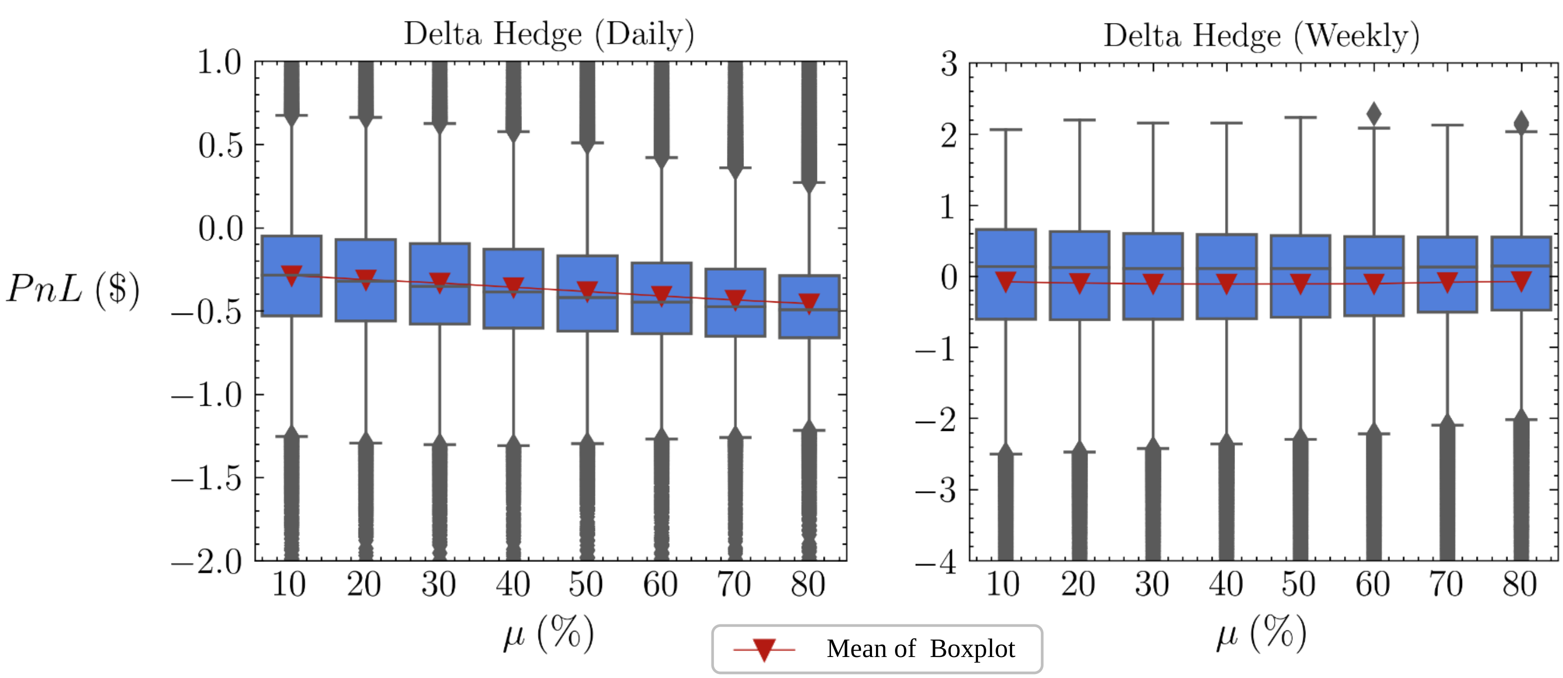


Figure: *Results based upon **weekly** re-hedging and* $\mu =$ 10%

Mean ≈ $0.19, Median ≈ $0.45, Standard Deviation ≈ $1.02

Skewness = -1.23, sample size (n) = 10,000

The previous results are based upon a specific drift factor. The boxplots below show the distribution of PnL for discrete values of μ in 10% increments (i.e. $\mu_i \epsilon \{10\ \%,\ 20\ \%, \ldots, 80\ \%\}$). The first plot is daily re-hedging and the second is weekly. (r = 5% for the boxplots)



Lastly, a regression model is generated with varying re-hedging intervals (Δt) with respect to the error (PnL). Every dot in the scatter plot represents the median PnL for 100,000 simulations. As $\Delta t \rightarrow 0$, the error approaches the y-intercept -.23.

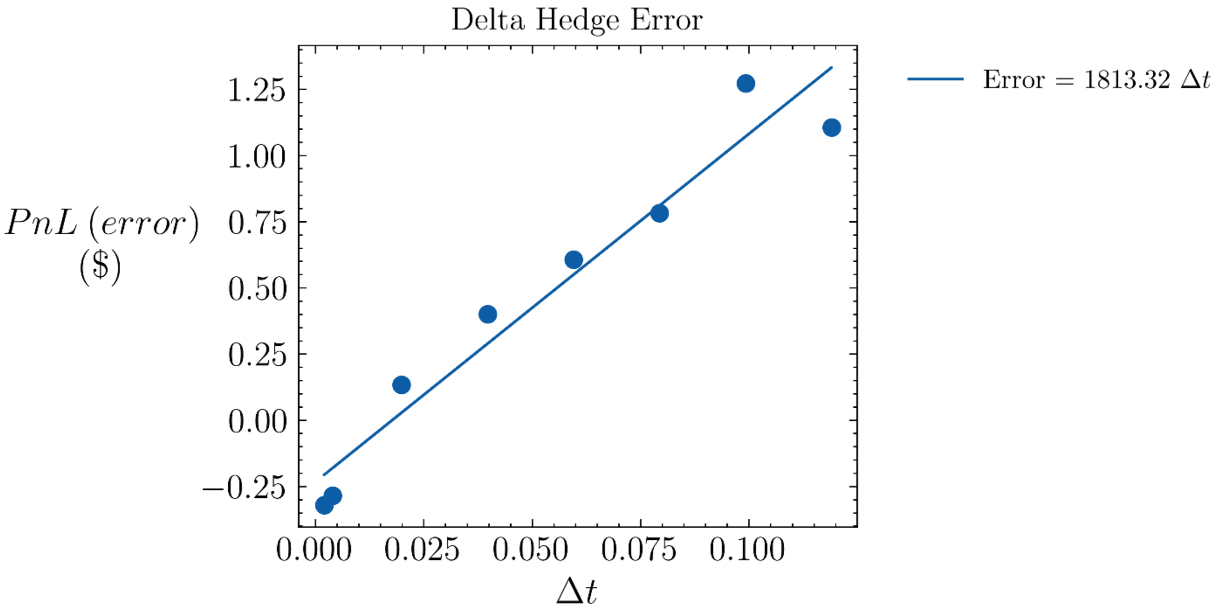


Figure: *Scatter Plot with Simple Linear Regression*

Linear Regression Model Output

Call:
lm(formula = df$y ~ df$x)

Residuals:

Min	1Q	Median	3Q	Max
-0.22682	-0.10779	0.01418	0.10697	0.20161

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-0.23177	0.08915	-2.600	0.0407 *
df$x	13.1420	1.33186	9.867	6.25e-05 ***

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.1549 on 6 degrees of freedom
Multiple R-squared: 0.942, Adjusted R-squared: 0.9323
F-statistic: 97.37 on 1 and 6 DF, p-value: 6.249e-05

2. Stop Loss

This problem revises the original delta hedge strategy. Begin by selling one call option at the price ($C_0 = (S_0 - Ke^{-rT})^+$). At every point in time, we determine if the stock price S_T is less then or greater than the Strike price (K).

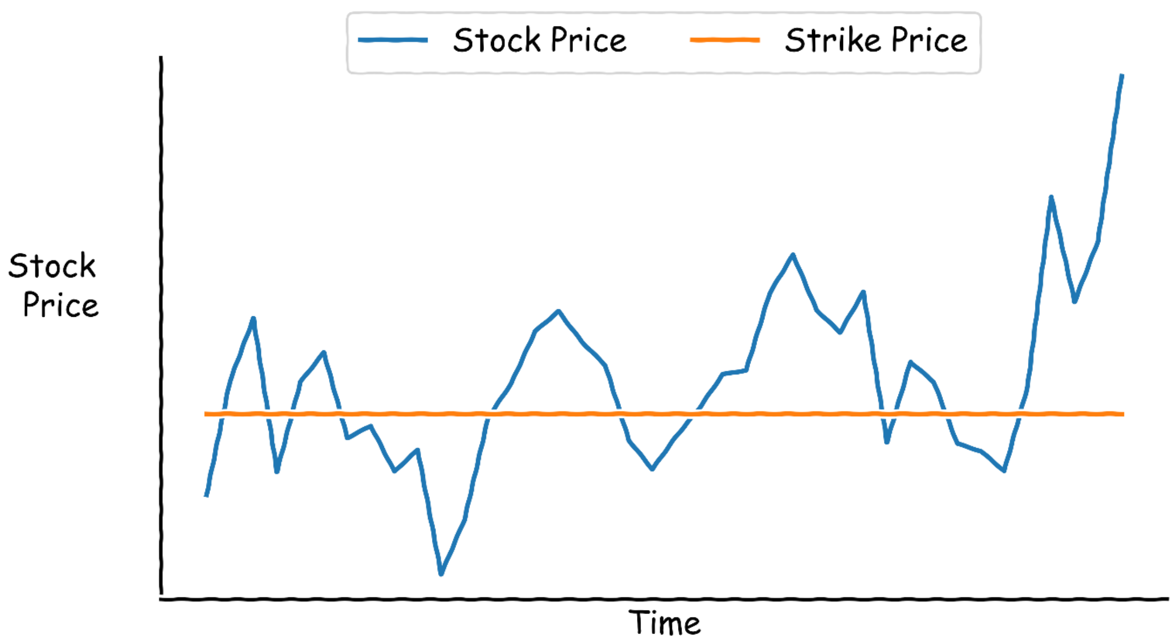
If $S_T > K,$ then:

-Purchase one share

If $S_T \leq K,$ then:

- Sell all shares (if applicable)

In either case, the investor will own 1 share or 0 shares. The graph below is an abstract representation of this process. Every intersection indicates a buy or sell point and impacts the profit of this strategy.



This strategy was implemented using C++ by generating 1,000,000 simulations for daily $\left(\Delta t = \frac{1}{252}\right)$ and weekly $\left(\Delta t = \frac{5}{252}\right)$ re-hedging intervals. The histograms below show the resulting Profit and Loss (PnL).

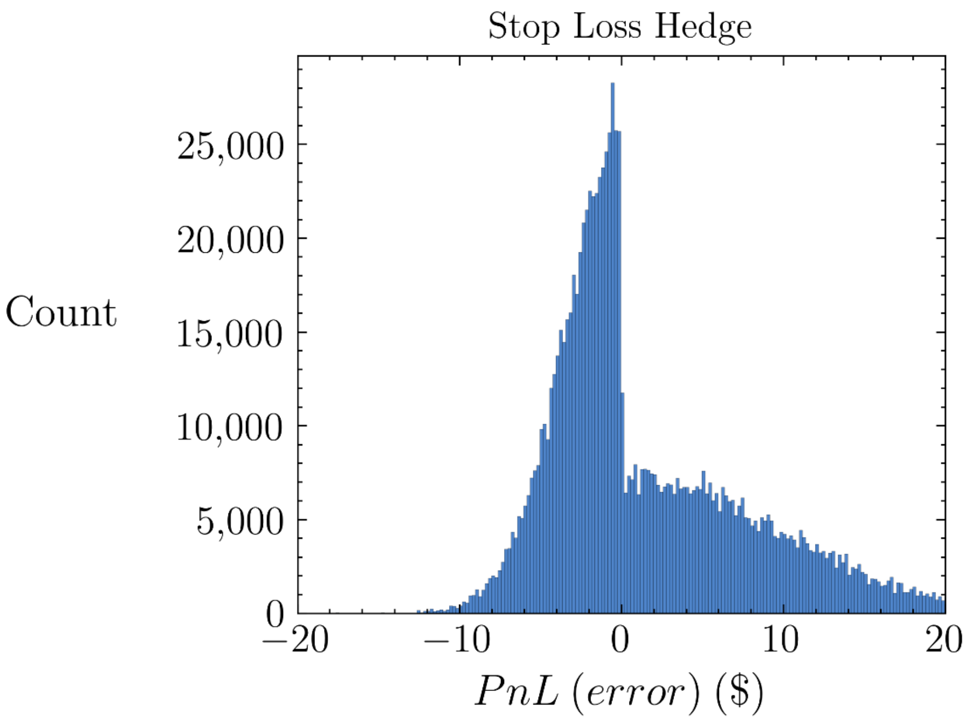


Figure: Results are based upon ***daily*** re-hedging *and* $\mu =$ 10%

Mean = 1.9, Median = -0.44,

Standard Deviation = 6.7

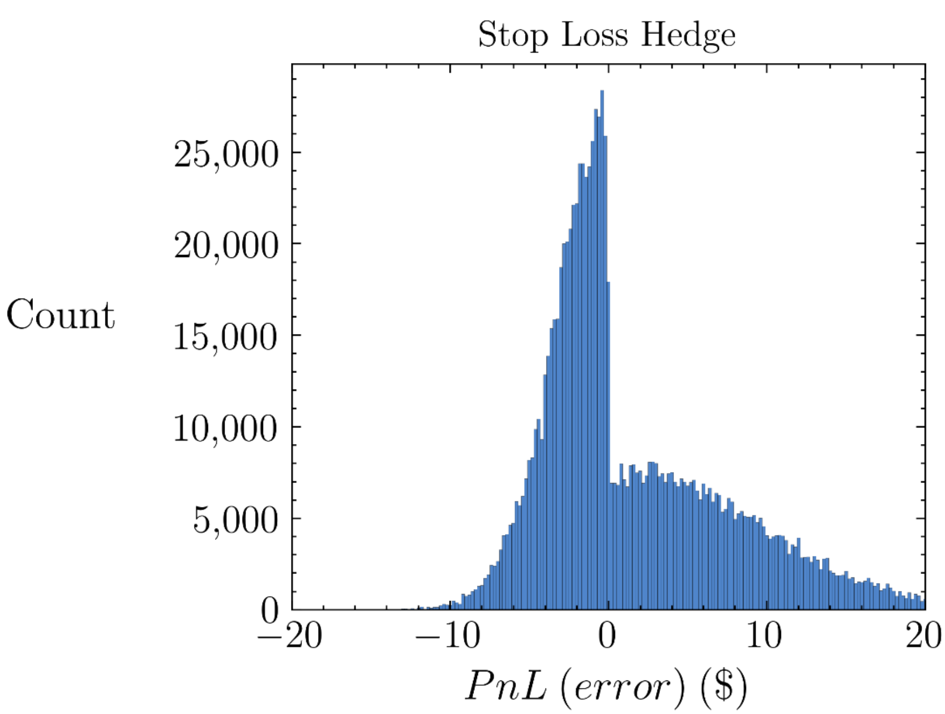


Figure: Results are based upon ***weekly*** re-hedging *and* $\mu =$ 10%

Mean = 1.7, Median = -0.4,

Standard Deviation = 6.2

3. Effect on Non-Constant Volatility

Consider the situation where variance is ***not*** constant. Using the Euler Discretization scheme.

$$\hat{S}_{i+1} = \hat{S}_i(1 + \mu\Delta t + \alpha\hat{S}_i^{\beta}\sqrt{\Delta t}Z_{i+1})$$

Let:

$$S_0 = \$\,50 \qquad \mu = 10\%$$

$$K = \$50 \qquad r = 0\,\%$$

$$T = .25 \qquad \beta = -0.8$$

Modify your simulation in Problem 1 but keep ***hedging the option with the Black-Scholes delta at constant volatility 30%.*** Let $\Delta t = \frac{1}{252}$

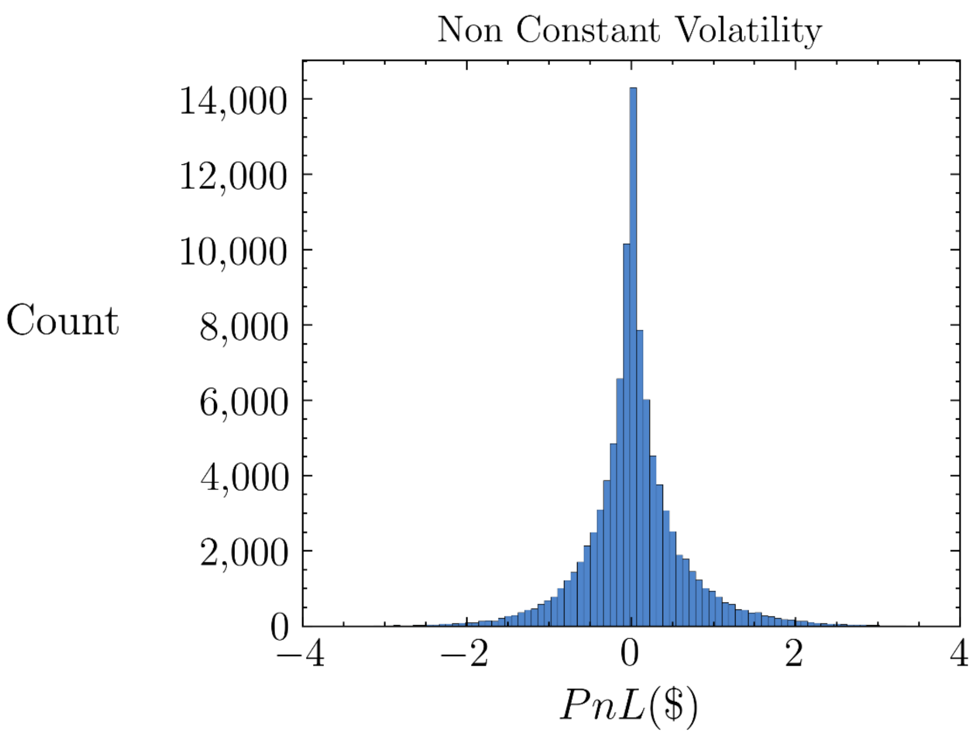


Figure: Results are based upon

daily re-hedging *and* $\mu = 10\%$

Mean = .038, Median = .005,

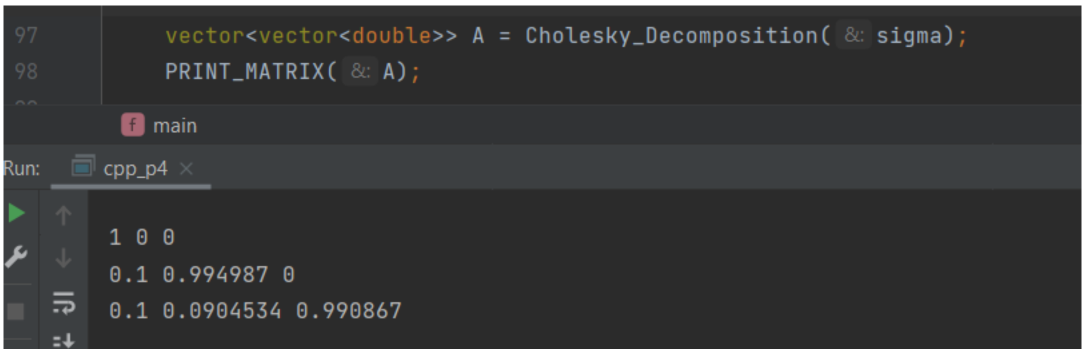
Standard Deviation = .62

4. Basket Options

Part a.) Write a function that uses Cholesky decomposition to find a lower triangular matrix A such that Σ = AAT . ***Demonstrate that your code works for d = 3 and ρ = 0.1.***

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{pmatrix} \quad \Rightarrow \quad \Sigma = \begin{pmatrix} 1 & .1 & .1 \\ .1 & 1 & .1 \\ .1 & .1 & 1 \end{pmatrix}$$

Below is a screenshot of C++ results. Based upon the Sigma matrix, the algorithm calculates the correct result.



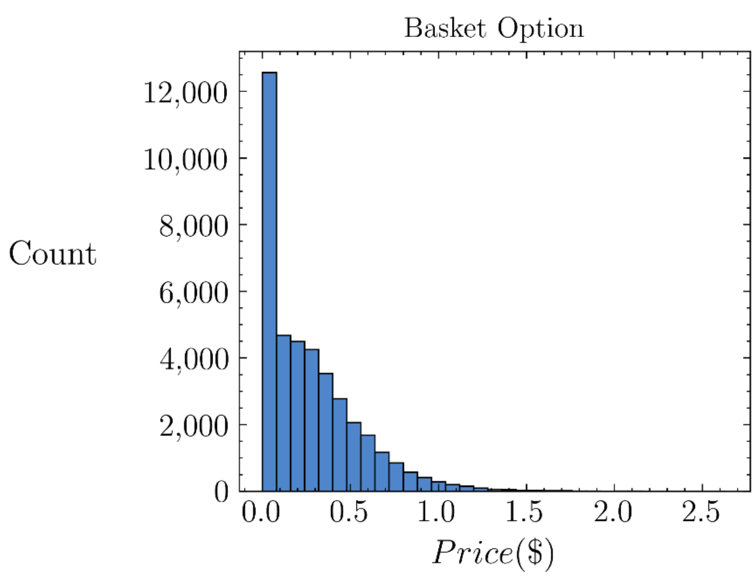
Part b.) Use Monte Carlo simulation to price a European basket option with payoff

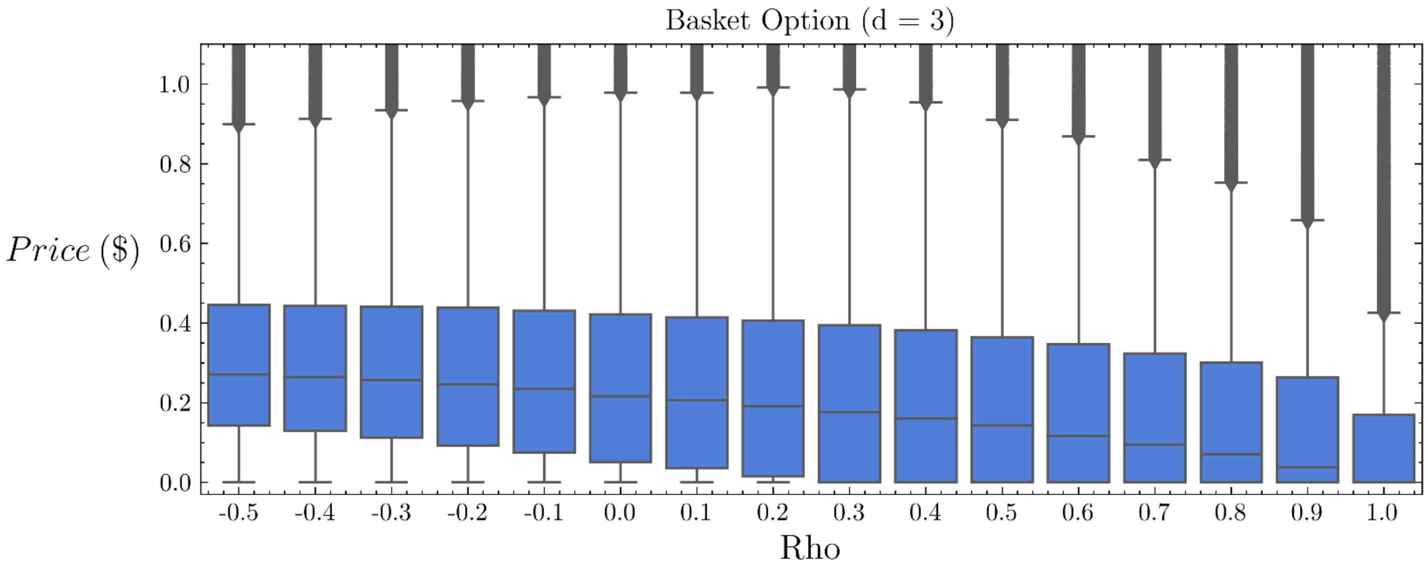
$$f(S_T) = \left(\max_{1 \le i \le d} S_T^i - K \right)^+$$

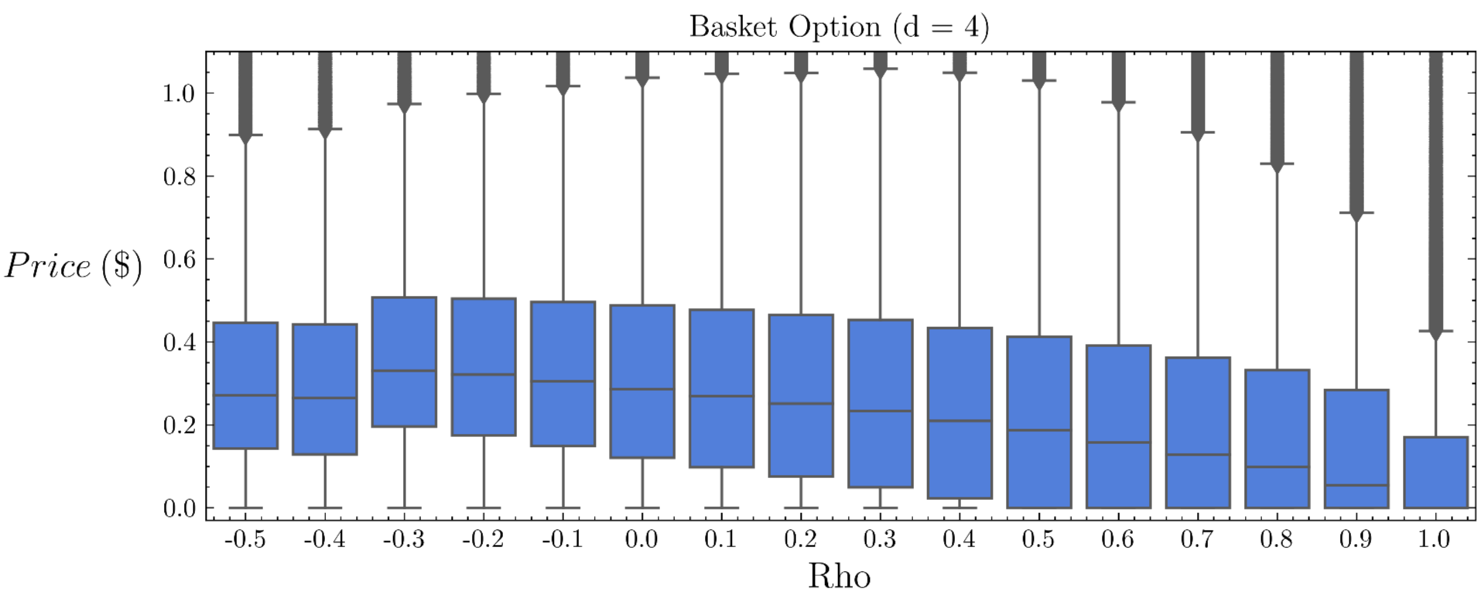
Use parameters S = 1, r = 0, σ = 0.3, T = 1 and K = 1. Do this for values of ρ between $-\frac{1}{d-1}$ and 1. Report your results graphically for d = 3 and d = 4.

The graph to the right is a histogram by running 400,000 simulations. The Median = $.2 and **Mean = $.27.** (d = 3 & Rho = .1

Below are the results with varying values of Rho and dimensions, d.




Basket Option (d = 3)
Price ($)
1.0
0.8
0.6
0.4
0.2
0.0
-0.5 -0.4 -0.3 -0.2 -0.1 0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
Rho


Basket Option (d = 4)
Price ($)
1.0
0.8
0.6
0.4
0.2
0.0
-0.5 -0.4 -0.3 -0.2 -0.1 0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
Rho

Appendix – Basket Option Method

Problem 4 – part a ($\boldsymbol{d \geq 3}$)

<u>Step 1</u>- Generate Correlation Matrix (Σ)

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{pmatrix}, \qquad choose\ \ 0 \leq \rho \leq 1$$

$\boxed{\text{Choose } \boldsymbol{\rho} = \mathbf{.1}}$

$$\Sigma = \begin{pmatrix} 1 & .1 & .1 \\ .1 & 1 & .1 \\ .1 & .1 & 1 \end{pmatrix} \ (sigma\ is\ provided)$$

<u>Step 2</u>- Solve for A (Cholesky Decomposition)

$$\mathrm{A\,A^T} = \Sigma$$

$$\mathrm{A} = \begin{pmatrix} 1 & 0 & 0 \\ .1 & .99 & 0 \\ .1 & .09 & .99 \end{pmatrix}, \qquad \mathrm{A^T} = \begin{pmatrix} 1 & .1 & .1 \\ 0 & .99 & .09 \\ 0 & 0 & .99 \end{pmatrix}$$

<u>Step 3</u> - Calculate the matrix, W.

$$\sqrt{T}\ \mathrm{A} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}$$

$$\sqrt{T} = Time\ Variable \left(i.e. \sqrt{T} = \sqrt{\frac{1}{252}} \Rightarrow daily\ time\ step \right)$$

Z_i = Random Number from normal ~ N(0,1)

<u>Step 4</u> - Solve for S_T^i (Plug in W_i into the Geometric Brownian Motion)

$$S_T^i = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma^i W_i}$$

References

[1] J. Hull, *Options, Futures and Other Derivatives*, Pearson 9th Edition, 2015