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American chooser options

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ABSTRACT

This paper examines the valuation of American chooser options, i.e., American-style contracts written on the maximum of an American put and an American call. The structure of the immediate exercise region is examined. The early exercise premium representation of the chooser's price is derived and used to construct a system of coupled recursive integral equations for a pair of boundary components. Numerical implementations of the model based on this system are carried out and used to examine the boundary properties and the price behavior.

1. Introduction

Advances in pricing theory and computational power over the last two decades have allowed financial institutions to design and trade customized derivatives designed to manage increasingly complex risk exposures. Examples of contracts filling particular needs include capped options, compound options, options on multiple underlying assets (outperformance options, spread options, etc.) and path-dependent contracts (Asian options, lookbacks, occupation time derivatives, etc.). The vast majority of contracts require the holder to commit to a particular payoff profile at the outset. This feature often proves unattractive for some potential users, especially those who do not hold a specific view about the course of future events. Flexible products, such as chooser options, have attracted interest as they allow the holder to wait before committing to a particular contractual form. This paper focuses on the valuation and properties of these contracts.

A chooser option is a particular type of multiasset option: it gives the holder the right to choose between a put option or a call option at or before some given maturity date T_1. The underlying options are typically written on a common asset and carry the same strike price K and maturity date $T_2 \geqslant T_1$. A European chooser option gives the holder the right to choose, at the maturity date T_1, the best of a European put and a European call. Its exercise payoff is $\max(C^e_{T_1}, P^e_{T_1})$, where $C^e_{T_1}$ (resp. $P^e_{T_1}$) is the European call (resp. put) price. It can be valued by using the put/call parity relation for European options (see Rubinstein, 1991). A variation of this contract gives the holder the right to choose the type of underlying European option at any date before the maturity date T_1. Given that exercise is costless and that the underlying options do not pay dividends it is intuitively clear that early exercise, prior to the maturity date T_1, is suboptimal. The value of this contract, with an early exercise feature, is therefore the same as the value of a

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European chooser option (see Rubinstein, 1991) and, for all practical purposes, the contract is equivalent to a European-style chooser.[1]

An American-style chooser is a chooser option written on a pair of American-style options. The American chooser option can be exercised at any time before its own maturity date T_1. In the event of exercise the holder selects an American put or call with maturity date $T_2 \geqslant T_1$. Various contracts of this type were written by Bankers Trust during the 1990s. Assets underlying the options have included commodities (oil) and indices (DAX and BCI). For this contract the underlying options carry implicit dividends as they may be optimally exercised before their own maturity date T_2. Early exercise of the American chooser option may therefore become beneficial and, in that event, the contract value will differ from the European chooser value.

It is perhaps tempting to conjecture that an American chooser should be exercised at or prior to the smallest of the optimal exercise dates of the underlying options. Indeed, doing so would permit the holder of the chooser to fully extract the early exercise premia embedded in the underlying American options. An even more tempting conjecture is that it should be optimal to exercise the chooser at times very close to the maturity date T_1, if it is optimal to exercise one of the underlying options. Indeed, suppose for instance that it is optimal to exercise the underlying call at $T_1 - \varepsilon$ where $\varepsilon > 0$ is an arbitrarily small number. Delaying exercise of the chooser option, in this event, would entail a loss equal to the instantaneous exercise premium on the call, which is positive. Moreover, potential benefits of this waiting policy are not apparent, as the underlying put is out of the money and the maturity date is extremely close. A goal of the present paper is to identify the immediate exercise region of the contract and, in particular, to examine these two conjectures.

The analysis carried out is based on the early exercise premium (EEP) representation of an American-style derivative contract. The EEP representation was initially developed by Kim (1990), Jacka (1991) and Carr et al. (1992) for plain vanilla options in the context of the Black–Scholes financial market, i.e. the standard model with constant coefficients. An extension of this representation to financial models where prices are driven by continuous semimartingales and derivatives have payoffs adapted to a continuous filtration can be found in Rutkowski (1994). Martingale methods are instrumental in the derivation of this general version of the decomposition formula. Versions of the formula for exotic payoff structures, such as multiasset options, barrier options and strangles appear, respectively, in Broadie and Detemple (1997), Gao et al. (2000) and Chiarella and Ziogas (2005). In these papers the underlying financial market is the Black–Scholes market model.

As is now well known, the EEP representation leads to a simple characterization of the exercise boundaries delineating the immediate exercise region. This characterization takes the form of recursive integral equations for the boundary components. When the derivative under consideration is a plain vanilla option the integral equation is unique (e.g. Carr et al., 1992). For certain multiasset options, such as max-options, the exercise region is described by multiple boundaries. In these situations multiple, coupled recursive integral equations may be needed to describe the behavior of all the boundary components (e.g. Broadie and Detemple, 1997). Likewise, when the underlying payoff is a mixture of piecewise linear components, as in the case of strangles, several boundary components are needed to describe the event of exercise and a system of coupled recursive equations will characterize the boundaries (e.g. Chiarella and Ziogas, 2005).

Our analysis of chooser options is rooted in the methods developed by Rutkowski (1994) and Broadie and Detemple (1997). After characterizing the broad geometric structure and properties of the chooser's exercise region, the EEP representation will be derived. This formula will lead to a system of coupled recursive integral equations for a pair of exercise boundary components. Implementations of the model, based on this system of equations, will be carried out.[2]

Section 2 presents the model. Section 3 examines the structure and properties of the optimal exercise region. A valuation formula for the American chooser and a system of recursive equations for the boundary components are derived in Section 4. Greeks, such as the delta and the gamma of the option, are provided in Section 5. Implementation of the model is discussed and carried out in Sections 6 and 7. Relations with other exotic options, such as straddles, and extensions to more complex contractual forms can be found in Sections 8 and 9. Conclusions are in Section 10. Detailed proofs are in the Appendix.

2. Setup and notation

We examine an American-style chooser option consisting of a right to choose an American call on an asset with price S, exercise price K, and expiry date T_2 or an American put on the same asset with exercise price K, and expiry date T_2. Let T_1 be the expiration date of the chooser, where $T_1 \leqslant T_2$. The chooser payoff, if exercised at some time $t \leqslant T_1$, is the maximum of the underlying American call and put options. When the expiry date for the American-style chooser coincides with the expiry date of the underlying options, i.e. $T_1 = T_2$, the chooser is identical to a self-closing American straddle. In its general form, the American chooser contract can be viewed as a multiasset derivative written on the maximum of two assets, an American put and an American call, both defined on the same underlying asset and with identical expiry dates and strike

[1] A European-style chooser, with given maturity (exercise) date T_1, can also be written on a pair of underlying American-style options. The exercise payoff of this contract is $\max(C_{T_1}, P_{T_1})$, where C_{T_1} (resp. P_{T_1}) is the American call (resp. put) price at T_1. This type of contract emerges as a term in the decomposition formula established in this paper.

[2] Various numerical schemes have been proposed to approximate the solution of the integral equation. Schemes for the case of plain vanilla options can, for instance, be found in Huang et al. (1996), Broadie and Detemple (1996), Ju (1998) and Kallast and Kivinukk (2003). Schemes for certain exotic options, such as strangles, are described in Chiarella and Ziogas (2005).

prices. It can also be viewed as a derivative written on a portfolio of American options with endogenous portfolio weights: at exercise of the chooser the portfolio weights can be selected so as to put the full weight on either one of the two possible components. Finally, the chooser option can be understood as a right to get an American call option along with an option to immediately exchange (at the exercise time of the chooser) the American call for an American put with the same characteristics.

The economic setting is the standard financial market model with constant coefficients. The underlying asset price follows the geometric Brownian motion process

$$dS_t = S_t(r-\delta)\,dt + S_t\sigma\,dW_t \tag{1}$$

where r is the riskfree rate of interest, assumed to be positive, δ is the dividend rate on the underlying asset, assumed to be nonnegative, and σ is the volatility of the asset's return. The dynamics in (1) describes the risk neutralized evolution of the asset price. The process W is a Brownian motion under the risk neutral measure $\mathbb{Q}$.

Let $V(S,t)$ be the value of the chooser contract when the asset price is S at time $t \in [0,T_1]$. Let $C(S,t)$ (resp. $P(S,t)$), or simply C_t (resp. P_t), be the value at time t of the American call (resp. put) option. Standard results can be invoked to establish that the American chooser price is

$$V(S,t) = \sup_{\tau\in\mathscr{S}_{t,T_1}} \mathbb{E}_t[e^{-r(\tau-t)}\max\{C(S_\tau,\tau), P(S_\tau\tau)\}]$$

where $\mathscr{S}_{t,T_1}$ is the set of stopping times of the Brownian filtration, with values in $[t,T_1]$ and $\mathbb{E}_t[\cdot]$ is the conditional expectation at t under the risk neutral measure (see Karatzas, 1988, Theorem 5.4). In addition, it is easy to show that $C(S,t), P(S,t)$ and $V(S,t)$ are continuous functions of (S,t) and convex functions of S (see Jaillet et al., 1990). The convexity of $V(S,t)$ follows from the convexity of the payoff function $\max\{x,y\}$ and the convexity of each of the price functions $C(S,t), P(S,t)$, for all $t \in [0,T_1]$.

3. Immediate exercise region

Before describing the properties of the immediate exercise region of the American-style chooser, it is useful to recall fundamental features of the exercise regions of American-style calls and puts. Let

$$\mathscr{E}^c \equiv \{(S,t) \in \mathbb{R}_+ \times [0,T_2] : C(S,t) = \max\{S-K,0\}\}$$

$$\mathscr{E}^p \equiv \{(S,t) \in \mathbb{R}_+ \times [0,T_2] : P(S,t) = \max\{K-S,0\}\}$$

be the immediate exercise regions of an American call and an American put option, respectively. It is well known that $\mathscr{E}^c$ is up-connected ($(S,t)\in\mathscr{E}^c$ implies $(\lambda S,t)\in\mathscr{E}^c$ for all $\lambda\geqslant 1$) and right-connected, i.e., connected as time moves forward ($(S,t_1)\in\mathscr{E}^c$ implies $(S,t_2)\in\mathscr{E}^c$ for $t_1\leqslant t_2$). In addition the call continuation region, $\mathscr{C}^c \equiv \mathbb{R}_+\times[0,T_2]\backslash\mathscr{E}^c$, is bounded above when the underlying asset pays dividends (for $\delta>0$, there exists $M>0$ such that $(S,t)\in\mathscr{E}^c$ for all $S\geqslant M$). By put-call symmetry (see Schroder, 1999; also Detemple, 2006, Proposition 36), the exercise region of an American put option is down-connected ($(S,t)\in\mathscr{E}^p$ implies $(\lambda S,t)\in\mathscr{E}^p$ for $0\leqslant\lambda\leqslant 1$) and right-connected ($(S,t_1)\in\mathscr{E}^p$, implies $(S,t_2)\in\mathscr{E}^p$ for $t_1\leqslant t_2$). Moreover, for t strictly less than the expiry date, the continuation region $\mathscr{C}^p \equiv \mathbb{R}_+\times[0,T_2]\backslash\mathscr{E}^p$ is bounded away from zero (there exists $b_t^p\in(0,K)$ such that $(b_t^p,t)\in\mathscr{E}^p$ for all $t\in[0,T_2]$). Fig. 1 illustrates the structure of these exercise regions in the time-to-maturity dimension.

Let $\mathscr{E}^{ch}$ be the immediate exercise region of the chooser option

$$\mathscr{E}^{ch} \equiv \{(S,t)\in\mathbb{R}_+\times[0,T_1] : V(S,t) = \max\{C(S,t),P(S,t)\}\}$$

Its complement $\mathscr{C}^{ch} \equiv \{(S,t)\in\mathbb{R}_+\times[0,T_1] : V(S,t) > \max\{C(S,t),P(S,t)\}\}$ is the continuation region, where immediate exercise is suboptimal. By the continuity of the value function $V(S,t)$ the exercise region $\mathscr{E}^{ch}$ is a closed set; the continuation region $\mathscr{C}^{ch}$ is open. In addition, $(0,t)\in\mathscr{E}^{ch}$ for any $t\in[0,T_1]$. Given that the underlying asset price follows a geometric Brownian motion process, the level $S=0$ is an absorbing state. In that state, it is optimal to exercise the underlying put option immediately as it attains its greatest exercise value, whereas the call option value is null. The optimality of immediate exercise for the chooser option follows.

The following relation characterizes the continuation regions $\mathscr{C}^{ch}, \mathscr{C}^c$ and $\mathscr{C}^p$.

Proposition 1. *Suppose that $t\leqslant T_1$ and $(S,t)\in\mathscr{C}^c\cap\mathscr{C}^p$. Then $(S,t)\in\mathscr{C}^{ch}$. That is, if immediate exercise is suboptimal for both the American call and put options, then it is also suboptimal for the American chooser option.*

Intuition for this result follows from the fact that the holder of a chooser option can pursue, among other strategies, the optimal exercise policy of the underlying call option or that of the underlying put option and attain the corresponding payoffs. A chooser option is therefore at least as valuable as each of the underlying derivatives, i.e.,

$$V(S,t)\geqslant\max\{C(S,t),P(S,t)\}$$

At the maturity date it is clear that $V(S,T_1)=\max\{C(S,T_1),P(S,T_1)\}$, by no arbitrage. Prior to maturity, if immediate exercise of the call option and immediate exercise of the put option are suboptimal policies, the chooser option must be worth

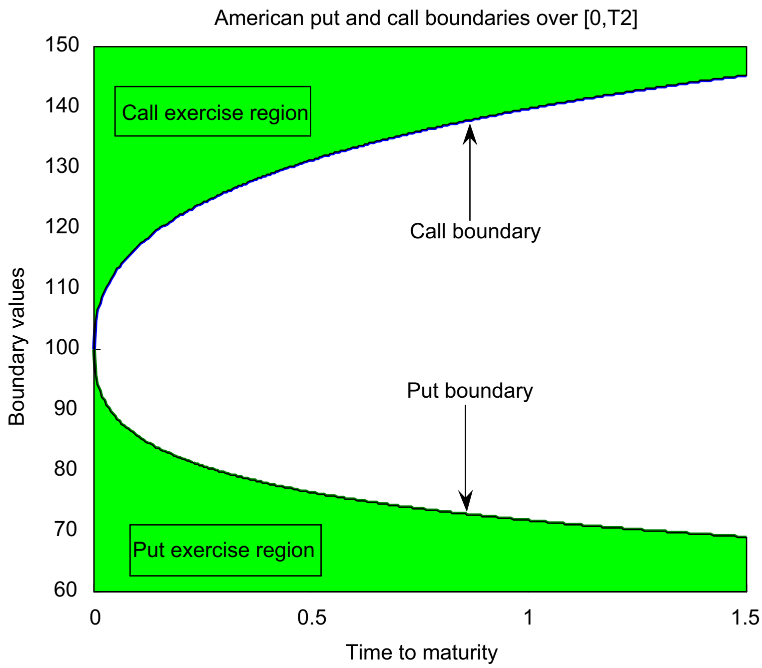

Fig. 1. This figure displays the immediate exercise regions of a call option and a put option with common strike price $K = 100$ and maturity date $T_2 = 1.5$ years. Parameter values are $r = \delta = 0.06$ and $\sigma = 0.20$. The exercise boundaries are computed using the integral equation method with $n = 150$ discretization points.

strictly more than each of the underlying derivatives. If not, an arbitrage portfolio can be constructed (see the proof in Appendix A for details). Thus, $V(S,t) > \max\{C(S,t), P(S,t)\}$ for $t < T_1$.

The suboptimality of immediate exercise in the region $\mathscr{C}^c \cap \mathscr{C}^p$ implies that the exercise region of the chooser option consists of two subsets

$$\mathscr{E}^{ch} = \mathscr{E}_1^{ch} \cup \mathscr{E}_2^{ch}$$

which do not intersect prior to the maturity date. The set $\mathscr{E}_1^{ch}$ (resp. $\mathscr{E}_2^{ch}$) corresponds to the event in which it is optimal to exercise and choose the call (resp. put). Moreover, in this case $V(S,t) = C(S,t) = S - K$ because $\mathscr{E}_1^{ch} \subseteq \mathscr{E}^c$. For $t \leqslant T_1$ let

$$\mathscr{E}^{ch}(t) \equiv \{S \in \mathbb{R}_+ : (S,t) \in \mathscr{E}^{ch}\}$$

be the t-section of the exercise region $\mathscr{E}^{ch}$, i.e. the set of asset prices at which it is optimal to exercise at time t. Similarly, let $\mathscr{E}_i^{ch}(t)$ be the t-section of $\mathscr{E}_i^{ch}$, $i = 1, 2$. Finally, let $S^*(t)$ be the unique solution of the equation $C(S,t) = P(S,t)$. The curve $(t, S^*(t))$ can be viewed as a diagonal: it represents the set of points at which the underlying instruments have equal values. With these definitions, it holds that $\mathscr{E}_1^{ch}(t) \cap \mathscr{E}_2^{ch}(t) = \emptyset$ for $t < T_1$ and $\mathscr{E}_1^{ch}(T_1) \cap \mathscr{E}_2^{ch}(T_1) = \{S^*(T_1)\}$ for $t = T_1$. The exercise subregions, thus, have a common point $S^*(T_1)$ at the maturity date T_1. An application of the American put–call symmetry property (e.g. Detemple, 2006, Proposition 36) shows that $S^*(T_1) = K$ in the special cases where $r = \delta$ and $T_1 < T_2$ or where $T_1 = T_2$.

The immediate exercise region $\mathscr{E}^{ch}$ has the following properties:

Proposition 2. *The region $\mathscr{E}^{ch}$ satisfies*

(i) *Non-emptyness*: $(0,t) \in \mathscr{E}^{ch}$ *for all* $t \in [0, T_1]$.
(ii) *Right-connectedness*: $(S,t) \in \mathscr{E}^{ch}$ *implies* $(S,s) \in \mathscr{E}^{ch}$ *for all* $t \in [0, T_1]$, $s \in [t, T_1]$.
(iii) *Up-connectedness*: $(S,t) \in \mathscr{E}_1^{ch}$ *implies* $(\lambda S, t) \in \mathscr{E}_1^{ch}$ *for all* $\lambda \geqslant 1$.
(iv) *Down-connectedness*: $(S,t) \in \mathscr{E}_2^{ch}$ *implies* $(\lambda S, t) \in \mathscr{E}_2^{ch}$ *for all* $\lambda \leqslant 1$.
(v) *Diagonal behavior*: $(S^*(t), t) \notin \mathscr{E}^{ch}$ *for all* $t \in [0, T_1)$.

Fig. 2 illustrates the typical shape of the exercise region, reflecting Properties (i)–(iv). Fig. 3 highlights the impact of the diagonal boundary for some parameter values.

Non-emptyness of the exercise set, in Property (i), follows from the fact that it is optimal to exercise the underlying put option immediately when the asset price is null, while the call has no value. Property (ii) is standard. Property (iii) states that immediate exercise remains optimal at any price greater than S if it is optimal at (S,t) and the optimal policy is to

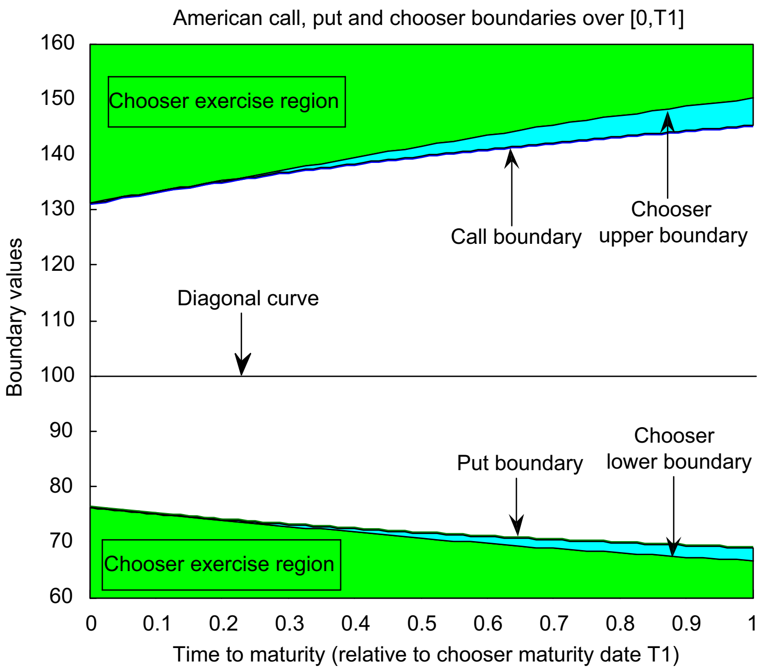

Fig. 2. This figure displays the immediate exercise region of a chooser option with maturity date $T_1 = 1$ year when the underlying American options have strike price $K = 100$ and maturity date $T_2 = 1.5$ years. Time to maturity, on the horizontal axis, is measured relative to T_1. Parameter values are $r = \delta = 0.06$ and $\sigma = 0.20$. The exercise boundaries of the chooser, which satisfy Eqs. (5)–(6), are computed using the integral equation method with $n = 100$ discretization points (see Section 6). The integral in the European chooser option (Eq. (9)) is approximated using $N = 1,000$ discretization points.

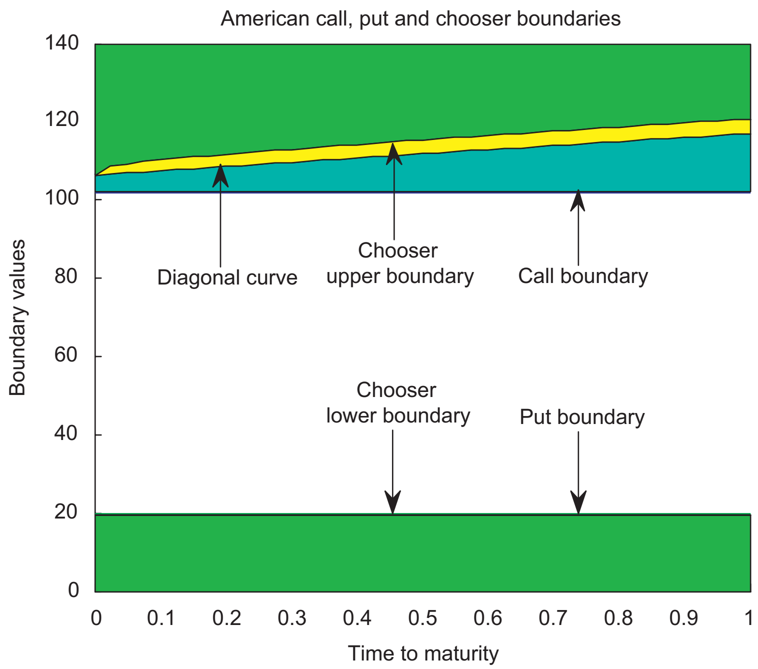

Fig. 3. This figure displays the immediate exercise region of a chooser option with maturity date $T_1 = 1$ year when the underlying American options have strike price $K = 100$ and maturity date $T_2 = 1.5$ years. Parameter values are $r = 0.06, \delta = 0.30$ and $\sigma = 0.10$. The exercise boundaries of the chooser, which satisfy Eqs. (5)–(6), are computed using the integral equation method with $n = 100$ discretization points (see Section 6). The integral in the European chooser option (Eq. (9)) is approximated using $N = 1,000$ discretization points.

choose the call. Likewise, Property (iv) establishes that it remains optimal to exercise at any price smaller than S if it is optimal to exercise at (S,t) and the optimal policy is to choose the put option.

The last property is particularly interesting and goes to the heart of the conjectures formulated in the introduction. The property states that immediate exercise, prior to maturity, is suboptimal for the chooser along the curve $S^*(t)$. This holds even if it is optimal to exercise one of the underlying options (i.e. if $(S^*(t),t) \in \mathscr{E}^{\mathrm{c}} \cup \mathscr{E}^{\mathrm{p}}$) and no matter how close the maturity date T_1 is. A simple no-arbitrage argument establishes the result. Suppose, to the contrary, that immediate exercise is optimal at $(S^*(t),t)$ for some $t<T_1$: $V(S^*(t),t) = C(S^*(t),t) = P(S^*(t),t)$. Suppose moreover that immediate exercise of the put is suboptimal at that point, i.e., $P(S^*(t),t) > K - S^*(t)$. Under these circumstances, selling the put and using the proceeds to buy the chooser is a costless strategy. If the put is optimally exercised at or before T_1 the chooser can be exercised as well to produce a net cash flow equal to zero. The same is true if the put is still alive at T_1 and is more valuable than the call. In the event that the call is more valuable than the put at T_1 the chooser can be exercised as a call and the proceeds used to buy back the American put. The net cash flow, $C(S(T_1),T_1) - P(S(T_1),T_1)$, is strictly positive. The strategy outlined, therefore, produces an arbitrage profit. The absence of arbitrage opportunities precludes the conditions leading to this arbitrage from occurring. A straightforward modification of this argument applies when $C(S^*(t),t) > S^*(t) - K$.

Property (v) shows that the intuitive conjectures formulated in the introductory section are invalid. Exercising the chooser prior to the maturity date is always suboptimal along the curve $(S^*(t),t)$. This is true even if it is optimal to exercise one of the two underlying options. It is also true at times extremely close to maturity. In all these cases the benefits of waiting, dominate the potential costs (see Fig. 3 for an example). The intuition for the property is reminiscent of a result for American max-options (see Broadie and Detemple, 1997), which states that immediate exercise of a max-option is suboptimal when the underlying asset prices are equal. Likewise, Property (v) shows that immediate exercise of the chooser option is also suboptimal when the underlying option values are the same. In this event the likelihood of an increase in the max of the two contract prices, over the next increment of time, is close to one so that waiting-to-exercise dominates the alternative.

Given Propositions 1 and 2 it is apparent that two boundaries are needed to describe the exercise region of the chooser option. Accordingly, define for $t<T_1$, the functions

$$B_t^1 = \inf\{S : S \in \mathscr{E}_1^{\mathrm{ch}}(t)\}$$

$$B_t^2 = \sup\{S : S \in \mathscr{E}_2^{\mathrm{ch}}(t)\}$$

and note that, by Proposition 2(ii), B_t^1 is a nonincreasing function of time, while B_t^2 is a nondecreasing function of time. It is also clear that $B_t^1 \geqslant B_t^2$ for all $t \in [0,T_1]$. The next proposition provides additional properties of the boundaries. The notation $b^{\mathrm{c}}(\cdot)$ (resp. $b^{\mathrm{p}}(\cdot)$) stands for the exercise boundary of the American-style call (resp. put) with maturity date T_2 and strike K.

Proposition 3. *The upper exercise boundary B^1 is a continuous, nonincreasing function that satisfies the lower bound $B^1 \geqslant b^{\mathrm{c}}$ and converges to $\max\{b_{T_1}^{\mathrm{c}}, S^*(T_1)\}$ as t approaches the maturity date T_1. The lower exercise boundary B^2 is a continuous, nondecreasing function that satisfies the upper bound $B^2 \leqslant b^{\mathrm{p}}$ and converges to $\min\{b_{T_1}^{\mathrm{p}}, S^*(T_1)\}$ as maturity approaches. Finally, $B^1 > S^* > B^2$.*

The behaviors of the exercise boundary components B^1, B^2 parallel the properties of the exercise subregions $\mathscr{E}_1^{\mathrm{ch}}, \mathscr{E}_2^{\mathrm{ch}}$. The limiting values follow from the fact that the chooser becomes either a call or a put as the maturity date approaches. For increasingly short maturities, the nature of the underlying option eventually becomes one-sided and the switching feature (i.e. the option to instantaneously exchange one underlying option for the other one) loses all value.

4. EEP representation

The payoff, at date t, of the American chooser option is $Y_t = \max\{C_t, P_t\}$, where C_t is the American call price and P_t is the American put price. Results from Rutkowski (1994) (see also Detemple, 2006, Theorem 21) can be applied to show that the value of the American chooser has the EEP representation

$$V(S_t,t;B^1,B^2) = v(S_t,t) + \pi(S_t,t;B^1,B^2) \tag{2}$$

where

$$v(S_t,t) = \mathbb{E}_t[\mathrm{e}^{-r(T_1-t)} \max\{C_{T_1}, P_{T_1}\}] \tag{3}$$

$$\pi(S_t,t;B^1,B^2) = \mathbb{E}_t\left[\int_t^{T_1} \mathrm{e}^{-r(s-t)} 1_{\{\tau_s = s\}} (r \max\{C_s,P_s\}\,\mathrm{d}s - \mathrm{d}A_s^Y)\right] \tag{4}$$

and $Y_t = Y_0 + M_t^Y + A_t^Y$, $\mathbb{Q}$ a.s. In this decomposition $v(S_t,t)$ is the value of a European-style chooser, with maturity date T_1, written on a pair of underlying American options and $\pi(S_t,t,B^1,B^2)$ is the early exercise premium. Equivalently,

$$\pi(S_t,t;B^1,B^2)=\mathbb{E}_t\left[\int_t^{T_1}\mathrm{e}^{-r(s-t)}1_{\{S_s\geqslant B_s^1\}}(r\max\{C_s,P_s\}\,\mathrm{d}s-\mathrm{d}A_s^Y)\right]+\mathbb{E}_t\left[\int_t^{T_1}\mathrm{e}^{-r(s-t)}1_{\{S_s\leqslant B_s^2\}}(r\max\{C_s,P_s\}\,\mathrm{d}s-\mathrm{d}A_s^Y)\right]$$

On $\{S_s\geqslant B_s^1\}$, the payoff is $Y_s=\max\{C_s,P_s\}=C_s=S_s-K$, where the last equality follows because $B_s^1\geqslant b_s^{\mathrm{c}}$. The dynamics of the underlying price, in (1), implies

$$Y_s=S_0-K+\int_0^s(r-\delta)S_u\,\mathrm{d}u+\int_0^s\sigma S_u\,\mathrm{d}W_u$$

and $A_s^Y=\int_0^s(r-\delta)S_u\,\mathrm{d}u$ on this event. Similarly, on $\{S_s\leqslant B_s^2\}$, $Y_s=\max(C_s,P_s)=P_s=K-S$, where the last equality follows because $B_s^2\leqslant b_s^{\mathrm{p}}$. By the same argument as above it follows that $A_s^Y=\int_0^s(\delta-r)S_u\,\mathrm{d}u$ on $\{S_s\leqslant B_s^2\}$. The exercise premium in the EEP decomposition can then be rewritten as

$$\begin{aligned}\pi(S_t,t;B^1,B^2)&=\mathbb{E}_t\left[\int_t^{T_1}\mathrm{e}^{-r(s-t)}1_{\{S_s\geqslant B_s^1\}}(r(S_s-K)-(r-\delta)S_s)\,\mathrm{d}s\right]\\&\quad+\mathbb{E}_t\left[\int_t^{T_1}\mathrm{e}^{-r(s-t)}1_{\{S_s\leqslant B_s^2\}}(r(K-S_s)-(\delta-r)S_s)\,\mathrm{d}s\right]\\&=\mathbb{E}_t\left[\int_t^{T_1}\mathrm{e}^{-r(s-t)}1_{\{S_s\geqslant B_s^1\}}(\delta S_s-rK)\,\mathrm{d}s\right]\\&\quad+\mathbb{E}_t\left[\int_t^{T_1}\mathrm{e}^{-r(s-t)}1_{\{S_s\leqslant B_s^2\}}(rK-\delta S_s)\,\mathrm{d}s\right]\end{aligned}$$

The assumption that the asset price follows a geometric Brownian motion enables us to simplify this valuation formula. The following holds:

Theorem 4. *Suppose that the underlying asset price follows the geometric Brownian motion process in* (1). *The value of the American chooser option has the EEP representation* (2)–(4) *for* $t\in[0,T_1]$. *The value of the European chooser written on a pair of American options is given in Remark* 1 *below. The early exercise premium is the sum of two components,* $\pi(S_t,t;B^1,B^2)=\pi_1(S_t,t;B^1)+\pi_2(S_t,t;B^2)$, *where*

$$\begin{aligned}\pi_1(S_t,t;B^1)&=\mathbb{E}_t\left[\int_t^{T_1}\mathrm{e}^{-r(s-t)}1_{\{S_s\geqslant B_s^1\}}(\delta S_s-rK)\,\mathrm{d}s\right]\\&=\int_t^{T_1}\phi_1(S_t;B_s^1,s-t)\,\mathrm{d}s\end{aligned}$$

$$\begin{aligned}\pi_2(S_t,t;B^2)&=\mathbb{E}_t\left[\int_t^{T_1}\mathrm{e}^{-r(s-t)}1_{\{S_s\leqslant B_s^2\}}(rK-\delta S_s)\,\mathrm{d}s\right]\\&=\int_t^{T_1}\phi_2(S_t;B_s^2,s-t)\,\mathrm{d}s\end{aligned}$$

with

$$\phi_1(S_t,B_s^1,v)\equiv\delta S_t\mathrm{e}^{-\delta v}N(d^{\mathrm{U}}(S_t,B_s^1,v))-rK\mathrm{e}^{-rv}N(d_1^{\mathrm{U}}(S_t,B_s^1,v))$$

$$\phi_2(S_t;B_s^2,v)\equiv rK\mathrm{e}^{-rv}N(d_1^{\mathrm{L}}(S_t,B_s^2,v))-\delta S_t\mathrm{e}^{-\delta v}N(d^{\mathrm{L}}(S_t,B_s^2,v))$$

and

$$d^{\mathrm{U}}(S_t,B_s^1,v)=\frac{\log(S_t/B_s^1)+(r-\delta+\frac{1}{2}\sigma^2)v}{\sigma\sqrt{v}}$$

$$d_1^{\mathrm{U}}(S_t,B_s^1,v)=d^{\mathrm{U}}(S_t,B_s^1,v)-\sigma\sqrt{v}$$

$$d^{\mathrm{L}}(S_t,B_s^2,v)=-\frac{\log(S_t/B_s^2)+(r-\delta+\frac{1}{2}\sigma^2)v}{\sigma\sqrt{v}}$$

$$d_1^{\mathrm{L}}(S_t,B_s^2,v)=d^{\mathrm{L}}(S_t,B_s^2,v)+\sigma\sqrt{v}$$

The immediate exercise boundaries B_t^1, B_t^2 *solve the system of coupled integral equations*

$$B_t^1-K=V(B_t^1,t;B^1,B^2)\tag{5}$$

$$K-B_t^2=V(B_t^2,t;B^1,B^2)\tag{6}$$

for $t\in[0,T_1)$, *subject to the boundary conditions* $\lim_{t\uparrow T_1}B_t^1=\max\{b_{T_1}^{\mathrm{c}},S^*\}$, $\lim_{t\uparrow T_1}B_t^2=\min\{b_{T_1}^{\mathrm{p}},S^*\}$.

The structure of the EEP representation of the chooser option provides several interesting insights. First, it shows that the EEP can be decomposed in two parts, each corresponding to the benefits in the event that one of the underlying assets (i.e. the call or the put) achieves the maximum payoff. Second, it reveals the nature of the benefits associated with each underlying asset. For the call option these benefits consist of the dividends on the underlying asset net of the cost of exercising the call. For the put option the benefits are given by the interest cost associated with the exercise of the put net of the dividends on the underlying asset. Given that immediate exercise of the chooser option is costless, there are no additional costs that are specific to the chooser. Third, the structure shows that the exercise premium, hence the option value, is a function of both boundary components. This property is quite natural given that immediate exercise will involve the choice of either instrument when appropriate conditions are met.

As usual the EEP representation leads to a characterization for the exercise boundary taking the form of a recursive integral equation. For the chooser option the exercise region is described by two boundary components (see Section 3). The EEP formula applies to each boundary component, leading to the system of integral equations (5)–(6). Each of these equations describes the behavior of one boundary component as a function of the future values taken by the boundary: these equations are recursive in nature. In addition, because the exercise premium and the option price depend on the future values of both boundary components, the integral equations are coupled. The current value of each boundary component depends on the future values of both boundary components.

Remark 1. Under the assumptions of Theorem 4, the value of the European chooser written on a pair of American options is given by $v(S_t, t) = v_1(S_t, t) + v_2(S_t, t) + v_3(S_t, t)$ where

$$\begin{aligned} v_1(S_t,t) &= \mathbb{E}_t[\mathrm{e}^{-r(T_1-t)} 1_{\{S_{T_1} \geqslant b^{\mathrm{c}}_{T_1} \vee S^*(T_1)\}} \max(C_{T_1}, P_{T_1})] \\ &= S_t \mathrm{e}^{-\delta(T_1-t)} N(d^{\mathrm{U}}(S_t, b^{\mathrm{c}}_{T_1} \vee S^*(T_1), T_1-t)) \\ &\quad - K\mathrm{e}^{-r(T_1-t)} N(d_1^{\mathrm{U}}(S_t, b^{\mathrm{c}}_{T_1} \vee S^*(T_1), T_1-t)) \end{aligned} \tag{7}$$

$$\begin{aligned} v_2(S_t,t) &= \mathbb{E}_t[\mathrm{e}^{-r(T_1-t)} 1_{\{S_{T_1} \leqslant b^{\mathrm{p}}_{T_1} \wedge S^*(T_1)\}} \max(C_{T_1}, P_{T_1})] \\ &= K\mathrm{e}^{-r(T_1-t)} N(d_1^{\mathrm{L}}(S_t, b^{\mathrm{p}}_{T_1} \wedge S^*(T_1), T_1-t)) \\ &\quad - S_t \mathrm{e}^{-\delta(T_1-t)} N(d^{\mathrm{L}}(S_t, b^{\mathrm{p}}_{T_1} \wedge S^*(T_1), T_1-t)) \end{aligned} \tag{8}$$

$$\begin{aligned} v_3(S_t,t) &= \mathbb{E}_t[\mathrm{e}^{-r(T_1-t)} 1_{\{b^{\mathrm{c}}_{T_1} \vee S^*(T_1) \geqslant S_{T_1} \geqslant b^{\mathrm{p}}_{T_1} \wedge S^*(T_1)\}} \max\{C_{T_1}, P_{T_1}\}] \\ &= \mathrm{e}^{-r(T_1-t)} \int_{b^{\mathrm{p}}_{T_1} \wedge S^*(T_1)}^{b^{\mathrm{c}}_{T_1} \vee S^*(T_1)} \max\{C(w,T_1), P(w,T_1)\} \times \frac{n(d_1^{\mathrm{L}}(S_t, w, T_1-t))}{w\sigma\sqrt{T_1-t}} \,\mathrm{d}w \end{aligned} \tag{9}$$

In these expressions, C_{T_1} (resp. P_{T_1}) is the value of the underlying American call (resp. put) option at the date T_1 and

$$d_1^{\mathrm{L}}(S_t, w, T_1-t) = \frac{\log(w/S_t) - (r - \delta - \frac{1}{2}\sigma^2)(T_1-t)}{\sigma\sqrt{T_1-t}}$$

where $n(\cdot)$ stands for the standard normal density function.

5. Hedging

For risk management purposes it is important to understand the behavior of the option price when the underlying asset price changes. The option delta Δ measures the sensitivity of the option price with respect to the underlying asset price. It is the first derivative of the option price

$$\Delta = V_S(S_t, t; B^1, B^2) \equiv \frac{\partial V(S_t, t; B^1, B^2)}{\partial S}$$

The option gamma Γ measures the sensitivity of the option delta with respect to the underlying asset price. It is the second derivative of the option price

$$\Gamma = V_{SS}(S_t, t; B^1, B^2) \equiv \frac{\partial^2 V(S_t, t; B^1, B^2)}{\partial S^2}$$

The EEP formula in Theorem 4 permits the identification of these derivatives.

Theorem 5. *Suppose that the underlying asset price follows the geometric Brownian motion process in* (1). *The delta and the gamma of the American chooser option are*

$$\Delta = \Delta^e + \pi_S(S_t, t; B^1, B^2)$$

$$\Gamma = \Gamma^e + \pi_{SS}(S_t, t; B^1, B^2)$$

where Δ^e (*resp.* Γ^e) *is the delta* (*resp. gamma*) *of the European chooser written on a pair of American options, given in Remark* 2 *below, and*

$$\pi_S(S_t, t; B^1, B^2) = \sum_{i=1}^{2} \int_t^{T_1} \phi_{iS}(S_t; B_s^i, s-t)\,\mathrm{d}s$$

$$\pi_{SS}(S_t, t; B^1, B^2) = \sum_{i=1}^{2} \int_t^{T_1} \phi_{iSS}(S_t; B_s^i, s-t)\,\mathrm{d}s$$

The functions $\{\phi_{iS}, \phi_{iSS} : i = 1,2\}$, *in these expressions are*

$$\begin{aligned}\phi_{1S}(S_t, B_s^1, v) \equiv{} & \delta \mathrm{e}^{-\delta v} N(d^{\mathrm{U}}(S_t, B_s^1, v)) + [\delta S_t \mathrm{e}^{-\delta v} n(d^{\mathrm{U}}(S_t, B_s^1, v)) \\ & - rK\mathrm{e}^{-rv} n(d_1^{\mathrm{U}}(S_t, B_s^1, v))] \frac{1}{S_t \sigma \sqrt{v}}\end{aligned}$$

$$\begin{aligned}\phi_{2S}(S_t; B_s^2, v) \equiv{} & -\delta \mathrm{e}^{-\delta v} N(d^{\mathrm{L}}(S_t, B_s^2, v)) - [rK\mathrm{e}^{-rv} n(d_1^{\mathrm{L}}(S_t, B_s^2, v)) \\ & - \delta S_t \mathrm{e}^{-\delta v} n(d^{\mathrm{L}}(S_t, B_s^2, v))] \frac{1}{S_t \sigma \sqrt{v}}\end{aligned}$$

for the first derivatives, and

$$\begin{aligned}\phi_{1SS}(S_t, B_s^1, v) \equiv{} & \delta \mathrm{e}^{-\delta v} n(d^{\mathrm{U}}(S_t, B_s^1, v)) \frac{1}{S_t \sigma \sqrt{v}} + rK\mathrm{e}^{-rv} n(d_1^{\mathrm{U}}(S_t, B_s^1, v)) \frac{1}{S_t^2 \sigma \sqrt{v}} \\ & - [\delta S_t \mathrm{e}^{-\delta v} n(d^{\mathrm{U}}(S_t, B_s^1, v)) d^{\mathrm{U}}(S_t, B_s^1, v) \\ & - rK\mathrm{e}^{-rv} n(d_1^{\mathrm{U}}(S_t, B_s^1, v)) d_1^{\mathrm{U}}(S_t, B_s^1, v)] \left(\frac{1}{S_t \sigma \sqrt{v}}\right)^2\end{aligned}$$

$$\begin{aligned}\phi_{2SS}(S_t; B_s^2, v) \equiv{} & \delta \mathrm{e}^{-\delta v} n(d^{\mathrm{L}}(S_t, B_s^2, v)) \frac{1}{S_t \sigma \sqrt{v}} \\ & + rK\mathrm{e}^{-rv} n(d_1^{\mathrm{L}}(S_t, B_s^2, v)) \frac{1}{S_t^2 \sigma \sqrt{v}} \\ & - [rK\mathrm{e}^{-rv} n(d_1^{\mathrm{L}}(S_t, B_s^2, v)) d_1^{\mathrm{L}}(S_t, B_s^2, v) \\ & - \delta S_t \mathrm{e}^{-\delta v} n(d^{\mathrm{L}}(S_t, B_s^2, v)) d^{\mathrm{L}}(S_t, B_s^2, v)] \left(\frac{1}{S_t \sigma \sqrt{v}}\right)^2\end{aligned}$$

for the second derivatives. In these expressions $n(\cdot)$ *is the standard normal density function.*

The decomposition of the American chooser price, as the sum of two components, naturally carries over to the delta and the gamma of the option. The first part of these Greeks is the corresponding Greek of the European chooser on a pair of American options. The second part is the corresponding Greek of the EEP. The second part is parametrized by the two boundaries characterizing the immediate exercise region of the contract.

Remark 2. Under the assumptions of Theorem 5 the delta and gamma of the European chooser on American options can be written as

$$\Delta^e = v_S(S_t, t) \quad \text{and} \quad \Gamma^e = v_{SS}(S_t, t)$$

where

$$v_S(S_t, t) = \sum_{i=1}^{3} v_{iS}(S_t, t), \quad v_{SS}(S_t, t) = \sum_{i=1}^{3} v_{iSS}(S_t, t)$$

with components

$$\begin{aligned}
v_{1S}(S_t,t) = {} & e^{-\delta(T_1-t)}N(d^{U}(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t)) \\
& + e^{-\delta(T_1-t)}n(d^{U}(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t))\frac{1}{\sigma\sqrt{T_1-t}} \\
& - Ke^{-r(T_1-t)}n(d^{U}_1(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t))\frac{1}{S_t\sigma\sqrt{T_1-t}}
\end{aligned}$$

$$\begin{aligned}
v_{2S}(S_t,t) = {} & -e^{-\delta(T_1-t)}N(d^{L}(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t)) \\
& - Ke^{-r(T_1-t)}n(d^{L}_1(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t))\frac{1}{S_t\sigma\sqrt{T_1-t}} \\
& + e^{-\delta(T_1-t)}n(d^{L}(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t))\frac{1}{\sigma\sqrt{T_1-t}}
\end{aligned}$$

$$\begin{aligned}
v_{3S}(S_t,t) = {} & \left(\frac{1}{S_t\sigma\sqrt{T_1-t}}\right)e^{-r(T_1-t)}\int_{b^{p}_{T_1}\wedge S^*(T_1)}^{b^{c}_{T_1}\vee S^*(T_1)} \max\{C(w,T_1),P(w,T_1)\} \\
& \times n(d^{L}_1(S_t,w,T_1-t))d^{L}_1(S_t,w,T_1-t)\frac{1}{w\sigma\sqrt{T_1-t}}\,dw
\end{aligned}$$

$$\begin{aligned}
v_{1SS}(S_t,t) = {} & e^{-\delta(T_1-t)}n(d^{U}(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t))\frac{1}{S_t\sigma\sqrt{T_1-t}} \\
& - e^{-\delta(T_1-t)}n(d^{U}(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t)) \\
& \times d^{U}(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t)\frac{1}{S_t}\left(\frac{1}{\sigma\sqrt{T_1-t}}\right)^2 \\
& + Ke^{-r(T_1-t)}n(d^{U}_1(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t))\frac{1}{S_t^2\sigma\sqrt{T_1-t}} \\
& + Ke^{-r(T_1-t)}n(d^{U}_1(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t)) \\
& \times d^{U}_1(S_t,b^{c}_{T_1}\vee S^*(T_1),T_1-t)\left(\frac{1}{S_t\sigma\sqrt{T_1-t}}\right)^2
\end{aligned}$$

$$\begin{aligned}
v_{2SS}(S_t,t) = {} & e^{-\delta(T_1-t)}n(d^{L}(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t))\frac{1}{S_t\sigma\sqrt{T_1-t}} \\
& + Ke^{-r(T_1-t)}n(d^{L}_1(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t))\frac{1}{S_t^2\sigma\sqrt{T_1-t}} \\
& - Ke^{-r(T_1-t)}n(d^{L}_1(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t)) \\
& \times d^{L}_1(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t)\left(\frac{1}{S_t\sigma\sqrt{T_1-t}}\right)^2 \\
& + e^{-\delta(T_1-t)}n(d^{L}(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t)) \\
& \times d^{L}(S_t,b^{p}_{T_1}\wedge S^*(T_1),T_1-t)\frac{1}{S_t}\left(\frac{1}{\sigma\sqrt{T_1-t}}\right)^2
\end{aligned}$$

$$\begin{aligned}
v_{3SS}(S_t,t) = {} & \left(-\frac{1}{S_t^2\sigma\sqrt{T_1-t}}\right)e^{-r(T_1-t)}\int_{b^{p}_{T_1}\wedge S^*(T_1)}^{b^{c}_{T_1}\vee S^*(T_1)} \max\{C(w,T_1),P(w,T_1)\} \\
& \times n(d^{L}_1(S_t,w,T_1-t))d^{L}_1(S_t,w,T_1-t)\frac{1}{w\sigma\sqrt{T_1-t}}\,dw \\
& + \left(\frac{1}{S_t\sigma\sqrt{T_1-t}}\right)e^{-r(T_1-t)}\int_{b^{p}_{T_1}\wedge S^*(T_1)}^{b^{c}_{T_1}\vee S^*(T_1)} \max\{C(w,T_1),P(w,T_1)\} \\
& \times n(d^{L}_1(S_t,w,T_1-t))(d^{L}_1(S_t,w,T_1-t))^2\frac{1}{S_t w\sigma^2(T_1-t)}\,dw \\
& - \left(\frac{1}{S_t\sigma\sqrt{T_1-t}}\right)e^{-r(T_1-t)}\int_{b^{p}_{T_1}\wedge S^*(T_1)}^{b^{c}_{T_1}\vee S^*(T_1)} \max\{C(w,T_1),P(w,T_1)\} \\
& \times n(d^{L}_1(S_t,w,T_1-t))\frac{1}{S_t w\sigma^2(T_1-t)}\,dw
\end{aligned}$$

The function $n(\cdot)$ is the standard normal density function.

6. Implementation

Implementation of the valuation formula is carried out in two steps. In the first step the exercise boundary components are calculated. In the second step the EEP formula is used to value the option parametrized by the exercise boundary. The second step is straightforward. The first step is described next.

In order to compute the boundary components the solution of the system

$$B_t^1 - K = v(B_t^1, t) + \int_t^{T_1} \phi_1(B_t^1; B_s^1, s-t)\,ds + \int_t^{T_1} \phi_2(B_t^1; B_s^2, s-t)\,ds \tag{10}$$

$$K - B_t^2 = v(B_t^2, t) + \int_t^{T_1} \phi_1(B_t^2; B_s^1, s-t)\,ds + \int_t^{T_1} \phi_2(B_t^2; B_s^2, s-t)\,ds \tag{11}$$

is approximated using step functions. The algorithm proceeds as follows. Divide the period $[0, T_1]$ into n equal subintervals of length $h = T_1/n$. For $j = 0, \ldots, n$ set $t(j) = jh$. The objective is to calculate the step function approximations $\{B_{(n)}^1(j) \equiv B_{t(j)}^1 : j = 0, \ldots, n\}$, $\{B_{(n)}^2(j) \equiv B_{t(j)}^2 : j = 0, \ldots, n\}$ using a recursive procedure.

Given the limiting conditions $\lim_{t\uparrow T_1} B_t^1 = \max\{b_{T_1}^c, S^*(T_1)\}$, $\lim_{t\uparrow T_1} B_t^2 = \min\{b_{T_1}^p, S^*(T_1)\}$, set $B_{(n)}^1(n) = \max\{b_{T_1}^c, S^*(T_1)\}$ and $B_{(n)}^2(n) = \min\{b_{T_1}^p, S^*(T_1)\}$ for $j = n$. Next, let $j<n$ and suppose that the boundary approximations $B_{(n)}^1(l), B_{(n)}^2(l)$ are known for all $l>j$. Consider (10). A non-linear equation for $B_{(n)}^1(j)$, parametrized by

$$\{(B_{(n)}^1(l), B_{(n)}^2(l)) : l = j+1, \ldots, n\}$$

is obtained by replacing both integrals on the right-hand side of (10) by discrete approximations. The discretization points for these approximations correspond to the time discretization points $\{j, \ldots, n\}$. The integral approximations are calculated based on the trapezoidal rule. Given that the equation for $B_{(n)}^1(j)$ is independent of $B_{(n)}^2(j)$ a solution can be computed using standard fixed point algorithms. A bisection method is used for that purpose. The same operation can be performed to handle (11), leading to an estimate $B_{(n)}^2(j)$. Proceeding recursively from $j = n-1$ to $j = 0$ yields an estimate $\{(B_{(n)}^1(j), B_{(n)}^2(j)) : j = 0, \ldots, n\}$ of the exercise boundary components.

For the implementation of this algorithm one needs to compute the component $v_3(S_t, t)$ of the European chooser on American options in Remark 1. This integral can be approximated by discretizing the interval $[\min\{b_{T_1}^p, S^*(T_1)\}, \max\{b_{T_1}^c, S^*(T_1)\}]$ into N subintervals. A variety of quadrature schemes can be used for that purpose. Our implementation uses the trapezoidal rule.[3] Computations can be accelerated by calculating the integrand at a small number of points N and then interpolating between these points using splines.[4]

The algorithm described above shares common elements with several other procedures employed in the literature. The overall approach is similar to Chiarella and Ziogas (2005), and takes advantage of the fact that, for the discretized equations, the interdependence of (10)–(11) vanishes at the current point t. The approximation of integrals based on the trapezoidal rule parallels the approach followed by Kallast and Kivinukk (2003). The computation of fixed points based on a bisection method is similar to Broadie and Detemple (1996).

7. Numerical results

This section provides numerical results pertaining to the American chooser option. Base parameter values are $r = \delta = 0.06, \sigma = 0.2, K = 100, T_1 = 1$ and $T_2 = 1.5$. Computations are performed using the integral equation method described above, with $n = 100$ time discretization points and $N = 1,000$ space discretization points for the integral in the value of the European chooser written on American options.

Fig. 4 displays the behavior of the value of the chooser contract. As expected the chooser price displays a V-shaped pattern with a curved convex bottom, with respect to the asset price S.[5] When S is sufficiently large or small, immediate exercise is optimal for both the chooser and the underlying options leading to values satisfying $V(S, t; B^1, B^2) = \max\{C(S, t), P(S, t)\} = \max\{S - K, K - S\}$. When S is in the continuation region the chooser value is a convex function, bounded below by the maximum of the American call and put prices. In this region, the premium in the chooser relative to the underlying options, i.e. the delayed exercise premium (DEP), can be substantial. It attains $16.8626 - 9.1164 = 7.7462$

[3] For extreme parameter values the integrand in $v_3(S_t, t)$ becomes null over most of the domain of integration and exhibits rapid growth near the edges of that domain. The algorithm implemented identifies the region $\mathscr{D}(\varepsilon)$ over which the integrand exceeds a minimal value (say $\varepsilon = 10^{-6}$) and uses a denser grid in that area. In implementations with $N = 1,000$, the grid selected uses 900 points, equally spaced, over $\mathscr{D}(\varepsilon)$ and 100 points, equally spaced, over the complement.

[4] Implementations in this paper are typically based on $n = 100$ time discretization points and $N = 1,000$ space discretization points. Fixed point calculations are performed using a tolerance of 10^{-7}. No attempt has been made to optimize the program code. Computations are carried out in MATLAB on a computer with a Pentium 4 CPU 3.06 GHz and 2.00 GB of RAM. Typical computation times are about 30 s in the examples underlying the figures. A spline interpolation with $N_s \times N$ points of the integrand in $v_3(S, t)$, when the integrand is calculated at N points, can be used to cut computation times. For $N_s = 20$ and $N = 50$ the spline interpolation gives computation times of about 15 s.

[5] As shown in the graph, the price function is asymmetric about the strike K. This reflects the asymmetry, with respect to K, of the underlying American put and call prices.

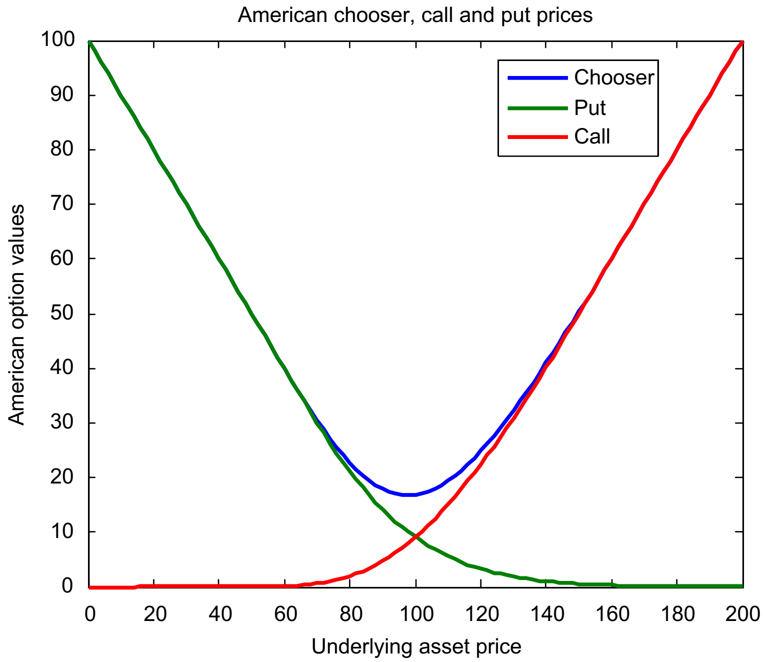

Fig. 4. This figure displays the prices of the chooser with maturity date $T_1 = 1$ and of the underlying put and call options with common maturity date $T_2 = 1.5$. All options have strikes $K = 100$. Parameter values are $r = \delta = 0.06$ and $\sigma = 0.20$. Computations are performed using the integral equation method with $n = 100$ discretization points (see Section 6). The integral in the European chooser option (Eq. (9)) is approximated using 1,000 discretization points.

Table 1
American and European chooser prices, exercise premia and chooser delta

Asset	Am-chooser	Payoff	Put	Call	Euro-chooser	DEP	EEP	Delta
20	80.0000	80.0000	80.0000	0.0000	75.3411	0.0000	4.6588	−1.0000
40	60.0000	60.0000	60.0000	0.0003	56.5059	0.0000	3.4940	−1.0000
60	40.0000	40.0000	40.0000	0.1167	37.7891	0.0000	2.2108	−1.0000
70	30.1331	30.0128	30.0128	0.6080	29.0378	0.1203	1.4693	−0.9207
80	22.2628	21.1879	21.1879	1.9854	21.8528	1.5232	0.8583	−0.6396
90	17.5913	14.2452	14.2452	4.7390	17.4412	3.6680	0.4720	−0.2860
100	16.5937	9.1164	9.1164	9.1164	16.5050	7.7462	0.3576	0.0829
110	19.0814	15.0768	5.5754	15.0768	18.9292	4.3479	0.4954	0.4038
120	24.3776	22.3985	3.2782	22.3985	24.0279	2.4697	0.8402	0.6439
130	31.6935	30.8127	1.8649	30.8127	30.9665	1.4893	1.3355	0.8098
140	40.3798	40.0902	1.0327	40.0902	39.0407	0.8726	1.9221	0.9214
160	60.0000	60.0000	0.2982	60.0000	56.8304	0.0000	3.1695	1.0000
180	80.0000	80.0000	0.0816	80.0000	75.4132	0.0000	4.5867	1.0000

Note: Column 1 is the underlying asset price S. Columns 2 and 3 are, respectively, the American chooser price and its immediate exercise payoff. Columns 4 and 5 provide the underlying American put and call prices. Column 6 is the price of the European chooser on the max of an American put price and an American call price. Columns 7 and 8 give the delayed exercise premium (DEP) and the Early Exercise premium (EEP). Column 9 provides the delta of the American chooser. Parameter values are $r = \delta = 0.06, \sigma = 0.2, K = 100, T_1 = 1$ and $T_2 = 1.5$. Computations are based on the integral equation method (see Section 7) with $n = 100$ time discretization points and $N = 1,000$ space discretization points for the integral in the value of the European chooser on American options.

when $S = 100$ (see Table 1). In contrast the EEP is large at the extremes, when immediate exercise is optimal. Over the region of prices displayed in Table 1 it reaches 4.6588 when $S = 20$. This represents nearly 6% of the option value.

The effects of volatility on chooser prices can be substantial (see Table 2). Because choosers have convex payoffs their prices are non-decreasing with respect to volatility. The behavior of the EEP is more intricate. In the exercise region the EEP is a decreasing function of volatility: the price of the European chooser on American options increases while the price of the American chooser, equal to the exercise value, is unchanged. As shown in the table, when the chooser is near the tipping point between a put and a call (i.e. at the point $S = K$), the behavior of the exercise premium with respect to volatility can be reversed.

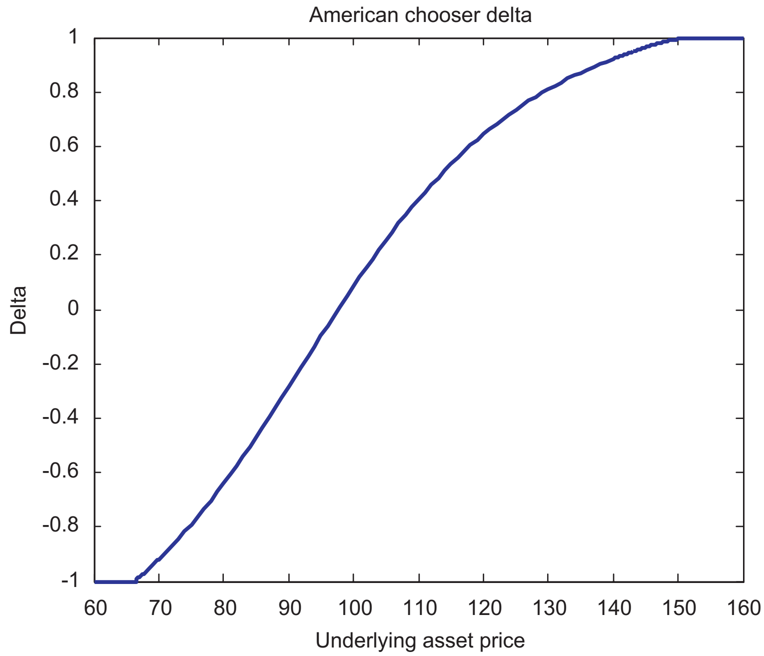

Fig. 5. This figure displays the delta of the American chooser with maturity date $T_1 = 1$. All underlying options have strikes $K = 100$ and maturity dates $T_2 = 1.5$. Parameter values are $r = \delta = 0.06$ and $\sigma = 0.20$. Computations are performed using the integral equation method with $n = 100$ discretization points (see Section 6). The integral in the European chooser option (Eq. (9)) is approximated using 1,000 discretization points.

Fig. 5 illustrates the behavior of the chooser delta. Numerical values can be found in the last column of Table 1. In the lower exercise subregion delta is -1. In the upper exercise subregion it is $+1$. In between, delta increases, in a continuous fashion, with the underlying asset price. This structure reflects the two conflicting incentives embedded in the chooser payoff. In areas where the incentives associated with the put (call) part of the payoff dominate, the delta is negative (positive). Table 2 shows the effects of volatility on delta. An increase in return volatility increases the chooser price and raises (lowers) the upper (lower) boundary components. At prices sufficiently close to the lower boundary this implies an increase in delta. At prices sufficiently close to the upper boundary, the reverse behavior occurs and delta decreases. As shown in the table, the magnitude of these variations can be important.

8. Some bounds on chooser prices

Upper and lower bounds on chooser prices can be constructed from certain combinations of puts and calls. Combinations are broadly defined as portfolios consisting of put and call options. Straddles, combine a single put and a single call, both with the same strike price. Strips and straps involve different numbers of puts and calls. Strangles consist of a put and a call, but with different strike prices.

An American-style straddle pays off $\max\{S_\tau - K, K - S_\tau\}$ upon exercise. Denote by $V^{\text{straddle}}(S_t, t; T)$ the price at t if the straddle matures at T. As $C(S_\tau, \tau; T_2) \geqslant (S_\tau - K)^+$ and $P(S_\tau, \tau; T_2) \geqslant (K - S_\tau)^+$ for all $\tau \leqslant T_1$ it is immediate to see that an American chooser with maturity date T_1 is more valuable than an American straddle with the same maturity date T_1: $V(S_t, t; T_1) \geqslant V^{\text{straddle}}(S_t, t; T_1)$. Likewise, as $V^{\text{straddle}}(S_{T_1}, T_1; T_2) \geqslant \max\{C(S_{T_1}, T_1; T_2), P(S_{T_1}, T_1; T_2)\}$ and as the straddle can implement the optimal exercise policy and reach the payoff of the chooser prior to T_1 it is also clear that an American chooser with maturity date T_1 is less valuable than an American straddle with maturity date T_2: $V^{\text{straddle}}(S_t, t; T_2) \geqslant V(S_t, t; T_1)$. Hence, American chooser prices are bounded above and below by American straddle prices (see Fig. 6 for an illustration).

Table 3 provides perspective on the price differentials between chooser options and straddles in the benchmark case $r = \delta = 0.06, \sigma = 0.2, K = 100, T_1 = 1$ and $T_2 = 1.5$. The differentials are greatest when the chooser is on the tipping point between the put and the call payoff, namely $S = K$. In this event $V^{\text{straddle}}(S_t, t; T_2) - V(S_t, t; T_1) = 18.1756 - 16.5937 = 1.5819$, while $V(S_t, t; T_1) - V^{\text{straddle}}(S_t, t; T_1) = 16.5937 - 15.1876 = 1.4061$.

From the price bounds it also follows that the exercise boundary of the chooser option envelopes the exercise boundary of the T_1-straddle (see Fig. 7). Likewise, the exercise boundary of the T_2-straddle is an envelope for the boundary of the chooser.

Table 2
American and European chooser prices, exercise premia and chooser delta

Asset	Volatility	Am-chooser	Payoff	DEP	Euro-chooser	EEP	Delta
60	0.1	40.0000	40.0000	0.0000	37.6705	2.3294	−1.0000
	0.2	40.0000	40.0000	0.0000	37.7891	2.2108	−1.0000
	0.3	40.3229	40.0669	0.2560	38.9530	1.3699	−0.8834
80	0.1	20.0000	20.0000	0.0000	18.9889	1.0110	−1.0000
	0.2	22.2628	21.1879	1.5232	21.8528	0.8583	−0.6396
	0.3	27.3831	24.1037	3.2794	27.0657	0.3174	−0.3917
100	0.1	8.3096	4.5664	3.7431	8.2654	0.0442	0.0415
	0.2	16.5937	9.1164	7.7462	16.5050	0.0887	0.0829
	0.3	24.8271	13.6333	11.1937	24.6931	0.1339	0.1241
120	0.1	20.0622	20.0022	0.0599	19.3091	0.7530	0.9509
	0.2	24.3776	22.3985	2.4697	24.0279	0.3497	0.6439
	0.3	31.4720	26.4071	5.0649	31.1692	0.3028	0.5151
140	0.1	40.0000	40.0000	0.0000	37.6795	2.3204	1.0000
	0.2	40.3798	40.0902	0.8726	39.0407	1.9221	0.9214
	0.3	44.3970	42.2997	2.0972	43.5680	0.8289	0.7579
160	0.1	60.0000	60.0000	0.0000	56.5059	3.4940	1.0000
	0.2	60.0000	60.0000	0.0000	56.8304	3.1695	1.0000
	0.3	61.0919	60.3787	0.7131	59.2651	1.8267	0.9002

Note: Columns 1 and 2 give the underlying asset price S and its return volatility σ. Columns 3 and 4 provide the American chooser price and its immediate exercise payoff. Column 6 is the delayed exercise premium. Column 7 is the price of the European chooser on American options and column 8 is the EEP. Column 9 is the Delta of the chooser. Parameter values are $r = \delta = 0.06, K = 100, T_1 = 1$ and $T_2 = 1.5$. Computations are based on the integral equation method (see Section 7) with $n = 100$ time discretization points and $N = 1{,}000$ space discretization points for the integral in the value of the European chooser on American options.

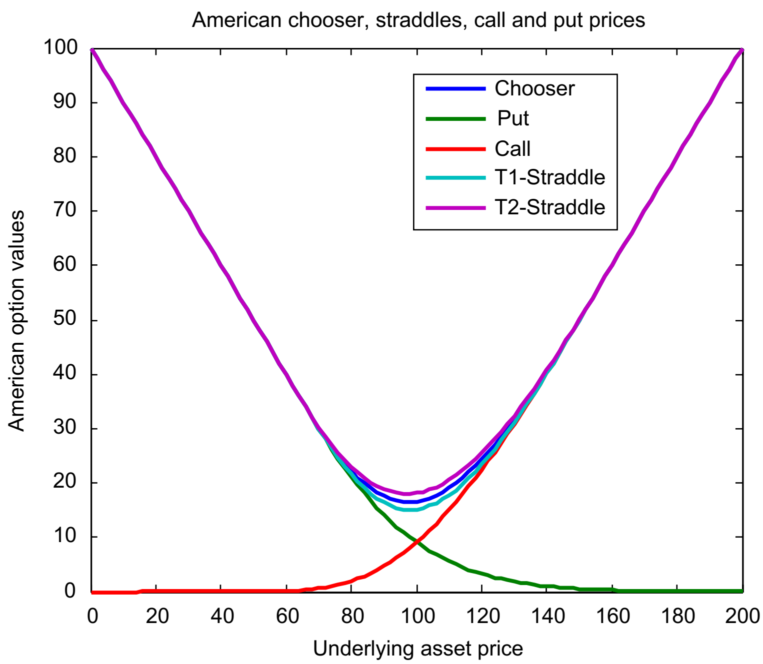

Fig. 6. This figure displays the prices of the chooser with maturity date $T_1 = 1$, the underlying put and call with common maturity date $T_2 = 1.5$ and the straddles with maturity dates $T_1 = 1$ and $T_2 = 1.5$. All options have strikes $K = 100$. Parameter values are $r = \delta = 0.06$ and $\sigma = 0.20$. Computations are performed using the integral equation method with $n = 100$ discretization points (see Section 6). The integral in the European chooser option (Eq. (9)) is approximated using 1,000 discretization points.

Table 3
American chooser, put, call and straddle prices

Asset	Chooser	Payoff	Put	Call	T_1-straddle	T_2-straddle
20	80.0000	80.0000	80.0000	0.0000	80.0000	80.0000
40	60.0000	60.0000	60.0000	0.0003	60.0000	60.0000
60	40.0000	40.0000	40.0000	0.1167	40.0000	40.0000
70	30.1331	30.0128	30.0128	0.6080	30.0322	30.3228
80	22.2628	21.1879	21.1879	1.9854	21.6377	23.0327
90	17.5913	14.2452	14.2452	4.7390	16.4384	18.9107
100	16.5937	9.1164	9.1164	9.1164	15.1876	18.1756
110	19.0814	15.0768	5.5754	15.0768	17.7721	20.5749
120	24.3776	22.3985	3.2782	22.3985	23.3975	25.5436
130	31.6935	30.8127	1.8649	30.8127	31.1103	32.4482
140	40.3798	40.0902	1.0327	40.0902	40.1500	40.7487
160	60.0000	60.0000	0.2982	60.0000	60.0000	60.0000
180	80.0000	80.0000	0.0816	80.0000	80.0000	80.0000

Note: Column 1 is the underlying asset price S. Columns 2 and 3 are, respectively, the American chooser price and its immediate exercise payoff. Columns 4 and 5 provide the underlying American put and call prices. Columns 6 and 7 give the prices of American straddles with maturity dates T_1 and T_2, respectively. Parameter values are $r = \delta = 0.06$, $\sigma = 0.2$, $K = 100, T_1 = 1$ and $T_2 = 1.5$. Computations are based on the integral equation method (see Section 7) with $n = 100$ time discretization points and $N = 1,000$ space discretization points for the integral in the value of the European chooser on American options.

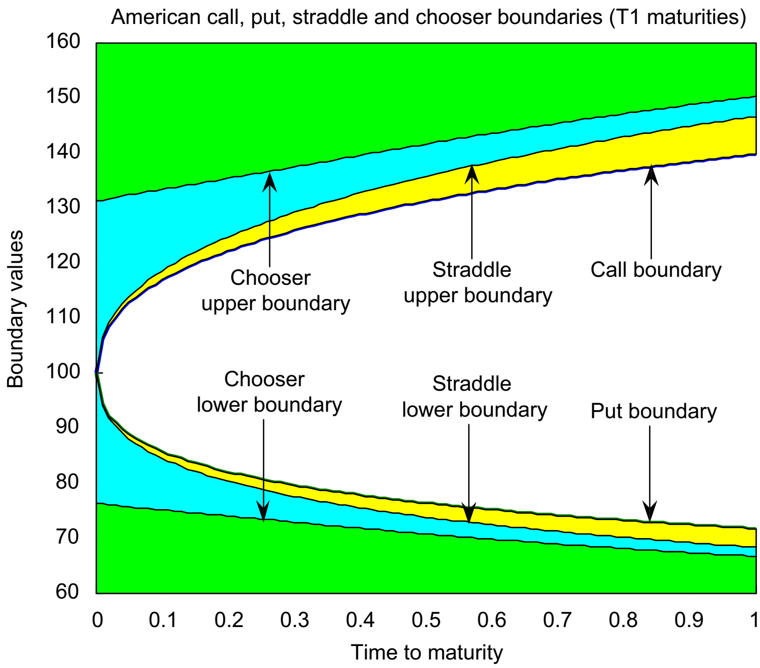

Fig. 7. This figure displays the immediate exercise regions of chooser, straddle, put and call options with common maturity dates $T_1 = 1$. The American options underlying the chooser have strike price $K = 100$ and maturity date $T_2 = 1.5$ years. The straddle, call and put options have strike price $K = 100$ and maturity date $T_1 = 1$ year. Parameter values are $r = \delta = 0.06$ and $\sigma = 0.20$. The exercise boundaries are computed using the integral equation method with $n = 100$ discretization points (see Section 6). The integral in the European chooser option (Eq. (9)) is approximated using 1,000 discretization points.

A straddle is a special case of a strangle, where the put and call have different strike prices (the strangle payoff is $\max\{S_\tau - K_2, K_1 - S_\tau\}$ with $K_1 < K_2$). American strangles are examined by Chiarella and Ziogas (2005). Their analysis is based on a PDE approach to valuation. It uses McKean's incomplete Fourier transform method to derive a system of coupled integral equations for the strangle's upper and lower exercise boundaries. Implementations of their model are based on the numerical resolution of this coupled system. The structure of the integral equations for American strangles is simpler than the structure obtained for American choosers. The main difference is due to the payoff at the maturity date T_1. In the case of a strangle the terminal payoff is a piecewise linear function of the underlying asset price. In the case of the chooser the terminal payoff is only piecewise linear above and below the (endogenous) exercise boundaries at T_1. In between these boundaries the payoff is the maximum of an American put price and an American call price, each of which solves a stopping time problem. This payoff difference affects the European-style component of the EEP formula of the claim (see Remark 1).

9. Complex choosers

This section extends the analysis to complex choosers, where the underlying put and call have different strike prices and/or maturity dates. Suppose that the call option has maturity date T_2^{c} and strike K^{c}, while the put option has maturity date T_2^{p} and strike K^{p}. Assume that $\min\{T_2^{\mathrm{c}}, T_2^{\mathrm{p}}\} > T_1$, where T_1 is the maturity date of the chooser. The strikes $K^{\mathrm{c}}, K^{\mathrm{p}} > 0$ can be in any order.

As for the simple choosers analyzed in the previous sections immediate exercise is suboptimal when it is optimal to continue holding both the underlying American call and put options.

Proposition 6. *Suppose that $t \leqslant T_1$ and $(S,t) \in \mathscr{C}^{\mathrm{c}} \cap \mathscr{C}^{\mathrm{p}}$. Then $(S,t) \in \mathscr{C}^{\mathrm{ch}}$. That is, if immediate exercise is suboptimal for both the American call and put options, then it is also suboptimal for the American chooser option.*

The arguments for suboptimality of exercise parallel those in the case $K^{\mathrm{c}} = K^{\mathrm{p}}$, $T_2^{\mathrm{c}} = T_2^{\mathrm{p}}$ and will not be repeated. Likewise the properties outlined in Proposition 2 hold.

Proposition 7. *The region $\mathscr{E}^{\mathrm{ch}}$ satisfies*

(i) *Non-emptyness*: $(0,t) \in \mathscr{E}^{\mathrm{ch}}$ *for all* $t \in [0, T_1]$.
(ii) *Right-connectedness*: $(S,t) \in \mathscr{E}^{\mathrm{ch}}$ *implies* $(S,s) \in \mathscr{E}^{\mathrm{ch}}$ *for all* $t \in [0,T_1]$, $s \in [t, T_1]$.
(iii) *Up-connectedness*: $(S,t) \in \mathscr{E}_1^{\mathrm{ch}}$ *implies* $(\lambda S, t) \in \mathscr{E}_1^{\mathrm{ch}}$ *for all* $\lambda \geqslant 1$.
(iv) *Down-connectedness*: $(S,t) \in \mathscr{E}_2^{\mathrm{ch}}$ *implies* $(\lambda S, t) \in \mathscr{E}_2^{\mathrm{ch}}$ *for all* $\lambda \leqslant 1$.
(v) *Diagonal behavior*: $(S^*(t), t) \notin \mathscr{E}^{\mathrm{ch}}$ *for all* $t \in [0, T_1)$.

The diagonal behavior, in (v), is the most interesting aspect of this complex contract (for details see the proof in the Appendix). Suppose that[6]

$$b_{T_1}^{\mathrm{c}} < b_{T_1}^{\mathrm{p}}$$

This configuration of boundaries at T_1 implies that $\max\{K^{\mathrm{c}}, (r/\delta)K^{\mathrm{c}}\} < \min\{K^{\mathrm{p}}, (r/\delta)K^{\mathrm{p}}\}$. Assuming that T_1 is sufficiently large, it also implies that there exists a time $T^* < T_1$ at which the boundaries cross, $b_{T^*}^{\mathrm{c}} = b_{T^*}^{\mathrm{p}}$. This follows because the call (put) boundary is nonincreasing (nondecreasing) in time. In the interval $[T^*, T_1]$ there exist a region, denoted $\mathscr{E}^{\mathrm{c,p}} \equiv \mathscr{E}^{\mathrm{c}} \cap \mathscr{E}^{\mathrm{p}}$, in which it is optimal to exercise both underlying contracts. The region $\mathscr{E}^{\mathrm{c,p}}$ may or may not contain part of the diagonal. Suppose that it does not contain any part of the diagonal (see Fig. 8 for an illustration). Even though it is optimal to exercise both underlying contracts in $\mathscr{E}^{\mathrm{c,p}}$, it nevertheless remains suboptimal to exercise the American chooser, as long as the chooser lower boundary has not been reached (i.e., as long as $S > B^2$). In the case where the diagonal cuts through $\mathscr{E}^{\mathrm{c,p}}$, its value is $S^*(t) = \frac{1}{2}(K^{\mathrm{c}} + K^{\mathrm{p}})$ for all t such that

$$b_t^{\mathrm{c}} \leqslant \tfrac{1}{2}(K^{\mathrm{c}} + K^{\mathrm{p}}) \leqslant b_t^{\mathrm{p}}$$

In this scenario, it remains suboptimal to exercise the American chooser along the diagonal component $S^*(t)$ in $\mathscr{E}^{\mathrm{c,p}}$ even at times arbitrarily close to maturity T_1. This result is striking, but the intuition is clear. Along this diagonal, any movement in the price S will increase the chooser payoff: if S increases the call exercise value increases, otherwise it is the put exercise value that increases. Given that the chooser payoff increases for sure in the next infinitesimal time interval, a waiting policy is optimal.

The boundary behavior becomes:

Proposition 8. *The upper exercise boundary B^1 is a continuous, nonincreasing function that satisfies the lower bound $B^1 \geqslant b^{\mathrm{c}} \vee S^*$ and converges to $\max\{b_{T_1}^{\mathrm{c}}, S^*(T_1)\}$ as t approaches the maturity date T_1. The lower exercise boundary B^2 is a continuous, nondecreasing function that satisfies the upper bound $B^2 \leqslant b^{\mathrm{p}} \wedge S^*$ and converges to $\min\{b_{T_1}^{\mathrm{p}}, S^*(T_1)\}$ as maturity approaches. Finally, $B^1 > S^* > B^2$.*

Given these properties the EEP decomposition takes the form:

Theorem 9. *Suppose that the underlying asset price follows the geometric Brownian motion process in* (1) *and consider an American chooser written on an American call and an American put with respective provisions $(K^{\mathrm{c}}, T_2^{\mathrm{c}})$ and $(K^{\mathrm{p}}, T_2^{\mathrm{p}})$. The value of the American chooser option has the EEP representation* (2)–(4) *for $t \in [0, T_1]$. The value of the European chooser written on the underlying American options is given in Remark* 1 *below. The early exercise premium is the sum of two components, $\pi(S_t, t; B^1, B^2) = \pi_1(S_t, t; B^1) + \pi_2(S_t, t; B^2)$, where*

$$\pi_1(S_t, t; B^1) = \mathbb{E}_t\left[\int_t^{T_1} \mathrm{e}^{-r(s-t)} 1_{\{S_s \geqslant B_s^1\}} (\delta S_s - rK^{\mathrm{c}})\,\mathrm{d}s\right] = \int_t^{T_1} \phi_1(S_t; B_s^1, s-t)\,\mathrm{d}s$$

$$\pi_2(S_t, t; B^2) = \mathbb{E}_t\left[\int_t^{T_1} \mathrm{e}^{-r(s-t)} 1_{\{S_s \leqslant B_s^2\}} (rK^{\mathrm{p}} - \delta S_s)\,\mathrm{d}s\right] = \int_t^{T_1} \phi_2(S_t; B_s^2, s-t)\,\mathrm{d}s$$

[6] The configuration $b_{T_1}^{\mathrm{c}} \geqslant b_{T_1}^{\mathrm{p}}$ is essentially similar to the case of common strikes and maturities studied in prior sections.

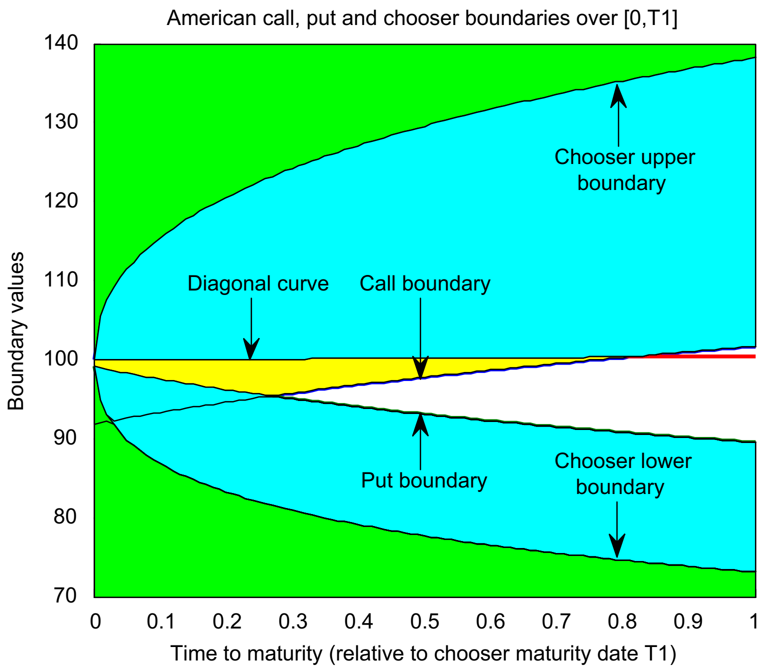

Fig. 8. This figure displays the immediate exercise regions of chooser, put and call options with maturity dates $T_1 = 1$. The American options underlying the chooser have strike prices $K^c = 70, K^p = 130$ and common maturity dates $T_2^c = T_2^p = 1.5$ years. Parameter values are $r = \delta = 0.06$ and $\sigma = 0.20$. The exercise boundaries are computed using the integral equation method with $n = 100$ discretization points (see Section 6). The integral in the European chooser option (Eq. (9)) is approximated using 1,000 discretization points.

with

$$\phi_1(S_t, B_s^1, v) \equiv \delta S_t e^{-\delta v} N(d^U(S_t, B_s^1, v)) - rK^c e^{-rv} N(d_1^U(S_t, B_s^1, v))$$

$$\phi_2(S_t; B_s^2, v) \equiv rK^p e^{-rv} N(d_1^L(S_t, B_s^2, v)) - \delta S_t e^{-\delta v} N(d^L(S_t, B_s^2, v))$$

and

$$d^U(S_t, B_s^1, v) = \frac{\log(S_t/B_s^1) + (r - \delta + \frac{1}{2}\sigma^2)v}{\sigma\sqrt{v}}$$

$$d_1^U(S_t, B_s^1, v) = d^U(S_t, B_s^1, v) - \sigma\sqrt{v}$$

$$d^L(S_t, B_s^2, v) = -\frac{\log(S_t/B_s^2) + (r - \delta + \frac{1}{2}\sigma^2)v}{\sigma\sqrt{v}}$$

$$d_1^L(S_t, B_s^2, v) = d^L(S_t, B_s^2, v) + \sigma\sqrt{v}$$

The immediate exercise boundaries B_t^1, B_t^2 *solve the system of coupled integral equations*

$$B_t^1 - K^c = V(B_t^1, t; B^1, B^2) \tag{12}$$

$$K^p - B_t^2 = V(B_t^2, t; B^1, B^2) \tag{13}$$

for $t \in [0, T_1)$, *subject to the boundary conditions* $\lim_{t\uparrow T_1} B_t^1 = \max\{b_{T_1}^c, S^*(T_1)\}$, $\lim_{t\uparrow T_1} B_t^2 = \min\{b_{T_1}^p, S^*(T_1)\}$.

The European-style component in the EEP formula is:

Remark 3. Under the assumptions of Theorem 9 the value of the European chooser on the underlying American options can be written as $v(S_t,t) = v_1(S_t,t) + v_2(S_t,t) + v_3(S_t,t)$ where

$$\begin{aligned} v_1(S_t,t) &= \mathbb{E}[\mathrm{e}^{-r(T_1-t)} 1_{\{S_{T_1} \geqslant b^{\mathrm{c}}_{T_1} \vee S^*(T_1)\}} \max(C_{T_1}, P_{T_1})] \\ &= S\mathrm{e}^{-\delta(T_1-t)} N(d^{\mathrm{U}}(S_t, b^{\mathrm{c}}_{T_1} \vee S^*(T_1), T_1 - t)) \\ &\quad - K^{\mathrm{c}} \mathrm{e}^{-r(T_1-t)} N(d^{\mathrm{U}}_1(S_t, b^{\mathrm{c}}_{T_1} \vee S^*(T_1), T_1 - t)) \end{aligned}$$

$$\begin{aligned} v_2(S_t,t) &= \mathbb{E}[\mathrm{e}^{-r(T_1-t)} 1_{\{S_{T_1} \leqslant b^{\mathrm{p}}_{T_1} \wedge S^*(T_1)\}} \max(C_{T_1}, P_{T_1})] \\ &= K^{\mathrm{p}} \mathrm{e}^{-r(T_1-t)} N(d^{\mathrm{L}}_1(S_t, b^{\mathrm{p}}_{T_1} \wedge S^*(T_1), T_1 - t)) \\ &\quad - S_t \mathrm{e}^{-\delta(T_1-t)} N(d^{\mathrm{L}}(S_t, b^{\mathrm{p}}_{T_1} \wedge S^*(T_1), T_1 - t)) \end{aligned}$$

$$\begin{aligned} v_3(S_t,t) &= \mathbb{E}_t[\mathrm{e}^{-r(T_1-t)} 1_{\{b^{\mathrm{c}}_{T_1} \vee S^*(T_1) \geqslant S_{T_1} \geqslant b^{\mathrm{p}}_{T_1} \wedge S^*(T_1)\}} \max\{C_{T_1}, P_{T_1}\}] \\ &= \mathrm{e}^{-r(T_1-t)} \int_{b^{\mathrm{p}}_{T_1} \wedge S^*(T_1)}^{b^{\mathrm{c}}_{T_1} \vee S^*(T_1)} \max\{C(w,T_1), P(w,T_1)\} \frac{n(d^{\mathrm{L}}_1(S_t, w, T_1 - t))}{w\sigma\sqrt{T_1 - t}} \, \mathrm{d}w \end{aligned}$$

In these expressions, C_{T_1} (resp. P_{T_1}) is the value of the underlying American call (resp. put) option at the maturity date T_1 and

$$d^{\mathrm{L}}_1(S_t, w, T_1 - t) = \frac{\log(w/S_t) - (r - \delta - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}$$

where $n(\cdot)$ stands for the standard normal density function.

10. Conclusion

This paper has examined the valuation of American-style chooser options, giving the holder the right to choose the best of an American put or an American call before some prescribed maturity date. The optimal exercise decision for the contract was shown to be the first hitting time of a pair of boundaries that solve a system of coupled recursive integral equations. Implementations of the pricing formulas, based on this characterization, were carried out.

Chooser options constitute an example of a flexible product that does not require the holder to have a predetermined view about the course of future events. Simple variations of the chooser products examined in this paper include ratios of puts-to-calls that differ from one. More complex versions can be structured by using exotic derivatives instead of plain vanilla options. The methodology used in this paper can be applied to examine the properties of these alternative contractual forms.

Acknowledgment

We thank three anonymous referees and the editor for helpful comments.

Appendix A. Proofs

Proof of Proposition 1. We need to show that $V(S,t) > \max\{C(S,t), P(S,t)\}$, i.e. $(S,t) \in \mathscr{C}^{\mathrm{ch}}$. Suppose $V(S,t) = \max\{C(S,t), P(S,t)\}$ for some $t \in [0, T_1]$. For example, say $V(S,t) = C(S,t) > P(S,t)$. Consider the portfolio consisting of a long position in the chooser and a short position in the call. Note that the chooser is at least as valuable as the call, at any point up to and including expiry, as it can always be exercised as a call. If the holder of the call exercises for $C(S,t')$ at $t' \in [t, T_1]$, the chooser can be exercised as a call to cover the short position. At expiry, by no arbitrage, $V(S,T_1) = \max\{C(S,T_1), P(S,T_1)\}$. If $P(S,T_1) > C(S,T_1)$, the chooser can be exercised as a put, sold for $P(S,T_1)$, and a call can be bought for $C(S,T_1)$. The profit realized is $P(S,T_1) - C(S,T_1) > 0$. If $C(S,T_1) \geqslant P(S,T_1)$, the chooser can be exercised as a call which offsets the short position. Given that $\mathbb{Q}[P(S,T_1) > C(S,T_1)] > 0$, the portfolio selected represents an arbitrage opportunity.

Similarly, if $V(S,t) = P(S,t) > C(S,t)$ for some $t \in [0, T_1]$, then a long position in the chooser and a short position in the put represents an arbitrage opportunity.

The absence of arbitrage opportunities implies $V(S,t) > \max\{C(S,t), P(S,t)\}$, i.e. $(S,t) \in \mathscr{C}^{\mathrm{ch}}$. □

Proof of Proposition 2. (i) Suppose $S = 0$ for some $t \in [0, T_1]$. As $S = 0$ is an absorbing state of the geometric Brownian motion process, i.e. $\mathbb{Q}[S_{t'} = 0, \forall t' \in [t, T_1] | S_t = 0] = 1$, it follows that $C(0,t) = 0$. In addition, $P(0,t) = K - 0$ because K is the maximal payoff of the American put. Now suppose $V(0,t) > P(0,t) = K$ and consider a short position in the American chooser along with an investment of K at the riskfree rate. This is an arbitrage portfolio, because the proceeds from the riskless investment can be used to cover any payoff from the short position. The absence of arbitrage opportunities implies that $V(0,t) = P(0,t)$, i.e. $(0,t) \in \mathscr{E}^{\mathrm{ch}}$.

(ii) Fix S and let $s \geqslant t$. For $s \geqslant t$ define the new Brownian motion $\widetilde{W}$ such that $\widetilde{W}_s = W_s - W_t$. As the filtration generated by $W_s - W_t$ is identical to the filtration generated by $W_{s-t} - W_0$ the chooser value at time t is

$$V(S,t) = \sup_{\tau \in \mathscr{S}_{0,T_1-t}} \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(SN_\tau,\tau), P(SN_\tau\tau)\}]$$

where $N_\tau \equiv \exp((r - \delta - \frac{1}{2}\sigma^2)\tau + \sigma W_\tau)$. As $\mathscr{S}_{0,T_1-s} \subset \mathscr{S}_{0,T_1-t}$, we have

$$V(S,t) \geqslant V(S,s)$$

Also, by hypothesis, immediate exercise is optimal at t. Suppose $(S,t) \in \mathscr{E}_1^{\mathrm{ch}}$, i.e. $V(S,t) = C(S,t)$. The inclusion $\mathscr{E}_1^{\mathrm{ch}} \subseteq \mathscr{E}^{\mathrm{c}}$, implies $V(S,t) = C(S,t) = S - K$. Therefore, $S - K \geqslant V(S,s)$ from the inequality above. But $V(S,s) \geqslant \max\{C(S,s), P(S,s)\} \geqslant S - K$. Hence, $V(S,s) = S - K$. As $C(S,s) \geqslant S - K$ it also follows that $V(S,s) = C(S,K) = S - K$. We conclude that $(S,s) \in \mathscr{E}_1^{\mathrm{ch}}$. A symmetric argument can be made when $(S,t) \in \mathscr{E}_2^{\mathrm{ch}}$, to show $(S,s) \in \mathscr{E}_2^{\mathrm{ch}}$.

(iii) Suppose $(S,t) \in \mathscr{E}_1^{\mathrm{ch}}$, i.e. $V(S,t) = C(S,t) = S - K$. We seek to show that $(\lambda S,t) \in \mathscr{E}_1^{\mathrm{ch}}$ for all $\lambda \geqslant 1$. Given that $(S,t) \in \mathscr{E}^{\mathrm{c}}$ and that $\mathscr{E}^{\mathrm{c}}$ is up-connected, it follows that $C(\lambda S,t) = \lambda S - K$ for $\lambda \geqslant 1$. Hence, $V(\lambda S,t) \geqslant C(\lambda S,t) = \lambda S - K$ for $\lambda \geqslant 1$. To show that $V(\lambda S,t) \leqslant \lambda S - K$ recall that

$$V(\lambda S,t) = \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(\lambda SN_\tau,\tau), P(\lambda SN_\tau,\tau)\}]$$

where τ is the optimal stopping time in $\mathscr{S}_{0,T_1-t}$. The following sequence of relations holds

$$\begin{aligned}
V(\lambda S,t) &= \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(\lambda SN_\tau,\tau), P(\lambda SN_\tau,\tau)\}] \\
&\leqslant \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(\lambda SN_\tau,\tau), P(SN_\tau,\tau)\}] \\
&= \mathbb{E}[\mathrm{e}^{-r\tau} \max\{P(SN_\tau,\tau), C(SN_\tau,\tau) + (C(\lambda SN_\tau,\tau) - C(SN_\tau,\tau))\}] \\
&\leqslant \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(SN_\tau,\tau), P(SN_\tau,\tau)\}] + \mathbb{E}[\mathrm{e}^{-r\tau}(C(\lambda SN_\tau,\tau) - C(SN_\tau,\tau))] \\
&\leqslant V(S,t) + \mathbb{E}[\mathrm{e}^{-r\tau}(C(\lambda SN_\tau,\tau) - C(SN_\tau,\tau))] \\
&= S - K + \mathbb{E}[\mathrm{e}^{-r\tau}(C(\lambda SN_\tau,\tau) - C(SN_\tau,\tau))] \\
&\leqslant S - K + \mathbb{E}[\mathrm{e}^{-r\tau}(\lambda S - S)N_\tau] \\
&= S - K + (\lambda S - S)\mathbb{E}[\mathrm{e}^{-r\tau}N_\tau] \\
&\leqslant \lambda S - K
\end{aligned}$$

The second line results from the fact that, for $\lambda \geqslant 1$, and fixed $\omega \in \Omega$, we know $SN_\tau \leqslant \lambda SN_\tau$ and thus $P(\lambda SN_\tau,\tau) \leqslant P(SN_\tau,\tau)$. The third line is simply a rewrite of the previous line. The fourth line follows because of the upper bound $\max\{p_1, c_1 + (c_2 - c_1)\} \leqslant \max\{p_1, c_1\} + (c_2 - c_1)$, which holds for $c_2 \geqslant c_1$. The fifth line follows because τ is suboptimal at (S,t). The sixth line follows by hypothesis. The seventh line results from the proof of Proposition 32(iv), p. 74 Detemple (2006). The eighth line holds because, for $\lambda S - S > 0$, $(\lambda S - S)\mathrm{e}^{-rt}N_t$ is a $\mathbb{Q}$-supermartingale.

(iv) Suppose $(S,t) \in \mathscr{E}_2^{\mathrm{ch}}$, i.e. $V(S,t) = P(S,t) = K - S$. We seek to show that $(\lambda S,t) \in \mathscr{E}_2^{\mathrm{ch}}$ for all $\lambda \leqslant 1$. Given that $\mathscr{E}_2^{\mathrm{ch}} \subseteq \mathscr{E}^{\mathrm{p}}$, it follows that $P(\lambda S,t) = K - \lambda S$ for $0 \leqslant \lambda \leqslant 1$. Therefore $V(\lambda S,t) \geqslant P(\lambda S,t) = K - \lambda S$. It remains to show $V(\lambda S,t) \leqslant K - \lambda S$. Recall that

$$V(\lambda S,t) = \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(\lambda SN_\tau,\tau), P(\lambda SN_\tau,\tau)\}]$$

where τ is the optimal stopping time in $\mathscr{S}_{0,T_1-t}$. The following sequence of relations holds

$$\begin{aligned}
V(\lambda S,t) &= \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(\lambda SN_\tau,\tau), P(\lambda SN_\tau,\tau)\}] \\
&\leqslant \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(SN_\tau,\tau), P(\lambda SN_\tau,\tau)\}] \\
&= \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(SN_\tau,\tau), P(SN_\tau,\tau) + (P(\lambda SN_\tau,\tau) - P(SN_\tau,\tau)\}] \\
&\leqslant \mathbb{E}[\mathrm{e}^{-r\tau} \max\{C(SN_\tau,\tau), P(SN_\tau,\tau)\}] \\
&\quad + \mathbb{E}[\mathrm{e}^{-r\tau}(P(\lambda SN_\tau,\tau) - P(SN_\tau,\tau))] \\
&\leqslant V(S,t) + \mathbb{E}[\mathrm{e}^{-r\tau}(P(\lambda SN_\tau,\tau) - P(SN_\tau,\tau))] \\
&= K - S + \mathbb{E}[\mathrm{e}^{-r\tau}(P(\lambda SN_\tau,\tau) - P(SN_\tau,\tau))] \\
&\leqslant K - S + \mathbb{E}[\mathrm{e}^{-r\tau}(S - \lambda S)N_\tau] \\
&= K - S + (S - \lambda S)\mathbb{E}[\mathrm{e}^{-r\tau}N_\tau] \\
&\leqslant K - \lambda S
\end{aligned}$$

The inequality on the second line follows from $SN_\tau \geqslant \lambda SN_\tau$ for $0 \leqslant \lambda \leqslant 1$ and fixed $\omega \in \Omega$, which implies $C(\lambda SN_\tau,\tau) \leqslant C(SN_\tau,\tau)$. The third line is simply a rewrite of the previous line. The fourth line follows from $\max\{c_1, p_1 + (p_2 - p_1)\} \leqslant \max\{c_1, p_1\} + (p_2 - p_1)$, for $p_2 \geqslant p_1$. The fifth line is a consequence of the suboptimality of τ at (S,t). The sixth line follows by hypothesis. The seventh line uses an argument in the proof of Proposition 32(iv) p. 74 in Detemple (2006) for the American put option. The eighth line is obtained because, $S - \lambda S > 0$, $(S - \lambda S)\mathrm{e}^{-rt}N_t$ is a $\mathbb{Q}$-supermartingale.

(v) The proof is by contradiction. Suppose $(S^*(t),t) \in \mathscr{E}^{ch}$ for some $t \in [0,T_1)$. Recall, $\mathscr{E}^{ch}(t) = \mathscr{E}_1^{ch}(t) \cup \mathscr{E}_2^{ch}(t)$ and $\mathscr{E}_1^{ch}(t) \cup \mathscr{E}_2^{ch}(t) = \emptyset$ for all $t \in [0,T_1)$, i.e. $\mathscr{E}^{ch}(t)$ is a disjoint union of sets for each fixed t. As a result, suppose first that $(S^*(t),t) \in \mathscr{E}_1^{ch}$ for some $t \in [0,T_1)$. As $\mathscr{E}_1^{ch}(t) \subset \mathscr{E}^{c}(t) \subset \mathscr{C}^{p}(t)$, we know $P(S^*(t),t) > K - S^*(t)$. Now, using the assumption $(S^*(t),t) \in \mathscr{E}^{ch}$ and the definition of $S^*(t)$, gives

$$\begin{aligned} V(S^*(t),t) &= \max\{C(S^*(t),t), P(S^*(t),t)\} \\ &= C(S^*(t),t) \\ &= P(S^*(t),t) > K - S^*(t) \end{aligned}$$

As $V(S^*(t),t) = P(S^*(t),t)$, selling the put and buying the chooser at time t is a costless strategy. In addition, it is known that for all $t \in [0,T_1]$,

$$V(S(t),t) \geqslant \max\{C(S(t),t), P(S(t),t)\} \geqslant P(S(t),t)$$

Therefore, the portfolio consisting of a long chooser and a short put has the following properties for $v \in [t,T_1]$,

$$\begin{aligned} &V(S(v),v) - P(S(v),v) \geqslant 0 \\ &V(S(v),v) - P(S(v),v) > 0 \quad \text{when } S(v) > S^*(v) \end{aligned}$$

This represents an arbitrage opportunity. If it is now assumed that $(S^*(t),t) \in \mathscr{E}_2^{ch}$ for some $t \in [0,T_1)$ the relations

$$V(S^*(t),t) = P(S^*(t),t) = C(S^*(t),t) > S^*(t) - K$$

hold. Now selling a call and buying a chooser is a costless strategy and for $v \in [t,T_1]$

$$\begin{aligned} &V(S(v),v) - C(S(v),v) \geqslant 0 \\ &V(S(v),v) - C(S(v),v) > \quad \text{when } S(v) < S^*(v) \end{aligned}$$

Again, an arbitrage opportunity. The absence of arbitrage opportunities therefore requires $(S^*(t),t) \notin \mathscr{E}_1^{ch} \cup \mathscr{E}_2^{ch} \equiv \mathscr{E}^{ch}$ for all $t \in [0,T_1)$. $\square$

Proof of Proposition 3. First, we show B^1 is a continuous, nonincreasing function. The proof that B^2 is a continuous, nondecreasing function follows using analogous arguments. Recall from Proposition 2(ii) that $V(S,\cdot)$ is nonincreasing on $[0,T_1]$, i.e. $V(S,t) \geqslant V(S,s)$ for $t \leqslant s$. Suppose that the boundary B^1 fails to be nonincreasing, i.e., $B_t^1 < B_s^1$ for some $s > t$. In other words, suppose $(S,t) \in \mathscr{E}_1^{ch}$ and $(S,s) \notin \mathscr{E}_1^{ch}$ for some $S \geqslant B_t^1$. Now $(S,t) \in \mathscr{E}_1^{ch}$ implies $V(S,t) = S - K$ and $(S,s) \notin \mathscr{E}_1^{ch}$ implies $V(S,s) > S - K = V(S,t)$. But $V(S,t) \geqslant V(S,s)$ because $V(S,\cdot)$ is nonincreasing, a contradiction. We conclude that B^1 must be nonincreasing. As B^1 is nonincreasing, it only remains to show that $B_{t+}^1 \geqslant B_t^1$ in order to have $B_{t+}^1 = B_t^1$. For $v > t$, consider $(B_v^1, v) \in \mathscr{E}_1^{ch}$. As $\mathscr{E}_1^{ch}$ is closed, it follows that $\lim_{v \downarrow t}(B_v^1, v) = (B_{t+}^1, t) \in \mathscr{E}_1^{ch}$. The definition $B_t^1 \equiv \inf\{S : S \in \mathscr{E}_1^{ch}(t)\}$ then implies $B_{t+}^1 \geqslant B_t^1$. Hence, $B_{t+}^1 = B_t^1$, i.e., B^1 is right-continuous.

We now show left-continuity. By the nonincreasing property of B^1, we only need to show $B_{t-}^1 \leqslant B_t^1$. Since $\mathscr{E}_1^{ch} \subseteq \mathscr{E}^{c}$, we can directly apply the argument used to prove $b_{t-}^c \leqslant b_t^c$ (Detemple, 2006, Proposition 33) to show $B_{t-}^1 \leqslant B_t^1$. Thus, we conclude B^1 is continuous.

Now show $\lim_{t \uparrow T_1} B_t^1 = \max(b^c(T_1), S^*(T_1))$ and $\lim_{t \uparrow T_1} B_t^2 = \min(b^p(T_1), S^*(T_1))$.

Case $r = \delta$: Using put/call symmetry, we know $b_{T_1}^c \geqslant K = S^*(T_1)$ and from Proposition 2, $B_{T_1-}^1 \geqslant b_{T_1}^c$. Using these facts, let us suppose that $B_{T_1-}^1 > b_{T_1}^c$ and attempt to find a contradiction. Let $x \in (b_{T_1}^c, B_{T_1-}^1)$. Therefore, $(x,t) \in \mathscr{C}^{ch}$ for $t < T_1$ and

$$V_t(x,t) + V_S(x,t)x(r-\delta) + \tfrac{1}{2}V_{SS}(x,t)x^2\sigma^2 - rV(x,t) = 0$$

Since $V(x,t) \in \mathscr{C}^{2,1}(\mathbb{R}_+ \times [0,T_1])$, taking the limit as $t \uparrow T_1$, we have

$$\begin{aligned} \tfrac{1}{2}V_{SS}(x,T_1)x^2\sigma^2 &= -V_t(x,T_1) - V_S(x,T_1)x(r-\delta) + rV(x,T_1) \\ &= -x(r-\delta) + r(x-K) = \delta x - rK \end{aligned}$$

But, $x \in (b_{T_1}^c, B_{T_1-}^1) \Rightarrow x > b_{T_1}^c \geqslant (r/\delta)K \Rightarrow \delta x > rK$. Therefore, $V_{SS}(x,T_1) > 0$, $\forall x \in (b_{T_1}^c, B_{T_1-}^1)$. Two applications of the fundamental theorem of calculus and the identity $V_S(b_{T_1}^c, T_1) = 1$ yield

$$V(B_{T_1-}^1, T_1) - V(b_{T_1}^c, T_1) - (B_{T_1-}^1 - b_{T_1}^c) = \int_{b_{T_1}^c}^{B_{T_1-}^1} \int_{b_{T_1}^c}^{y} V_{SS}(x,T_1)\,\mathrm{d}x\,\mathrm{d}y$$

However, the left-hand side of the above equation is 0 and the right-hand side is strictly greater than 0, a contradiction. It follows that $B_{T_1-}^1 = b_{T_1}^c$. Analogous arguments show that B^2 converges to $b_{T_1}^p$ as maturity approaches in the case when $r = \delta$.

Case $r < \delta$: We first show $b_{T_1}^p \leqslant S^*(T_1)$. Suppose $b_{T_1}^p > S^*(T_1)$. Let τ_c denote the optimal stopping time for the American call option at $(b_{T_1}^p, T_1)$. The following holds:

$$\begin{aligned} \mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)}(K - b_{T_1}^p N_{T_1,\tau_c})^+] &= C(b_{T_1}^p, T_1) + \mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)}(K - b_{T_1}^p N_{T_1,\tau_c})] \\ &< P(b_{T_1}^p, T_1) \end{aligned}$$

where $N_{t,\tau} \triangleq \exp((r-\delta-\frac{1}{2}\sigma^2)(\tau-t)+\sigma(z_\tau - z_t))$ and z_t is a Brownian motion process. If $b^{\mathrm{p}}_{T_1} > S^*(T_1)$, then $P(b^{\mathrm{p}}_{T_1}, T_1) - C(b^{\mathrm{p}}_{T_1}, T_1) < 0$. This would then imply

$$\mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)}(K - b^{\mathrm{p}}_{T_1} N_{T_1,\tau_c})] < 0 \Leftrightarrow K\mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)}] < b^{\mathrm{p}}_{T_1}\mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)} N_{T_1,\tau_c}]$$
$$\Leftrightarrow b^{\mathrm{p}}_{T_1} > \frac{\mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)}]}{\mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)} N_{T_1,\tau_c}]} K$$

Now, consider the process $\mathrm{e}^{-r(t-T_1)} - \mathrm{e}^{-r(t-T_1)} N_{T_1,t}$ for $T_1 \leqslant t \leqslant T_2$. Note that $\mathbb{E}_{T_1}[\mathrm{e}^{-r(t-T_1)} - \mathrm{e}^{-r(t-T_1)} N_{T_1,t}] = \mathrm{e}^{-r(t-T_1)} - \mathrm{e}^{-\delta(t-T_1)} > 0$, since $r < \delta$. This shows that the process is a submartingale. Then, the optional sampling theorem (Karatzas and Shreve, 1988, p.19). implies

$$b^{\mathrm{p}}_{T_1} > \frac{\mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)}]}{\mathbb{E}_{T_1}[\mathrm{e}^{-r(\tau_c - T_1)} N_{T_1,\tau_c}]} K > K$$

a contradiction. Therefore, $b^{\mathrm{p}}_{T_1} \leqslant S^*(T_1)$. With this, we can now apply the same argument as when $r=\delta$ to prove $B^2_{T_1 -} = b^{\mathrm{p}}_{T_1}$. Thus, $\lim_{t \uparrow T_1} B^2_t = \min(b^{\mathrm{p}}(T_1), S^*(T_1))$.

When $r < \delta$, it is possible that $b^{\mathrm{c}}_{T_1} < S^*(T_1)$. If this is not the case and $b^{\mathrm{c}}_{T_1} \geqslant S^*(T_1)$, the same argument as when $r=\delta$ can be used to conclude $B^1_{T_1 -} = b^{\mathrm{c}}_{T_1}$. We now prove $B^1_{T_1 -} = S^*(T_1)$ if, indeed, $b^{\mathrm{c}}_{T_1} < S^*(T_1)$. First, we show that it cannot be the case that $B^1_{T_1 -} < S^*(T_1)$ and then we show that it also cannot be the case that $B^1_{T_1 -} > S^*(T_1)$. Recall, by no arbitrage, we know that $B^1_{T_1 -} \geqslant b^{\mathrm{c}}_{T_1}$. As a result, let us consider the case $b^{\mathrm{c}}_{T_1} \leqslant B^1_{T_1 -} < S^*(T_1)$. Note we have the following:

$$V(B^1_{T_1 -}, T_1) = P(B^1_{T_1 -}, T_1) > B^1_{T_1 -} - K$$

where the first equality follows since $B^1_{T_1 -} < S^*(T_1)$ and the second inequality follows since $C(B^1_{T_1 -}, T_1) = B^1_{T_1 -} - K$. On the other hand we have

$$V(B^1_{T_1 -}, T_1) = \lim_{t \uparrow T_1} V(B^1_t, t) = \lim_{t \uparrow T_1} B^1_t - K = B^1_{T_1 -} - K$$

This contradiction implies that it cannot be the case that $b^{\mathrm{c}}_{T_1} \leqslant B^1_{T_1 -} < S^*(T_1)$. At this point, we know that $B^1_{T_1 -} \geqslant S^*(T_1)$. We now suppose $B^1_{T_1 -} > S^*(T_1)$ and attempt to find a contradiction. Since $S^*(T_1) < B^1_{T_1 -}$, there exists an $\varepsilon > 0$ such that $S^*(T_1) + \varepsilon < B^1_{T_1 -}$. For $x \in (S^*(T_1)+\varepsilon, B^1_{T_1 -})$, we have, by a continuity argument and $V_t(x, T_1) \leqslant 0$,

$$\begin{aligned}\tfrac{1}{2}\sigma^2 x^2 V_{SS}(x, T_1) &\geqslant -V_S(x, T_1)x(r-\delta) + rV(x, T_1) \\ &= -x(r-\delta) + r(x-K) \\ &= \delta x - rK > 0\end{aligned}$$

where $\delta x - rK > 0$ since $x > S^*(T_1) + \varepsilon > b^{\mathrm{c}}_{T_1} \geqslant (r/\delta)K$. Therefore, we know $V_{SS}(x, T_1) > 0$ for all $x \in (S^*(T_1)+\varepsilon, B^1_{T_1 -})$. Using two applications of the fundamental theorem of calculus and the identity $V_S(S^*(T_1)+\varepsilon, T_1) = 1$ we have:

$$\begin{aligned}&V(B^1_{T_1 -}, T_1) - V(S^*(T_1)+\varepsilon, T_1) - (B^1_{T_1 -} - (S^*(T_1)+\varepsilon)) \\ &\quad = \int_{S^*(T_1)+\varepsilon}^{B^1_{T_1 -}} \int_{S^*(T_1)+\varepsilon}^{y} V_{SS}(s, T_1)\,\mathrm{d}s\,\mathrm{d}y\end{aligned}$$

The left-hand side of the equation above is equal to zero. However, the right-hand side equals zero if and only if $V_{SS}(x, T_1) = 0$ almost everywhere in $(S^*(T_1)+\varepsilon, B^1_{T_1 -})$. But we already showed that $V_{SS}(x, T_1) > 0$ for all points $x \in (S^*(T_1)+\varepsilon, B^1_{T_1 -})$. This contradiction implies there must not exist an $\varepsilon > 0$ such that $S^*(T_1) + \varepsilon < B^1_{T_1 -}$, i.e. $S^*(T_1) \nless B^1_{T_1 -}$. At this point, we conclude $S^*(T_1) = B^1_{T_1 -}$. Therefore, when $r < \delta$, we have $\lim_{t \uparrow T_1} B^1_t = \max(b^{\mathrm{c}}(T_1), S^*(T_1))$ and $\lim_{t \uparrow T_1} B^2_t = \min(b^{\mathrm{p}}(T_1), S^*(T_1))$.

Case $r > \delta$: Analogous arguments used when $r < \delta$ can be employed in order to show $B^1_{T_1 -} = b^{\mathrm{c}}_{T_1}$, $B^2_{T_1 -} = S^*(T_1)$, i.e. $\lim_{t \uparrow T_1} B^1_t = \max(b^{\mathrm{c}}(T_1), S^*(T_1))$ and $\lim_{t \uparrow T_1} B^2_t = \min(b^{\mathrm{p}}(T_1), S^*(T_1))$. □

Proof of Theorem 4. The arguments above Theorem 4 establish that $\pi(S_t, t; B^1, B^2) = \pi_1(S_t, t; B^1) + \pi_2(S_t, t; B^2)$, where

$$\begin{aligned}\pi_1(S_t, t; B^1) &= \mathbb{E}_t\left[\int_t^{T_1} \mathrm{e}^{-r(s-t)} 1_{\{S_s \geqslant B^1_s\}}(\delta S_s - rK)\,\mathrm{d}s\right] \\ &= \int_t^{T_1} \mathrm{e}^{-r(s-t)} \mathbb{E}_t[1_{\{S_s \geqslant B^1_s\}}(\delta S_s - rK)]\,\mathrm{d}s \\ &\equiv \int_t^{T_1} \phi_1(S_t; B^1_s, s-t)\,\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
\pi_2(S_t, t; B^2) &= \mathbb{E}_t\left[\int_t^{T_1} \mathrm{e}^{-r(s-t)} 1_{\{S_s \leqslant B_s^2\}}(rK - \delta S_s)\,\mathrm{d}s\right] \\
&= \int_t^{T_1} \mathrm{e}^{-r(s-t)} \mathbb{E}_t[1_{\{S_s \leqslant B_s^2\}}(rK - \delta S_s)]\,\mathrm{d}s \\
&\equiv \int_t^{T_1} \phi_2(S_t; B_s^2, s-t)\,\mathrm{d}s
\end{aligned}$$

Calculating the expectation in $\phi_1(S_t; B_s^1, s-t)$ gives

$$\begin{aligned}
&\phi_1(S_t, B_s^1, s-t) \\
&\quad = \mathrm{e}^{-r(s-t)} \mathbb{E}_t[1_{\{S_s \geqslant B_s^1\}}(\delta S_s - rK)] \\
&\quad = \mathrm{e}^{-r(s-t)} \int_{-d_1^{\mathrm{U}}(S_t, B_s^1, s-t)}^{\infty} (\delta S_t \mathrm{e}^{(r-\delta-(1/2)\sigma^2)(s-t)+\sigma z\sqrt{s-t}} - rK) n(z)\,\mathrm{d}z \\
&\quad = \delta S_t \mathrm{e}^{-\delta(s-t)} \int_{-d_1^{\mathrm{U}}(S_t, B_s^1, s-t)}^{\infty} \mathrm{e}^{-(1/2)\sigma^2(s-t)+\sigma z\sqrt{s-t}} n(z)\,\mathrm{d}z \\
&\qquad - rK\mathrm{e}^{-r(s-t)} \int_{-d_1^{\mathrm{U}}(S_t, B_s^1, s-t)}^{\infty} n(z)\,\mathrm{d}z \\
&\quad = \delta S_t \mathrm{e}^{-\delta(s-t)} N(d_1^{\mathrm{U}}(S_t, B_s^1, s-t) + \sigma\sqrt{s-t}) - rK\mathrm{e}^{-r(s-t)} N(d_1^{\mathrm{U}}(S_t, B_s^1, s-t)) \\
&\quad = \delta S_t \mathrm{e}^{-\delta v} N(d^{\mathrm{U}}(S_t, B_s^1, s-t)) - rK\mathrm{e}^{-rv} N(d_1^{\mathrm{U}}(S_t, B_s^1, s-t))
\end{aligned}$$

with

$$d^{\mathrm{U}}(S_t, B_s^1, v) = \frac{\log(S_t/B_s^1) + (r - \delta + \frac{1}{2}\sigma^2)v}{\sigma\sqrt{v}}$$

$$d_1^{\mathrm{U}}(S_t, B_s^1, v) = d^{\mathrm{U}}(S_t, B_s^1, v) - \sigma\sqrt{v}$$

Similar calculations show that

$$\phi_2(S_t; B_s^2, v) \equiv rK\mathrm{e}^{-rv} N(d_1^{\mathrm{L}}(S_t, B_s^2, v)) - \delta S_t \mathrm{e}^{-\delta v} N(d^{\mathrm{L}}(S_t, B_s^2, v))$$

with

$$d^{\mathrm{L}}(S_t, B_s^2, v) = -\frac{\log(S_t/B_s^2) + (r - \delta + \frac{1}{2}\sigma^2)v}{\sigma\sqrt{v}}$$

$$d_1^{\mathrm{L}}(S_t, B_s^2, v) = d^{\mathrm{L}}(S_t, B_s^2, v) + \sigma\sqrt{v}$$

The integral Eqs. (5)–(6) follow from the fact that immediate exercise is optimal along the boundaries. □

Proof of Remark 1. The derivations of these formulas are similar to those in the proof of Theorem 4. To find the formula for $v_1(S_t, t)$ note that

$$\begin{aligned}
v_1(S_t, t) &= \mathbb{E}_t[\mathrm{e}^{-r(T_1-t)} 1_{\{S_{T_1} \geqslant b_{T_1}^{\mathrm{c}} \vee S^*(T_1)\}} \max\{C_{T_1}, P_{T_1}\}] \\
&= \mathbb{E}_t[\mathrm{e}^{-r(T_1-t)} 1_{\{S_{T_1} \geqslant b_{T_1}^{\mathrm{c}} \vee S^*(T_1)\}} (S_{T_1} - K)] \\
&= \mathrm{e}^{-r(T_1-t)} S_t \int_{-d_1^{\mathrm{U}}(S_t, b_{T_1}^{\mathrm{c}} \vee S^*(T_1), T_1-t)}^{\infty} \mathrm{e}^{(r-\delta-(1/2)\sigma^2)(T_1-t)+\sigma z\sqrt{T_1-t}} n(z)\,\mathrm{d}z \\
&\quad - \mathrm{e}^{-r(T_1-t)} K \int_{-d_1^{\mathrm{U}}(S_t, b_{T_1}^{\mathrm{c}} \vee S^*(T_1), T_1-t)}^{\infty} n(z)\,\mathrm{d}z \\
&= \mathrm{e}^{-\delta(T_1-t)} S_t N(d^{\mathrm{U}}(S_t, b_{T_1}^{\mathrm{c}} \vee S^*(T_1), T_1-t)) \\
&\quad - K\mathrm{e}^{-r(T_1-t)} N(d_1^{\mathrm{U}}(S_t, b_{T_1}^{\mathrm{c}} \vee S^*(T_1), T_1-t))
\end{aligned}$$

The formula for $v_2(S_t, t)$ is obtained by using similar arguments. For $v_3(S_t, t)$ note that

$$\begin{aligned}
v_3(S_t, t) &= \mathbb{E}_t[\mathrm{e}^{-r(T_1-t)} 1_{\{b_{T_1}^{\mathrm{c}} \vee S^*(T_1) \geqslant S_{T_1} \geqslant b_{T_1}^{\mathrm{p}} \wedge S^*(T_1)\}} \max\{C_{T_1}, P_{T_1}\}] \\
&= \mathrm{e}^{-r(T_1-t)} \int_{b_{T_1}^{\mathrm{p}} \wedge S^*(T_1)}^{b_{T_1}^{\mathrm{c}} \vee S^*(T_1)} \max\{C(w, T_1), P(w, T_1)\} \times \frac{n(d_1^{\mathrm{L}}(S_t, w, T_1-t))}{w\sigma\sqrt{T_1-t}}\,\mathrm{d}w
\end{aligned}$$

where $C_{T_1} = C(w, T_1), P_{T_1} = P(w, T_1)$ and where the second line follows from

$$\begin{aligned} w &= S_t \exp((r - \delta - \tfrac{1}{2}\sigma^2)(T_1 - t) + \sigma z\sqrt{T_1 - t}) \\ \Leftrightarrow z &= \frac{\log(w/S_t) - (r - \delta - 1/2\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \equiv d_1^{\mathrm{L}}(S_t, w, T_1 - t) \end{aligned}$$

and

$$n(z)\,\mathrm{d}z = n(d_1^{\mathrm{L}}(S_t, w, T_1 - t))\frac{1}{w\sigma\sqrt{T_1 - t}}\,\mathrm{d}w$$

This completes the proof. $\square$

Proof of Theorem 5. Formulas for the delta and the gamma of the American chooser are obtained by direct differentiation of the components in the EEP representation. $\square$

Proof of Proposition 7. (i)–(iv) The arguments provided in the proofs of these properties for the standard American chooser still hold for the complex chooser examined here.

(v) Recall, in the standard American chooser we had

$$\begin{aligned} &\mathscr{E}^{\mathrm{ch}}(t) = \mathscr{E}_1^{\mathrm{ch}}(t) \cup \mathscr{E}_2^{\mathrm{ch}}(t) \\ &\mathscr{E}_1^{\mathrm{ch}}(t) \cap \mathscr{E}_2^{\mathrm{ch}}(t) = \emptyset, \quad \forall t \in [0, T_1) \end{aligned}$$

Disjointedness of the exercise region is not guaranteed in the complex chooser setting. Thus, to complete the proof, we must examine the case when there exists $T^* < T_1$ such that $b_{T^*}^{\mathrm{c}} = b_{T^*}^{\mathrm{p}}$ and show that the diagonal curve $(S^*(\cdot), \cdot)$ is not contained in $\mathscr{E}^{\mathrm{c,p}}$.

Suppose $b_{T_1}^{\mathrm{c}} > S^*(T_1)$ or $b_{T_1}^{\mathrm{p}} < S^*(T_1)$. In either of these situations, there cannot exist a $t < T_1$ such that $(S^*(t), t) \in \mathscr{E}^{\mathrm{c,p}}$ since it would then require $b_{T_1}^{\mathrm{c}} \leqslant S^*(T_1) \leqslant b_{T_1}^{\mathrm{p}}$ (if $(S^*(t), t) \in \mathscr{E}^{\mathrm{c,p}}$ then $S^*(t) = K^{\mathrm{c}} + K^{\mathrm{p}}/2 \in [b_t^{\mathrm{c}}, b_t^{\mathrm{p}}] \subseteq [b_{T_1}^{\mathrm{c}}, b_{T_1}^{\mathrm{p}}]$). As a result, we only have to consider the case when

$$b_{T_1}^{\mathrm{c}} \leqslant S^*(T_1) \leqslant b_{T_1}^{\mathrm{p}}$$

In this situation, we know that $S^*(T_1) = K^{\mathrm{c}} + K^{\mathrm{p}}/2$ and moreover we know $S^*(t) = K^{\mathrm{c}} + K^{\mathrm{p}}/2$ for all t such that $b_t^{\mathrm{c}} \leqslant K^{\mathrm{c}} + K^{\mathrm{p}}/2 \leqslant b_t^{\mathrm{p}}$. Thus, the proof will be complete if we can show that for all t satisfying $b_t^{\mathrm{c}} \leqslant K^{\mathrm{c}} + K^{\mathrm{p}}/2 \leqslant b_t^{\mathrm{p}}$, we have $(S^*(t), t) \notin \mathscr{E}^{\mathrm{ch}}$.

Let $t \in [0, T_1]$ be such that $b_t^{\mathrm{c}} \leqslant K^{\mathrm{c}} + K^{\mathrm{p}}/2 \leqslant b_t^{\mathrm{p}}$. For some fixed time $s > t$ the following holds:

$$\begin{aligned} V(S^*(T_1), t) &\geqslant \mathbb{E}_t[\mathrm{e}^{-r(s-t)} \max\{C(S^*(T_1)N_{t,s}, s), P(S^*(T_1)N_{t,s}, s)\}] \\ &\geqslant \mathbb{E}_t[\mathrm{e}^{-r(s-t)} \max\{S^*(T_1)N_{t,s} - K^{\mathrm{c}}, K^{\mathrm{p}} - S^*(T_1)N_{t,s}\}] \\ &= \mathbb{E}_t[\mathrm{e}^{-r(s-t)}\{S^*(T_1)N_{t,s} - K^{\mathrm{c}}\}] \\ &\quad + \mathbb{E}_t[\mathrm{e}^{-r(s-t)}\{(K^{\mathrm{p}} - S^*(T_1)N_{t,s}) - (S^*(T_1)N_{t,s} - K^{\mathrm{c}})\}^+] \\ &= S^*(T_1)\mathrm{e}^{-\delta(s-t)} - K^{\mathrm{c}}\mathrm{e}^{-r(s-t)} \\ &\quad + \mathbb{E}_t[\mathrm{e}^{-r(s-t)}\{(K^{\mathrm{p}} - S^*(T_1)N_{t,s}) - (S^*(T_1)N_{t,s} - K^{\mathrm{c}})\}^+] \end{aligned}$$

where $N_{t,s} \triangleq \exp((r - \delta - \sigma^2/2)(s - t) + \sigma z\sqrt{s - t})$. Let $\widetilde{V}(t, s)$ denote the right-hand side of the last equation above. Now note, as $s \to t$ the limits

$$S^*(T_1)\mathrm{e}^{-\delta(s-t)} - K^{\mathrm{c}}\mathrm{e}^{-r(s-t)} \to S^*(T_1) - K^{\mathrm{c}}$$

$$\mathbb{E}_t[\mathrm{e}^{-r(s-t)}\{(K^{\mathrm{p}} - S^*(T_1)N_{t,s}) - (S^*(T_1)N_{t,s} - K^{\mathrm{c}})\}^+] \to 0$$

hold. We next show that convergence is from above. If this holds, then there exists a time $s \in [t, T_1]$ such that

$$V(S^*(T_1), t) \geqslant \widetilde{V}(t, s) > S^*(T_1) - K^{\mathrm{c}} = C(S^*(T_1), t) = P(S^*(T_1), t)$$

This implies $(S^*(t), t) \notin \mathscr{E}^{\mathrm{ch}}$. We now show there exists a time $s \in [t, T_1]$ such that

$$\begin{aligned} 0 &< S^*(T_1)(\mathrm{e}^{-\delta(s-t)} - 1) - K^{\mathrm{c}}(\mathrm{e}^{-r(s-t)} - 1) \\ &\quad + \mathbb{E}_t[\mathrm{e}^{-r(s-t)}\{(K^{\mathrm{p}} - S^*(T_1)N_{t,s}) - (S^*(T_1)N_{t,s} - K^{\mathrm{c}})\}^+] \end{aligned}$$

For $h \in [0, T_1 - t]$, let

$$\begin{aligned} &\Gamma(h) \equiv S^*(T_1)(\mathrm{e}^{-\delta h} - 1) - K^{\mathrm{c}}(\mathrm{e}^{-rh} - 1) \\ &\Psi(h) \equiv \mathrm{e}^{-rh}\mathbb{E}[\{(K^{\mathrm{p}} - S^*(T_1)N_{0,h}) - (S^*(T_1)N_{0,h} - K^{\mathrm{c}})\}^+] \end{aligned}$$

Straightforward calculations show $\Gamma(0) = 0$ and $\Gamma'(0) = rK^{c} - \delta S^{*}(T_1)$. For $\Psi(h)$, we have the following

$$\begin{aligned}
\Psi(h) &\equiv \mathrm{e}^{-rh}\mathbb{E}[\{(K^{p} - S^{*}(T_1)N_{0,h}) - (S^{*}(T_1)N_{0,h} - K^{c})\}^{+}] \\
&= \mathrm{e}^{-rh}\int_{-\infty}^{d(S^{*}(T_1),(K^{p}+K^{c})/2,h)} \{(K^{p} - S^{*}(T_1)N_{0,h}) - (S^{*}(T_1)N_{0,h} - K^{c})\} n(z)\,\mathrm{d}z \\
&= \mathrm{e}^{-rh}(K^{p} + K^{c})N\left(d\left(S^{*}(T_1), \frac{K^{p}+K^{c}}{2}, h\right)\right) \\
&\quad - 2S^{*}(T_1)\mathrm{e}^{-\delta h}N\left(d\left(S^{*}(T_1), \frac{K^{p}+K^{c}}{2}, h\right) - \sigma\sqrt{h}\right)
\end{aligned}$$

where

$$d\left(S^{*}(T_1), \frac{K^{p}+K^{c}}{2}, h\right) \triangleq \frac{\log\left(\dfrac{K^{p}+K^{c}}{2S^{*}(T_1)}\right) - \left(r - \delta - \dfrac{\sigma^2}{2}\right)h}{\sigma\sqrt{h}} = -\frac{\left(r - \delta - \frac{\sigma^2}{2}\right)\sqrt{h}}{\sigma}$$

Note, $\Psi(h) = 0$. Also, we have

$$\begin{aligned}
\Psi'(h) = &-r\mathrm{e}^{-rh}(K^{p} + K^{c})N\left(d\left(S^{*}(T_1), \frac{K^{p}+K^{c}}{2}, h\right)\right) \\
&+ \mathrm{e}^{-rh}(K^{p} + K^{c})n\left(d\left(S^{*}(T_1), \frac{K^{p}+K^{c}}{2}, h\right)\right)\left(-\frac{r - \delta - \dfrac{\sigma^2}{2}}{2\sigma\sqrt{h}}\right) \\
&+ 2\delta S^{*}(T_1)\mathrm{e}^{-\delta h}N\left(d\left(S^{*}(T_1), \frac{K^{p}+K^{c}}{2}, h\right) - \sigma\sqrt{h}\right) \\
&- 2S^{*}(T_1)\mathrm{e}^{-\delta h}n\left(d\left(S^{*}(T_1), \frac{K^{p}+K^{c}}{2}, h\right) - \sigma\sqrt{h}\right)\left(-\frac{r - \delta + \dfrac{\sigma^2}{2}}{2\sigma\sqrt{h}}\right)
\end{aligned}$$

From this, it is easy to verify $\Psi'(0) = \infty$. Therefore, there exists an $h > 0$ such that for $s = t + h$ we have

$$\begin{aligned}
0 &< S^{*}(T_1)(\mathrm{e}^{-\delta(s-t)} - 1) - K^{c}(\mathrm{e}^{-r(s-t)} - 1) \\
&\quad + \mathbb{E}_t[\mathrm{e}^{-r(s-t)}\{(K^{p} - S^{*}(T_1)N_{t,s}) - (S^{*}(T_1)N_{t,s} - K^{c})\}^{+}].
\end{aligned}$$

Thus, we have shown

$$V(S^{*}(T_1), t) > C(S^{*}(T_1), t) = P(S^{*}(T_1), t)$$

This completes the proof of the proposition. □

Proof of Proposition 8. The argument that the upper (resp. lower) exercise boundary B^1 (resp. B^2) is a continuous, nonincreasing (resp. nondecreasing) function follows exactly as in the standard chooser setting. To complete the proof, we focus on establishing the limits $B^1_{T_1-}$ and $B^2_{T_1-}$. The primary complicating feature for the complex chooser is that the exercise boundaries of the underlying call and put options can intersect prior to expiry T_1. If this does not happen, the limits $B^1_{T_1-}$ and $B^2_{T_1-}$ follow as in the standard chooser setting. We, therefore, concentrate on the case where the underlying call and put exercise boundaries intersect prior to T_1, that is $b^{c}_{T_1} < b^{p}_{T_1}$.

Case $r = \delta$: Since $b^{c}_{T_1} \geqslant K^{c}$ and $b^{p}_{T_1} \leqslant K^{p}$, we have that $K^{c} < K^{p}$. Using put/call symmetry and the monotonicity of the American call price with respect to the strike, we have

$$P(K^{p}, K^{p}, r, r, T_2 - T_1) = C(K^{p}, K^{p}, r, r, T_2 - T_1) < C(K^{p}, K^{c}, r, r, T_2 - T_1)$$

where $P(S, K, r, \delta, T)$ (resp. $C(S, K, r, \delta, T)$) is the value of an American Put (resp. American Call) when the current stock price is S, the strike price is K, the interest rate is r, the dividend rate δ and the time-to-maturity T. From this inequality, we know that $S^{*}(T_1) \leqslant K^{p}$. Similarly, the following also holds:

$$P(K^{c}, K^{p}, r, r, T_2 - T_1) > P(K^{c}, K^{c}, r, r, T_2 - T_1) = C(K^{c}, K^{c}, r, r, T_2 - T_1)$$

This implies that $S^{*}(T_1) \geqslant K^{c}$. Therefore, we know

$$K^{c} \leqslant S^{*}(T_1) \leqslant K^{p}$$

Suppose now that $b^{c}_{T_1} < S^{*}(T_1)$. We would like to show that $B^1_{T_1-} = S^{*}(T_1)$, if indeed $b^{c}_{T_1} < S^{*}(T_1)$. First, consider the case $b^{c}_{T_1} \leqslant B^1_{T_1-} < S^{*}(T_1)$. The following holds:

$$V(B^1_{T_1-}, T_1) = P(B^1_{T_1-}, T_1) > B^1_{T_1-} - K^{c}$$

where the first equality follows since $B^1_{T_1-} < S^*(T_1)$ and the second inequality follows since $C(B^1_{T_1-}, T_1) = B^1_{T_1-} - K^c$. On the other hand we have

$$V(B^1_{T_1-}, T_1) = \lim_{t \uparrow T_1} V(B^1_t, t) = \lim_{t \uparrow T_1} B^1_t - K^c = B^1_{T_1-} - K^c$$

This contradiction implies $B^1_{T_1-} \geqslant S^*(T_1)$. Now suppose $B^1_{T_1-} > S^*(T_1)$. Since $S^*(T_1) < B^1_{T_1-}$, there exists $\varepsilon > 0$ such that $S^*(T_1) + \varepsilon < B^1_{T_1-}$. For $x \in (S^*(T_1) + \varepsilon, B^1_{T_1-})$, we have by a continuity argument and $V_t(x, T_1) \leqslant 0$,

$$\begin{aligned} \tfrac{1}{2}\sigma^2 x^2 V_{SS}(x, T_1) &\geqslant -V_S(x, T_1)x(r-\delta) + rV(x, T_1) \\ &= -x(r-\delta) + r(x - K^c) \\ &= \delta x - rK^c > 0 \end{aligned}$$

where $\delta x - rK^c > 0$ since $x > S^*(T_1) + \varepsilon > b^c_{T_1} \geqslant r/\delta K^c$. Therefore, we know $V_{SS}(x, T_1) > 0$ for all $x \in (S^*(T_1) + \varepsilon, B^1_{T_1-})$. Using two applications of the fundamental theorem of calculus and the identity $V_S(S^*(T_1) + \varepsilon, T_1) = 1$ we have

$$\begin{aligned} &V(B^1_{T_1-}, T_1) - V(S^*(T_1) + \varepsilon, T_1) - (B^1_{T_1-} - (S^*(T_1) + \varepsilon)) \\ &\quad = \int_{S^*(T_1)+\varepsilon}^{B^1_{T_1-}} \int_{S^*(T_1)+\varepsilon}^{y} V_{SS}(s, T_1)\,\mathrm{d}s\,\mathrm{d}y \end{aligned}$$

The left-hand side of the equation above is equal to zero. However, the right-hand side equals zero if and only if $V_{SS}(x, T_1) = 0$ almost everywhere in $(S^*(T_1) + \varepsilon, B^1_{T_1-})$. But we already showed $V_{SS}(x, T_1) > 0$ for all $x \in (S^*(T_1) + \varepsilon, B^1_{T_1-})$. This contradiction implies there cannot exist an $\varepsilon > 0$ such that $S^*(T_1) + \varepsilon < B^1_{T_1-}$. Therefore, we conclude that $S^*(T_1) = B^1_{T_1-}$ if $b^c_{T_1} < S^*(T_1)$.

If $b^c_{T_1} \geqslant S^*(T_1)$, then the argument used to prove $B^1_{T_1-} = b^c_{T_1}$ when $r = \delta$ in the standard American chooser case (i.e. $K^c = K^p$, $T^c_2 = T^p_2$) can be used to show $B^1_{T_1-} = b^c_{T_1}$. Therefore, when $r = \delta$, $\lim_{t \uparrow T_1} B^1_t = \max(b^c_{T_1}, S^*(T_1))$.

An entirely symmetric argument as above can be applied to prove $\lim_{t \uparrow T_1} B^2_t = \min(b^p_{T_1}, S^*(T_1))$.

Case $r \neq \delta$: Recall, we are considering the case when $K^c \leqslant b^c_{T_1} < b^p_{T_1} \leqslant K^p$. We first consider the limiting behavior of the boundary for the put side of the chooser. Therefore, suppose $S^*(T_1) < b^p_{T_1}$. We show $B^2_{T_1-} = S^*(T_1)$ via contradiction. Suppose first that $B^2_{T_1-} > S^*(T_1)$. Note the following holds:

$$\begin{aligned} V(B^2_{T_1-}, T_1) &= C(B^2_{T_1-}, K^c, T_1) > P(B^2_{T_1-}, K^p, T_1) \geqslant K^p - B^2_{T_1-} \\ V(B^2_{T_1-}, T_1) &= \lim_{t \uparrow T_1} V(B^2_t, t) = K^p - B^2_{T_1-} \end{aligned}$$

This contradiction implies that $B^2_{T_1-} \leqslant S^*(T_1)$. Suppose now that $B^2_{T_1-} < S^*(T_1)$. For any $\varepsilon > 0$, let $x \in (B^2_{T_1-}, S^*(T_1) - \varepsilon)$. The following holds:

$$\begin{aligned} \tfrac{1}{2}\sigma^2 x^2 V_{SS}(x, T_1) &= -V_t(x, T_1) - V_S(x, T_1)x(r-\delta) + rV(x, T_1) \\ &= x(r-\delta) + r(K^p - x) \\ &= rK^p - \delta x > 0 \end{aligned}$$

where $rK^p - \delta x > 0$ since $x < b^p_{T_1} \leqslant r/\delta K^p$. Therefore, we know $V_{SS}(x, T_1) > 0$ for all $x \in (B^2_{T_1-}, S^*(T_1) - \varepsilon)$. Using two applications of the fundamental theorem of calculus and the identity $V_S(S^*(T_1) - \varepsilon, T_1) = -1$ we have

$$\begin{aligned} &V(B^2_{T_1-}, T_1) - V(S^*(T_1) - \varepsilon, T_1) + (B^2_{T_1-} - (S^*(T_1) - \varepsilon)) \\ &\quad = \int_{B^2_{T_1-}}^{S^*(T_1)-\varepsilon} \int_{y}^{S^*(T_1)-\varepsilon} V_{SS}(s, T_1)\,\mathrm{d}s\,\mathrm{d}y \end{aligned}$$

The left-hand side of the equation above is equal to zero. However, the right-hand side equals zero if and only if $V_{SS}(x, T_1) = 0$ almost everywhere in $(B^2_{T_1-}, S^*(T_1) - \varepsilon)$. But we already showed $V_{SS}(x, T_1) > 0$ for all $x \in (B^2_{T_1-}, S^*(T_1) - \varepsilon)$. This contradiction implies there must not exist an $\varepsilon > 0$ such that $S^*(T_1) - \varepsilon > B^2_{T_1-}$. Thus, we have shown $B^2_{T_1-} = S^*(T_1)$.

Suppose $b^p_{T_1} \leqslant S^*(T_1)$. If $B^2_{T_1-} < b^p_{T_1}$, one can consider $x \in (B^2_{T_1-}, b^p_{T_1})$, and apply the argument used throughout relying upon the Black–Scholes PDE in the continuation region of the chooser and the fundamental theorem of calculus with $V_S(b^p_{T_1}, T_1) = -1$ to prove that $B^2_{T_1-} \geqslant b^p_{T_1}$ and thus $B^2_{T_1-} = b^p_{T_1}$.

The limiting behavior of the call side of the chooser when $r \neq \delta$ can be shown to satisfy $B^1_{T_1-} = \max(b^c_{T_1}, S^*(T_1))$ using symmetric arguments to the put side of the chooser shown above and thus will not be repeated here. □

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