FE - 680 Assignment 1

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```
import os
import numpy as np
import pandas as pd
from scipy.stats import norm
import seaborn as sns

import matplotlib.pyplot as plt
```

```
os.chdir(r'D:\Documents\Stevens\FE-680\HW1')
os.getcwd()

#print(os.listdir())
```

```
'D:\\Documents\\Stevens\\FE-680\\HW1'
```

Problem 1

- Fill in forward curve, discount curve, zero curve

Forward Rate

$r_0(t_1, t_2) =$ Forward rate between time t_1 and t_2

$[1 + r_0(0,1)][1 + r_0(1,2)] = [1 + r_0(0,2)^2]$

$[1 + r_0(1,2)] = \frac{[1+r_0(0,2)^2]}{[1+r_0(0,1)]}$

$r_0(1,2) = \frac{[1+r_0(0,2)^2]}{[1+r_0(0,1)]} - 1$

$f(t) = \frac{d(t-1)}{d(t)} - 1$

Discount Rate

$d_1 = \frac{1}{(1+f_1)}$

$d_2 = \frac{1}{(1+f_1)(1+f_2)}$

$d_n = \frac{1}{(1+f_1)(1+f_2)...(1+f_n)}$

Zero Rate

$z(t) = (\frac{1}{d(t)})^{\frac{1}{t}} - 1$

Par Coupon Rate

$c = \frac{1-d(t)}{\Sigma d(i)}$

```
def forward(dt_1, dt):
    return(  ((dt_1 / dt)-1) * 100  )

def discount_rate(list_of_rates):
    new_d = [((i/100)+1) for i in list_of_rates]
    denominator = np.prod(new_d)
    return( round( 1/denominator , ndigits = 4) )

def zero(dt,t):
    return( ((1/dt)**(1/t) - 1)*100 )

def coupon(list_of_dts):
    l = len(list_of_dts)
    number = (1 - list_of_dts[(l-1)]) / np.sum(list_of_dts)
    return( number*100 )
```

```
dn = [5, 5.2]
discount_rate(dn)
```

```
0.9053
```

```
df = pd.DataFrame({'TYPE': ['OVERNIGHT', 'CASH', 'CASH', 'FORWARDS', 'FORWARDS', 'FORWARDS', 'SWAPS', 'SWAPS','SWAPS','SWAPS','SWAPS',],
                   'YEARS': np.arange(0,11,1),
                   'INPUTS': [1.9, 2.2, 2.4, 2.5, 2.75, 2.8, 2.85, 3.1, 3.150, 3.3, 3.45] })
n_row, n_col = df.shape
df.head(45)
```

	TYPE	YEARS	INPUTS
0	OVERNIGHT	0	1.90
1	CASH	1	2.20
2	CASH	2	2.40
3	FORWARDS	3	2.50
4	FORWARDS	4	2.75
5	FORWARDS	5	2.80
6	SWAPS	6	2.85
7	SWAPS	7	3.10
8	SWAPS	8	3.15
9	SWAPS	9	3.30
10	SWAPS	10	3.45

```
DISCOUNT = []
DISCOUNT.append(1.0)
print(DISCOUNT)

for i in range(1, n_row):
    DISCOUNT.append( discount_rate( df['INPUTS'].iloc[1:i] ) )
df.insert(3, 'DISCOUNT_RATE', DISCOUNT)
```

```
[1.0]
```

```
FORWARD = []
FORWARD.append(1.9)
FORWARD.append(2.2)

for i in range(2, n_row):
    FORWARD.append( forward( df['DISCOUNT_RATE'].iloc[i-1],  df['DISCOUNT_RATE'].iloc[i]))

df.insert(4, 'FORWARD_RATE', FORWARD)
```

```
COUPON = []
COUPON.append(1.9)
COUPON.append(2.2)

for i in range(2, n_row):
    COUPON.append( coupon(df['DISCOUNT_RATE'].iloc[0:i])  )

df.insert(5, 'COUPON_RATE', COUPON)
```

```
ZERO = []
ZERO.append(1.0)

for i in range(1,n_row):
    ZERO.append( zero( df['DISCOUNT_RATE'].iloc[i], df['YEARS'].iloc[i] ) )

df.insert( 6, 'ZERO_RATE', ZERO )
```

```
df.head(45)
```

	TYPE	YEARS	INPUTS	DISCOUNT_RATE	FORWARD_RATE	COUPON_RATE	ZERO_RATE
0	OVERNIGHT	0	1.90	1.0000	1.900000	1.900000	1.000000
1	CASH	1	2.20	1.0000	2.200000	2.200000	0.000000
2	CASH	2	2.40	0.9785	2.197241	0.000000	1.092651
3	FORWARDS	3	2.50	0.9555	2.407117	0.721840	1.528921
4	FORWARDS	4	2.75	0.9322	2.499464	1.131164	1.770691
5	FORWARDS	5	2.80	0.9073	2.744406	1.393284	1.964693
6	SWAPS	6	2.85	0.8826	2.798550	1.605612	2.103198
7	SWAPS	7	3.10	0.8581	2.855145	1.763796	2.210282
8	SWAPS	8	3.15	0.8323	3.099844	1.888425	2.321056
9	SWAPS	9	3.30	0.8069	3.147850	2.009225	2.412594
10	SWAPS	10	3.45	0.7811	3.303034	2.109599	2.501291

b.) Compute the PV of the bond cashflows

$$PV = \Sigma d(t_i) * C(i)$$


```
PV = np.sum(   df['DISCOUNT_RATE'] * df['COUPON_RATE'] )
PV = np.round(PV, decimals = 2)
print("The Present Value = $" + str(PV) )
```

```
The Prent Value = $15.24
```

b.) Change the forward curve by +0.5% (at each maturity one at a time)

- Compute the discount factors and
  - PV
  - DV01
  - Duration.
    - Which forward change has the highest DV01?

```
dfc = df.copy()
dfc['INPUTS'] = dfc['INPUTS'] + .5

DISCOUNT = []
DISCOUNT.append(0)

for i in range(1, n_row):
    DISCOUNT.append( discount_rate( dfc['INPUTS'].iloc[1:i] ) )
dfc['DISCOUNT_RATE'] = DISCOUNT


FORWARD = []
FORWARD.append(1.9)
FORWARD.append(2.2)

for i in range(2, n_row):
    FORWARD.append( forward( dfc['DISCOUNT_RATE'].iloc[i-1],  dfc['DISCOUNT_RATE'].iloc[i]))

dfc['FORWARD'] = FORWARD

COUPON = []
COUPON.append(1.9)
COUPON.append(2.2)

for i in range(2, n_row):
    COUPON.append( coupon(dfc['DISCOUNT_RATE'].iloc[0:i])  )

dfc['COUPON_RATE'] = COUPON


PV = np.sum(   dfc['DISCOUNT_RATE'] * dfc['COUPON_RATE'] )
PV = np.round(PV, decimals = 2)
print("The Present Value = $" + str(PV) )
```

```
The Present Value = $17.39
```

- Compute the PV of the bond when increasing simultaneously all the forward rates by 1%,2%, and 3%

- What is the forward price of the bond 18 months from today?

Problem 2

Consider an eight-month European put option on a Treasury bond that currently has **14.25 years to maturity**. The current cash bond price is **$908**, the exercise price is $900, andthevolatilityforthebondpriceis **1025$ **will be paid by the bond in** three months**. The risk-free interest rate is** 1.5%** for all maturities up to one year. Use Black's model to determine the price of the option. Consider both the case where the strike price corresponds to the cash price of the bond and the case where it corresponds to the quoted price.

This is a great problem!

Summary:

Price a *put* option, with 8 months untill expiration.

The Bond has the following conditions:

- Years to maturity = 14.25
- Bond Price = $908
- Exercise Price = $900
- Volatility of Bond Price = 10%
- Coupon = $25 (paid in 3 months)
- risk-free rate (r) = 1.5%

Week 2 lecture says:

$F_0 = E[F_T]$

$$c = P(0,t)[F_B N(d_1) - kN_(d_2)]$$

$$p = P(0,t)[kN(-d_2) - F_B N(-d_1)]$$

where:

$$d_1 = \frac{ln(\frac{F_B}{k}) + \sigma_B^2 \frac{T}{2}}{\sigma_B \sqrt{T}}$$

$$d_2 = d_1 - \sigma_B \sqrt{T}$$

F_B = forward bond price

$$F_B = \frac{B_0 - I}{P(0,t)}$$

$B_0 =$ Bond at time 0

$I =$ present value of coupons in $(0, t)$

$P(t,T) =$ price at the t of a zero-coupon bond that pays $1 at T

$$P(t,T) = e^{-r(T-t)}$$

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```
# INPUTS:
B0 = 908

c = 25
r = .015
time_to_expiration = 8/12
K = 900
sigma = .10

# Step 1: Price I
I = c*np.exp(-r*.25)

# Step 2: Price ZCB (P(0,T))
P0t = np.exp(-r*time_to_expiration)

# Step 3: Price Forward bond price (F_B)
FB = (B0 - I)/P0t


# Step 5: Price nd1
d1 = ((np.log(FB/K)) + .1**2 * time_to_expiration/2 )
d1= d1/(sigma*np.sqrt(time_to_expiration))
nd1 = norm.cdf(d1,loc=0, scale=1)

# Step 6: Price nd2
d2 = d1 - sigma*np.sqrt(time_to_expiration)
nd2 = norm.cdf(d2,loc=0, scale=1)

# Step 7: Price c
c = P0t*(FB*nd1 - K*d2)
c = np.round(c, decimals = 2)

# Step 8: Price p
p = P0t*(K*norm.cdf(-d2,loc=0, scale=1) - FB*norm.cdf(-d1,loc=0, scale=1))
p = np.round(p, decimals = 2)

# print(c)
print(p)
```

```
33.04
```

The final price is listed above (33.04)

Problem 3

Consider the following data:

Use the data provided to build the yield curve using the **cubic spline model**. Report the value for the estimated coefficients and write the final expression for the rate as

$$R(0,t) = a + b(t - t_1) + c(t - t_1)^2 + \Sigma_{k=1}^{n-1} d_k (t - t_k)_+^3$$

a) Plot the fitted model and the original yield rates on the same graph. Compare the results.

b) Calculate the yield rate for tt= 4 years.

Inputs = x, and y

x = time to maturity

y = yield rate

This is a 4 step process

Step 1: For the matrix A, and R fill out the matrix with values

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ . \\ . \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} a \\ b \\ c \\ d_1 \\ . \\ . \\ d_{n-1} \end{bmatrix}$$

Step 2: Multiply the matrix on the right hand side

$$\begin{bmatrix} a \\ b \\ c \\ d_1 \\ . \\ . \\ d_{n-1} \end{bmatrix} = A^{-1} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ . \\ . \\ 0 \\ 0 \end{bmatrix}$$

Step 3: Based upon coefficient matrix, plug in x values to generate a line.

Step 4: Plot it!


```
df = pd.DataFrame({
    'TIME_TO_MATURITY':[.25,.5,1,2,3,5,7,10,20],
    'YIELD_RATE':[1.2,1.23, 1.3, 1.5, 1.77, 2.4, 3.14, 3.45, 3.8]
})

df.head()
```

	TIME_TO_MATURITY	YIELD_RATE
0	0.25	1.20
1	0.50	1.23
2	1.00	1.30
3	2.00	1.50
4	3.00	1.77

```
x =  np.asarray([2,3,4])
y =  np.asarray([3,2,4])

x = np.asarray(df['TIME_TO_MATURITY'])
y = np.asarray(df['YIELD_RATE'])

n = len(x)

sub_sets = n-1; print(sub_sets)   # This means we have N equations


A = np.zeros((4*(sub_sets), 4*(sub_sets)))
b = np.concatenate([y[:1], np.repeat(y[1:-1], 2), y[-1:], np.zeros(2*(sub_sets))])

xi = 0

for i in range(sub_sets):
    r = i*2
    c = i*4

    A[r][c:c+4] = [x[xi]**3, x[xi]**2, x[xi], 1]
    xi += 1
    A[r+1][c:c+4] = [x[xi]**3, x[xi]**2, x[xi], 1]

c = 0
xi = 0

for i in range(sub_sets-1):
    c = i*4
    r += 2
    xi += 1

    A[r][c:c+3] = [3*(x[xi]**2), 2*x[xi], 1]
    A[r][c+4:c+7] = [-(3*(x[xi]**2)), -(2*x[xi]), -1]

    A[r+1][c:c+2] = [6*(x[xi]), 2]
    A[r+1][c+4:c+6] = [-(6*(x[xi])), -2]

A[-2][:2] = [6*x[0], 2]
A[-1][-4:-2] = [6*x[-1], 2]

print(A)
print(b)
```

```
8
[[ 0.015625  0.0625    0.25     ...  0.        0.        0.      ]
 [ 0.125     0.25      0.5      ...  0.        0.        0.      ]
 [ 0.        0.        0.       ...  0.        0.        0.      ]
 ...
 [ 0.        0.        0.       ... -2.        0.        0.      ]
 [ 1.5       2.        0.       ...  0.        0.        0.      ]
 [ 0.        0.        0.       ...  2.        0.        0.      ]]
[1.2  1.23 1.23 1.3  1.3  1.5  1.5  1.77 1.77 2.4  2.4  3.14 3.14 3.45
 3.45 3.8  0.   0.   0.   0.   0.   0.   0.   0.   0.   0.   0.   0.
 0.   0.   0.   0.  ]
```

```
### Based upon the input data, X & Y, now we can do the matrix multiplication to sovle for matrix [a ,b, c, d1, d2, d3]

A_INVERSE = np.linalg.inv(A)

c = np.dot(A_INVERSE, b)
c = c.reshape(-1 , 4)

#  lines = np.dot(np.linalg.inv(A), b).reshape(-1, 4)
#  expanded = np.hstack([c[i] * (values ** i) for i in range(0, len(coeffs))])
#  np.sum(expanded, axis=1)

print(c)
```

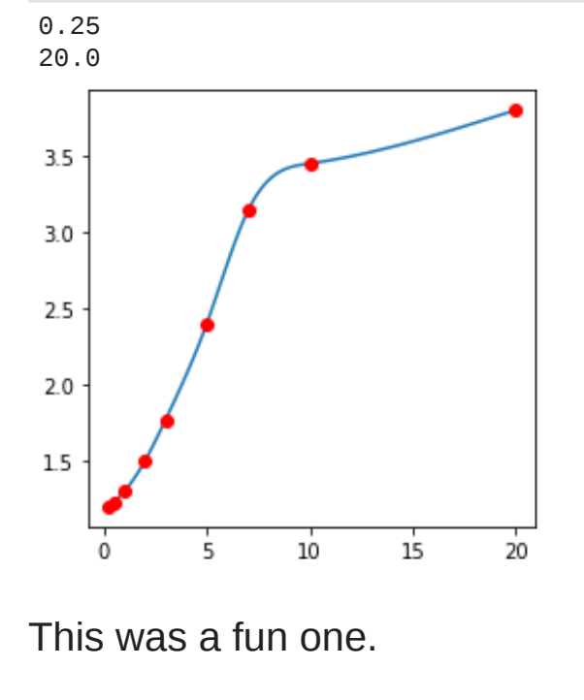
```
[[ 3.47808384e-02 -2.60856288e-02  1.24347605e-01  1.17000000e+00]
 [ 1.04383232e-02  1.04281440e-02  1.06090718e-01  1.17304281e+00]
 [-5.08958097e-06  4.17583823e-02  7.47604801e-02  1.18348623e+00]
 [-1.34607793e-02  1.22492520e-01 -8.67077961e-02  1.29113174e+00]
 [ 6.87567490e-03 -6.05355670e-02  4.62376466e-01  7.42047482e-01]
 [-2.19738815e-02  3.72207780e-01 -1.70134027e+00  4.34824204e+00]
 [ 1.01840524e-02 -3.03108832e-01  3.02587602e+00 -6.68192929e+00]
 [-8.04246134e-05  4.82547680e-03 -5.34670747e-02  3.58254768e+00]]
```

```
c[0,1]
```

```
-0.026085628804966454
```

```
min_x = np.min(x)
max_x = np.max(x)

print(min_x)
print(max_x)

x_steps = np.arange(min_x, max_x, step = (max_x -min_x)/100 )


y_line = []
final_y = []


for count, value in enumerate(x):

    if count >=1:
        y_line = x_steps[ (x_steps<=value) & (x_steps>=x[count-1])]

        for j_count, j_value in enumerate(y_line):
            final_y.append( c[count-1,3] + c[count-1,2]* j_value**1 + c[count-1,1]* j_value**2 + c[count-1,0]* j_value**3  )



#plt.plot(x_steps, final_y)


fig, ax = plt.subplots(figsize=(4,4))
plt.plot(x_steps, final_y)
plt.plot(df['TIME_TO_MATURITY'], df['YIELD_RATE'], 'o', color = 'red')
plt.show()
```

```
0.25
20.0
```



This was a fun one.

How can this method be applied to other problems in a variety of industries ?

Problem 4

The European Central Bank reports the Euro yield curve by providing the Nelson-Siegel parameters. Use the functional form of Nelson-Siegel model to estimate the parameters $\beta0$, $\beta1$, $\beta2$, $\tau1$ for the Treasury Coupon Bonds provided below. The Nelson-Siegel model assumes that:

$$R(0,t) = \beta_0 + \beta_1(\frac{1 - e^{-t\lambda}}{t\lambda}) + \beta_2(\frac{1 - e^{-t\lambda}}{t\lambda} - e^{-t\lambda})$$

Let

$$MSE = \frac{1}{n}\Sigma_{i=1}^{n}(Y_i - Y_{ihat})^2$$

```
df = pd.read_excel('HW1_data_for_Problem_4_F22-Riley.xlsx')
df.head(5)
```

	Time to next payment	Payment frequency	Time to maturity	Coupon rate	Clean Price	YTM
0	0.4356	2	0.4356	0.0088	100.30	0.010796
1	0.2644	2	0.7644	0.0088	100.48	0.010674
2	0.2658	2	1.2658	0.0075	100.50	0.009040
3	0.4342	2	1.9342	0.0063	100.31	0.007669
4	0.0192	2	2.0192	0.0038	99.78	0.004893

```
def rt(t, beta0, beta1, beta2, LAMBDA):
     r_t = beta0 + beta1*(1-np.exp(-t*LAMBDA) / (t*LAMBDA) )  +  beta2*( (1-np.exp(-t*LAMBDA) / (t*LAMBDA) ) - np.exp(-t*LAMBDA) )
     r_t = np.array(r_t)
     return(r_t)


def MSE(actual_values, estimate):
     sub_set = actual_values - estimate
     sub_set = np.power(sub_set,2)
     return( np.mean(sub_set) )
```

```
YEILD = np.array(df['YTM'])
time = np.array(df['Time to maturity'])
```

```
BETA0 = np.arange( .00001 , .01, .001)
BETA1 = np.arange( .00001 , .01, .001)
BETA2 = np.arange( .00001 , .01, .001)
LAMBDA = np.arange(1 , 55, 1)

l1 = len(BETA0)
l2 = len(BETA1)
l3 = len(BETA2)
l4 = len(LAMBDA)

print('TOTAL LOOPS = ' + str(l1*l2*l3*l4 ))
```

```
TOTAL LOOPS = 54000
```

```
b0 = []
b1 = []
b2 = []
lam = []
MSE_LIST = []


for beta0 in BETA0:
    for beta1 in BETA1:
        for beta2 in BETA2:
            for lambd_a in LAMBDA:

                YC = rt(time , beta0, beta1, beta2, lambd_a)

                b0.append(beta0)
                b1.append(beta1)
                b2.append(beta2)
                lam.append(lambd_a)
                MSE_LIST.append(MSE( YEILD , YC ))
```

```
print(len(b0))
print(len(b1))
print(len(b2))
print(len(lam))
print(len(MSE_LIST))
```

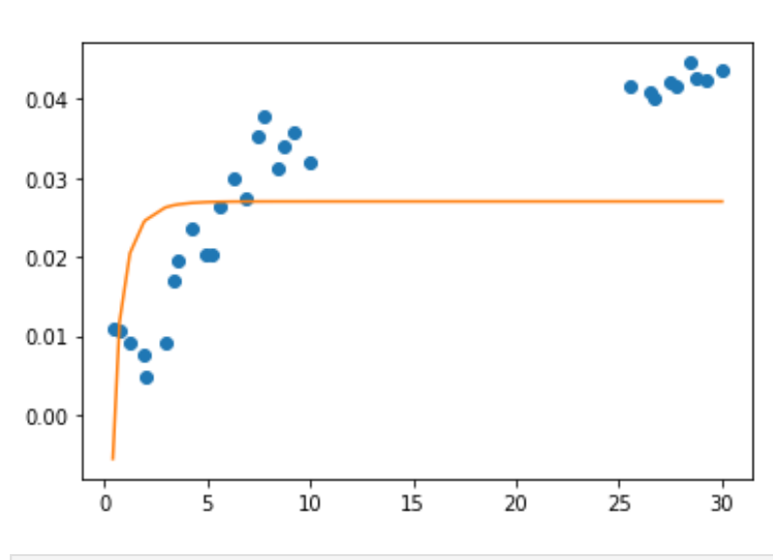
```
54000
54000
54000
54000
54000
```

```
results_df = pd.DataFrame({ 'B0':b0,
                            'B1':b1,
                            'B2':b2,
                            'LAMBDA':lam,
                            'MSE':MSE_LIST  } )

top_10_model = results_df.sort_values(by = 'MSE')
top_10_model.head(10)
```

	B0	B1	B2	LAMBDA	MSE
53946	0.00901	0.00901	0.00901	1	0.000139
53892	0.00901	0.00901	0.00801	1	0.000146
53406	0.00901	0.00801	0.00901	1	0.000146
48546	0.00801	0.00901	0.00901	1	0.000147
53947	0.00901	0.00901	0.00901	2	0.000148
48547	0.00801	0.00901	0.00901	2	0.000153
53407	0.00901	0.00801	0.00901	2	0.000153
53893	0.00901	0.00901	0.00801	2	0.000154
53838	0.00901	0.00901	0.00701	1	0.000154
53352	0.00901	0.00801	0.00801	1	0.000154

```
YC = rt(time , 0.00901, 0.00901 ,0.00901, 1)

plt.plot(df['Time to maturity'], df['YTM'], 'o')
plt.plot(time, YC)
```

```
[<matplotlib.lines.Line2D at 0x137c5762940>]
```



```
## The model above can be improved to fit a better model.

# Some Ideas include an optimzation engine, or running more for loops!
```

Thank you for offering some amazing problems to work on.

Any and all feedback is appreciated. Thank you!