Sample Project: American Options on AMZN

FE-620

October 13, 2021

1 Problem setting and choice of underlying stock

We study the pricing of American options on AMZN as of 2-Dec-2019. We will price the American options in the Black-Scholes model using a binomial tree implementation. Recall the Itô process for the stock price in the Black-Scholes model

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{1}$$

The model has three parameters: S_0, σ, r.

Pricing the options in the Black-Scholes model requires that we determine/estimate these 3 parameters of the BS model.

- *Spot stock price* S_0. On 2-Dec-2019 the AMZN stock closed at $S_0 = 1781.60$.
- *Volatility* σ. We estimate it from historical stock price data. This will give the so-called historical volatility.
- *Risk-free rate* r. We proxy this rate with the 13-week T-bill rate.

2 Market Data Analysis

We download market data for daily prices of the AMZN stock and the 13-week T-bill rate from Yahoo Finance. The R code is shown in the Appendix.

2.1 Analyzing the time series of the stock price S_i

The plot in Fig. 1 (left) shows the closing stock price of AMZN over the past 6 months ending on 2-Dec-2019, obtained from Yahoo Finance.

Using the daily closing prices S_i we compute the daily log-returns $u_i = \log(S_i/S_{i-1})$. Under the Black-Scholes model the log-returns are iid normally distributed random variables with mean $\mu = r - \frac{1}{2}\sigma^2$ and variance $Var(u) = \sigma^2\tau$ where $\tau = 1/252$ the one-day time interval.

The right plot in Fig. 1 shows the QQ plot of the log-returns. A straight QQ plot indicates a normal distribution. We observe deviations from log-normality in the tails, which agree with expectations of heavy tailed distributions and deviations from log-normality.

Estimating the historical volatility. Using the time series of daily closing prices, we estimate the historical volatility using lookback windows with several lengths: 6, 3, 1 months. This is obtained using

$$\hat{\sigma} = \sqrt{\frac{1}{\tau}Var(u_i)} = \sqrt{252} \cdot \mathrm{stdev}(u_i) \tag{2}$$

and the statistical error of the estimate is

$$\frac{\delta\hat{\sigma}}{\hat{\sigma}} = \frac{1}{\sqrt{2n_{days}}} \tag{3}$$

with n_{days} the number of days in the lookback window.

The results are shown in Table 1. We observe that the estimation error decreases with the length of the lookback window, as expected. However a longer window introduces also systematic error as it includes data points from a potentially different economic regime.

Estimating the risk-free rate. The final parameter of the Black-Scholes model is the risk-free rate r. We proxy this rate with the 13-week Treasury bill rate, which is accessible in Yahoo Finance as $\hat{}IRX$. The daily values of this rate are shown in Figure 2. The closing value of this rate on 2-Dec-2019 is

$$r = 1.538\% \tag{4}$$

We will use this as the risk-free rate in our pricing of the American options.

Table 1: Estimates for the historical volatility of the AMZN stock on 2-Dec-2019 using three lookback windows: 6m, 3m, 1m.

Lookback window	Days	$\hat{\sigma}$
3-Jun-2019 to 2-Dec-2019	128	$(18.70 \pm 1.17)\%$
3-Sep-2019 to 2-Dec-2019	64	$(15.75 \pm 1.39)\%$
4-Oct-2019 to 2-Dec-2019	20	$(13.15 \pm 2.08)\%$

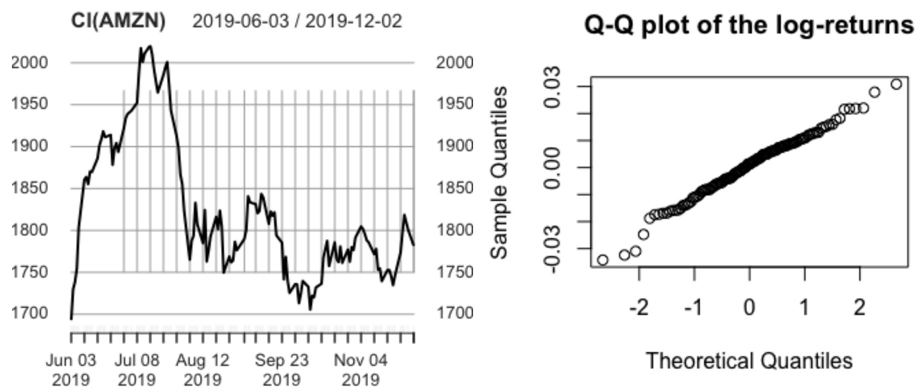


Figure 1: Left: Historical price of the AMZN closing price for the 6m preceding 2-Dec-2019. Right: Q-Q plot of the log-returns.

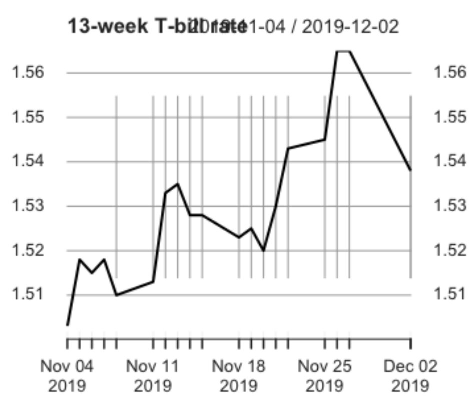
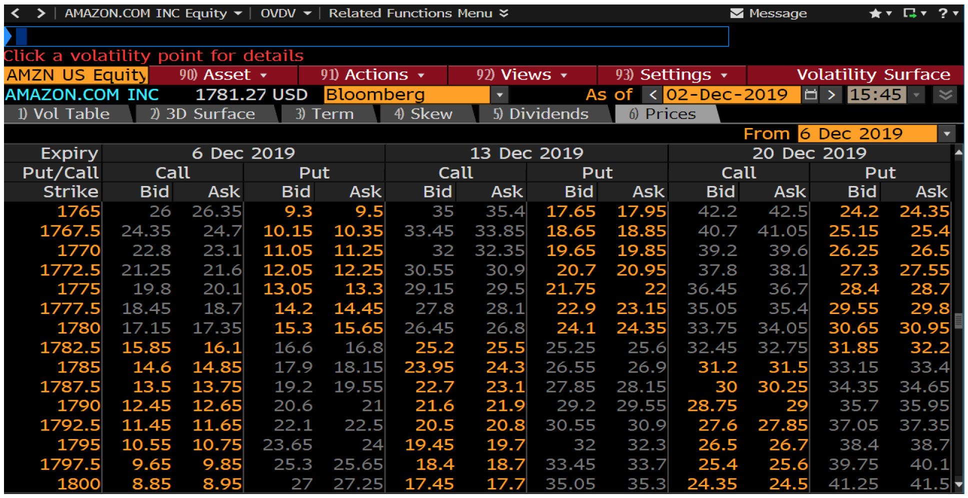


Figure 2: Plot of the 13-week T-bill rate for the 1 month period ending on 2-Dec-2019.



Expiry	6 Dec 2019				13 Dec 2019				20 Dec 2019			
Put/Call	Call		Put		Call		Put		Call		Put	
Strike	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
1765	26	26.35	9.3	9.5	35	35.4	17.65	17.95	42.2	42.5	24.2	24.35
1767.5	24.35	24.7	10.15	10.35	33.45	33.85	18.65	18.85	40.7	41.05	25.15	25.4
1770	22.8	23.1	11.05	11.25	32	32.35	19.65	19.85	39.2	39.6	26.25	26.5
1772.5	21.25	21.6	12.05	12.25	30.55	30.9	20.7	20.95	37.8	38.1	27.3	27.55
1775	19.8	20.1	13.05	13.3	29.15	29.5	21.75	22	36.45	36.7	28.4	28.7
1777.5	18.45	18.7	14.2	14.45	27.8	28.1	22.9	23.15	35.05	35.4	29.55	29.8
1780	17.15	17.35	15.3	15.65	26.45	26.8	24.1	24.35	33.75	34.05	30.65	30.95
1782.5	15.85	16.1	16.6	16.8	25.2	25.5	25.25	25.6	32.45	32.75	31.85	32.2
1785	14.6	14.85	17.9	18.15	23.95	24.3	26.55	26.9	31.2	31.5	33.15	33.4
1787.5	13.5	13.75	19.2	19.55	22.7	23.1	27.85	28.15	30	30.25	34.35	34.65
1790	12.45	12.65	20.6	21	21.6	21.9	29.2	29.55	28.75	29	35.7	35.95
1792.5	11.45	11.65	22.1	22.5	20.5	20.8	30.55	30.9	27.6	27.85	37.05	37.35
1795	10.55	10.75	23.65	24	19.45	19.7	32	32.3	26.5	26.7	38.4	38.7
1797.5	9.65	9.85	25.3	25.65	18.4	18.7	33.45	33.7	25.4	25.6	39.75	40.1
1800	8.85	8.95	27	27.25	17.45	17.7	35.05	35.3	24.35	24.5	41.25	41.5

Figure 3: American option prices on AMZN as of 2-Dec-2019. (Bloomberg)

3 American options on AMZN

Several options are traded on AMZN. As of 2-Dec-2019, the first expiries are 6-Dec-2019, 13-Dec-2019, 20-Dec-2019. These dates are the Fridays of each week.

The prices of these options for several strikes are shown in Figure 3, taken from Bloomberg. This image is also shown in Canvas on the page with information about extracting option prices from Yahoo and Bloomberg.

Using the mid-prices $m(K) = \frac{1}{2}(a(K) + b(K))$ we can test the put-call parity inequalities for American option prices. Recall the bounds on the prices of the American options on stocks which do not pay a dividend[1]

$$S_0 - K < C(K) - P(K) < S_0 - Ke^{-rT} . \tag{5}$$

For short maturity options the discount factor e^{-rT} is almost 1 so the upper and lower bounds are very close. Figure 4 shows the difference of the mid-prices for the 6-Dec-2019 maturity options (points) vs K, and the red line shows the difference $S_0 - K$. They agree indeed very well.

We would like to price these American options in the Black-Scholes model using a binomial tree approach. The first two maturities of the options in

[1] See the lecture week 6, Eq. (117) on page 240 in Hull [1].

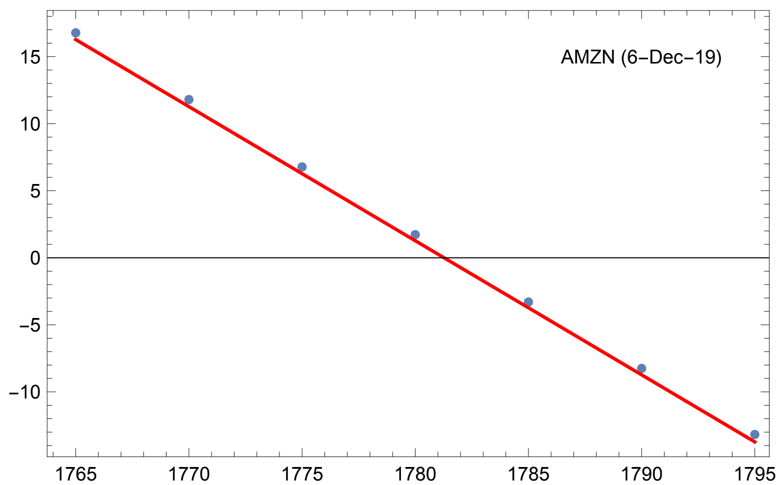


Figure 4: Put-call parity bounds on American option prices on AMZN with expiry 6-Dec-2019. The dots show $C(K) - P(K)$ vs K, and the red line is $S_0 - K$.

Fig. 3 are

$$\text{6-Dec-2019}: \qquad T = \frac{4}{252} \text{ years} \tag{6}$$

$$\text{13-Dec-2019}: \qquad T = \frac{9}{252} \text{ years} \tag{7}$$

Note that we count only business days when determining the maturity of options on stocks.

We need to know how many time steps n we should use in the tree. The result will depend very sensitively on this choice. We determine the number of steps by examining the convergence of the prices as n becomes very large.

3.1 Sensitivity study

Let us price one of the American options and study the dependence of its price on the number of time steps n, while keeping fixed the option maturity $T = n\tau$.

Consider the put option with expiry 6-Dec-19 and strike $K = 1780$ which is closest to the at-the-money point $K = S_0$. The binomial tree price with $n = 4$ time steps (daily time steps) gives

$$P(K = 1780; n = 4) = 12.56$$

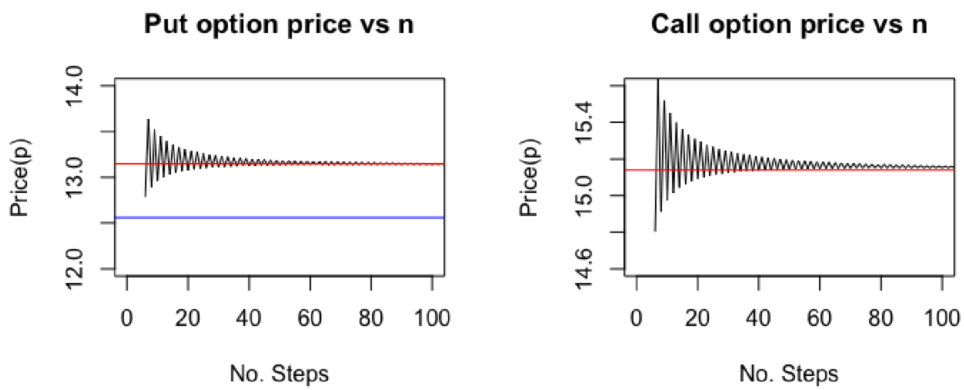


Figure 5: Convergence of the American option prices in the binomial tree model as the number of time steps n increases. Left: put option with $n = 4$ time steps (blue) and $n = 100$ time steps (red). Right: AMZN does not pay dividends so the American call price should converge to the European call price (red line).

How accurate is this? The left plot in Fig. 5 shows the price of this option as n increases from $n = 5$ to 100. We observe that prices converge for sufficiently large n. For example with $n = 100$ steps we have

$$P(K = 1780; n = 100) = 13.146$$

The put option price with $n = 100$ is shown as the red line, and the $n = 4$ value as the blue line. Clearly the $n = 100$ value is more accurate.

Another test for the binomial tree pricing can be done if we recall that American call options on a stock which does not pay dividends have the same price as the European call options with the same maturity.

The table below shows the binomial tree price of the American call option with strike $K = 1780$ for several values of n, the time steps of the tree. We observe that as n increases, the binomial tree prices approach the European option price, as expected.

n	4	10	20	30	40	50	100	BS
$C(K = 1780)$	14.58	14.97	15.09	15.12	15.14	15.14	15.15	15.14

This test is also shown in graphical form in the right plot in Fig. 5.

Table 2: American option prices on AMZN as of 2-Dec-2019 with expiry 6-Dec-2019. The *Num* columns show the results of pricing using the binomial tree method with $n = 100$ steps and the historical volatility estimate $\hat{\sigma} = 15.75\%$ (central value for 3m lookback window). The risk-free rate is $r = 1.538\%$. The *BS* column shows the price of the European call option obtained from the Black-Scholes formula with the same parameters. The AMZN closing price on this day is $S_0 = 1781.60$. Options closest to the ATM point are shown in red.

Strike K	Call option			Put option	
	Num	BS	[bid, ask]	Num	[bid, ask]
1770	20.91	20.88	[22.8, 23.1]	8.90	[11.05,11.25]
1772.5	19.37	19.34	[21.25,21.6]	9.85	[12.05,12.25]
1775	17.86	17.87	[19.8,20.1]	10.84	[13.05,13.3]
1777.5	16.50	16.47	[18.45,18.7]	11.99	[14.2,14.45]
1780	15.15	15.14	[17.15,17.35]	13.14	[15.3,15.65]
1782.5	13.87	13.87	[15.85,16.1]	14.36	[16.6,16.8]
1785	12.72	12.68	[14.6,14.85]	15.71	[17.9,18.15]
1787.5	11.57	11.56	[13.5,13.75]	17.06	[19.2,19.55]
1790	10.52	10.51	[12.45,12.65]	18.52	[20.6,21.0]

3.2 Options pricing: model prices vs market prices

Table 2 shows the numerical results for the American options with expiry 6-Dec-2019 using a binomial tree implementation of the Black-Scholes model (column *Num*). The binomial tree prices for the American options with expiry 13-Dec-2019 are shown in Tables 8. All these results are obtained using $n = 100$ time steps in the tree.

The binomial tree prices in Tables 2 and 8 are consistently below the market prices of the American options. What is the explanation for this discrepancy?

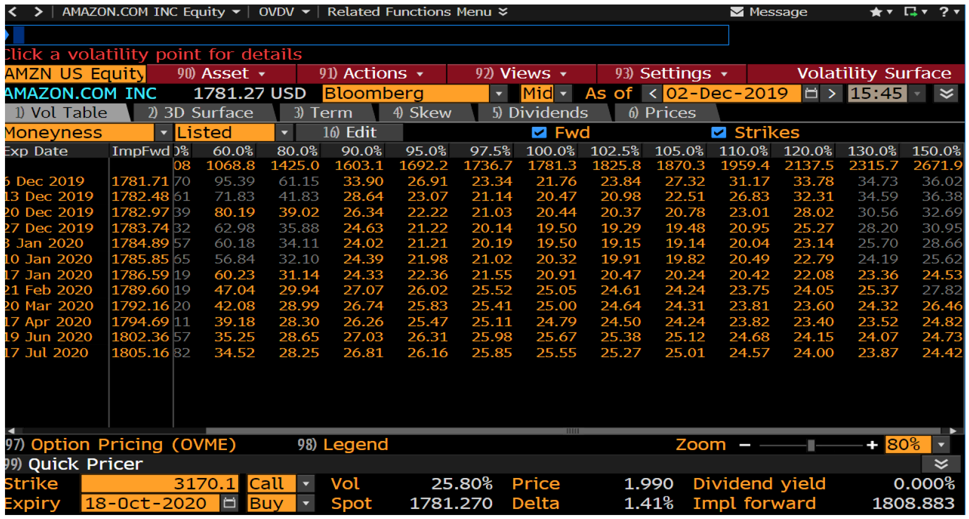
We can understand the discrepancy by examining the implied volatilities of the American options, which are shown in Figure 6 as function of moneyness K/S_0. Recall that the implied volatility is that value of σ which reproduces the market option price when pricing it under the Black-Scholes model.

The implied volatilities of the options with expiry 6-Dec-2019 are shown in Fig. 7 vs K/S_0. From this plot we note:

- The implied volatility $\sigma_{\rm BS}(K,T)$ depends on strike. This is the so-called *implied volatility smile* effect, and implies that we cannot hope to reproduce all market option prices with a common volatility.
- Our binomial tree pricing in the Black-Scholes model assumes a common volatility for all strikes. We estimated this volatility as the historical volatility. This is shown in Fig. 7 as the grey band. This band is below the implied volatility at all strikes, which explains why our Black-Scholes pricing underestimates the market option prices.
- The historical volatility is typically lower than the implied volatility. This is a manifestation of *risk aversion* of the option market participants, who are willing to overpay for options as portfolio insurance against market crashes.

4 Greeks calculation

The binomial tree pricer can be used to compute also the Greeks of the American options using finite difference methods.



Exp Date	ImpFwd	)%	60.0%	80.0%	90.0%	95.0%	97.5%	100.0%	102.5%	105.0%	110.0%	120.0%	130.0%	150.0%
		08	1068.8	1425.0	1603.1	1692.2	1736.7	1781.3	1825.8	1870.3	1959.4	2137.5	2315.7	2671.9
5 Dec 2019	1781.71	70	95.39	61.15	33.90	26.91	23.34	21.76	23.84	27.32	31.17	33.78	34.73	36.02
13 Dec 2019	1782.48	61	71.83	41.83	28.64	23.07	21.14	20.47	20.98	22.51	26.83	32.31	34.59	36.38
20 Dec 2019	1782.97	39	80.19	39.02	26.34	22.22	21.03	20.44	20.37	20.78	23.01	28.02	30.56	32.69
27 Dec 2019	1783.74	32	62.98	35.88	24.63	21.22	20.14	19.50	19.29	19.48	20.95	25.27	28.20	30.95
3 Jan 2020	1784.89	57	60.18	34.11	24.02	21.21	20.19	19.50	19.15	19.14	20.04	23.14	25.70	28.66
10 Jan 2020	1785.85	65	56.84	32.10	24.39	21.98	21.02	20.32	19.91	19.82	20.49	22.79	24.19	25.62
17 Jan 2020	1786.59	19	60.23	31.14	24.33	22.36	21.55	20.91	20.47	20.24	20.42	22.08	23.36	24.53
21 Feb 2020	1789.60	19	47.04	29.94	27.07	26.02	25.52	25.05	24.61	24.24	23.75	24.05	25.37	27.82
20 Mar 2020	1792.16	20	42.08	28.99	26.74	25.83	25.41	25.00	24.64	24.31	23.81	23.60	24.32	26.46
17 Apr 2020	1794.69	11	39.18	28.30	26.26	25.47	25.11	24.79	24.50	24.24	23.82	23.40	23.52	24.82
19 Jun 2020	1802.36	57	35.25	28.65	27.03	26.31	25.98	25.67	25.38	25.12	24.68	24.15	24.07	24.73
17 Jul 2020	1805.16	82	34.52	28.25	26.81	26.16	25.85	25.55	25.27	25.01	24.57	24.00	23.87	24.42

Figure 6: Implied volatilities of the American option prices on AMZN as of 2-Dec-2019 vs moneyness K/S_0. (Bloomberg)

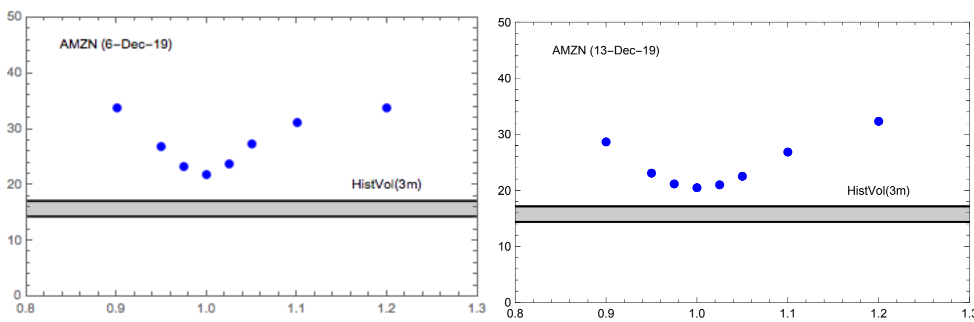


Figure 7: Implied volatilities of the American option prices with expiry 6-Dec-2019 (left) and 13-Dec-2019 (right) (blue dots), plotted vs moneyness K/S_0. The historical volatility $\hat{\sigma} = 15.75 \pm 1.39\%$ with lookback window 3m used in the binomial model is shown as the grey band.

```
> callData
  kStrikes callPrice callDelta     callGamma
1   1770.0  27.94820 0.6305030 -1.421085e-12
2   1772.5  26.51618 0.5530359 -5.684342e-12
3   1775.0  25.16389 0.5530359 -4.263256e-12
4   1777.5  23.81160 0.5530359 -3.552714e-12
5   1780.0  22.45931 0.5530359 -3.552714e-12
6   1782.5  21.17861 0.4734909 -1.421085e-12
7   1785.0  20.02519 0.4734909 -7.105427e-13
8   1787.5  18.87176 0.4734909  7.105427e-13
9   1790.0  17.71833 0.4734909  7.105427e-13
```

```
> putData
  kStrikes putPrice  putDelta      putGamma
1   1770.0 15.41422 -0.3717610  3.992645e-04
2   1772.5 16.48681 -0.4475422  8.913131e-05
3   1775.0 17.63497 -0.4477666 -1.421085e-12
4   1777.5 18.78376 -0.4481873  1.923623e-04
5   1780.0 19.93411 -0.4489289 -1.421085e-12
6   1782.5 21.16021 -0.5272866  1.068818e-04
7   1785.0 22.50780 -0.5275672  5.466894e-06
8   1787.5 23.85616 -0.5279638  1.064746e-04
9   1790.0 25.20613 -0.5289584  3.033728e-06
```

Figure 8: Prices and Greeks (Delta and Gamma) of the American put options with maturity 13-Dec-2019 vs strike K obtained using the binomial tree method with $n = 100$ time steps. The Gamma is very noisy.

Delta is computed using central finite differences with step $S_0 \to S_0 \pm 0.1$.

$$\Delta = \frac{P(S_0 + 0.1) - P(S_0 - 0.1)}{0.2} \tag{8}$$

Gamma is computed on the same grid of points $(S_0 - 0.1, S_0, S_0 + 0.1)$

$$\Gamma = \frac{P(S_0 + 0.1) - 2P(S_0) + P(S_0 - 0.1)}{0.1^2} \tag{9}$$

For this test we choose the American options with maturity 13-Dec-2019 ($T = 9/252$ years). The Delta and Gamma of these options are shown in the tables of Fig. 8. Fig. 9 plots the Delta and Gamma for the put options.

The Greeks are very noisy due to the discontinuity inherent in the binomial tree as the stock price crosses certain discrete values imposed by tree structure. For example, Gamma turns out to be negative for certain strikes although it must be always positive! This is a major shortcoming of the binomial tree and motivates the introduction of trinomial trees which have a denser coverage of the stock price dimension.

5 Hedging exercise

The closing prices of AMZN for the first 2 weeks of Dec-2019 are shown below.

For the hedging exercise, consider the put option with strike $K = 1780$ for each of the last 4 days of its existence: 2-Dec to 6-Dec. For each day

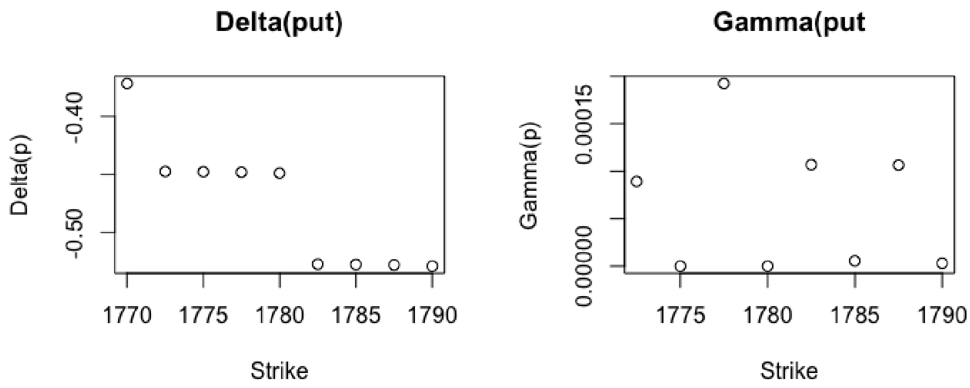


Figure 9: The Delta and Gamma of the American put options with maturity 13-Dec-2019 vs strike K.

```
> head(Cl(AMZN),10)
           AMZN.Close
2019-12-02    1781.60
2019-12-03    1769.96
2019-12-04    1760.69
2019-12-05    1740.48
2019-12-06    1751.60
2019-12-09    1749.51
2019-12-10    1739.21
2019-12-11    1748.72
2019-12-12    1760.33
2019-12-13    1760.94
```

Figure 10: The closing prices of AMZN for the first 2 weeks of Dec. 2019. The call option which is ATM on 2-Dec, with strike $K = 1780$, ends up out of the money on 6-Dec, when the stock price is 1751.60. How can we hedge it?

Table 3: Hedging strategy for the put option on AMZN with expiry 6-Dec-19 and strike $K = 1780$. The last two columns show the daily price change of the option (unhedged) and of the option hedged with Δ shares of stock.

Day	Maturity	S_0	Put Price	Delta	$P(t) - P(t-1)$	$\Pi(t) - \Pi(t-1)$
2-Dec-2019	4/252	1781.60	13.14	-0.4523		
3-Dec-2019	3/252	1769.96	17.69	-0.6116	4.28	-0.786
4-Dec-2019	2/252	1760.69	22.274	-0.7554	4.584	-1.086
5-Dec-2019	1/252	1740.48	39.340	-0.9916	17.066	1.800
6-Dec-2019	-	1751.60	28.40	-1.0	-10.94	0.087

compute the option price and its Delta using the binomial tree model. The results are shown in the table below.

Using the Delta values in the table, we construct a dynamically hedged portfolio: for one put option, we purchase Δ shares of stock. Denote the price of the option + stock hedged portfolio $\Pi(t)$.

The daily price change of the hedged portfolio is

$$\Pi(t) - \Pi(t-1) = P(t) - P(t-1) + \Delta(t-1)(S(t) - S(t-1))$$

The last two columns in Table 3 show the daily price changes of the unhedged put option, and of the Delta hedged position of option plus stock. The hedge is adjusted daily according to the Delta of the put option on each day, computed using the binomial tree model. We see that the hedged position has much smaller daily price volatility than the "naked" put option.

The hedge is not perfect because of Theta (time change Greek) and Gamma contributions, and also because of time discretization errors (Delta hedging is perfect only when performed continuously in time).

6 Other topics

If there is extra time one could explore further along different directions.

- Improve the tree pricing using the European option as control variate

- What should the Greeks of the American option look like, if we could compute them with very high precision? They differ in some surprising ways from the Greeks of European options. Study the Greeks dependence on the tweak size.

- What is the cost of the hedge?

A R code for data analysis

```
source("AmericanOptionPricer.R")
library(quantmod)

# get historical stock prices, and compute the historical volatility

getSymbols("AMZN", src="yahoo", from="2019-09-02", to="2019-12-03")

# visualise the data downloaded

names(AMZN)

plot(Cl(AMZN))

# estimate the historical volatility

prices <- Cl(AMZN)
head(prices)
tail(prices)
ndays <- length(prices)

#compute daily log-returns

logret <- periodReturn(prices, period="daily", type="log")

plot(logret, main="AMZN daily log returns")

# check the normality of the log-returns

qqnorm(logret, main="Q-Q plot of the log-returns")

# compute annualized volatility

yearvol = sqrt(252.0)*sd(logret)  # stdev(log-ret) STDEV

print(yearvol)   #sigma = 15.75% +- 1.39% (3m)
histvol <- yearvol
```

```
histvolerr <- yearvol/sqrt(2*ndays)

# Risk-free interest rate ^IRX = 13-week T-bill rate
getSymbols("^IRX", src="yahoo", from="2019-09-03", to="2019-12-03")

names(IRX)

rfrate <- na.omit(Cl(IRX))
nrf <- length(rfrate)

plot(rfrate, main="13-week T-bill rate")

tail(rfrate)
rf <- 1.538/100
```

The convergence test in Fig. 5 was obtained using this R code.

```
# convergence study
nSteps <- numeric()
putPriceVsN <- numeric()

for (i in 1:100) {

  n <- i + 5
  poptn <- binomial_option("put", histvol, 4/252, rf, 1780, S0, n, TRUE)
  nSteps <- c(nSteps,n)
  putPriceVsN <- c(putPriceVsN,poptn$price)
}

plot(nSteps,putPriceVsN,xlim=c(0,100), ylim=c(12.0, 14.0),
     xlab="No. Steps", ylab="Price(p)", type='l', main="Put option price vs n")
abline(h=12.56, col="blue")
abline(h=13.146, col="red")

putPriceVsN[100]
```

The Greeks have been computed with this code.

```
kStrikes <- numeric()
callPrice <- numeric()
putPrice <- numeric()

callDelta <- numeric()
callGamma <- numeric()
putDelta <- numeric()
putGamma <- numeric()


for (i in 1:9) {
  k <- 1770 + 2.5*(i-1)
  S0up <- S0 + 0.1
  S0down <- S0 - 0.1
#  copt <- binomial_option("call", histvol, 4/252, rf, k, S0, 40, TRUE)
  copt <- binomial_option("call", histvol, 9/252, rf, k, S0, 100, TRUE)
  coptup <- binomial_option("call", histvol, 9/252, rf, k, S0up, 100, TRUE)
  coptdown <- binomial_option("call", histvol, 9/252, rf, k, S0down, 100, TRUE)

  popt <- binomial_option("put", histvol, 9/252, rf, k, S0, 100, TRUE)
  poptup <- binomial_option("put", histvol, 9/252, rf, k, S0up, 100, TRUE)
  poptdown <- binomial_option("put", histvol, 9/252, rf, k, S0down, 100, TRUE)

  kStrikes <- c(kStrikes,k)
  callPrice <- c(callPrice,copt$price)
  callDelta <- c(callDelta,(coptup$price - coptdown$price)/0.2)
  callGamma <- c(callGamma,(coptup$price + coptdown$price - 2*copt$price)/0.01)

  putPrice <- c(putPrice,popt$price)
  putDelta <- c(putDelta,(poptup$price - poptdown$price)/0.2)
  putGamma <- c(putGamma,(poptup$price + poptdown$price - 2*popt$price)/0.01)

}

callData <- data.frame(kStrikes,callPrice, callDelta,callGamma)
putData <- data.frame(kStrikes,putPrice, putDelta,putGamma)

putData
```

```
callData
```

References

[1] J. Hull, Options, Futures and Other Derivatives, Pearson 10th Edition