(1) Prove that $C = \{w \mid w$ has an equal number of 0's and 1's$\}$ is not regular.

Consider the string $s = 0^p 1^p$

Proof. Assume that C is regular and let p be the pumping length for C.

Choose the string $s = 0^p 1^p \in C$ so that $|s| > p$. By the PL s can be partitioned into 3 pieces $s = xyz$ such that for all $i \geq 0$, $xy^i z \in C$.

- Consider the 3 following cases.

1. The partition y consists of only 0's. Then $s' = xyyz = xy^2z$ has more 0's than 1's. Therefore $s' \notin C$ and Condition 1 of the PL is violated. This is a contradiction for our assumption that C is regular.
2. The partition y consists of only 1's. Using the same argument as the previous case, we obtain another contradiction.
3. The partition y consists of 0's and 1's. Then $s' = xy^2z$ can have the same number of 0's and 1's, but some of the 1's will come before some of the 0's. and violates the terms of the language C. This yields another contradiction with our assumption of regularity.

Therefore, we cannot avoid a contradiction with any y partition. Thus we conclude that C is not a regular language. $\square$

(2) Prove that $F = \{ww \mid w \text{ is a string from } \{0,1\}^*\}$ is not regular.

Consider the string $s = 0^p 1 0^p 1$.

<u>Proof</u>. Assume that F is regular and let p be the pumping length for F.

Choose the string $s = 0^p 1 0^p 1 \in F$ so that $|s| < p$. By the PL s can be partitioned into 3 pieces $s = xyz$ such that for all $i \geq 0$, $xy^iz \in F$.

- Let $p = 4$, and the string $s = xyz = \underset{x}{0}|\underset{y}{0}|\underset{z}{00100001} \in F$.
- Consider the following cases.

1. Let $i = 2$. Then, $s' = xyyz = xy^2z$. By pumping up, the number of 0's before the first 1 is greater than the number of 0's before the second 1. Clearly $s' \notin F$, and Condition 1 of the PL is violated. This is a contradiction for our assumption that F is regular.
2. Let $i = 0$. Then $s' = xz$. By pumping down, the number of 0's before the first 1 is less than the number of 0's before the second 1. Using the same argument as the previous case, we obtain another contradiction.

Therefore, the string s cannot be pumped up or down without avoiding contradiction. Thus we conclude that F cannot be a regular language. $\square$

(3) Prove that $A = \{www \mid w \text{ is a string from } \{a,b\}^*\}$ is not regular.

Consider the string $s = a^p b a^p b a^p b$.

<u>Proof.</u> Assume that A is regular and let p be the pumping length for A.

Choose the string $s = a^p b a^p b a^p b \in A$ so that $|s| > p$. By the PL s can be partitioned into 3 pieces $s = xyz$ such that for all $i \geq 0$, $xy^iz \in A$.

- Let $p = 4$, and the string $s = xyz = \underset{x}{a} \mid \underset{y}{a} \mid \underset{z}{a\ a\ b\ a\ a\ a\ a\ b\ a\ a\ a\ a\ b}$

- Consider the following cases.

1. Let $i = 2$. Then $s' = xyyz = xy^2z$. By pumping up, the number of a's before the first b is greater than the number of a's before the second and third b's. Clearly $s' \notin A$, and Condition 1 of the PL is violated. This is a contradiction for our assumption that A is regular.

2. Let $i = 0$. Then $s' = xz$. By pumping down, the number of a's before the first b is less than the number of a's before the second and third b's. Using the same argument as the previous case, we obtain another contradiction.

Therefore, the string s cannot be pumped up or down without avoiding contradiction. Thus we conclude that A cannot be a regular language. □

(4) Prove that $L = \{0^n 1^m 0^n \mid m, n \geq 0\}$ is not regular.

Consider the string $s = 0^p 1 0^p$.

Proof. Assume that L is regular and let p be the pumping length for L.

Choose the string $s = 0^p 1 0^p \in L$ so that $|s| < P$. By the PL s can be partitioned into 3 pieces $s = xyz$ such that for all $i \geq 0$, $xy^iz \in L$.

- Let $p = 4$, and the string $S = xyz = \underset{x}{0}|\underset{y}{0}|\underset{z}{00\,1\,0000}$.

- Consider the following cases.

1. Let $i = 2$. Then $s' = xyyz = xy^2z$. By pumping up, the number of 0's before the 1 is greater than the number of 0's after the 1. Clearly $s' \notin L$, and condition 1 of the PL is violated. This is a contradiction for our assumption that L is regular.

2. Let $i = 0$. Then $s' = xz$. By pumping down, the number of 0's before the 1 is less than the number of 0's after the 1. Using the same argument as the previous case, we obtain another contradiction.

Therefore, the string s cannot be pumped up or down without avoiding contradiction. Thus we conclude that L cannot be a regular language. $\square$