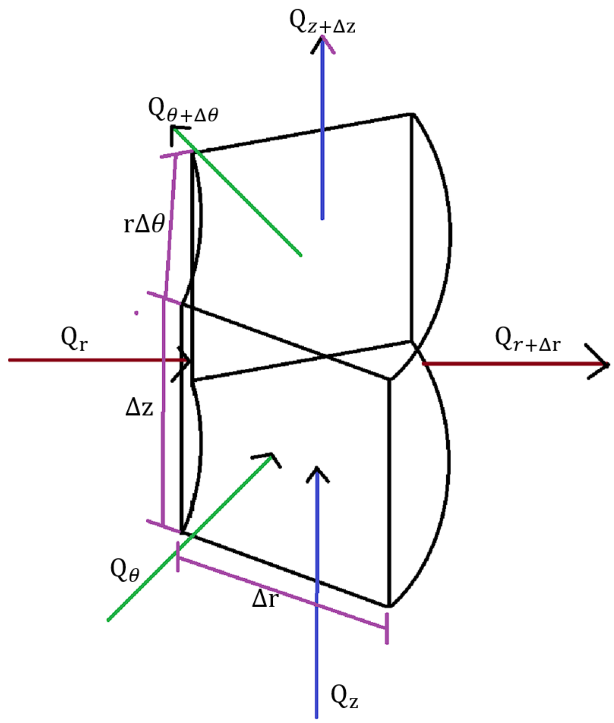
1. Derive the general heat conduction equation in cylindrical coordinates systems.



Start with a first law analysis of a single cylindrical element. This includes the energy entering the element, the energy leaving the element, and the energy generated within the element. Putting these together can give us the change in internal energy with respect to time:

$$\left(\dot{Q}_r - \dot{Q}_{r+\Delta r}\right) + \left(\dot{Q}_\theta - \dot{Q}_{\theta+\Delta\theta}\right) + \left(\dot{Q}_z - \dot{Q}_{z+\Delta z}\right) + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

Where:

$$\Delta E_{element} = E_{t+\Delta T} - E_t = mc\,(T_{t+\Delta t} + T_t) = \rho\Delta V c\,(T_{t+\Delta t} + T_t) = \rho\,[r\Delta r\Delta\theta\Delta z]\,c\,(T_{t+\Delta t} + T_t)$$

$$\lim_{\Delta T\to 0}\frac{\Delta E_{element}}{\Delta T} = \rho\,[r\Delta r\Delta\theta\Delta z]\,c\frac{\partial T}{\partial t}$$

$$\dot{E}_{gen,element} = \frac{\dot{e}_{gen}}{\Delta V} = \frac{\dot{e}_{gen}}{r\Delta r\Delta\theta\Delta z}$$

Substituting gives:

$$\left(\dot{Q}_r - \dot{Q}_{r+\Delta r}\right) + \left(\dot{Q}_\theta - \dot{Q}_{\theta+\Delta\theta}\right) + \left(\dot{Q}_z - \dot{Q}_{z+\Delta z}\right) + \frac{\dot{e}_{gen}}{r\Delta r\Delta\theta\Delta z} = \rho\,[r\Delta r\Delta\theta\Delta z]\,c\frac{\partial T}{\partial t}$$

Now divide both sides by volume:

$$\frac{1}{r\Delta\theta\Delta z}\frac{\dot{Q}_r - \dot{Q}_{r+\Delta r}}{\Delta r} + \frac{1}{r\Delta r\Delta z}\frac{\dot{Q}_\theta - \dot{Q}_{\theta+\Delta\theta}}{\Delta\theta} + \frac{1}{r\Delta r\Delta\theta}\frac{\dot{Q}_z - \dot{Q}_{z+\Delta z}}{\Delta z} + \dot{e}_{gen} = \rho c\frac{\partial T}{\partial t}$$

Now by taking the limit in each direction and applying Fourier's law of conduction:

$$\lim_{\Delta r \to 0} \frac{1}{r\Delta\theta\Delta z}\frac{\dot{Q}_r - \dot{Q}_{r+\Delta r}}{\Delta r} = \frac{1}{r\Delta\theta\Delta z}\frac{\partial \dot{Q}_r}{\partial r} = \frac{1}{r\Delta\theta\Delta z}\frac{\partial}{\partial r}\left[-kr\Delta\theta\Delta z\frac{\partial T}{\partial r}\right] = \frac{1}{r}\frac{\partial}{\partial r}\left[-kr\frac{\partial T}{\partial r}\right]$$

$$\lim_{\Delta \theta \to 0} \frac{1}{r\Delta r\Delta\theta}\frac{\dot{Q}_\theta - \dot{Q}_{\theta+\Delta \theta}}{\Delta \theta} = \frac{1}{\Delta r\Delta z}\frac{\partial \dot{Q}_\theta}{r\partial \theta} = \frac{1}{\Delta r\Delta z}\frac{\partial}{r\partial \theta}\left[-k\Delta r\Delta z\frac{\partial T}{r\partial \theta}\right] = \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[-k\frac{\partial T}{\partial \theta}\right]$$

$$\lim_{\Delta z \to 0} \frac{1}{r\Delta r\Delta\theta}\frac{\dot{Q}_z - \dot{Q}_{z+\Delta z}}{r\Delta z} = \frac{1}{r\Delta r\Delta\theta}\frac{\partial \dot{Q}_z}{\partial z} = \frac{1}{r\Delta r\Delta\theta}\frac{\partial}{\partial z}\left[-kr\Delta r\Delta\theta\frac{\partial T}{\partial z}\right] = \frac{\partial}{\partial z}\left[-k\frac{\partial T}{\partial z}\right]$$

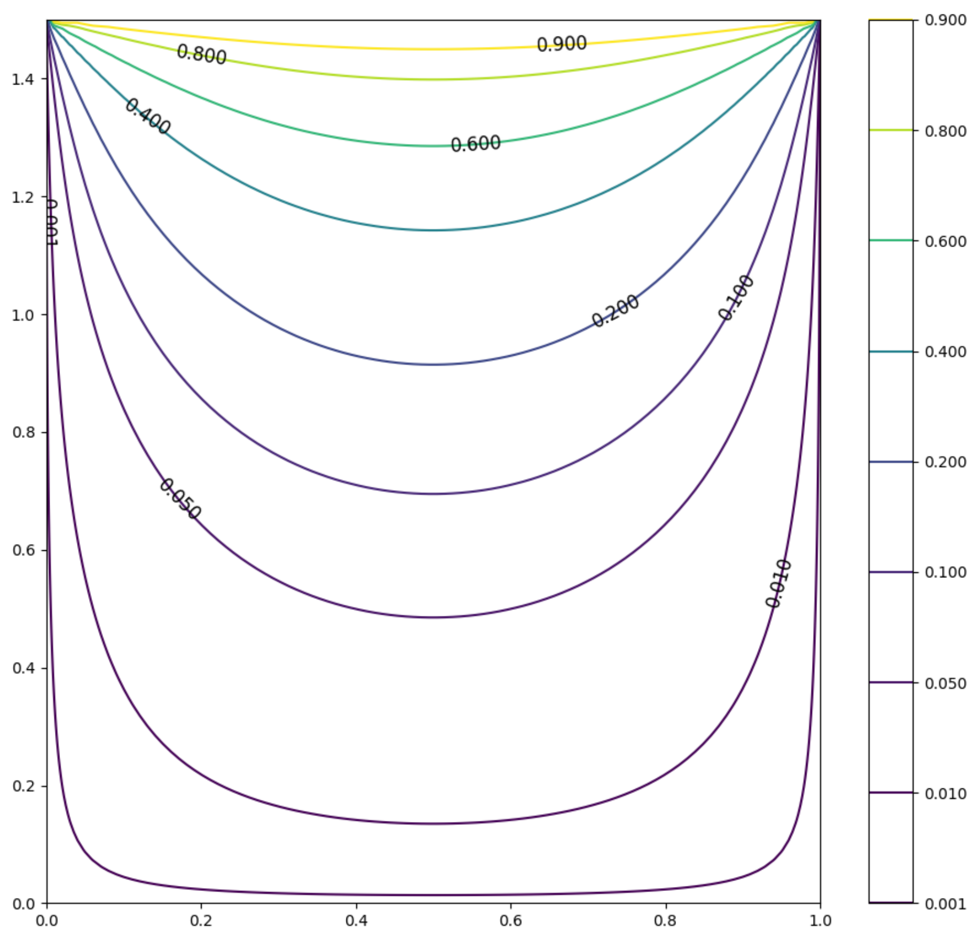
Substituting back into the main equation gives:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[-kr\frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[-k\frac{\partial T}{\partial \theta}\right] + \frac{\partial}{\partial z}\left[-k\frac{\partial T}{\partial z}\right] + \dot{e}_{gen} = \rho c\frac{\partial T}{\partial t}$$

Now dividing by the thermal conductivity and simplifying gives:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial Z^2} + \frac{e_{\dot{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

2. Consider a long rectangular bar of length a in the x-direction and width b in the y-direction. The temperatures of the surfaces at $x = 0$, $y = 0$, and $x = a$ are all equal to T_1, while the temperature of the surface at $y = b$ is equal to T_2. Assuming constant thermal conductivity and steady state heat conduction with no heat generation, (a) determine the general relation for temperature distribution inside the rectangular bar; (b) draw isotherms in the bar if ($b = 1.5a$).



3. Consider a water pipe of length L = 17 m, inner radius r_1 = 15 cm, outer radius r_2 = 20 cm, and thermal conductivity k = 14 W/m·K. Heat is generated in the pipe material uniformly by a 25-kW electric resistance heater. The inner and outer surfaces of the pipe are at T_1 = 60°C and T_2 = 80°C, respectively. Obtain a general relation for temperature distribution inside the pipe under steady conditions and determine the temperature at the center plane of the pipe.

Energy generation rate per unit volume is:

$$\dot{e}_{gen} = \frac{\dot{E}_{gen}}{V} = \frac{\dot{E}_{gen}}{\pi\left(r_2^2 - r_1^2\right)L} = \frac{25000}{\pi\left(0.20^2 - 0.15^2\right)17} = 26748.730\frac{W}{m^3}$$

1D heat transfer equation for a long tube at steady state:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{gen}}{k} = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -r\frac{\dot{e}_{gen}}{k}$$

Integrating with respect to radius gives:

$$r\frac{\partial T}{\partial r} = -r^2\frac{\dot{e}_{gen}}{2k} + c_1$$

$$\Rightarrow \frac{\partial T}{\partial r} = -r\frac{\dot{e}_{gen}}{2k} + \frac{c_1}{r}$$

Integrating again with respect to the radius gives:

$$T(r) = -r^2\frac{\dot{e}_{gen}}{4k} + c_1\ln r + c_2$$

Apply the boundary conditions:

$$T(r_1) = T_1 = -r_1{}^2\frac{\dot{e}_{gen}}{4k} + c_1\ln r_1 + c_2$$

$$\Rightarrow c_2 = T_1 + r_1{}^2\frac{\dot{e}_{gen}}{4k} - c_1\ln r_1$$

$$T(r_2) = T_2 = -r_2{}^2\frac{\dot{e}_{gen}}{4k} + c_1\ln r_2 + c_2$$

$$\Rightarrow c_2 = T_2 + r_2{}^2\frac{\dot{e}_{gen}}{4k} - c_1\ln r_2$$

$$\Rightarrow T_1 + r_1{}^2\frac{\dot{e}_{gen}}{4k} - c_1\ln r_1 = T_2 + r_2{}^2\frac{\dot{e}_{gen}}{4k} - c_1\ln r_2$$

$$\Rightarrow c_1 = \frac{T_2 - T_1 + \frac{\dot{e}_{gen}}{4k}\left(r_2^2 - r_1^2\right)}{\ln\frac{r_2}{r_1}} = \frac{80 - 60 + \frac{26748.730}{4\times14}\left(0.20^2 - 0.15^2\right)}{\ln\frac{0.20}{0.15}} = 98.577$$

$$\Rightarrow c_2 = (80 + 273) + 0.2^2\frac{26748.730}{4\times14} - 98.577\ln 0.2 = 530.911$$

Now solving for the temperature at the midplane:

$$T(0.175) = -0.175^2$$

$$4\times14 + 98.577\ln 0.175 + 530.911 = \boxed{344.46K = 71.46^\circ C}$$

4. Heat is generated uniformly at a rate of 4.2×10^6 W/m^3 in a spherical ball ($k = 45$ W/m·K) of diameter 24 cm. The ball is exposed to iced-water at 0°C with a heat transfer coefficient of 1200 W/m^2·K. (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the spherical ball, (*b*) obtain a relation for the variation of temperature in the ball by solving the differential equation, and (*c*) Determine the temperatures at the center and the surface of the ball.