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FRONT PROPAGATION IN UNIAXIAL FERROMAGNETS

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This work is a theoretical study of fronts propagating into an uniaxial ferromagnetic medium with velocity v, when the magnetization, $\mathbf{M}$, is driven by a DC applied magnetic field, H_0, from the demagnetized to the magnetized state. It is assumed that the dynamics of $\mathbf{M}$ is govern by the Landau–Lifshitz–Gilbert equation (LLGE). We also consider an effective field that includes in-plane uniaxial, H_U, and shape anisotropy fields, H_D. We show that, in the particular case of uniformly translating profiles, this equation reduces to a damped-forced Duffing's equation, with a family of solutions that describe periodic oscillating (PO), damped oscillating (DO) or exponential front profiles (EF). Of particular interest is the existence of a critical speed, v_0, below which there are no stable states for the magnetization. When the velocity of the profile exactly equals $v_{\rm osc}$, the only stable states are the PO profiles. Above $v_{\rm osc}$, there is an asymptotic speed, v^*, that separates the DO profiles from EF profiles. This velocity (v^*) is connected with the existence of a non-linear marginal stability point for front propagation. Both v_0 and v^* depend on the applied field and on the anisotropy constants of the material. A stability diagram for front propagation and magnetization curves are also calculated.

Keywords: Fronts; magnetic waves; Duffing's equation; non-linear dynamics.

1. Introduction

The study of front propagation in a physical system has been the subject of considerable attention in the last few years. Intuitively, a front is imagined as an interface between two regions, and that this interface moves from one side to another. If suddenly one region invades the other, then we say that a front is propagating into that region. In general, a stable phase always moves towards an unstable phase. This dynamics is commonly related with a kind of non-linear partial differential equations called in the literature *diffusion-reaction equations* (DRE) [2, 11]. There

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are many different types of fronts: solitons, shocks, dissipative, uniformly translating. These fronts arise in many fields of science such as biology [10], population dynamics [13], chemistry [6], magnetism [14], mathematics [8], and so on. In all these, fronts develop as domain walls separating an unstable state from some other stable state. As it propagates, the velocity of the front approaches an asymptotic value, so the shape of the profile remains unchanged. Numerical studies indicate that the velocity of the profile of a front is selected via some common dynamical mechanism, and that these selection mechanisms are always related to marginal stability [15].

Traditionally, a conventional ferromagnetic material is treated as an arrangement of homogeneously magnetized regions, called domains, interacting through an interface or domain wall. Due to this interaction and below the Curie point, the magnetization of the local domains align in some direction giving rise to a macroscopic magnetization moment. In the absence of an applied field, this direction is a preferred one and is called an easy axis. Hence, a magnetic field must be applied in order to push the magnetization away from this axis. When the magnetic field is applied, the initially magnetized state becomes unstable and the material is magnetized to saturation. The energy absorbed during this process is characterized as uniaxial, and is measured by observing hysteresis loops or magnetization curves. These materials are also called uniaxial ferromagnets.

During the magnetization rotation, and due to domain wall motion, magnetic pulses, fronts and other profiles may propagate through a magnetic medium with a well-defined dynamics, which leads to various non-linear phenomena. On the other hand, a magnetic material exhibits a non-linear relation between the magnetization and the applied magnetic field that affects not only the switching process, but also distorts magnetic pulses that propagate through the medium. Since magnetic materials are commonly used for technical and practical applications, it is clear that the study of non-linearities in these systems is of great importance.

In this work, front propagation in a ferromagnetic system with uniaxial anisotropy is studied. Based on the Landau–Lifshitz–Gilbert equation (LLGE) [7], our calculations show that the dynamics of the magnetization is governed by a non-linear diffusion-reaction equation [8, 14], and the velocity of the magnetic front is determined by a selection mechanism mainly due to the magnetic anisotropy of the medium. This study enables us to obtain the magnetization curves and stability diagrams for front propagation in a ferromagnetic medium.

The paper is organized as follows. In Sec. 2 we derive the non-linear partial differential equation that governs the dynamic of the magnetization in ferromagnetic material with uniaxial anisotropy. In Sec. 3 we apply this equation to the case of profiles uniformly translating through the system, and then we analyze the asymptotic solutions. It is also shown that these particular states lead to the damped-forced Duffing's equation. Section 4 is devoted to solving numerically the equation resulting from the analysis presented in Sec. 3. In Sec. 5 we summarize our results.

2. Modeling a Ferromagnet

We start by assuming that the dynamics of the magnetization in a ferromagnet (FM) is governed by the Landau–Lifschitz–Gilbert equation (LLGE) [7],

$$\frac{\partial}{\partial t}\mathbf{M} = -\boldsymbol{\gamma}\mathbf{M} \times \mathbf{H} - \lambda\mathbf{M} \times \mathbf{M} \times \mathbf{H}\,, \tag{1}$$

where γ and λ, are the gyromagnetic ratio and the Gilbert damping parameter, respectively, $\mathbf{M}$ is the magnetization, $\mathbf{H} = -\nabla_M E$ is the effective magnetic field, and E is the magnetic free energy. The first term on the right-hand side represents the precessional torque exerted by the magnetic field and the second term is a damping torque due to viscous forces acting on the magnetization. For the magnetic energy we used the Landau–Ginsburg free energy expanded in powers of the magnetization near a critical point, which may be expressed as

$$E = -\mathbf{H}_0 \cdot \mathbf{M} + \sum_i \frac{K_{ui}}{M^2}\, M_i^2 - \sum_i \frac{D_i}{M^2}\, M_i^2 + \frac{J}{M^2}(\nabla\mathbf{M})^2 \quad (i = x, y, z)\,. \tag{2}$$

The first term on the right-hand side is the Zeeman energy, the second term represents the uniaxial energy, the third the demagnetizing energy, and the fourth the exchange energy, with K_{ui}, D_i and J denoting the uniaxial anisotropy, demagnetizing and exchange constants, respectively. $\mathbf{H}_0$ is an external DC applied field.

With the help of expression (2) and the definition of the effective field, Eq. (1) transforms into the 3×3 system:

$$\dot{M}_x = -\gamma(M_y H_z - M_z H_y) - \lambda[M_y(M_x H_y - M_y H_x) - M_z(M_z H_x - M_x H_z)]\,, \tag{3}$$

$$\dot{M}_y = -\gamma(M_z H_x - M_x H_z) - \lambda[M_z(M_y H_z - M_z H_y) - M_x(M_x H_y - M_y H_x)]\,, \tag{4}$$

$$\dot{M}_z = -\gamma(M_x H_y - M_y H_x) - \lambda[M_x(M_z H_x - M_x H_z) - M_y(M_y H_z - M_z H_y)]\,, \tag{5}$$

where the effective field components are given by

$$H_i = H_{0i} - \frac{2}{M^2}\,(K_{ui} - D_i)M_i - \frac{J}{M^2}\nabla^2 M_i \quad (i = x, y, z)\,. \tag{6}$$

For the sake of simplicity, we will consider a FM of length, L, large compared to its width, $w(w \ll L)$, and the thickness, t, sufficiently small ($t \ll w$), so that $\nabla \to \partial/\partial x$. The plane of the FM is in the (x, y)-plane and an external DC magnetic field, $\mathbf{H}_0$, is applied along the axis of the material (x-axis). It is also considered that the magnetization is completely in-plane ($M_z^2 \ll M_x^2 + M_y^2 \cong M^2$), and both demagnetizing and uniaxial fields are parallel to the x-axis. With all these assumptions, Eqs. (3)–(5) are rewritten as

$$\dot{M}_x = \lambda(M_x H_y - M_y H_x)M_y\,, \tag{7}$$

$$\dot{M}_y = \lambda(M_y H_x - M_x H_y)M_x\,, \tag{8}$$

$$\dot{M}_z = \gamma(M_y H_x - M_x H_y) \tag{9}$$

with the effective field components:

$$H_x = H_{0x} - \frac{2}{M^2}(K_u - D)M_x - \frac{J}{M^2}\frac{\partial^2}{\partial x^2}M_x \tag{10}$$

and

$$H_y = -\frac{J}{M^2}\frac{\partial^2}{\partial x^2}M_y\,. \tag{11}$$

Combining Eqs. (7)–(9), it is easy to show that

$$\dot{M}_z = \frac{\gamma}{\lambda}\frac{\dot{M}_x}{M_y}\,. \tag{12}$$

Although the FM is magnetized in-plane, there is always a torque that pushes the magnetization out-of-plane. To avoid this effect, let us consider a damping sufficiently high that the torque part can be neglected, so that $\gamma \ll \lambda$ and $\dot{M}_z \approx 0$. Since Eqs. (7) and (8) are equivalent, our problem is reduced to solving Eq. (7) together with the relation

$$M^2 = M_x(x,t)^2 + M_y(x,t)^2 = \text{const}\,. \tag{13}$$

Substitution of Eqs. (10), (11) and (13) into (7), lead to the non-linear partial differential equation for the magnetization:

$$\begin{aligned}\dot{\varphi}(x,t) = -a\frac{\partial^2}{\partial x^2}\varphi(x,t) - a\frac{\varphi(x,t)}{1-\varphi(x,t)^2}\left(\frac{\partial}{\partial x}\varphi(x,t)\right)^2 \\ + c_3\varphi(x,t)^3 - c_2\varphi(x,t)^2 - c_1\varphi(x,t) + c_0\,,\end{aligned} \tag{14}$$

where $\varphi(x,t) = M_x(x,t)/M$, $H_u = 2K_u/M$, $H_D = 2D/M$, $H_E = J/M$, $a = \lambda H_E M$, $c_0 = c_2 = \lambda H_0 M$, and $c_1 = c_3 = \lambda(H_u - H_D)M$. The resulting model describes a one-dimensional mean-field XY model for a ferromagnet. Within the mean-field approximation, we assume that the only important configurations near the critical point are those of uniform magnetization density, i.e. $\partial\varphi/\partial x = 0$, so the second term in Eq. (14) can be neglected, and it can be expressed in the final form:

$$\dot{\varphi}(x,t) \cong -a\frac{\partial^2}{\partial x^2}\varphi(x,t) + c_3\varphi(x,t)^3 - c_2\varphi(x,t)^2 - c_1\varphi(x,t) + c_0\,. \tag{15}$$

Equation (15) belong to a class of equations termed diffusion-reaction. The case $H_0 = 0$, has no sense in the context of a ferromagnet, however when $H_0 \neq 0$ the demagnetized state ($\varphi = 0$) becomes unstable, and the magnetization stabilizes around some equilibrium state ($\varphi_S \leq 1$). Here, we are interested in the case when the magnetization is driven from the demagnetized state to a saturated state, by a applied DC field.

3. Wave Propagation and the Duffing's Equation

The mathematical and physical theory of front propagation into magnetic media, presents a number of unsolved problems. The most relevant are

- How do magnetic fronts form in the material?
- What is the stability of such a front?
- How fast will a wave move through a magnetic medium?

All these questions become more clear by considering solutions of the type

$$\varphi(x,t) = \varphi(\xi = x - vt)\,, \tag{16}$$

which connect the unstable state $\varphi(\xi) = 0$ (demagnetized) and $\varphi(\xi) = \varphi_S \leq 1$ (magnetized), or

$$\lim_{\xi\to-\infty} \varphi(\xi) = 0\,, \tag{17}$$

$$\lim_{\xi\to+\infty} \varphi(\xi) = \varphi_S\,. \tag{18}$$

These solutions are shape-preserving and move with constant velocity, v, and are referred to in the literature as uniformly translating profiles [2]. These fronts occur when the state emerging behind is homogeneous. Substitution of (16) into (15) gives the ordinary non-linear equation:

$$\frac{d^2}{d\xi^2}\,\varphi(\xi) + \frac{v}{a}\,\frac{d}{d\xi}\,\varphi(\xi) + c_3\varphi(\xi)^3 - c_2\varphi(\xi)^2 - c_1\varphi(\xi) + c_0 = 0\,. \tag{19}$$

This is the damped Duffing's equation with a forced term [9]. It is apparent that this equation has acceptable solutions for any value of v. However, according to other authors [15, 17] there are some natural conditions for which dynamical velocity selection takes place, and then the velocity of all acceptable fronts converge asymptotically to one particular value.

One of the most common methods for studying stability of fronts is the linearization method [4]. For this, let us perturb the magnetized state (stable solution) with a small term $\delta\varphi$ such that, $\varphi \to \varphi_S - \delta\varphi$ as $\xi \to \infty$, where $|\delta\varphi| \ll \varphi_S$. Under this perturbation the magnetization will relax at some rate around it equilibrium state. Introducing the solution $\varphi = \varphi_S - \delta\varphi$ into Eq. (19), and retaining only terms up to first order in $\delta\varphi$, we obtain the linear equation:

$$\frac{d^2}{d\xi^2}\,\delta\varphi + \frac{v}{a}\,\frac{d}{d\xi}\,\delta\varphi + \varepsilon\delta\varphi = 0\,, \tag{20}$$

where

$$\varepsilon = (3c_3 - 2c_2 - c_1)/a = 2(H_u - H_D - H_0)/H_E \tag{21}$$

with the corresponding characteristic polynomial:

$$\lambda^2 + \frac{v}{a}\,\lambda + \varepsilon = 0\,. \tag{22}$$

The roots of this polynomial are given by

$$\lambda_{\pm} = -\frac{v}{2a} \pm \sqrt{\left(\frac{v}{2a}\right)^2 - \left(\frac{v^*}{2a}\right)^2}\,, \tag{23}$$

where we have defined $v^* = 2a\varepsilon^{1/2}$. This yields the following solutions:

$$\delta\varphi = e^{-\frac{v}{2a}\xi}\left(Ae^{-\sqrt{v^2-v^{*2}}\xi/2a} + Be^{\sqrt{v^2-v^{*2}}\xi/2a}\right). \tag{24}$$

Since we require that $\delta\varphi \to 0$ as $\xi \to \infty$, we get the asymptotic form:

$$\delta\varphi \propto e^{-K\xi}\,, \tag{25}$$

where

$$K = \frac{v}{2a} + \sqrt{\left(\frac{v}{2a}\right)^2 - \left(\frac{v^*}{2a}\right)^2} \tag{26}$$

or, equivalently,

$$\frac{v}{a} = K + \frac{\varepsilon}{K}\,. \tag{27}$$

This expression gives the branch for asymptotic behavior, and is plotted in Fig. 1 for $\gamma = 0.1$. Every front profile in this branch is purely exponential and propagates with velocity $v \geq v^* = 2a\varepsilon^{1/2}$. Following van Saarloos's argument [15], this region is governed by a non-linear marginal-stability scenario, and the speed $v = v^*$ is

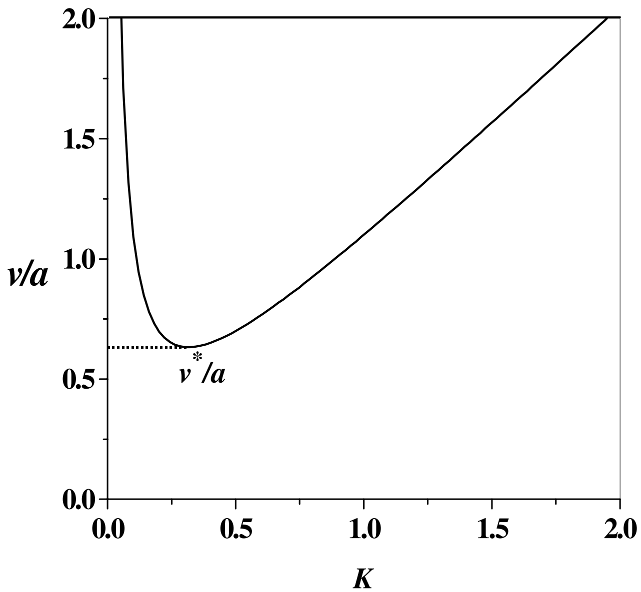

Fig. 1. Branch of asymptotic behavior for front profiles as given by Eq. (27). All solutions in this branch represent purely exponential fronts that propagate with velocity $v \geq v^* = 2a\varepsilon^{1/2}$. The minimum of this curve is a transition point for marginal stability.

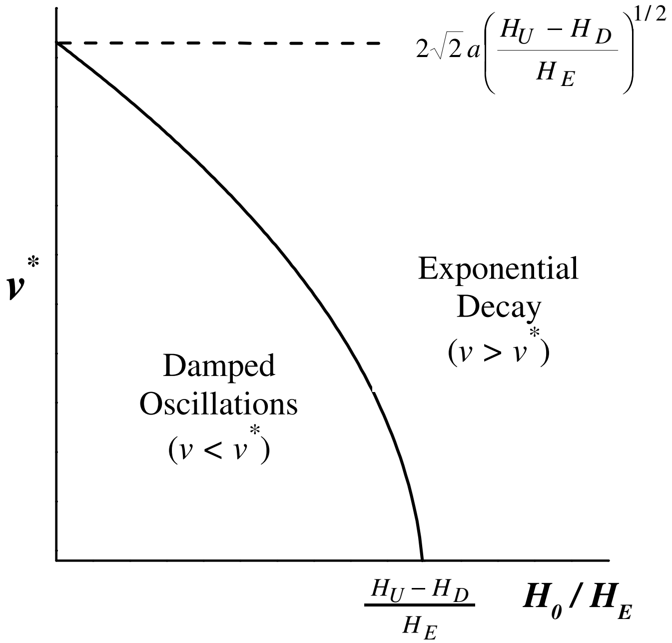

Fig. 2. Dependence of v^* with respect to the applied magnetic field, H_0, for a representative value of $(H_U - H_D)/H_E$. This curves separate the two regions of asymptotic behavior.

a transition point of marginal stability. In the case when $v < v^*$, $K = 1/2a[v + i(v^{*2} - v^2)^{1/2}]$, and the front oscillates with "natural frequency" $v^*/2a = \varepsilon^{1/2}$, and the damping parameter $v/2a$. These results are summarized in Fig. 2, where the relation $v^* = 2a\varepsilon^{1/2}$ is plotted for some value of the parameter $(H_U - H_D)/H_E$. This curve clearly differentiates the oscillating fronts from the exponential front profiles. Each point on this curve corresponds to a minimum in Eq. (27) for each value of H_0/H_E and $(H_U - H_D)/H_E$, as shown in Fig. 1. Also, from relation (21) it is noted that, as $(H_U - H_D)/H_E$ increases the region of oscillatory profiles grows, while for decreasing $(H_U - H_D)/H_E$ the oscillations disappear and the only stable states under perturbations are purely exponential front profiles.

4. Numerical Results

In order to understand the problem of wave propagation through a FM medium, we must go back to Eq. (19). By means of a Runge–Kutta–Fehlberg scheme [5] the damped-forced Duffing's equation is solved numerically with the boundary conditions (17) and (18), for several values of v, and magnetic fields in the range $0 \leq H_0 \leq H_E$, and $0 \leq H_U \leq H_D \leq H_E$. We obtain a family of solutions that describe periodic oscillations (PO), damped oscillations (DO) or exponential front profiles (EF), depending on the relative value of $(H_U - H_D)/H_E$, and the wave speed, v. Representative results of these calculations performed on a tape of

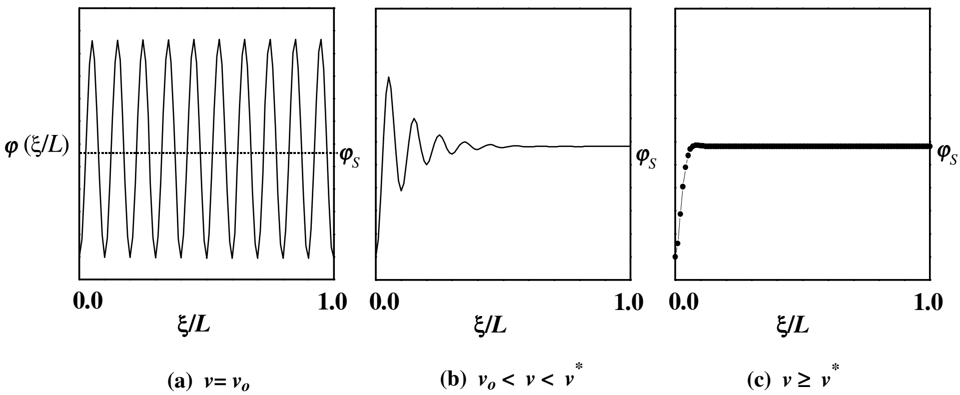

Fig. 3. Representative solutions of Eq. (19) on a ferromagnetic material of normalized length ξ/L. For (a) $v = v_0$, the only stable states are harmonic oscillations; for (b) $v_0 < v < v^*$ the fronts are damped oscillations; and for (c) $v \geq v^*$ the fronts are exponential. For speeds $v < v_0$ there are no stable fronts.

normalized length ξ/L are shown in Fig. 3. We found a critical velocity, v_0 (H_0), below which there are no stable states for the magnetization. At $v = v_0(H_0)$, only PO states are possible. For $v > v_0(H_0)$, **M** stabilizes around the state $\varphi_S(\leq 1)$, through damped oscillations (DO) up to the asymptotic value $v^*(H_0)$, where the magnetized state rapidly invades the demagnetized state and an exponential front begins to propagate. The dashed line correspond to the value of φ_S around which the magnetization stabilizes asymptotically. The different stability regions for a magnetic tape are presented in Fig. 4, as a function of the DC magnetic field. The closed squares indicate the velocities observed in the stable solutions of Eq. (19). The dashed curve is the asymptotic value determined by the relation $v^*(H_0)$. As noted, the regions of DO and EF profiles are strongly dependent on $(H_U - H_D)/H_E$. There also exists a region in which all states are unstable (US) and grows as $(H_U - H_D)/H_E$ decreases. A small region of unstability is also observed with fronts propagating with velocity $v < v_{\text{osc}}$ (shadowed). The region labeled as NA is physically inadmissible since $\varphi_S > 1$. Although linearly stable, these fronts are not accessible to the system. The dependence of the magnetized state, φ_S, with respect to the magnetic field and the parameter $(H_U - H_D)/H_E$ was also studied. For this, the value of $(H_U - H_D)/H_E$ was fixed and the magnetic field varied until the value of φ_S was reached. In this manner, the magnetization curves for the ferromagnet were obtained as a function of H_0 for several values of $(H_U - H_D)/H_E$ (closed symbols in Fig. 5). The solid lines correspond to the solutions of the cubic equation:

$$\frac{H_0}{H_E}(1-\varphi^2) + \left(\frac{H_U - H_D}{H_E}\right)(1-\varphi^2)\varphi = 0\,, \tag{28}$$

which results from setting $\dot{\varphi}(x,t) = \partial^2\varphi(x,t)/\partial x^2 = 0$, and are given by $\varphi_S = \pm 1$, $H_0/(H_U - H_D)$. The magnetization increases linearly with respect to the applied

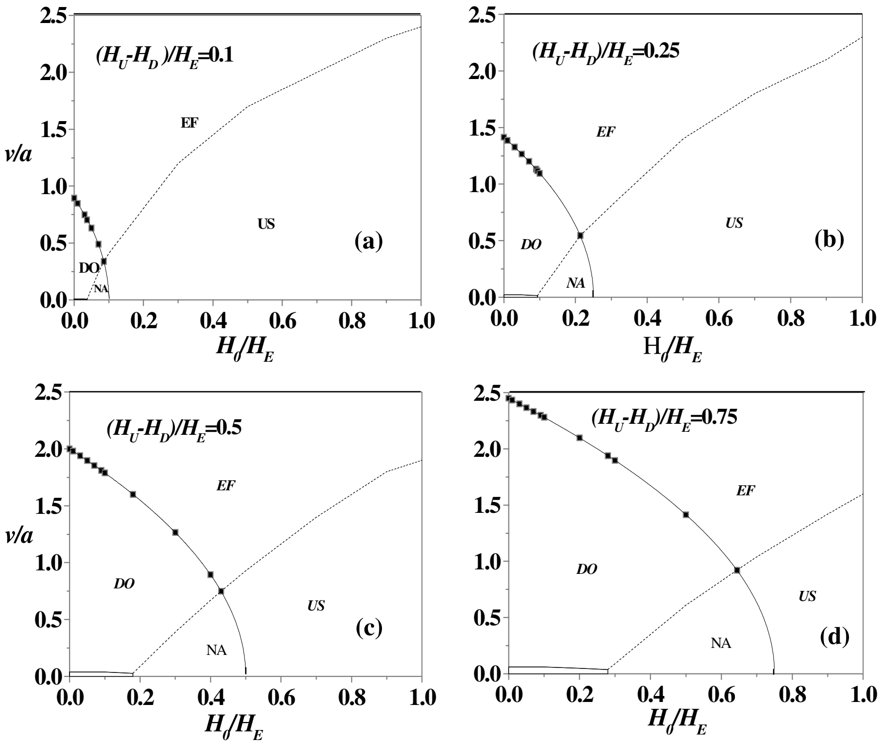

Fig. 4. Stability diagrams for front propagation in a ferromagnetic material, as obtained from solving numerically Eq. (19). All stability regions are clearly demarcated, as explained in the text. The fronts in the shadow region are unstable. As $H_U - H_D$ increases the oscillating fronts invade the exponential profiles.

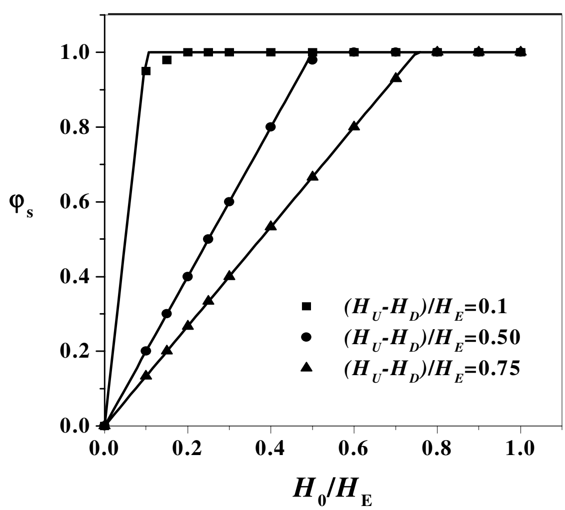

Fig. 5. Magnetization curves of the ferromagnet for several values of the parameter $(H_U - H_D)/H_E$. Note that the saturation field is equal to $H_U - H_D$.

field up to saturation at a saturation field $H_S \cong H_U - H_D$. As $(H_U - H_D)/H_E$ is increased, the system becomes magnetically harder. This shows that the dynamics of the magnetization between states of different stability is controlled by the anisotropy fields.

5. Concluding Remarks

In conclusion, we have shown that the dynamics of the magnetization in a ferromagnetic material with uniaxial anisotropy is governed by a non-linear diffusion-reaction equation. In the important case of uniformly translating profiles, the propagation of magnetic waves is governed by the damped-forced Duffing's equation, and the velocity of the profiles is determined by a selection mechanism mainly due to the magnetic anisotropies of the material. When the initially demagnetized system is perturbed by a DC applied field damped oscillations and exponential front profiles can propagate through the medium, depending on the value of the applied field and the internal anisotropy fields. This study allowed us to plot stability diagrams for front propagation and magnetization curves in a ferromagnet.

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