Results from “Dynamics of a SIR epidemic model with limited medical resources, revisited again”

This Mathematica Notebook is a supplemntary material to the paper “Dynamics of a SIR epidemic model with limited medical
resources, revisited again”. It contains some of the calculations and illustrations appearing in the paper .

I)SIR with B(s,i)=β s /(1+μ i) and T(i)=η w i/(i+w) preliminaries. Output: R0

In[1]:=
```
SetDirectory[NotebookDirectory[]];
AppendTo[$Path, Directory];
Clear["Global`*"];
<< "def.m"; (*trG*)
Params = {Λ, δ, γ, β, ξ, μ, ω, α};
Paramsp = {Λ, δ, γ, β, ξ, μ, ω, α, η};
X = {s, i};
cet0 = {η → 0};
cetaBp = {η → (β Λ + (γ + μ) (μ - ω v_1) + 2 Sqrt[β Λ (γ + μ) (μ - ω v_1)]) / (ω v_1)};
cetaBm = {η → (β Λ + (γ + μ) (μ - ω v_1) - 2 Sqrt[β Λ (γ + μ) (μ - ω v_1)]) / (ω v_1)};
cdel = {δ → 0};

Lgn = (γ + Λ + δ); cir = {ir → γ - is};
sd = Λ / μ; R1 = β / V_2; R0 = sd R1 ; (*basic reproduction number*)
η0 = sd β - v_2 ; ωH = (μ (β Λ - μ (γ + δ + μ))) / (β Λ (β + μ ξ));

(*Conditions de positivité*)
cp = Thread[Paramsp > 0];
(*conditions for switching between
 the parameters and particular cases*)
cv2 = {v_2 → (μ + γ + δ) };
cv1 = {v_1 → (β + μ ξ) };
cV2 = {V_2 → v_2 + η, v_2 → (μ + γ + δ) };
cv = {ξ → (v_1 - β) / μ, δ → v_2 - (μ + γ) , η → V_2 - v_2};
ceta = {η → α / ω};
calv1 = {α → ω (v_1 - β) / μ};
Print["R0  =", R0 //. cV2, " ,η0=", η0 , " ,ωH=", ωH,
 " and critical β is ", b0 = β /. Solve[R0 == 1, β][[1]] // FullSimplify]


(*Numerical conditions used in first tests*)
Param = Thread[{Λ, δ, γ, β, ξ, μ, v_2, v_1, V_2} →
    {16, 2 / 10, 12 / 100, 1 / 100, 1 / 1000, 12 / 100, μ + γ + δ, β + μ ξ, μ + γ + δ + η}];
(*Save["def.m",Param];*)
ParamF = Thread[{Λ, δ, γ, β, ξ, μ, v_2, v_1, V_2} →
    {16, 2 / 10, 12 / 100, 1 / 100, 1 / 1000, 1 / 10, μ + γ + δ, β + μ ξ, μ + γ + δ + η}];
paramGc = Thread[{Λ, δ, γ, β, ξ, μ, v_2, v_1, V_2} →
    {1 / 2, 2 / 10, 1 / 10, 2 / 10, 7 / 100, 1 / 10, (μ + γ + δ), (β + μ ξ), μ + γ + δ + η}];
 (*Parameters of Figure 4 in Zhou Fan*)
ParNumCheck = Thread[{ω, α, η} → {7/64, 179/64, α / ω}];
Print["test cNN="]
cNN = Join[ParNumCheck, Param]
```

R0 $=\frac{\beta\Lambda}{\mu(\gamma+\delta+\eta+\mu)}$,η0=$\frac{\beta\Lambda}{\mu}-v_2$,ωH=$\frac{\mu(\beta\Lambda-\mu(\gamma+\delta+\mu))}{\beta\Lambda(\beta+\mu\xi)}$ and critical β is $\frac{\mu V_2}{\Lambda}$

test cNN=

Out[20]= $\left\{\omega\to\frac{7}{64},\ \alpha\to\frac{179}{64},\ \eta\to\frac{\alpha}{\omega},\ \Lambda\to 16,\ \delta\to\frac{1}{5},\ \gamma\to\frac{3}{25},\ \beta\to\frac{1}{100},\ \xi\to\frac{1}{1000},\ \mu\to\frac{3}{25},\ v_2\to\gamma+\delta+\mu,\ v_1\to\beta+\mu\xi,\ V_2\to\gamma+\delta+\eta+\mu\right\}$

0) Model, eqF, sol, Jacobian, jacE0, equation for endemic i, ie ,se2,detE2. Outputs: trE2, Discriminant

In[21]:=
```
(*SIR  epidemic model of Fan:*)
s1=Λ -   β s  i/(1+ξ i)-s μ ;(*λ s (1-s/K)*)
ifac=β s   /(1+ξ i)-v2  -η ω /(i+ω);
i1= i ifac;
r1=η  ω i/(i+ω)-μ r + γ i ;

dyn={s1,i1,r1}//.cv2;(*field*)
vars={s,i,r};
equi=Solve[Thread[dyn==0],vars];

 Print["(s' i' r')= ",dyn//MatrixForm]

(*Diff.sys and Numerical test:*)
varst=Through[vars[t]];(*Map[#[t]&, vars];Revarst = Thread[vars→varst]*);
diff= D[varst,t] - (dyn/.Thread[vars→varst]);
diffN=diff//.cNN;
initcond = (varst/.t→0)-{1.5, 0.5, 0.1};
eqs=Thread[Flatten[{diffN, initcond}] == 0];
ndesoln = NDSolveValue[eqs,varst,{t, 0, 1000}];
Print["simulation test when sd=",Λ/μ//.cNN]
Chop[ndesoln/.t→1000]

(*Two dimensional Fan:*)
dyn2={s1,i1}/.cv2;(*we may reduce to this dyntem since these two equations
 do not depend on r*)
 Print["For 2-dim case, we have (s' i')=",dyn2//MatrixForm]
 eqF=Thread[Flatten[dyn2==0]];
 equi2=Solve[eqF,{s,i}];
 Print[" fixed points are"]
equi2//.cNN//N
 dyn2E={s1,Simplify[i1/i]}/.cv2;

(*Computation of the Jacobian, Trace, Det  *)

jac=Grad[dyn2,{s,i}]//FullSimplify;
det=Det[jac]//FullSimplify;
Print["2 dim jac=",jac//MatrixForm]
```

```
jacE0=(jac/.i→0/.s→sd)//FullSimplify;
Print["jac(DFE)=", jacE0//MatrixForm]
tr=Tr[jac];
Print[" detG is"]
Timing[detG=GroebnerBasis[{s1,ifac,det},Params,X][[1]]//FullSimplify]
(*Timing[trG=GroebnerBasis[{s1,ifac,tr},Params,X][[1]]](*32 secs, long*)(*Do not simplify*)
Save["def.m",trG]*)

detE0=Det[jacE0]//FullSimplify;trE0=Tr[jacE0]//FullSimplify;

Print["Det(J(E0)=",detE0, ", and Tr(J(E0))=", Apart[trE0]]

(*Elimination of s via plugging, for trace and det*)
Print["s formula cs from first and sec eqs are"]
cs=Flatten[Solve[dyn2[[1]]==0,s]]
cs2=Flatten[Solve[(dyn2[[2]]/i)==0,s]]
se02=s/.cs2;
(*detEn=det/.s→se02 /.i→ic0;*)

deti=(det/.cs2)//.cv//Factor;
Print[" det after elim of s=", deti]
nudet=Factor[Numerator[deti]];

(*det1=(det/.cs)//FullSimplify*)
(*Print["Det i"]
deti=Simplify[(detF/.cs)( 1+ξ i)(( ω + i)^2)( μ+(μ ξ+β) i)]/.cv//FullSimplify
Print["PolynomialRemainder detR"]
detR=PolynomialRemainder[deti,poli,i];
Timing[icd=i/.Solve[detR==0,i][[1]]//FullSimplify]*)

(*Factor[Numerator[deti]]*)
Print["tr= ",tr]
tri=Together[(tr/.cs2)]//Factor;
(*trs=Together[(tr/.cs)]//Factor;*)
Print[" tr after elim of s is ", tri]
Print[" numer is nutr="]
nutr=Factor[Numerator[tri]]

(*Equation for endemic i *)
poli=Collect[Factor[Numerator[Together[dyn[[2]]/.cs]]]/(-i),i];
Print["sec. order A i^2 + B i + C=0 for endemic i is"]
polv= Collect[ poli/.cv//FullSimplify,i]
(*Mathematica Eliminate*)
el=Eliminate[eqF,{s}];
poliF=Collect[Numerator[Factor[el[[1,1]]-el[[1,2]]]/(-i)]//FullSimplify,i];
Print["check  poli/poliF="]
poli/poliF//FullSimplify

Print["coeffs are"]
cf=CoefficientList[poli,i];
Aa=cf[[3]];Bb=cf[[2]]; Cc=cf[[1]];
Print["{A,B,C} =",{Aa,Bb,Cc}//FullSimplify//.cv2]

Print["dis "]
dis=Discriminant[poli,i];
```

```
Print[" H  has coords "]
{η0,ωH}
(*Print["  and trH="]
Timing[trH=trE2/.{η→η0,ω→ωH}//.cv2//FullSimplify]*)
Print["The discriminant Δ = B^2 - 4 A C at H  is"]
(dis/.{η→η0,ω→ωH})//.cv2//FullSimplify

ie=i/.Solve[poli==0,i];ic0=(-Bb/(2 Aa));

trE2=tri/.i→ie[[2]];
trE1=tri/.i→ie[[1]];
(*Save["def.m",dis];Save["def.m",trE2]*)
se1=Λ ( 1+ξ ie[[1]])/(μ+ ie[[1]] v1)(*endemic s*);
se2=Λ ( 1+ξ ie[[2]])/(μ+ ie[[2]] v1);
se0=Λ ( 1+ξ ic0)/(μ+ ic0 v1);


detE2=deti/.i→ie[[2]];
detE1=deti/.i→ie[[1]];
detE=deti /.i→ic0(* det ωhen $i=-B/(2A)**);


(*Print["i1,  icd, i2"]
Chop[{ie[[1]],icd,ie[[2]]}//.cNN//N]*)
rest=Resultant[poli,nutr,i];
Print["similified restr"]
Timing[rests=FullSimplify[μ^2 rest//.cv]](*quicker than trG*)
Length[rests]
rest3=ω^3 v1^4 (μ v2-(μ+v2) V2) (-β Λ μ-β Λ V2+μ V2^2)+
β v2 (β^3 Λ^3 (μ+β ω)+β Λ μ (-β Λ μ+μ^3-β^2 Λ ω) V2+μ^3 v2^2 (β Λ+μ^2-μ V2)-
μ v2 (β Λ (2 β Λ μ+μ^3+2 β^2 Λ ω)+μ V2 ×
(-2 β Λ μ+μ^3-3 β^2 Λ ω+β μ ω V2)))+v1 (-μ^2 (μ+2 β ω) v2^3 (β Λ+μ^2-μ V2)+
β Λ μ (-β^2 Λ^2 (μ+β ω)+μ (β Λ μ-μ^3+β^2 Λ ω) V2)+
β v2 (Λ (β Λ μ^3+μ^5-2 β^3 Λ^2 ω+β^2 Λ μ (-Λ+μ ω))+Λ (-2 μ^4-2 β^3 Λ ω^2+β μ^2 (Λ-5 μ ω)) V2
+μ ω (2 β Λ μ+μ^3+2 β^2 Λ ω) V2^2)+
v2^2 (β Λ (μ^4+8 β^2 Λ μ ω+4 β^3 Λ ω^2+2 β μ^2 (Λ+2 μ ω))+μ V2 ×
(μ^4-10 β^2 Λ μ ω-6 β^3 Λ ω^2+β μ^2 (-2 Λ+μ ω)+2 β μ ω (μ+β ω) V2)))
+ω v1^2 (β^3 Λ^3 μ+μ^2 v2^3 (β Λ+μ^2-μ V2)+β Λ μ V2 (β Λ μ+3 μ^3+2 β^2 Λ ω-2 μ (μ+β ω) V2)
+v2^2 (-β Λ (5 μ^3+8 β^2 Λ ω+6 β μ (Λ+μ ω))+
μ V2 (8 β Λ μ+μ^3+3 β (4 β Λ+μ^2) ω-2 μ (μ+2 β ω) V2))+v2 (β Λ (-5 β Λ μ^2-3 μ^4+β^2 Λ (Λ-4 μ ω))
V2 (β Λ (10 μ^3+4 β^2 Λ ω+β μ (Λ+10 μ ω))+V2 (-2 μ^2 (β Λ+μ^2)-3 β μ (β Λ+μ^2) ×
ω+β^3 Λ ω^2-β μ ω (μ+β ω) V2))))+ω^2 v1^3 (2 v2^2 (2 β Λ-μ V2) (β Λ+μ^2-μ V2)
+μ V2 (-β Λ (2 β Λ+3 μ^2)+β Λ (μ-β ω) V2+μ (μ+β ω) V2^2)+
v2 (β Λ μ (4 β Λ+3 μ^2)+V2 (-2 β Λ (β Λ+4 μ^2)+
V2 (β Λ μ+μ^3-β (2 β Λ+μ^2) ω+μ (μ+2 β ω) V2))));
resd=Resultant[poli,nudet,i];

Print["HP"]
HP=Solve[Join[{η==η0&&Bb==0},cp,{η==α/ω}]//.Param,{ω,α,η}];
HP//N
```

$$\begin{pmatrix} s' \\ i' \\ r' \end{pmatrix} = \begin{pmatrix} \Lambda - s\,\mu - \frac{i\,s\,\beta}{1+i\,\xi} \\ i\left(-\gamma-\delta-\mu+\frac{s\,\beta}{1+i\,\xi}-\frac{\eta\,\omega}{i+\omega}\right) \\ i\,\gamma - r\,\mu + \frac{i\,\eta\,\omega}{i+\omega} \end{pmatrix}$$

simulation test when $sd = \frac{400}{3}$

Out[36]= {133.333, 0, 0}

For 2-dim case, ωe have $\begin{pmatrix} s' \\ i' \end{pmatrix} = \begin{pmatrix} \Lambda - s\,\mu - \frac{i\,s\,\beta}{1+i\,\xi} \\ i\left(-\gamma-\delta-\mu+\frac{s\,\beta}{1+i\,\xi}-\frac{\eta\,\omega}{i+\omega}\right) \end{pmatrix}$

fixed points are

Out[41]= $\{\{s\to 133.333, i\to 0.\}, \{s\to 86.1361, i\to 6.61878\}, \{s\to 69.9589, i\to 10.99\}\}$

$$\text{2 dim jac} = \begin{pmatrix} -\mu - \frac{i\,\beta}{1+i\,\xi} & -\frac{s\,\beta}{(1+i\,\xi)^2} \\ \frac{i\,\beta}{1+i\,\xi} & -\gamma-\delta-\mu+\frac{s\,\beta}{(1+i\,\xi)^2}-\frac{\eta\,\omega^2}{(i+\omega)^2} \end{pmatrix}$$

$$\text{jac}(\text{DFE}) = \begin{pmatrix} -\mu & -\frac{\beta\,\Lambda}{\mu} \\ 0 & \frac{\beta\,\Lambda-\mu\,(\gamma+\delta+\eta+\mu)}{\mu} \end{pmatrix}$$

detG is

Out[50]= $\Big\{3.8125,$
$(\gamma+\delta+\mu)\,\left(\beta\,\Lambda\,\left(\beta^2\,\Lambda^2-\beta\,\eta\,\Lambda\,\mu+\eta\,\mu^2\,(\gamma+\delta+\mu)\right)-2\,\beta\,\eta\,\Lambda\,(\beta\,\Lambda+\mu\,(\gamma+\delta-\eta+\mu))\,(\beta+\mu\,\xi)\,\omega+\right.$
$\left.\eta\,\left(-\eta^2\,\mu+\beta\,\Lambda\,(\gamma+\delta+\eta+\mu)\right)\,(\beta+\mu\,\xi)^2\,\omega^2\right)+$
$v_2\,\left(-2\,(\gamma+\delta+\mu)\,\left(-\beta^2\,\Lambda^2+\eta^2\,\mu\,(\beta+\mu\,\xi)\,\omega\right)\,(\beta\,\omega+\mu\,(-1+\xi\,\omega))+v_2\,\Big((\beta\,\Lambda-\eta\,\mu)\right.$
$\left(\mu\,(-\beta\,\Lambda+\mu\,(\gamma+\delta+\mu))-2\,\mu\,(\gamma+\delta+\mu)\,(\beta+\mu\,\xi)\,\omega+(\gamma+\delta+\eta+\mu)\,(\beta+\mu\,\xi)^2\,\omega^2\right)+$
$\left.\mu\,(\beta\,\omega+\mu\,(-1+\xi\,\omega))\,v_2\,(-2\,(\beta\,\Lambda+\eta\,(\beta+\mu\,\xi)\,\omega)+(\mu-(\beta+\mu\,\xi)\,\omega)\,v_2)\Big)\right)\Big\}$

Det(J(E0)) $=-\beta\,\Lambda+\mu\,(\gamma+\delta+\eta+\mu)$, and Tr(J(E0)) $=-\gamma-\delta-\eta+\frac{\beta\,\Lambda}{\mu}-2\,\mu$

s formula cs from first and sec eqs are

Out[54]= $\left\{s\to\frac{\Lambda\,(1+i\,\xi)}{i\,\beta+\mu+i\,\mu\,\xi}\right\}$

Out[55]= $\left\{s\to\frac{(1+i\,\xi)\,\left(\gamma+\delta+\mu+\frac{\eta\,\omega}{i+\omega}\right)}{\beta}\right\}$

det after elim of $s = \frac{i\,\mu\,\left(\mu\,\omega\,v_2+i^2\,v_1\,v_2+2\,i\,\omega\,v_1\,v_2-\mu\,\omega\,V_2+\omega^2\,v_1\,V_2\right)}{(i+\omega)^2\,(-i\,\beta+\mu+i\,v_1)}$

tr $= -\gamma-\delta-2\,\mu+\frac{s\,\beta}{(1+i\,\xi)^2}-\frac{i\,\beta}{1+i\,\xi}-\frac{\eta\,\omega^2}{(i+\omega)^2}$

tr after elim of s is

$$-\frac{1}{(1+i\,\xi)\,(i+\omega)^2}\left(i^3\,\beta+i^2\,\mu+i^3\,\gamma\,\xi+i^3\,\delta\,\xi+2\,i^3\,\mu\,\xi+2\,i^2\,\beta\,\omega-i\,\eta\,\omega+2\,i\,\mu\,\omega+2\,i^2\,\gamma\,\xi\,\omega+2\,i^2\,\delta\,\xi\,\omega+4\,i^2\,\mu\,\xi\,\omega+i\,\beta\,\omega^2+\mu\,\omega^2+i\,\gamma\,\xi\,\omega^2+i\,\delta\,\xi\,\omega^2+i\,\eta\,\xi\,\omega^2+2\,i\,\mu\,\xi\,\omega^2\right)$$

numer is nutr=

Out[64]= $-\mathbf{i}^3\beta-\mathbf{i}^2\mu-\mathbf{i}^3\gamma\xi-\mathbf{i}^3\delta\xi-2\mathbf{i}^3\mu\xi-2\mathbf{i}^2\beta\omega+\mathbf{i}\eta\omega-2\mathbf{i}\mu\omega-2\mathbf{i}^2\gamma\xi\omega-2\mathbf{i}^2\delta\xi\omega-4\mathbf{i}^2\mu\xi\omega-\mathbf{i}\beta\omega^2-\mu\omega^2-\mathbf{i}\gamma\xi\omega^2-\mathbf{i}\delta\xi\omega^2-\mathbf{i}\eta\xi\omega^2-2\mathbf{i}\mu\xi\omega^2$

sec. order A i^2 + B i + C=0 for endemic i is

Out[67]= $-\beta\Lambda\omega+\mathbf{i}^2 v_1 v_2+\mu\omega V_2+\mathbf{i}\,(-\beta\Lambda+\mu v_2+\omega v_1 V_2)$

check poli/poliF=

Out[71]= 1

coeffs are

{A,B,C} =
$\{(\gamma+\delta+\mu)(\beta+\mu\xi),\ -\beta\Lambda+\mu(\gamma+\delta+\mu)+(\gamma+\delta+\eta+\mu)(\beta+\mu\xi)\omega,\ (-\beta\Lambda+\mu(\gamma+\delta+\eta+\mu))\omega\}$

dis

H has coords

Out[79]= $\left\{\frac{\beta\Lambda}{\mu}-v_2,\ \frac{\mu(\beta\Lambda-\mu(\gamma+\delta+\mu))}{\beta\Lambda(\beta+\mu\xi)}\right\}$

The discriminant △ = B^2 - 4 A C at H is

Out[81]= 0

simplified restr

Out[93]= $\Big\{6.70313,\ \omega^2(v_2-V_2)\Big(\omega^3 v_1^4(\mu v_2-(\mu+v_2)V_2)\left(-\beta\Lambda\mu-\beta\Lambda V_2+\mu V_2^2\right)+\beta v_2\left(\beta^3\Lambda^3(\mu+\beta\omega)+\beta\Lambda\mu\left(-\beta\Lambda\mu+\mu^3-\beta^2\Lambda\omega\right)V_2+\mu^3 v_2^2\left(\beta\Lambda+\mu^2-\mu V_2\right)-\mu v_2\left(\beta\Lambda\left(2\beta\Lambda\mu+\mu^3+2\beta^2\Lambda\omega\right)+\mu V_2\left(-2\beta\Lambda\mu+\mu^3-3\beta^2\Lambda\omega+\beta\mu\omega V_2\right)\right)\right)+v_1\Big(-\mu^2(\mu+2\beta\omega)v_2^3\left(\beta\Lambda+\mu^2-\mu V_2\right)+\beta\Lambda\mu\left(-\beta^2\Lambda^2(\mu+\beta\omega)+\mu\left(\beta\Lambda\mu-\mu^3+\beta^2\Lambda\omega\right)V_2\right)+\beta v_2\left(\Lambda\left(\beta\Lambda\mu^3+\mu^5-2\beta^3\Lambda^2\omega+\beta^2\Lambda\mu(-\Lambda+\mu\omega)\right)+\Lambda\left(-2\mu^4-2\beta^3\Lambda\omega^2+\beta\mu^2(\Lambda-5\mu\omega)\right)V_2+\mu\omega\left(2\beta\Lambda\mu+\mu^3+2\beta^2\Lambda\omega\right)V_2^2\right)+v_2^2\left(\beta\Lambda\left(\mu^4+8\beta^2\Lambda\mu\omega+4\beta^3\Lambda\omega^2+2\beta\mu^2(\Lambda+2\mu\omega)\right)+\mu V_2\left(\mu^4-10\beta^2\Lambda\mu\omega-6\beta^3\Lambda\omega^2+\beta\mu^2(-2\Lambda+\mu\omega)+2\beta\mu\omega(\mu+\beta\omega)V_2\right)\right)\Big)+\omega v_1^2\Big(\beta^3\Lambda^3\mu+\mu^2 v_2^3\left(\beta\Lambda+\mu^2-\mu V_2\right)+\beta\Lambda\mu V_2\left(\beta\Lambda\mu+3\mu^3+2\beta^2\Lambda\omega-2\mu(\mu+\beta\omega)V_2\right)+v_2^2\left(-\beta\Lambda\left(5\mu^3+8\beta^2\Lambda\omega+6\beta\mu(\Lambda+\mu\omega)\right)+\mu V_2\left(8\beta\Lambda\mu+\mu^3+3\beta\left(4\beta\Lambda+\mu^2\right)\omega-2\mu(\mu+2\beta\omega)V_2\right)\right)+v_2\left(\beta\Lambda\left(-5\beta\Lambda\mu^2-3\mu^4+\beta^2\Lambda(\Lambda-4\mu\omega)\right)+V_2\left(\beta\Lambda\left(10\mu^3+4\beta^2\Lambda\omega+\beta\mu(\Lambda+10\mu\omega)\right)+V_2\left(-2\mu^2\left(\beta\Lambda+\mu^2\right)-3\beta\mu\left(\beta\Lambda+\mu^2\right)\omega+\beta^3\Lambda\omega^2-\beta\mu\omega(\mu+\beta\omega)V_2\right)\right)\right)\Big)+\omega^2 v_1^3\Big(2v_2^2(2\beta\Lambda-\mu V_2)\left(\beta\Lambda+\mu^2-\mu V_2\right)+\mu V_2\left(-\beta\Lambda\left(2\beta\Lambda+3\mu^2\right)+\beta\Lambda(\mu-\beta\omega)V_2+\mu(\mu+\beta\omega)V_2^2\right)+v_2\left(\beta\Lambda\mu\left(4\beta\Lambda+3\mu^2\right)+V_2\left(-2\beta\Lambda\left(\beta\Lambda+4\mu^2\right)+V_2\left(\beta\Lambda\mu+\mu^3-\beta\left(2\beta\Lambda+\mu^2\right)\omega+\mu(\mu+2\beta\omega)V_2\right)\right)\right)\Big)\Big)\Big\}$

Out[94]= 3

Out[95]= $\omega^3 v_1^4(\mu v_2-(\mu+v_2)V_2)\left(-\beta\Lambda\mu-\beta\Lambda V_2+\mu V_2^2\right)+\beta v_2\left(\beta^3\Lambda^3(\mu+\beta\omega)+\beta\Lambda\mu\left(-\beta\Lambda\mu+\mu^3-\beta^2\Lambda\omega\right)V_2+\mu^3 v_2^2\left(\beta\Lambda+\mu^2-\mu V_2\right)-\mu v_2\left(\beta\Lambda\left(2\beta\Lambda\mu+\mu^3+2\beta^2\Lambda\omega\right)+\mu V_2\left(-2\beta\Lambda\mu+\mu^3-3\beta^2\Lambda\omega+\beta\mu\omega V_2\right)\right)\right)+v_1\Big(-\mu^2(\mu+2\beta\omega)v_2^3\left(\beta\Lambda+\mu^2-\mu V_2\right)+\beta\Lambda\mu\left(-\beta^2\Lambda^2(\mu+\beta\omega)+\mu\left(\beta\Lambda\mu-\mu^3+\beta^2\Lambda\omega\right)V_2\right)+\beta v_2\left(\Lambda\left(\beta\Lambda\mu^3+\mu^5-2\beta^3\Lambda^2\omega+\beta^2\Lambda\mu(-\Lambda+\mu\omega)\right)+\Lambda\left(-2\mu^4-2\beta^3\Lambda\omega^2+\beta\mu^2(\Lambda-5\mu\omega)\right)V_2+\mu\omega\left(2\beta\Lambda\mu+\mu^3+2\beta^2\Lambda\omega\right)V_2^2\right)+v_2^2\left(\beta\Lambda\left(\mu^4+8\beta^2\Lambda\mu\omega+4\beta^3\Lambda\omega^2+2\beta\mu^2(\Lambda+2\mu\omega)\right)+\mu V_2\left(\mu^4-10\beta^2\Lambda\mu\omega-6\beta^3\Lambda\omega^2+\beta\mu^2(-2\Lambda+\mu\omega)+2\beta\mu\omega(\mu+\beta\omega)V_2\right)\right)\Big)$

HP

Out[100]= $\{\{\omega \to 7.94466,\ \alpha \to 7.09723,\ \eta \to 0.893333\}\}$

In[101]:=
```
eq=Flatten[Join[{dis==0&&trE2==0},cp]]//.Join[{η→α/ω},Param];
Print["BTP Symb is", BTP=Solve[eq,{ω,α},Reals][[1]]//FullSimplify," =",BTP//N]

Print["This is BogdanovTP"]
eq=Join[BTP,Param,{η→α/ω}]
cS=NSolve[(dyn2//.eq)==0,X,WorkingPrecision→60];
Chop[N[det//.Join[eq,cS[[2]]],60]]
Print["w of BT"]
wBT=ω/.BTP
Print["Two B pts?"]
BP=Solve[Join[{η==η0&&trE2==0&&dis>0},cp,{η==α/ω}]//.Param,{ω,α,η},
Reals]//FullSimplify;
BP//N
(*Save["def.m",BTP];Save["def.m",BP]*)
```

BTP Symb is $\left\{\omega \to \boxed{6.84\ldots},\ \alpha \to \frac{100\times\left(4960+\boxed{-3.88\ldots\times 10^{3}}\right)}{17\,457}\right\}$ $=\{\omega \to 6.84183,\ \alpha \to 6.20319\}$

This is BogdanovTP

Out[104]= $\left\{\omega \to \boxed{6.84\ldots},\ \alpha \to \frac{100\times\left(4960+\boxed{-3.88\ldots\times 10^{3}}\right)}{17\,457},\ \Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma+\delta+\mu,\ v_1 \to \beta+\mu\,\xi,\ V_2 \to \gamma+\delta+\eta+\mu,\ \eta \to \frac{\alpha}{\omega}\right\}$

Out[106]= 0

w of BT

Out[108]= $\boxed{6.84\ldots}$

Two B pts?

Out[111]= $\{\{\omega \to 5.15735,\ \alpha \to 4.60724,\ \eta \to 0.893333\},\ \{\omega \to 7.35966,\ \alpha \to 6.57463,\ \eta \to 0.893333\}\}$

In[112]:=
```
Print["B points "]
resn=rest/.η→η0//.cv//FullSimplify
ress=CoefficientList[resn⟦4⟧,ω];
Length[ress]
cB=Solve[(resn//.Param)==0,ω](*resn is defined in def.m*)
nu=N[ω/.cB,15]
Print["These are BTP"]
{detG//.Join[cB⟦3⟧,Param,{η→η0}]//N,
Chop[detG//.Join[cB⟦4⟧,Param,{η→η0}]//N],
Chop[detG//.Join[cB⟦5⟧,Param,{η→η0}]//N]}
```

B points

Out[113]=
$$\frac{1}{\mu^3}\omega^2\,(-\beta\,\Lambda+\mu\,v_2)\,\left(-\beta^2\,\Lambda^2\,\mu\,v_1\,(\mu-\omega\,v_1)^3+\beta\,\Lambda\,\left(\beta^2\,\Lambda+\left(-\beta\,\Lambda+\mu^2\right)\,v_1\right)\,(\mu-\omega\,v_1)^3\,v_2+\beta\,\Lambda\,\mu^2\,(\mu-\omega\,v_1)\,(-2\,\beta\,\mu+v_1\,(2\,\mu+3\,\beta\,\omega-2\,\omega\,v_1))\,v_2^2+\mu^4\,\left(\beta\,\mu-(\mu+2\,\beta\,\omega)\,v_1+\omega\,v_1^2\right)\,v_2^3\right)$$

Out[115]= 4

Out[116]= {{ω → 0}, {ω → 0}, {ω → 5.16…}, {ω → 7.36…}, {ω → 42.5…}}

Out[117]= {0, 0, 5.15735404465674, 7.35965704025757, 42.4504362201850}

These are BTP

Out[119]= {0., 0, 0}

In[120]:=
```
(*Numeric approach, using Param*)
test[RI]=Join[FindInstance[Join[{dis>0,R0<1,(trE2)>0, Bb<0,ω>2},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
test[II]=Join[FindInstance[Join[{dis>0,R0>1,(trE2)>0},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
test[III]=Join[FindInstance[Join[{dis>0,R0>1,(trE2)<0},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
test[IV]=Join[FindInstance[Join[{dis>0&&R0<1&&Bb>0,ω<11},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
test[V]=Join[FindInstance[Join[{dis<0,R0<1},cp,{η==α/ω}]//.Param,{ω,α,η}]⟦1⟧
,Drop[Param,-3]];
test[VI]=Join[FindInstance[Join[{dis>0,R0<1,(trE2)<0},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
test[VIa]=Join[FindInstance[Join[{dis>0,R0<1,(trE2)<0},cp,{η==α/ω}]//.Param,
{ω,α,η},3]⟦2⟧,Drop[Param,-3]];

testBoIIandIII=Join[FindInstance[Join[{dis>0&&trE2==0 &&Bb<0&&R0>1},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
testBoIandVI=Join[FindInstance[Join[{dis>0 &&Bb<0&&trE2==0&&R0<1,ω<wBT},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
testBoIandVII=Join[FindInstance[Join[{dis>0 &&Bb<0&&trE2==0&&R0<1,ω>wBT},cp,{η==α/ω}]//.Param
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
testBoIIandI=Join[FindInstance[Join[{dis>0&&trE2>0 &&Bb<0&&R0==1},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
testBoIIandIV=Join[FindInstance[Join[{dis>0 &&Bb>0&&R0==1},cp,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Drop[Param,-3]];
testBoIandVIa=Join[FindInstance[Join[{trE2==0&& dis>0 && R0<1},cp,{η==α/ω}]//.Param,
{ω,α,η},6]⟦6⟧,Drop[Param,-3]];
```

```
testHP=Join[HP[[1]],Drop[Param,-3]];
testBTP=Join[BTP,Drop[Param,-3]];
testBP=Join[BP[[1]],Drop[Param,-3]]

Print["between I and II at R0=1 downward B1"]
testR1=Join[Param,Thread[{ω,α}→{901/195, 60367/14625}]]
testR2=Join[Param,Thread[{ω,α}→(({ω,α}//.BP[[1]])+({ω,α}//.BP[[2]]))/2]]
testR3=Join[Param,Thread[{ω,α}→(({ω,α}//.HP[[1]])+({ω,α}//.BP[[2]]))/2]]
testR4=Join[Param,Thread[{ω,α}→{8139/1028, 181771/25700}]]



(*Now on Gupta parameters*)

testGca=Join[paramGc,Thread[{ω,α}→{10/99, 9/99}]];
testGcb=Join[paramGc,Thread[{ω,α}→{10000/103387, 9000/103387}]];
testGcc=Join[paramGc,Thread[{ω,α}→{10/108, 9/108}]];
testGcd=Join[paramGc,Thread[{ω,α}→{1/11, 9/110}]];
testGc4=Join[paramGc,Thread[{ω,α}→{1000000000/1604038240, 900000000/1604038240}]];
testG3a=Join[paramGc,Thread[{ω,α}→{100000000/1063265757, 90000000/1063265757}]];



R0TD={R0-1, trE2, dis, Bb};
Print["R0-1, Tr,Dis, B for region I is "]
R0TD//.Join[test[RI],ceta]//N
Print["R0-1, Tr,Dis, B for the boundary between II and III is "]
Chop[Evaluate[R0TD//.Join[testBoIIandIII,ceta]//N]]
Print["R0-1, Tr,Dis, B of Gupta (Fig 1a) is : "]
R0TD//.Join[testGca,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1b) is : "]
R0TD//.Join[testGcb,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1c) is : "]
R0TD//.Join[testGcc,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1d) is : "]
R0TD//.Join[testGcd,ceta]//N
Print["R0-1, Tr,Dis, B, of Gupta (Fig 3a) is : "]
R0TD//.Join[testG3a,ceta]//N

Print["at H, dis is"]
dis//.testHP//FullSimplify

Print["TrE2 at H when μ=1/12 is "]
trE2//.testHP//N
```

Out[135]= $\left\{\omega \to 5.16\ldots,\ \alpha \to \frac{67\ 7.14\ldots\times 10^{5}}{10\,379\,325},\ \eta \to \frac{67}{75},\ \Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25}\right\}$

between I and II at R0=1 downward B1

Out[137]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to \frac{901}{195},\ \alpha \to \frac{60367}{14625}\right\}$

Out[138]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to \frac{1}{2}\left(5.16\ldots + 7.36\ldots\right),\ \alpha \to \frac{1}{2}\left(\frac{67\ 7.14\ldots\times 10^{5}}{10379325} + \frac{67\ 1.02\ldots\times 10^{6}}{10379325}\right)\right\}$

Out[139]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to \frac{1}{2}\times\left(\frac{2010}{253} + 7.36\ldots\right),\ \alpha \to \frac{1}{2}\times\left(\frac{8978}{1265} + \frac{67\ 1.02\ldots\times 10^{6}}{10379325}\right)\right\}$

Out[140]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to \frac{8139}{1028},\ \alpha \to \frac{181771}{25700}\right\}$

R0-1, Tr,Dis, B for region I is

Out[149]= $\left\{-1. + \frac{1.33333}{V_2},\ 0.0139738,\ 0.000608759,\ -0.0624949\right\}$

R0-1, Tr,Dis, B for the boundary betωeen II and III is

Out[151]= $\left\{-1. + \frac{1.33333}{V_2},\ 0,\ 0.0016453,\ -0.0298199\right\}$

R0-1, Tr,Dis, B of Gupta (Fig 1a) is :

Out[153]= {-0.230769, 0.024036, 0.0000733967, -0.0328182}

R0-1, Tr,Dis, B of Gupta (Fig 1b) is :

Out[155]= {-0.230769, 0.0000111711, 0.000193019, -0.0339716}

R0-1, Tr,Dis, B of Gupta (Fig 1c) is :

Out[157]= {-0.230769, -0.0183133, 0.00031084, -0.0350833}

R0-1, Tr,Dis, B of Gupta (Fig 1d) is :

Out[159]= {-0.230769, -0.025093, 0.00035956, -0.0355364}

R0-1, Tr,Dis, B, of Gupta (Fig 3a) is :

Out[161]= {-0.230769, -0.0121655, 0.000268999, -0.0346912}

at H, dis is

Out[163]= 0

TrE2 at H when μ=1/12 is

Out[165]= -0.12

In[ ]:=
```
(*checks*)
Print["Tr at region I "]
trE2 /. η → α / ω //. test[RI] // N
Print["Tr at region II "]
trE2 /. η → α / ω //. test[II] // N
Print["Tr at region III "]
trE2 /. η → α / ω //. test[III] // N
Print["Tr at region IV "]
trE2 /. η → α / ω //. test[IV] // N
Print["Tr at region V "]
trE2 /. η → α / ω //. test[V] // N
Print["Tr at region VI "]
trE2 /. η → α / ω //. test[VI] // N
Print["Tr at HP "]
trE2 /. η → α / ω //. testHP // N
trE2 /. η → α / ω //. testR1 // N
R0 /. cV2 /. η → α / ω //. testR1 // N
trE2 /. η → α / ω //. test[VIa] // N
```

```
Tr at region I
```
Out[ ]= 0.0139738

```
Tr at region II
```
Out[ ]= 0.0177594

```
Tr at region III
```
Out[ ]= -0.357115

```
Tr at region IV
```
Out[ ]= -1.42512

```
Tr at region V
```
Out[ ]= 0.328201 - 0.648173 i

```
Tr at region VI
```
Out[ ]= -0.35369

```
Tr at HP
```
Out[ ]= -0.12

Out[ ]= -0.034338

Out[ ]= 1.

Out[ ]= -0.0335335

Bifurcation Map:

In[166]:=
```
(**Fig 6ns/3, Fig62/4 *)
cn=Param
xm=0;ym=0;xM=14;yM=14;
(*μ/(v1)//.cn//N*)
p1g=Graphics[{Thick,Orange,Dashed,Line[{{μ/(v1)//.cn,0},{μ/(v1)//.cn,45}}]}];
```

```
R0ωa=(R0//.cn/.η→α/ω);
disωa=(dis/.ceta//.cn);
Bbωa=(Bb/.ceta//.cn);
trωa=(trE2/.ceta//.cn);
trω2=((rest)/.ceta//.cn);

trω=((trG//.cv1)/.ceta//.cn);
ptr=ContourPlot[trωa==0,{ω,xm,xM},{α,ym,yM},ContourStyle→{Red},PlotPoints → 200,
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"}];
  ptr2=ContourPlot[trω2==0,{ω,xm,xM},{α,ym,yM},PlotPoints → 290,
 MaxRecursion → 2, WorkingPrecision → 35, ContourStyle→{Red},
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"[Tr[J(E2)]=0]⋃[Tr[J(E1)]=0]"}];


Print["dis at BTP is "]
Chop[Evaluate[disωa//.testBTP//N]]
Print["Dis at H ="]
dis//.testHP//N

pR0=ContourPlot[R0ωa==1,{ω,xm,xM},{α,ym,yM},ContourStyle→{Black,Dotted},
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"R0=1"}];

pD=ContourPlot[disωa==0,{ω,xm,xM},{α,ym,yM},
ContourStyle→{ Blue,Dashed}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},
PlotLegends→{"Δ=0"}];

pB=ContourPlot[Bbωa==0,{ω,xm,xM},{α,ym,yM},
ContourStyle→{Dashed,Cyan}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"B=0"}];

 epi= {Black,Style[Text["V)R0<1,Δ<0",{5.4,11}],13],Style[Text["0 EnP",{5,10.5}],13],
Style[Text["Tr[J(E2)]>0",{7.8,6}],6],Style[Text["II)R0>1",{8,6.5}],7],Style
[Text["1 EnP",{6.9,5.9}],6],
Style[Text["IV)R0<1,Δ>0",{13,13.6}],10],Style[Text["0EnP",{13,13.2}],12],
Style[Text["B>0",{13,12.8}],12],
Style[Text[" VI)R0<1,Δ>0",{1.2,2}],10],Style[Text["Bistablity",{1,1.5}],10],
Style[Text[" I)2 EnP",{4,4.2}],7],
Style[Text[" III)R0>1,Tr[J(E2)]<0, B<0",{9,3}],13],
Style[Text["1 stable EnP",{8,1.5}],13]};

PH=Text["H",Offset[{-5,10},{ω,η ω}//.testHP]];PHp={PointSize[Medium],Style[Point[{ω,η ω}
//.testHP],Yellow]};
PBT=Text["BT",Offset[{-3,6},{ω,α}//.testBTP//N]];PBTp={PointSize[Medium],Style[Point[
{ω,α}//.testBTP//N],Red]};
BP1=Text["B1",Offset[{10,-7},{ω,α}//.BP[[1]]]];BP1p={PointSize[Medium],Style[Point[
{ω,α}//.BP[[1]]],Green]};
BP2=Text["B2",Offset[{-5,10},{ω,α}//.BP[[2]]]];BP2p={PointSize[Medium],Style[Point[
{ω,α}//.BP[[2]]],Blue]};
P1=Text["R1",Offset[{10,-7},{ω,α}//.testR1]];P1p={PointSize[Medium],Style[Point[
{ω,α}//.testR1],Purple]};
(*P4=Text["R4",Offset[{10,-7},{ω,α}//.testR4]];P4p={PointSize[Medium],
Style[Point[{ω,α}//.testR4],Purple]};*)
P3=Text["R3",Offset[{-4,5},({ω,α}//.testR3)]];
P3p={PointSize[Medium],Style[Point[{ω,α}//.testR3],Purple]};
P2=Text["R2",Offset[{-4,5},{ω,α}//.testR2]];
P2p={PointSize[Medium],Style[Point[{ω,α}//.testR2],Purple]};
```

```
QI=Text["QI",Offset[{8,5},{ω,α}//.test[RI]]];QIp={PointSize[Medium],Style[Point[
{ω,α}//.test[RI]],Magenta]};
QII=Text["QII",Offset[{8,5},{ω,α}//.test[II]]];QIIp={PointSize[Medium],Style[Point[
{ω,α}//.test[II]],Magenta]};
QIII=Text["QIII",Offset[{8,5},{ω,α}//.test[III]]];QIIIp={PointSize[Medium],Style[Point[
{ω,α}//.test[III]],Magenta]};
QIV=Text["QIV",Offset[{8,5},{ω,α}//.test[IV]]];QIVp={PointSize[Medium],Style[Point[
{ω,α}//.test[IV]],Magenta]};
QV=Text["QV",Offset[{-5,10},{ω,α}//.test[V]]];QVp={PointSize[Medium],Style[Point[
{ω,α}//.test[V]],Magenta]};
QVI=Text["QVI",Offset[{8,5},{ω,α}//.test[VI]]];QVIp={PointSize[Medium],
Style[Point[{ω,α}//.test[VI]],Magenta]};
QVIa=Text["QVIa",Offset[{8,5},{ω,α}//.test[VIa]]];QVIap={PointSize[Medium],
Style[Point[{ω,α}//.test[VIa]],Magenta]};
T1=Text["T1",Offset[{10,-7},{ω,α}//.testBoIIandIII]];T1a={PointSize[Medium],
Style[Point[{ω,α}//.testBoIIandIII],Black]};
T2=Text["T2",Offset[{8,-7},{ω,α}//.testBoIandVIa]];T2a={PointSize[Medium],
Style[Point[{ω,α}//.testBoIandVIa],Black]};
regions={RI,II,III,IV,V,VI};
pt=Table[{ω,α}//.test[j],{j,regions}];
pG=Table[Text[P[j],Offset[{-5,10},pt[[j]]]],{j,6}];

epiP={PH,PHp,PBT,PBTp,BP1,BP1p,BP2,BP2p,P1,P1p,P3,P3p,P2,P2p,(*P4,P4p,*)QI,QIp,QII,
QIIp,QIII,QIIIp,QIV,QIVp,QV,QVp,QVI,QVIp,T1,T1a,T2,T2a}//N;
epiP1={PH,PHp,PBT,PBTp,BP1,BP1p,BP2,BP2p,P1,P1p,P3,P3p,P2,P2p,T1,T1a,T2,T2a}//N;
fig6F=Show[{pR0,ptr,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yellow
,Epilog→{epi,epiP},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]
fig62=Show[{pR0,ptr2,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yell
,Epilog→{epi,epiP1},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]

Export["fig6ns.pdf",fig6F]
Export["fig62.pdf",fig62]
```

Out[166]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ \mathrm{v}_2 \to \gamma + \delta + \mu,\ \mathrm{v}_1 \to \beta + \mu\,\xi,\ \mathrm{V}_2 \to \gamma + \delta + \eta + \mu\right\}$

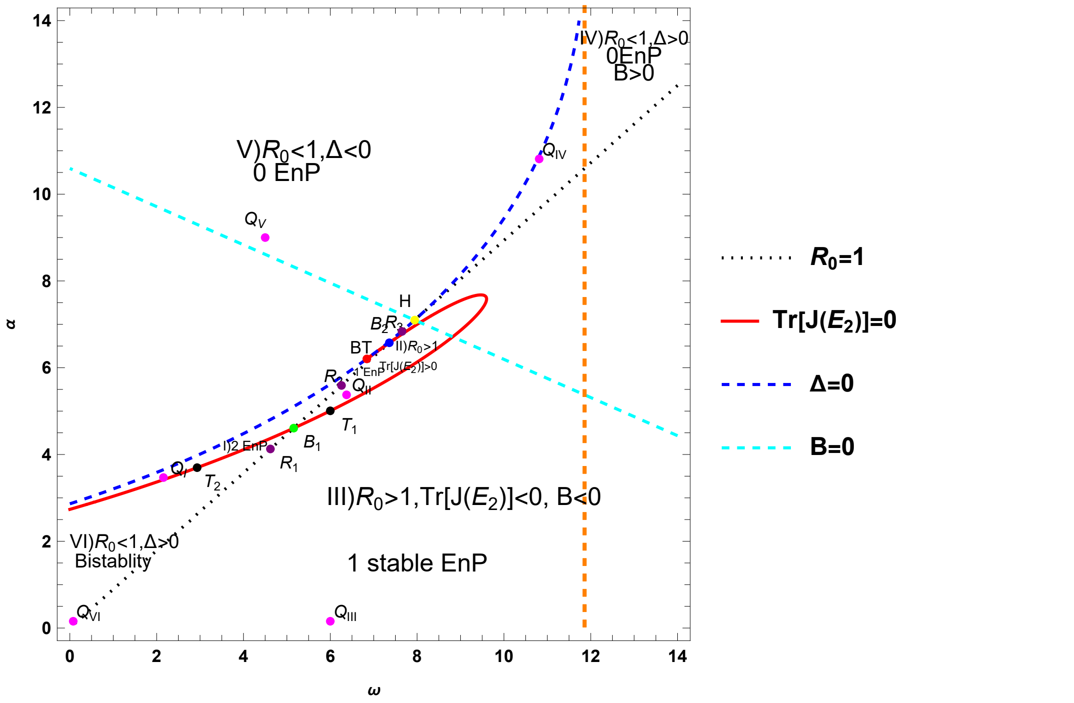
dis at BTP is

Out[178]= 0

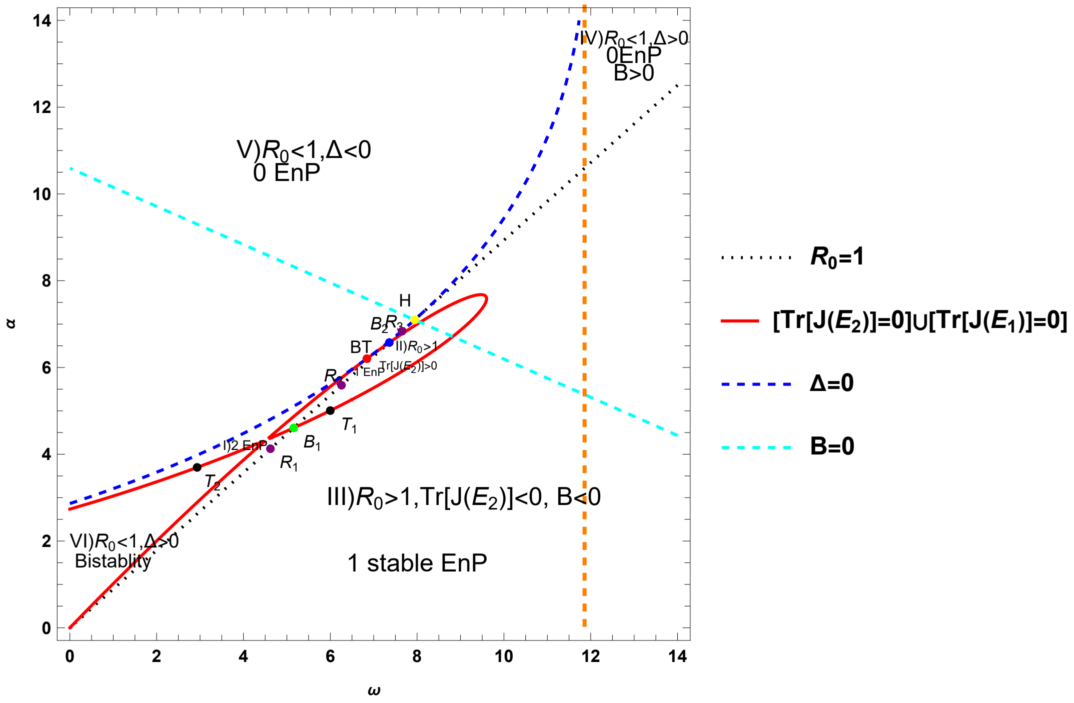
Dis at H =

Out[180]= 0.

Out[208]=



Out[209]=



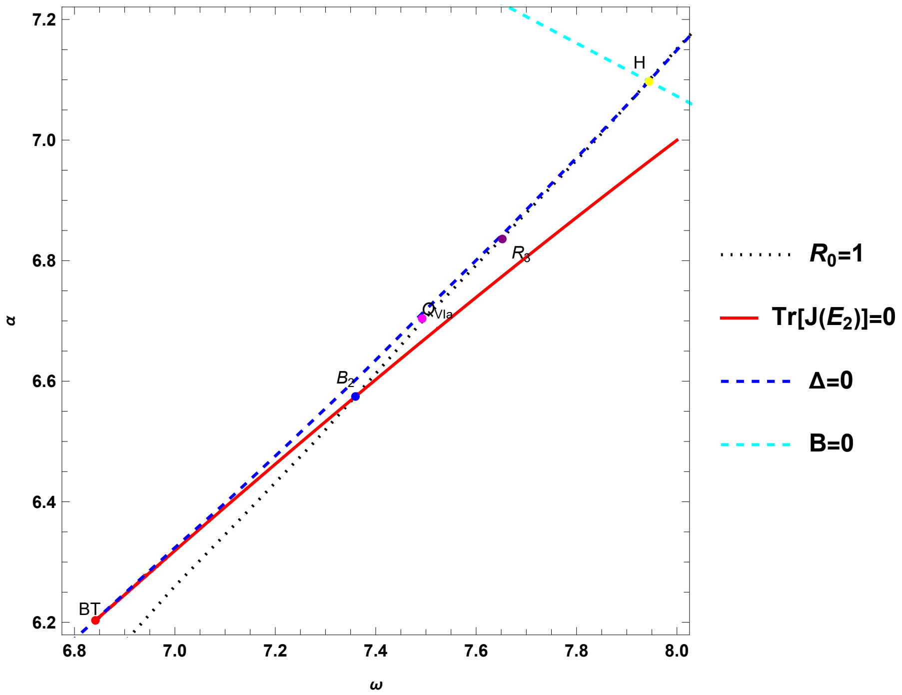
Out[210]= fig6ns.pdf

Out[211]= fig62.pdf

Blow-up of the Map:

In[220]:=
```
(*xm=6.8;ym=6.25;xM=7.8;yM=6.9;*)
xm=6.8;ym=6.2;xM=8;yM=7.2;
trωa=((trE2//.cv1)/.ceta//.cn);
ptr=ContourPlot[trωa==0,{ω,xm,xM},{α,ym,yM},ContourStyle→{Red},PlotPoints → 200,
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"}];
 P3=Text["R3",Offset[{10,-7},{ω,α}//.testR3]];
 P3p={PointSize[Medium],Style[Point[{ω,α}//.testR3],Purple]};
 epiP1={PH,PHp,PBT,PBTp,BP1,BP1p,BP2,BP2p,P1,P1p,P3,P3p,QVIa,QVIap}//N;
fig6F=Show[{pR0,ptr,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yellow
,Epilog→{epiP1},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]
Export["fig6BT.pdf",fig6F]
```

Out[226]=

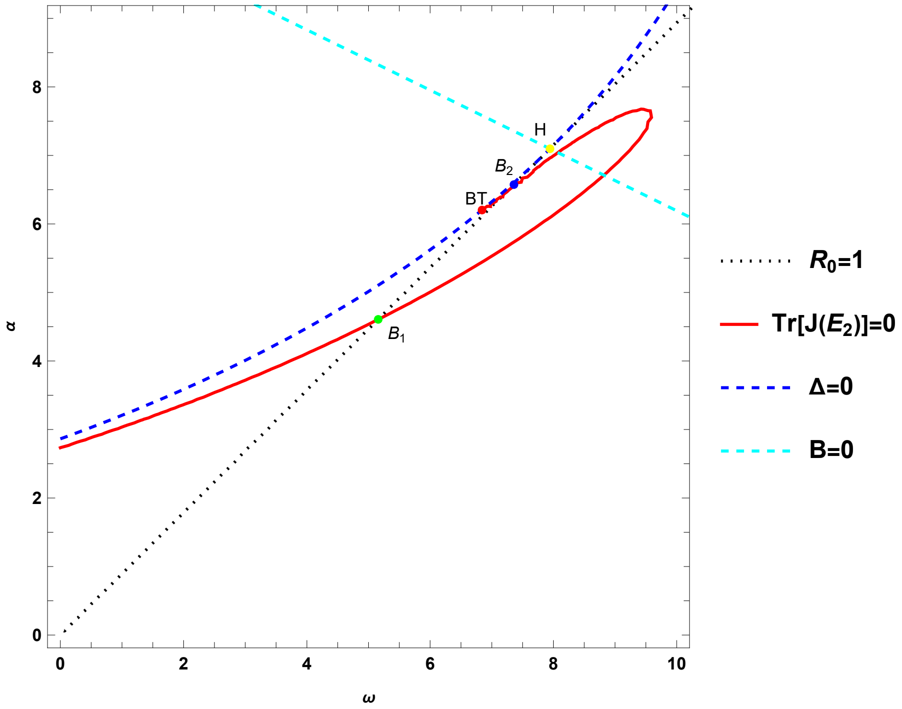


Out[227]= fig6BT.pdf

In[∘]:=
```
(*xm=6.8;ym=6.25;xM=7.8;yM=6.9;*)
xm=0;ym=0;xM=10;yM=9;
epiP={PH,PHp,PBT,PBTp,BP1,BP1p,BP2,BP2p}//N;
trωa=((trE2//.cv1)/.ceta//.cn);
ptr=ContourPlot[trωa==0,{ω,xm,xM},{α,ym,yM},ContourStyle→{Red},
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"}];

fig6F=Show[{pR0,ptr,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yellow
,Epilog→{epiP},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]
Export["fig6n.pdf",fig6F]
```

Out[∘]=



Out[∘]= fig6n.pdf

```
FindInstance[
   Join[{R0 < 1 && dis > 0 && Bb > 0 && trE2 > 0}, cp, {η == α / ω}] //. Param, {ω, α, η}] // N
Solve[Join[{η == η0 && Bb == 0 && trE2 > 0}, cp, {η == α / ω}] //. Param, {ω, α, η}]
```

Out[∘]= { }

In[∘]:=
```
(*At the boundary tr(E2)=0 between I and VIa**)
FindInstance[Join[{trE2==0&& dis>0 && R0ωa<1},Drop[cp,{7,9}],{η0>0}]/.ceta//.Param,{ω,α},6][[(
```

Out[∘]= $\left\{\omega \to \frac{251}{71},\ \alpha \to 3.93\ldots\right\}$

```
ωBTP=ω/.testBTP;
ωBP=ω/.testBP;
aBTP=α/.testBTP;
aBP=α/.testBP;
BR01=FindInstance[Join[{α==ω η0, ωBP<ω<ωBTP ,aBP <α<aBTP},Drop[cp,{7,9}],{η0>0}]//.Param,{ω,
Drop[BR01,30];
Print["Points betωeen BP and BTP "]
Drop[BR01,30]//N
Print["Points betωeen 0 and BP "]
BR01a=FindInstance[Join[{α==ω η0, 0<ω<ωBP ,0<α<aBP},Drop[cp,{7,9}],{η0>0}]//.Param,{ω,α},10]
BR01a
BR01a//N
Print["α and ω at BTP and boundary R0==1 are respectively "]
alBTP= ω η0//.testBTP//N
ω//.BTP//N
Print["α and ω at BP are respectively "]
alBP= ω η0//.testBP//N
ω//.BP//N

R0//.testBP//N

FindInstance[Join[{α==ω η0, (ω/.BR⟦2⟧)<ω<(ω/.HP) ,(α/.BR⟦2⟧) <α<(α/.HP)},Drop[cp,{7,9}],{η0
%//N
```

Points betωeen BP and BTP

Out[∘]= $\{\{\omega\to 6.18693, \alpha\to 5.52699\}, \{\omega\to 6.18963, \alpha\to 5.5294\}, \{\omega\to 6.20209, \alpha\to 5.54053\}, \{\omega\to 6.21421, \alpha\to 5.55136\}, \{\omega\to 6.23779, \alpha\to 5.57243\}, \{\omega\to 6.27989, \alpha\to 5.61004\}, \{\omega\to 6.28697, \alpha\to 5.61636\}, \{\omega\to 6.30145, \alpha\to 5.62929\}, \{\omega\to 6.30549, \alpha\to 5.6329\}, \{\omega\to 6.32031, \alpha\to 5.64614\}, \{\omega\to 6.48535, \alpha\to 5.79358\}, \{\omega\to 6.57494, \alpha\to 5.87361\}, \{\omega\to 6.58774, \alpha\to 5.88505\}, \{\omega\to 6.6029, \alpha\to 5.89859\}, \{\omega\to 6.61165, \alpha\to 5.90641\}, \{\omega\to 6.65005, \alpha\to 5.94071\}, \{\omega\to 6.69586, \alpha\to 5.98163\}, \{\omega\to 6.71708, \alpha\to 6.00059\}, \{\omega\to 6.76625, \alpha\to 6.04452\}, \{\omega\to 6.79993, \alpha\to 6.07461\}\}$

Points betωeen 0 and BP

Out[∘]= $\left\{\left\{\omega\to\frac{46}{195}, \alpha\to\frac{3082}{14625}\right\}, \left\{\omega\to\frac{16}{65}, \alpha\to\frac{1072}{4875}\right\}, \left\{\omega\to\frac{27}{65}, \alpha\to\frac{603}{1625}\right\}, \left\{\omega\to\frac{89}{39}, \alpha\to\frac{5963}{2925}\right\}, \left\{\omega\to\frac{178}{65}, \alpha\to\frac{11926}{4875}\right\}, \left\{\omega\to\frac{574}{195}, \alpha\to\frac{38458}{14625}\right\}, \left\{\omega\to\frac{709}{195}, \alpha\to\frac{47503}{14625}\right\}, \left\{\omega\to\frac{713}{195}, \alpha\to\frac{47771}{14625}\right\}, \left\{\omega\to\frac{901}{195}, \alpha\to\frac{60367}{14625}\right\}, \left\{\omega\to\frac{190}{39}, \alpha\to\frac{2546}{585}\right\}\right\}$

Out[∘]= $\{\{\omega\to 0.235897, \alpha\to 0.210735\}, \{\omega\to 0.246154, \alpha\to 0.219897\}, \{\omega\to 0.415385, \alpha\to 0.371077\}, \{\omega\to 2.28205, \alpha\to 2.03863\}, \{\omega\to 2.73846, \alpha\to 2.44636\}, \{\omega\to 2.94359, \alpha\to 2.62961\}, \{\omega\to 3.6359, \alpha\to 3.24807\}, \{\omega\to 3.65641, \alpha\to 3.26639\}, \{\omega\to 4.62051, \alpha\to 4.12766\}, \{\omega\to 4.87179, \alpha\to 4.35214\}\}$

α and ω at BTP and boundary R0==1 are respectively

Out[∘]= 6.11203

Out[∘]= 6.84183

α and ω at BP are respectively

Out[∘]= 4.60724

Out[•]= {5.15735, 7.35966}

Out[•]= 1.

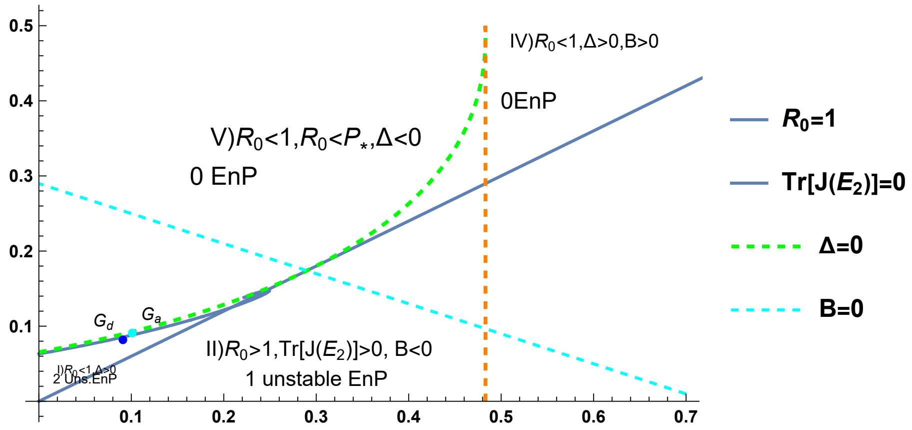
Out[•]= $\left\{\left\{\omega\to\frac{7611}{1028},\ \alpha\to\frac{169\,979}{25\,700}\right\},\ \left\{\omega\to\frac{7715}{1028},\ \alpha\to\frac{103\,381}{15\,420}\right\},\ \left\{\omega\to\frac{1943}{257},\ \alpha\to\frac{130\,181}{19\,275}\right\},\ \left\{\omega\to\frac{2027}{257},\ \alpha\to\frac{135\,809}{19\,275}\right\},\ \left\{\omega\to\frac{8139}{1028},\ \alpha\to\frac{181\,771}{25\,700}\right\},\ \left\{\omega\to\frac{8153}{1028},\ \alpha\to\frac{546\,251}{77\,100}\right\}\right\}$

Out[•]= $\{\{\omega\to 7.4037,\ \alpha\to 6.61397\},\ \{\omega\to 7.50486,\ \alpha\to 6.70435\},\ \{\omega\to 7.56031,\ \alpha\to 6.75388\},\ \{\omega\to 7.88716,\ \alpha\to 7.04586\},\ \{\omega\to 7.91732,\ \alpha\to 7.0728\},\ \{\omega\to 7.93093,\ \alpha\to 7.08497\}\}$

In[ ]:=
```
(**Fig Gupta*)
paramGG=Thread[{Λ,δ,γ,β,ξ,μ,v2,v1}→{1/2,2/10,1/10,2/10,7/100,1/10,(μ+γ+δ),(β+ μ ξ)}];
cn=paramGG;
xM=.7;La=.5;
p1g=Graphics[{Thick,Orange,Dashed,Line[{{μ/(v1)//.cn,0},{μ/(v1)//.cn,La}}]}];
R0ωa=(R0/.cV2//.cn/.η→α/ω);
aω=Solve[R0ωa==1,α][[1]]
pR0=Plot[α/.aω,{ω,0,20},PlotLegends→{"R0=1"},LabelStyle→{Black,Bold}];
 disωa=(dis/.ceta//.cn);
pD=ContourPlot[disωa==0,{ω,0,xM},{α,0,La},
ContourStyle→{Thick, Dashed,Green}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},
PlotLegends→{"Δ=0"}];
Bbωa=(Bb/.ceta//.cn);
pB=ContourPlot[Bbωa==0,{ω,0,xM},{α,0,La},
ContourStyle→{Dashed,Cyan}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"B=0"}];
trωa=((trE2//.cv1)/.ceta//.cn);
pTr=ContourPlot[trωa==0,{ω,0,xM},{α,0,La},ContourStyle→Brown,
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"},MaxRecursion→5];
epi= {Black,Style[Text["V)R0<1,R0<P*,Δ<0",{0.3,0.35}],13],Style[Text["0 EnP",{0.2,0.3}],13],
Style[Text["Tr[J(E2)]<0",{11.3,10.5}],12],Style[Text["III)R0>1,1 EnP",{13,12}],12],
Style[Text["IV)R0<1,Δ>0,B>0",{0.59,0.48}],10],Style[Text["0EnP",{0.53,0.4}],12],

Style[Text[" I)R0<1,Δ>0",{0.05,0.04}],6],Style[Text["2 Uns.EnP",{0.05,0.03}],7],
Style[Text[" VI)Bistability",{3,4.7}],6],
Style[Text[" II)R0>1,Tr[J(E2)]>0, B<0",{0.3,0.07}],11],
Style[Text["1 unstable EnP",{0.3,0.03}],11]};
Gca1=Text["Ga",Offset[{10,10},{ω,α}//.ParGca]];GcaS1={PointSize[Medium],Style[Point[{ω,α}//.
Gcd1=Text["Gd",Offset[{-10,10},{ω,α}//.ParGcd]];GcdS1={PointSize[Medium],Style[Point[{ω,α}//
epiG={Gca1,GcaS1,Gcd1,GcdS1};
mapG=Show[{pR0,pTr,pD,pB,p1g},Epilog→{epi,epiG},FrameLabel→{ω,"α"},
PlotRange→{{0,xM},{0,La}}]
```

Out[ ]= $\left\{\alpha \to \frac{3\,\omega}{5}\right\}$

Out[ ]=


Bifurcation diagram :

In[ ]:=

```
(**Bifurcation Diagram of Region VI*)
cn=Drop[test[VI],{4}]
xm=0;ym=0;xM=3/100;yM=2/10;

dis0=β/.Solve[dis==0,β];

Print["β*=",b0n= b0/.cV2/.cv2/.η→α/ω//.cn//N, " ,β1*=", bc1=dis0[[1]]/.η→α/ω//.cn//N, " ,β2*='

lin1=Line[{{ bc1,0},{ bc1,yM}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,yM}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
pE2a=Plot[{ie[[2]]/.η→α/ω}//.cn,{β,0,b0n},PlotStyle→{Dashed,Thick,Green},PlotRange→All,PlotLeg
pE2b=Plot[{ie[[2]]/.η→α/ω}//.cn,{β,b0n,xM},PlotStyle→{Thick,Purple},
PlotRange→All,PlotLegends→{"E2 stable"}];
pE1a=Plot[{ie[[1]]/.η→α/ω}//.cn,{β,0,b0n},PlotStyle→{Dashed,Thick,Brown},PlotRange→All,PlotLeg
pE1b=Plot[{ie[[1]]/.η→α/ω}//.cn,{β,b0n,xM},PlotStyle→{Thick,Orange},
PlotRange→All,PlotLegends→{"E1 stable"}];
pdfea=Plot[0,{β,0, b0n},PlotStyle→{Thick,Red},PlotRange→All,
PlotLegends→{"E0 stable"}];
pdfeb=Plot[0,{β,b0n, xM},PlotStyle→{Dashed,Thick,Blue},PlotRange→All,
PlotLegends→{"E0 unstable"}];
bifNO=Show[{pE2a, pE2b,pE1a, pE1b,pdfea,pdfeb,li1,li2},PlotRange→{{0,xM},{0,yM}},Epilog→
{{Text["β*",Offset[{-8,10},{ b0n,0}]],{PointSize[Large],
Style[Point[{ b0n,0}],Orange]},Text["β2*",Offset[{10,10},{ bc2,0}]],
{PointSize[Large],Style[Point[{ bc2,0}],Yellow]},Text["β1*",Offset[{10,10},{ bc1,0}]],
{PointSize[Large],Style[Point[{ bc1,0}],Black]}}
},AxesLabel→{"β","iee"}]
```

Out[ ]=

$$\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma+\delta+\mu,\ v_1 \to \beta+\mu\,\xi,\ V_2 \to \gamma+\delta+\eta+\mu,\ \omega \to \frac{5}{64},\ \alpha \to \frac{5}{32}\right\}$$

$\beta{*}$=0.0183 ,β_{1*}=0.00271739 ,β_{2*}=0.0040654

Out[ ]=