I)SIR with B(s,i)=β s /(1+μ i) and T(i)=η w i/(i+w) preliminaries

```
In[1]:= SetDirectory[NotebookDirectory[]];
AppendTo[$Path, Directory];
Clear["Global`*"];
Params = {Λ, δ, γ, β, ξ, ω, μ, α};
cet0 = {η → 0};

Lgn = (γ + Λ + δ); cir = {ir → γ - is};
sd = Λ / μ;
R1 = β / V2;
R0 = sd R1 (*basic reproduction number*);
η0 = sd β - v2 ;
ωH = μ (β Λ - μ (γ + δ + μ)) / (β Λ (β + μ ξ));
Print["R0  =", R0, " ,η0=", η0 , " ,ωH=", ωH,
 " and critical β is ", b0 = β /. Solve[R0 == 1, β][[1]] // FullSimplify]
(*Conditions de positivité*)
cp = {Λ > 0, δ > 0, γ > 0, β > 0, ξ > 0, μ / v1 > ω > 0, η > 0, μ > 0, α > 0, v1 > 0, v2 > 0};
<< "def.m";(*Contains icd,tre,tre11,tre12,tre2,trR,trE2*)
(*Print["Bautin"]
 Timing[trz=FullSimplify[Reduce[Join[{trR==0(*&&dis==0*)},cp]]]]*)
(*Numerical conditions used in first tests*)
Param = Thread[{Λ, δ, γ, β, ξ, μ, v2, v1, V2} →
    {16, 2 / 10, 12 / 100, 1 / 100, 1 / 1000, 12 / 100, μ + γ + δ, β + μ ξ, μ + γ + δ + η}];
ParamF = Thread[{Λ, δ, γ, β, ξ, μ, v2, v1, V2} →
    {16, 2 / 10, 12 / 100, 1 / 100, 1 / 1000, 1 / 10, μ + γ + δ, β + μ ξ, μ + γ + δ + η}];
paramGc = Thread[{Λ, δ, γ, β, ξ, μ, v2, v1, V2} →
    {1 / 2, 2 / 10, 1 / 10, 2 / 10, 7 / 100, 1 / 10, (μ + γ + δ), (β + μ ξ), μ + γ + δ + η}];
 (*Parameters of Figure 4 in Zhou Fan*)
ParNumCheck = Thread[{ω, α, η} → {7/64, 179/64, α / ω}];
Print["test cNN="]
cNN = Join[ParNumCheck, Param]


(*conditions for switching between
 the parameters and particular cases*)
cv2 = {v2 → (μ + γ + δ)}; cv1 = {v1 → (β + μ ξ)}; cV2 = {V2 → (v2 + η)};
cv = {ξ → (v1 - β) / μ, δ → v2 - (μ + γ), η → V2 - v2};
ceta = {η → α / ω};
calv1 = {α → ω (v1 - β) / μ};
```

R0 $=\frac{\beta\Lambda}{\mu V_2}$,$\eta 0=\frac{\beta\Lambda}{\mu}-v_2$,$\omega H=\frac{\mu(\beta\Lambda-\mu(\gamma+\delta+\mu))}{\beta\Lambda(\beta+\mu\xi)}$ and critical β is $\frac{\mu V_2}{\Lambda}$

test cNN=

Out[16]= $\{\omega\to\frac{7}{64},\ \alpha\to\frac{179}{64},\ \eta\to\frac{\alpha}{\omega},\ \Lambda\to 16,\ \delta\to\frac{1}{5},\ \gamma\to\frac{3}{25},\ \beta\to\frac{1}{100},\ \xi\to\frac{1}{1000},\ \mu\to\frac{3}{25},\ v_2\to\gamma+\delta+\mu,\ v_1\to\beta+\mu\xi,\ V_2\to\gamma+\delta+\eta+\mu\}$

0) Model, eqF, sol, Jacobian, jacE0, equation for endemic i, Discriminant, ie ,se2,trE2,detE2

In[19]:=
```
(*SIR  epidemic model of Fan:*)
s1=Λ -   β s  i/(1+ξ i)-s μ ;(*λ s (1-s/K)*)
i1= β s   i/(1+ξ i)-v2 i -η ω i/(i+ω);
r1=η  ω i/(i+ω)-μ r + γ i ;

dyn={s1,i1,r1}//.cv2;(*field*)
vars={s,i,r};
equi=Solve[Thread[dyn==0],vars];

 Print["( s' i' r')= ",dyn//MatrixForm]

(*Diff. sys and Numerical test:*)
varst=Through[vars[t]];(*Map[#[t]&, vars];Revarst = Thread[vars→varst]*);
diff= D[varst,t] - (dyn/.Thread[vars→varst]);
diffN=diff//.cNN;
initcond = (varst/.t→0)-{1.5, 0.5, 0.1};
eqs=Thread[Flatten[{diffN, initcond}] == 0];
ndesoln = NDSolveValue[eqs,varst,{t, 0, 1000}];
Print["simulation test"]
ndesoln/.t→1000

(*Two dimensional Fan:*)
dyn2={s1,i1}/.cv2;(*we may reduce to this dyntem since these two equations
 do not depend on r*)
 Print["For 2-dim case, we have ( s' i')=",dyn2//MatrixForm]
 eqF=Thread[Flatten[dyn2==0]];
 equi2=Solve[eqF,{s,i}];
 Print[" numeric equil are"]
equi2//.cNN//N
 dyn2E={s1,Simplify[i1/i]}/.cv2;

(*Computation of the Jacobian  *)

jac=Grad[dyn2,{s,i}]//FullSimplify;
det=Det[jac]//FullSimplify;
Print["2 dim jac=",jac//MatrixForm]
jacE0=(jac/.i→0/.s→sd)//FullSimplify;
```

```
Print["jac(DFE)=", jacE0//MatrixForm]
tr=Tr[jac]//FullSimplify;
Print["trF= ",tr]
detE0=Det[jacE0]//FullSimplify;trE0=Tr[jacE0]//FullSimplify;

Print["Det(J(E0)=",detE0, ", and Tr(J(E0))=", Apart[trE0]]

(*Elimination of s via plugging, for trace and det*)
Print["simple det formula: detF=",det]
Print["s formula cs is"]
cs=Flatten[Solve[dyn2⟦1⟧==0,s]]


(*Equation for endemic i *)
poli=Collect[Factor[Numerator[Together[dyn⟦2⟧/.cs]]]/(-i),i];
Print["sec. order A i^2 + B i + C=0 for endemic i is"]
polv= Collect[ poli/.cv//FullSimplify,i]
(*Mathematica Eliminate*)
el=Eliminate[eqF,{s}]//FullSimplify;
poliF=Collect[Numerator[Factor[el⟦1,1⟧-el⟦1,2⟧]/(-i)]//FullSimplify,i];
Print["check  poli/poliF="]
poli/poliF//FullSimplify

Print["coeffs are"]
cf=CoefficientList[poli,i];
Aa=cf⟦3⟧;Bb=cf⟦2⟧; Cc=cf⟦1⟧;
Print["{A,B,C} =",{Aa,Bb,Cc}//FullSimplify//.cv2]

ie=i/.Solve[poli==0,i];

ic0=(-Bb/(2 Aa));
se1=Λ ( 1+ξ ie⟦1⟧)/(μ+ ie⟦1⟧ v1)(*endemic s*);
se2=Λ ( 1+ξ ie⟦2⟧)/(μ+ ie⟦2⟧ v1);
se0=Λ ( 1+ξ ic0)/(μ+ ic0 v1);

detE2=det/.s→se2/.i→ie⟦2⟧;
detE1=det/.s→se1/.i→ie⟦1⟧;
detE=det/.s→se0 /.i→ic0(* det ωhen $i=-B/(2A)**);

trE2=tr/.i→ie⟦2⟧/.s→se2;
trE1=tr/.i→ie⟦1⟧/.s→se1;

(*Print["Det i"]
deti=Simplify[(detF/.cs)( 1+ξ i)(( ω + i)^2)( μ+(μ ξ+β) i)]/.cv//FullSimplify
Print["PolynomialRemainder detR"]
detR=PolynomialRemainder[deti,poli,i];
Timing[icd=i/.Solve[detR==0,i]⟦1⟧//FullSimplify]*)
Print["i1,  icd, i2"]
Chop[{ie⟦1⟧,icd,ie⟦2⟧}//.cNN//N]

dis=Discriminant[poli,i]//FullSimplify;
Save["def.m",dis]
(*ωH=ω/.Flatten[Solve[(Bb/.η→η0)==0,ω]/.cv2//FullSimplify]*)
Print[" H  has coords "]
{η0,ωH}
```

```
Print["  and trH="]
Timing[trH=trE2/.{η→η0,ω→ωH}//.cv2//FullSimplify]
Print["The discriminant Δ = B^2 - 4 A C= ",dis," at H it is"]
(dis/.{η→η0,ω→ωH})//.cv2//FullSimplify

p1=Plot[{detE1}//.Drop[cNN,{1}]//N,{ω,0,ωH}//.Drop[cNN,{1}]//N,PlotStyle→{Green}];
p0=Plot[{detE}//.Drop[cNN,{1}]//N,{ω,0,ωH}//.Drop[cNN,{1}]//N,PlotStyle→{Blue}];
(*p0n=Plot[{detEn}//.Drop[cNN,{1}]//N,{ω,0,ωH}//.Drop[cNN,{1}]//N,PlotStyle→{Yellow}];*)
p2=Plot[{detE2}//.Drop[cNN,{1}]//N,{ω,0,ωH}//.Drop[cNN,{1}]//N,PlotStyle→{Red}];
cs2=Flatten[Solve[(dyn2[[2]]/i)==0,s]];
se02=s/.cs2;
detEn=det/.s→se02 /.i→ic0;
p102=Show[p1,p0,p2,PlotRange→All,AxesOrigin→{0,0}]
(*Export["p102.pdf",p102]*)
```

$$\begin{pmatrix} s' \\ i' \\ r' \end{pmatrix} = \begin{pmatrix} \Lambda - s\mu - \frac{i s \beta}{1+i\xi} \\ -i(\gamma+\delta+\mu) + \frac{i s \beta}{1+i\xi} - \frac{i\eta\omega}{i+\omega} \\ i\gamma - r\mu + \frac{i\eta\omega}{i+\omega} \end{pmatrix}$$

simulation test

Out[33]= $\{133.333, 4.9801\times10^{-27}, -8.73471\times10^{-13}\}$

For 2-dim case, ωe have $\begin{pmatrix} s' \\ i' \end{pmatrix} = \begin{pmatrix} \Lambda - s\mu - \frac{i s \beta}{1+i\xi} \\ -i(\gamma+\delta+\mu) + \frac{i s \beta}{1+i\xi} - \frac{i\eta\omega}{i+\omega} \end{pmatrix}$

numeric equil are

Out[39]= $\{\{s\to133.333, i\to0.\}, \{s\to86.1361, i\to6.61878\}, \{s\to69.9589, i\to10.99\}\}$

2 dim jac= $\begin{pmatrix} -\mu - \frac{i\beta}{1+i\xi} & -\frac{s\beta}{(1+i\xi)^2} \\ \frac{i\beta}{1+i\xi} & -\gamma-\delta-\mu+\frac{s\beta}{(1+i\xi)^2} - \frac{\eta\omega^2}{(i+\omega)^2} \end{pmatrix}$

jac(DFE) = $\begin{pmatrix} -\mu & -\frac{\beta\Lambda}{\mu} \\ 0 & \frac{\beta\Lambda-\mu(\gamma+\delta+\eta+\mu)}{\mu} \end{pmatrix}$

trF= $-\gamma-\delta-2\mu-\frac{\beta\left(i-s+i^2\xi\right)}{(1+i\xi)^2}-\frac{\eta\omega^2}{(i+\omega)^2}$

Det(J(E0)) = $-\beta\Lambda+\mu(\gamma+\delta+\eta+\mu)$, and Tr(J(E0)) = $-\gamma-\delta-\eta+\frac{\beta\Lambda}{\mu}-2\mu$

simple det formula: detF=

$$\mu^2-\frac{s\beta\mu}{(1+i\xi)^2}+\frac{i\beta\mu}{1+i\xi}+\gamma\left(\mu+\frac{i\beta}{1+i\xi}\right)+\delta\left(\mu+\frac{i\beta}{1+i\xi}\right)+\frac{\eta\mu\omega^2}{(i+\omega)^2}+\frac{i\beta\eta\omega^2}{(1+i\xi)(i+\omega)^2}$$

s formula cs is

Out[52]= $\left\{s\to\frac{\Lambda(1+i\xi)}{i\beta+\mu+i\mu\xi}\right\}$

sec. order A i^2 + B i + C=0 for endemic i is

Out[55]= $-\beta\Lambda\omega+i^2 v_1 v_2+\mu\omega V_2+i\left(-\beta\Lambda+\mu v_2+\omega v_1 V_2\right)$

check poli/poliF=

Out[59]= 1

coeffs are

{A,B,C} =
$\{(\gamma+\delta+\mu)\ (\beta+\mu\,\xi),\ -\beta\,\Lambda+\mu\ (\gamma+\delta+\mu)+(\gamma+\delta+\eta+\mu)\ (\beta+\mu\,\xi)\ \omega,\ (-\beta\,\Lambda+\mu\ (\gamma+\delta+\eta+\mu))\ \omega\}$

i1, icd, i2

Out[75]= {6.61878, 7.61514, 10.99}

H has coords

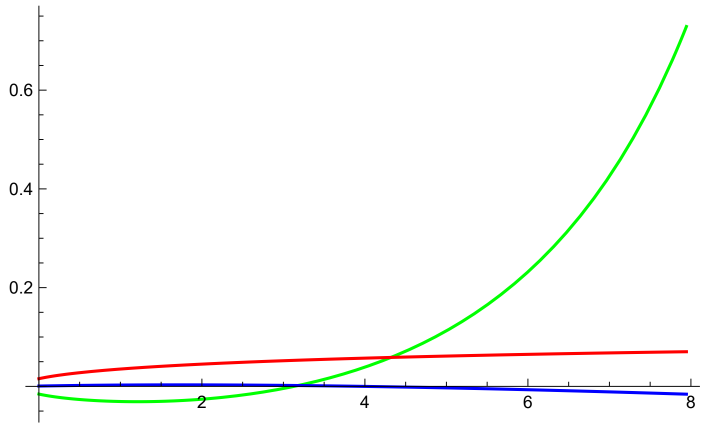
Out[79]= $\left\{\frac{\beta\,\Lambda}{\mu}-v_2,\ \frac{\mu\ (\beta\,\Lambda-\mu\ (\gamma+\delta+\mu))}{\beta\,\Lambda\ (\beta+\mu\,\xi)}\right\}$

and trH=

Out[81]= $\{0.0625,\ -\mu\}$

The discriminant △ = B^2 - 4 A C=
$\mu^2\ (\gamma+\delta+\mu-(\gamma+\delta+\eta+\mu)\ \xi\,\omega)^2+\beta^2\left(\Lambda^2+2\,\Lambda\ (\gamma+\delta-\eta+\mu)\ \omega+(\gamma+\delta+\eta+\mu)^2\,\omega^2\right)+2\,\beta\,\mu\left(-\Lambda\ (\gamma+\delta+\mu)-(\gamma+\delta+\mu)\ (\gamma+\delta+\eta+\mu)\ \omega+\Lambda\ (\gamma+\delta-\eta+\mu)\ \xi\,\omega+(\gamma+\delta+\eta+\mu)^2\,\xi\,\omega^2\right)$ at H it is

Out[83]= 0

Out[90]=


In[91]:=
```
RI=FindInstance[Join[{dis>0,R0<1,(trE2)>0, Bb<0},cp,{η==α/ω}]//.Param,{ω,α,η}];
RVI=FindInstance[Join[{dis>0,R0<1,(trE2)<0},cp,{η==α/ω}]//.Param,{ω,α,η}];
RII=FindInstance[Join[{dis>0,R0>1,(trE2)>0},cp,{η==α/ω}]//.Param,{ω,α,η}];
RIII=FindInstance[Join[{dis>0,R0>1,(trE2)<0},cp,{η==α/ω}]//.Param,{ω,α,η}];
RIV=FindInstance[Join[{dis>0&&R0<1&&Bb>0},cp,{η==α/ω}]//.Param,{ω,α,η}];
RV=FindInstance[Join[{dis<0,R0<1},cp,{η==α/ω}]//.Param,{ω,α,η}];
BoIIandIII=FindInstance[Join[{dis>0&&trE2==0 &&Bb<0&&R0>1},cp,{η==α/ω}]//.Param,{ω,α,η}];
BoIIandI=FindInstance[Join[{dis>0&&trE2>0 &&Bb<0&&R0==1},cp,{η==α/ω}]//.Param,{ω,α,η}];
BoIIandIV=FindInstance[Join[{dis>0 &&Bb>0&&R0==1},cp,{η==α/ω}]//.Param,{ω,α,η}];
BoIandVI=FindInstance[Join[{dis>0 &&Bb<0&&trE2==0&&R0<1},cp,{η==α/ω}]//.Param,{ω,α,η}];

Print["HP"]
HP=NSolve[Join[{η==η0&&Bb==0},cp,{η==α/ω}]//.Param,{ω,α,η}]
Print["BTP"]
eq=Flatten[Join[{dis==0&&trE2==0},cp,{α==η ω}]]//.Param;
BTP=Solve[eq,{ω,α,η},Reals]
BTP//N

Print["BP"]
BP=Solve[Join[{η==η0&&trE2==0&&dis>0},cp,{η==α/ω}]//.Param,{ω,α,η},Reals]
```

```
BP//N



ParRI=Thread[{ω,α}→({ω,α}//.RI[[1]])];
ParRVI=Thread[{ω,α}→({ω,α}//.RVI[[1]])];
ParRII=Thread[{ω,α}→({ω,α}//.RII[[1]])];
ParRIII=Thread[{ω,α}→({ω,α}//.RIII[[1]])];
ParRIV=Thread[{ω,α}→({ω,α}//.RIV[[1]])];
ParRV=Thread[{ω,α}→({ω,α}//.RV[[1]])];
ParHP=Thread[{ω,η}→({ω,η}//.HP[[1]])];
ParBTP=Thread[{ω,α,η}→({ω,α,η}//.BTP[[1]])];
ParBP=Thread[{ω,α,η}→({ω,α,η}//.BP[[1]])];
ParBoIIandIII=Thread[{ω,α}→({ω,α}//.BoIIandIII[[1]])];
ParBoIIandI=Thread[{ω,α}→({ω,α}//.BoIIandI[[1]])];
ParBoIIandIV=Thread[{ω,α}→({ω,α}//.BoIIandIV[[1]])];
ParBoIandVI=Thread[{ω,α}→({ω,α}//.BoIandVI[[1]])];
ParGca=Thread[{ω,α}→{10/99, 9/99}];
ParGcb=Thread[{ω,α}→{10000/103387, 9000/103387}];
ParGcc=Thread[{ω,α}→{10/108, 9/108}];
ParGcd=Thread[{ω,α}→{1/11, 9/110}];
ParGc4=Thread[{ω,α}→{1000000000/1604038240, 900000000/1604038240}];
ParG3a=Thread[{ω,α}→{100000000/1063265757, 90000000/1063265757}];

testGca=Join[paramGc,ParGca];
testGcb=Join[paramGc,ParGcb];
testGcc=Join[paramGc,ParGcc];
testGcd=Join[paramGc,ParGcd];
testGc4=Join[paramGc,ParGc4];
Print["testG3a"]
testG3a=Join[paramGc,ParG3a]
Print["testI"]
test["I"]=Join[Param,ParRI]
test[VI]=Join[Param,ParRVI];
Print["testII"]
test[II]=Join[Param,ParRII]

Print["testIII"]
test[III]=Join[Param,ParRIII]

test[IV]=Join[Param,ParRIV];
test[V]=Join[Param,ParRV];

testBoIIandIII=Join[Param,ParBoIIandIII];
Print["testBoIIandIII"]
testBoIIandIII
testBoIIandIV=Join[Param,ParBoIIandIV];
testBoIIandI=Join[Param,ParBoIIandI];
Print["testBoIIandI"]
testBoIIandI//N
R0//.ceta//.testBoIIandI//N
trE2//.ceta//.testBoIIandI//N

testBoIandVI=Join[Param,ParBoIandVI];testBoIandVI//N;
R0TD={R0-1, trE2, dis, Bb};
```

```
Print["R0-1, Tr,Dis, B for region I is "]
R0TD//.Join[test["I"],ceta]//N
Print["R0-1, Tr,Dis, B for the boundary between II and III is "]
Chop[Evaluate[R0TD//.Join[testBoIIandIII,ceta]//N]]
Print["R0-1, Tr,Dis, B of Gupta (Fig 1a) is : "]
R0TD//.Join[testGca,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1b) is : "]
R0TD//.Join[testGcb,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1c) is : "]
R0TD//.Join[testGcc,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1d) is : "]
R0TD//.Join[testGcd,ceta]//N
Print["R0-1, Tr,Dis, B, of Gupta (Fig 3a) is : "]
R0TD//.Join[testG3a,ceta]//N

Print["at H, dis is"]
testHP=Join[Param,ParHP,{η→α/ω}];
testBTP=Join[Param,ParBTP];
testBP=Join[Param,ParBP];
dis//.testHP//FullSimplify

Print["TrE2 at H when μ=1/12 is "]
trE2//.testHP//N
```

HP

Out[102]= $\{\{\eta \to 0.893333,\ \omega \to 7.94466,\ \alpha \to 7.09723\}\}$

BTP

Out[105]= $\{\{\omega \to \boxed{6.84\ldots},\ \alpha \to \boxed{0.907\ldots}\ \boxed{6.84\ldots},\ \eta \to \boxed{0.907\ldots}\}\}$

Out[106]= $\{\{\omega \to 6.84183,\ \alpha \to 6.20319,\ \eta \to 0.906657\}\}$

BP

Out[108]= $\left\{\left\{\omega \to \boxed{5.16\ldots},\ \alpha \to \frac{67}{75}\ \boxed{5.16\ldots},\ \eta \to \frac{67}{75}\right\},\ \left\{\omega \to \boxed{7.36\ldots},\ \alpha \to \frac{67}{75}\ \boxed{7.36\ldots},\ \eta \to \frac{67}{75}\right\}\right\}$

Out[109]= $\{\{\omega \to 5.15735,\ \alpha \to 4.60724,\ \eta \to 0.893333\},\ \{\omega \to 7.35966,\ \alpha \to 6.57463,\ \eta \to 0.893333\}\}$

testG3a

Out[135]= $$\left\{\Lambda \to \frac{1}{2},\ \delta \to \frac{1}{5},\ \gamma \to \frac{1}{10},\ \beta \to \frac{1}{5},\ \xi \to \frac{7}{100},\ \mu \to \frac{1}{10},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to \frac{100\,000\,000}{1\,063\,265\,757},\ \alpha \to \frac{30\,000\,000}{354\,421\,919}\right\}$$

testI

Out[137]= $$\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to \frac{7}{64},\ \alpha \to \frac{179}{64}\right\}$$

testII

Out[140]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to \frac{51}{8},\ \alpha \to \frac{43}{8}\right\}$

testIII

Out[142]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to 6,\ \alpha \to \frac{5}{32}\right\}$

testBoIIandIII

Out[147]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to 6,\ \alpha \to \boxed{5.01\ldots}\right\}$

testBoIIandI

Out[151]= $\{\Lambda \to 16.,\ \delta \to 0.2,\ \gamma \to 0.12,\ \beta \to 0.01,\ \xi \to 0.001,\ \mu \to 0.12,\ v_2 \to \gamma + \delta + \mu,\ v_1 \to \beta + \mu\,\xi,\ V_2 \to \gamma + \delta + \eta + \mu,\ \omega \to 6.25,\ \alpha \to 5.58333\}$

Out[152]= $\frac{1.33333}{0.44 + \eta}$

Out[153]= 0.0453481

R0-1, Tr,Dis, B for region I is

Out[157]= {-0.94874, 0.0132728, 0.000378861, -0.0784086}

R0-1, Tr,Dis, B for the boundary betωeen II and III is

Out[159]= {0.046264, 0, 0.0016453, -0.0298199}

R0-1, Tr,Dis, B of Gupta (Fig 1a) is :

Out[161]= {-0.230769, 0.024036, 0.0000733967, -0.0328182}

R0-1, Tr,Dis, B of Gupta (Fig 1b) is :

Out[163]= {-0.230769, 0.0000111711, 0.000193019, -0.0339716}

R0-1, Tr,Dis, B of Gupta (Fig 1c) is :

Out[165]= {-0.230769, -0.0183133, 0.00031084, -0.0350833}

R0-1, Tr,Dis, B of Gupta (Fig 1d) is :

Out[167]= {-0.230769, -0.025093, 0.00035956, -0.0355364}

R0-1, Tr,Dis, B, of Gupta (Fig 3a) is :

Out[169]= {-0.230769, -0.0121655, 0.000268999, -0.0346912}

at H, dis is

Out[174]= 0.

TrE2 at H when μ=1/12 is

Out[176]= -0.12

Bifurcation Map:

In[177]:=

```
(**Fig 6n of ZF*)
cn=Param
xm=0;ym=0;xM=14;yM=14;
μ/(v1)//.cn//N
p1g=Graphics[{Thick,Orange,Dashed,Line[{{μ/(v1)//.cn,0},{μ/(v1)//.cn,45}}]}];
R0ωa=(R0//.cn/.η→α/ω);
disωa=(dis/.ceta//.cn)//FullSimplify;
Bbωa=(Bb/.ceta//.cn);

Solve[disωa==0,ω]//FullSimplify;
Solve[Bbωa==0,ω]//FullSimplify;

Print["R0-1 at BTP is "]
-1+R0ωa//.testBTP//N
Print["dis at BTP is "]
Chop[Evaluate[disωa//.testBTP//N]]
Print["Dis at H ="]
dis//.testHP//N

pR0=ContourPlot[R0ωa==1,{ω,xm,xM},{α,ym,yM},ContourStyle→Black,
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"R0=1"}];

pD=ContourPlot[disωa==0,{ω,xm,xM},{α,ym,yM},
ContourStyle→{ Blue}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},
PlotLegends→{"Δ=0"}];

pB=ContourPlot[Bbωa==0,{ω,xm,xM},{α,ym,yM},
ContourStyle→{Dashed,Cyan}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"B=0"}];
trωa=((trE2//.cv1)/.ceta//.cn);

pmu=ContourPlot[trωa==0,{ω,xm,xM},{α,ym,yM},ContourStyle→{Red},
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"}];
 epi= {Black,Style[Text["V)R0<1,R0<P*,Δ<0",{5.4,11}],13],Style[Text["0 EnP",{5,9}],13],
Style[Text["Tr[J(E2)]<0",{7.8,6}],6],Style[Text["III)R0>1",{8,6.5}],6],Style[Text["1 EnP",{6.
Style[Text["IV)R0<1,Δ>0",{13,13}],10],Style[Text["0EnP",{12,12}],12],
Style[Text["B>0",{11.6,11}],12],
Style[Text[" I)R0<1,Δ>0",{1.2,2}],10],Style[Text["2 Uns.EnP",{1,1.5}],10],
Style[Text[" VI)Bistability",{4,4.2}],6],
Style[Text[" II)R0>1,Tr[J(E2)]>0, B<0",{9,3}],13],
Style[Text["1 unstable EnP",{8,1.5}],13]};

PH=Text["H",Offset[{-5,10},{ω,η ω}//.ParHP]];PHp={PointSize[Medium],Style[Point[{ω,η ω}//.Pa
PBT=Text["BT",Offset[{-5,10},{ω,α}//.ParBTP//N]];PBTp={PointSize[Medium],Style[Point[{ω,α}//
BP1=Text["B1",Offset[{10,-7},{ω,α}//.BP⟦1⟧]];BP1p={PointSize[Medium],Style[Point[{ω,α}//.BP⟦
BP2=Text["B2",Offset[{-5,10},{ω,α}//.BP⟦2⟧]];BP2p={PointSize[Medium],Style[Point[{ω,α}//.BP⟦
regions={"I",II,III,IV,V,VI};
pt=Table[{ω,α}//.test[j],{j,regions}];
pG=Table[Text[P[j],Offset[{-5,10},pt⟦j⟧]],{j,6}];

epiP={PH,PHp,PBT,PBTp,BP1,,BP1p,BP2,BP2p}//N;
fig6F=Show[{pR0,pmu,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yellow
```

```
,Epilog→{epi,epiP},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]
Export["fig6n.pdf",fig6F]
```

Out[177]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ v_2 \to \gamma+\delta+\mu,\ v_1 \to \beta+\mu\,\xi,\ V_2 \to \gamma+\delta+\eta+\mu\right\}$

Out[179]= 11.8577

R0-1 at BTP is

Out[187]= -0.00989389

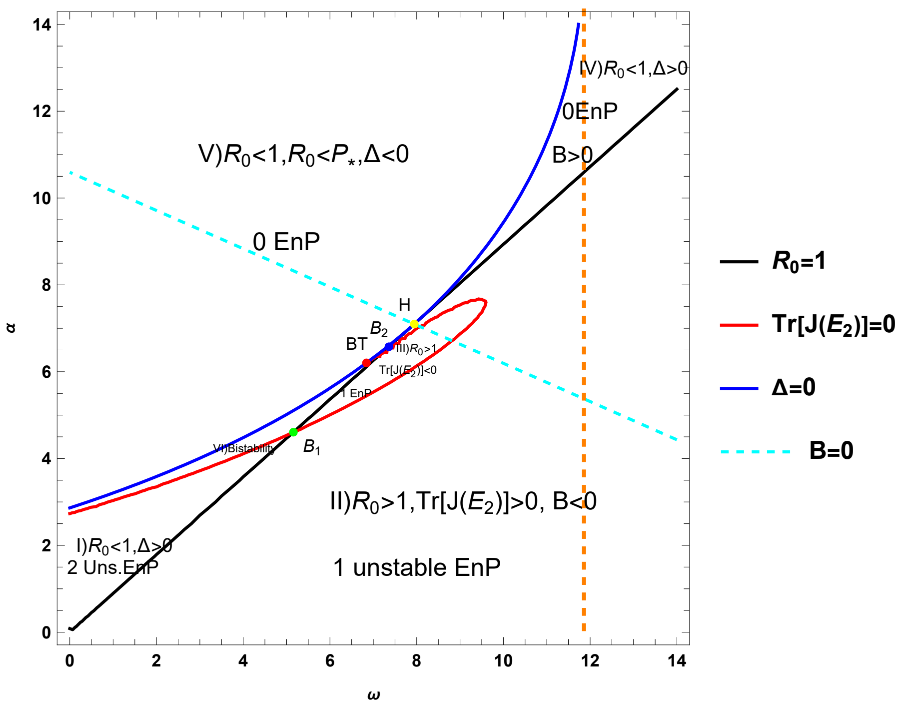
dis at BTP is

Out[189]= 0

Dis at H =

Out[191]= 0.

Out[206]=



Out[207]= fig6n.pdf

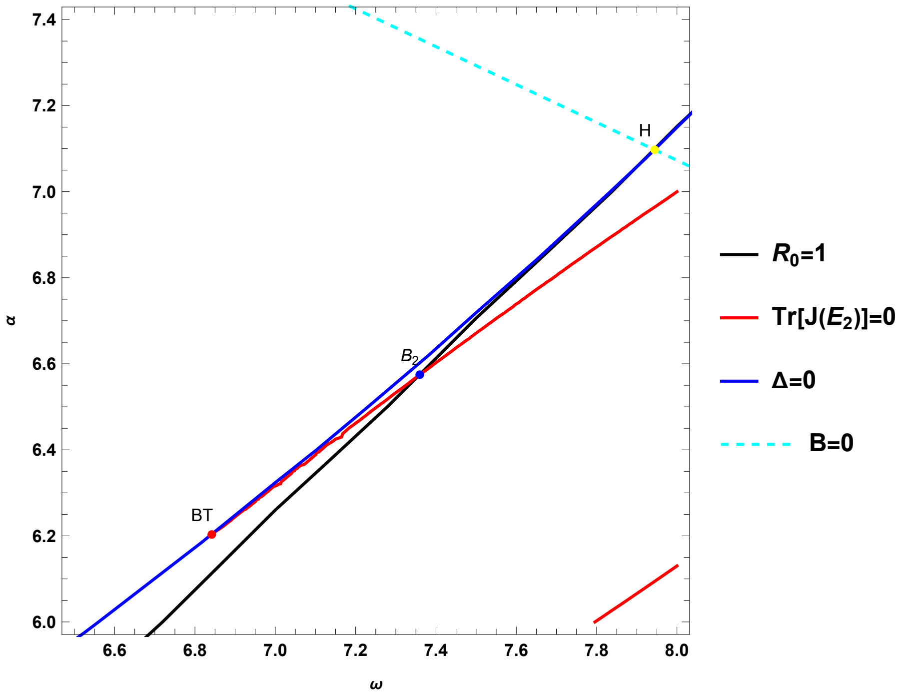
Blow-up of the Map:

In[255]:=

```
(*xm=6.8;ym=6.25;xM=7.8;yM=6.9;*)
xm=6.5;ym=6;xM=8;yM=7.4;
trωa=((trE2//.cv1)/.ceta//.cn);
pmu=ContourPlot[trωa==0,{ω,xm,xM},{α,ym,yM},ContourStyle→{Red},
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"}];

fig6F=Show[{pR0,pmu,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yellow
,Epilog→{epiP},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]
Export["fig6ns.pdf",fig6F]
```

Out[258]=



Out[259]= fig6ns.pdf

In[ ]:=

```
ωBTP=ω/.testBTP;
ωBP=ω/.testBP;
aBTP=α/.testBTP;
aBP=α/.testBP;
BR01=FindInstance[Join[{α==ω η0, ωBP<ω<ωBTP ,aBP <α<aBTP},Drop[cp,{7,9}],{η0>0}]//.Param,{ω,
Drop[BR01,30];
Print["Points betωeen BP and BTP "]
Drop[BR01,30]//N
Print["Points betωeen 0 and BP "]
BR01a=FindInstance[Join[{α==ω η0, 0<ω<ωBP ,0<α<aBP},Drop[cp,{7,9}],{η0>0}]//.Param,{ω,α},10]
BR01a
BR01a//N
Print["α and ω at BTP and boundary R0==1 are respectively "]
alBTP= ω η0//.testBTP//N
ω//.BTP//N
Print["α and ω at BP are respectively "]
alBP= ω η0//.testBP//N
ω//.BP//N

R0//.testBP//N

FindInstance[Join[{α==ω η0, (ω/.BP⟦2⟧)<ω<(ω/.HP) ,(α/.BP⟦2⟧) <α<(α/.HP)},Drop[cp,{7,9}],{η0
```

Points betωeen BP and BTP

Out[ ]= $\{\{\omega \to 6.18457, \alpha \to 5.52489\}, \{\omega \to 6.19064, \alpha \to 5.5303\}, \{\omega \to 6.19636, \alpha \to 5.53542\}, \{\omega \to 6.2287, \alpha \to 5.5643\}, \{\omega \to 6.23476, \alpha \to 5.56972\}, \{\omega \to 6.29269, \alpha \to 5.62147\}, \{\omega \to 6.31694, \alpha \to 5.64313\}, \{\omega \to 6.32974, \alpha \to 5.65457\}, \{\omega \to 6.44257, \alpha \to 5.75537\}, \{\omega \to 6.4675, \alpha \to 5.77763\}, \{\omega \to 6.59751, \alpha \to 5.89377\}, \{\omega \to 6.60997, \alpha \to 5.90491\}, \{\omega \to 6.63826, \alpha \to 5.93018\}, \{\omega \to 6.63894, \alpha \to 5.93078\}, \{\omega \to 6.7292, \alpha \to 6.01142\}, \{\omega \to 6.75042, \alpha \to 6.03038\}, \{\omega \to 6.79387, \alpha \to 6.06919\}, \{\omega \to 6.79724, \alpha \to 6.0722\}, \{\omega \to 6.82082, \alpha \to 6.09326\}, \{\omega \to 6.82957, \alpha \to 6.10108\}\}$

Points betωeen 0 and BP

Out[ ]= $\left\{\left\{\omega \to \frac{46}{195}, \alpha \to \frac{3082}{14625}\right\}, \left\{\omega \to \frac{16}{65}, \alpha \to \frac{1072}{4875}\right\}, \left\{\omega \to \frac{27}{65}, \alpha \to \frac{603}{1625}\right\}, \left\{\omega \to \frac{89}{39}, \alpha \to \frac{5963}{2925}\right\}, \left\{\omega \to \frac{178}{65}, \alpha \to \frac{11926}{4875}\right\}, \left\{\omega \to \frac{574}{195}, \alpha \to \frac{38458}{14625}\right\}, \left\{\omega \to \frac{709}{195}, \alpha \to \frac{47503}{14625}\right\}, \left\{\omega \to \frac{713}{195}, \alpha \to \frac{47771}{14625}\right\}, \left\{\omega \to \frac{901}{195}, \alpha \to \frac{60367}{14625}\right\}, \left\{\omega \to \frac{190}{39}, \alpha \to \frac{2546}{585}\right\}\right\}$

Out[ ]= $\{\{\omega \to 0.235897, \alpha \to 0.210735\}, \{\omega \to 0.246154, \alpha \to 0.219897\}, \{\omega \to 0.415385, \alpha \to 0.371077\}, \{\omega \to 2.28205, \alpha \to 2.03863\}, \{\omega \to 2.73846, \alpha \to 2.44636\}, \{\omega \to 2.94359, \alpha \to 2.62961\}, \{\omega \to 3.6359, \alpha \to 3.24807\}, \{\omega \to 3.65641, \alpha \to 3.26639\}, \{\omega \to 4.62051, \alpha \to 4.12766\}, \{\omega \to 4.87179, \alpha \to 4.35214\}\}$

α and ω at BTP and boundary R0==1 are respectively

Out[ ]= 6.11203

Out[ ]= {6.84183}

α and ω at BP are respectively

Out[ ]= 4.60724

Out[∘]= {5.15735, 7.35966}

Out[∘]= 1.

Out[∘]= $\{\{\omega \to 7.39223, \alpha \to 6.60373\}, \{\omega \to 7.54757, \alpha \to 6.7425\}, \{\omega \to 7.91456, \alpha \to 7.07034\}\}$

In[∘]:=

```
(**Fig Gupta*)
paramGG=Thread[{Λ,δ,γ,β,ξ,μ,v2,v1}→{1/2,2/10,1/10,2/10,7/100,1/10,(μ+γ+δ),(β+ μ ξ)}];
cn=paramGG;
xM=.7;La=.5;
p1g=Graphics[{Thick,Orange,Dashed,Line[{{μ/(v1)//.cn,0},{μ/(v1)//.cn,La}}]}];
R0ωa=(R0/.cV2//.cn/.η→α/ω);
aω=Solve[R0ωa==1,α][[1]]
pR0=Plot[α/.aω,{ω,0,20},PlotLegends→{"R0=1"},LabelStyle→{Black,Bold}];
 disωa=(dis/.ceta//.cn);
pD=ContourPlot[disωa==0,{ω,0,xM},{α,0,La},
ContourStyle→{Thick, Dashed,Green}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},
PlotLegends→{"Δ=0"}];
Bbωa=(Bb/.ceta//.cn);
pB=ContourPlot[Bbωa==0,{ω,0,xM},{α,0,La},
ContourStyle→{Dashed,Cyan}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"B=0"}];
trωa=((trE2//.cv1)/.ceta//.cn);
pTr=ContourPlot[trωa==0,{ω,0,xM},{α,0,La},ContourStyle→Brown,
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"},MaxRecursion→5];
epi= {Black,Style[Text["V)R0<1,R0<P*,Δ<0",{0.3,0.35}],13],Style[Text["0 EnP",{0.2,0.3}],13],
Style[Text["Tr[J(E2)]<0",{11.3,10.5}],12],Style[Text["III)R0>1,1 EnP",{13,12}],12],
Style[Text["IV)R0<1,Δ>0,B>0",{0.59,0.48}],10],Style[Text["0EnP",{0.53,0.4}],12],

Style[Text[" I)R0<1,Δ>0",{0.05,0.04}],6],Style[Text["2 Uns.EnP",{0.05,0.03}],7],
Style[Text[" VI)Bistability",{3,4.7}],6],
Style[Text[" II)R0>1,Tr[J(E2)]>0, B<0",{0.3,0.07}],11],
Style[Text["1 unstable EnP",{0.3,0.03}],11]};
Gca1=Text["Ga",Offset[{10,10},{ω,α}//.ParGca]];GcaS1={PointSize[Medium],Style[Point[{ω,α}//.
Gcd1=Text["Gd",Offset[{-10,10},{ω,α}//.ParGcd]];GcdS1={PointSize[Medium],Style[Point[{ω,α}//
epiG={Gca1,GcaS1,Gcd1,GcdS1};
mapG=Show[{pR0,pTr,pD,pB,p1g},Epilog→{epi,epiG},FrameLabel→{ω,"α"},
PlotRange→{{0,xM},{0,La}}]
```

Out[∘]= $\left\{\alpha \to \frac{3\,\omega}{5}\right\}$

Out[∘]=