Zhou-Fan Model (SIR):(Sections 5 and 6 in the manuscript)

In[ ]:=

```
ClearAll["Global`*"]
SetDirectory[NotebookDirectory[]];
<< Epid.wl; << mods.wl; (*<<"def.m"; (*trG*) *)

(*The Format commands bridge between Mathematica and Latex notations *)
Format[γr] := Subscript[γ, r]; Format[dr] := Subscript[d, r];
Format[μi] := Subscript[μ, i];

(*Numerical conditions: Param is close to the
   numerical condition used in ZF for codimension 2 bifurcation maps;
cN is used for simulations*)
Param = Thread[{Λ, δ, γ, β, ξ, μ} → {16, 2 / 10, 12 / 100, 1 / 100, 1 / 1000, 12 / 100}];
ParNum = Thread[{ω, α, η} → {7/64, 179/64, α / ω}];
Print["Numerical condition cN used for simulations"]
cN = Join[ParNum, Param]

(*Numerical conditions: ParamGc corresponding to the one given by Gupta,
they are used in the checking of his computations*)
paramGc = Thread[{Λ, δ, γ, β, ξ, μ} → {1 / 2, 2 / 10, 1 / 10, 2 / 10, 7 / 100, 1 / 10}];

(*Conditions on g[i],T[i] for setting  the model,
simplifying notations (μi),important constants (sd),
relations between parametrizations, (ηH,ωH) are the coordinates
 of H when the two endemic points and DFE collide in the map *)

cZF = {g[i] → β i / (1 + ξ i), T[i] → i η ω / (ω + i), μi → μ + δ}; ceta = {η → α / ω};
 sd = Λ / μ; ηH = sd β - (γ + μ + δ) ; ωH = (μ (β Λ - μ (γ + δ + μ))) / (β Λ (β + μ ξ));

(*inputting the model defined in mods*)
mod = SIRgT; S = mod[[4]];
Vg = mod[[5]];
RHS = mod[[1]]; RHSclosed = RHS //. cZF // FullSimplify;
Print[" (s′, i′, r′) = ", RHSclosed // MatrixForm]
```

Numerical condition cN used for simulations

Out[ ]= $\left\{\omega \to \frac{7}{64}, \alpha \to \frac{179}{64}, \eta \to \frac{\alpha}{\omega}, \Lambda \to 16, \delta \to \frac{1}{5}, \gamma \to \frac{3}{25}, \beta \to \frac{1}{100}, \xi \to \frac{1}{1000}, \mu \to \frac{3}{25}\right\}$

$$\begin{pmatrix} s' \\ i' \\ r' \end{pmatrix} = \begin{pmatrix} \Lambda - s\,\mu - \frac{i\,s\,\beta}{1+i\,\xi} \\ i\left(-\gamma - \delta - \mu + \frac{s\,\beta}{1+i\,\xi} - \frac{\eta\,\omega}{i+\omega}\right) \\ -r\,\mu + i\left(\gamma + \frac{\eta\,\omega}{i+\omega}\right) \end{pmatrix}$$

Two dim case of ZF:

In[ ]:=

```
(*Setting of the 2dim- model, computation of R0, jac, det, tr, and FPs*)
var = {s, i};
RHS2 = Drop[RHSclosed, -1]; pars = Par[RHS2, var];
cpZ = Thread[Drop[Variables[RHSclosed], 3] > 0];
Print["Two dim ZF model is ", RHS2 // MatrixForm]

(*Computation of R0 using the script*)
SIRF = {RHS2, var, pars};
mode = SIRF; inf = {2};
cdfe = DFE[mode, inf] // FullSimplify // Flatten;
dfe = var /. Thread[var[[inf]] → 0] /. cdfe // FullSimplify;
ngm = NGM[mode, inf];
K = ngm[[6]];
Print[ "DFE =", dfe, ", and  R0=", R0 = Eigenvalues[K] [[1]] /. cdfe]
b0 = β /. Solve[R0 == 1, β] [[1]] // FullSimplify
  (*critical β used in the diagram of Bif*);

jac2 = Grad[RHS2, var]; det2 = Det[jac2]; tr2 = Tr[jac2];
Print["Two-dim Jacobian is ", jac2 // FullSimplify // MatrixForm]

eqF2 = Thread[Flatten[RHS2 == 0]];
 equi2 = Solve[eqF2, {s, i}];
 Print[" Two endemic fixed points !"]
FPn = equi2 //. cN ; FPn // N
 RHS2E = {RHS2[[1]], Simplify[RHS2[[2]] / i]};

(*Section 5 in the paper*)
Print[" detG factors"]
Timing[
  detG = GroebnerBasis[{RHS2[[1]], RHS2[[2]] / i, det2}, pars, var] [[1]] // FullSimplify]
(*Timing[trG=GroebnerBasis[{s1,ifac,tr},Params,X] [[1]]]
  (*32 secs, long*) (*Do not simplify*)
Save["def.m",trG]*)
```

Two dim ZF model is $\begin{pmatrix} \Lambda - s\,\mu - \frac{i\,s\,\beta}{1+i\,\xi} \\ i\left(-\gamma - \delta - \mu + \frac{s\,\beta}{1+i\,\xi} - \frac{\eta\,\omega}{i+\omega}\right) \end{pmatrix}$

DFE $=\left\{\frac{\Lambda}{\mu}, 0\right\}$, and R0$=\dfrac{\beta\,\Lambda}{\mu\,(\gamma+\delta+\eta+\mu)}$

Two-dim Jacobian is $\begin{pmatrix} -\mu - \frac{i\,\beta}{1+i\,\xi} & -\frac{s\,\beta}{(1+i\,\xi)^2} \\ \frac{i\,\beta}{1+i\,\xi} & -\gamma - \delta - \mu + \frac{s\,\beta}{(1+i\,\xi)^2} - \frac{\eta\,\omega^2}{(i+\omega)^2} \end{pmatrix}$

Two endemic fixed points !

Out[ ]= {{s → 133.333, i → 0.}, {s → 86.1361, i → 6.61878}, {s → 69.9589, i → 10.99}}

detG factors

Out[ ]= $\{7.84375,\ (\beta\,\Lambda - \mu\,(\gamma+\delta+\eta+\mu))\ \left(\mu^2\,(\gamma+\delta+\mu-(\gamma+\delta+\eta+\mu)\,\xi\,\omega)^2 + \beta^2\,\left(\Lambda^2 + 2\,\Lambda\,(\gamma+\delta-\eta+\mu)\,\omega + (\gamma+\delta+\eta+\mu)^2\,\omega^2\right) + 2\,\beta\,\mu\,\left(-\Lambda\,(\gamma+\delta+\mu) - (\gamma+\delta+\mu)\,(\gamma+\delta+\eta+\mu)\,\omega + \Lambda\,(\gamma+\delta-\eta+\mu)\,\xi\,\omega + (\gamma+\delta+\eta+\mu)^2\,\xi\,\omega^2\right)\right)\}$

In[ ]:=
```
(*Elimination of s via plugging, chP, dis, checking examples for Hopf*)

Print["s formula cs from first and sec eqs of RHS2 are"]
cs=Flatten[Solve[RHS2[[1]]==0,s]]
cs2=Flatten[Solve[(RHS2[[2]]/i)==0,s]]
se02=s/.cs2;

(*Equation for endemic i using the chP and FPs equations *)
poli=Collect[Factor[Numerator[Together[RHS2[[2]]/.cs]]]/(-i),i];
Print["sec. order A i^2 + B i + C=0 for endemic i is"]
pol2= Collect[ poli//FullSimplify,i]

Print["coeffs are"]
cf=CoefficientList[poli,i];
Aa=cf[[3]];Bb=cf[[2]]; Cc=cf[[1]];
Print["{A,B,C} =",{Aa,Bb,Cc}//FullSimplify]

ie=Solve[poli==0,i](*endemic i*);
Print["Trace of jac2 in terms of i after substitution of solution of cs2 is "]
tri=tr2/.cs2//FullSimplify
nutr=Factor[Numerator[tri]];
rest=Resultant[poli,nutr,i](*it will be used in the numerical illustrations*);
tra2=tri/.ie[[2]](*Trace at the second endemic point*);
Print["dis "]
dis=Discriminant[poli,i]//FullSimplify

(*Computation of the coordinates corresponding to the HOpf point in the two dimensional map*)
Print[" Example of Hopf pnt in the case of one endemic point (Tr[E2]=0 , dis>0, R0>1, Bb<0)"]
bnd23=Join[FindInstance[Join[{dis>0&&tra2==0 &&Bb<0&&R0>1},Thread[Drop[Variables[RHS2],2]>0],
{ω,α,η}][[1]],Param];
%//N
Chop[Eigenvalues[jac2/.cs2/.ie[[2]]/.Param/.bnd23]//N]
```

s formula cs from first and sec eqs of RHS2 are

Out[ ]= $\left\{s \to \frac{\Lambda\,(1+\mathtt{i}\,\xi)}{\mathtt{i}\,\beta+\mu+\mathtt{i}\,\mu\,\xi}\right\}$

Out[ ]= $\left\{s \to \frac{(1+\mathtt{i}\,\xi)\,\left(\gamma+\delta+\mu+\frac{\eta\,\omega}{\mathtt{i}+\omega}\right)}{\beta}\right\}$

sec. order A i^2 + B i + C=0 for endemic i is

Out[ ]= $\mathtt{i}^2\,(\gamma+\delta+\mu)\,(\beta+\mu\,\xi)-\beta\,\Lambda\,\omega+\mu\,(\gamma+\delta+\eta+\mu)\,\omega+$
$\mathtt{i}\,(-\beta\,\Lambda+\mu\,(\gamma+\delta+\mu)+(\gamma+\delta+\eta+\mu)\,(\beta+\mu\,\xi)\,\omega)$

coeffs are

{A,B,C} =
$\{(\gamma+\delta+\mu)\,(\beta+\mu\,\xi),\ -\beta\,\Lambda+\mu\,(\gamma+\delta+\mu)+(\gamma+\delta+\eta+\mu)\,(\beta+\mu\,\xi)\,\omega,\ (-\beta\,\Lambda+\mu\,(\gamma+\delta+\eta+\mu))\,\omega\}$

Trace of jac2 in terms of i after substitution of solution of cs2 is

Out[ ]= $-\mu-\frac{\mathtt{i}\,\beta}{1+\mathtt{i}\,\xi}+\frac{\mathtt{i}\,\eta\,\omega}{(\mathtt{i}+\omega)^2}-\frac{\mathtt{i}\,\xi\,\left(\gamma+\delta+\mu+\frac{\eta\,\omega}{\mathtt{i}+\omega}\right)}{1+\mathtt{i}\,\xi}$

dis

Out[ ]= $\mu^2\,(\gamma+\delta+\mu-(\gamma+\delta+\eta+\mu)\,\xi\,\omega)^2+\beta^2\,\left(\Lambda^2+2\,\Lambda\,(\gamma+\delta-\eta+\mu)\,\omega+(\gamma+\delta+\eta+\mu)^2\,\omega^2\right)+$
$2\,\beta\,\mu\,\left(-\Lambda\,(\gamma+\delta+\mu)-(\gamma+\delta+\mu)\,(\gamma+\delta+\eta+\mu)\,\omega+\Lambda\,(\gamma+\delta-\eta+\mu)\,\xi\,\omega+(\gamma+\delta+\eta+\mu)^2\,\xi\,\omega^2\right)$

Example of Hopf pnt in the case of one endemic point (Tr[E2]=0 , dis>0, R0>1, Bb<0)

Out[ ]= $\{\omega\to 6.,\ \alpha\to 5.00625,\ \eta\to 0.834376,\ \Lambda\to 16.,$
$\delta\to 0.2,\ \gamma\to 0.12,\ \beta\to 0.01,\ \xi\to 0.001,\ \mu\to 0.12\}$

Out[ ]= $\{0.+0.15125\,\mathbb{i},\ 0.-0.15125\,\mathbb{i}\}$

Computation of some important points in the codim 2 map:

In[∘]:=
```
Print["The point HP from the Map"]

HP=Solve[Join[{η==ηH&&Bb==0},cpZ,{η==α/ω}]//.Param,{ω,α,η}][[1]]//FullSimplify;
HP//N
Print["Check of Eigenvalues at H "]
Eigenvalues[jac2/.equi2[[2]]/.Param/.HP]//N


eq=Flatten[Join[{dis==0&&tra2==0},cpZ]]//.Join[{η→α/ω},Param];
Print["BTP Symb is", BTP=Solve[eq,{ω,α},Reals][[1]]//FullSimplify," =",BTP//N]

Print["This is BogdanovTP"]
eq=Join[BTP,Param,{η→α/ω}]
cS=NSolve[(RHS2//.eq)==0,var,WorkingPrecision→60];
Print["Check of Det at the BTP at E2 "]
Chop[N[det2//.Join[eq,cS[[2]]],60]]
(*w of BT*)
wBT=ω/.BTP;
η0=sd β-(μ+γ+δ)(*it comes from Solve[R0==1,η]*);
Print["Two B pts?"]
BP=Solve[Join[{η==η0&&tra2==0&&dis>0},cpZ,{η==α/ω}]//.Param,{ω,α,η},
Reals]//FullSimplify;
BP//N
(*Save["def.m",BTP];Save["def.m",BP]*)
```

The point HP from the Map

Out[∘]= $\{\omega \to 7.94466,\ \alpha \to 7.09723,\ \eta \to 0.893333\}$

Check of Eigenvalues at H

Out[∘]= $\{-0.12,\ 0.\}$

BTP Symb is $\left\{\omega \to \boxed{6.84\ldots},\ \alpha \to \dfrac{100 \times \left(4960 + \boxed{-3.88\ldots \times 10^3}\right)}{17457}\right\}$ $=\{\omega \to 6.84183,\ \alpha \to 6.20319\}$

This is BogdanovTP

Out[∘]= $\left\{\omega \to \boxed{6.84\ldots},\ \alpha \to \dfrac{100 \times \left(4960 + \boxed{-3.88\ldots \times 10^3}\right)}{17457},\ \Lambda \to 16,\ \delta \to \dfrac{1}{5},\ \gamma \to \dfrac{3}{25},\ \beta \to \dfrac{1}{100},\ \xi \to \dfrac{1}{1000},\ \mu \to \dfrac{3}{25},\ \eta \to \dfrac{\alpha}{\omega}\right\}$

Check of Det at the BTP at E2

Out[∘]= 0

Two B pts?

Out[∘]= $\{\{\omega \to 5.15735,\ \alpha \to 4.60724,\ \eta \to 0.893333\},\ \{\omega \to 7.35966,\ \alpha \to 6.57463,\ \eta \to 0.893333\}\}$

In[∘]:=
```
(*Numeric approach, using Param*)
test[RI]=Join[FindInstance[Join[{dis>0,R0<1,(tra2)>0, Bb<0,ω>2},cpZ,{η==α/ω}]//.Param,
```

```
{ω,α,η}]⟦1⟧,Param];
test[II]=Join[FindInstance[Join[{dis>0,R0>1,(tra2)>0},cpZ,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Param];
test[III]=Join[FindInstance[Join[{dis>0,R0>1,(tra2)<0},cpZ,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Param];
test[IV]=Join[FindInstance[Join[{dis>0&&R0<1&&Bb>0,ω<11},cpZ,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Param];
test[V]=Join[FindInstance[Join[{dis<0,R0<1},cpZ,{η==α/ω}]//.Param,{ω,α,η}]⟦1⟧
,Param];
test[VI]=Join[FindInstance[Join[{dis>0,R0<1,(tra2)<0},cpZ,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Param];
test[VIa]=Join[FindInstance[Join[{dis>0,R0<1,(tra2)<0},cpZ,{η==α/ω}]//.Param,
{ω,α,η},3]⟦2⟧,Param];

testBoIIandIII=Join[FindInstance[Join[{dis>0&&tra2==0 &&Bb<0&&R0>1},cpZ,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Param];
testBoIandVI=Join[FindInstance[Join[{dis>0 &&Bb<0&&tra2==0&&R0<1,ω<wBT},cpZ,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Param];
testBoIandVII=Join[FindInstance[Join[{dis>0 &&Bb<0&&tra2==0&&R0<1,ω>wBT},cpZ,{η==α/ω}]//.Param
{ω,α,η}]⟦1⟧,Param];
testBoIIandI=Join[FindInstance[Join[{dis>0&&tra2>0 &&Bb<0&&R0==1},cpZ,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Param];
testBoIIandIV=Join[FindInstance[Join[{dis>0 &&Bb>0&&R0==1},cpZ,{η==α/ω}]//.Param,
{ω,α,η}]⟦1⟧,Param];
testBoIandVIa=Join[FindInstance[Join[{tra2==0&& dis>0 && R0<1},cpZ,{η==α/ω}]//.Param,
{ω,α,η},6]⟦6⟧,Param];

testHP=Join[HP,Param];
testBTP=Join[BTP,Param];
testBP=Join[BP⟦1⟧,Param]

Print["between I and II at R0=1 downward B1"]
testR1=Join[Param,Thread[{ω,α}→{901/195,60367/14625}]]
testR2=Join[Param,Thread[{ω,α}→(({ω,α}//.BP⟦1⟧)+({ω,α}//.BP⟦2⟧))/2]]
testR3=Join[Param,Thread[{ω,α}→(({ω,α}//.HP)+({ω,α}//.BP⟦2⟧))/2]]
testR4=Join[Param,Thread[{ω,α}→{8139/1028,181771/25700}]]



(*Now on Gupta parameters*)

testGca=Join[paramGc,Thread[{ω,α}→{10/99, 9/99}]];
testGcb=Join[paramGc,Thread[{ω,α}→{10000/103387, 9000/103387}]];
testGcc=Join[paramGc,Thread[{ω,α}→{10/108, 9/108}]];
testGcd=Join[paramGc,Thread[{ω,α}→{1/11, 9/110}]];
testGc4=Join[paramGc,Thread[{ω,α}→{1000000000/1604038240, 900000000/1604038240}]];
testG3a=Join[paramGc,Thread[{ω,α}→{100000000/1063265757, 90000000/1063265757}]];


R0TD={R0-1, tra2, dis, Bb};
Print["R0-1, Tr,Dis, B for region I is "]
R0TD//.Join[test[RI],ceta]//N
```

```
Print["R0-1, Tr,Dis, B for the boundary between II and III is "]
Chop[Evaluate[R0TD//.Join[testBoIIandIII,ceta]//N]]
Print["R0-1, Tr,Dis, B of Gupta (Fig 1a) is : "]
R0TD//.Join[testGca,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1b) is : "]
R0TD//.Join[testGcb,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1c) is : "]
R0TD//.Join[testGcc,ceta]//N
Print["R0-1, Tr,Dis, B of Gupta (Fig 1d) is : "]
R0TD//.Join[testGcd,ceta]//N
Print["R0-1, Tr,Dis, B, of Gupta (Fig 3a) is : "]
R0TD//.Join[testG3a,ceta]//N

Print["at H, dis is"]
dis//.testHP//FullSimplify

Print["tra2 at H when μ=1/12 is "]
tra2//.testHP//N
```

Out[•]= $\left\{\omega \to 5.16\ldots,\ \alpha \to \frac{67\ 7.14\ldots\times 10^{5}}{10\,379\,325},\ \eta \to \frac{67}{75},\ \Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25}\right\}$

between I and II at R0=1 downward B1

Out[•]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ \omega \to \frac{901}{195},\ \alpha \to \frac{60\,367}{14\,625}\right\}$

Out[•]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ \omega \to \frac{1}{2}\left(5.16\ldots + 7.36\ldots\right),\ \alpha \to \frac{1}{2}\left(\frac{67\ 7.14\ldots\times 10^{5}}{10\,379\,325} + \frac{67\ 1.02\ldots\times 10^{6}}{10\,379\,325}\right)\right\}$

Out[•]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ \omega \to \frac{1}{2}\times\left(\frac{2010}{253} + 7.36\ldots\right),\ \alpha \to \frac{1}{2}\times\left(\frac{8978}{1265} + \frac{67\ 1.02\ldots\times 10^{6}}{10\,379\,325}\right)\right\}$

Out[•]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25},\ \omega \to \frac{8139}{1028},\ \alpha \to \frac{181\,771}{25\,700}\right\}$

R0-1, Tr,Dis, B for region I is

Out[•]= {-0.349179, 0.0139738, 0.000608759, -0.0624949}

R0-1, Tr,Dis, B for the boundary between II and III is

Out[•]= {0.046264, 0, 0.0016453, -0.0298199}

R0-1, Tr,Dis, B of Gupta (Fig 1a) is :

Out[•]= {-0.230769, 0.024036, 0.0000733967, -0.0328182}

R0-1, Tr,Dis, B of Gupta (Fig 1b) is :

Out[•]= {-0.230769, 0.0000111711, 0.000193019, -0.0339716}

R0-1, Tr,Dis, B of Gupta (Fig 1c) is :

Out[•]= {-0.230769, -0.0183133, 0.00031084, -0.0350833}

R0-1, Tr,Dis, B of Gupta (Fig 1d) is :

Out[•]= {-0.230769, -0.025093, 0.00035956, -0.0355364}

R0-1, Tr,Dis, B, of Gupta (Fig 3a) is :

Out[•]= {-0.230769, -0.0121655, 0.000268999, -0.0346912}

at H, dis is

Out[•]= 0

tra2 at H when μ=1/12 is

Out[•]= -0.12

Bifurcation Map:

In[•]:=

```
(**Fig 6ns/3, Fig62/4 *)
cn=Param
xm=0;ym=0;xM=14;yM=14;
(*μ/(v1)//.cn//N*)
p1g=Graphics[{Thick,Orange,Dashed,Line[{{μ/((β+ μ ξ))//.cn,0},{μ/((β+ μ ξ))//.cn,45}}]}];
R0ωa=(R0//.cn/.η→α/ω);
disωa=(dis/.ceta//.cn);
Bbωa=(Bb/.ceta//.cn);
trωa=(tra2/.ceta//.cn);
trω2=((rest)/.ceta//.cn);

(*trω=((trG//.cv1)/.ceta//.cn);*)
ptr=ContourPlot[trωa==0,{ω,xm,xM},{α,ym,yM},ContourStyle→{Red},PlotPoints → 200,
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"}];
  ptr2=ContourPlot[trω2==0,{ω,xm,xM},{α,ym,yM},PlotPoints → 290,
 MaxRecursion → 2, WorkingPrecision → 35, ContourStyle→{Red},
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"[Tr[J(E2)]=0]⋃[Tr[J(E1)]=0]"}];


Print["dis at BTP is "]
Chop[Evaluate[disωa//.testBTP//N]]
Print["Dis at H ="]
dis//.testHP//N

pR0=ContourPlot[R0ωa==1,{ω,xm,xM},{α,ym,yM},ContourStyle→{Black,Dotted},
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"R0=1"}];

pD=ContourPlot[disωa==0,{ω,xm,xM},{α,ym,yM},
ContourStyle→{ Blue,Dashed}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},
PlotLegends→{"Δ=0"}];

pB=ContourPlot[Bbωa==0,{ω,xm,xM},{α,ym,yM},
ContourStyle→{Dashed,Cyan}, AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"B=0"}];

 epi= {Black,Style[Text["V)R0<1,Δ<0",{5.4,11}],13],Style[Text["0 EnP",{5,10.5}],13],
```

```
Style[Text["Tr[J(E₂)]>0",{7.8,6}],6],Style[Text["II)R₀>1",{8,6.5}],7],Style
[Text["1 EnP",{6.9,5.9}],6],
Style[Text["IV)R₀<1,Δ>0",{13,13.6}],10],Style[Text["0EnP",{13,13.2}],12],
Style[Text["B>0",{13,12.8}],12],
Style[Text[" VI)R₀<1,Δ>0",{1.2,2}],10],Style[Text["Bistablity",{1,1.5}],10],
Style[Text[" I)2 EnP",{4,4.2}],7],
Style[Text[" III)R₀>1,Tr[J(E₂)]<0, B<0",{9,3}],13],
Style[Text["1 stable EnP",{8,1.5}],13]};

PH=Text["H",Offset[{-5,10},{ω,η ω}//.testHP]];PHp={PointSize[Medium],Style[Point[{ω,η ω}
//.testHP],Yellow]};
PBT=Text["BT",Offset[{-3,6},{ω,α}//.testBTP//N]];PBTp={PointSize[Medium],Style[Point[
{ω,α}//.testBTP//N],Red]};
BP1=Text["B₁",Offset[{10,-7},{ω,α}//.BP⟦1⟧]];BP1p={PointSize[Medium],Style[Point[
{ω,α}//.BP⟦1⟧],Green]};
BP2=Text["B₂",Offset[{-5,10},{ω,α}//.BP⟦2⟧]];BP2p={PointSize[Medium],Style[Point[
{ω,α}//.BP⟦2⟧],Blue]};
P1=Text["R₁",Offset[{10,-7},{ω,α}//.testR1//N]];P1p={PointSize[Medium],Style[Point[
{ω,α}//.testR1//N],Purple]};
(*P4=Text["R₄",Offset[{10,-7},{ω,α}//.testR4]];P4p={PointSize[Medium],
Style[Point[{ω,α}//.testR4],Purple]};*)



P3=Text["R₃",Offset[{-4,5},({ω,α}//.testR3//N)]];
P3p={PointSize[Medium],Style[Point[{ω,α}//.testR3//N],Purple]};
P2=Text["R₂",Offset[{-4,5},{ω,α}//.testR2//N]];
P2p={PointSize[Medium],Style[Point[{ω,α}//.testR2//N],Purple]};
QI=Text["Q_I",Offset[{8,5},{ω,α}//.test[RI]]];QIp={PointSize[Medium],Style[Point[
{ω,α}//.test[RI]],Magenta]};
QII=Text["Q_II",Offset[{8,5},{ω,α}//.test[II]]];QIIp={PointSize[Medium],Style[Point[
{ω,α}//.test[II]],Magenta]};
QIII=Text["Q_III",Offset[{8,5},{ω,α}//.test[III]]];QIIIp={PointSize[Medium],Style[Point[
{ω,α}//.test[III]],Magenta]};
QIV=Text["Q_IV",Offset[{8,5},{ω,α}//.test[IV]]];QIVp={PointSize[Medium],Style[Point[
{ω,α}//.test[IV]],Magenta]};
QV=Text["Q_V",Offset[{-5,10},{ω,α}//.test[V]]];QVp={PointSize[Medium],Style[Point[
{ω,α}//.test[V]],Magenta]};
QVI=Text["Q_VI",Offset[{8,5},{ω,α}//.test[VI]]];QVIp={PointSize[Medium],
Style[Point[{ω,α}//.test[VI]],Magenta]};
QVIa=Text["Q_VIa",Offset[{8,5},{ω,α}//.test[VIa]]];QVIap={PointSize[Medium],
Style[Point[{ω,α}//.test[VIa]],Magenta]};
T1=Text["T₁",Offset[{10,-7},{ω,α}//.testBoIIandIII]];T1a={PointSize[Medium],
Style[Point[{ω,α}//.testBoIIandIII],Black]};
T2=Text["T₂",Offset[{8,-7},{ω,α}//.testBoIandVIa]];T2a={PointSize[Medium],
Style[Point[{ω,α}//.testBoIandVIa],Black]};
regions={RI,II,III,IV,V,VI};
pt=Table[{ω,α}//.test[j],{j,regions}];
pG=Table[Text[P[j],Offset[{-5,10},pt⟦j⟧]],{j,6}];

epiP={PH,PHp,PBT,PBTp,BP1,BP1p,BP2,BP2p,P1,P1p,P3,P3p,P2,P2p,(*P4,P4p,*)QI,QIp,QII,
QIIp,QIII,QIIIp,QIV,QIVp,QV,QVp,QVI,QVIp,T1,T1a,T2,T2a}//N;
epiP1={PH,PHp,PBT,PBTp,BP1,BP1p,BP2,BP2p,P1,P1p,P3,P3p,P2,P2p,T1,T1a,T2,T2a}//N;
fig6F=Show[{pR0,ptr,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yellow
,Epilog→{epi,epiP},FrameLabel→{ω,"α"},
```

```
PlotRange→{{xm,xM},{ym,yM}}]
fig62=Show[{pR0,ptr2,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yell
,Epilog→{epi,epiP1},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]

Export["fig6ns.pdf",fig6F]
Export["fig62.pdf",fig62]
```

Out[]= $\left\{\Lambda \to 16,\ \delta \to \frac{1}{5},\ \gamma \to \frac{3}{25},\ \beta \to \frac{1}{100},\ \xi \to \frac{1}{1000},\ \mu \to \frac{3}{25}\right\}$

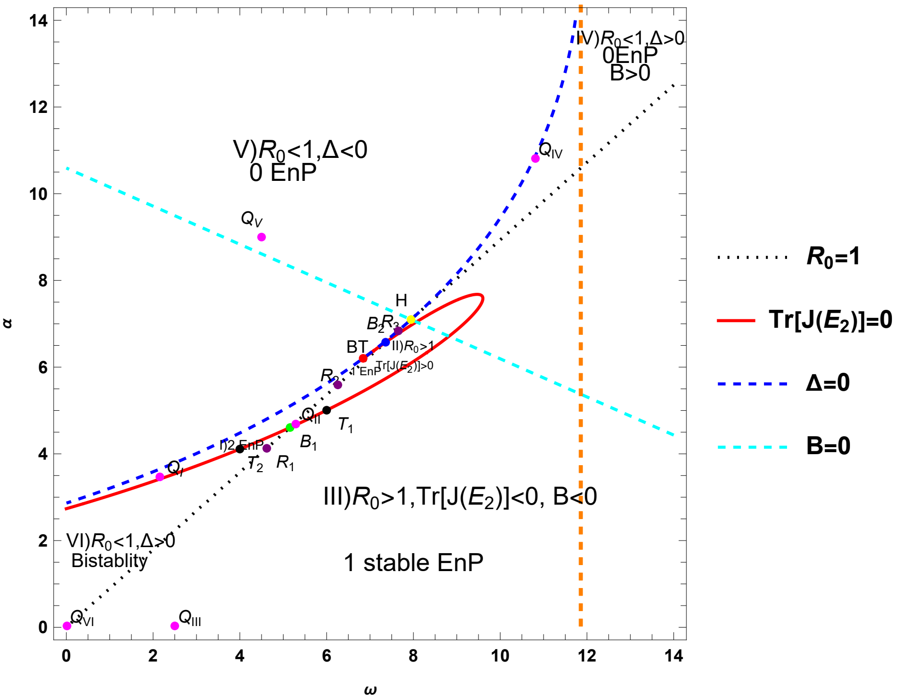
dis at BTP is

Out[]= 0

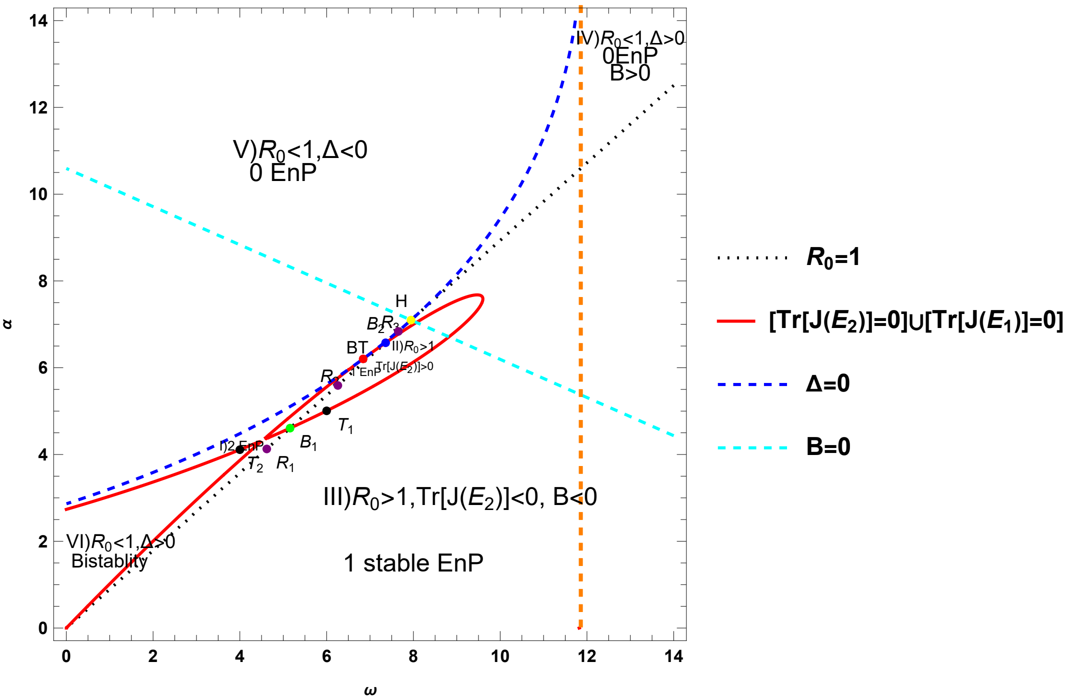
Dis at H =

Out[]= 0.

Out[]=



Out[∘]=



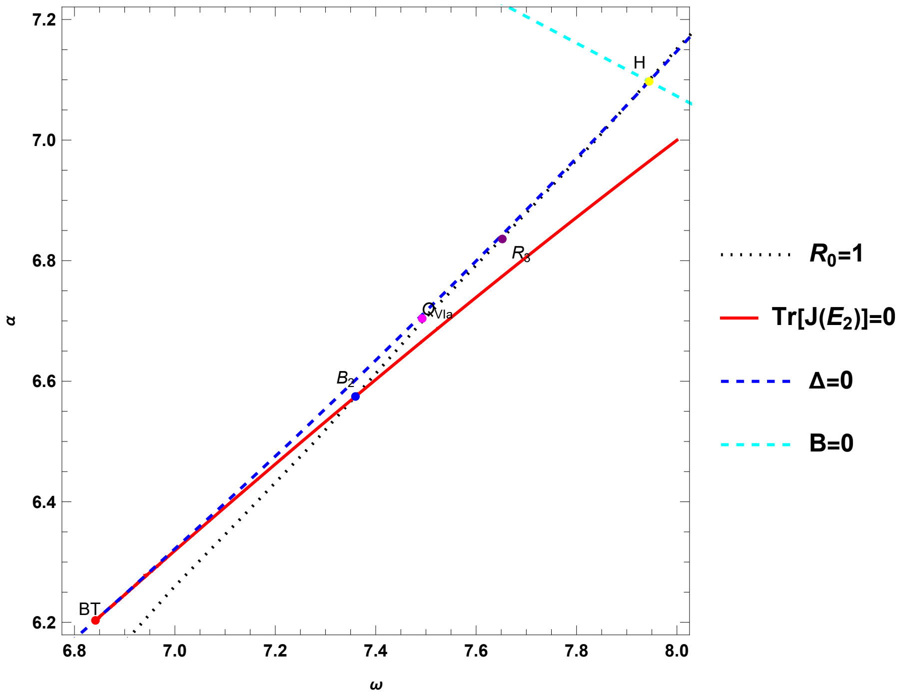
Out[∘]= `fig6ns.pdf`

Out[∘]= `fig62.pdf`

Blow-up of the Map:

In[∘]:=
```
(*xm=6.8;ym=6.25;xM=7.8;yM=6.9;*)
xm=6.8;ym=6.2;xM=8;yM=7.2;
trωa=((tra2)/.ceta//.cn);
ptr=ContourPlot[trωa==0,{ω,xm,xM},{α,ym,yM},ContourStyle→{Red},PlotPoints → 200,
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E₂)]=0"}];
 P3=Text["R₃",Offset[{10,-7},{ω,α}//.testR3//N]];
 P3p={PointSize[Medium],Style[Point[{ω,α}//.testR3//N],Purple]};
 epiP1={PH,PHp,PBT,PBTp,BP1,BP1p,BP2,BP2p,P1,P1p,P3,P3p,QVIa,QVIap}//N;
fig6F=Show[{pR0,ptr,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yello
,Epilog→{epiP1},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]
Export["fig6BT.pdf",fig6F]
```

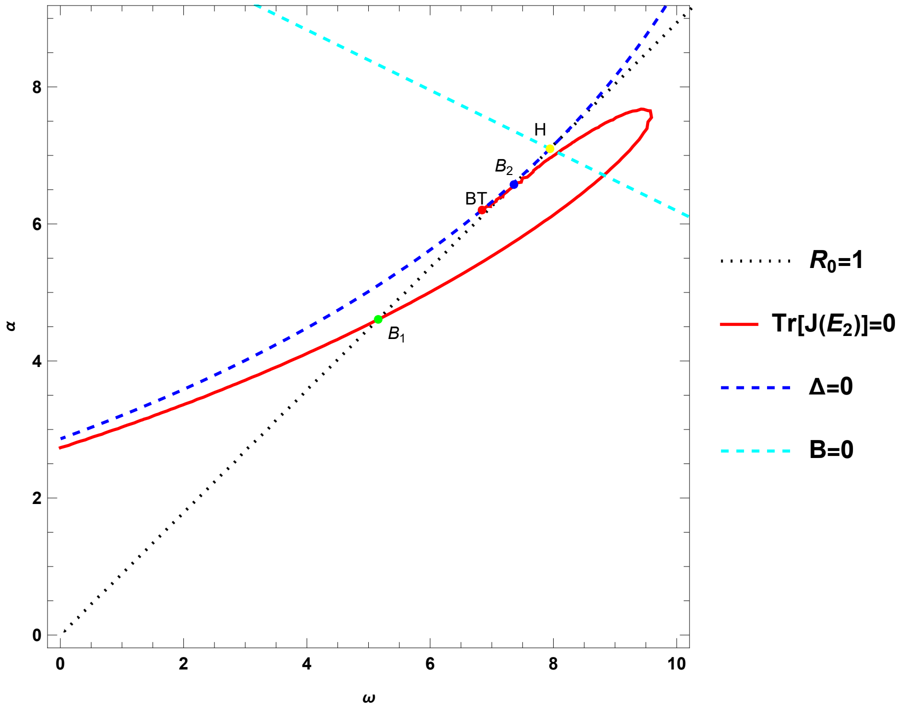
Out[∘]=



Out[∘]= fig6BT.pdf

In[ ]:=
```
(*xm=6.8;ym=6.25;xM=7.8;yM=6.9;*)
xm=0;ym=0;xM=10;yM=9;
epiP={PH,PHp,PBT,PBTp,BP1,BP1p,BP2,BP2p}//N;
trωa=((tra2)/.ceta//.cn);
ptr=ContourPlot[trωa==0,{ω,xm,xM},{α,ym,yM},ContourStyle→{Red},
 AxesLabel→{ω,"α"},LabelStyle→{Black,Bold},PlotLegends→{"Tr[J(E2)]=0"}];

fig6F=Show[{pR0,ptr,pD,pB,p1g},PlotStyle→Join[ColorData[97,"ColorList"]],Filling→{3→{0,Yellow
,Epilog→{epiP},FrameLabel→{ω,"α"},
PlotRange→{{xm,xM},{ym,yM}}]
Export["fig6n.pdf",fig6F]
```

Out[ ]=



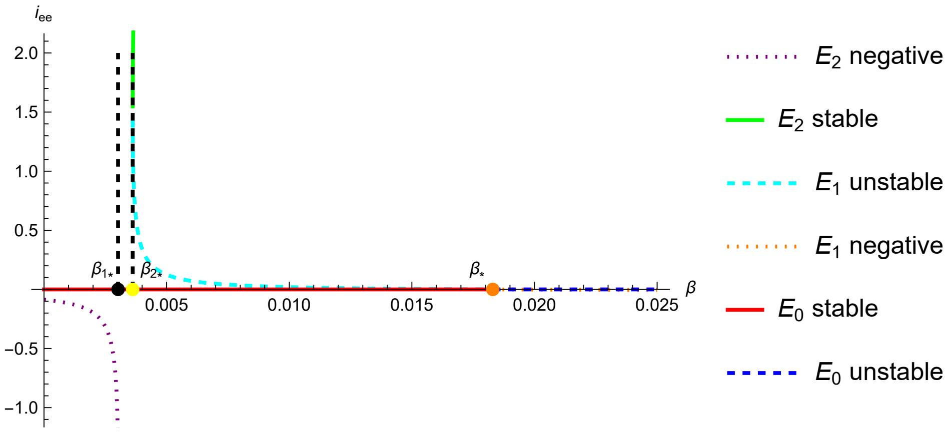
Out[ ]= fig6n.pdf

Bifurcation diagram :

In[ ]:=
```
(**Bifurcation Diagram of Region VI*)
cn=Drop[test[VI],{7}]
xm=0;ym=-1;xM=0.025;yM=2;

dis0=β/.Solve[dis==0,β];

Print["β*=",b0n= b0/.η→α/ω//.cn//N, " ,β1*=", bc1=dis0[[1]]/.η→α/ω//.cn//N, " ,β2*=", bc2=dis0

lin1=Line[{{ bc1,0},{ bc1,yM}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,yM}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
pE2a=Plot[{i/.ie[[2]]/.η→α/ω}//.cn,{β,0,bc1},PlotStyle→{Dotted,Thick,Purple},PlotRange→All,Pl
pE2b=Plot[{i/.ie[[2]]/.η→α/ω}//.cn,{β,bc1,xM},PlotStyle→{Thick,Green},
PlotRange→All,PlotLegends→{"E2 stable"}];
pE1a=Plot[{i/.ie[[1]]/.η→α/ω}//.cn,{β,0,b0n},PlotStyle→{Dashed,Thick,Cyan},PlotRange→All,Plot
pE1b=Plot[{i/.ie[[1]]/.η→α/ω}//.cn,{β,b0n,xM},PlotStyle→{Dotted,Thick,Orange},
PlotRange→All,PlotLegends→{"E1 negative"}];
pdfea=Plot[0,{β,0, b0n},PlotStyle→{Thick,Red},PlotRange→All,
PlotLegends→{"E0 stable"}];
pdfeb=Plot[0,{β,b0n, xM},PlotStyle→{Dashed,Thick,Blue},PlotRange→All,
PlotLegends→{"E0 unstable"}];
bifZF=Show[{pE2a, pE2b,pE1a, pE1b,pdfea,pdfeb,li1,li2},PlotRange→{{xm,xM},{ym,yM}},Epilog→
{{Text["β*",Offset[{-8,10},{ b0n,0}]],{PointSize[Large],
Style[Point[{ b0n,0}],Orange]},Text["β2*",Offset[{10,10},{ bc2,0}]],
{PointSize[Large],Style[Point[{ bc2,0}],Yellow]},Text["β1*",Offset[{-8,10},{ bc1,0}]],
{PointSize[Large],Style[Point[{ bc1,0}],Black]}}
},AxesLabel→{"β","iee"}]

Export["bifZF.pdf",bifZF]
```

Out[ ]= $\left\{\omega \to \frac{1}{64}, \alpha \to \frac{1}{32}, \eta \to 2, \Lambda \to 16, \delta \to \frac{1}{5}, \gamma \to \frac{3}{25}, \xi \to \frac{1}{1000}, \mu \to \frac{3}{25}\right\}$

$\beta*$=0.0183 ,β_{1*}=0.00302031 ,β_{2*}=0.00361597

Out[ ]=


Out[ ]= bifZF.pdf

Symbolic Analysis of the model:

In[ ]:=
```
Print["RHS of general gT model SIRgT is RHS=",RHS//MatrixForm, " and RHS of closed version RHS
RHSclosed//FullSimplify//MatrixForm]
Print["Check the sum of RHS ",Total[RHSclosed]//FullSimplify]

nu=NullSpace[S];di=nu//Length;
V=Sum[x[j]×nu[[j]],{j,di}];
cx={x[1]→r μ,x[2]→i μi,x[3]→s μ};
Vp=V/.cx;
Print["null space of ",S//MatrixForm," is",
V//MatrixForm," and in terms of pars Vp",Vp//MatrixForm];
V={v1,v2,v3,s μ,i μi,r μ};
so=Solve[S.V==0][[1]];
Print["null space of S  in terms of pars ",so]
```

RHS of general gT model SIRgT is RHS= $\begin{pmatrix} \Lambda - s\,\mu - s\,g[i] \\ -i\,\gamma - i\,\mu_i + s\,g[i] - T[i] \\ i\,\gamma - r\,\mu + T[i] \end{pmatrix}$

and RHS of closed version RHSclosed is $\begin{pmatrix} \Lambda - s\,\mu - \frac{i\,s\,\beta}{1+i\,\xi} \\ i\left(-\gamma - \delta - \mu + \frac{s\,\beta}{1+i\,\xi} - \frac{\eta\,\omega}{i+\omega}\right) \\ -r\,\mu + i\left(\gamma + \frac{\eta\,\omega}{i+\omega}\right) \end{pmatrix}$

Check the sum of RHS $-i\,\delta + \Lambda - (i + r + s)\,\mu$

null space of $\begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$ is

$\begin{pmatrix} x[1] + x[2] + x[3] \\ x[1] + x[2] \\ x[1] \\ x[3] \\ x[2] \\ x[1] \end{pmatrix}$ and in terms of pars Vp $\begin{pmatrix} r\,\mu + s\,\mu + i\,\mu_i \\ r\,\mu + i\,\mu_i \\ r\,\mu \\ s\,\mu \\ i\,\mu_i \\ r\,\mu \end{pmatrix}$

null space of S in terms of pars $\{v1 \to r\,\mu + s\,\mu + i\,\mu_i,\ v2 \to r\,\mu + i\,\mu_i,\ v3 \to r\,\mu\}$

Symbolic 3dim model:

```
(*computation of a Hopf example for the symbolic 3dim  model*)
var = {s, i, r}; jac = Grad[RHS, var]; det = Det[jac];
Print["Jacobian is ", jac // MatrixForm]
chp = -CharacteristicPolynomial[jac, ϵ];
cof = CoefficientList[chp, ϵ] // FullSimplify;
{co0, co1, co2, co3} = cof;

Print["CharacteristicPolynomial has last cof= ",
 co3 // FullSimplify, "  det ", co0, " and the other cofs have pars  "]
parc = Variables[cof]
cp = Thread[parc > 0];

Heq = {co1 > 0, co1 * co2 - co0 == 0, RHS[[3]] == 0, RHS[[2]] == 0, RHS[[1]] == 0};
(*added flux equations*)
cond = Union[Heq, cp, {i > 0}];
Print["Example satisfies Hopf Condition under cond: "]
Print["fiH=", fiH = FindInstance[cond, Join[parc, {i, r, Λ, T[i]}]][[1]]]
Print["Check RHS=0 under fiH ", RHS /. fiH]
jaf = jac /. fiH;
Print[" Eigenvalues of", jaf // MatrixForm, " are ", Eigenvalues[jaf] // N]
Tr[jac] /. fiH
```

Jacobian is $\begin{pmatrix} -\mu - g[i] & -s\, g'[i] & 0 \\ g[i] & -\gamma - \mu_i + s\, g'[i] - T'[i] & 0 \\ 0 & \gamma + T'[i] & -\mu \end{pmatrix}$

CharacteristicPolynomial has last cof= 1 det
$\mu\,(g[i]\,(\gamma + \mu_i + T'[i]) + \mu\,(\gamma + \mu_i - s\, g'[i] + T'[i]))$ and the other cofs have pars

Out[∘]= $\{s, \gamma, \mu, \mu_i, g[i], g'[i], T'[i]\}$

Example satisfies Hopf Condition under cond:

fiH= $\left\{s \to 8,\ \gamma \to \frac{1}{2},\ \mu \to 1,\ \mu_i \to 1,\ g[i] \to 1,\ g'[i] \to 1,\ T'[i] \to \frac{9}{2},\ i \to 1,\ r \to 7,\ \Lambda \to 16,\ T[i] \to \frac{13}{2}\right\}$

Check RHS=0 under fiH {0, 0, 0}

Eigenvalues of $\begin{pmatrix} -2 & -8 & 0 \\ 1 & 2 & 0 \\ 0 & 5 & -1 \end{pmatrix}$ are {0. + 2. ⅈ, 0. - 2. ⅈ, -1.}

Out[∘]= -1