On a three-dimensional tumor-virus compartmental model, and two four-dimensional oncolytic Virotherapy models

This Mathematica Notebook is a supplementary material to the paper which has the same title as this
document. It contains some of the calculations and illustrations appearing in the paper.

1) Section 2 (in paper): Deterministic model with Logistic growth [Tian2011]

0) Definition of the model [Tian11]:

In[644]:=
```
SetDirectory[NotebookDirectory[]];
AppendTo[$Path,Directory];
Clear["Global`*"];
(*Some aliases*)
Format[μv]:=Subscript[μ,v];Format[μy]:=Subscript[μ,y];
parT={β>0,λ>0,δ>0,b>1};
cparT={ μv→0,μy→0,γ→1, K→1};
cnTian={ μv→0, μy→0,K→1,γ→1,λ→0.36,β→0.11,δ→0.44}(*Numerical values of Tian*);

(****** Four dim Deterministic  epidemic model with Logistic growth ****)
x1=λ  x(1-(x+y)/K)-   β x v ;
y1=β x v -μy y z- γ y;
v1=-β x v - μv  v z+ b  γ y - δ v;

dyn={x1,y1,v1}/.μy→0/.μv→0(*Tian case with K>0*);

Print[" (x' y' v')=",dyn//FullSimplify//MatrixForm,

" , and the reparametrized dynamics [Tian 2011] are  (x' y' v')=",

dyn/.cparT//FullSimplify//MatrixForm]
```

$$\begin{pmatrix} x' \\ y' \\ v' \end{pmatrix} = \begin{pmatrix} -v\,x\,\beta + x\left(1 - \frac{x+y}{K}\right)\lambda \\ v\,x\,\beta - y\,\gamma \\ b\,y\,\gamma - v\,(x\,\beta + \delta) \end{pmatrix}$$

, and the reparametrized dynamics [Tian 2011] are $\begin{pmatrix} x' \\ y' \\ v' \end{pmatrix} = \begin{pmatrix} -x\,(v\,\beta + (-1 + x + y)\,\lambda) \\ -y + v\,x\,\beta \\ b\,y - v\,(x\,\beta + \delta) \end{pmatrix}$

Fixed points and analysis of the Stability via Routh Hurwitz:

In[656]:=

```
cfp=Solve[Thread[dyn=={0,0,0}],{x,y,v}]//FullSimplify;
fp={x,y,v}/.cfp;
Print[Length[fp]," fixed points, the third is E*="]
fp[[3]]//FullSimplify
(*"Jacobian is"*)
Jac=Grad[dyn,{x,y,v}]//FullSimplify;
J0=Jac/.cfp[[1]];J0//MatrixForm;
Print["J(E_K) is"]
J1=Jac/.cfp[[2]];J1//MatrixForm
Eigenvalues[J1]
R0=b β K/(β K+δ);bcrit=1+δ/(β K);(*Reduce[Join[{R0>1},pars],δ]*)
Print["J(E_*) is"]
Jst=Jac/.cfp[[3]]//FullSimplify;Jst//MatrixForm
Jstcr=Jst/.b→bcrit//FullSimplify;
Print["J(E*)/.b->b0 is",Jstcr//MatrixForm," eigvals are ",Eigenvalues[Jstcr]]

(*Routh Hurwitz conditions for the stability of E_**)
pc=Collect[Det[ψ IdentityMatrix[3]-Jst],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Print["a_1=",a1=Apart[coT[[3]]], ", a_2=",a2=coT[[2]], ", a_3=",a3=coT[[1]]]
H2=a1*a2-a3;
Print["H2(b0)=",H2/.b→bcrit//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H2]]/.K→1//FullSimplify]
Together[H2//FullSimplify];
ϕb=Collect[Numerator[Together[H2]]/(δ λ),b]/.K→1//FullSimplify;
Print["Coefficients of ϕ(b)are:",cofi=CoefficientList[ϕb,b]//FullSimplify]
(*ϕb/.b→1//FullSimplify*)
Print["value at crit b is "]
ϕb/.b→bcrit/.K→1//FullSimplify
```

3 fixed points, the third is E*=

Out[659]= $\left\{\frac{\delta}{(-1+b)\,\beta},\ \frac{((-1+b)\,K\,\beta-\delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)},\ \frac{\gamma\,((-1+b)\,K\,\beta-\delta)\,\lambda}{\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)}\right\}$

J(E_K) is

Out[663]//MatrixForm=

$$\begin{pmatrix} -\lambda & -\lambda & -K\,\beta \\ 0 & -\gamma & K\,\beta \\ 0 & b\,\gamma & -K\,\beta-\delta \end{pmatrix}$$

Out[664]= $\left\{\frac{1}{2}\left(-K\,\beta-\gamma-\delta-\sqrt{(K\,\beta+\gamma+\delta)^2-4\,(K\,\beta\,\gamma-b\,K\,\beta\,\gamma+\gamma\,\delta)}\right),\right.$
$\left.\frac{1}{2}\left(-K\,\beta-\gamma-\delta+\sqrt{(K\,\beta+\gamma+\delta)^2-4\,(K\,\beta\,\gamma-b\,K\,\beta\,\gamma+\gamma\,\delta)}\right),\ -\lambda\right\}$

J(E_*) is

Out[667]//MatrixForm=

$$\begin{pmatrix} \frac{\delta\,\lambda}{K\,\beta-b\,K\,\beta} & \frac{\delta\,\lambda}{K\,\beta-b\,K\,\beta} & -\frac{\delta}{-1+b} \\ \frac{\gamma\,((-1+b)\,K\,\beta-\delta)\,\lambda}{(-1+b)\,K\,\beta\,\gamma+\delta\,\lambda} & -\gamma & \frac{\delta}{-1+b} \\ \frac{\gamma\,(K\,(\beta-b\,\beta)+\delta)\,\lambda}{(-1+b)\,K\,\beta\,\gamma+\delta\,\lambda} & b\,\gamma & \frac{b\,\delta}{1-b} \end{pmatrix}$$

J(E*)/.b->b0 is $\begin{pmatrix} -\lambda & -\lambda & -K\beta \\ 0 & -\gamma & K\beta \\ 0 & \gamma+\frac{\gamma\delta}{K\beta} & -K\beta-\delta \end{pmatrix}$ eigvals are $\{0, -K\beta-\gamma-\delta, -\lambda\}$

$a_1=\frac{-\gamma+b\gamma+b\delta}{-1+b}+\frac{\delta\lambda}{(-1+b)K\beta}$, $a_2=$

$$\frac{\delta\lambda\left((-1+b)K\beta\gamma(K(\beta-b\beta)+(-1+b)\gamma+\delta+b\delta)+\left((-1+b)^2K\beta\gamma+b\delta^2\right)\lambda\right)}{(-1+b)^2K\beta((-1+b)K\beta\gamma+\delta\lambda)}$$

, $a_3=\gamma\delta\left(1+\frac{\delta}{K\beta-bK\beta}\right)\lambda$

$H2(b0)=(K\beta+\gamma+\delta)\lambda(K\beta+\gamma+\delta+\lambda)$

Denominator of H2 is $(-1+b)^3\beta^2((-1+b)\beta\gamma+\delta\lambda)$

Coefficients of ϕ(b)are:

$\{-\beta\gamma(\gamma+\lambda)(\beta\gamma-\delta\lambda), \beta^3\gamma(\gamma-\delta)+\delta^3\lambda^2-\beta\gamma\delta\lambda(2\gamma+3\delta+2\lambda)+\beta^2\gamma\left(3\gamma^2-\delta^2+3\gamma(\delta+\lambda)\right),$
$\beta\left(\beta^2\gamma(-3\gamma+2\delta)-3\beta\gamma^2(\gamma+2\delta+\lambda)+\delta\lambda\left(\gamma^2+\delta^2+\gamma(3\delta+\lambda)\right)\right),$
$\beta^2\gamma\left(3\beta\gamma+\gamma^2-\beta\delta+3\gamma\delta+\delta^2+\gamma\lambda\right), -\beta^3\gamma^2\}$

value at crit b is

Out[680]= $$\frac{\delta^3(\beta+\gamma+\delta)(\gamma+\lambda)(\beta+\gamma+\delta+\lambda)}{\beta}$$

Computations of the Jacobians and Eigenvalues using EcoEvo package:

In[695]:=
```
<<EcoEvo`
(*EcoEvoDocs;*)
(********Analysis of the Model, K=γ=1***)
dynKeq1=dyn/.cparT;
ClearParameters;
UnsetModel;
SetModel[{Pop[x]→{Equation:→dynKeq1[[1]],Color→Red},Pop[y]→
{Equation:→(dynKeq1[[2]]),Color→Green},
Pop[v]→{Equation:→(dynKeq1[[3]]),Color→Purple},
 Parameters:→(cp=parT)}]

fpT=SolveEcoEq[]//FullSimplify;

J0T=EcoJacobian[fpT[[1]]]//FullSimplify;
J1T=EcoJacobian[fpT[[2]]]//FullSimplify;
Jst=EcoJacobian[fpT[[3]]]//FullSimplify;
Print["Jac(E0)=",J0T//MatrixForm]
Print["Jac(E1)=",J1T//MatrixForm]
Print["Jac(E*)=",Jst//MatrixForm]
Print["Eigenvalues of E1 are:",eiT=EcoEigenvalues[fpT[[2]]]//FullSimplify]

Print["b0=bs1=",bs1=Apart[Last[Last[Reduce[Join[{eiT[[2]]>0},parT],b]]]]]
```

Out[695]= EcoEvo Package Version 1.6.4 (November 5, 2021)
Christopher A. Klausmeier <christopher.klausmeier@gmail.com>

$$\text{Jac}(E_0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -1 & 0 \\ 0 & b & -\delta \end{pmatrix}$$

$$\text{Jac}(E_1) = \begin{pmatrix} -\lambda & -\lambda & -\beta \\ 0 & -1 & \beta \\ 0 & b & -\beta-\delta \end{pmatrix}$$

$$\text{Jac}(E_*) = \begin{pmatrix} \frac{\delta\lambda}{\beta - b\beta} & \frac{\delta\lambda}{\beta - b\beta} & -\frac{\delta}{-1+b} \\ \frac{((-1+b)\beta-\delta)\lambda}{(-1+b)\beta+\delta\lambda} & -1 & \frac{\delta}{-1+b} \\ \frac{(\beta-b\beta+\delta)\lambda}{(-1+b)\beta+\delta\lambda} & b & \frac{b\delta}{1-b} \end{pmatrix}$$

Eigenvalues of E_1 are:

$$\left\{\frac{1}{2}\times\left(-1-\beta-\delta-\sqrt{(1+\beta+\delta)^2-4(\beta-b\beta+\delta)}\right), \frac{1}{2}\times\left(-1-\beta-\delta+\sqrt{(1+\beta+\delta)^2-4(\beta-b\beta+\delta)}\right), -\lambda\right\}$$

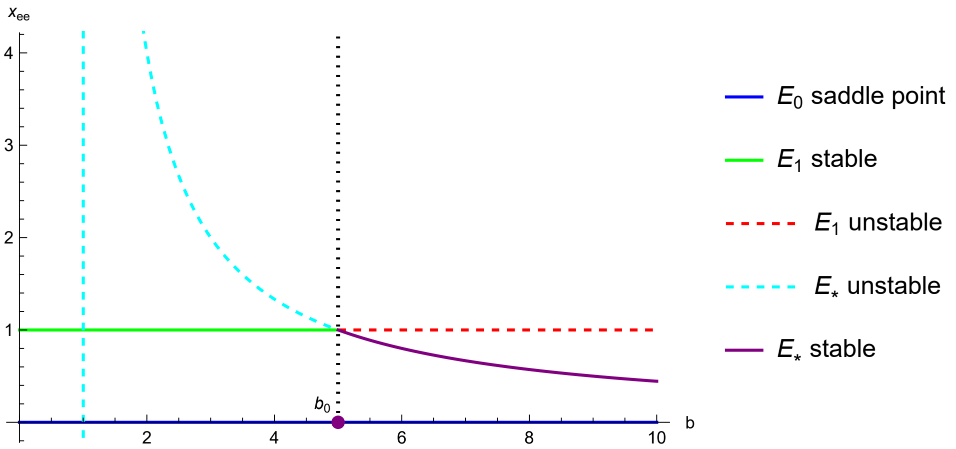
$$b_0 = b_{s1} = 1 + \frac{\delta}{\beta}$$

1) Numerical simulations:

Bifurcation Diagram:

In[709]:=
```
ClearParameters;
λ=0.36;β=0.11;δ=0.44;bL=10;
Print["b0=",bs1//N]
fpT//N;
linb0=Line[{{ bs1,0},{ bs1,10}}];
lib0=Graphics[{Thick,Black,Dotted,linb0}];
px0=Plot[0,{b,0,bL},PlotStyle→{Blue},PlotRange→All,PlotLegends→{"E0 saddle point"}];
px1a=Plot[{1},{b,0,bs1},PlotStyle→{Green},PlotRange→All,PlotLegends→{"E1 stable"}];
px1b=Plot[1,{b,bs1,bL},PlotStyle→{Red,Dashed},PlotRange→All,
PlotLegends→{"E1 unstable"}];
pxe1=Plot[{x/.fpT〚3〛},{b,0, bs1},PlotStyle→{Cyan,Dashed},PlotRange→All,
PlotLegends→{"E* unstable"}];
pxe2=Plot[{x/.fpT〚3〛},{b,bs1, bL},PlotStyle→{Purple},PlotRange→All,
PlotLegends→{"E* stable"}];
bif11T=Show[{px0,px1a,px1b,pxe1,pxe2,lib0},Epilog→{Text["b0",Offset[{-8,10},{ bs1,0}]],
{PointSize[Large],Style[Point[{ bs1,0}],Purple]}},
PlotRange→{{0,10},{0,4}},AxesLabel→{"b","xee"}]
Export["bif11T.pdf",bif11T]
```

b0=5.

Out[720]=


Out[721]= bif11T.pdf

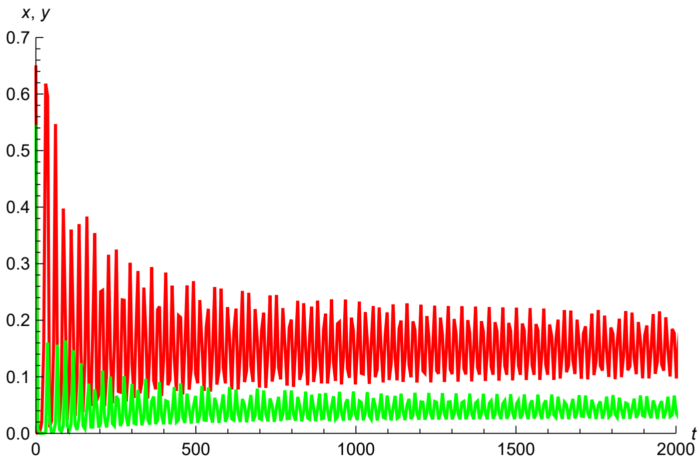
Periodic x,y values when b=28:

In[722]:=
```
ClearParameters;
λ=0.36;β=0.11;δ=0.44;b=28;
in={x→0.5,y→0.5,v→1.5};
fpT//N
EcoEigenvalues[fpT[[3]]] (*Eigenvalues corresponding to E* **)
solE3=EcoSim[RuleListAdd[fpT[[3]],in],20000];

Fig5T=PlotDynamics[{solE3[[1]],solE3[[2]]},PlotRange→{{0,2000},{0,0.7}}]
Export["Fig5T.pdf",Fig5T]
```

Out[725]= {{x → 0., y → 0., v → 0.}, {x → 1., y → 0., v → 0.},
{x → 0.148148, y → 0.0431317, v → 2.64672}}

Out[726]= {-1.51022, 0.000296187 + 0.298909 ⅈ, 0.000296187 - 0.298909 ⅈ}

Out[728]=



Out[729]= Fig5T.pdf

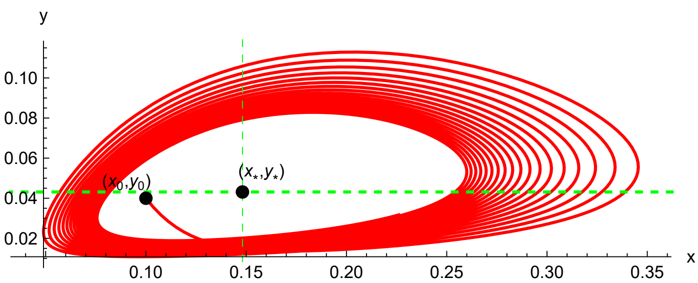
Numerical illustrations (Parametric plot) when b=28:

In[730]:=

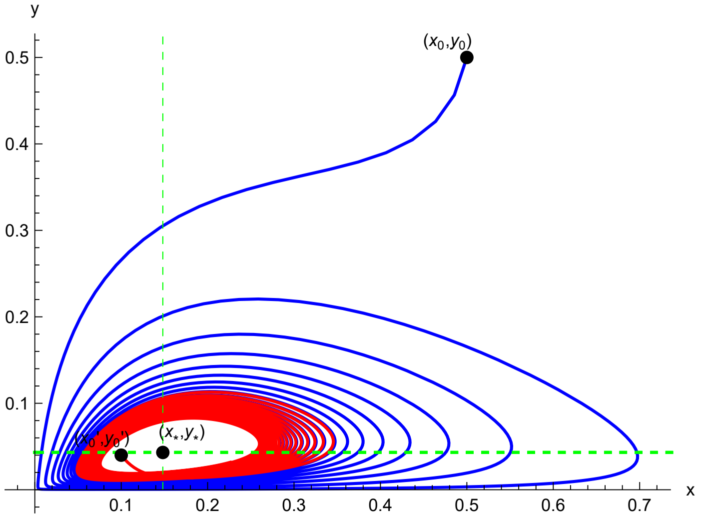
```
ClearParameters;
λ=0.36;β=0.11;δ=0.44;b=28;K=1;γ=1;
Print["E*",fp⟦3⟧//N]
x1=λ  x[t] (1-(x[t]+y[t]))-   β x[t]×v[t] ;
y1=β x[t]×v[t] -  y[t];
v1=-β x[t]×v[t] + b   y[t] - δ v[t];

ode={x'[t]==x1,y'[t]==y1,v'[t]==v1,x[0]==0.5,y[0]==0.5,v[0]==1.5};
sol=NDSolve[ode,{x,y,v},{t,0,400}];

x0=0.5; y0=0.5;v0=1.5;
ppb28=ParametricPlot[{  x[t],(y[t])}/.sol,{t,0,400}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py=Plot[(y/.y→fp⟦3,2⟧),{t,0,400},PlotStyle→{Dashed,Green}];
pb28=Show[{ppb28,py, Graphics[{Green,Dashed,
Line[{{x/.x→fp⟦3,1⟧,0},{x/.x→fp⟦3,1⟧,1}}]}] },
Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→fp⟦3,1⟧),(y/.y→fp⟦3,2⟧)}]],
{PointSize[Large],Style[Point[{(x/.x→fp⟦3,1⟧),(y/.y→fp⟦3,2⟧)}],Black]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}];
(****************b=28; different initial values   ******************)
ClearParameters;
λ=0.36;β=0.11;δ=0.44;b=28;K=1;γ=1;

ode={x'[t]==x1,y'[t]==y1,v'[t]==v1,x[0]==0.1,y[0]==0.04,v[0]==0.01};
sol=NDSolve[ode,{x,y,v},{t,0,400}];

(*****..Parametric plot conditions***)
x0=0.1; y0=0.04;v0=0.01;
ppb28n=ParametricPlot[{  x[t],(y[t])}/.sol,{t,0,400}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Red}];
pyn=Plot[(y/.y→fp⟦3,2⟧),{t,0,200},PlotStyle→{Dashed,Green}];
pb28n=Show[{ppb28n,pyn, Graphics[{Green,Dashed,
Line[{{x/.x→fp⟦3,1⟧,0},{x/.x→fp⟦3,1⟧,1}}]}] },
Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→fp⟦3,1⟧),(y/.y→fp⟦3,2⟧)}]],
{PointSize[Large],Style[Point[{(x/.x→fp⟦3,1⟧),(y/.y→fp⟦3,2⟧)}],Black]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}]
cy11=Show[{pb28,pb28n},Epilog→{Text["(x*,y*)",
Offset[{10,10},{(x/.x→fp⟦3,1⟧),(y/.y→fp⟦3,2⟧)}]],
{PointSize[Large],Style[Point[{(x/.x→fp⟦3,1⟧),(y/.y→fp⟦3,2⟧)}],Black]},
{PointSize[Large],Point[{0.1,0.04}]},Text["(x0',y0')",Offset[{-10,8},{0.1,0.04}]],
{PointSize[Large],Point[{0.5,0.5}]},Text["(x0,y0)",Offset[{-10,8},{0.5,0.5}]]}]
Export["cy11.pdf",cy11]
```

E*{0.148148, 0.0431317, 2.64672}

Out[749]=



Out[750]=



Out[751]= cy11.pdf

2)Sections 3 and 4(in paper): Deterministic model with Logistic growth (4 dim when ϵ=0)

In[752]:=
```
SetDirectory[NotebookDirectory[]];
AppendTo[$Path, Directory];
Clear["Global`*"];
(*Some aliases*)
Format[μv] := Subscript[μ, v]; Format[μy] := Subscript[μ, y];
parE = {β > 0, λ > 0, γ > 0, δ > 0, μ > 0, μv > 0, b > 1, K > 0, s > 0, c > 0};
cKga1 = {ϵ → 0, K → 1, γ → 1};
cE = {ϵ → 1};
R0 = b β K / (β K + δ) (* Reproduction number*);
cnT17 = {μv → 0.16, μy → 0.48, K → 1, γ → 1, b → 9, β → 0.11,
   λ → 0.36, δ → 0.2, s → 0.6, c → 0.036} (*Numerical values of Tian17*);
```

2-1)Description of the model when ϵ=0 and analysis of the stability of the fixed point when z->0

```
In[•]:= (****** Four dim Deterministic  epidemic model with Logistic growth ****)
x1=λ  x(1-(x+y)/K)-   β x v ;
y1=β x v -μy y z- γ y;
v1=-β x v - μv  v z+ b  γ y - δ v;
z1=z(s  y - c );
ye=c/s; vM=λ(1-ye)/β;vMN=vM/.cnT17;
dyn={x1,y1,v1,z1};
dyn3={x1,y1,v1}/.z→0;(*3dim case*)
Print[" (x' y' v' z')=",dyn//FullSimplify//MatrixForm, "  the reparametrized model is (x' y' v' z')=",
 dyn//.cKga1//FullSimplify//MatrixForm]
Print["b0=",b0=b/.Apart[Solve[R0==1,b][[1]]//FullSimplify]]
(*****Fixed points of Tian17 using the elimination when K=1, γ=1***)

fv=(ye (b-1)-v δ); gv=(ye μy+v μv); hv=(1-ye-v β/λ);

Print["xe= ",xe=hv," ze =", ze=fv/gv]

Pv=Numerator[Together[v β xe-ye (1+μy fv/gv)]]/(-s^2 β^2 μv);
Print["P(v)=",pc=Collect[Together[Pv],v]," coefs are"]
coP=CoefficientList[pc,v]//Simplify

(***Fixed point when z→0**)
eq=Thread[dyn3=={0,0,0}];
sol=Solve[eq,{x,y,v}]//FullSimplify;
Es={x,y,v}/.sol[[3]];(*Endemic point with z=0*);
Print["when K=γ=1, E*=",Est=Es/.cKga1//FullSimplify(* E* when K=γ=1***)]

bn=b/.Solve[Est[[2]]==ye,b]; bnn=bn/.cnT17;
bcn=b/.Solve[Est[[2]][[1]]==ye,b];
bcnn=bn/.cnT17;

Dis=Chop[Collect[Discriminant[Numerator[Pv],v],b]];
solb=Solve[Dis==0,b];
Jac=Grad[dyn/.cKga1,{x,y,v,z}]//FullSimplify;
```

$$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -v\,x\,\beta + x\left(1 - \frac{x+y}{K}\right)\lambda \\ v\,x\,\beta - y\left(\gamma + z\,\mu_y\right) \\ b\,y\,\gamma - v\,(x\,\beta + \delta + z\,\mu_v) \\ (-c + s\,y)\,z \end{pmatrix}$$ the reparametrized model is $$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -x\,(v\,\beta + (-1 + x + y)\,\lambda) \\ v\,x\,\beta - y\left(1 + z\,\mu_y\right) \\ b\,y - v\,(x\,\beta + \delta + z\,\mu_v) \\ (-c + s\,y)\,z \end{pmatrix}$$

$$b0 = 1 + \frac{\delta}{K\,\beta}$$

$$xe = 1 - \frac{c}{s} - \frac{v\,\beta}{\lambda} \quad ze = \frac{\frac{(-1+b)\,c}{s} - v\,\delta}{v\,\mu_v + \frac{c\,\mu_y}{s}}$$

$$P(v) = v^3 + \frac{b\,c^2\,\lambda\,\mu_y}{s^2\,\beta^2\,\mu_v} + \frac{v^2\left(c\,s\,\beta\,\lambda\,\mu_v - s^2\,\beta\,\lambda\,\mu_v + c\,s\,\beta^2\,\mu_y\right)}{s^2\,\beta^2\,\mu_v} + \frac{v\left(c\,s\,\lambda\,\mu_v + c^2\,\beta\,\lambda\,\mu_y - c\,s\,\beta\,\lambda\,\mu_y - c\,s\,\delta\,\lambda\,\mu_y\right)}{s^2\,\beta^2\,\mu_v}$$ coefs are

Out[∘]= $$\left\{\frac{b\,c^2\,\lambda\,\mu_y}{s^2\,\beta^2\,\mu_v}, \frac{c\,\lambda\left(c\,\beta\,\mu_y + s\left(\mu_v - (\beta + \delta)\,\mu_y\right)\right)}{s^2\,\beta^2\,\mu_v}, \frac{c\,\lambda\,\mu_v - s\,\lambda\,\mu_v + c\,\beta\,\mu_y}{s\,\beta\,\mu_v}, 1\right\}$$

when K=γ=1, $E* = \left\{\frac{\delta}{(-1+b)\,\beta}, \frac{((-1+b)\,\beta - \delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,\beta + \delta\,\lambda)}, \frac{((-1+b)\,\beta - \delta)\,\lambda}{\beta\,((-1+b)\,\beta + \delta\,\lambda)}\right\}$

2-2)Interior equilibrium

Analysis of the stability of the interior point E_*:

In[∘]:=
```
Jac3=Grad[dyn3/.cKga1,{x,y,v}]//FullSimplify;
bcrit=1+δ/(β );(*Reduce[Join[{R0>1},pars],δ]*)
Print["J(E_*) is"]
Jst=(Jac3/.x→Est[[1]]/.y→Est[[2]]/.v→Est[[3]])//FullSimplify;Jst//MatrixForm
Trs=Tr[Jst];
pc=Collect[Det[ψ IdentityMatrix[3]-Jst],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Print["a₁=",a1=coT[[3]], ", a₂=",a2=coT[[2]], ", a₃=",a3=coT[[1]]]
H2=a1*a2-a3;
Print["H2(b0)=",H2/.b→bcrit//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H2]]/.cKga1//FullSimplify]
ϕb=Collect[Numerator[Together[H2]]/(δ λ),b]/.cKga1;
cofi=CoefficientList[ϕb,b](*Coefficients of ϕ(b)*);
Print["value of ϕ(b) at crit b is "]
ϕb/.b→bcrit/.cKga1//FullSimplify
```

J(E_*) is

Out[∘]//MatrixForm=
$$\begin{pmatrix} \frac{\delta\,\lambda}{\beta - b\,\beta} & \frac{\delta\,\lambda}{\beta - b\,\beta} & -\frac{\delta}{-1+b} \\ \frac{((-1+b)\,\beta - \delta)\,\lambda}{(-1+b)\,\beta + \delta\,\lambda} & -1 & \frac{\delta}{-1+b} \\ \frac{(\beta - b\,\beta + \delta)\,\lambda}{(-1+b)\,\beta + \delta\,\lambda} & b & \frac{b\,\delta}{1-b} \end{pmatrix}$$

$$a_1=\frac{\beta\,(-1+b+b\,\delta)+\delta\,\lambda}{(-1+b)\,\beta},\quad a_2=\frac{\delta\,\lambda\,\left((-1+b)\,\beta\,(-1+\beta+\delta+b\,(1-\beta+\delta))+\left((-1+b)^2\,\beta+b\,\delta^2\right)\lambda\right)}{(-1+b)^2\,\beta\,((-1+b)\,\beta+\delta\,\lambda)},\quad a_3=\delta\left(1+\frac{\delta}{\beta-b\,\beta}\right)\lambda$$

H2(b0) = $(1+\beta+\delta)\,\lambda\,(1+\beta+\delta+\lambda)$

Denominator of H2 is $(-1+b)^3\,\beta^2\,((-1+b)\,\beta+\delta\,\lambda)$

value of ϕ(b) at crit b is

Out[∘]=
$$\frac{\delta^3\,(1+\beta+\delta)\times(1+\lambda)\times(1+\beta+\delta+\lambda)}{\beta}$$

Numerical values:

In[∘]:=
```
cn=cnT17;
cb=NSolve[(ϕb//.Drop[cnT17,{5}])==0,b,WorkingPrecision→20]
bM=Max[Table[Re[b/.cb[[i]]],{i,Length[cb]}]];
Print["bH=",bH=N[bM,30]]
PR=Solve[Pv==0,v];
 vn=v/.PR[[2]];vi=v/.PR[[3]]; vp=v/.PR[[1]];
Chop[{vn,vi,vp}/.cnT17];
PR=Solve[Pv==0,v];
 vn=v/.PR[[2]]; vp=v/.PR[[1]];vi=v/.PR[[3]];
Print["b0=",b0/.cn//N, " , b1=", bnn[[1]],   ", b2=", bnn[[2]], "  ,bH=",bH]
PRN=Chop[PR//.cn//N](*values of the roots v*);
Print["E*",Es=Es/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

Jiv=Jac//.{x→xe,y→ye,z→ze};
JEi=Jiv/.v→vi//.cn//N//FullSimplify;
JEp=Jiv/.v→vp//.cn//N//FullSimplify;
JEif=Jac/.x→1/.y→0/.z→0/.v→0//.cn//N//FullSimplify;
Print["Eigv of E*:",Append[Eigenvalues[Jst]//.cn//N,Es[[2]]-ye/.cnT17]," Eigv of E+:",
Re[Eigenvalues[JEp]//N], " Eigv of Ei:",Re[Eigenvalues[JEi]//N],
"   , Eigv of Eif:",Eigenvalues[JEif]//N]
```

NSolve: The precision of the argument function ({$-0.0056848+0.0319512\,b-0.0515086\,b^2+0.0279268\,b^3-0.001331\,b^4$}) is less than WorkingPrecision (20).

Out[∘]= {{b → 0.29905192792090222818}, {b → 0.83532939126460210381 − 0.23115561298178191540 i},
{b → 0.83532939126460210381 + 0.23115561298178191540 i}, {b → 19.012107471368075382}}

bH=19.012107471368075382

b0=2.81818 , b1=3.58676, b2=8.66779 ,bH=19.012107471368075382

E*{0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

Eigv of E*:{-1.25056, -0.0281268 - 0.20904 i, -0.0281268 + 0.20904 i, -0.00155844}
Eigv of E+:{-1.29833, -0.0332218, -0.0332218, 0.000834552}
Eigv of Ei:{-1.69849, -0.076539, -0.076539, -0.00803072}
, Eigv of Eif:{-1.7081, 0.398103, -0.36, -0.036}

3D- Plot of the dynamic:

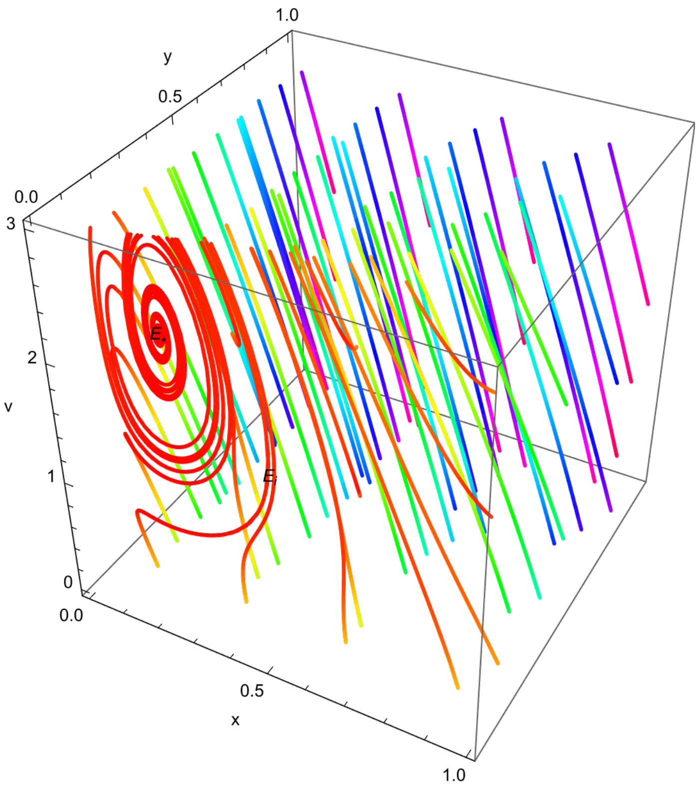
In[ ]:=
```
cn=cnT17;
Print["E*",Es=Es/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]
epi={Text["E*",Offset[{-10,10},{Es[[1]],Es[[2]]}]]//.cn,
{PointSize[Large],Style[Point[{Es[[1]],Es[[2]]}],Blue]//.cn},Text["Ei",Offset[{0,10},
{Ei[[1]],Ei[[2]]}]],{PointSize[Large],Style[Point[{Ei[[1]],Ei[[2]]}],Purple]},
Text["E+",Offset[{0,10},{Ep[[1]],Ep[[2]]}]],{PointSize[Large],
Style[Point[{Ep[[1]],Ep[[2]]}],Orange]}};
sp3=StreamPlot3D[dyn3//.cnT17,{x,0,1},{y,0,1},{v,0,3},AxesLabel→{"x","y","v"},
 StreamColorFunction→Hue,PlotRange→All];
sp3D=Show[{sp3},Graphics3D[Text[Style["E*",Black,Thick],Es//.cn],
{PointSize[0.06],Style[Point[Es],Black]}],Graphics3D[Text[Style["Ei",Black,Thick],
Drop[Ei,{4}]//.cn],{PointSize[0.06],Style[Point[Drop[Ei,{4}]],Black]}]]
Export["sp3D.pdf",sp3D]
```

E*{0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

Out[ ]=



Out[ ]= sp3D.pdf

3)Sections 3 and 4(in paper):Figures used in the manuscript (*Run the previous cell*)

Numerical illustrations when ϵ=0 (Bifurcations diagrams, parametric plots,

and 3D plot)

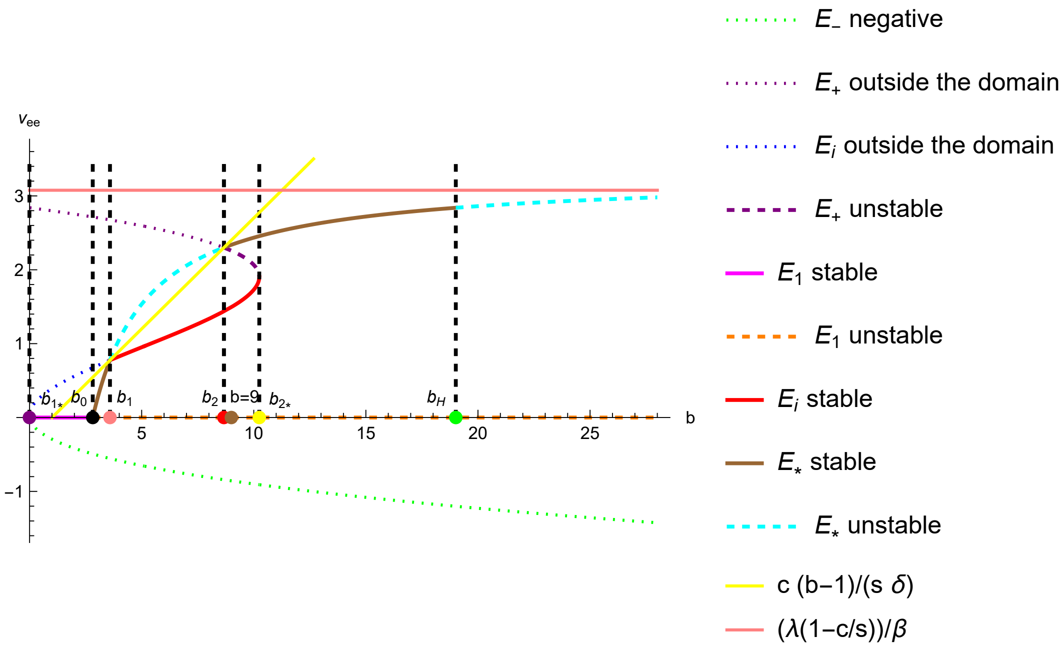
Bifurcation diagram when b varies:

In[ ]:=
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=9;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;
Print["{b2*,b1*}=",{b/.solb⟦1⟧,b/.solb⟦2⟧} ,", and b0=",b0, "  , bH=",bH,
 " ,b1=", bnn⟦1⟧,   ", b2=", bnn⟦2⟧]
Clear["b"];
vs= ((-1+b) β-δ) λ / (β ((-1+b) β+δ λ)) (*v of E* *  ****);
bL=28; max=3.5;
bc2=b/.solb⟦1⟧;
bc1=b/.solb⟦2⟧;
lin1=Line[{{ bc1,0},{ bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{ b0,0},{ b0,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
linH=Line[{{ bH,0},{ bH,max}}];
liH=Graphics[{Thick,Black,Dashed,linH}];
linb1=Line[{{ bnn⟦1⟧,0},{ bnn⟦1⟧,max}}];
lib1=Graphics[{Thick,Black,Dashed,linb1}];
linb2=Line[{{ bnn⟦2⟧,0},{ bnn⟦2⟧,max}}];
lib2=Graphics[{Thick,Black,Dashed,linb2}];
linb9=Line[{{ 9,0},{9,max}}];
lib9=Graphics[{Thick,Black,Dashed,linb2}];
pn=Plot[{vn},{b,0,bL},PlotStyle→{Green,Dotted},PlotRange→All,PlotLegends→{"E_ negative"}];
p0=Plot[0,{b,0,bL},PlotStyle→{Brown,Thick},PlotRange→All,PlotLegends→{"E0 saddle point"}];
ppa=Plot[{vp},{b,0, bnn⟦2⟧},PlotStyle→{Purple,Dotted},PlotRange→All,
PlotLegends→{"E+ outside the domain"}];
ppb=Plot[{vp},{b,bnn⟦2⟧, bL},PlotStyle→{Purple,Thick,Dashed},
PlotRange→All,PlotLegends→{"E+ unstable"}];
pi1=Plot[{vi},{b,0,bnn⟦1⟧},PlotStyle→{Blue,Dotted},
PlotRange→All,PlotLegends→{"Ei outside the domain"}];
pi2=Plot[{vi},{b,bnn⟦1⟧,bL},PlotStyle→{Red,Thick},
PlotRange→All,PlotLegends→{"Ei stable"}];
pi3=Plot[{vi},{b,bnn⟦2⟧,bL},PlotStyle→{Blue,Thick,Dashed},
PlotRange→All(*,PlotLegends→{"Ei unstable"}*)];
ps1=Plot[{vs},{b,b0,bnn⟦1⟧},PlotStyle→{Brown,Thick},
PlotRange→{{0,bL},{0,max}},PlotLegends→{"E* stable"}];
ps2=Plot[{vs},{b,bnn⟦1⟧,bnn⟦2⟧},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{0,bL},{0,max}},PlotLegends→{"E* unstable"}];
ps3=Plot[{vs},{b,bnn⟦2⟧,bH},PlotStyle→{Brown,Thick},
PlotRange→{{0,bL},{0,max}}(*,PlotLegends→{"E* stable"}*)];
ps4=Plot[{vs},{b,bH,bL},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{0,bL},{0,max}}(*,PlotLegends→{"E* unstable"}*)];
pdf1=Plot[{0},{b,0,b0},PlotStyle→{Magenta, Thick},
PlotRange→{{0,200},{0,max}},PlotLegends→{"E1 stable"}];
pdf2=Plot[{0},{b,b0,bL},PlotStyle→{Orange, Thick,Dashed},
PlotRange→{{0,200},{0,max}},PlotLegends→{"E1 unstable"}];
pvmax=Plot[{c (b-1)/(s δ),(λ(1-c/s))/β},{b,0,bL},PlotStyle→{Yellow, Pink},
PlotRange→{{0,200},{0,max}},
```

```
PlotLegends→{"c (b-1)/(s δ)","(λ(1-c/s))/β"}];

bifT=Show[{pn,ppa,pi1,ppb,pdf1,pdf2,pi2,ps1,ps2,ps3,ps4,li1,li2,li3,
lib2,lib1,lib9,pvmax,liH},
Epilog→{Text["b1*",Offset[{12,10},{ bc1,0}]],{PointSize[Large],
Style[Point[{ bc1,0}],Purple]},
Text["b2*",Offset[{11,10},{ bc2,0}]],{PointSize[Large],Style[Point[{ bc2,0}],Yellow]},
 Text["b0",Offset[{-7,11},{ b0,0}]],{PointSize[Large],Style[Point[{ b0,0}],Black]},
Text["bH",Offset[{-10,11},{ bH,0}]],{PointSize[Large],Style[Point[{ bH,0}],Green]},
 Text["b1",Offset[{8,11},{ bnn[[1]],0}]],{PointSize[Large],Style[Point[{ bnn[[1]],0}],Pink]},
Text["b2",Offset[{-7,11},{ bnn[[2]],0}]],{PointSize[Large],Style[Point[{ bnn[[2]],0}],Red]},
Text["b=9",Offset[{7,11},{9,0}]],{PointSize[Large],Style[Point[{ 9,0}],Brown]}},
PlotRange→All,AxesLabel→{"b","vee"}]
Export["BiifT17.pdf",bifT]
```

$\{b_{2*},b_{1*}\}=\{10.2462, -0.00697038\}$, and $b_0=$
2.81818 , b_H=19.012107471368075382 ,b1=3.58676, b2=8.66779

Out[=]=



Out[=]= BiifT17.pdf

Parametric plots at the intervals of stability:

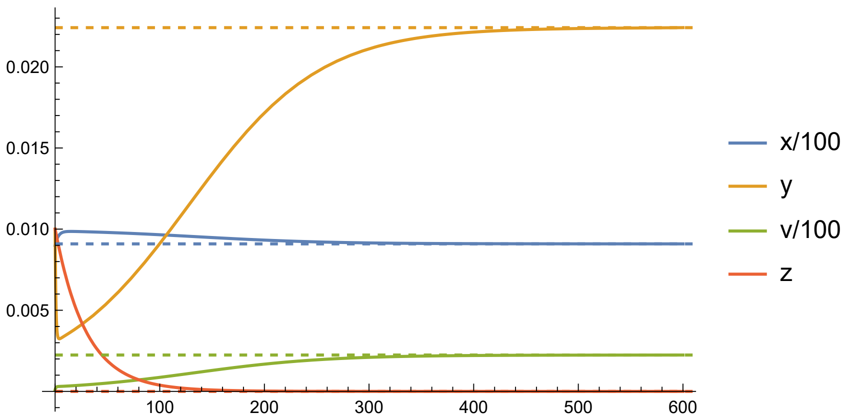
When b0<b=3<b1:

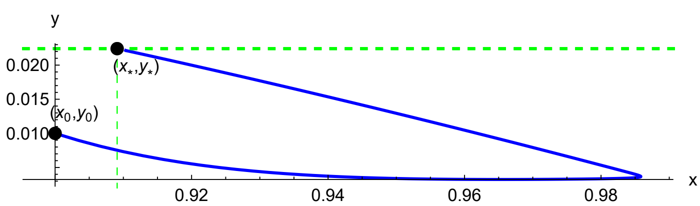
In[∘]:=
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=3;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;

Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]
x1=λ  x[t](1-(x[t]+y[t])/K)-   β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv  v[t]×z[t]+ b  γ y[t] - δ v[t];
z1=z[t](s  y[t] - c );
ode1={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol01=NDSolve[ode1,{x,y,v,z},{t,0,500}];
pdy1=Plot[{x[t]/100/. sol01,y[t]/. sol01,v[t]/100/. sol01,z[t]/. sol01},
{t,0,600},PlotLegends→{"x/100","y","v/100","z"}];
pEs1=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
Dyn01=Show[pdy1,pEs1]

(*****..Parametric plot conditions***)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb3=ParametricPlot[{  x[t],(y[t])}/.sol01,{t,0,400}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py3=Plot[y/.y→Es[[2]],{t,0,400},PlotStyle→{Dashed,Green}];
pb3=Show[{ppb3,py3, Graphics[{Green,Dashed,Line[{{x/.x→Es[[1]],0},{x/.x→Es[[1]],1}}]}] },
Epilog→{{Thick,Text["(x*,y*)",Offset[{10,-10},{x/.x→Es[[1]],y/.y→Es[[2]]}]],
{PointSize[Large],Style[Point[{x/.x→Es[[1]],y/.y→Es[[2]]}],Black]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{10,10},{x0,y0}]]}]
Export["pb3.pdf",pb3]
Export["Dyn01.pdf",Dyn01]
```

E*{0.909091, 0.0224159, 0.224159}

E+={0.113348, 0.06, 2.70541, -0.912093}

Ei={0.726116, 0.06, 0.699984, -0.142026}

Out[∘]=


Out[∘]=


Out[ ]= pb3.pdf

Out[ ]= Dyn01.pdf

When b1<b=6<b2:

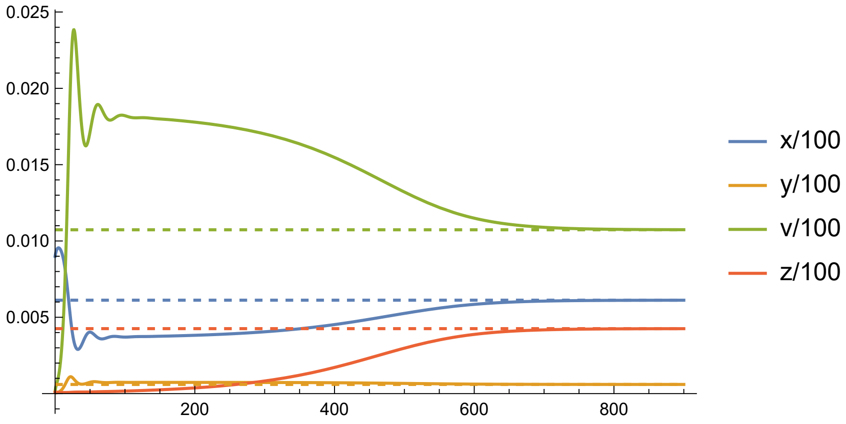
In[ ]:=
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=6;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ  x[t](1-(x[t]+y[t])/K)-   β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv  v[t]×z[t]+ b  γ y[t] - δ v[t];
z1=z[t](s  y[t] - c );
ode2={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol12=NDSolve[ode2,{x,y,v,z},{t,0,1000}];
pdy1=Plot[{x[t]/100/. sol12,y[t]/100/. sol12,v[t]/100/. sol12,z[t]/100/. sol12},
{t,0,900},PlotLegends→{"x/100","y/100","v/100","z/100"}];
pEs1=Plot[{x/100/.x→Ei[[1]],y/100/.y→Ei[[2]],v/100/.v→Ei[[3]],z/100/.z→Ei[[4]]},{t,0,900},
PlotStyle→{Dashed}];
Dyn12=Show[pdy1,pEs1]
(*****..Parametric plot conditions***)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
startP=Epilog→{{PointSize[Large],Point[{0.9,0.01}]},Text["(x0,y0)",
Offset[{0,10},{0.9,0.01}]]};
bnd =Thread[{x[0],y[0],v[0],z[0]}=={x0,y0,v0,z0}](*Starting point of the Paramateric Plot**);
ppb6=ParametricPlot[{  x[t],(y[t])}/.sol12,{t,0,900}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py6=Plot[(y/.y→Ei[[2]]),{t,0,1000},PlotRange→{{0,0.612},{0,0.08}},PlotStyle→{Dashed,Green}];
pb6=Show[{ppb6,py6, Graphics[{Green,Dashed,Line[{{x/.x→Ei[[1]],0},{x/.x→Ei[[1]],1}}]}] },
Epilog→{{Text["(xi,yi)",Offset[{10,10},{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}]],
{PointSize[Large],Style[Point[{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}],Black]}},
{PointSize[Large],Point[{0.9,0.01}]},Text["(x0,y0)",Offset[{-10,8},{0.9,0.01}]]}}]
Export["Dyn12.pdf",Dyn12]
Export["pb6.pdf",pb6]
```

E*{0.363636, 0.0736627, 1.84157}

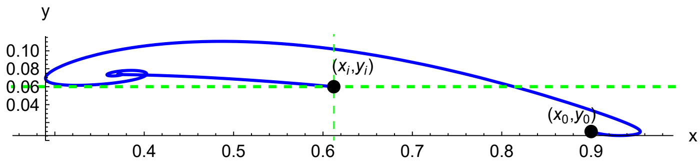
E+={0.166196, 0.06, 2.53245, -0.475792}

Ei={0.612063, 0.06, 1.07325, 0.425644}

Out[∘]=



Out[∘]=



Out[∘]= Dyn12.pdf

Out[∘]= pb6.pdf

When b2<b=9<b2*:

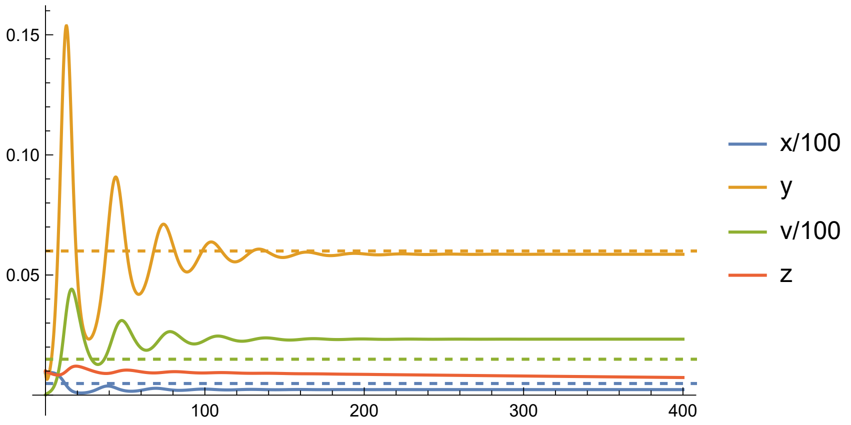
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=9;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ  x[t](1-(x[t]+y[t])/K)-   β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv  v[t]×z[t]+ b  γ y[t] - δ v[t];
z1=z[t](s  y[t] - c );
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,400}];
pdy2=Plot[{x[t]/100/. sol22,y[t]/. sol22,v[t]/100/. sol22,z[t]/. sol22},{t,0,400},
PlotLegends→{"x/100","y","v/100","z"}];
pEs2=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
pEi2=Plot[{x/100/.x→Ei[[1]],y/.y→Ei[[2]],v/100/.v→Ei[[3]],z/.z→Ei[[4]]},{t,0,1000},
PlotStyle→{Dashed}];
Dyni22=Show[pdy2,pEi2]
Dyns22=Show[pdy2,pEs2]

(*****.Parametric plot conditions***)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb9=ParametricPlot[{  x[t],(y[t])}/.sol22,{t,0,200}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py9=Plot[(y/.y→Es[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb9=Show[{ppb9,py9, Graphics[{Green,Dashed,Line[{{x/.x→Es[[1]],0},{x/.x→Es[[1]],1}}]}] },
Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),(y/.y→Es[[2]])}]],{PointSize[Large],
Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Black]}},{PointSize[Large],Point[{0.9,0.01}]},
Text["(x0,y0)",Offset[{-10,8},{0.9,0.01}]]}]
Export["pb9.pdf",pb9]
Export["Dyni22.pdf",Dyni22]
Export["Dyns22.pdf",Dyns22]
```

E*{0.227273, 0.0584416, 2.33766}

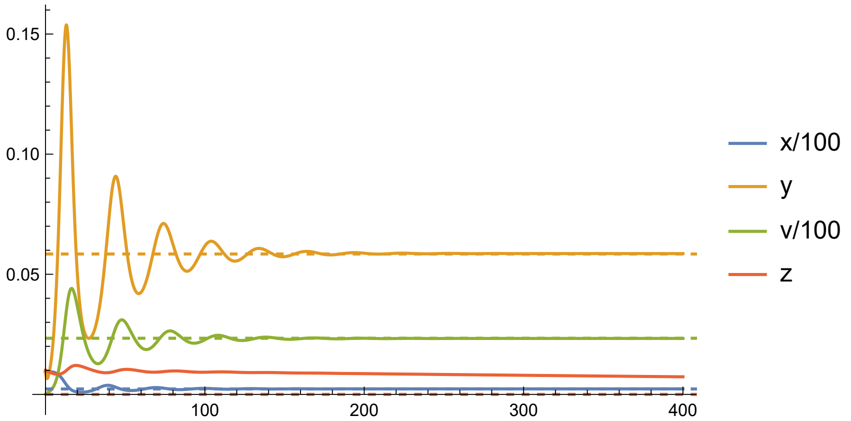
E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

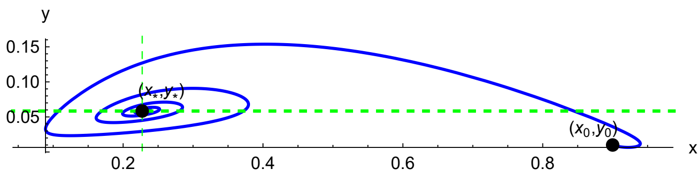
Out[•]=



Out[•]=



Out[•]=



Out[•]= `pb9.pdf`

Out[•]= `Dyni22.pdf`

Out[•]= `Dyns22.pdf`

When b2<b=9<b2* and different initial values :

In[ ]:=

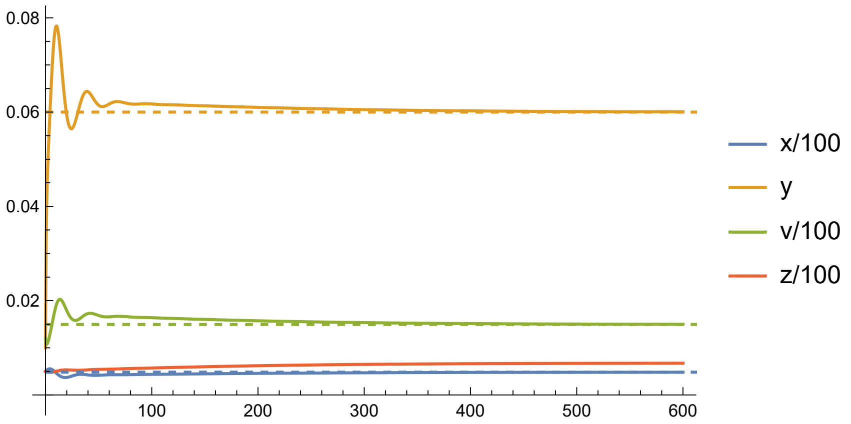
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=9;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ  x[t](1-(x[t]+y[t])/K)-   β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv  v[t]×z[t]+ b  γ y[t] - δ v[t];
z1=z[t](s  y[t] - c );
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.5,y[0]==0.01,v[0]==1.2,
z[0]==0.5};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,1000}];
pdy2=Plot[{x[t]/100/. sol22,y[t]/. sol22,v[t]/100/. sol22,z[t]/100/. sol22},{t,0,600},
PlotLegends→{"x/100","y","v/100","z/100"}];
pEs2=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
pEi2=Plot[{x/100/.x→Ei[[1]],y/.y→Ei[[2]],v/100/.v→Ei[[3]],z/.z→Ei[[4]]},{t,0,1000},
PlotStyle→{Dashed}];
Dyni22b=Show[pdy2,pEi2]
Dyns22=Show[pdy2,pEs2]

(*****..Parametric plot conditions***)
x0=0.5; y0=0.01;v0=1.2;z0=0.5;
ppb9=ParametricPlot[{  x[t],(y[t])}/.sol22,{t,0,500}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py9=Plot[(y/.y→Ei[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb9i=Show[{ppb9,py9, Graphics[{Green,Dashed,Line[{{x/.x→Ei[[1]],0},{x/.x→Ei[[1]],1}}]}] },
Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}]],{PointSize[Large],
Style[Point[{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}],Black]}},{PointSize[Large],Point[{0.9,0.01}]},
Text["(x0,y0)",Offset[{-10,8},{0.9,0.01}]]}}]
Export["pb9i.pdf",pb9i]
Export["Dyni22b.pdf",Dyni22b]
```

E*{0.227273, 0.0584416, 2.33766}

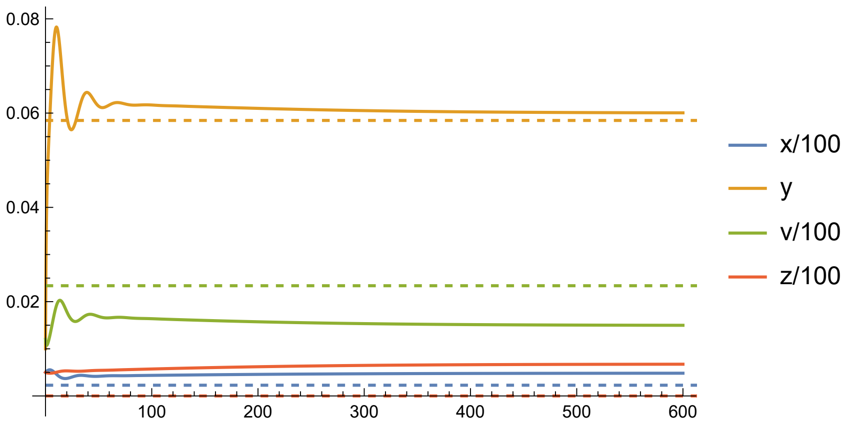
E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

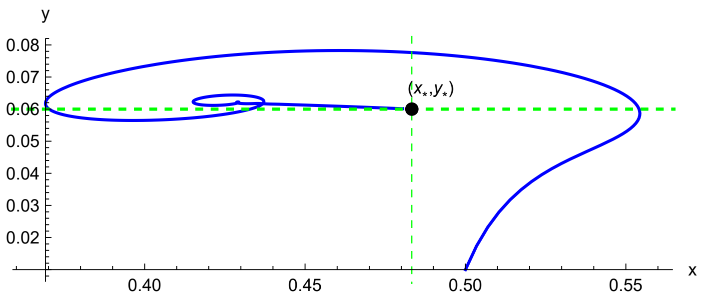
Out[ ]=



Out[ ]=



Out[ ]=



Out[ ]= pb9i.pdf

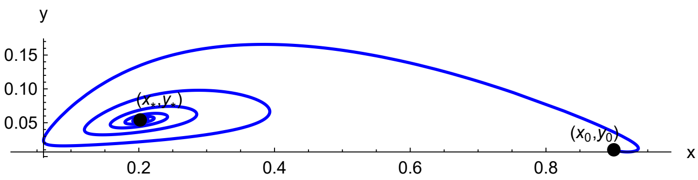
Out[ ]= Dyni22b.pdf

When b2<b=10<b2*:

```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=10;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ  x[t](1-(x[t]+y[t])/K)-   β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv  v[t]×z[t]+ b  γ y[t] - δ v[t];
z1=z[t](s  y[t] - c );

ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,400}];


(*****.Parametric plot conditions***)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb10=ParametricPlot[{  x[t],(y[t])}/.sol22,{t,0,200}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py10=Plot[(y/.y→Es[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb10=Show[{ppb10},Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),
(y/.y→Es[[2]])}]],{PointSize[Large],Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Black]}},
{PointSize[Large],Point[{0.9,0.01}]},Text["(x0,y0)",Offset[{-10,8},{0.9,0.01}]]}]

Export["pb10.pdf",pb10]
```

```
E*{0.20202, 0.0541003, 2.43451}

E+={0.30876, 0.06, 2.06588, 0.352938}

Ei={0.411479, 0.06, 1.72971, 0.635108}
```

Out[•]=



Out[•]= pb10.pdf

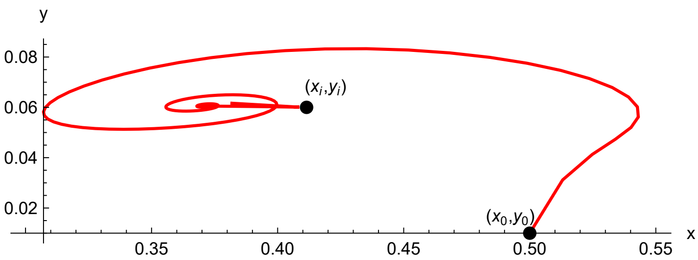
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=10;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ  x[t](1-(x[t]+y[t])/K)-   β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv  v[t]×z[t]+ b  γ y[t] - δ v[t];
z1=z[t](s  y[t] - c );
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.5,y[0]==0.01,v[0]==1.2,
z[0]==0.5};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,1000}];
x0=0.5; y0=0.01;v0=1.2;z0=0.5;
ppb10i=ParametricPlot[{  x[t],(y[t])}/.sol22,{t,0,1900}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Red}];
py10i=Plot[(y/.y→Ei[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb10i=Show[{ppb10i },Epilog→{{Text["(xi,yi)",Offset[{10,10},{(x/.x→Ei[[1]]),
(y/.y→Ei[[2]])}]],
{PointSize[Large],Style[Point[{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}],Black]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}]
pp10si=Show[{pb10,pb10i},Epilog→{{Text["(xi,yi)",Offset[{10,10},{(x/.x→Ei[[1]]),
(y/.y→Ei[[2]])}]],{PointSize[Large],Style[Point[{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}],Black]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0i,y0i)",Offset[{-10,8},{x0,y0}]],
{Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),(y/.y→Es[[2]])}]],{PointSize[Large],
Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Black]}},{PointSize[Large],Point[{0.9,0.01}]},
Text["(x0*,y0*)",Offset[{-10,8},{0.9,0.01}]]}]
Export["pp10si.pdf",pp10si]
Export["pb10i.pdf",pb10i]
```

E⋆{0.20202, 0.0541003, 2.43451}

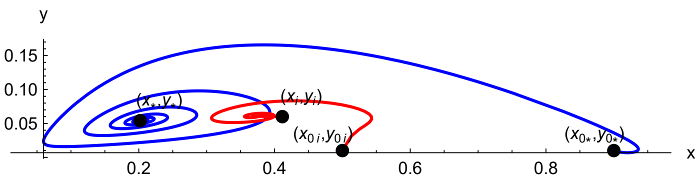
E+={0.30876, 0.06, 2.06588, 0.352938}

Ei={0.411479, 0.06, 1.72971, 0.635108}

Out[•]=



Out[•]=



Out[•]= pp10si.pdf

Out[•]= pb10i.pdf

When b2*<b=15<bH:

In[•]:=

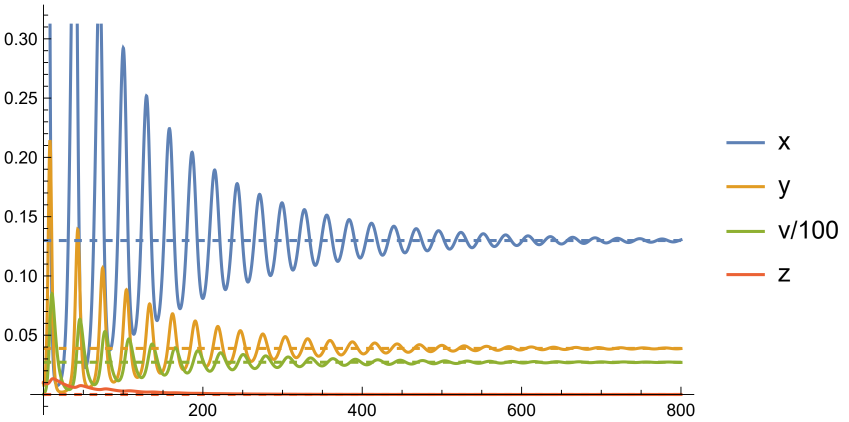
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=15;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;
Print["E⋆",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ  x[t](1-(x[t]+y[t])/K)-   β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv  v[t]×z[t]+ b  γ y[t] - δ v[t];
z1=z[t](s  y[t] - c );
ode4={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol2H=NDSolve[ode4,{x,y,v,z},{t,0,800}];
pdy2H=Plot[{x[t]/. sol2H,y[t]/. sol2H,v[t]/100/. sol2H,z[t]/. sol2H},{t,0,800},
PlotLegends→{"x","y","v/100","z"}];
pEs2H=Plot[{x/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,800},
PlotStyle→{Dashed}];
Dyn2H=Show[pdy2H,pEs2H,PlotRange→All]

(*****..Parametric plot conditions***)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb15=ParametricPlot[{  x[t],(y[t])}/.sol2H,{t,0,500}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py15=Plot[(y/.y→Es[[2]]),{t,0,400},PlotStyle→{Dashed,Green}];
pb15=Show[{ppb15,py15, Graphics[{Green,Dashed,Line[{{x/.x→Es[[1]],0},
{x/.x→Es[[1]],1}}]}] },Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),
(y/.y→Es[[2]])}]],{PointSize[Large],Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Black]}}
,{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}]
Export["pb15.pdf",pb15]
Export["Dyn2H.pdf",Dyn2H]
```

E⋆{0.12987, 0.0388644, 2.72051}

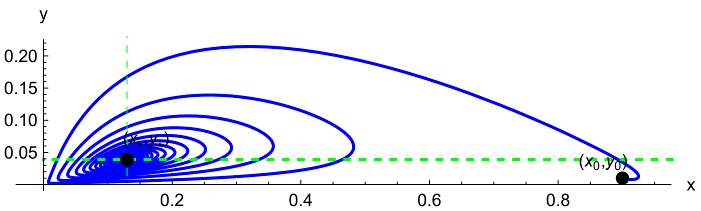
E+={0.332366 + 0.217051 ⅈ, 0.06, 1.98862 - 0.710347 ⅈ, 1.03002 + 0.746836 ⅈ}

Ei={0.332366 - 0.217051 ⅈ, 0.06, 1.98862 + 0.710347 ⅈ, 1.03002 - 0.746836 ⅈ}

Out[ ]=



Out[ ]=



Out[ ]= pb15.pdf

Out[ ]= Dyn2H.pdf

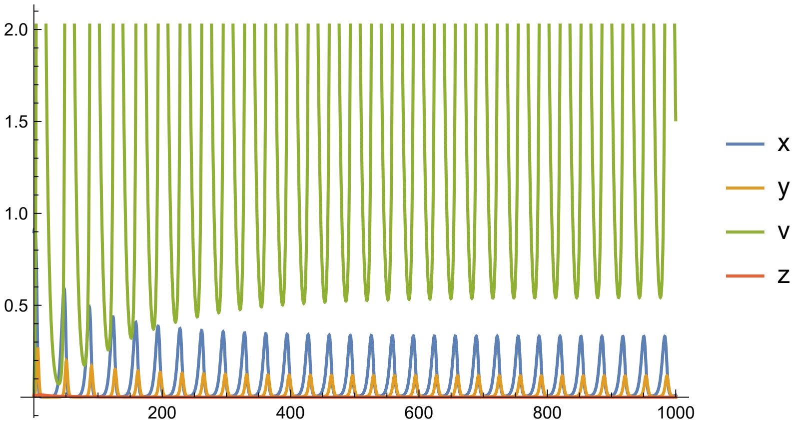
When bH<b=23<b∞:

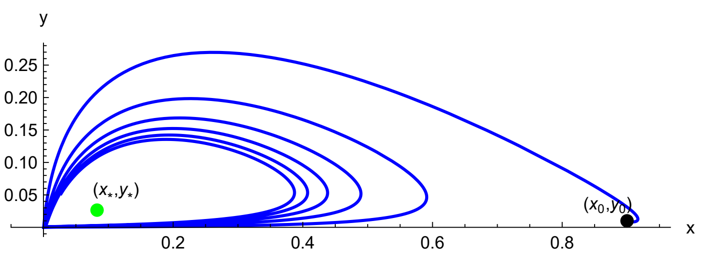
In[ ]:=
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=23;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ϵ=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]
x1=λ  x[t](1-(x[t]+y[t])/K)-   β x[t]×v[t]  ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv  v[t]×z[t]+ b  γ y[t] - δ v[t];
z1=z[t](s  y[t] - c );
ode5={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
solHI=NDSolve[ode5,{x,y,v,z},{t,0,10000}];
pdyHI=Plot[{x[t]/. solHI,y[t]/. solHI,v[t]/. solHI,z[t]/. solHI},{t,0,1000},
PlotLegends→{"x","y","v","z"}];
DynHI=Show[pdyHI,PlotRange→All]

(*****..Parametric plot conditions***)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb23=ParametricPlot[{  x[t],(y[t])}/.solHI,{t,0,200}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
NDSolve[{y'[t]==y[t]×(x[t]-1),x'[t]==x[t] (2-y[t]),x[0]==1,y[0]==2.7},{x,y},{t,0,10}];
py23=Plot[(y/.y→Es[[2]]),{t,0,400},PlotStyle→{Dashed,Green}];
pb23=Show[{ppb23},Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),
(y/.y→Es[[2]])}]],{PointSize[Large],Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Green]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}}]
Export["pb23.pdf",pb23]
Export["DynHI.pdf",DynHI]
```

E*{0.0826446, 0.0265046, 2.91551}

E+={0.297813 + 0.339863 i, 0.06, 2.1017 - 1.11228 i, 1.75115 + 1.463 i}

Ei={0.297813 - 0.339863 i, 0.06, 2.1017 + 1.11228 i, 1.75115 - 1.463 i}

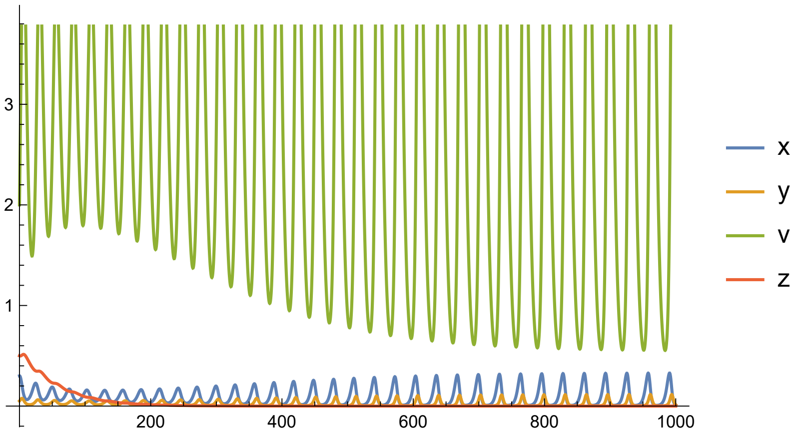
Out[ ]=


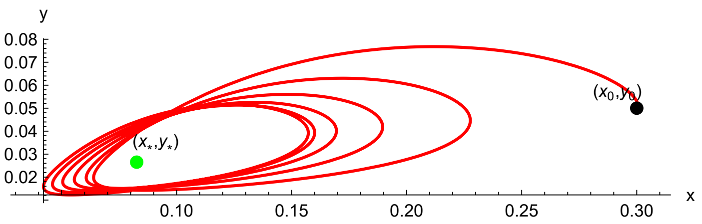
Out[ ]=


Out[ ]= pb23.pdf

Out[ ]= DynHI.pdf

In[ ]:=
```
ode5={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.3,y[0]==0.05,v[0]==2,
z[0]==0.5};
solHI=NDSolve[ode5,{x,y,v,z},{t,0,10000}];
pdyHI=Plot[{x[t]/. solHI,y[t]/. solHI,v[t]/. solHI,z[t]/. solHI},{t,0,1000},
PlotLegends→{"x","y","v","z"}];
DynHIc=Show[pdyHI,PlotRange→All]
(*New initial conditions**)
x0=0.3; y0=0.05;v0=2;z0=0.5;
ppb23c=ParametricPlot[{  x[t],(y[t])}/.solHI,{t,0,150}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Red}];
pb23c=Show[{ppb23c },Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},
{(x/.x→Es⟦1⟧),(y/.y→Es⟦2⟧)}]],{PointSize[Large],Style[Point[{(x/.x→Es⟦1⟧),
(y/.y→Es⟦2⟧)}],Green]}},{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",
Offset[{-10,8},{x0,y0}]]}]
Export["pb23b.pdf",pb23c]
Export["DynHIb.pdf",DynHIc]
pp23cs=Show[{pb23,pb23c},Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},{(x/.x→Es⟦1⟧),
(y/.y→Es⟦2⟧)}]],{PointSize[Large],Style[Point[{(x/.x→Es⟦1⟧),(y/.y→Es⟦2⟧)}],Green]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0',y0')",Offset[{16,6},{x0,y0}]],
{PointSize[Large],Point[{0.9,0.01}]},Text["(x0*,y0*)",Offset[{-10,8},{0.9,0.01}]]}]
Export["pp23s.pdf",pp23cs]
```

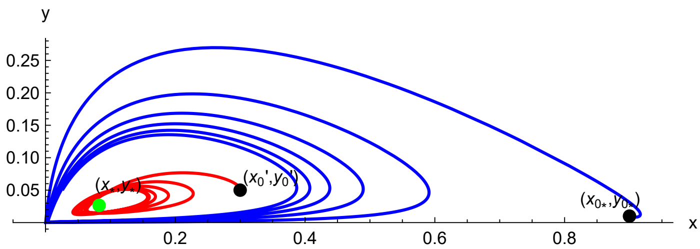
Out[ ]=


Out[ ]=


Out[ ]= pb23b.pdf

Out[ ]= DynHIb.pdf

Out[ ]=



Out[ ]= pp23s.pdf

4)Section 5(in paper): 4-Dim.Viro-therapy model when ϵ=1

4-1)Definition of the model and fixed points when ϵ=1

In[∘]:=
```
SetDirectory[NotebookDirectory[]];
AppendTo[$Path,Directory];
Clear["Global`*"];
Clear["K"];
Format[μv]:=Subscript[μ,v];Format[μy]:=Subscript[μ,y];Format[La]:=Λ;(*La >0*)
pars={β,λ,γ,δ,μy, μv,b,K,s,c};
cpos={β>0,λ>0,γ>0,δ>0,μy>0, μv>0,b>1,K>0,s>0,c>0};
La=0; bL=100;cnb={b→50};
cEri={μy→1/48,K→2139.258, β→.0002,λ→.2062,γ→1/18,δ→.025, μv→2*10^(-8),c→10^(-3),s→.027};
(****** Four dim Deterministic  epidemic model with Logistic growth ****)
x1=La+λ  x(1-(x+y)/K)-   β x v ;
y1=β x v -μy y z- γ y;
v1=-β x v - μv  v z+ b  γ y - δ v;
z1=z(s  y - c z);
x1s=λ  (1-(x+y)/K)-   β  v ;
z1s=s y - c z;
dyns={x1s,y1,v1,z1s};
dyn={x1,y1,v1,z1};
dyn3={x1,y1,v1}/.z→0;
          x'
Print[" ( y' ) =",dyn//FullSimplify//MatrixForm]
          v'
          z'

(*Jacobian*)
Jac=Grad[(dyn),{x,y,v,z}]//FullSimplify;
Jac3=Grad[(dyn3),{x,y,v}]//FullSimplify;
det=Det[Jac]//FullSimplify; tr=Tr[Jac]//FullSimplify;
R0=b β K/(β K+δ);bcrit=1+δ/(β K);(*Reduce[Join[{R0>1},pars],δ]*)
(****Endemic points in 3dims **)
eqE=Thread[dyn3=={0,0,0}];
Print["Three Fixed points in 3-dim case:",solE=Solve[eqE,{x,y,v}]//FullSimplify]

el=Eliminate[Thread[dyns=={0,0,0,0}],{x,v,z}];
Qybyelim=Factor[el[[1,1]]-el[[1,2]]]/y//FullSimplify;
Print["Coefficients of  Qy by elim polynomial are:"]
cof=CoefficientList[Qybyelim,y]//FullSimplify
so=y/.Solve[Qybyelim==0,y];(*Third order roots*)

(*****Fixed points of 4-dim model using P(y)***)
fy=(c γ(b-1)-y μy s);
gy=( μv s y+c δ); hy=(γ +y s μy/c);
xe=hy gy/(β fy); ve=y fy/gy; ze= s y /c;
ys=y/.solE[[3]](* y of E*     ****);

Py=λ(1-y/K)-β y fy/gy-λ hy gy/(β K fy); yb=c γ (b-1)/(μy s);
Qy=λ fy gy(1-  y /K)- λ hy gy^2/(β K)-y β fy^2//FullSimplify;
Qycol=Collect[Qy,y];
Qycoef=CoefficientList[Qycol,y];
(*Print["Check Eric Qy -elim Qy=",Qybyelim+c K β Qy//FullSimplify]*)
Dis=Collect[Discriminant[Qy,y],b];
Discoef=CoefficientList[Dis,b];Length[Discoef];
DisE=Dis//.cEri//N;
```

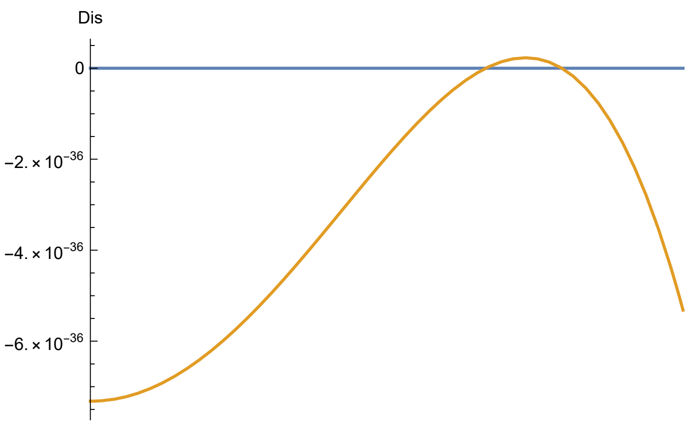
$$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -v\,x\,\beta + x\left(1 - \frac{x+y}{K}\right)\lambda \\ v\,x\,\beta - y\left(\gamma + z\,\mu_y\right) \\ b\,y\,\gamma - v\,(x\,\beta + \delta + z\,\mu_v) \\ z\,(s\,y - c\,z) \end{pmatrix}$$

Three Fixed points in 3-dim case: $\left\{\{x \to 0, y \to 0, v \to 0\}, \{x \to K, y \to 0, v \to 0\}, \left\{x \to \frac{\delta}{(-1+b)\,\beta}, y \to \frac{((-1+b)\,K\,\beta - \delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,K\,\beta\,\gamma + \delta\,\lambda)}, v \to \frac{\gamma\,((-1+b)\,K\,\beta - \delta)\,\lambda}{\beta\,((-1+b)\,K\,\beta\,\gamma + \delta\,\lambda)}\right\}\right\}$

Coefficients of Qy by elim polynomial are:

Out[ ]=
$$\left\{c^3\,\gamma\,\delta\,(K\,(\beta - b\,\beta) + \delta)\,\lambda,\right.$$
$$c^2\left((-1+b)\,c\,\beta\,\gamma\,((-1+b)\,K\,\beta\,\gamma + \delta\,\lambda) + s\,\lambda\left(\gamma\,(K\,(\beta - b\,\beta) + 2\,\delta)\,\mu_v + \delta\,(K\,\beta + \delta)\,\mu_y\right)\right),$$
$$c\,s\left(s\,\lambda\,\mu_v\left(\gamma\,\mu_v + K\,\beta\,\mu_y + 2\,\delta\,\mu_y\right) + c\,\beta\left(-\delta\,\lambda\,\mu_y + (-1+b)\,\gamma\left(\lambda\,\mu_v - 2\,K\,\beta\,\mu_y\right)\right)\right),$$
$$\left.s^2\,\mu_y\left(s\,\lambda\,\mu_v^2 + c\,\beta\left(-\lambda\,\mu_v + K\,\beta\,\mu_y\right)\right)\right\}$$

In[ ]:=
```
Plot[{0,DisE},{b,0,2 bL},AxesLabel→{"b","Dis"}]
Print["Roots of Dis[b]=0 are: ",solbE=Solve[DisE==0,b]]
QR=Solve[Qy==0,y];
Print["critical b from R0=1 is:", bcrit//.cEri//N]
ym= y/.QR[[1]];
yp=  y/.QR[[2]];yi=  y/.QR[[3]];

{Chop[yp],Chop[ym]}//.cEri//Simplify;

jacEK=Jac/.x→K/.y→0/.v→0/.z→0;
Print["J(EK)=", jacEK//MatrixForm]
Print["Eig.val of J(EK) are:", Eigenvalues[jacEK]//FullSimplify]
```

Out[ ]=


Roots of Dis[b]=0 are:
{{b → -468749.}, {b → -468749.}, {b → -159.797}, {b → -132.005}, {b → 133.421}, {b → 159.122}}

critical b from R0=1 is:1.05843

$$J(EK) = \begin{pmatrix} -\lambda & -\lambda & -K\,\beta & 0 \\ 0 & -\gamma & K\,\beta & 0 \\ 0 & b\,\gamma & -K\,\beta - \delta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eig.val of J(EK) are: $\left\{0, \frac{1}{2}\left(-K\,\beta - \gamma - \delta - \sqrt{-4\,\gamma\,(K\,(\beta - b\,\beta) + \delta) + (K\,\beta + \gamma + \delta)^2}\right), \frac{1}{2}\left(-K\,\beta - \gamma - \delta + \sqrt{-4\,\gamma\,(K\,(\beta - b\,\beta) + \delta) + (K\,\beta + \gamma + \delta)^2}\right), -\lambda\right\}$

4-2)Stability of the interior points using Routh Hurwitz:

In[ ]:=

```
jacE=Jac/.x→xe/.v→ve/.z→ze/.y→y;
jacE //FullSimplify//MatrixForm
Print["Det[Jac]=",Det[jacE]//FullSimplify, " , Trace[Jac]=",
Tr[jacE]//FullSimplify]
(*Reduce[Join[{Det[Jac]>0&&x>0&&y>0&&z>0&&v>0},cpos],β](*take so long time**)*)
(*Reduce[Join[{Tr[jacE]<0&&x>0&&y>0&&z>0&&v>0},cpos],β];
poly=Collect[Together[Det[ψ IdentityMatrix[4]-jacE]],ψ];
coe=CoefficientList[poly,ψ];*)
(*Print["Coefficients of the Characteristic polynomial are:",coe]*)
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \lambda + \frac{y\beta\,(c\,(\gamma-b\gamma)+s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v} - \frac{\lambda\left(y+\frac{2\,(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)}\right)}{K} & -\frac{\lambda\,(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{K\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)} & -\frac{(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{(-1+b)\,c\,\gamma-s\,y\,\mu_y} & 0 \\ \frac{y\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v} & -\frac{c\,\gamma+s\,y\,\mu_y}{c} & \frac{(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{(-1+b)\,c\,\gamma-s\,y\,\mu_y} & -y\,\mu_y \\ \frac{y\beta\,(c\,(\gamma-b\gamma)+s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v} & b\,\gamma & \frac{b\,\gamma\,(c\,\delta+s\,y\,\mu_v)}{c\,(\gamma-b\gamma)+s\,y\,\mu_y} & \frac{y\,\mu_v\,(c\,(\gamma-b\gamma)+s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v} \\ 0 & \frac{s^2\,y}{c} & 0 & -s\,y \end{pmatrix}$$

Det[Jac] =

$$-\frac{1}{c^3 K\beta\left((-1+b)\,c\,\gamma-s\,y\,\mu_y\right)^2}\, s\,y^2 \Big(-(-1+b)^2 c^5 \beta\gamma^3 \left((-1+b)\,K\beta\gamma+\delta\lambda\right) + 2\,s^5 y^4 \lambda\,\mu_v^2 \mu_y^3 + c\,s^4$$
$$y^3 \mu_y^2 \left(2\times(3-2\,b)\,\gamma\lambda\,\mu_v^2 + 2\,(-y\beta+\delta)\,\lambda\,\mu_v\,\mu_y + K\beta\,\mu_y\left(\lambda\,\mu_v + 2\,y\beta\,\mu_y\right)\right) -$$
$$c^2 s^3 y^2 \mu_y \Big(6\times(-1+b)\,\gamma^2\lambda\,\mu_v^2 + y\beta\delta\lambda\,\mu_y^2 +$$
$$\gamma\,\mu_y\left((3\times(-1+b)\,K\beta + (6-5\,b)\,y\beta + 6\times(-1+b)\,\delta)\,\lambda\,\mu_v + (-7+6\,b)\,K\,y\,\beta^2\mu_y\right)\Big) -$$
$$c^3 s^2 y\,\gamma \Big(2\times(-1+b)\,\gamma^2\lambda\,\mu_v^2 + \delta\,(3\,y\beta + b\,(K\beta-3\,y\beta+2\,\delta))\,\lambda\,\mu_y^2 +$$
$$\gamma\,\mu_y\left((3\times(-2+b)\times(-1+b)\,y\beta - 6\,\delta + 8\,b\,\delta)\,\lambda\,\mu_v - (-1+b)\times(-3+2\,b)\,K\beta\left(\lambda\,\mu_v + 3\,y\beta\,\mu_y\right)\right)\Big) +$$
$$c^4 s\,\gamma^2 \Big(-2\times(-1+b)\,\gamma\,((-1+b)\,y\beta+\delta)\,\lambda\,\mu_v - \delta\,((-1+b)\times(-3+2\,b)\,y\beta + 2\,b\,\delta)\,\lambda\,\mu_y +$$
$$(-1+b)\,K\beta\left(b\,\delta\lambda\,\mu_y + (-1+b)\,\gamma\left(\lambda\,\mu_v + (5-2\,b)\,y\beta\,\mu_y\right)\right)\Big)\Big)$$

, Trace[Jac] =

$$-s\,y - \gamma - \delta + \lambda - \frac{s\,y\,\mu_v}{c} - \frac{s\,y\,\mu_y}{c} + \frac{y\beta\left(c\,(\gamma-b\gamma)+s\,y\,\mu_y\right)}{c\,\delta+s\,y\,\mu_v} - \frac{(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{(-1+b)\,c\,\gamma-s\,y\,\mu_y} - \frac{\lambda\left(y+\frac{2\,(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)}\right)}{K}$$

4-3)Stability of E* using Routh Hurwitz:

In[ ]:=
```
jacEs=Jac/.x→xe/.v→ve/.z→0/.y→y;
Print["y*=",y/.solE⟦3⟧//FullSimplify]
jacEs //FullSimplify//MatrixForm
Print["Det[Jac]=",Det[jacEs]//FullSimplify, " , Trace[Jac]=",
Tr[jacEs]//FullSimplify]
(*Reduce[Join[{Tr[jacEs]<0&&x>0&&y>0&&v>0},cpos],β];*)
poly=Collect[Together[Det[ψ IdentityMatrix[4]-jacEs]],ψ];
coe=CoefficientList[poly,ψ];
```

$$y* = \frac{((-1+b)\,K\,\beta-\delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} \lambda+\frac{y\,\beta\,(c\,(\gamma-b\,\gamma)+s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v}-\frac{\lambda\left(y+\frac{2\,(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)}\right)}{K} & -\frac{\lambda\,(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{K\,\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)} & -\frac{(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{(-1+b)\,c\,\gamma-s\,y\,\mu_y} & 0 \\ \frac{y\,\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v} & -\gamma & \frac{(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{(-1+b)\,c\,\gamma-s\,y\,\mu_y} & -y\,\mu_y \\ \frac{y\,\beta\,(c\,(\gamma-b\,\gamma)+s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v} & b\,\gamma & \frac{s\,y\,\mu_v}{c}+\frac{b\,\gamma\,(c\,\delta+s\,y\,\mu_v)}{c\,(\gamma-b\,\gamma)+s\,y\,\mu_y} & \frac{y\,\mu_v\,(c\,(\gamma-b\,\gamma)+s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v} \\ 0 & 0 & 0 & s\,y \end{pmatrix}$$

$$\begin{aligned} \mathrm{Det}[\mathrm{Jac}] = \Big(\Big(& s\,y\,\big(y\,\beta\,(c\,\delta+s\,y\,\mu_v)\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)\,(c\,\gamma+s\,y\,\mu_y) \\ & (c\,K\,\beta\,\gamma\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)+\lambda\,(c\,\delta+s\,y\,\mu_v)\,(c\,\gamma+s\,y\,\mu_y))-b\,\gamma\,\lambda\,(c\,\delta+s\,y\,\mu_v)^2\,(c\,\gamma+s\,y\,\mu_y) \\ & (c\,K\,\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)-2\,(c\,\delta+s\,y\,\mu_v)\,(c\,\gamma+s\,y\,\mu_y)+c\,y\,\beta\,(c\,(\gamma-b\,\gamma)+s\,y\,\mu_y))- \\ & (b\,c^2\,\gamma\,\delta+s\,y\,\mu_v\,(c\,\gamma+s\,y\,\mu_y))\,\big(y\,\beta\,\lambda\,(c\,\delta+s\,y\,\mu_v)\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)\,(c\,\gamma+s\,y\,\mu_y)+ \\ & \gamma\,(-c\,K\,\beta\,\lambda\,(c\,\delta+s\,y\,\mu_v)\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)+c\,K\,y\,\beta^2\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)^2+ \\ & \lambda\,(c\,\delta+s\,y\,\mu_v)\,(c\,y\,\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)+2\,(c\,\delta+s\,y\,\mu_v)\,(c\,\gamma+s\,y\,\mu_y)))\big)\big)\Big)\Big)\Big/ \\ & \big(c^2\,K\,\beta\,(c\,\delta+s\,y\,\mu_v)\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)^2\big)\ ,\ \mathrm{Trace}[\mathrm{Jac}]=s \end{aligned}$$

$$y-\gamma-\delta+\lambda+\frac{y\,\beta\,(c\,(\gamma-b\,\gamma)+s\,y\,\mu_y)}{c\,\delta+s\,y\,\mu_v}-\frac{(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{(-1+b)\,c\,\gamma-s\,y\,\mu_y}-\frac{\lambda\left(y+\frac{2\,(c\,\delta+s\,y\,\mu_v)\left(\gamma+\frac{s\,y\,\mu_y}{c}\right)}{\beta\,((-1+b)\,c\,\gamma-s\,y\,\mu_y)}\right)}{K}$$

4-4)Stability of the interior points numerically :

Routh Hurwitz conditions for the stability of E_- (4 dim)

In[ ]:=
```
Jac4=Grad[dyn//.cEri,{x,y,v,z}]//FullSimplify;
bcrit=1+δ/(β K);
Jst=(Jac4/.x→xe/.v→ve/.z→ze/.y→ym)//.cEri;Jst//N//MatrixForm;
Trs=Tr[Jst];
pc=Collect[Det[ψ IdentityMatrix[4]-Jst],ψ];
coT=CoefficientList[pc,ψ](*So long computations*);
```

Routh Hurwitz conditions for the stability of E_*

In[ ]:=
```
cF1={β→87/2,λ→1,γ→1/128,δ→1/2,μy→1,μv→1,K→1/2,s→1,c→1};
cEri=cF1;
Jac3=Grad[dyn3//.cEri,{x,y,v}]//FullSimplify;
bcrit=1+δ/(β K);(*Reduce[Join[{R0>1},pars],δ]*)
Print["J(E_*) is"]
Jst=(Jac3/.x→x/.solE⟦3⟧/.y→ys/.v→v/.solE⟦3⟧)//.cEri//FullSimplify;Jst//MatrixForm

Trs=Tr[Jst];
pc=Collect[Det[ψ IdentityMatrix[3]-Jst],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Print["a1=",a1=coT⟦3⟧//.cEri, ", a2=",a2=coT⟦2⟧//.cEri, ", a3=",a3=coT⟦1⟧//.cEri]
H2=a1*a2-a3;
Print["H2(b0)=",H2/.b→bcrit//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H2]]/.cEri//FullSimplify]
ϕb=Collect[Numerator[Together[H2]]/(δ λ),b]/.cEri;
cofi=CoefficientList[ϕb,b](*Coefficients of ϕ(b)*);
Print["value of ϕ(b) at crit b is "]
ϕb/.b→bcrit/.cEri//N
cb=NSolve[(H2//.cEri)==0,b,WorkingPrecision→20]
bM=Max[Table[Re[b/.cb⟦i⟧],{i,Length[cb]}]];
Print["bH=",bH=N[bM,30]]
```

J(E_*) is

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{2}{87-87\,b} & \frac{2}{87-87\,b} & \frac{1}{2-2\,b} \\ 1-\frac{258}{169+87\,b} & -\frac{1}{128} & \frac{1}{2\times(-1+b)} \\ -1+\frac{258}{169+87\,b} & \frac{b}{128} & \frac{b}{2-2\,b} \end{pmatrix}$$

$$a_1=\frac{65}{128}+\frac{91}{174\times(-1+b)},\quad a_2=\frac{-236\,553+(478\,354-225\,417\,b)\,b}{5568\,(-1+b)^2\,(169+87\,b)},\quad a_3=\frac{87-\frac{2}{-1+b}}{22\,272}$$

$$\mathrm{H2}(\mathrm{b0})=-\frac{1}{256}+\frac{K\,\beta\left(-\frac{13\,695\,976\,827}{256\,K\,\beta+87\,\delta}+\frac{5\,963\,776\,K^2\,\beta^2+13\,781\,248\,K\,\beta\,\delta-76\,919\,571\,\delta^2}{\delta^3}\right)}{992\,083\,968}$$

Denominator of H2 is $62\,005\,248\,(-1+b)^3\,(169+87\,b)$

value of ϕ(b) at crit b is

Out[ ]= 2.01216×10^8

Out[ ]= {{b → -61.53327023855142092}, {b → 1.3349402618015729010}, {b → 0.7834975171549898868}, {b → 0.0013985515488811205258}}

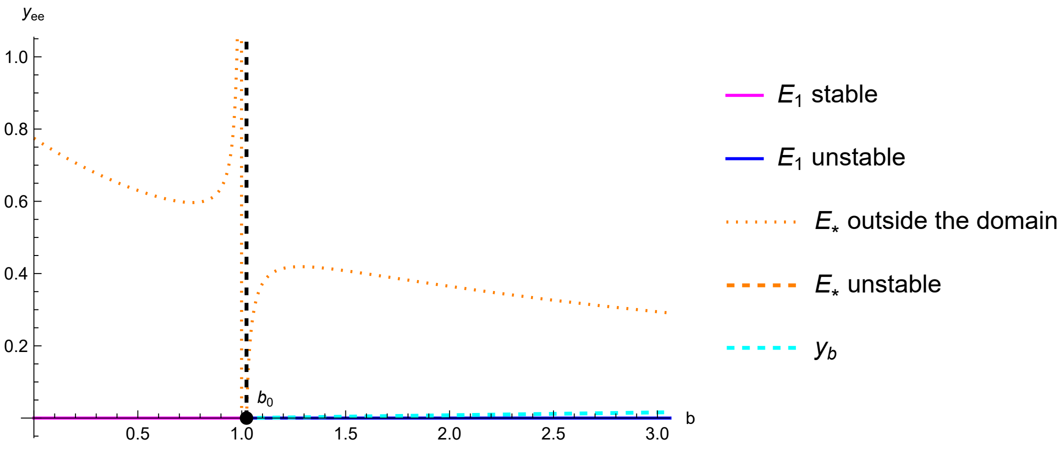
bH=1.3349402618015729010

Numerical solution of the stability (Bifurcation diagram)

In[ ]:=
```
cond=cF1;
Print["roots of Dis[b]=0:", bcE=NSolve[(Dis//.cond)==0,b]]
Print["b0=",bc=b/.Solve[R0==1,b][[1]]//.cond//N]
bL=100; max=2;
bc1=b/.bcE[[5]];
bc2=b/.bcE[[6]];
lin1=Line[{{ bc1,0},{ bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{ bc,0},{ bc,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
p1a=Plot[{ym}//.cond,{b,0,bc1},PlotStyle→{Dashed,Thick,Green},
PlotRange→All,PlotLegends→{"E_ unstable"}];
p1b=Plot[{ym}//.cond,{b,bc1,bL},PlotStyle→{Green},PlotRange→All,
PlotLegends→{"E_ stable"}];
p01=Plot[0,{b,0,bc},PlotStyle→{Magenta},PlotRange→All,PlotLegends→{"E1 stable"}];
p02=Plot[0,{b,bc,bL},PlotStyle→{Blue},PlotRange→All,PlotLegends→{"E1 unstable"}];
pp=Plot[{yi}//.cond,{b,0, bL},PlotStyle→{Purple},PlotRange→All,
PlotLegends→{"Ei unstable"}];
pm=Plot[{yp}//.cond,{b,0,bL},PlotStyle→{Red},PlotRange→All,
PlotLegends→{"E+ unstable"}];
ps1=Plot[{ys}//.cond,{b,0,13.45},PlotStyle→{Orange, Dotted},
PlotRange→{{0,200},{0,max}},
PlotLegends→{"E* outside the domain"}];
ps2=Plot[{ys}//.cond,{b,13.45,bL},PlotStyle→{Orange,Thick,Dashed},
PlotRange→{{0,200},{0,max}},
PlotLegends→{"E* unstable"}];

pyb=Plot[{yb}//.cond,{b,0,bL},PlotStyle→{Dashed,Thick,Cyan},
PlotRange→{{0,200},{0,max}},PlotLegends→{"yb "}];
Print["y*'(b0)=",D[ys,b]/.b→bc//.cond//N//FullSimplify]
Chop[ys/.b→bc//.cond//N](*Check*);
bifE2=Show[{p01,p02,ps1,ps2,pyb,li3},PlotRange→{{0,3},{0,1}},
Epilog→{Text["b0",Offset[{10,11},{ bc//.cond,0}]],{PointSize[Large],
Style[Point[{ bc//.cond,0}],Black]}},
PlotRange→All,AxesLabel→{"b","yee"}]
Export["bifEE.pdf",bifE2]
```

```
roots of Dis[b]=0:
 {{b → -152.24}, {b → -63.}, {b → -63.}, {b → -25.6124}, {b → 27.2273}, {b → 30.5559}}

b0=1.02299

y*'(b0)=21.5814
```

Out[ ]=

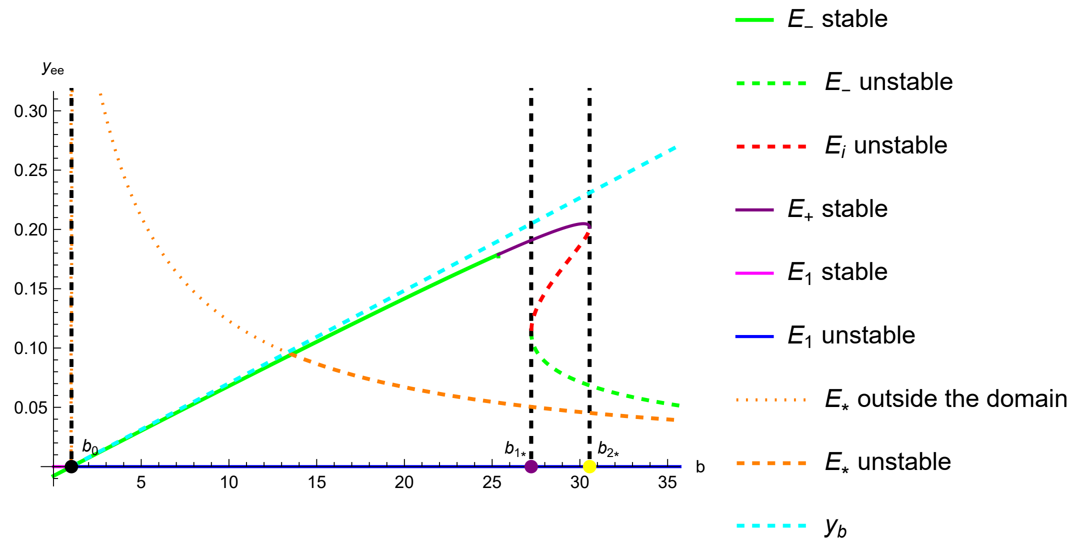


Out[ ]= bifEE.pdf

In[∘]:=

```
p1a=Plot[{ym}//.cond,{b,0,13.45},PlotStyle→{Thick,Green},PlotRange→All];
p1c=Plot[{ym}//.cond,{b,13.45,bc1},PlotStyle→{Thick,Green},
PlotRange→All,PlotLegends→{"E_ stable"}];
p1d=Plot[{ym}//.cond,{b,bc1,bL},PlotStyle→{Dashed,Thick,Green},PlotRange→All,
PlotLegends→{"E_ unstable"}];
pi=Plot[{yi}//.cond,{b,0, bL},PlotStyle→{Dashed,Thick,Red},PlotRange→All,
PlotLegends→{"Ei unstable"}];
pp1=Plot[{yp}//.cond,{b,0,bL},PlotStyle→{Purple},PlotRange→All,
PlotLegends→{"E+ stable"}];
shon=Show[{p1a,p1c,p1d,li1,li2},PlotRange→{{0,60},{0,0.2}},Epilog→
{{Text["b1*",Offset[{-8,10},{ bc1,0}]],{PointSize[Large],
Style[Point[{ bc1,0}],Purple]},Text["b2*",Offset[{10,10},{ bc2,0}]],
{PointSize[Large],Style[Point[{ bc2,0}],Yellow]}}
},AxesLabel→{"b","yee"}];
shoip=Show[{pi,pp1},PlotRange→All,AxesLabel→{"b","yee"}];
Bnip=Show[shon,shoip];
EriB=Show[shon,shoip,bifE2,PlotRange→{{0,35},{0,0.3}},Epilog→{
 Text["b0",Offset[{10,11},{ bc//.cond,0}]],{PointSize[Large],
Style[Point[{ bc//.cond,0}],Black]},
Text["b1*",Offset[{-8,10},{ bc1,0}]],{PointSize[Large],
Style[Point[{ bc1,0}],Purple]},
Text["b2*",Offset[{10,10},{ bc2,0}]],{PointSize[Large],
Style[Point[{ bc2,0}],Yellow]}}]
Export["Bnip.pdf",Bnip]
Export["EriB.pdf",EriB]
```

Out[∘]=



Out[∘]= Bnip.pdf

Out[∘]= EriB.pdf