On a four-dimensional oncolytic Virotherapy model (ϵ=0)

This Mathematica Notebook is a supplementary material to the paper "On a three-dimensional and two four-dimensional
oncolytic viro-therapy models". It contains some of the calculations and illustrations appearing in the paper concerning the section on 4 dimensional model.

Section 3 : Four-dim viro-therapy model with Logistic growth when ϵ>0 and ϵ=0

In[ ]:=
```
SetDirectory[NotebookDirectory[]];
AppendTo[$Path, Directory];
Clear["Global`*"];
Clear["K"];
(*Some aliases*)
Format[βy] = Subscript[β, y];
Format[βv] = Subscript[β, v];
Format[βz] = Subscript[β, z];


Unprotect[Power];
Power[0, 0] = 1;
Protect[Power];

par = {b, β, λ, δ, βy, βv, βz, c, γ, K, ϵ};
cp = Join[Thread[Drop[par, {1}] > 0], {b > 1}];
cKga1 = {K → 1, γ → 1};
cep1 = {ϵ → 1}; cep0 = {ϵ → 0};
R0 = b β K / (β K + δ) (* Reproduction number*);
cnT17 = {βv → 0.16, βy → 0.48, K → 1,
    γ → 1, b → 9, β → 0.11, λ → 0.36, δ → 0.2, βz → 0.6, c → 0.036};
(*Numerical values of Tian 17 after dimensionalization*)

(****** Four dim Deterministic  epidemic model with Logistic growth ****)
x1 = λ x (1 - (x + y) / K) -  β x v ;
y1 = β x v - βy y z - γ y;
v1 = -β x v - βv v z + b γ y - δ v;
z1 = z (βz y - c z^ϵ);
ye = c / βz; vM = λ (1 - ye) / β; vMN = vM /. cnT17;
dyn = {x1, y1, v1, z1};
dynS = dyn //. cKga1;
```

```
dyn3 = {x1, y1, v1} /. z → 0;(*3dim case used for E* *)
dyn3S = dyn3 //. cKga1;
          x'
Print[" (y' )=", dyn // FullSimplify // MatrixForm,
          v'
          z'

                                       x'
 "  the reparametrized model is (y' )=",
                                       v'
                                       z'

 dynS // FullSimplify // MatrixForm]
Print["b0=", b0 = b /. Apart[Solve[R0 == 1, b][[1]] // FullSimplify]]
b0S = b0 //. cKga1;
(*****Fixed points of Tian17 using the scaling when K=1, γ=1***)

fv = (ye (b - 1) - v δ); gv = (ye βy + v βv); hv = (1 - ye - v β / λ);

Print["xe= ", xe = hv, " , ye=", ye, " , ze =", ze = fv / gv]

Pv = Numerator[Together[v β xe - ye (1 + βy fv / gv)]] / (-βz^2 β^2 βv);
Print["P(v)=", pc = Collect[Together[Pv], v], " coefs are"]
coP = CoefficientList[pc, v] // Simplify

(***Fixed point when z→0**)
eq = Thread[dyn3 == {0, 0, 0}];
sol = Solve[eq, {x, y, v}] // FullSimplify;
Es = {x, y, v} /. sol[[3]]; (*Endemic point with z=0*);
Print[" Endemic point with z=0 is E*=", Es // FullSimplify]
EsS = Es /. cKga1 // FullSimplify(* E* when K=γ=1***)
Print["Check: when K=γ=1, E*=", EsS]

bn = b /. Solve[EsS[[2]] == ye, b]; bnn = bn /. cnT17;
Print["y*/ye="]
EsS[[2]] / ye // FullSimplify
Print["(b1,b2)="]
bn // FullSimplify

bcn = b /. Solve[EsS[[2]][[1]] == ye, b];
bcnn = bn /. cnT17;

Dis = Chop[Collect[Discriminant[Numerator[Pv], v], b]];
solb = Solve[Dis == 0, b];
jacS = Grad[dynS, {x, y, v, z}] // FullSimplify;
Print["Jacobian"]
jacS // MatrixForm
Print["Jacobian at E_K "]
jacK = FullSimplify[jacS /. Join[sol[[2]], {z → 0}], Assumptions → ϵ ≥ 0];
jacK // MatrixForm
Print["Jacobian at E_* when K>1"]
```

```
jacEs = FullSimplify[jacS /. Join[sol[[3]], {z → 0}], Assumptions → ϵ ≥ 0](*when K>0*);
jacEs // MatrixForm

PR = Solve[Pv == 0, v, Cubics → False]
  (*//ComplexExpand[#,TargetFunctions→{Re,Im}]&*);
 vn = v /. PR[[1]]; vp = v /. PR[[3]]; vi = v /. PR[[2]];
Ep = {xe, ye, v, ze} /. v → vp;
Ei = {xe, ye, v, ze} /. v → vi;

JEi = jacS //. {x → xe, y → ye, v → vi, z → ze};
JEp = jacS //. {x → xe, y → ye, v → vp, z → ze};
JEm = jacS //. {x → xe, y → ye, v → vn, z → ze};
jacEs1 = jacEs /. cKga1;
Print["Tr[J[E*]] corresponding to 4 dim model when K=γ=1 is "]
Tr[jacEs1]
Print["Tr[J[E*]] >0 iff"]
Reduce[Join[{(Tr[jacEs1] /. ϵ → 0) > 0}, Drop[cp, {9, 10}]],
  Drop[par, {10, 11}]] // FullSimplify
bc2 = b /. solb[[1]] // FullSimplify;
bc1 = b /. solb[[2]] // FullSimplify;
Print["b1*=", bc1]
Print[" b2*=", bc2]
```

$(* \left(-1+\frac{b\ \delta}{1-b}-c-\frac{\delta\ \lambda}{(-1+b)\ \beta}+\frac{\delta}{(-1+b)\ \beta}\left(\frac{b\ \lambda(Rz-1)\ \beta z}{\lambda(b-Rz)+(-1+b)\ \gamma\ Rz}\right)\right) *)$

$$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -v\, x\, \beta + x\left(1-\frac{x+y}{K}\right)\lambda \\ v\, x\, \beta - y\left(z\, \beta_y + \gamma\right) \\ b\, y\, \gamma - v\,(x\, \beta + z\, \beta_v + \delta) \\ z\,(-c\, z^{\epsilon} + y\, \beta_z) \end{pmatrix} \quad \text{the reparametrized model is} \quad \begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -x\,(v\, \beta + (-1+x+y)\, \lambda) \\ v\, x\, \beta - y\left(1 + z\, \beta_y\right) \\ b\, y - v\,(x\, \beta + z\, \beta_v + \delta) \\ z\,(-c\, z^{\epsilon} + y\, \beta_z) \end{pmatrix}$$

$$b0=1+\frac{\delta}{K\,\beta}$$

$$xe= 1-\frac{c}{\beta_z}-\frac{v\,\beta}{\lambda}\ ,\ ye=\frac{c}{\beta_z}\ ,\ ze=\frac{\frac{(-1+b)\,c}{\beta_z}-v\,\delta}{v\,\beta_v+\frac{c\,\beta_y}{\beta_z}}$$

$$P(v)=v^3+\frac{b\,c^2\,\beta_y\,\lambda}{\beta^2\,\beta_v\,\beta_z^2}+\frac{v^2\left(c\,\beta^2\,\beta_y\,\beta_z+c\,\beta\,\beta_v\,\beta_z\,\lambda-\beta\,\beta_v\,\beta_z^2\,\lambda\right)}{\beta^2\,\beta_v\,\beta_z^2}+$$
$$\frac{v\left(c^2\,\beta\,\beta_y\,\lambda+c\,\beta_v\,\beta_z\,\lambda-c\,\beta\,\beta_y\,\beta_z\,\lambda-c\,\beta_y\,\beta_z\,\delta\,\lambda\right)}{\beta^2\,\beta_v\,\beta_z^2}\ \text{coefs are}$$

Out[∘]= $\left\{\frac{b\,c^2\,\beta_y\,\lambda}{\beta^2\,\beta_v\,\beta_z^2},\ \frac{c\left(c\,\beta\,\beta_y+\beta_z\left(\beta_v-\beta_y\,(\beta+\delta)\right)\right)\lambda}{\beta^2\,\beta_v\,\beta_z^2},\ \frac{c\,\beta\,\beta_y+c\,\beta_v\,\lambda-\beta_v\,\beta_z\,\lambda}{\beta\,\beta_v\,\beta_z},\ 1\right\}$

Endemic point with z=0 is E*=

$$\left\{\frac{\delta}{(-1+b)\,\beta},\ \frac{((-1+b)\,K\,\beta-\delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)},\ \frac{\gamma\,((-1+b)\,K\,\beta-\delta)\,\lambda}{\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)}\right\}$$

Out[∘]= $\left\{\frac{\delta}{(-1+b)\,\beta},\ \frac{((-1+b)\,\beta-\delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,\beta+\delta\,\lambda)},\ \frac{((-1+b)\,\beta-\delta)\,\lambda}{\beta\,((-1+b)\,\beta+\delta\,\lambda)}\right\}$

Check: when K=γ=1, $E*=\left\{\frac{\delta}{(-1+b)\,\beta},\ \frac{((-1+b)\,\beta-\delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,\beta+\delta\,\lambda)},\ \frac{((-1+b)\,\beta-\delta)\,\lambda}{\beta\,((-1+b)\,\beta+\delta\,\lambda)}\right\}$

y*/ye=

Out[∘]=
$$\frac{\beta_z\,((-1+b)\,\beta-\delta)\,\delta\,\lambda}{(-1+b)\,c\,\beta\,((-1+b)\,\beta+\delta\,\lambda)}$$

(b1,b2) =

Out[∘]=
$$\left\{\frac{2\,c\,\beta-c\,\delta\,\lambda+\beta_z\,\delta\,\lambda-\delta\,\sqrt{\lambda}\,\sqrt{-4\,c\,\beta_z+(c-\beta_z)^2\,\lambda}}{2\,c\,\beta},\right.$$
$$\left.\frac{2\,c\,\beta-c\,\delta\,\lambda+\beta_z\,\delta\,\lambda+\delta\,\sqrt{\lambda}\,\sqrt{-4\,c\,\beta_z+(c-\beta_z)^2\,\lambda}}{2\,c\,\beta}\right\}$$

Jacobian

Out[∘]//MatrixForm=
$$\begin{pmatrix} -v\,\beta-(-1+2\,x+y)\,\lambda & -x\,\lambda & -x\,\beta & 0 \\ v\,\beta & -1-z\,\beta_y & x\,\beta & -y\,\beta_y \\ -v\,\beta & b & -x\,\beta-z\,\beta_v-\delta & -v\,\beta_v \\ 0 & z\,\beta_z & 0 & y\,\beta_z-c\,z^{\epsilon}\,(1+\epsilon) \end{pmatrix}$$

Jacobian at E_K

Out[∘]//MatrixForm=
$$\begin{pmatrix} \lambda-2\,K\,\lambda & -K\,\lambda & -K\,\beta & 0 \\ 0 & -1 & K\,\beta & 0 \\ 0 & b & -K\,\beta-\delta & 0 \\ 0 & 0 & 0 & -0^{\epsilon}\,c\,(1+\epsilon) \end{pmatrix}$$

Jacobian at E_* when K>1

Out[∘]//MatrixForm=
$$\begin{pmatrix} -\frac{\delta\,\lambda\,(\delta\,\lambda+(-1+b)\,\beta\,((-1+2\,K)\,\gamma+(-1+K)\,\lambda))}{(-1+b)\,\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)} & \frac{\delta\,\lambda}{\beta-b\,\beta} & -\frac{\delta}{-1+b} & 0 \\ \frac{\gamma\,((-1+b)\,K\,\beta-\delta)\,\lambda}{(-1+b)\,K\,\beta\,\gamma+\delta\,\lambda} & -1 & \frac{\delta}{-1+b} & \frac{\beta_y\,\delta\,(K\,(\beta-b\,\beta)+\delta)\,\lambda}{(-1+b)\,\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)} \\ \frac{\gamma\,(K\,(\beta-b\,\beta)+\delta)\,\lambda}{(-1+b)\,K\,\beta\,\gamma+\delta\,\lambda} & b & \frac{b\,\delta}{1-b} & \frac{\beta_v\,\gamma\,(K\,(\beta-b\,\beta)+\delta)\,\lambda}{\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)} \\ 0 & 0 & 0 & -0^{\epsilon}\,c\,(1+\epsilon)+\frac{\beta_z\,((-1+b)\,K\,\beta-\delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,K\,\beta\,\gamma+\delta\,\lambda)} \end{pmatrix}$$

Tr[J[E*]] corresponding to 4 dim model when K=γ=1 is

Out[∘]=
$$-1+\frac{b\,\delta}{1-b}-0^{\epsilon}\,c\,(1+\epsilon)-\frac{\delta\,\lambda}{(-1+b)\,\beta}+\frac{\beta_z\,((-1+b)\,\beta-\delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,\beta+\delta\,\lambda)}$$

Tr[J[E*]] >0 iff

Out[∘]=
$$\beta>0\ \&\&\ \lambda>0\ \&\&\ 0<\delta<(-1+b)\,\beta\ \&\&\ \beta_y>0\ \&\&\ \beta_v>0\ \&\&$$
$$((-1+b)\,\beta-\delta)\,\delta\,\lambda\,\left((-1+b)\,\beta^2\,(-1+b+b\,\delta)+\beta\,\delta\,(-2+\beta_z+b\,(2-\beta_z+\delta))\,\lambda+\delta^2\,\lambda\,(\beta_z+\lambda)\right)<$$
$$0\ \&\&\ 0<c<-\frac{(-1+b)\,\beta^2\,(-1+b+b\,\delta)+\beta\,\delta\,(-2+\beta_z+b\,(2-\beta_z+\delta))\,\lambda+\delta^2\,\lambda\,(\beta_z+\lambda)}{(-1+b)\,\beta\,((-1+b)\,\beta+\delta\,\lambda)}\ \&\&\ \gamma>0$$

b1∗=

$$\frac{1}{27\,c^2\,\beta^2\,\beta_v^2\,\beta_y^2\,\beta_z^2\,\lambda}\left(-2\,\sqrt{\beta^2\,\beta_y^2\,\beta_z^2\,\left(c^2\,\beta^2\,\beta_y^2+c\,\beta_v\,\left(-c\,\beta\,\beta_y-3\,\beta_v\,\beta_z+\beta_y\,\beta_z\,(\beta+3\,\delta)\right)\,\lambda+\beta_v^2\,(c-\beta_z)^2\,\lambda^2\right)^3}+\beta\,\beta_y\,\beta_z\,\left(2\,c^2\,\beta^2\,\beta_y^2+c\,\beta_v\,\left(-5\,c\,\beta\,\beta_y+\beta_z\,\left(-9\,\beta_v+5\,\beta\,\beta_y+9\,\beta_y\,\delta\right)\right)\,\lambda+2\,\beta_v^2\,(c-\beta_z)^2\,\lambda^2\right)\left(\beta_v\,\beta_z\,\lambda-c\,\left(\beta\,\beta_y+\beta_v\,\lambda\right)\right)\right)$$

b2∗=

$$\frac{1}{27\,c^2\,\beta^2\,\beta_v^2\,\beta_y^2\,\beta_z^2\,\lambda}\left(2\,\sqrt{\beta^2\,\beta_y^2\,\beta_z^2\,\left(c^2\,\beta^2\,\beta_y^2+c\,\beta_v\,\left(-c\,\beta\,\beta_y-3\,\beta_v\,\beta_z+\beta_y\,\beta_z\,(\beta+3\,\delta)\right)\,\lambda+\beta_v^2\,(c-\beta_z)^2\,\lambda^2\right)^3}+\beta\,\beta_y\,\beta_z\,\left(2\,c^2\,\beta^2\,\beta_y^2+c\,\beta_v\,\left(-5\,c\,\beta\,\beta_y+\beta_z\,\left(-9\,\beta_v+5\,\beta\,\beta_y+9\,\beta_y\,\delta\right)\right)\,\lambda+2\,\beta_v^2\,(c-\beta_z)^2\,\lambda^2\right)\left(\beta_v\,\beta_z\,\lambda-c\,\left(\beta\,\beta_y+\beta_v\,\lambda\right)\right)\right)$$

2-2)Interior equilibrium

Analysis of the stability of the interior point E_* when K=γ=1:

In[∘]:=
```
(*Since z=0, we reduce our analysis for this point to 3 dimensions*)
jac3=Grad[dyn3/.cKga1,{x,y,v}]//FullSimplify;
Print["J(E_*) is when K=1"](*in three dimension*)
JsS=(jac3/.sol〚3〛/.cKga1)//FullSimplify;
JsS//MatrixForm
Trs=Tr[JsS](*the trace of J[E*] in 3 dimension*);
Print["Tr(J(E*))=",Trs//FullSimplify]
Print["Det(J(E*))=",Det[JsS]//FullSimplify]


pc=Collect[Det[ψ IdentityMatrix[3]-JsS],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Length[coT]
Print["a₁=",a1=coT〚3〛, ", a₂=",a2=coT〚2〛, ", a₃=",a3=coT〚1〛]
H2=a1*a2-a3;
Print["H2(b0)=",H2/.b→b0S//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H2]]/.cKga1//FullSimplify]
ϕb=Collect[Numerator[Together[H2]]/(δ λ),b]/.cKga1;
cofi=CoefficientList[ϕb,b](*Coefficients of ϕ(b)*);
Print["value of ϕ(b) at crit b is "]
ϕb/.b→b0S/.cKga1//FullSimplify
```

J(E_∗) is when K=1

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{\delta\,\lambda}{\beta-b\,\beta} & \frac{\delta\,\lambda}{\beta-b\,\beta} & -\frac{\delta}{-1+b} \\ \frac{((-1+b)\,\beta-\delta)\,\lambda}{(-1+b)\,\beta+\delta\,\lambda} & -1 & \frac{\delta}{-1+b} \\ \frac{(\beta-b\,\beta+\delta)\,\lambda}{(-1+b)\,\beta+\delta\,\lambda} & b & \frac{b\,\delta}{1-b} \end{pmatrix}$$

$$\mathrm{Tr}(J(E*))=-1+\frac{\delta\,(b\,\beta+\lambda)}{\beta-b\,\beta}$$

$$\mathrm{Det}(J(E*))=\frac{\delta\,(\beta-b\,\beta+\delta)\,\lambda}{(-1+b)\,\beta}$$

Out[∘]= 4

$$a_1 = \frac{\beta\,(-1+b+b\,\delta)+\delta\,\lambda}{(-1+b)\,\beta},\ a_2 = \frac{\delta\,\lambda\left((-1+b)\,\beta\,(-1+\beta+\delta+b\,(1-\beta+\delta))+\left((-1+b)^2\,\beta+b\,\delta^2\right)\lambda\right)}{(-1+b)^2\,\beta\,((-1+b)\,\beta+\delta\,\lambda)},\ a_3 = \delta\left(1+\frac{\delta}{\beta-b\,\beta}\right)\lambda$$

H2(b0) = $(1+\beta+\delta)\,\lambda\,(1+\beta+\delta+\lambda)$

Denominator of H2 is $(-1+b)^3\,\beta^2\,((-1+b)\,\beta+\delta\,\lambda)$

value of ϕ(b) at crit b is

Out[ ]=
$$\frac{\delta^3\,(1+\beta+\delta)\times(1+\lambda)\times(1+\beta+\delta+\lambda)}{\beta}$$

In[ ]:=
```
(*Analysis of the stability of E* in 4 dim when ϵ=0*)

pc=Collect[Det[ψ IdentityMatrix[4]-(jacEs1/.cep0)],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Length[coT]
a1=coT[[4]]//FullSimplify ;a2=coT[[3]]//FullSimplify;a3=coT[[2]]//FullSimplify ; a4=coT[[1]]//FullSi
Print["a1=",a1, ", a2=",a2, ", a3=",a3, " ,a4=", a4 ]
H4=a1*a2*a3-a3^2+a1^2 a4;
Print["H2(b0)=",H4/.b→b0S//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H4]]//FullSimplify]
ϕb4=Collect[Numerator[Together[H4]],b];
cofi=CoefficientList[ϕb4,b];
Length[cofi]
Print["value of ϕ(b) at crit b is "]
ϕb4/.b→b0S//FullSimplify
```

Out[ ]= 5

$$a_1 = c+\frac{(-1+b)\,\beta^2\,(-1+b+b\,\delta)+\beta\,\delta\,(-2+\beta_z+b\,(2-\beta_z+\delta))\,\lambda+\delta^2\,\lambda\,(\beta_z+\lambda)}{(-1+b)\,\beta\,((-1+b)\,\beta+\delta\,\lambda)}$$

$$,\ a_2 = \frac{1}{(-1+b)^2\,\beta^2}\Bigg((-1+b)\,c\,\beta\,(\beta\,(-1+b+b\,\delta)+\delta\,\lambda)+\frac{1}{(-1+b)\,\beta+\delta\,\lambda}\,\delta\,\lambda\,\Big(-(-1+b)^2\,\beta^3+\beta_z\,\delta^2\,\lambda-(-1+b)\,\beta^2\,(1-\beta_z-\delta+b\,(-1+\beta_z)\times(1+\delta)+\lambda-b\,\lambda)+\beta\,\delta\,(b\,\delta\,\lambda+\beta_z\,(-1+b+b\,\delta+\lambda-b\,\lambda))\Big)\Bigg),\ a_3 = \frac{1}{(-1+b)^3\,\beta^2\,((-1+b)\,\beta+\delta\,\lambda)^2}$$

$$\delta\,\lambda\,\Big(((-1+b)\,\beta-\delta)\,\Big((-1+b)^4\,\beta^3+(-1+b)\,\beta\,\delta\,(\beta_z+(-1+b)\,\beta\,(-2+2\,b+\beta_z)-\beta_z\,(b+\delta+b\,\delta))\,\lambda-\delta\,\left((-1+b)^2\,\beta\,(\beta_z-\delta)+b\,\beta_z\,\delta^2\right)\lambda^2\Big)-(-1+b)\,c\,\beta\,((-1+b)\,\beta+\delta\,\lambda)\,\left((-1+b)^2\,\beta^2-b\,\delta^2\,\lambda-(-1+b)\,\beta\,(-1+b+\delta+b\,\delta+(-1+b)\,\lambda)\right)\Big)$$

$$,a_4 = \frac{((-1+b)\,\beta-\delta)\,\delta\,\lambda\,(\beta_z\,\delta\,(\beta-b\,\beta+\delta)\,\lambda+(-1+b)\,c\,\beta\,((-1+b)\,\beta+\delta\,\lambda))}{(-1+b)^2\,\beta^2\,((-1+b)\,\beta+\delta\,\lambda)}$$

H2(b0) = $c\,(1+\beta+\delta)\times(1+c+\beta+\delta)\,\lambda\,(c+\lambda)\,(1+\beta+\delta+\lambda)$

Denominator of H2 is $(-1+b)^6\,\beta^5\,((-1+b)\,\beta+\delta\,\lambda)^4$

Out[ ]= 11

value of ϕ(b) at crit b is

Out[ ]=
$$\frac{c\,\delta^{10}\,(1+\beta+\delta)\times(1+c+\beta+\delta)\,\lambda\,(1+\lambda)^4\,(c+\lambda)\,(1+\beta+\delta+\lambda)}{\beta}$$

```
FindInstance[Join[{Tr[jacEs/.cKga1/.ϵ→0]>0},Drop[cp,{9,10}]],Drop[par,{10,11}]]
```

Out[ ]= $\{\{b \to 2,\ \beta \to 2,\ \lambda \to 1,\ \delta \to 1,\ \beta_y \to 1,\ \beta_v \to 1,\ \beta_z \to 28,\ c \to 1,\ \gamma \to 1\}\}$

Numerical values ϵ=0:

In[ ]:=
```
cn=Join[cnT17,cep0];
cb=NSolve[(ϕb//.Drop[cnT17,{5}])==0,b,WorkingPrecision→10]
bM=Max[Table[Re[b/.cb[[i]]],{i,Length[cb]}]];
Print["bH=",bH=N[bM,30]]
Chop[{vn,vi,vp}/.cnT17];

Print["b0=",b0/.cn//N, " , b1=", bnn[[1]],   ", b2=", bnn[[2]], "  ,bH=",bH]
PRN=Chop[PR//.cn//N](*values of the roots v*);
Print["E*",Chop[Es/.cn]//N]
Print["E+=",Chop[Ep/.cn]//N]
Print["Ei=",Chop[Ei/.cn]//N]
Print["Eigv of E*:",Append[Eigenvalues[JsS],Es[[2]]-ye]//.cn," , Eigv of E+:",
Chop[Eigenvalues[JEp//.cn]]//N//FullSimplify, " Eigv of Eim:",Chop[Eigenvalues[JEi//.cn]]//N,
"   , Eigv of EK:",Eigenvalues[jacK//.cn]//N]

Print["b1*=",  bc1//.cn]
Print[" b2*=",  bc2//.cn]
```

Out[ ]= {{b → 0.299052}, {b → 0.835329 - 0.231156 i}, {b → 0.835329 + 0.231156 i}, {b → 19.0121}}

```
bH=19.0121

b0=2.81818 , b1=3.58676, b2=8.66779  ,bH=19.0121

E*{0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

Eigv of E*:{-1.25056, -0.0281268 - 0.20904 i, -0.0281268 + 0.20904 i, -0.00155844}
  , Eigv of E+:{-1.29833, -0.0332218 + 0.21255 i, -0.0332218 - 0.21255 i, 0.000834552}
  Eigv of Eim:{-1.69849, -0.076539 + 0.209952 i, -0.076539 - 0.209952 i, -0.00803072}
    , Eigv of EK:{-1.7081, 0.398103, -0.36, -0.036}

b1*=-0.00697038

 b2*=10.2462
```

Determination of the endemic points with respect to x:

In[ ]:=
```
Print["ve(x)="]
vee=Solve[(x1/x)==0,v][[1]]/.cKga1
Print["ze(x)="]
zee=Solve[v1+(x1/x)==0,z][[1]]/.cKga1//FullSimplify
(*(y1/.zee/.vee/.y→ye);*)
xee=Solve[Collect[Numerator[Together[(y1/.zee/.vee/.y→ye)]],x]==0,x] (*The expressions are so
Print["Number of endemic x"]
Length[xee]
Chop[Evaluate[xee//.cnT17]] (*Numerical check*)
```

ve(x) =

Out[ ]= $\left\{v \to \frac{(1-x-y)\ \lambda}{\beta}\right\}$

ze(x) =

Out[ ]= $\left\{z \to \frac{b\,y - v\,(\beta + x\,\beta + \delta) + \lambda - (x+y)\,\lambda}{v\,\beta_v}\right\}$

Number of endemic x

Out[ ]= 3

Out[ ]= {{x → 0.249944}, {x → 1.20177}, {x → 0.483284}}

3)Sections 3.1---3.3(in paper):Figures used in the manuscript (*Run the previous cell*)
Numerical illustrations when ϵ=0 (Bifurcations diagrams, parametric plots, and 3D plot)

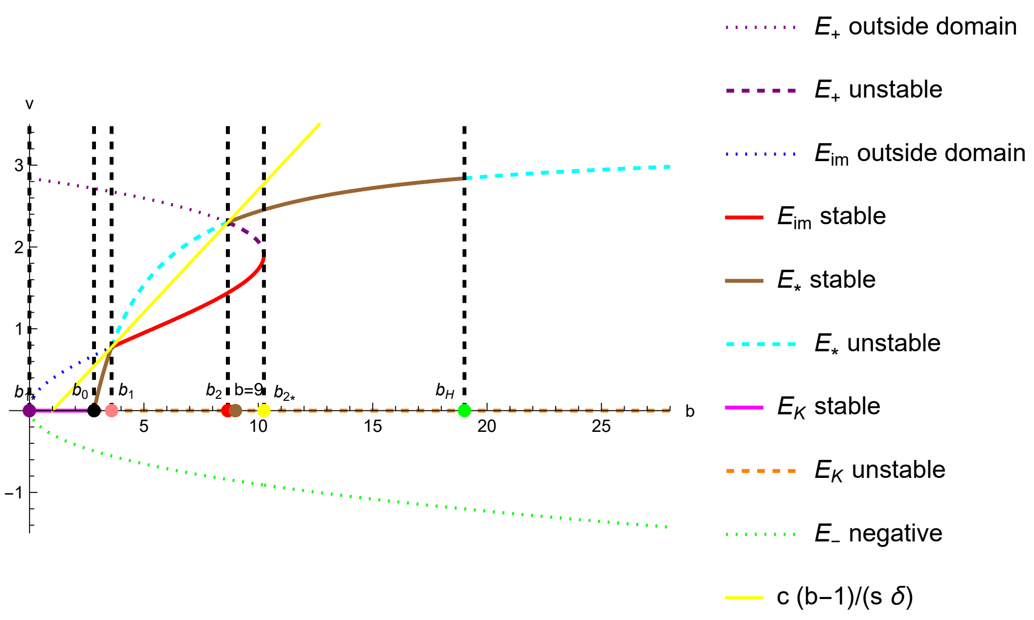
Bifurcation diagram when b varies:

In[ ]:=
```
ClearParameters;
βv=0.16; βy=0.48;K=1;b=9;γ=1;λ=0.36;β=0.11;δ=0.2;βz=0.6; c=0.036;ϵ=0;
Print["{b2*,b1*}=",{b/.solb[[1]],b/.solb[[2]]} ,", and b0=",b0, "  , bH=",bH,
 " ,b1=", bnn[[1]],   ", b2=", bnn[[2]]]
Clear["b"];
vs= ((-1+b) β-δ) λ / (β ((-1+b) β+δ λ)) (*v of E* *  ****);
bL=28; max=3.5;bmin=-0.9;
lin1=Line[{{ bc1,0},{ bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{ b0,0},{ b0,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
linH=Line[{{ bH,0},{ bH,max}}];
liH=Graphics[{Thick,Black,Dashed,linH}];
linb1=Line[{{ bnn[[1]],0},{ bnn[[1]],max}}];
lib1=Graphics[{Thick,Black,Dashed,linb1}];
linb2=Line[{{ bnn[[2]],0},{ bnn[[2]],max}}];
lib2=Graphics[{Thick,Black,Dashed,linb2}];
linb9=Line[{{ 9,0},{9,max}}];
```

```
lib9=Graphics[{Thick,Black,Dashed,linb2}];
pn=Plot[{vn},{b,0,bL},PlotStyle→{Green,Dotted},PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"
p0=Plot[0,{b,0,bL},PlotStyle→{Brown,Thick},PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E0 s
ppa=Plot[{vp},{b,0, bnn⟦2⟧},PlotStyle→{Purple,Dotted},PlotRange→{{bmin,bL},{-2,max}},
PlotLegends→{"E+ outside domain"}];
ppb=Plot[{vp},{b,bnn⟦2⟧, bL},PlotStyle→{Purple,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E+ unstable"}];
pi1=Plot[{vi},{b,0,bnn⟦1⟧},PlotStyle→{Blue,Dotted},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"Eim outside domain"}];
pi2=Plot[{vi},{b,bnn⟦1⟧,bL},PlotStyle→{Red,Thick},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"Eim stable"}];
pi3=Plot[{vi},{b,bnn⟦2⟧,bL},PlotStyle→{Blue,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}}(*,PlotLegends→{"Ei unstable"}*)];
ps1=Plot[{vs},{b,b0,bnn⟦1⟧},PlotStyle→{Brown,Thick},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E* stable"}];
ps2=Plot[{vs},{b,bnn⟦1⟧,bnn⟦2⟧},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E* unstable"}];
ps3=Plot[{vs},{b,bnn⟦2⟧,bH},PlotStyle→{Brown,Thick},
PlotRange→{{bmin,bL},{0,max}}(*,PlotLegends→{"E* stable"}*)];
ps4=Plot[{vs},{b,bH,bL},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}}(*,PlotLegends→{"E* unstable"}*)];
pdf1=Plot[{0},{b,0,b0},PlotStyle→{Magenta, Thick},
PlotRange→{{bmin,bL},{0,max}},PlotLegends→{"EK stable"}];
pdf2=Plot[{0},{b,b0,bL},PlotStyle→{Orange, Thick,Dashed},
PlotRange→{{bmin,bL},{0,max}},PlotLegends→{"EK unstable"}];
pvmax=Plot[{c (b-1)/(βz δ)(*,(λ(1-c/βz))/β*)},{b,0,bL},PlotStyle→{Yellow},
PlotRange→{{0,bL},{0,max}},
PlotLegends→{"c (b-1)/(s δ)"}];

bifT=Show[{ppa,ppb,pi1,pi2,ps1,ps2,ps3,ps4,pdf1,pdf2,pn,li1,li2,li3,
lib2,lib1,lib9,pvmax,liH},AxesLabel→{"b","v"},PlotRange→{{bmin,bL},{-3/2,max}},
Epilog→{Text["b1*",Offset[{-2,10},{ bc1,0}]],{PointSize[Large],
Style[Point[{ bc1,0}],Purple]},
Text["b2*",Offset[{11,10},{ bc2,0}]],{PointSize[Large],Style[Point[{ bc2,0}],Yellow]},
 Text["b0",Offset[{-7,11},{ b0,0}]],{PointSize[Large],Style[Point[{ b0,0}],Black]},
Text["bH",Offset[{-10,11},{ bH,0}]],{PointSize[Large],Style[Point[{ bH,0}],Green]},
 Text["b1",Offset[{8,11},{ bnn⟦1⟧,0}]],{PointSize[Large],Style[Point[{ bnn⟦1⟧,0}],Pink]},
Text["b2",Offset[{-7,11},{ bnn⟦2⟧,0}]],{PointSize[Large],Style[Point[{ bnn⟦2⟧,0}],Red]},
Text["b=9",Offset[{7,11},{9,0}]],{PointSize[Large],Style[Point[{ 9,0}],Brown]}}]
Export["BiifT17.pdf",bifT]
```

$\{b_{2*},b_{1*}\}=\{10.2462, -0.00697038\}$, and b_0=2.81818 , b_H=19.0121 ,b1=3.58676, b2=8.66779

Out[∘]=



Out[∘]= BiifT17.pdf

Numerical solution of the stability (Bifurcation diagram) wrt v:

In[∘]:=

```
(*Checks on the stability of the fixed points**)
Print["Eigenvalues of E∗:"]
Print["E∗ between b0 and b1"]
Eigenvalues[jacEs1] /. b → 3 // N
Print["E∗ between b1 and b2"]
Eigenvalues[jacEs1] /. b → 5 // N
Print["E∗ between b2 and bH"]
Eigenvalues[jacEs1] /. b → 15 // N
Print["E∗ after bH"]
Eigenvalues[jacEs1] /. b → 25 // N
Print["Eigenvalues of Eim:"]
Print["Eim between 0 and b1"]
Chop[Eigenvalues[JEi] /. b → 2 // N]
Print["E∗ between b1 and b2∗"]
Chop[Eigenvalues[JEi] /. b → 9 // N]
Print["Eigenvalues of E+:"]
Print["Eim between 0 and b2"]
Chop[Eigenvalues[JEp] /. b → 2 // N]
Print["E∗ between b2 and b2∗"]
Chop[Eigenvalues[JEp] /. b → 9 // N]
```

```
Eigenvalues of E⋆:

E⋆ between b0 and b1

Out[•]= {-0.0225504, -1.29942, -0.311695, -0.0161607}

E⋆ between b1 and b2

Out[•]= {0.0100227, -1.26091, -0.0763636 - 0.159107 i, -0.0763636 + 0.159107 i}

E⋆ between b2 and bH

Out[•]= {-0.0126814, -1.24799, -0.00652554 - 0.223959 i, -0.00652554 + 0.223959 i}

E⋆ after bH

Out[•]= {-0.0212776, -1.24705, 0.00572283 - 0.230932 i, 0.00572283 + 0.230932 i}

Eigenvalues of Eim:

Eim between 0 and b1

Out[•]= {-1.00272, -0.249728, -0.0735624, 0.0456119}

E⋆ between b1 and b2⋆

Out[•]= {-1.69849, -0.00803072, -0.076539 - 0.209952 i, -0.076539 + 0.209952 i}

Eigenvalues of E+:

Eim between 0 and b2

Out[•]= {-0.559828, -0.123746, 0.053141 - 0.0959514 i, 0.053141 + 0.0959514 i}

E⋆ between b2 and b2⋆

Out[•]= {-1.29833, 0.000834552, -0.0332218 - 0.21255 i, -0.0332218 + 0.21255 i}
```

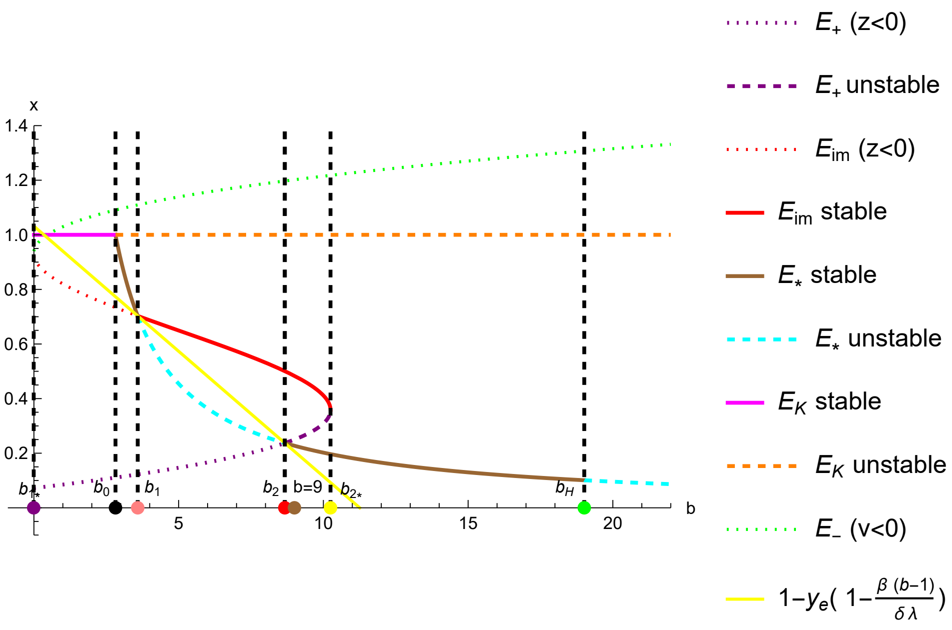
(x-b)-Bifurcation diagram :

```
In[•]:= Clear["K"];
K=1;
Print["{b2⋆,b1⋆}=",{b/.solb[[1]],b/.solb[[2]]} ,", and b0=",b0, "  , bH=",bH,
 " ,b1=", bnn[[1]],   ", b2=", bnn[[2]]]
Clear["b"];
xs=EsS[[1]](*x of E* *  ****);
bL=22; max=1.4;bmin=-0.9;
lin1=Line[{{ bc1,0},{ bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{ b0,0},{ b0,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
linH=Line[{{ bH,0},{ bH,max}}];
liH=Graphics[{Thick,Black,Dashed,linH}];
linb1=Line[{{ bnn[[1]],0},{ bnn[[1]],max}}];
lib1=Graphics[{Thick,Black,Dashed,linb1}];
linb2=Line[{{ bnn[[2]],0},{ bnn[[2]],max}}];
lib2=Graphics[{Thick,Black,Dashed,linb2}];
linb9=Line[{{ 9,0},{9,max}}];
lib9=Graphics[{Thick,Black,Dashed,linb2}];
(*pn=Plot[{x/.xee[[1]]},{b,0,b0},PlotStyle→{Purple,Dotted,Thick},PlotRange→{{bmin,bL},{-2,max}
pn1=Plot[{x/.xee[[1]]},{b,0,bnn[[2]]},PlotStyle→{Purple,Dotted,Thick},PlotRange→{{bmin,bL},{-2,m
PlotLegends→{"E+ (z<0)"}];
```

```
pn2=Plot[{x/.xee⟦1⟧},{b,bnn⟦2⟧,bL},PlotStyle→{Purple,Dashed,Thick},PlotRange→{{bmin,bL},{-2,
PlotLegends→{"E₊ unstable"}];
p0=Plot[0,{b,0,bL},PlotStyle→{Brown,Thick},PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E0 s
ppa=Plot[{x/.xee⟦2⟧},{b,0, bL},PlotStyle→{Green,Dotted},PlotRange→{{bmin,bL},{-2,max}},
PlotLegends→{"E₋ (v<0)"}];
pi1=Plot[{x/.xee⟦3⟧},{b,0,bnn⟦1⟧},PlotStyle→{Red,Dotted},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"Eim (z<0)"}];
pi2=Plot[{x/.xee⟦3⟧},{b,bnn⟦1⟧,bL},PlotStyle→{Red,Thick},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"Eim stable"}];
ps1=Plot[{xs},{b,b0,bnn⟦1⟧},PlotStyle→{Brown,Thick},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E* stable"}];
ps2=Plot[{xs},{b,bnn⟦1⟧,bnn⟦2⟧},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E* unstable"}];
ps3=Plot[{xs},{b,bnn⟦2⟧,bH},PlotStyle→{Brown,Thick},
PlotRange→{{bmin,bL},{0,max}}(*,PlotLegends→{"E* stable"}*)];
ps4=Plot[{xs},{b,bH,bL},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}}(*,PlotLegends→{"E* unstable"}*)];
pdf1=Plot[K,{b,0,b0},PlotStyle→{Magenta, Thick},
PlotRange→{{bmin,bL},{0,max}},PlotLegends→{"EK stable"}];
pdf2=Plot[K,{b,b0,bL},PlotStyle→{Orange, Thick,Dashed},
PlotRange→{{bmin,bL},{0,max}},PlotLegends→{"EK unstable"}];
pxmax=Plot[{1-ye-β/λ (ye (b-1)/δ)},{b,0,bL},PlotStyle→{Yellow},
PlotRange→{{0,bL},{0,max}},
PlotLegends→{"1-ye( 1-(β (b-1))/(δ λ))"}];

bifx=Show[{pn1,pn2,pi1,pi2,ps1,ps2,ps3,ps4,pdf1,pdf2,ppa,pxmax,li1,li2,li3,
lib2,lib1,lib9,liH},AxesLabel→{"b","x"},PlotRange→{{bmin,bL},{-0.1,max}},
Epilog→{Text["b1*",Offset[{-2,10},{ bc1,0}]],{PointSize[Large],
Style[Point[{ bc1,0}],Purple]},
Text["b2*",Offset[{11,10},{ bc2,0}]],{PointSize[Large],Style[Point[{ bc2,0}],Yellow]},
 Text["b0",Offset[{-7,11},{ b0,0}]],{PointSize[Large],Style[Point[{ b0,0}],Black]},
Text["bH",Offset[{-10,11},{ bH,0}]],{PointSize[Large],Style[Point[{ bH,0}],Green]},
 Text["b1",Offset[{8,11},{ bnn⟦1⟧,0}]],{PointSize[Large],Style[Point[{ bnn⟦1⟧,0}],Pink]},
Text["b2",Offset[{-7,11},{ bnn⟦2⟧,0}]],{PointSize[Large],Style[Point[{ bnn⟦2⟧,0}],Red]},
Text["b=9",Offset[{7,11},{9,0}]],{PointSize[Large],Style[Point[{ 9,0}],Brown]}}]
Export["bifx.pdf",bifx]
```

$\{b_{2*},b_{1*}\}=\{10.2462, -0.00697038\}$, and b_0=2.81818 , b_H=19.0121 ,b1=3.58676, b2=8.66779

Out[∘]=



Out[∘]= bifx.pdf

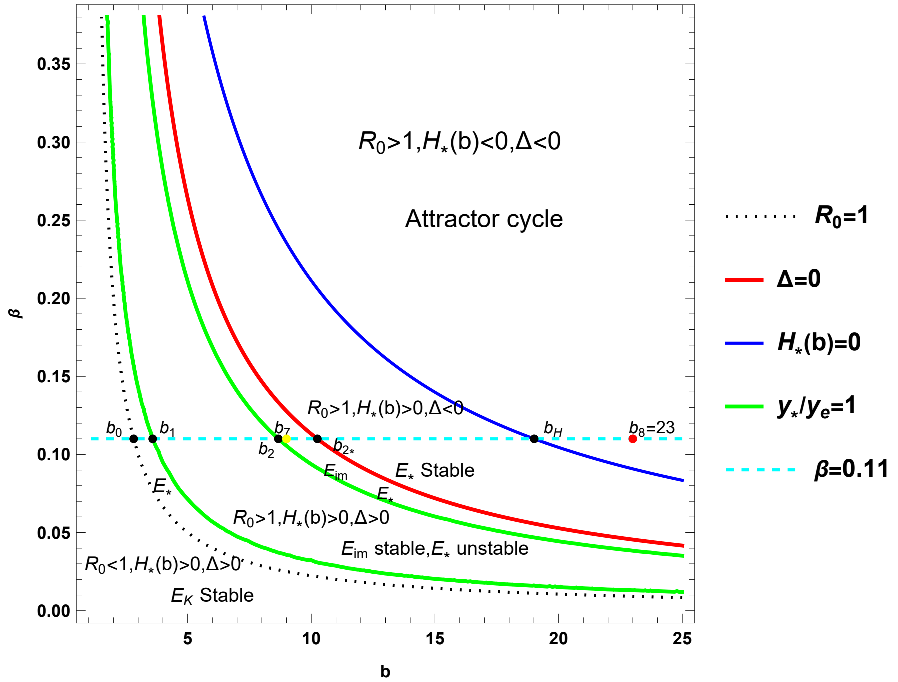
Bifurcation diagram of codimension 2:

In[∘]:=

```
Clear["b"];
Clear["β"];
(*Here we modified a slightly the values from FindInstance when Tr[jacEs]>0*)
(*λ=1;δ=0.2;βy=0.48;βv=0.16;βz=28;c=1;γ=1;K=1; *)

bm=1; bM=25; ym=0; yM=0.38;
trEp=Tr[JEp];
trEim=Tr[JEi];
detEs=Det[jacEs1];

R1=Style[Text["R0>1,H*(b)<0,Δ<0",{16,0.30}],13];
R1a=Style[Text["Attractor cycle",{17,0.25}],13];
R2=Style[Text["R0>1,H*(b)>0,Δ<0",{13,0.13}],10];
R2a=Style[Text["E* Stable",{15,0.09}],10];
RIII=Style[Text["R0>1,H*(b)>0,Δ>0",{10,0.06}],10];
RIIIa=Style[Text["Eim stable,E* unstable",{15,0.04}],10];
RII=Style[Text["E*",{4,0.08}],10];
RI=Style[Text["EK Stable",{6,0.01}],10];
RIa=Style[Text["R0<1,H*(b)>0,Δ>0",{4,0.03}],10];
Rbi=Style[Text["Eim",{11,0.09}],9];
Rbia=Style[Text["E*",{13,0.075}],9];

pt0s=Text["b0",Offset[{-10,6},{b0/.β→0.11,0.11}]];
pt0={PointSize[Medium],Style[Point[{b0/.β→0.11,0.11}],Black]};
ptHs=Text["bH",Offset[{10,6},{bH,0.11}]];
ptH1={PointSize[Medium],Style[Point[{bH,0.11}],Black]};
ptb2as=Text["b2*",Offset[{14,-3},{bc2/.β→0.11,0.11}]];
ptb2a={PointSize[Medium],Style[Point[{bc2/.β→0.11,0.11}],Black]};
ptb2bs=Text["b2",Offset[{-6,-5},{bnn[[2]],0.11}]];
ptb2b={PointSize[Medium],Style[Point[{bnn[[2]],0.11}],Black]};
ptb1s=Text["b1",Offset[{8,6},{bnn[[1]],0.11}]];
```

```
ptb1={PointSize[Medium],Style[Point[{bnn[[1]],0.11}],Black]};
ptbas=Text["b7",Offset[{-2,6},{9,0.11}]];
ptba={PointSize[Medium],Style[Point[{9,0.11}],Yellow]};
ptbbs=Text["b8=23",Offset[{10,6},{23,0.11}]];
ptbb={PointSize[Medium],Style[Point[{23,0.11}],Red]};
epi={R1,R1a,R2,R2a,RIII,RIIIa,RII,RI,RIa,Rbi,Rbia,pt0s,pt0,ptHs,ptH1,ptb2as,ptb2a,ptb2bs,ptb2b
ptbas,ptba,ptbbs,ptbb,ptb1s,ptb1};

pR0=ContourPlot[R0==1,{b,bm,bM},{β,ym,yM},ContourStyle→{Black,Dotted},
 FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"R0=1"}];
(*ptrEs=ContourPlot[Tr[jacEs]==0,{b,bm,bM},{β,ym,yM},ContourStyle→{Green,Bold},
 FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Tr[J(E*)]=0"}];*)
 pphi=ContourPlot[H2(*or ϕb*)==0,{b,bm,bM},{β,ym,yM},ContourStyle→{Blue,Bold},PlotPoints → 18
 FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"H*(b)=0"}];
 pdetEs=ContourPlot[detEs==0,{b,bm,bM},{β,ym,yM},ContourStyle→{Green,Bold},
 FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Det[J(E*)]=0"}];
  pdis=ContourPlot[Dis==0,{b,bm,bM},{β,ym,yM},ContourStyle→{Red,Thick},PlotPoints→180,
 FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Δ=0"}];
(*ptrEi=ContourPlot[trEim==0,{b,bm,bM},{β,ym,yM},ContourStyle→{Red,Dashed},PlotPoints → 180,
 FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Tr[J(Eim)]=0"}];*)
 pbeta=ContourPlot[β==0.11,{b,bm,bM},{β,ym,yM},ContourStyle→{Cyan,Dashed},
 FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"β=0.11"}];
 ppar=ContourPlot[ EsS[[2]]/ye==1,{b,bm,bM},{β,ym,yM},ContourStyle→{Green,Thick},
 FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"y*/ye=1"}];



 peq={pR0,pdis,pphi,ppar(*pdetEs,ptrEi*)(*ptrEs,*),pbeta};
pcut=Show[peq,Epilog→epi]
Export["map4.pdf",pcut]
```

Out[•]=



Out[•]= map4.pdf

```
In[•]:= (***Some checks***)
(*For Ein**)
Chop[Evaluate[xee〚3〛 /. b → 2]] // N
Chop[Evaluate[zee /. vee /. xee〚3〛 /. y → ye /. b → 2]] // N
Chop[vee /. zee /. xee〚3〛 /. y → ye /. b → 2] // N
(*For E+**)
Chop[Evaluate[xee〚1〛 /. b → 3]] // N
Chop[zee /. vee /. xee〚1〛 /. y → ye /. b → 3] // N
Chop[vee /. zee /. xee〚1〛 /. y → ye /. b → 3] // N

Out[•]= {x → 0.768927}
Out[•]= {z → -0.439052}
Out[•]= {v → 0.559875}
Out[•]= {x → 0.113348}
Out[•]= {z → -0.912093}
Out[•]= {v → 2.70541}

In[•]:= Chop[Evaluate[xee〚2〛 /. b → 6]] // N
zee /. vee /. xee〚2〛 /. y → ye /. b → 6 // N
vee /. zee /. xee〚2〛 /. y → ye /. b → 6 // N

Out[•]= {x → 1.15674}
```

Out[•]= $\{z \to -5.21725 + 3.87119 \times 10^{-16}\, i\}$

Out[•]= $\{v \to -0.709334 - 9.08364 \times 10^{-17}\, i\}$

Parametric plots at the intervals of stability:

When b2<b=9<b2*:

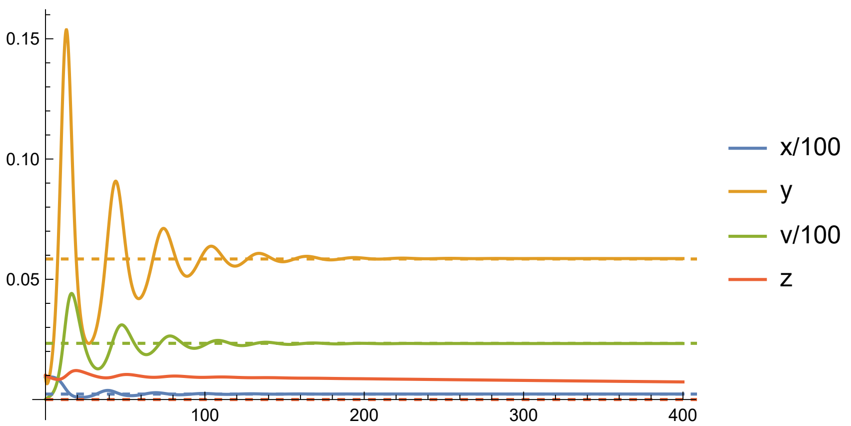
In[]:=
```
b=9;
Print["E*",Es=EsS//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi//N]]

X={x,y,v,z};
Xt=Map[#[t]&,X ];
Thread[X→Xt];
dynt=dyn/.Thread[X→Xt];
x1=dynt[[1]] ;
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,z[0]==z0};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,400}];
pdy2=Plot[{x[t]/100/. sol22,y[t]/. sol22,v[t]/100/. sol22,z[t]/. sol22},{t,0,400},
PlotLegends→{"x/100","y","v/100","z"}];
pEs2=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
pEi2=Plot[{x/100/.x→Ei[[1]],y/.y→Ei[[2]],v/100/.v→Ei[[3]],z/.z→Ei[[4]]},{t,0,1000},
PlotStyle→{Dashed}];
Dyni22=Show[pdy2,pEi2];
Dyns22=Show[pdy2,pEs2]

(*****..Parametric plot conditions***)
ppb9=ParametricPlot[{  x[t],(y[t])}/.sol22,{t,0,200}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py9=Plot[(y/.y→Es[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb9=Show[{ppb9,py9, Graphics[{Green,Dashed,Line[{{x/.x→Es[[1]],0},{x/.x→Es[[1]],1}}]}] },
Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),(y/.y→Es[[2]])}]],{PointSize[Large],
Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Black]}},{PointSize[Large],Point[{x0,y0}]},
Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}]
Export["pb9.pdf",pb9]
Export["Dyni22.pdf",Dyni22]
Export["Dyns22.pdf",Dyns22]
```

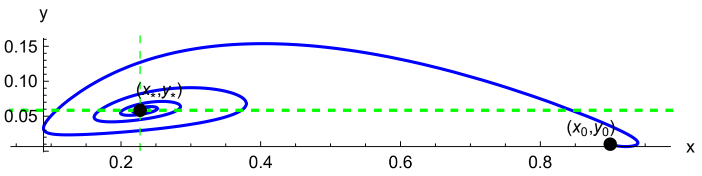
E*{0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

Out[]=



Out[ ]= 

Out[ ]= pb9.pdf

Out[ ]= Dyni22.pdf

Out[ ]= Dyns22.pdf

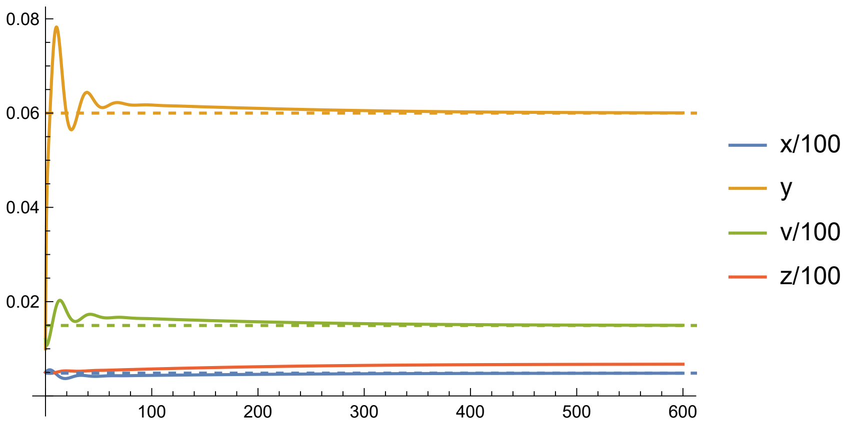
When b2<b=9<b2* and different initial values :

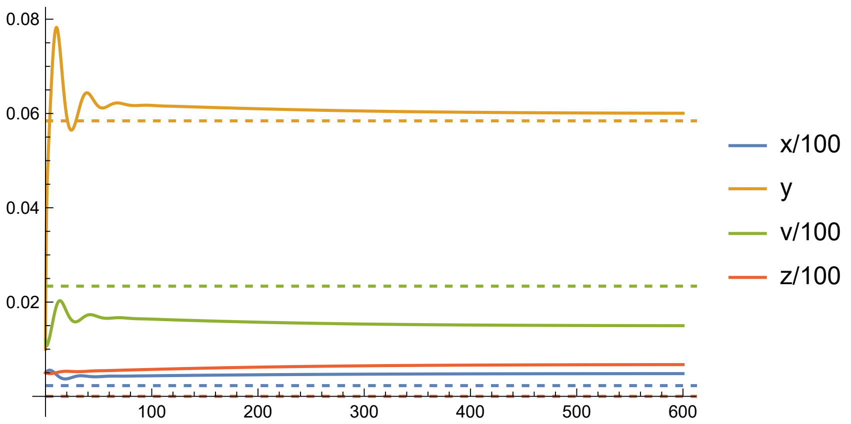
In[ ]:=
```
b=9;
Print["E*",Es=EsS//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi//N]]

X={x,y,v,z};
Xt=Map[#[t]&,X ];
Thread[X→Xt];
dynt=dyn/.Thread[X→Xt];
x1=dynt[[1]] ;
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x0=0.5; y0=0.01;v0=1.2;z0=0.5;
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,z[0]==z0};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,1000}];
pdy2=Plot[{x[t]/100/. sol22,y[t]/. sol22,v[t]/100/. sol22,z[t]/100/. sol22},{t,0,600},
PlotLegends→{"x/100","y","v/100","z/100"}];
pEs2=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
pEi2=Plot[{x/100/.x→Ei[[1]],y/.y→Ei[[2]],v/100/.v→Ei[[3]],z/.z→Ei[[4]]},{t,0,1000},
PlotStyle→{Dashed}];
Dyni22b=Show[pdy2,pEi2]
Dyns22=Show[pdy2,pEs2]

(*****..Parametric plot conditions***)
ppb9=ParametricPlot[{  x[t],(y[t])}/.sol22,{t,0,500}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py9=Plot[(y/.y→Ei[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb9i=Show[{ppb9,py9, Graphics[{Green,Dashed,Line[{{x/.x→Ei[[1]],0},{x/.x→Ei[[1]],1}}]}] },
Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}]],{PointSize[Large],
Style[Point[{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}],Black]}},{PointSize[Large],Point[{x0,y0}]},
Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}]
Export["pb9i.pdf",pb9i]
Export["Dyni22b.pdf",Dyni22b]
```

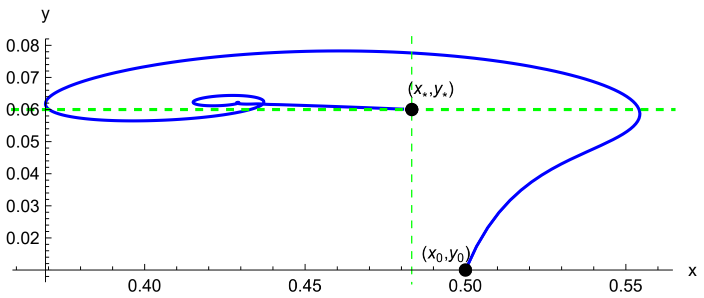
E*{0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

Out[ ]= 

Out[ ]= 

Out[ ]= 

Out[ ]= pb9i.pdf

Out[ ]= Dyni22b.pdf

When b2<b=10<b2*:

In[ ]:=

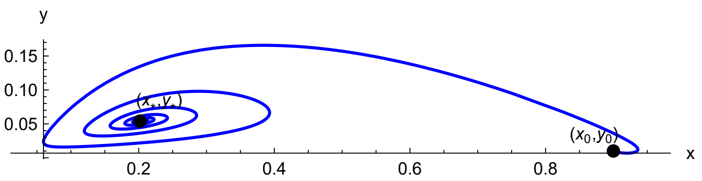
```
Clear["b"];
b=10;
Print["E*",Es=EsS//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi//N]]

X={x,y,v,z};
Xt=Map[#[t]&,X ];
Thread[X→Xt];
dynt=dyn/.Thread[X→Xt];
x1=dynt[[1]] ;
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,z[0]==z0};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,400}];


(*****..Parametric plot conditions***)
ppb10=ParametricPlot[{  x[t],(y[t])}/.sol22,{t,0,200}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py10=Plot[(y/.y→Es[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb10=Show[{ppb10},Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),
(y/.y→Es[[2]])}]],{PointSize[Large],Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Black]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}]

Export["pb10.pdf",pb10]
```

E*{0.20202, 0.0541003, 2.43451}

E+={0.30876, 0.06, 2.06588, 0.352938}

Ei={0.411479, 0.06, 1.72971, 0.635108}

Out[ ]=



Out[ ]= pb10.pdf

In[ ]:=

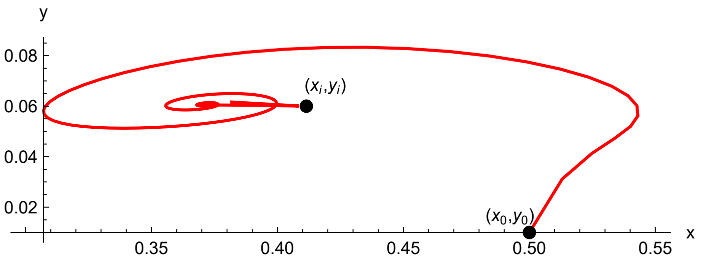
```
Print["E*",Es=EsS//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi//N]]

X={x,y,v,z};
Xt=Map[#[t]&,X ];
Thread[X→Xt];
dynt=dyn/.Thread[X→Xt];
x1=dynt[[1]] ;
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x0=0.5; y0=0.01;v0=1.2;z0=0.5;
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,
z[0]==z0};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,1000}];

ppb10i=ParametricPlot[{  x[t],(y[t])}/.sol22,{t,0,1900}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Red}];
py10i=Plot[(y/.y→Ei[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb10i=Show[{ppb10i },Epilog→{Text["(xi,yi)",Offset[{10,10},{(x/.x→Ei[[1]]),
(y/.y→Ei[[2]])}]],
{PointSize[Large],Style[Point[{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}],Black]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}]
pp10si=Show[{pb10,pb10i},Epilog→{Text["(xi,yi)",Offset[{10,10},{(x/.x→Ei[[1]]),
(y/.y→Ei[[2]])}]],{PointSize[Large],Style[Point[{(x/.x→Ei[[1]]),(y/.y→Ei[[2]])}],Black]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0i,y0i)",Offset[{-10,8},{x0,y0}]],
{Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),(y/.y→Es[[2]])}]],{PointSize[Large],
Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Black]}},{PointSize[Large],Point[{0.9,0.01}]},
Text["(x0*,y0*)",Offset[{-10,8},{0.9,0.01}]]}]
Export["pp10si.pdf",pp10si]
Export["pb10i.pdf",pb10i]
```

E*{0.20202, 0.0541003, 2.43451}

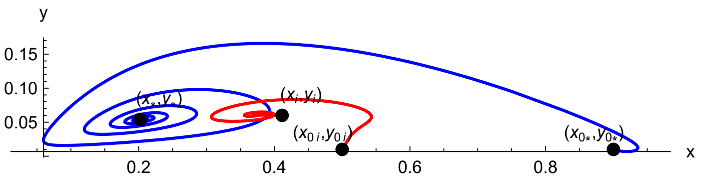
E+={0.30876, 0.06, 2.06588, 0.352938}

Ei={0.411479, 0.06, 1.72971, 0.635108}

Out[ ]=



Out[ ]=



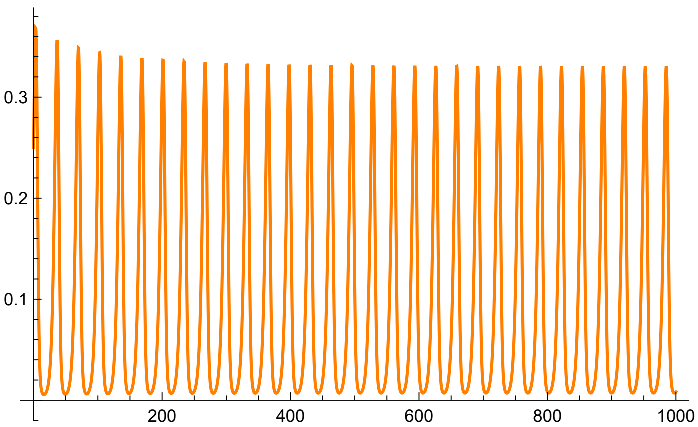
Out[ ]= pp10si.pdf

Out[•]= pb10i.pdf

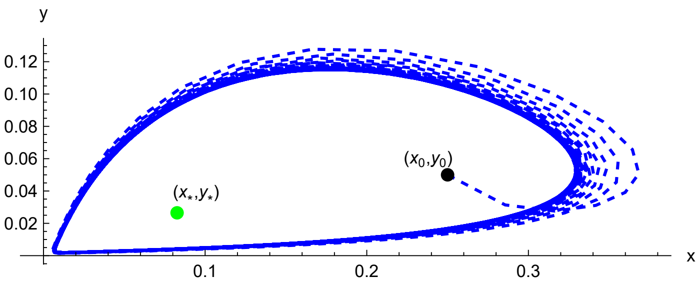
When bH<b=23<b∞:

In[•]:=
```
Clear["b"];
b=23;T=2000;
Print["E*",Es=EsS//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi//N]]
X={x,y,v,z};
Xt=Map[#[t]&,X ];
Thread[X→Xt];
dynt=dyn/.Thread[X→Xt];
x1=dynt[[1]] ;
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x01=0.25; y01=0.05;v01=0.01;z01=0.01;
ode5={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x01,y[0]==y01,v[0]==v01,
z[0]==z01};
solHI=NDSolve[ode5,{x,y,v,z},{t,0,T}];
pdyHI1=Plot[{x[t]/. solHI(*,y[t]/. solHI,v[t]/. solHI,z[t]/. solHI*)},{t,0,T/2},PlotStyle→Or
DynHI1=Show[pdyHI1,PlotRange→All]

(*****..Parametric plot conditions***)
ppb23=ParametricPlot[{  x[t],(y[t])}/.solHI,{t,0,T}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue,Dashed}];
NDSolve[{y'[t]==y[t]×(x[t]-1),x'[t]==x[t] (2-y[t]),x[0]==1,y[0]==2.7},{x,y},{t,0,10}];
py23=Plot[(y/.y→Es[[2]]),{t,0,400},PlotStyle→{Dashed,Green}];
pb23=Show[{ppb23},Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1]]),
(y/.y→Es[[2]])}]],{PointSize[Large],Style[Point[{(x/.x→Es[[1]]),(y/.y→Es[[2]])}],Green]}},
{PointSize[Large],Point[{x01,y01}]},Text["(x0,y0)",Offset[{-10,8},{x01,y01}]]}]
Export["pb23.pdf",pb23]
Export["DynHI.pdf",DynHI]
```

```
E*{0.0826446, 0.0265046, 2.91551}
E+={0.297813 + 0.339863 i, 0.06, 2.1017 - 1.11228 i, 1.75115 + 1.463 i}
Ei={0.297813 - 0.339863 i, 0.06, 2.1017 + 1.11228 i, 1.75115 - 1.463 i}
```

Out[•]=


Out[∘]=



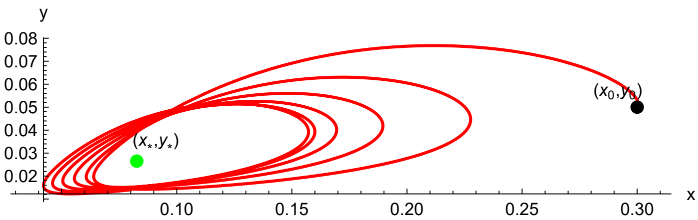
Out[∘]= pb23.pdf

Out[∘]= DynHI.pdf

In[∘]:=

```
x0=0.3; y0=0.05;v0=2;z0=0.5;T=2000;
ode5={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,z[0]==z0};
solHI=NDSolve[ode5,{x,y,v,z},{t,0,10000}];
pdyHI=Plot[{x[t]/. solHI(*,y[t]/. solHI,v[t]/. solHI,z[t]/. solHI*)},{t,0,T/2},PlotStyle→Gre
AxesLabel→{"t","x"}];
DynHIc=Show[pdyHI,PlotRange→All];
(*New initial conditions**)
ppb23c=ParametricPlot[{  x[t],(y[t])}/.solHI,{t,0,150}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Red}];
pb23c=Show[{ppb23c },Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},
{(x/.x→Es⟦1⟧),(y/.y→Es⟦2⟧)}]],{PointSize[Large],Style[Point[{(x/.x→Es⟦1⟧),
(y/.y→Es⟦2⟧)}],Green]}},{PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",
Offset[{-10,8},{x0,y0}]]}]
Export["pb23b.pdf",pb23c]
Export["DynHIb.pdf",DynHIc]
pp23cs=Show[{pb23,pb23c},Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},{(x/.x→Es⟦1⟧),
(y/.y→Es⟦2⟧)}]],{PointSize[Large],Style[Point[{(x/.x→Es⟦1⟧),(y/.y→Es⟦2⟧)}],Green]}},
{PointSize[Large],Point[{x0,y0}]},Text["(x0',y0')",Offset[{16,6},{x0,y0}]],
{PointSize[Large],Point[{x01,y01}]},Text["(x0*,y0*)",Offset[{-10,8},{x01,y01}]]}]
xtH=Show[{DynHI1,DynHIc},AxesLabel→{"t","x"}]
Export["xtH.pdf",xtH]
Export["pp23s.pdf",pp23cs]
```

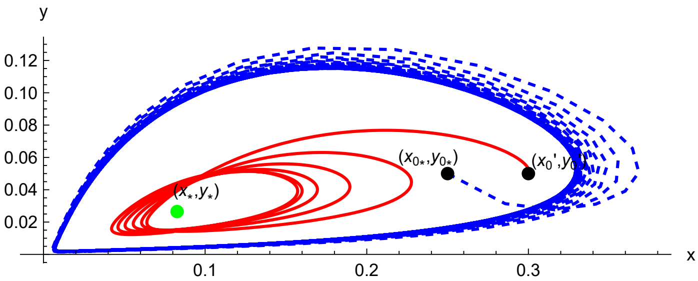
Out[∘]=

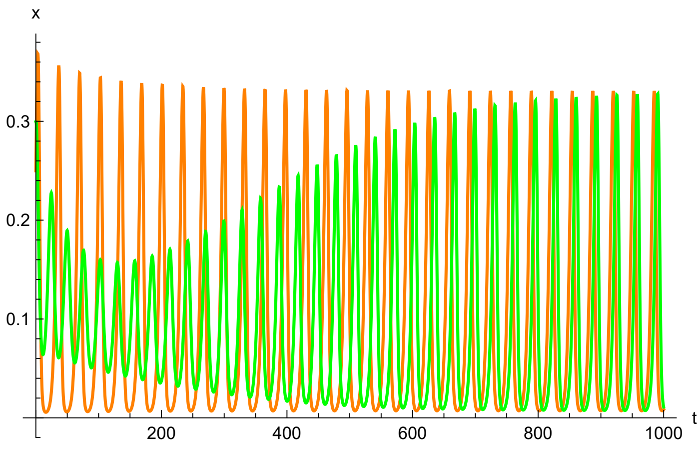


Out[∘]= pb23b.pdf

Out[∘]= DynHIb.pdf

Out[∘]=



Out[ ]= 

Out[ ]= xtH.pdf

Out[ ]= pp23s.pdf

Illustration of the limit cycle by combining EcoEvo package with Parametric plot:

In[ ]:=
```
<<EcoEvo`
(*EcoEvoDocs;*)
UnsetModel; T=10000;x0=0.9;y0=0.01;
SetModel[{Pop[x]→{Equation:→dyn[[1]],Color→Red},Pop[y]→
{Equation:→(dyn[[2]]),Color→Green},
Pop[v]→{Equation:→(dyn[[3]]),Color→Purple},Pop[z]→{Equation:→(dyn[[4]]),Color→Blue}}]

fpT=SolveEcoEq[]//FullSimplify
Print["Eigenvalues of E*:",EcoEigenvalues[fpT[[3]]]]
Print["We simulate around "]
Esn=RuleListTweak[fpT[[3]],{x,y},{0.1,0.01}]
sol=EcoSim[Esn,T];
psol=PlotDynamics[sol](*Time plot around E* *);

lc=FindEcoCycle[FinalSlice[sol]](*Finding the cycle using time slice*);
Print["Final time slice of the cycle is"]
FinalTime[lc]

pyn=Plot[(y/.fpT[[3,2]]),{t,0,T},PlotStyle→{Dashed,Green}];
cyH=Show[{ppb23,pyn, Graphics[{Green,Dashed,
Line[{{x/.fpT[[3,1]],0},{x/.fpT[[3,1]],1}}]}] },RuleListPlot[lc,{x,y},PlotStyle→{Red}],
Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.fpT[[3,1]]),(y/.fpT[[3,2]])}]],
{PointSize[Large],Style[Point[{(x/.fpT[[3,1]]),(y/.fpT[[3,2]])}],Black]}},
{PointSize[Large],Point[{x01,y01}]},Text["(x0,y0)",Offset[{10,8},{x01,y01}]]}]
Export["cyH.pdf",cyH]
```

Out[ ]= EcoEvo Package Version 1.6.4 (November 5, 2021)
Christopher A. Klausmeier <christopher.klausmeier@gmail.com>

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[•]= {{x → 0, y → 0, v → 0, z → 0}, {x → 1., y → 0, v → 0, z → 0},
{x → 0.0826446, y → 0.0265046, v → 2.91551, z → 0},
{x → 0, y → 0.06, v → -10.35, z → -2.08333},
{x → 1.33937, y → 0.06, v → -1.30704, z → -8.7697},
{x → 0.297813 + 0.339863 i, y → 0.06, v → 2.1017 - 1.11228 i, z → 1.75115 + 1.463 i},
{x → 0.297813 - 0.339863 i, y → 0.06, v → 2.1017 + 1.11228 i, z → 1.75115 - 1.463 i}}

Eigenvalues of E*:{-1.24715, 0.00415397 + 0.230094 i, 0.00415397 - 0.230094 i, -0.0200972}

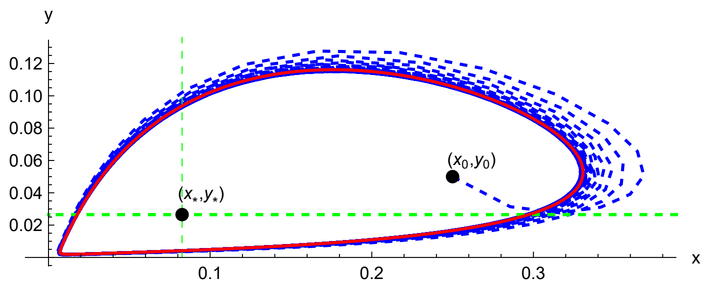
We simulate around

Out[•]= {v → 2.91551, z → 0, x → 0.182645, y → 0.0365046}

Final time slice of the cycle is

Out[•]= 32.613

Out[•]=



Out[•]= cyH.pdf