

On a four-dimensional oncolytic Virotherapy model ($\epsilon=0$)

This Mathematica Notebook is a supplementary material to the paper “On a three-dimensional and two four-dimensional oncolytic viro-therapy models”. It contains some of the calculations and illustrations appearing in the paper concerning the section on 4 dimensional model.

Section 3 : Four-dim viro-therapy model with Logistic growth when $\epsilon>0$ and $\epsilon=0$

```
In[ ]:= SetDirectory[NotebookDirectory[]];
AppendTo[$Path, Directory];
Clear["Global`*"];
Clear["K"];
(*Some aliases*)
Format[ $\beta y$ ] = Subscript[ $\beta$ , y];
Format[ $\beta v$ ] = Subscript[ $\beta$ , v];
Format[ $\beta z$ ] = Subscript[ $\beta$ , z];

Unprotect[Power];
Power[0, 0] = 1;
Protect[Power];

par = {b,  $\beta$ ,  $\lambda$ ,  $\delta$ ,  $\beta y$ ,  $\beta v$ ,  $\beta z$ , c,  $\gamma$ , K,  $\epsilon$ };
cp = Join[Thread[Drop[par, {1}] > 0], {b > 1}];
cKga1 = {K → 1,  $\gamma$  → 1};
cep1 = { $\epsilon$  → 1}; cep0 = { $\epsilon$  → 0};
R0 = b  $\beta$  K / ( $\beta$  K +  $\delta$ ) (* Reproduction number*);
cnT17 = { $\beta v$  → 0.16,  $\beta y$  → 0.48, K → 1,
 $\gamma$  → 1, b → 9,  $\beta$  → 0.11,  $\lambda$  → 0.36,  $\delta$  → 0.2,  $\beta z$  → 0.6, c → 0.036};
(*Numerical values of Tian 17 after dimensionalization*)

(***** Four dim Deterministic epidemic model with Logistic growth *****)
x1 =  $\lambda$  x (1 - (x + y) / K) -  $\beta$  x v;
y1 =  $\beta$  x v -  $\beta y$  y z -  $\gamma$  y;
v1 = - $\beta$  x v -  $\beta v$  v z + b  $\gamma$  y -  $\delta$  v;
z1 = z ( $\beta z$  y - c z $\epsilon$ );
ye = c /  $\beta z$ ; vM =  $\lambda$  (1 - ye) /  $\beta$ ; vMN = vM / . cnT17;
dyn = {x1, y1, v1, z1};
dynS = dyn /. cKga1;
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```

dyn3 = {x1, y1, v1} /. z -> 0; (*3dim case used for E* *)
dyn3S = dyn3 //. cKga1;

Print[" $\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} =$ ", dyn // FullSimplify // MatrixForm,

" the reparametrized model is  $\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} =$ ",

dynS // FullSimplify // MatrixForm]
Print["b0=", b0 = b /. Apart[Solve[R0 == 1, b][[1]] // FullSimplify]]
b0S = b0 //. cKga1;
(*****Fixed points of Tian17 using the scaling when K=1,  $\gamma=1$ ****)

fv = (ye (b - 1) - v  $\delta$ ); gv = (ye  $\beta$ y + v  $\beta$ v); hv = (1 - ye - v  $\beta$  /  $\lambda$ );

Print["xe= ", xe = hv, " , ye=", ye, " , ze =", ze = fv / gv]

Pv = Numerator[Together[v  $\beta$  xe - ye (1 +  $\beta$ y fv / gv)]] / (- $\beta$ z2  $\beta$ 2  $\beta$ v);
Print["P(v)=", pc = Collect[Together[Pv], v], " coefs are"]
coP = CoefficientList[pc, v] // Simplify

(***Fixed point when z->0***)
eq = Thread[dyn3 == {0, 0, 0}];
sol = Solve[eq, {x, y, v}] // FullSimplify;
Es = {x, y, v} /. sol[[3]]; (*Endemic point with z=0*);
Print[" Endemic point with z=0 is E*=", Es // FullSimplify]
EsS = Es /. cKga1 // FullSimplify(* E* when K= $\gamma=1$ ****)
Print["Check: when K= $\gamma=1$ , E*=", EsS]

bn = b /. Solve[EsS[[2]] == ye, b]; bnn = bn /. cnT17;
Print["y*/ye="]
EsS[[2]] / ye // FullSimplify
Print["(b1,b2)="]
bn // FullSimplify

bcn = b /. Solve[EsS[[2]][[1]] == ye, b];
bcnn = bn /. cnT17;

Dis = Chop[Collect[Discriminant[Numerator[Pv], v], b]];
solb = Solve[Dis == 0, b];
jacS = Grad[dynS, {x, y, v, z}] // FullSimplify;
Print["Jacobian"]
jacS // MatrixForm
Print["Jacobian at E_K "]
jack = FullSimplify[jacS /. Join[sol[[2]], {z -> 0}], Assumptions ->  $\epsilon \geq 0$ ];
jack // MatrixForm
Print["Jacobian at E_* when K>1"]

```

```
jacEs = FullSimplify[jacS /. Join[sol[[3]], {z → 0}], Assumptions →  $\epsilon \geq 0$ ] (*when  $K > 0$ *) ;
jacEs // MatrixForm
```

```
PR = Solve[Pv == 0, v, Cubics → False]
(*//ComplexExpand[#, TargetFunctions → {Re, Im}] &*) ;
vn = v /. PR[[1]]; vp = v /. PR[[3]]; vi = v /. PR[[2]];
Ep = {xe, ye, v, ze} /. v → vp;
Ei = {xe, ye, v, ze} /. v → vi;
```

```
JEi = jacS /. {x → xe, y → ye, v → vi, z → ze};
JEp = jacS /. {x → xe, y → ye, v → vp, z → ze};
JEm = jacS /. {x → xe, y → ye, v → vn, z → ze};
jacEs1 = jacEs /. cKga1;
Print["Tr[J[E*]] corresponding to 4 dim model when  $K=\gamma=1$  is "]
Tr[jacEs1]
Print["Tr[J[E*]] > 0 iff"]
Reduce[Join[{Tr[jacEs1] /.  $\epsilon \rightarrow 0$ ] > 0}, Drop[cp, {9, 10}]],
Drop[par, {10, 11}]] // FullSimplify
bc2 = b /. solb[[1]] // FullSimplify;
bc1 = b /. solb[[2]] // FullSimplify;
Print["b1*=", bc1]
Print[" b2*=", bc2]
```

$$\left(* \left(-1 + \frac{b \delta}{1-b} - c - \frac{\delta \lambda}{(-1+b) \beta} + \frac{\delta}{(-1+b) \beta} \left(\frac{b \lambda (Rz-1) \beta z}{\lambda (b-Rz) + (-1+b) \gamma Rz} \right) \right) * \right)$$

$$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -v x \beta + x \left(1 - \frac{x+y}{K} \right) \lambda \\ v x \beta - y (z \beta_y + \gamma) \\ b y \gamma - v (x \beta + z \beta_v + \delta) \\ z (-c z^\epsilon + y \beta_z) \end{pmatrix} \quad \text{the reparametrized model is} \quad \begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -x (v \beta + (-1+x+y) \lambda) \\ v x \beta - y (1 + z \beta_y) \\ b y - v (x \beta + z \beta_v + \delta) \\ z (-c z^\epsilon + y \beta_z) \end{pmatrix}$$

$$b\theta = 1 + \frac{\delta}{K \beta}$$

$$xe = 1 - \frac{c}{\beta_z} - \frac{v \beta}{\lambda}, \quad ye = \frac{c}{\beta_z}, \quad ze = \frac{\frac{(-1+b) c}{\beta_z} - v \delta}{v \beta_v + \frac{c \beta_y}{\beta_z}}$$

$$P(v) = v^3 + \frac{b c^2 \beta_y \lambda}{\beta^2 \beta_v \beta_z^2} + \frac{v^2 (c \beta^2 \beta_y \beta_z + c \beta \beta_v \beta_z \lambda - \beta \beta_v \beta_z^2 \lambda)}{\beta^2 \beta_v \beta_z^2} +$$

$$\frac{v (c^2 \beta \beta_y \lambda + c \beta_v \beta_z \lambda - c \beta \beta_y \beta_z \lambda - c \beta_y \beta_z \delta \lambda)}{\beta^2 \beta_v \beta_z^2} \quad \text{coefs are}$$

$$Out[4]= \left\{ \frac{b c^2 \beta_y \lambda}{\beta^2 \beta_v \beta_z^2}, \frac{c (c \beta \beta_y + \beta_z (\beta_v - \beta_y (\beta + \delta))) \lambda}{\beta^2 \beta_v \beta_z^2}, \frac{c \beta \beta_y + c \beta_v \lambda - \beta_v \beta_z \lambda}{\beta \beta_v \beta_z}, 1 \right\}$$

Endemic point with $z=0$ is E^* =

$$\left\{ \frac{\delta}{(-1+b) \beta}, \frac{((-1+b) K \beta - \delta) \delta \lambda}{(-1+b) \beta ((-1+b) K \beta \gamma + \delta \lambda)}, \frac{\gamma ((-1+b) K \beta - \delta) \lambda}{\beta ((-1+b) K \beta \gamma + \delta \lambda)} \right\}$$

$$Out[5]= \left\{ \frac{\delta}{(-1+b) \beta}, \frac{((-1+b) \beta - \delta) \delta \lambda}{(-1+b) \beta ((-1+b) \beta + \delta \lambda)}, \frac{((-1+b) \beta - \delta) \lambda}{\beta ((-1+b) \beta + \delta \lambda)} \right\}$$

Check: when $K=\gamma=1$, $E^* = \left\{ \frac{\delta}{(-1+b)\beta}, \frac{((-1+b)\beta - \delta)\delta\lambda}{(-1+b)\beta((-1+b)\beta + \delta\lambda)}, \frac{((-1+b)\beta - \delta)\lambda}{\beta((-1+b)\beta + \delta\lambda)} \right\}$

$y^*/y_e =$

$$Out[*]= \frac{\beta_z((-1+b)\beta - \delta)\delta\lambda}{(-1+b)\beta((-1+b)\beta + \delta\lambda)}$$

$(b1, b2) =$

$$Out[*]= \left\{ \frac{2c\beta - c\delta\lambda + \beta_z\delta\lambda - \delta\sqrt{\lambda}\sqrt{-4c\beta_z + (c - \beta_z)^2\lambda}}{2c\beta}, \frac{2c\beta - c\delta\lambda + \beta_z\delta\lambda + \delta\sqrt{\lambda}\sqrt{-4c\beta_z + (c - \beta_z)^2\lambda}}{2c\beta} \right\}$$

Jacobian

$Out[*]//MatrixForm=$

$$\begin{pmatrix} -v\beta - (-1+2x+y)\lambda & -x\lambda & -x\beta & 0 \\ v\beta & -1-z\beta_y & x\beta & -y\beta_y \\ -v\beta & b & -x\beta - z\beta_v - \delta & -v\beta_v \\ 0 & z\beta_z & 0 & y\beta_z - c z^\epsilon(1+\epsilon) \end{pmatrix}$$

Jacobian at E_K

$Out[*]//MatrixForm=$

$$\begin{pmatrix} \lambda - 2K\lambda & -K\lambda & -K\beta & 0 \\ 0 & -1 & K\beta & 0 \\ 0 & b & -K\beta - \delta & 0 \\ 0 & 0 & 0 & -\theta^\epsilon c(1+\epsilon) \end{pmatrix}$$

Jacobian at E_* when $K>1$

$Out[*]//MatrixForm=$

$$\begin{pmatrix} -\frac{\delta\lambda(\delta\lambda + (-1+b)\beta((-1+2K)\gamma + (-1+K)\lambda))}{(-1+b)\beta((-1+b)\beta\gamma + \delta\lambda)} & \frac{\delta\lambda}{\beta-b\beta} & -\frac{\delta}{-1+b} & 0 \\ \frac{\gamma((-1+b)K\beta - \delta)\lambda}{(-1+b)K\beta\gamma + \delta\lambda} & -1 & \frac{\delta}{-1+b} & \frac{\beta_y\delta(K(\beta-b\beta) + \delta)\lambda}{(-1+b)\beta((-1+b)K\beta\gamma + \delta\lambda)} \\ \frac{\gamma(K(\beta-b\beta) + \delta)\lambda}{(-1+b)K\beta\gamma + \delta\lambda} & b & \frac{b\delta}{1-b} & \frac{\beta_v\gamma(K(\beta-b\beta) + \delta)\lambda}{\beta((-1+b)K\beta\gamma + \delta\lambda)} \\ 0 & 0 & 0 & -\theta^\epsilon c(1+\epsilon) + \frac{\beta_z((-1+b)K\beta - \delta)\delta\lambda}{(-1+b)\beta((-1+b)K\beta\gamma + \delta\lambda)} \end{pmatrix}$$

$Tr[J[E_*]]$ corresponding to 4 dim model when $K=\gamma=1$ is

$$Out[*]= -1 + \frac{b\delta}{1-b} - \theta^\epsilon c(1+\epsilon) - \frac{\delta\lambda}{(-1+b)\beta} + \frac{\beta_z((-1+b)\beta - \delta)\delta\lambda}{(-1+b)\beta((-1+b)\beta + \delta\lambda)}$$

$Tr[J[E_*]] > 0$ iff

$$Out[*]= \beta > 0 \ \&\& \ \lambda > 0 \ \&\& \ 0 < \delta < (-1+b)\beta \ \&\& \ \beta_y > 0 \ \&\& \ \beta_v > 0 \ \&\&$$

$$((-1+b)\beta - \delta)\delta\lambda \left((-1+b)\beta^2(-1+b+b\delta) + \beta\delta(-2+\beta_z+b(2-\beta_z+\delta))\lambda + \delta^2\lambda(\beta_z+\lambda) \right) <$$

$$0 \ \&\& \ 0 < c < - \frac{(-1+b)\beta^2(-1+b+b\delta) + \beta\delta(-2+\beta_z+b(2-\beta_z+\delta))\lambda + \delta^2\lambda(\beta_z+\lambda)}{(-1+b)\beta((-1+b)\beta + \delta\lambda)} \ \&\& \ \gamma > 0$$

b1*=

$$\frac{1}{27 c^2 \beta^2 \beta_v^2 \beta_y^2 \beta_z^2 \lambda} \left(-2 \sqrt{\beta^2 \beta_y^2 \beta_z^2 (c^2 \beta^2 \beta_y^2 + c \beta_v (-c \beta \beta_y - 3 \beta_v \beta_z + \beta_y \beta_z (\beta + 3 \delta)) \lambda + \beta_v^2 (c - \beta_z)^2 \lambda^2)}^3 + \right. \\ \left. \beta \beta_y \beta_z (2 c^2 \beta^2 \beta_y^2 + c \beta_v (-5 c \beta \beta_y + \beta_z (-9 \beta_v + 5 \beta \beta_y + 9 \beta_y \delta)) \lambda + 2 \beta_v^2 (c - \beta_z)^2 \lambda^2) \right. \\ \left. (\beta_v \beta_z \lambda - c (\beta \beta_y + \beta_v \lambda)) \right)$$

b2*=

$$\frac{1}{27 c^2 \beta^2 \beta_v^2 \beta_y^2 \beta_z^2 \lambda} \left(2 \sqrt{\beta^2 \beta_y^2 \beta_z^2 (c^2 \beta^2 \beta_y^2 + c \beta_v (-c \beta \beta_y - 3 \beta_v \beta_z + \beta_y \beta_z (\beta + 3 \delta)) \lambda + \beta_v^2 (c - \beta_z)^2 \lambda^2)}^3 + \right. \\ \left. \beta \beta_y \beta_z (2 c^2 \beta^2 \beta_y^2 + c \beta_v (-5 c \beta \beta_y + \beta_z (-9 \beta_v + 5 \beta \beta_y + 9 \beta_y \delta)) \lambda + 2 \beta_v^2 (c - \beta_z)^2 \lambda^2) \right. \\ \left. (\beta_v \beta_z \lambda - c (\beta \beta_y + \beta_v \lambda)) \right)$$

2-2)Interior equilibrium

Analysis of the stability of the interior point E_* when $K=\gamma=1$:

In[]:=

```
(*Since z=0, we reduce our analysis for this point to 3 dimensions*)
jac3=Grad[dyn3/.cKga1,{x,y,v}]/FullSimplify;
Print["J(E_*) is when K=1"](*in three dimension*)
JsS=(jac3/.sol[[3]]/.cKga1)/FullSimplify;
JsS//MatrixForm
Trs=Tr[JsS](*the trace of J[E*] in 3 dimension*);
Print["Tr(J(E*))=",Trs//FullSimplify]
Print["Det(J(E*))=",Det[JsS]/FullSimplify]

pc=Collect[Det[IdentityMatrix[3]-JsS],psi];
coT=CoefficientList[pc,psi]/FullSimplify;
Length[coT]
Print["a1=",a1=coT[[3]], ", a2=",a2=coT[[2]], ", a3=",a3=coT[[1]]]
H2=a1*a2-a3;
Print["H2(b0)=",H2/.b->b0S//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H2]]/.cKga1//FullSimplify]
phiB=Collect[Numerator[Together[H2]]/(delta lambda),b]/.cKga1;
cofi=CoefficientList[phiB,b](*Coefficients of phi(b)*);
Print["value of phi(b) at crit b is "]
phiB/.b->b0S/.cKga1//FullSimplify
```

J(E_*) is when K=1

Out[]//MatrixForm=

$$\begin{pmatrix} \frac{\delta \lambda}{\beta - b \beta} & \frac{\delta \lambda}{\beta - b \beta} & -\frac{\delta}{-1+b} \\ \frac{((-1+b) \beta - \delta) \lambda}{(-1+b) \beta + \delta \lambda} & -1 & \frac{\delta}{-1+b} \\ \frac{(\beta - b \beta + \delta) \lambda}{(-1+b) \beta + \delta \lambda} & b & \frac{b \delta}{1-b} \end{pmatrix}$$

$$\text{Tr}(J(E_*)) = -1 + \frac{\delta (b \beta + \lambda)}{\beta - b \beta}$$

$$\text{Det}(J(E_*)) = \frac{\delta (\beta - b \beta + \delta) \lambda}{(-1+b) \beta}$$

Out[]= 4

$$a_1 = \frac{\beta (-1 + b + b \delta) + \delta \lambda}{(-1 + b) \beta}, \quad a_2 = \frac{\delta \lambda \left((-1 + b) \beta (-1 + \beta + \delta + b (1 - \beta + \delta)) + ((-1 + b)^2 \beta + b \delta^2) \lambda \right)}{(-1 + b)^2 \beta ((-1 + b) \beta + \delta \lambda)}, \quad a_3 = \delta \left(1 + \frac{\delta}{\beta - b \beta} \right) \lambda$$

$$H2(b0) = (1 + \beta + \delta) \lambda (1 + \beta + \delta + \lambda)$$

$$\text{Denominator of H2 is } (-1 + b)^3 \beta^2 ((-1 + b) \beta + \delta \lambda)$$

value of $\phi(b)$ at crit b is

$$\text{Out[*]} = \frac{\delta^3 (1 + \beta + \delta) \times (1 + \lambda) \times (1 + \beta + \delta + \lambda)}{\beta}$$

In[*]:=

(*Analysis of the stability of E* in 4 dim when $\epsilon=0$ *)

```
pc=Collect[Det[ψ IdentityMatrix[4]-(jacEs1/.cep0)],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Length[coT]
a1=coT[[4]]//FullSimplify;a2=coT[[3]]//FullSimplify;a3=coT[[2]]//FullSimplify;a4=coT[[1]]//FullSimplify
Print["a1=",a1," ",a2=",",a2," ",a3=",",a3," ",a4=",",a4]
H4=a1*a2*a3-a3^2+a1^2 a4;
Print["H2 (b0)=",H4/.b->b0S//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H4]]//FullSimplify]
φb4=Collect[Numerator[Together[H4]],b];
cofi=CoefficientList[φb4,b];
Length[cofi]
Print["value of φ(b) at crit b is "]
φb4/.b->b0S//FullSimplify
```

Out[*]= 5

$$a_1 = c + \frac{(-1 + b) \beta^2 (-1 + b + b \delta) + \beta \delta (-2 + \beta_z + b (2 - \beta_z + \delta)) \lambda + \delta^2 \lambda (\beta_z + \lambda)}{(-1 + b) \beta ((-1 + b) \beta + \delta \lambda)}$$

$$, \quad a_2 = \frac{1}{(-1 + b)^2 \beta^2} \left((-1 + b) c \beta (\beta (-1 + b + b \delta) + \delta \lambda) + \frac{1}{(-1 + b) \beta + \delta \lambda} \right.$$

$$\delta \lambda \left(-(-1 + b)^2 \beta^3 + \beta_z \delta^2 \lambda - (-1 + b) \beta^2 (1 - \beta_z - \delta + b (-1 + \beta_z) \times (1 + \delta) + \lambda - b \lambda) + \right.$$

$$\left. \beta \delta (b \delta \lambda + \beta_z (-1 + b + b \delta + \lambda - b \lambda)) \right), \quad a_3 = \frac{1}{(-1 + b)^3 \beta^2 ((-1 + b) \beta + \delta \lambda)^2}$$

$$\delta \lambda \left(((-1 + b) \beta - \delta) ((-1 + b)^4 \beta^3 + (-1 + b) \beta \delta (\beta_z + (-1 + b) \beta (-2 + 2 b + \beta_z) - \beta_z (b + \delta + b \delta)) \lambda - \right.$$

$$\delta ((-1 + b)^2 \beta (\beta_z - \delta) + b \beta_z \delta^2) \lambda^2) -$$

$$(-1 + b) c \beta ((-1 + b) \beta + \delta \lambda) ((-1 + b)^2 \beta^2 - b \delta^2 \lambda - (-1 + b) \beta (-1 + b + \delta + b \delta + (-1 + b) \lambda)) \left. \right)$$

$$, \quad a_4 = \frac{((-1 + b) \beta - \delta) \delta \lambda (\beta_z \delta (\beta - b \beta + \delta) \lambda + (-1 + b) c \beta ((-1 + b) \beta + \delta \lambda))}{(-1 + b)^2 \beta^2 ((-1 + b) \beta + \delta \lambda)}$$

$$H2(b0) = c (1 + \beta + \delta) \times (1 + c + \beta + \delta) \lambda (c + \lambda) (1 + \beta + \delta + \lambda)$$

$$\text{Denominator of H2 is } (-1 + b)^6 \beta^5 ((-1 + b) \beta + \delta \lambda)^4$$

Out[*]= 11

value of $\phi(b)$ at crit b is

$$\text{Out[*]} = \frac{c \delta^{10} (1 + \beta + \delta) \times (1 + c + \beta + \delta) \lambda (1 + \lambda)^4 (c + \lambda) (1 + \beta + \delta + \lambda)}{\beta}$$

```
FindInstance[Join[{Tr[jacEs/.cKga1/.ε→0]>0},Drop[cp,{9,10}],Drop[par,{10,11}]]
```

Out[]:= $\{ \{ b \rightarrow 2, \beta \rightarrow 2, \lambda \rightarrow 1, \delta \rightarrow 1, \beta_y \rightarrow 1, \beta_v \rightarrow 1, \beta_z \rightarrow 28, c \rightarrow 1, \gamma \rightarrow 1 \} \}$

Numerical values $\epsilon=0$:

```
In[ ]:= cn=Join[cnT17,cep0];
cb=NSolve[(ϕb//.Drop[cnT17,{5}])=0,b,WorkingPrecision→10]
bM=Max[Table[Re[b/.cb[[i]]],{i,Length[cb]}]];
Print["bH=",bH=N[bM,30]]
Chop[{vn,vi,vp}/.cnT17];

Print["b0=",b0/.cn//N," ",b1=", bnn[[1]], ", b2=", bnn[[2]], " ",bH=",bH]
PRN=Chop[PR//.cn//N](*values of the roots v*);
Print["E*",Chop[Es/.cn]//N]
Print["E+=",Chop[Ep/.cn]//N]
Print["Ei=",Chop[Ei/.cn]//N]
Print["Eigv of E*:",Append[Eigenvalues[JsS],Es[[2]]-ye]//.cn," ",Eigv of E+:",
Chop[Eigenvalues[JEp//.cn]]//N//FullSimplify," Eigv of Eim:",Chop[Eigenvalues[J Ei//.cn]]//N,
" ",Eigv of EK:",Eigenvalues[jack//.cn]//N]

Print["b1*=", bc1//.cn]
Print[" b2*=", bc2//.cn]
```

Out[]:= $\{ \{ b \rightarrow 0.299052 \}, \{ b \rightarrow 0.835329 - 0.231156 i \}, \{ b \rightarrow 0.835329 + 0.231156 i \}, \{ b \rightarrow 19.0121 \} \}$

bH=19.0121

b0=2.81818 , b1=3.58676, b2=8.66779 ,bH=19.0121

E*={0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

Eigv of E*:{-1.25056, -0.0281268 - 0.20904 i, -0.0281268 + 0.20904 i, -0.00155844}
 , Eigv of E+:{-1.29833, -0.0332218 + 0.21255 i, -0.0332218 - 0.21255 i, 0.000834552}
 Eigv of Eim:{-1.69849, -0.076539 + 0.209952 i, -0.076539 - 0.209952 i, -0.00803072}
 , Eigv of EK:{-1.7081, 0.398103, -0.36, -0.036}

b1*=-0.00697038

b2*=10.2462

Determination of the endemic points with respect to x:

```

In[ ]:= Print["ve(x) ="]
vee=Solve[(x1/x)==0,v][[1]]/.ckga1
Print["ze(x) ="]
zee=Solve[v1+(x1/x)==0,z][[1]]/.ckga1//FullSimplify
(* (y1/.zee/.vee/.y->ye); *)
xee=Solve[Collect[Numerator[Together[(y1/.zee/.vee/.y->ye)]],x]==0,x] (*The expressions are so
Print["Number of endemic x"]
Length[xee]
Chop[Evaluate[xee//.cnT17]] (*Numerical check*)

```

ve(x) =

$$\text{Out}[] = \left\{ v \rightarrow \frac{(1-x-y)\lambda}{\beta} \right\}$$

ze(x) =

$$\text{Out}[] = \left\{ z \rightarrow \frac{b y - v (\beta + x \beta + \delta) + \lambda - (x + y) \lambda}{v \beta v} \right\}$$

Number of endemic x

Out[] = 3

Out[] = { {x → 0.249944}, {x → 1.20177}, {x → 0.483284} }

3)Sections 3.1---3.3(in paper):Figures used in the manuscript (*Run the previous cell*)

Numerical illustrations when $\epsilon=0$ (Bifurcations diagrams, parametric plots, and 3D plot)

Bifurcation diagram when b varies:

```

In[ ]:= ClearParameters;
βv=0.16; βy=0.48;K=1;b=9;γ=1;λ=0.36;β=0.11;δ=0.2;βz=0.6; c=0.036;ε=0;
Print["{b2*,b1*}=",{b/.solb[[1]],b/.solb[[2]]} ," ,", and b0=",b0, " , bH=",bH,
" ,b1=", bnn[[1]], " , b2=", bnn[[2]]]
Clear["b"];
vs=-(((-1+b) β-δ) λ)/β ((-1+b) β+δ λ) (*v of E* * ****);
bL=28; max=3.5;bmin=-0.9;
lin1=Line[{{ bc1,0},{ bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{ b0,0},{ b0,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
linH=Line[{{ bH,0},{ bH,max}}];
liH=Graphics[{Thick,Black,Dashed,linH}];
linb1=Line[{{ bnn[[1]],0},{ bnn[[1]],max}}];
lib1=Graphics[{Thick,Black,Dashed,linb1}];
linb2=Line[{{ bnn[[2]],0},{ bnn[[2]],max}}];
lib2=Graphics[{Thick,Black,Dashed,linb2}];
linb9=Line[{{ 9,0},{ 9,max}}];

```



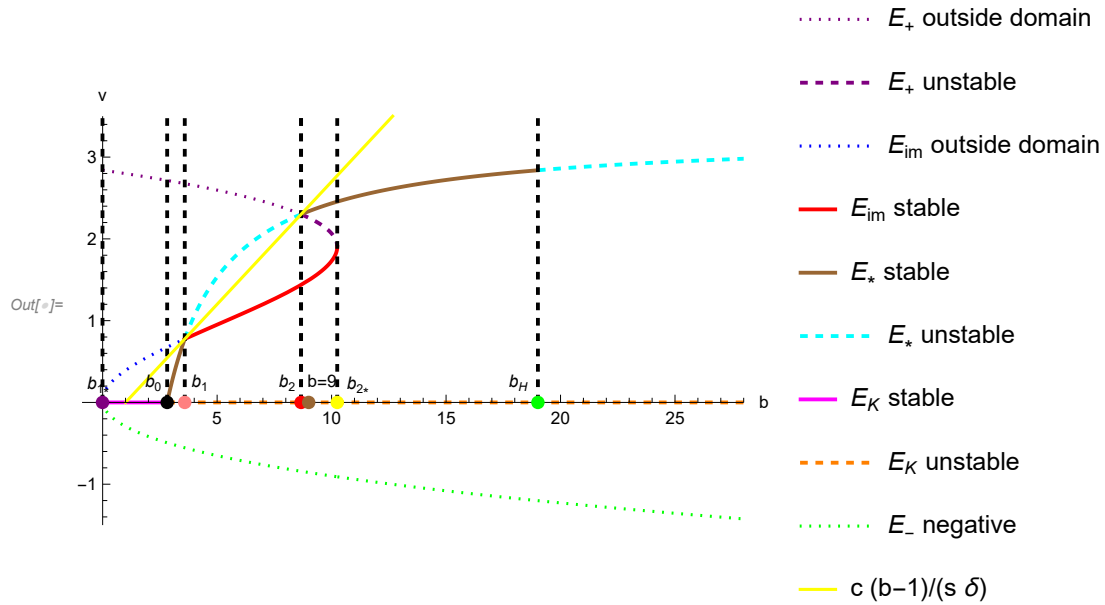
```

lib9=Graphics[{Thick,Black,Dashed,linb2}];
pn=Plot[{vn},{b,0,bL},PlotStyle→{Green,Dotted},PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"
p0=Plot[0,{b,0,bL},PlotStyle→{Brown,Thick},PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E0 s
ppa=Plot[{vp},{b,0,bnn[[2]]},PlotStyle→{Purple,Dotted},PlotRange→{{bmin,bL},{-2,max}},
PlotLegends→{"E* outside domain"}];
ppb=Plot[{vp},{b,bnn[[2]],bL},PlotStyle→{Purple,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E* unstable"}];
pi1=Plot[{vi},{b,0,bnn[[1]]},PlotStyle→{Blue,Dotted},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"Eim outside domain"}];
pi2=Plot[{vi},{b,bnn[[1]],bL},PlotStyle→{Red,Thick},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"Eim stable"}];
pi3=Plot[{vi},{b,bnn[[2]],bL},PlotStyle→{Blue,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}}(*,PlotLegends→{"Ei unstable"}*);
ps1=Plot[{vs},{b,b0,bnn[[1]]},PlotStyle→{Brown,Thick},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E* stable"}];
ps2=Plot[{vs},{b,bnn[[1]],bnn[[2]]},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}},PlotLegends→{"E* unstable"}];
ps3=Plot[{vs},{b,bnn[[2]],bH},PlotStyle→{Brown,Thick},
PlotRange→{{bmin,bL},{0,max}}(*,PlotLegends→{"E* stable"}*);
ps4=Plot[{vs},{b,bH,bL},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{bmin,bL},{-2,max}}(*,PlotLegends→{"E* unstable"}*);
pdf1=Plot[{0},{b,0,b0},PlotStyle→{Magenta,Thick},
PlotRange→{{bmin,bL},{0,max}},PlotLegends→{"Ek stable"}];
pdf2=Plot[{0},{b,b0,bL},PlotStyle→{Orange,Thick,Dashed},
PlotRange→{{bmin,bL},{0,max}},PlotLegends→{"Ek unstable"}];
pvmax=Plot[{c (b-1)/(βz δ) (*, (λ(1-c/βz)/β*))},{b,0,bL},PlotStyle→{Yellow},
PlotRange→{{0,bL},{0,max}},
PlotLegends→{"c (b-1)/(s δ)"}];

bifT=Show[{ppa,ppb,pi1,pi2,ps1,ps2,ps3,ps4,pdf1,pdf2,pn,li1,li2,li3,
lib2,lib1,lib9,pvmax,liH},AxesLabel→{"b","v"},PlotRange→{{bmin,bL},{-3/2,max}},
Epilog→{Text["b1*",Offset[{-2,10},{bc1,0}],{PointSize[Large],
Style[Point[{bc1,0}],Purple]},
Text["b2*",Offset[{11,10},{bc2,0}],{PointSize[Large],Style[Point[{bc2,0}],Yellow]},
Text["b0",Offset[{-7,11},{b0,0}],{PointSize[Large],Style[Point[{b0,0}],Black]},
Text["bH",Offset[{-10,11},{bH,0}],{PointSize[Large],Style[Point[{bH,0}],Green]},
Text["b1",Offset[{8,11},{bnn[[1]],0}],{PointSize[Large],Style[Point[{bnn[[1]],0}],Pink]},
Text["b2",Offset[{-7,11},{bnn[[2]],0}],{PointSize[Large],Style[Point[{bnn[[2]],0}],Red]},
Text["b=9",Offset[{7,11},{9,0}],{PointSize[Large],Style[Point[{9,0}],Brown]}}]
Export["BiifT17.pdf",bifT]

```

$\{b_{2*}, b_{1*}\} = \{10.2462, -0.00697038\}$, and $b_0 = 2.81818$, $b_H = 19.0121$, $b_1 = 3.58676$, $b_2 = 8.66779$



Out[*]= BiifT17.pdf

Numerical solution of the stability (Bifurcation diagram) wrt v:

In[*]:= **(*Checks on the stability of the fixed points*)**

```
Print["Eigenvalues of E*:"]  
Print["E* between b0 and b1"]  
Eigenvalues[jacEs1] /. b -> 3 // N  
Print["E* between b1 and b2"]  
Eigenvalues[jacEs1] /. b -> 5 // N  
Print["E* between b2 and bH"]  
Eigenvalues[jacEs1] /. b -> 15 // N  
Print["E* after bH"]  
Eigenvalues[jacEs1] /. b -> 25 // N  
Print["Eigenvalues of Eim:"]  
Print["Eim between 0 and b1"]  
Chop[Eigenvalues[J Ei] /. b -> 2 // N]  
Print["E* between b1 and b2*"]  
Chop[Eigenvalues[J Ei] /. b -> 9 // N]  
Print["Eigenvalues of E+:"]  
Print["Eim between 0 and b2"]  
Chop[Eigenvalues[J Ep] /. b -> 2 // N]  
Print["E* between b2 and b2*"]  
Chop[Eigenvalues[J Ep] /. b -> 9 // N]
```

Eigenvalues of E^* :

E^* between b_0 and b_1

```
Out[*]= {-0.0225504, -1.29942, -0.311695, -0.0161607}
```

E^* between b_1 and b_2

```
Out[*]= {0.0100227, -1.26091, -0.0763636 - 0.159107 i, -0.0763636 + 0.159107 i}
```

E^* between b_2 and b_H

```
Out[*]= {-0.0126814, -1.24799, -0.00652554 - 0.223959 i, -0.00652554 + 0.223959 i}
```

E^* after b_H

```
Out[*]= {-0.0212776, -1.24705, 0.00572283 - 0.230932 i, 0.00572283 + 0.230932 i}
```

Eigenvalues of E_{im} :

E_{im} between θ and b_1

```
Out[*]= {-1.00272, -0.249728, -0.0735624, 0.0456119}
```

E^* between b_1 and b_{2^*}

```
Out[*]= {-1.69849, -0.00803072, -0.076539 - 0.209952 i, -0.076539 + 0.209952 i}
```

Eigenvalues of E_+ :

E_{im} between θ and b_2

```
Out[*]= {-0.559828, -0.123746, 0.053141 - 0.0959514 i, 0.053141 + 0.0959514 i}
```

E^* between b_2 and b_{2^*}

```
Out[*]= {-1.29833, 0.000834552, -0.0332218 - 0.21255 i, -0.0332218 + 0.21255 i}
```

(x-b)-Bifurcation diagram :

```
In[*]:= Clear["K"];
K=1;
Print["{b2*,b1*}=", {b/.solb[[1]],b/.solb[[2]]} ,", and b0=",b0, " , bH=",bH,
" ,b1=", bnn[[1]], " , b2=", bnn[[2]]]
Clear["b"];
xs=EsS[[1]](*x of E* * ****);
bL=22; max=1.4;bmin=-0.9;
lin1=Line[{{ bc1,0},{ bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{ b0,0},{ b0,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
linH=Line[{{ bH,0},{ bH,max}}];
liH=Graphics[{Thick,Black,Dashed,linH}];
linb1=Line[{{ bnn[[1]],0},{ bnn[[1]],max}}];
lib1=Graphics[{Thick,Black,Dashed,linb1}];
linb2=Line[{{ bnn[[2]],0},{ bnn[[2]],max}}];
lib2=Graphics[{Thick,Black,Dashed,linb2}];
linb9=Line[{{ 9,0},{9,max}}];
lib9=Graphics[{Thick,Black,Dashed,linb2}];
(*pn=Plot[{x/.xee[[1]]},{b,0,b0},PlotStyle->{Purple,Dotted,Thick},PlotRange->{{bmin,bL},{-2,max}}
pn1=Plot[{x/.xee[[1]]},{b,0,bnn[[2]]},PlotStyle->{Purple,Dotted,Thick},PlotRange->{{bmin,bL},{-2,m
PlotLegends->{"E+ (z<0)"}];
```

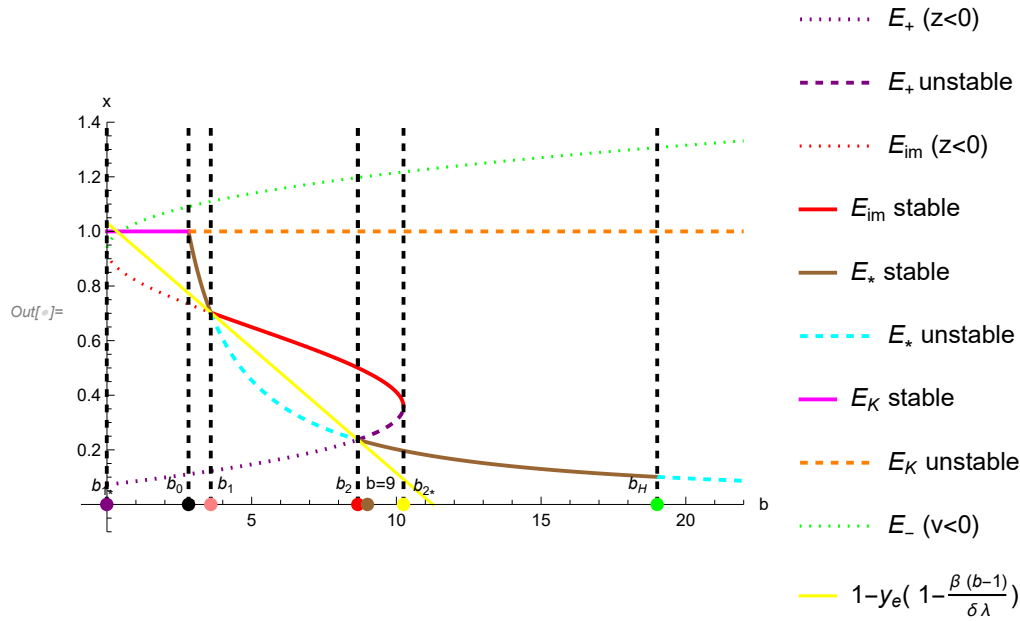
```

pn2=Plot[{x/.xee[[1]],{b,bnn[[2]],bL}},PlotStyle->{Purple,Dashed,Thick},PlotRange->{{bmin,bL},{-2,
PlotLegends->{"E+ unstable"}}];
p0=Plot[0,{b,0,bL},PlotStyle->{Brown,Thick},PlotRange->{{bmin,bL},{-2,max}},PlotLegends->{"E0 s
ppa=Plot[{x/.xee[[2]],{b,0,bL}},PlotStyle->{Green,Dotted},PlotRange->{{bmin,bL},{-2,max}},
PlotLegends->{"E- (v<0)"}];
pi1=Plot[{x/.xee[[3]],{b,0,bnn[[1]]}},PlotStyle->{Red,Dotted},
PlotRange->{{bmin,bL},{-2,max}},PlotLegends->{"Eim (z<0)"}];
pi2=Plot[{x/.xee[[3]],{b,bnn[[1]],bL}},PlotStyle->{Red,Thick},
PlotRange->{{bmin,bL},{-2,max}},PlotLegends->{"Eim stable"}];
ps1=Plot[{xs},{b,b0,bnn[[1]]},PlotStyle->{Brown,Thick},
PlotRange->{{bmin,bL},{-2,max}},PlotLegends->{"E+ stable"}];
ps2=Plot[{xs},{b,bnn[[1]],bnn[[2]]},PlotStyle->{Cyan,Thick,Dashed},
PlotRange->{{bmin,bL},{-2,max}},PlotLegends->{"E+ unstable"}];
ps3=Plot[{xs},{b,bnn[[2]],bH},PlotStyle->{Brown,Thick},
PlotRange->{{bmin,bL},{0,max}}(*,PlotLegends->{"E+ stable"}*);
ps4=Plot[{xs},{b,bH,bL},PlotStyle->{Cyan,Thick,Dashed},
PlotRange->{{bmin,bL},{-2,max}}(*,PlotLegends->{"E+ unstable"}*);
pdf1=Plot[K,{b,0,b0},PlotStyle->{Magenta,Thick},
PlotRange->{{bmin,bL},{0,max}},PlotLegends->{"Ek stable"}];
pdf2=Plot[K,{b,b0,bL},PlotStyle->{Orange,Thick,Dashed},
PlotRange->{{bmin,bL},{0,max}},PlotLegends->{"Ek unstable"}];
pxmax=Plot[{1-ye-β/λ (ye (b-1)/δ)},{b,0,bL},PlotStyle->{Yellow},
PlotRange->{{0,bL},{0,max}},
PlotLegends->{"1-ye(1- $\frac{\beta}{\delta \lambda} (b-1)$ )"}];

bifx=Show[{pn1,pn2,pi1,pi2,ps1,ps2,ps3,ps4,pdf1,pdf2,ppa,pxmax,li1,li2,li3,
lib2,lib1,lib9,liH},AxesLabel->{"b","x"},PlotRange->{{bmin,bL},{-0.1,max}},
Epilog->{Text["b1*",Offset[{-2,10},{bc1,0}],{PointSize[Large],
Style[Point[{bc1,0}],Purple]}],
Text["b2*",Offset[{11,10},{bc2,0}],{PointSize[Large],Style[Point[{bc2,0}],Yellow]}],
Text["b0",Offset[{-7,11},{b0,0}],{PointSize[Large],Style[Point[{b0,0}],Black]}],
Text["bH",Offset[{-10,11},{bH,0}],{PointSize[Large],Style[Point[{bH,0}],Green]}],
Text["b1",Offset[{8,11},{bnn[[1]],0}],{PointSize[Large],Style[Point[{bnn[[1]],0}],Pink]}],
Text["b2",Offset[{-7,11},{bnn[[2]],0}],{PointSize[Large],Style[Point[{bnn[[2]],0}],Red]}],
Text["b=9",Offset[{7,11},{9,0}],{PointSize[Large],Style[Point[{9,0}],Brown]}]}]
Export["bifx.pdf",bifx]

```

$\{b_{2*}, b_{1*}\} = \{10.2462, -0.00697038\}$, and $b_0 = 2.81818$, $b_H = 19.0121$, $b_1 = 3.58676$, $b_2 = 8.66779$



Out[*]= bifx.pdf

Bifurcation diagram of codimension 2:

In[*]=

```

Clear["b"];
Clear["β"];
(*Here we modified a slightly the values from FindInstance when Tr[jacEs]>0*)
(*λ=1;δ=0.2;βy=0.48;βv=0.16;βz=28;c=1;γ=1;K=1; *)

bm=1; bM=25; ym=0; yM=0.38;
trEp=Tr[JEp];
trEim=Tr[JEi];
detEs=Det[jacEs1];

R1=Style[Text["R_0>1,H_*(b)<0,Δ<0",{16,0.30}],13];
R1a=Style[Text["Attractor cycle",{17,0.25}],13];
R2=Style[Text["R_0>1,H_*(b)>0,Δ<0",{13,0.13}],10];
R2a=Style[Text["E_* Stable",{15,0.09}],10];
RIII=Style[Text["R_0>1,H_*(b)>0,Δ>0",{10,0.06}],10];
RIIIa=Style[Text["E_im stable,E_* unstable",{15,0.04}],10];
RII=Style[Text["E_*",{4,0.08}],10];
RI=Style[Text["E_K Stable",{6,0.01}],10];
RIa=Style[Text["R_0<1,H_*(b)>0,Δ>0",{4,0.03}],10];
Rbi=Style[Text["E_im",{11,0.09}],9];
Rbia=Style[Text["E_*",{13,0.075}],9];

pt0s=Text["b_0",Offset[{-10,6},{b0/.β→0.11,0.11}]];
pt0={PointSize[Medium],Style[Point[{b0/.β→0.11,0.11}],Black]};
ptHs=Text["b_H",Offset[{10,6},{bH,0.11}]];
ptH1={PointSize[Medium],Style[Point[{bH,0.11}],Black]};
ptb2as=Text["b_{2*}",Offset[{14,-3},{bc2/.β→0.11,0.11}]];
ptb2a={PointSize[Medium],Style[Point[{bc2/.β→0.11,0.11}],Black]};
ptb2bs=Text["b_2",Offset[{-6,-5},{bnn[[2]],0.11}]];
ptb2b={PointSize[Medium],Style[Point[{bnn[[2]],0.11}],Black]};
ptb1s=Text["b_1",Offset[{8,6},{bnn[[1]],0.11}]];

```

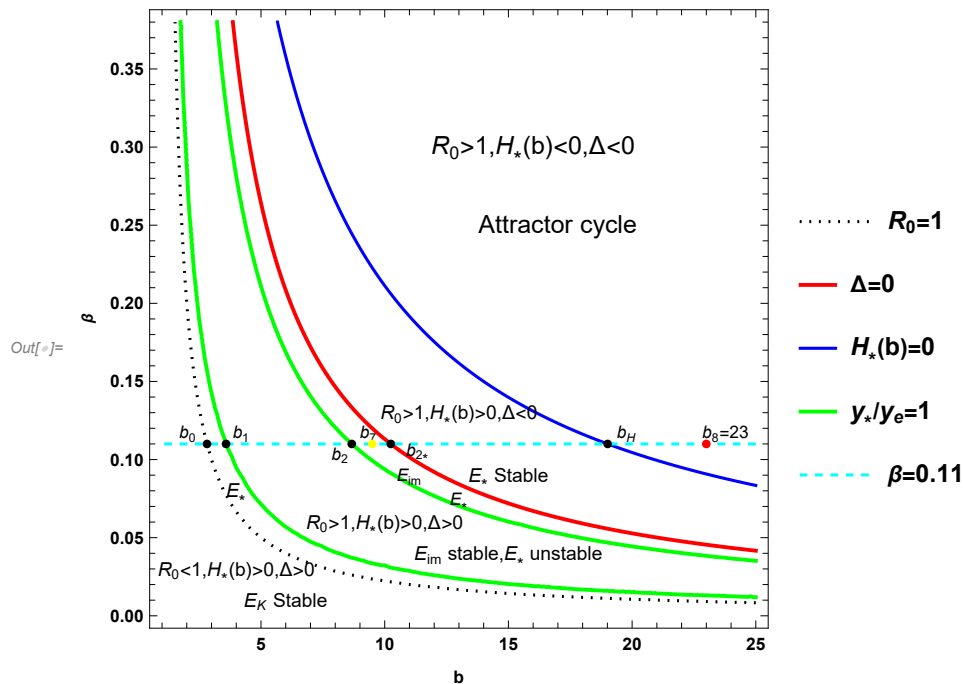
```

ptb1={PointSize[Medium],Style[Point[{bnn[[1]],0.11}],Black]};
ptbas=Text["b",Offset[{-2,6},{9.5,0.11}]];
ptba={PointSize[Medium],Style[Point[{9.5,0.11}],Yellow]};
ptbbs=Text["b8=23",Offset[{10,6},{23,0.11}]];
ptbb={PointSize[Medium],Style[Point[{23,0.11}],Red]};
epi={R1,R1a,R2,R2a,RIII,RIIIa,RII,RI,RIa,Rbi,Rbia,pt0s,pt0,ptHs,ptH1,ptb2as,ptb2a,ptb2bs,ptb2b,
ptbas,ptba,ptbbs,ptbb,ptb1s,ptb1};

pR0=ContourPlot[R0==1,{b,bm,bM},{β,ym,yM],ContourStyle→{Black,Dotted},
FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"R0=1"}];
(*ptrEs=ContourPlot[Tr[jacEs]==0,{b,bm,bM},{β,ym,yM],ContourStyle→{Green,Bold},
FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Tr[J(E*)]=0"}];*)
pphi=ContourPlot[H2(*or φb*)==0,{b,bm,bM},{β,ym,yM],ContourStyle→{Blue,Bold},PlotPoints→18
FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"H*(b)=0"}];
pdetEs=ContourPlot[detEs==0,{b,bm,bM},{β,ym,yM],ContourStyle→{Green,Bold},
FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Det[J(E*)]=0"}];
pdis=ContourPlot[Dis==0,{b,bm,bM},{β,ym,yM],ContourStyle→{Red,Thick},PlotPoints→180,
FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Δ=0"}];
(*ptrEi=ContourPlot[trEim==0,{b,bm,bM},{β,ym,yM],ContourStyle→{Red,Dashed},PlotPoints→180,
FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Tr[J(Eim)]=0"}];*)
pbeta=ContourPlot[β==0.11,{b,bm,bM},{β,ym,yM],ContourStyle→{Cyan,Dashed},
FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"β=0.11"}];
ppar=ContourPlot[ES[[2]]/ye==1,{b,bm,bM},{β,ym,yM],ContourStyle→{Green,Thick},
FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"y*/ye=1"}];

peq={pR0,pdis,pphi,ppar(*pdetEs,ptrEi*)(*ptrEs,*),pbeta};
pcut=Show[peq,Epilog→epi]
Export["map4.pdf",pcut]

```



Out[]= map4.pdf

```

In[ ]:= (**Some checks**)
(*For Ein**)
Chop[Evaluate[xee[[3]] /. b → 2]] // N
Chop[Evaluate[zee /. vee /. xee[[3]] /. y → ye /. b → 2]] // N
Chop[vee /. zee /. xee[[3]] /. y → ye /. b → 2] // N
(*For E**)
Chop[Evaluate[xee[[1]] /. b → 3]] // N
Chop[zee /. vee /. xee[[1]] /. y → ye /. b → 3] // N
Chop[vee /. zee /. xee[[1]] /. y → ye /. b → 3] // N

```

```
Out[ ]:= {x → 0.768927}
```

```
Out[ ]:= {z → -0.439052}
```

```
Out[ ]:= {v → 0.559875}
```

```
Out[ ]:= {x → 0.113348}
```

```
Out[ ]:= {z → -0.912093}
```

```
Out[ ]:= {v → 2.70541}
```

```

In[ ]:= Chop[Evaluate[xee[[2]] /. b → 6]] // N
zee /. vee /. xee[[2]] /. y → ye /. b → 6 // N
vee /. zee /. xee[[2]] /. y → ye /. b → 6 // N

```

```
Out[ ]:= {x → 1.15674}
```

```
Out[ ]:= {z → -5.21725 + 3.87119 × 10-16 i}
```

```
Out[ ]:= {v → -0.709334 - 9.08364 × 10-17 i}
```

Parametric plots at the intervals of stability:

When $b_2 < b = 9 < b_2^*$:

In[]:=

```

b=9;
Print["E*",Es=EsS//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi//N]]

X={x,y,v,z};
Xt=Map[#t]&,X];
Thread[X→Xt];
dynt=dyn/.Thread[X→Xt];
x1=dynt[[1]];
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,z[0]==z0};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,400}];
pdy2=Plot[{x[t]/100/.sol22,y[t]/.sol22,v[t]/100/.sol22,z[t]/.sol22},{t,0,400},
PlotLegends→{"x/100","y","v/100","z"}];
pEs2=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
pEi2=Plot[{x/100/.x→Ei[[1]],y/.y→Ei[[2]],v/100/.v→Ei[[3]],z/.z→Ei[[4]]},{t,0,1000},
PlotStyle→{Dashed}];
Dyni22=Show[pdy2,pEi2];
DyNs22=Show[pdy2,pEs2]

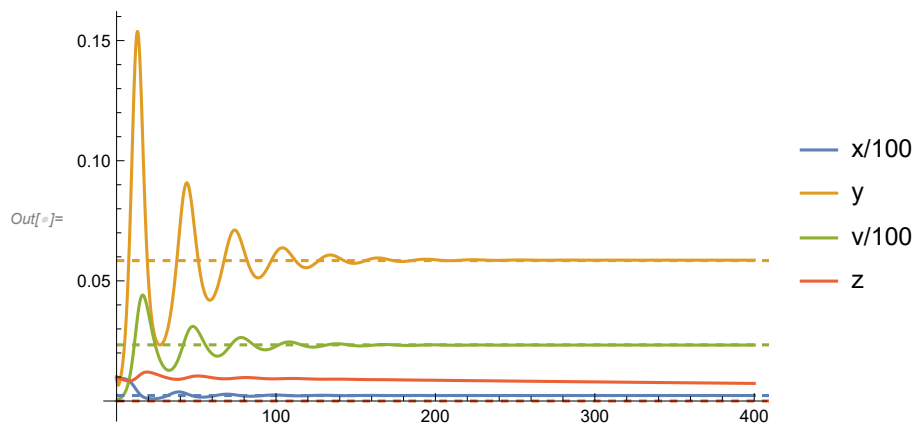
(*****Parametric plot conditions****)
ppb9=ParametricPlot[{x[t],(y[t])}/.sol22,{t,0,200},AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py9=Plot[(y/.y→Es[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb9=Show[{ppb9,py9,Graphics[{Green,Dashed,Line[{x/.x→Es[[1]],0},{x/.x→Es[[1]],1}]}]},
Epilog→{{Text["(x*,y*)",Offset[{10,10},{x/.x→Es[[1]],(y/.y→Es[[2]])}],{PointSize[Large],
Style[Point[{x/.x→Es[[1]],(y/.y→Es[[2]])}],Black]}},{PointSize[Large],Point[{x0,y0}]},
Text["(x0,y0)",Offset[{-10,8},{x0,y0}]}]}]
Export["pb9.pdf",pb9]
Export["Dyni22.pdf",Dyni22]
Export["DyNs22.pdf",DyNs22]

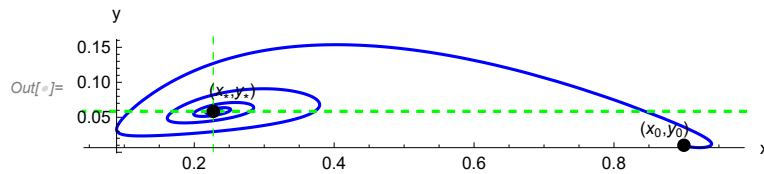
```

E*={0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}





Out[]:= pb9.pdf

Out[]:= Dyni22.pdf

Out[]:= Dyns22.pdf

When $b_2 < b = 9 < b_2^*$ and different initial values :

```
In[ ]:=
b=9;
Print["E*", Es=EsS//N]
Print["E+=", Ep=Chop[{xe,ye,v,ze}/.v->vp//N]]
Print["Ei=", Ei=Chop[{xe,ye,v,ze}/.v->vi//N]]

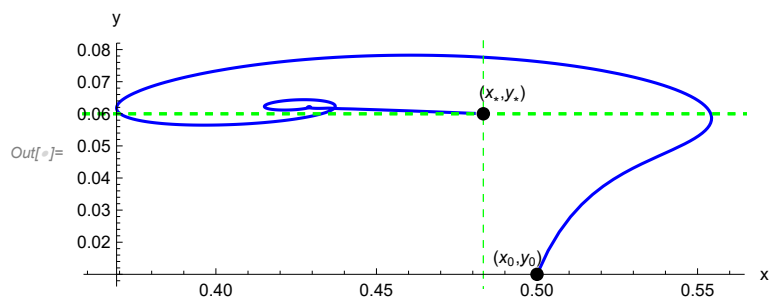
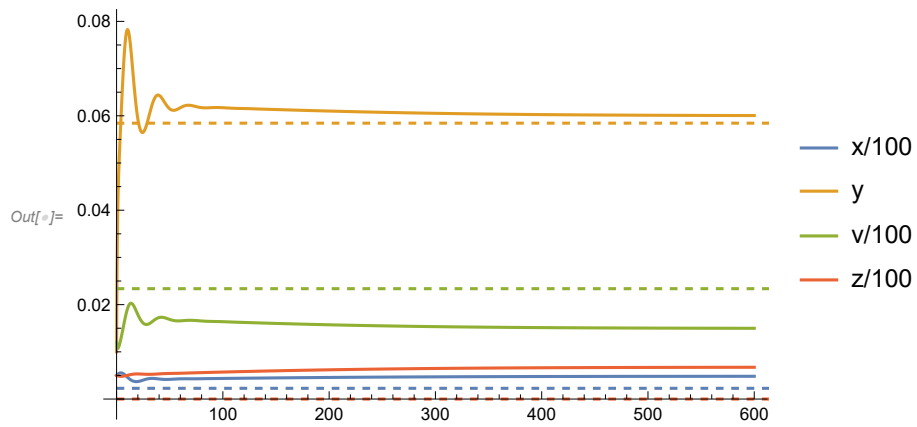
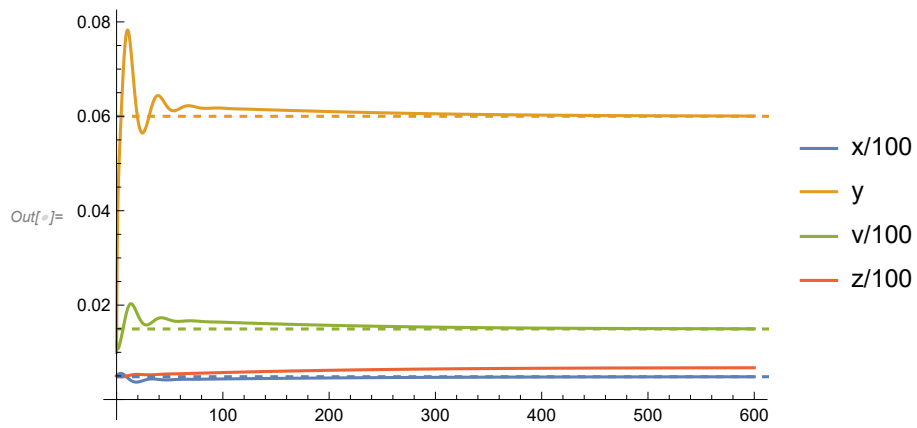
X={x,y,v,z};
Xt=Map[#t]&,X];
Thread[X->Xt];
dynt=dyn/.Thread[X->Xt];
x1=dynt[[1]];
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x0=0.5; y0=0.01; v0=1.2; z0=0.5;
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,z[0]==z0};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,1000}];
pdy2=Plot[{x[t]/100/.sol22,y[t]/.sol22,v[t]/100/.sol22,z[t]/100/.sol22},{t,0,600},
PlotLegends->{"x/100","y","v/100","z/100"}];
pEs2=Plot[{x/100/.x->Es[[1]],y/.y->Es[[2]],v/100/.v->Es[[3]],z/.z->0},{t,0,1000},
PlotStyle->{Dashed}];
pEi2=Plot[{x/100/.x->Ei[[1]],y/.y->Ei[[2]],v/100/.v->Ei[[3]],z/.z->Ei[[4]]},{t,0,1000},
PlotStyle->{Dashed}];
Dyni22b=Show[pdy2,pEi2]
Dyna22=Show[pdy2,pEs2]

(*****Parametric plot conditions***)
ppb9=ParametricPlot[{x[t],(y[t])}/.sol22,{t,0,500},AxesLabel->{"x","y"},
PlotRange->Full,PlotStyle->{Blue}];
py9=Plot[(y/.y->Ei[[2]]},{t,0,800},PlotStyle->{Dashed,Green}];
pb9i=Show[{ppb9,py9,Graphics[{Green,Dashed,Line[{(x/.x->Ei[[1]),0},{x/.x->Ei[[1]],1}]}]},
Epilog->{{Text["(x*,y*)",Offset[{10,10},{(x/.x->Ei[[1])},{(y/.y->Ei[[2])}]}],{PointSize[Large]},
Style[Point[{(x/.x->Ei[[1])},{(y/.y->Ei[[2])}]}],Black]}},{PointSize[Large],Point[{x0,y0}],
Text["(x0,y0)",Offset[{-10,8},{x0,y0}]}]}]
Export["pb9i.pdf",pb9i]
Export["Dyni22b.pdf",Dyni22b]
```

E*={0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}



$Out[t]=$ pb9i.pdf

$Out[t]=$ Dyni22b.pdf

When $b_2 < b = 10 < b_2^*$:

In[]:=

```

Clear["b"];
b=9.5;
Print["E*",Es=EsS//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi//N]]

X={x,y,v,z};
Xt=Map[#t]&,X];
Thread[X→Xt];
dynt=dyn/.Thread[X→Xt];
x1=dynt[[1]];
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,z[0]==z0};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,400}];
pdy2=Plot[{x[t]/100/.sol22,y[t]/.sol22,v[t]/100/.sol22,z[t]/.sol22},{t,0,400},
PlotLegends→{"x/100","y","v/100","z"}];
pEs2=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
Dyns22=Show[pdy2,pEs2]

(*****Parametric plot conditions****)
ppb10=ParametricPlot[{x[t],(y[t])}/.sol22,{t,0,200},AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py10=Plot[(y/.y→Es[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb10=Show[{ppb10,Epilog→{{Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1])],
(y/.y→Es[[2])}]}},{PointSize[Large],Style[Point[{(x/.x→Es[[1])],(y/.y→Es[[2])}],Black}}},
{PointSize[Large],Point[{x0,y0}],Text["(x0,y0)",Offset[{-10,8},{x0,y0}]}]}]

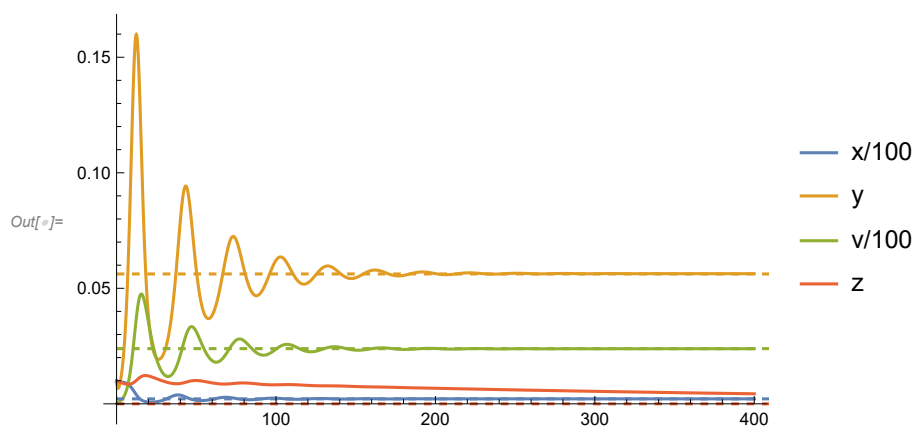
Export["pb10.pdf",pb10]
Export["Dyns22T.pdf",Dyns22]

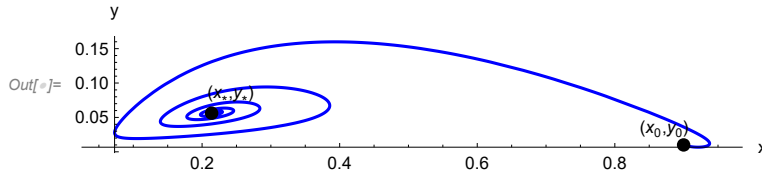
```

E*={0.213904, 0.0562055, 2.38873}

E+={0.273477, 0.06, 2.18135, 0.195148}

Ei={0.453156, 0.06, 1.59331, 0.67437}





Out[]:= pb10.pdf

Out[]:= Dyns22T.pdf

In[]:=

```
Print["E*", Es=EsS//N]
Print["E+=", Ep=Chop[{xe,ye,v,ze}/.v->vp//N]]
Print["Ei=", Ei=Chop[{xe,ye,v,ze}/.v->vi//N]]

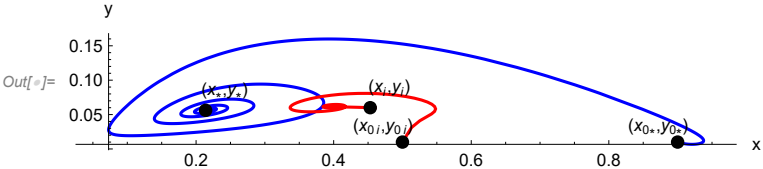
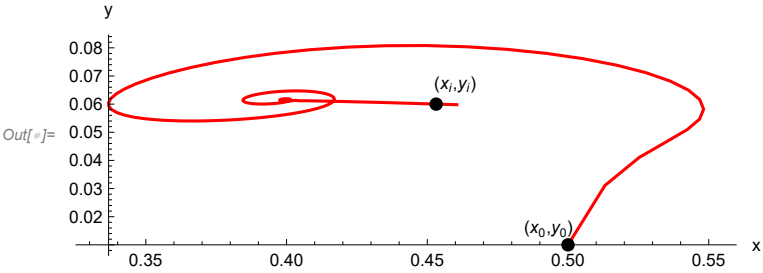
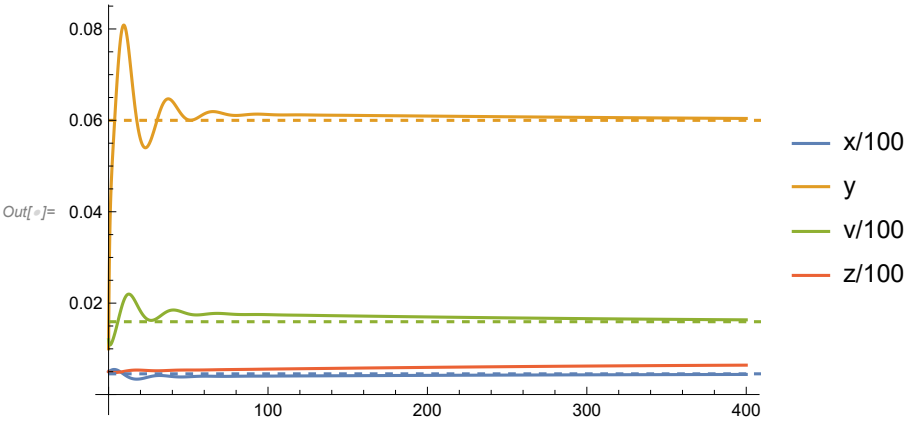
X={x,y,v,z};
Xt=Map[# [t]&, X];
Thread[X->Xt];
dynt=dyn/.Thread[X->Xt];
x1=dynt[[1]];
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x0=0.5; y0=0.01; v0=1.2; z0=0.5;
ode3={x'[t]==x1, y'[t]==y1, v'[t]==v1, z'[t]==z1, x[0]==x0, y[0]==y0, v[0]==v0,
z[0]==z0};
sol22=NDSolve[ode3, {x,y,v,z}, {t,0,1000}];
pdy2=Plot[{x[t]/100/.sol22, y[t]/.sol22, v[t]/100/.sol22, z[t]/100/.sol22}, {t,0,400},
PlotLegends->{"x/100", "y", "v/100", "z/100"}];
pEi2=Plot[{x/100/.x->Ei[[1]], y/.y->Ei[[2]], v/100/.v->Ei[[3]], z/.z->Ei[[4]]}, {t,0,1000},
PlotStyle->{Dashed}];
Dyni22=Show[pdy2, pEi2]

ppb10i=ParametricPlot[{x[t], (y[t])}/.sol22, {t,0,1900}, AxesLabel->{"x", "y"},
PlotRange->Full, PlotStyle->{Red}];
py10i=Plot[(y/.y->Ei[[2]]), {t,0,800}, PlotStyle->{Dashed, Green}];
pb10i=Show[{ppb10i}, Epilog->{{Text["(x_i, y_i)", Offset[{10,10}, {x/.x->Ei[[1]],
(y/.y->Ei[[2]])}],
{PointSize[Large], Style[Point[{x/.x->Ei[[1]], (y/.y->Ei[[2]])}], Black]}},
{PointSize[Large], Point[{x0,y0}], Text["(x_0, y_0)", Offset[{-10,8}, {x0,y0}]]}],
pp10si=Show[{pb10, pb10i}, Epilog->{{Text["(x_i, y_i)", Offset[{10,10}, {x/.x->Ei[[1]],
(y/.y->Ei[[2]])}],
{PointSize[Large], Style[Point[{x/.x->Ei[[1]], (y/.y->Ei[[2]])}], Black]}},
{PointSize[Large], Point[{x0,y0}], Text["(x_0i, y_0i)", Offset[{-10,8}, {x0,y0}]]},
{Text["(x_*, y_*)", Offset[{10,10}, {x/.x->Es[[1]], (y/.y->Es[[2]])}],
{PointSize[Large],
Style[Point[{x/.x->Es[[1]], (y/.y->Es[[2]])}], Black]}}, {PointSize[Large], Point[{0.9,0.01}],
Text["(x_0*, y_0*)", Offset[{-10,8}, {0.9,0.01}]]}],
Export["pp10si.pdf", pp10si]
Export["pb10i.pdf", pb10i]
Export["Dyni22T.pdf", Dyni22]
```

E*={0.213904, 0.0562055, 2.38873}

E+={0.273477, 0.06, 2.18135, 0.195148}

Ei={0.453156, 0.06, 1.59331, 0.67437}



Out[t]= pp10si.pdf

Out[t]= pb10i.pdf

Out[t]= Dyni22T.pdf

When $b_H < b = 23 < b_\infty$:

In[]:=

```

Clear["b"];
b=23;T=2000;
Print["E*",Es=EsS//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi//N]]
X={x,y,v,z};
Xt=Map[#&[t]&,X];
Thread[X→Xt];
dynt=dyn/.Thread[X→Xt];
x1=dynt[[1]];
y1=dynt[[2]];
v1=dynt[[3]];
z1=dynt[[4]];
x01=0.25; y01=0.05;v01=0.01;z01=0.01;
ode5={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x01,y[0]==y01,v[0]==v01,
z[0]==z01};
solHI=NDSolve[ode5,{x,y,v,z},{t,0,T}];
pdyHI1=Plot[{x[t]/.solHI(*,y[t]/.solHI,v[t]/.solHI,z[t]/.solHI*)},{t,0,T/2},PlotStyle→Orange,
DynHI1=Show[pdyHI1,PlotRange→All]

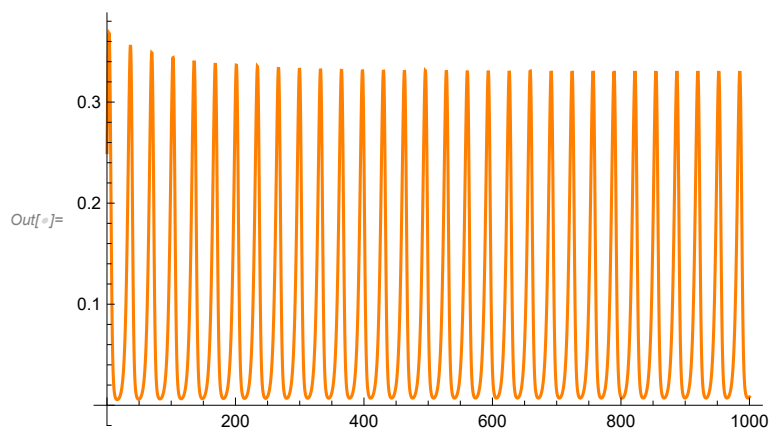
(*****Parametric plot conditions****)
ppb23=ParametricPlot[{x[t],(y[t])}/.solHI,{t,0,T}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue,Dashed}];
NDSolve[{y'[t]==y[t]×(x[t]-1),x'[t]==x[t] (2-y[t]),x[0]==1,y[0]==2.7},{x,y},{t,0,10}];
py23=Plot[(y/.y→Es[[2]]},{t,0,400},PlotStyle→{Dashed,Green}];
pb23=Show[{ppb23},Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},{x/.x→Es[[1]],
(y/.y→Es[[2]])}],{PointSize[Large],Style[Point[{(x/.x→Es[[1]),(y/.y→Es[[2]])}],Green}}},
{PointSize[Large],Point[{x01,y01}],Text["(x0,y0)",Offset[{-10,8},{x01,y01}]]}}]
Export["pb23.pdf",pb23]
Export["DynHI.pdf",DynHI]

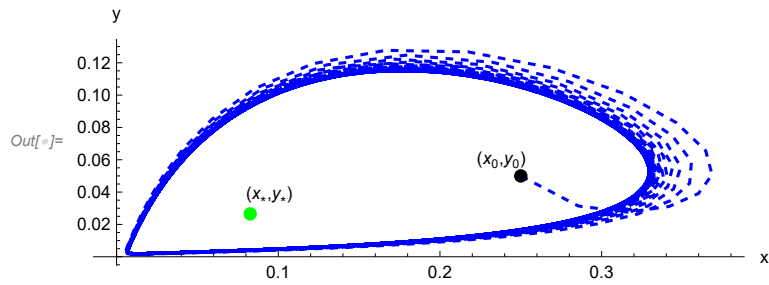
```

E*{0.0826446, 0.0265046, 2.91551}

E+={0.297813 + 0.339863 i, 0.06, 2.1017 - 1.11228 i, 1.75115 + 1.463 i}

Ei={0.297813 - 0.339863 i, 0.06, 2.1017 + 1.11228 i, 1.75115 - 1.463 i}

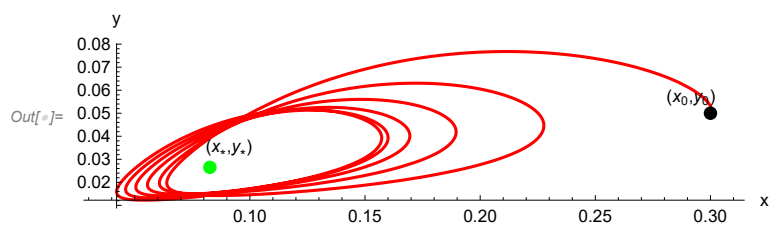




Out[]= pb23.pdf

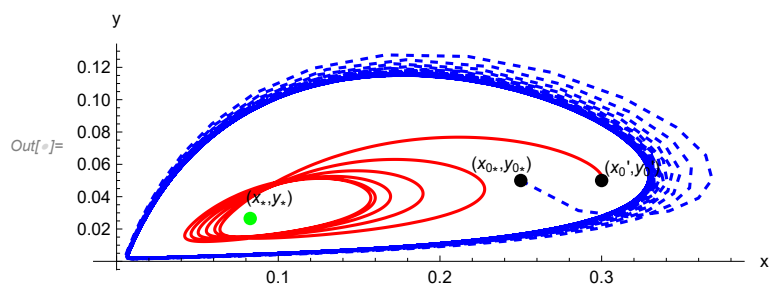
Out[]= DynHI.pdf

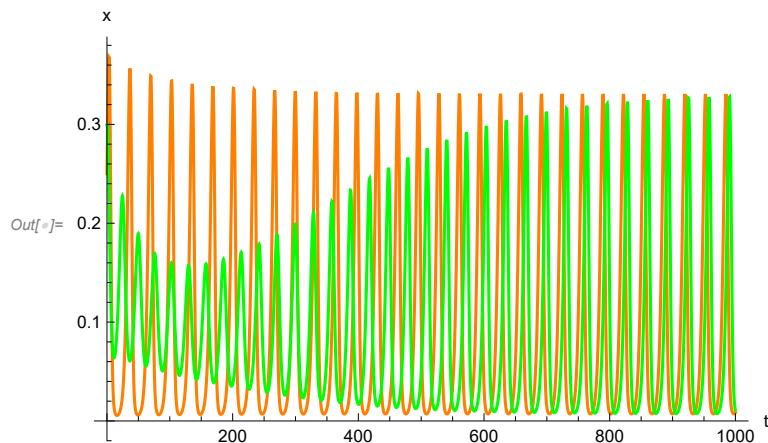
```
In[ ]:=
x0=0.3; y0=0.05;v0=2;z0=0.5;T=2000;
ode5={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==x0,y[0]==y0,v[0]==v0,z[0]==z0};
solHI=NDSolve[ode5,{x,y,v,z},{t,0,10000}];
pdyHI=Plot[{x[t]/.solHI(*,y[t]/.solHI,v[t]/.solHI,z[t]/.solHI*)},{t,0,T/2},PlotStyle->Green,
AxesLabel->{"t","x"}];
DynHIc=Show[pdyHI,PlotRange->All];
(*New initial conditions*)
ppb23c=ParametricPlot[{x[t],(y[t])}/.solHI,{t,0,150},AxesLabel->{"x","y"},
PlotRange->Full,PlotStyle->{Red}];
pb23c=Show[{ppb23c},Epilog->{{Thick,Text["(x*,y*)",Offset[{10,10},
{x/.x->Es[[1]],(y/.y->Es[[2]])}],{PointSize[Large],Style[Point[{(x/.x->Es[[1]],
(y/.y->Es[[2]])}],Green}],{PointSize[Large],Point[{x0,y0}],Text["(x0,y0)",
Offset[{-10,8},{x0,y0}]}]}]}];
Export["pb23b.pdf",pb23c]
Export["DynHIb.pdf",DynHIc]
pp23cs=Show[{pb23,pb23c},Epilog->{{Thick,Text["(x*,y*)",Offset[{10,10},
{x/.x->Es[[1]],(y/.y->Es[[2]])}],{PointSize[Large],Style[Point[{(x/.x->Es[[1]],(y/.y->Es[[2]])}],Green}],
{PointSize[Large],Point[{x0,y0}],Text["(x0',y0')",Offset[{16,6},{x0,y0}]}],
{PointSize[Large],Point[{x01,y01}],Text["(x0*,y0*)",Offset[{-10,8},{x01,y01}]}]}]}];
xtH=Show[{DynHI1,DynHIc},AxesLabel->{"t","x"}];
Export["xtH.pdf",xtH]
Export["pp23s.pdf",pp23cs]
```



Out[]= pb23b.pdf

Out[]= DynHIb.pdf





Out[]= xth.pdf

Out[]= pp23s.pdf

Illustration of the limit cycle by combining EcoEvo package with Parametric plot:

```
In[ ]:= <<EcoEvo`
(*EcoEvoDocs;*)
UnsetModel; T=10000;x0=0.9;y0=0.01;
SetModel[{Pop[x]→{Equation:→dyn[[1]],Color→Red},Pop[y]→
{Equation:→(dyn[[2]]),Color→Green},
Pop[v]→{Equation:→(dyn[[3]]),Color→Purple},Pop[z]→{Equation:→(dyn[[4]]),Color→Blue}}]

fpT=SolveEcoEq[]//FullSimplify
Print["Eigenvalues of E*:",EcoEigenvalues[fpT[[3]]]]
Print["We simulate around "]
Esn=RuleListTweak[fpT[[3]],{x,y},{0.1,0.01}]
sol=EcoSim[Esn,T];
psol=PlotDynamics[sol](*Time plot around E* *);

lc=FindEcoCycle[FinalSlice[sol]](*Finding the cycle using time slice*);
Print["Final time slice of the cycle is"]
FinalTime[lc]

pyn=Plot[(y/.fpT[[3,2]]),{t,0,T},PlotStyle→{Dashed,Green}];
cyH=Show[{ppb23,pyn,Graphics[{Green,Dashed,
Line[{x/.fpT[[3,1]],0},{x/.fpT[[3,1]],1}]}],RuleListPlot[lc,{x,y},PlotStyle→{Red}],
Epilog→{{Text["(x*,y*)",Offset[{10,10},{x/.fpT[[3,1]],(y/.fpT[[3,2]])}],
{PointSize[Large],Style[Point[{x/.fpT[[3,1]],(y/.fpT[[3,2]])}],Black]}},
{PointSize[Large],Point[{x01,y01}],Text["(x0,y0)",Offset[{10,8},{x01,y01}]]}}]
Export["cyH.pdf",cyH]
```

Out[]= EcoEvo Package Version 1.6.4 (November 5, 2021)

Christopher A. Klausmeier <christopher.klausmeier@gmail.com>

... **Solve:** Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numerizing the result.

$\text{Out}[*]= \{ \{x \rightarrow 0, y \rightarrow 0, v \rightarrow 0, z \rightarrow 0\}, \{x \rightarrow 1., y \rightarrow 0, v \rightarrow 0, z \rightarrow 0\},$
 $\{x \rightarrow 0.0826446, y \rightarrow 0.0265046, v \rightarrow 2.91551, z \rightarrow 0\},$
 $\{x \rightarrow 0, y \rightarrow 0.06, v \rightarrow -10.35, z \rightarrow -2.08333\},$
 $\{x \rightarrow 1.33937, y \rightarrow 0.06, v \rightarrow -1.30704, z \rightarrow -8.7697\},$
 $\{x \rightarrow 0.297813 + 0.339863 i, y \rightarrow 0.06, v \rightarrow 2.1017 - 1.11228 i, z \rightarrow 1.75115 + 1.463 i\},$
 $\{x \rightarrow 0.297813 - 0.339863 i, y \rightarrow 0.06, v \rightarrow 2.1017 + 1.11228 i, z \rightarrow 1.75115 - 1.463 i\} \}$

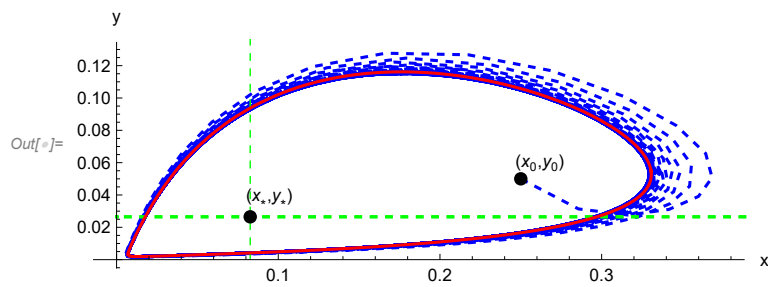
Eigenvalues of E^* : $\{-1.24715, 0.00415397 + 0.230094 i, 0.00415397 - 0.230094 i, -0.0200972\}$

We simulate around

$\text{Out}[*]= \{v \rightarrow 2.91551, z \rightarrow 0, x \rightarrow 0.182645, y \rightarrow 0.0365046\}$

Final time slice of the cycle is

$\text{Out}[*]= 32.613$



$\text{Out}[*]= \text{cyH.pdf}$