

On a four-dimensional oncolytic Virotherapy model ($\epsilon=1$)

This Mathematica Notebook is a supplementary material to the paper “On a three-dimensional and two four-dimensional oncolytic viro-therapy models”. It contains some of the calculations and illustrations appearing in the paper.

Ep1)Section 3.5(in paper): 4-Dim.Viro-therapy model when $\epsilon=1$

Ep1-1)Definition of the model and fixed points when $\epsilon=1$

```

In[61]:= SetDirectory[NotebookDirectory[]];
AppendTo[$Path, Directory];
Clear["Global`*"];
Clear["K"];
(*Some aliases*)
Format[ $\beta y$ ] = Subscript[ $\beta$ , y];
Format[ $\beta v$ ] = Subscript[ $\beta$ , v];
Format[ $\beta z$ ] = Subscript[ $\beta$ , z];

Unprotect[Power];
Power[0, 0] = 1;
Protect[Power];

par = {b,  $\beta$ ,  $\lambda$ ,  $\delta$ ,  $\beta y$ ,  $\beta v$ ,  $\beta z$ , c,  $\gamma$ , K,  $\epsilon$ };
cp = Join[Thread[Drop[par, {1}] > 0], {b > 1}];
cKga1 = {K → 1,  $\gamma$  → 1};
cep1 = { $\epsilon$  → 1};
R0 = b  $\beta$  K / ( $\beta$  K +  $\delta$ ) (* Reproduction number*);

(*cnb={b→50};
cE1ri=Join[{ $\beta y$ →1/48, K→2139.258,  $\beta$ →.0002,  $\lambda$ →.2062,
 $\gamma$ →1/18,  $\delta$ →.025,  $\beta v$ →2*10-8, c→10-3,  $\beta z$ →.027}, cep1];*)
cF1 = { $\beta$  →  $\frac{87}{2}$ ,  $\lambda$  → 1,  $\gamma$  →  $\frac{1}{128}$ ,  $\delta$  → 1/2,  $\beta y$  → 1,  $\beta v$  → 1, K → 1,  $\beta z$  → 1, c → 1,  $\epsilon$  → 1};

(***** Four dim Deterministic epidemic model with Logistic growth *****)
x1 =  $\lambda x (1 - (x + y) / K) - \beta x v$ ;
y1 =  $\beta x v - \beta y y z - \gamma y$ ;
v1 =  $-\beta x v - \beta v v z + b \gamma y - \delta v$ ;
z1 =  $z (\beta z y - c z^\epsilon)$ ;
dyn = {x1, y1, v1, z1};
dyn3 = {x1, y1, v1} /. z → 0; (*3dim case used for E* *)
Print[" $\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} =$ ", dyn // FullSimplify // MatrixForm]

Print["b0=", b0 = b /. Apart[Solve[R0 == 1, b][[1]] // FullSimplify]]

(**Fixed point when z→0**)
eq = Thread[dyn3 == {0, 0, 0}];
sol = Solve[eq, {x, y, v}] // FullSimplify;
Es = {x, y, v} /. sol[[3]]; (*Endemic point with z=0*);
Print[" Endemic point with z=0 is E=", Es // FullSimplify]
jacD = Grad[dyn /. cep1, {x, y, v, z}];
Print["J(x,y,v,z) ="]
jacD // MatrixForm
Print["Det(J(x,y,v,z)) ="]
Det[jacD] // FullSimplify

```

$$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -v x \beta + x \left(1 - \frac{x+y}{K}\right) \lambda \\ v x \beta - y (z \beta_y + \gamma) \\ b y \gamma - v (x \beta + z \beta_v + \delta) \\ z (-c z^e + y \beta_z) \end{pmatrix}$$

$$b_0 = 1 + \frac{\delta}{K \beta}$$

Endemic point with $z=0$ is $E^* =$

$$\left\{ \frac{\delta}{(-1+b) \beta}, \frac{((-1+b) K \beta - \delta) \delta \lambda}{(-1+b) \beta ((-1+b) K \beta \gamma + \delta \lambda)}, \frac{\gamma ((-1+b) K \beta - \delta) \lambda}{\beta ((-1+b) K \beta \gamma + \delta \lambda)} \right\}$$

$$J(x, y, v, z) =$$

Out[89]//MatrixForm=

$$\begin{pmatrix} -v \beta - \frac{x \lambda}{K} + \left(1 - \frac{x+y}{K}\right) \lambda & -\frac{x \lambda}{K} & -x \beta & 0 \\ v \beta & -z \beta_y - \gamma & x \beta & -y \beta_y \\ -v \beta & b \gamma & -x \beta - z \beta_v - \delta & -v \beta_v \\ 0 & z \beta_z & 0 & -2 c z + y \beta_z \end{pmatrix}$$

$$\text{Det}(J(x, y, v, z)) =$$

$$\text{Out[91]} = \frac{1}{K}$$

$$\begin{aligned} & (2 c z (K v \beta (z \beta_y + \gamma) (z \beta_v + \delta) + v x \beta (z \beta_v + \delta) \lambda - K (x \beta (z \beta_y + \gamma - b \gamma) + (z \beta_y + \gamma) (z \beta_v + \delta) \lambda) \\ & \quad + (2 x + y) (x \beta (z \beta_y + \gamma - b \gamma) + (z \beta_y + \gamma) (z \beta_v + \delta) \lambda) - \\ & \quad \beta_z (K y \gamma ((-1+b) x \beta - z \beta_v - \delta) \lambda - y (2 x + y) \gamma ((-1+b) x \beta - z \beta_v - \delta) \lambda + \\ & \quad v x \beta (-2 x z \beta_v + y \delta) \lambda + K v \beta (y \gamma (z \beta_v + \delta) + x z \beta_v \lambda)) \end{aligned}$$

In[92]:=

```
(****Fixed points of 4-dim model using P(y)****)
fy=(c γ(b-1)-y βy βz);
gy=( βv βz y+c δ); hy=(γ +y βz βy/c);
xey=hy gy/(β fy); vey=y fy/gy; zey= βz y /c;
ys=y/.sol[[3]] (* y of E* ****);

Py=λ(1-y/K)-β y fy/gy-λ hy gy/(β K fy); yb=c γ (b-1)/(βy βz);
Qy=λ fy gy(1- y /K)- λ hy gy^2/(β K)-y β fy^2;
Qycol=Collect[Together[Qy],y];
Qycoef=CoefficientList[Qycol,y];
Print["f(y)=", fy, " ,g(y)=", gy, " , h(y)=", hy]
Print["p(y)=", Py//FullSimplify]
Print["The polynomial Q(y) is of order ", Length[Qycoef]-1]
Print["Coefficients of Q(y) are ", Qycoef//FullSimplify]

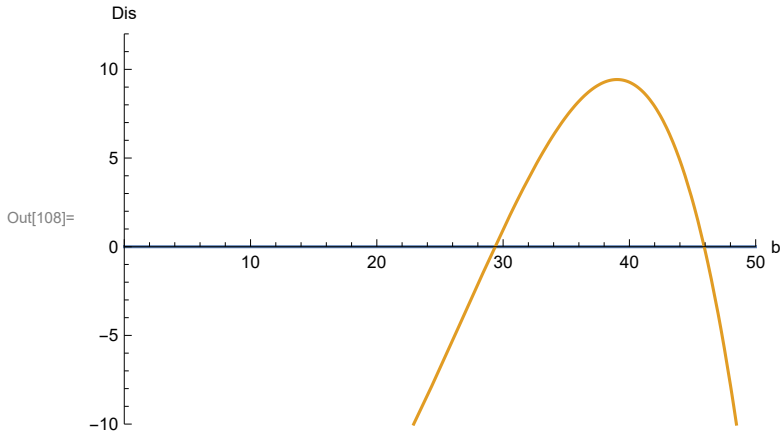
Dis=Collect[Discriminant[Qy,y],b];
Discoef=CoefficientList[Dis,b];Length[Discoef];
Disn=Dis//.cF1//N;
bL=50;
Plot[{0,Disn},{b,0,2 bL},AxesLabel->{"b","Dis"},PlotRange->{{0,bL},{-10,12}}]
Print["Roots of Dis[b]=0 are: ",solbE=Solve[Disn==0,b]]
```

$$f(y) = -y \beta_y \beta_z + (-1+b) c \gamma, g(y) = y \beta_v \beta_z + c \delta, h(y) = \frac{y \beta_y \beta_z}{c} + \gamma$$

$$p(y) = \frac{y \beta (y \beta_y \beta_z - (-1+b) c \gamma)}{y \beta_v \beta_z + c \delta} + \lambda - \frac{y \lambda}{K} - \frac{\left(\frac{y \beta_y \beta_z}{c} + \gamma\right) (y \beta_v \beta_z + c \delta) \lambda}{K \beta (-y \beta_y \beta_z + (-1+b) c \gamma)}$$

The polynomial $Q(y)$ is of order 3

$$\begin{aligned} \text{Coefficients of } Q(y) \text{ are } & \left\{ \frac{c^2 \gamma ((-1+b) K \beta - \delta) \delta \lambda}{K \beta}, \right. \\ & - \frac{c (\beta_z (\delta (2 \beta_v \gamma + \beta_y \delta) + K \beta (\beta_v \gamma - b \beta_v \gamma + \beta_y \delta)) \lambda + (-1+b) c \beta \gamma ((-1+b) K \beta \gamma + \delta \lambda))}{K \beta}, \\ & \frac{\beta_z (-\beta_v \beta_z (K \beta \beta_y + \beta_v \gamma + 2 \beta_y \delta) \lambda + c \beta (2 \times (-1+b) K \beta \beta_y \gamma + \beta_v (\gamma - b \gamma) \lambda + \beta_y \delta \lambda))}{K \beta}, \\ & \left. - \frac{\beta_y \beta_z^2 (\beta_v^2 \beta_z \lambda + c \beta (K \beta \beta_y - \beta_v \lambda))}{c K \beta} \right\} \end{aligned}$$



Roots of $\text{Dis}[b]=0$ are:

$$\{ \{b \rightarrow -126.518\}, \{b \rightarrow -63.\}, \{b \rightarrow -63.\}, \{b \rightarrow -24.5518\}, \{b \rightarrow 29.361\}, \{b \rightarrow 45.9232\} \}$$

In[110]:=

```

QR=Solve[Qy==0,y,Cubics->False]//ToRadicals(*casus irreducibilis*);
ym= y/.QR[[2]];
yp= y/.QR[[1]];yi= y/.QR[[3]];
Em1={xey,ym,vey,zey} /.y->ym;
Ep1={xey,yp,vey,zey} /.y->yp;
Eim1={xey,yi,vey,zey} /.y->yi;
Es1=Join[{x,y,v} /.sol[[3]],{0}];
{Chop[yp],Chop[ym]} /.cF1/N;

jacE1K=jacD/.x->K/.y->0/.v->0/.z->0;
Print["J(EK)=", jacE1K//MatrixForm]
(*Jacobians of the fixed points**)
jEs1=jacD/.sol[[3]] /.z->0;
jEm1=jacD/.x->xey/.v->vey/.z->zey/.y->ym;
jEim1=jacD/.x->xey/.v->vey/.z->zey/.y->yi;
jEp1=jacD/.x->xey/.v->vey/.z->zey/.y->yp;
Print["Eig.val of J(EK) are:", Eigenvalues[jacE1K]//FullSimplify]
jacE1=jacD/.x->xey/.v->vey/.z->zey/.y->y//FullSimplify;
jacE1 //MatrixForm;
Det[jacE1]//FullSimplify;
Print[" Trace of either Ei, E+ or E- is : ", Tr[jacE1]//FullSimplify]
Print["J(E_*) is"]
jEs1//FullSimplify//MatrixForm
bbs=b/.Solve[yb==(y/.sol[[3]]),b][[1]](*long expression*);

```

$$J(EK) = \begin{pmatrix} -\lambda & -\lambda & -K\beta & 0 \\ 0 & -\gamma & K\beta & 0 \\ 0 & b\gamma & -K\beta - \delta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Eig.val of } J(EK) \text{ are: } \left\{ 0, \frac{1}{2} \left(-K\beta - \gamma - \delta - \sqrt{-4\gamma(K(\beta - b\beta) + \delta) + (K\beta + \gamma + \delta)^2} \right), \right.$$

$$\left. \frac{1}{2} \left(-K\beta - \gamma - \delta + \sqrt{-4\gamma(K(\beta - b\beta) + \delta) + (K\beta + \gamma + \delta)^2} \right), -\lambda \right\}$$

$$\text{Trace of either } E_i, E_+ \text{ or } E_- \text{ is : } -\gamma\beta_z - \frac{\gamma\beta_y\beta_z + c\gamma}{c} + \frac{\gamma^2\beta\beta_y\beta_z}{\gamma\beta_v\beta_z + c\delta} -$$

$$\frac{(-1+b)c\gamma\beta\gamma}{\gamma\beta_v\beta_z + c\delta} + \frac{b\gamma(\gamma\beta_v\beta_z + c\delta)}{\gamma\beta_y\beta_z - (-1+b)c\gamma} + \lambda - \frac{\gamma\lambda}{K} - \frac{2\left(\frac{\gamma\beta_y\beta_z}{c} + \gamma\right)(\gamma\beta_v\beta_z + c\delta)\lambda}{K\beta(-\gamma\beta_y\beta_z + (-1+b)c\gamma)}$$

J(E_*) is

Out[130]//MatrixForm=

$$\begin{pmatrix} \frac{\delta\lambda}{K\beta - bK\beta} & \frac{\delta\lambda}{K\beta - bK\beta} & -\frac{\delta}{-1+b} & 0 \\ \frac{\gamma((-1+b)K\beta - \delta)\lambda}{(-1+b)K\beta\gamma + \delta\lambda} & -\gamma & \frac{\delta}{-1+b} & \frac{\beta_y\delta(K(\beta - b\beta) + \delta)\lambda}{(-1+b)\beta((-1+b)K\beta\gamma + \delta\lambda)} \\ \frac{\gamma(K(\beta - b\beta) + \delta)\lambda}{(-1+b)K\beta\gamma + \delta\lambda} & b\gamma & \frac{b\delta}{1-b} & \frac{\beta_v\gamma(K(\beta - b\beta) + \delta)\lambda}{\beta((-1+b)K\beta\gamma + \delta\lambda)} \\ 0 & 0 & 0 & \frac{\beta_z((-1+b)K\beta - \delta)\delta\lambda}{(-1+b)\beta((-1+b)K\beta\gamma + \delta\lambda)} \end{pmatrix}$$

Ep1-2) Trace, Det and third criterion of Routh Hurwitz applied to E*:

Det and Trace of E* and Analysis of the stability of E* in 4 dim when $\epsilon=1$:

In[132]:=

```

Print["Tr[J[E*]]="]
trEs1=Tr[jacD//.Join[sol[[3]],{z->0}]]//FullSimplify
Print["Det[J[E*]]="]
detEs1=Det[jacD//.Join[sol[[3]],{z->0}]]//FullSimplify

pc=Collect[Det[ψ IdentityMatrix[4]-(jEs1)],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Length[coT]
a1=coT[[4]]//FullSimplify;a2=coT[[3]]//FullSimplify;a3=coT[[2]]//FullSimplify;a4=coT[[1]]//FullSimplify
Print["a1=",a1," ",a2=",a2"," ",a3=",a3"," ",a4=",a4"]
H4=a1*a2+a3-a3^2+a1^2 a4;
Print["H2(b0)=",H4/.b->b0//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H4]]//FullSimplify]
φb4=Collect[Numerator[Together[H4]],b];
cofi=CoefficientList[φb4,b];
Length[cofi]
(*Print["value of φ(b) at crit b is "]
φb4/.b->b0//FullSimplify; (*so long expression*) *)

```

Tr[J[E*]] =

$$\text{Out[133]} = - \frac{K \delta (2 \times (-1+b) \beta \gamma + b \beta \delta + \beta_z \delta) \lambda + \delta^2 \lambda^2 + (-1+b) K^2 \beta (\beta \gamma ((-1+b) \gamma + b \delta) - \beta_z \delta \lambda)}{(-1+b) K \beta ((-1+b) K \beta \gamma + \delta \lambda)}$$

Det[J[E*]] =

$$\text{Out[135]} = - \frac{\beta_z \gamma \delta^2 (K (\beta - b \beta) + \delta)^2 \lambda^2}{(-1+b)^2 K \beta^2 ((-1+b) K \beta \gamma + \delta \lambda)}$$

Out[138]= 5

$$\begin{aligned}
 a_1 &= \frac{K \delta (2 \times (-1+b) \beta \gamma + b \beta \delta + \beta_z \delta) \lambda + \delta^2 \lambda^2 + (-1+b) K^2 \beta (\beta \gamma ((-1+b) \gamma + b \delta) - \beta_z \delta \lambda)}{(-1+b) K \beta ((-1+b) K \beta \gamma + \delta \lambda)} \\
 , a_2 &= \left(\delta \lambda \left(- \left((-1+b) K^2 \beta^2 ((-1+b) (\beta + \beta_z) \gamma + b \beta_z \delta) \right) + (b \beta + \beta_z) \delta^2 \lambda + \right. \right. \\
 &\quad \left. \left. K \beta (\beta_z \delta ((-1+b) \gamma + b \delta + \lambda - b \lambda) + (-1+b) \beta \gamma ((-1+b) \gamma + \delta - \lambda + b (\delta + \lambda))) \right) \right) / \\
 &\quad \left((-1+b)^2 K \beta^2 ((-1+b) K \beta \gamma + \delta \lambda) \right), a_3 = \\
 &\quad \left(\left((-1+b) K \beta - \delta \right) \delta \lambda \left(\delta^2 \left((-1+b)^2 \beta \gamma - b \beta_z \delta \right) \lambda^2 + (-1+b)^2 K^2 \beta^2 \gamma \left((-1+b)^2 \beta \gamma^2 + \beta_z \delta \lambda \right) + \right. \right. \\
 &\quad \left. \left. (-1+b) K \beta \gamma \delta \lambda \left(2 (-1+b)^2 \beta \gamma - \beta_z ((-1+b) \gamma + \delta - \lambda + b (\delta + \lambda)) \right) \right) \right) / \\
 &\quad \left((-1+b)^3 K \beta^2 ((-1+b) K \beta \gamma + \delta \lambda)^2 \right), a_4 = - \frac{\beta_z \gamma \delta^2 (K (\beta - b \beta) + \delta)^2 \lambda^2}{(-1+b)^2 K \beta^2 ((-1+b) K \beta \gamma + \delta \lambda)}
 \end{aligned}$$

H2(b0)=0

Denominator of H2 is $(-1+b)^6 K^3 \beta^5 ((-1+b) K \beta \gamma + \delta \lambda)^4$

Out[146]= 11

Numerical values $\epsilon=1$:

In[]:=

```

cn=Join[cF1]; cnb={b→40};
cb=NSolve[( $\phi$ b4//.cn)==0,b,WorkingPrecision→10]
bM=Max[Table[Re[b/.cb[[i]]],{i,Length[cb]}]];
Print["bH=",bH=N[bM,30]]
Print["b0=",b0/.cn//N]

Print["E*",Es1//.cn/.cnb//N]
Print["E+=",Em1//.cn/.cnb//N]
Print["Eim=",Eim1//.cn/.cnb//N]
Print["roots of Dis[b]=0:", bcE1=NSolve[(Dis//.cn)==0,b]]
bc1=Chop[Evaluate[b/.bcE1[[5]]]];
bc2=Chop[Evaluate[b/.bcE1[[6]]]];
Print["b1*=", bc1]
Print[" b2*=", bc2]

```

```

Out[ ]:= { {b → -258.7317419}, {b → -0.01615999598},
  {b → 0.060848063 - 10.686990738 i}, {b → 0.060848063 + 10.686990738 i},
  {b → 0.8448282668 - 0.9299250641 i}, {b → 0.8448282668 + 0.9299250641 i},
  {b → 0.8878046288}, {b → 1.011494253}, {b → 1.022996712}, {b → 1.152364263} }

```

```
bH=1.152364263
```

```
b0=1.01149
```

```
E*{0.000294724, 0.0363426, 0.0221463, 0.}
```

```

E+={0.00262408 + 5.51068×10-19 i, 0.0462057 + 8.67362×10-18 i,
  0.021866 + 3.02367×10-18 i, 0.0462057 + 8.67362×10-18 i}

```

```

Eim={0.114678 - 9.26732×10-17 i, 0.263221 - 2.77556×10-17 i,
  0.0143012 + 8.58446×10-18 i, 0.263221 - 2.77556×10-17 i}

```

```
roots of Dis[b]=0:
```

```
{ {b → -126.518}, {b → -63.}, {b → -63.}, {b → -24.5518}, {b → 29.361}, {b → 45.9232} }
```

```
b1*=29.361
```

```
b2*=45.9232
```

Ep1-3)Bifurcation diagrams :

Numerical solution of the stability (Bifurcation diagram) wrt y:


```

In[ ]:= (*Checks on the stability of the fixed points**)
Print["Eigenvalues of E* when b=20 and when b=40, respectively "]
Eigenvalues[jEs1] /. cF1 /. b → 20 // N
Eigenvalues[jEs1] /. cF1 /. b → 40 // N
Print["Eigenvalues of E+ when b=40"]
Chop[Eigenvalues[jEp1 /. cF1 /. b → 20 // N]]

Print["Eigenvalues of Eim when b=40"]
Chop[Eigenvalues[jEim1 /. cF1 /. b → 40 // N]]
Print["Eigenvalues of E- between b1* and b2* "]
Chop[Eigenvalues[jEm1 /. cF1 /. b → 35 // N]]
Print["Eigenvalues of E- between 0 and b1*"]
Chop[Eigenvalues[jEm1 /. cF1 /. b → 20 // N]]

Eigenvalues of E* when b=20 and when b=40, respectively
Out[ ]:= {0.0718263, -0.586224, 0.0257451 - 0.0774375 i, 0.0257451 + 0.0774375 i}

Eigenvalues of E+ when b=40
Out[ ]:= {0.0363426, -0.555066, 0.017069 - 0.082122 i, 0.017069 + 0.082122 i}

Eigenvalues of Eim when b=40
Out[ ]:= {-24.7847, -0.377551, -0.150826 + 0.260466 i, -0.150826 - 0.260466 i}

Eigenvalues of E- between b1* and b2*
Out[ ]:= {-4.10357, 0.346228, -0.0664887 + 0.180685 i, -0.0664887 - 0.180685 i}

Eigenvalues of E- between 0 and b1*
Out[ ]:= {-0.840551 + 1.00512 i, 0.242338 - 0.343832 i,
0.0933864 + 0.157103 i, -0.0937105 + 0.0144354 i}

```

In[]:=

```

cut=cF1;bL=50;max=0.37;
b0n=b0//.cn;
b1=Chop[bbs//.cF1//N];
Print["b0=",b0n//N," ",b0*=" ",b1," ",b1*=" ",bc1," ",b2*=" ",bc2]
lin1=Line[{{bc1,0},{bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{bc2,0},{bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{b0n,0},{b0n,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
lin4=Line[{{b1,0},{b1,max}}];
li4=Graphics[{Thick,Black,Dashed,lin4}];
pyb=Plot[{yb} /. cut, {b, 0, bL}, PlotStyle -> {Dashed, Thick, Cyan},
PlotRange -> {{0, 200}, {0, max}}, PlotLegends -> {"y_b " }];
(*pym1n=Plot[{ym} /. cut, {b, 0, bL}, PlotStyle -> {Green, Thick}, PlotRange -> All, PlotPoints -> 200,
PlotLegends -> {"E_ unstable"}];*)

pym=Plot[{ym} /. cut, {b, 0, bL}, PlotStyle -> {Green, Dashed, Thick}, PlotRange -> All, PlotPoints -> 180,
PlotLegends -> {"E_ unstable"}];
pyK1=Plot[0, {b, 0, b0n}, PlotStyle -> {Magenta}, PlotRange -> All, PlotLegends -> {"E_K stable"}];
pyK2=Plot[0, {b, b0n, bL}, PlotStyle -> {Blue}, PlotRange -> All, PlotLegends -> {"E_K unstable"}];
pyi=Plot[{yi} /. cut, {b, 0, bL}, PlotStyle -> {Red, Dashed, Thick}, PlotRange -> All,
PlotLegends -> {"E_im unstable"}];
pyp=Plot[{yp} /. cut, {b, b0n, bL}, PlotStyle -> {Purple, Thick}, PlotRange -> All,
PlotLegends -> {"E_ stable"}];
pypn=Plot[{yp} /. cut, {b, 0, b0n}, PlotStyle -> {Yellow, Thick}, PlotRange -> All,
PlotLegends -> {"E_ (y<0)"}];
N[{yp} /. cut /. b -> 40, 20] (*check*)
Print["Q(yp) at b0 is "]
Qy/.y->yp/.b->b0//FullSimplify
Show[pyp,pyb,li3,li1,li2,li4];
pys1=Plot[{ys} /. cut, {b, 0, b1}, PlotStyle -> {Orange, Dotted},
PlotRange -> {{0, 200}, {0, max}},
PlotLegends -> {"E_* outside domain"}];
pys2=Plot[{ys} /. cut, {b, b1, bL}, PlotStyle -> {Orange, Thick, Dashed},
PlotRange -> {{0, 200}, {0, max}},
PlotLegends -> {"E_* unstable"}];

Print["y*' (b0)=",D[ys,b] /. b->b0n//.cut//N//FullSimplify]
Chop[ys /. b->b0n//.cut//N] (*Check*)];
bifep1=Show[{pys2,pys1,pyi,pyp,pypn,pyK1,pyK2,pym,pyb,li3,li1,li2,li4},
Epilog->{Text["b0",Offset[{-2,10},{b0n//.cut,0}],{PointSize[Large],
Style[Point[{b0n//.cut,0}],Black]}],
Text["b1*",Offset[{-8,10},{bc1,0}],{PointSize[Large],
Style[Point[{bc1,0}],Purple]}],
Text["b2*",Offset[{10,10},{bc2,0}],{PointSize[Large],
Style[Point[{bc2,0}],Yellow]}],Text["b0*",Offset[{10,10},{b1,0}],{PointSize[Large],
Style[Point[{b1,0}],Magenta]}],AxesLabel->{"b","y"},PlotRange->{{-0.2,bL},{-0.1,max}}}
Export["EriB.pdf",bifep1]

```

b0=1.01149 ,b0*=14.0011, b1*=29.361 , b2*=45.9232

Out[]:= $\{0.29448140956728204616 + 0. \times 10^{-21} i\}$


$$y_e(x) = \frac{c z}{\beta_z}$$

$$v_e(x) =$$

$$Out[4]= \left\{ v \rightarrow \frac{(K - x - y) \lambda}{K \beta} \right\}$$

$$z_e(x) =$$

$$Out[4]= \left\{ z \rightarrow - \frac{c K \gamma + c x \lambda + K \beta_z \sqrt{\frac{c (4 K (K - x) x \beta_y \beta_z \lambda + c (K \gamma + x \lambda)^2)}{K^2 \beta_z^2}}}{2 c K \beta_y} \right\}$$

 **Solve:** Solutions may not be valid for all values of parameters.

Number of endemic x

Out[4]= 3

Numerical check

Out[4]= { {x → 0.00262408}, {x → 0.114678}, {x → 0.540959} }

```
In[4]:= (*Jacobians of the fixed points*)
jEsx = jacD /. cep1 /. sol[[3]] /. z → 0;
jEmx = jacD /. cep1 /. vex /. y → yex /. zex /. xex[[2]];
jEimx = jacD /. cep1 /. vex /. y → yex /. zex /. xex[[1]];
jEpx = jacD /. cep1 /. vex /. y → yex /. zex /. xex[[3]];
(*Checks on the stability of the fixed points*)
Print["Eigenvalues of E* : "]
Print[" between b0 and b1* "]
Eigenvalues[jEsx] /. cF1 /. b → 15 // N
Print[" between b1* and b2* "]
Eigenvalues[jEsx] /. cF1 /. b → 40 // N
Print["Eigenvalues of E+ between b1* and b2*"]
Chop[Eigenvalues[jEpx /. cF1 /. b → 42 // N]]
Print["Eigenvalues of E- between b1* and b2* "]
Chop[Eigenvalues[jEmx /. cF1 /. b → 40 // N]]
Print["Eigenvalues of Eim between b0 and b1*"]
Chop[Eigenvalues[jEimx /. cF1 /. b → 20 // N]]
Print["Eigenvalues of Eim between b1* and b2*"]
Chop[Eigenvalues[jEimx /. cF1 /. b → 40 // N]]
Print["Eigenvalues of Eim after b2*"]
Chop[Eigenvalues[jEimx /. cF1 /. b → 80 // N]]
```

Eigenvalues of E^* :

between b_0 and b_{1*}

$Out[*]= \{0.0950185, -0.606269, 0.0309604 - 0.074022 i, 0.0309604 + 0.074022 i\}$

between b_{1*} and b_{2*}

$Out[*]= \{0.0363426, -0.555066, 0.017069 - 0.082122 i, 0.017069 + 0.082122 i\}$

Eigenvalues of E_+ between b_{1*} and b_{2*}

$Out[*]= \{-22.7478, 0.942561 + 0.7947 i, 0.942561 - 0.7947 i, 0.792692\}$

Eigenvalues of E_- between b_{1*} and b_{2*}

$Out[*]= \{-6.37039, 0.799512, 0.558507 + 0.38198 i, 0.558507 - 0.38198 i\}$

Eigenvalues of E_{im} between b_0 and b_{1*}

$Out[*]= \{-36.7873, 1.03926 + 0.963099 i, 1.03926 - 0.963099 i, 0.329033\}$

Eigenvalues of E_{im} between b_{1*} and b_{2*}

$Out[*]= \{-0.802619, 0.144164 + 0.190903 i, 0.144164 - 0.190903 i, 0.0596342\}$

Eigenvalues of E_{im} after b_{2*}

$Out[*]= \{-0.601147, 0.062694 + 0.153094 i, 0.062694 - 0.153094 i, 0.027413\}$

```

In[ ]:= cut=cF1;bL=135;max=1.2;
b0n=b0//.cn;
b1=Chop[b/.Solve[(Es1[[1]]==(1-yb),b)[[2]]//.cF1//N]
Print["bH=",bH]
lin1=Line[{{bc1,0},{bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{bc2,0},{bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{b0n,0},{b0n,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
lin4=Line[{{b1,0},{b1,max}}];
li4=Graphics[{Thick,Black,Dashed,lin4}];
pxm2=Plot[{x/.xex[[3]]//.cut,{b,0,bL}],PlotStyle→{Purple,Dashed,Thick},PlotRange→All,PlotPoint:
PlotLegends→{"E+ unstable"}]
pxK1=Plot[K//.cut,{b,0,b0n},PlotStyle→{Magenta},PlotRange→All,PlotLegends→{"EK stable"}];
pxK2=Plot[K//.cut,{b,b0n,bL},PlotStyle→{Blue},PlotRange→All,PlotLegends→{"EK unstable"}];
pxi=Plot[{x/.xex[[1]]//.cut,{b,0,bL}],PlotStyle→{Red,Dashed,Thick},PlotRange→All,
PlotLegends→{"Eim unstable"}]
pxp=Plot[{x/.xex[[2]]//.cut,{b,0,bL}],PlotStyle→{Green,Dashed,Thick},PlotRange→All,
PlotLegends→{"E- unstable"}]

pxs2=Plot[{x/.sol[[3]]//.cut,{b,0,bL}],PlotStyle→{Orange,Thick,Dashed},
PlotRange→{{0,200},{0,max}},
PlotLegends→{"E* unstable"}]

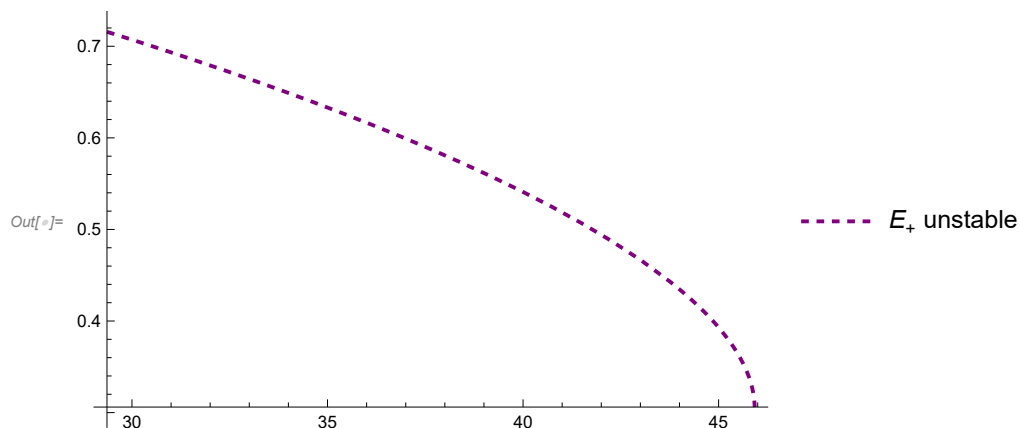
pxb=Plot[{1-yb)//.cut,{b,0,bL}],PlotStyle→{Dashed,Thick,Cyan},
PlotRange→{{0,200},{0,max}},PlotLegends→{"1-yb"}];

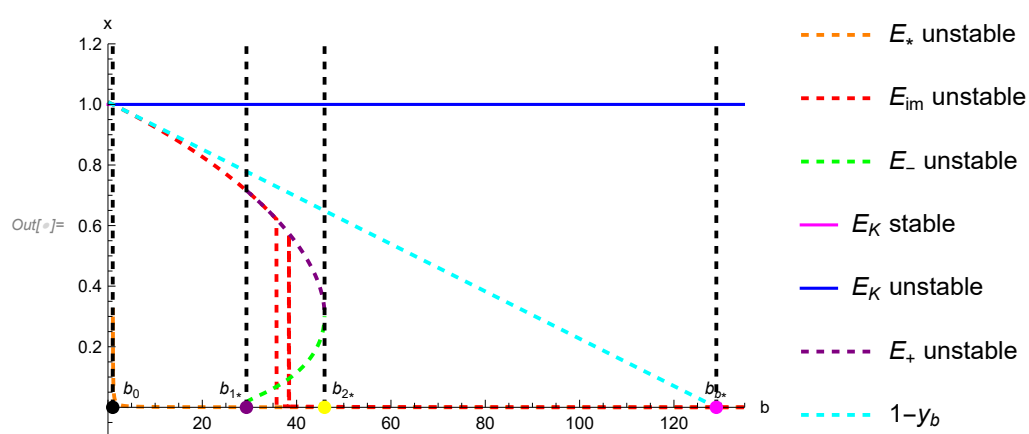
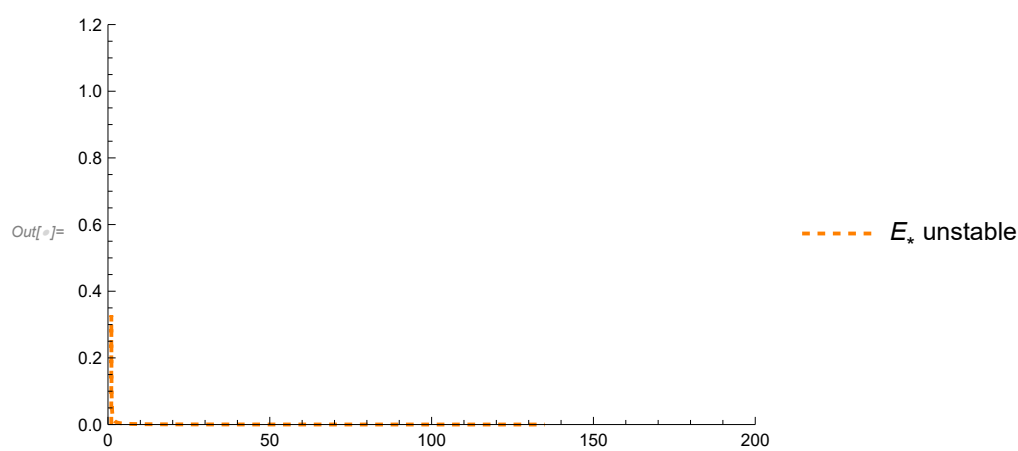
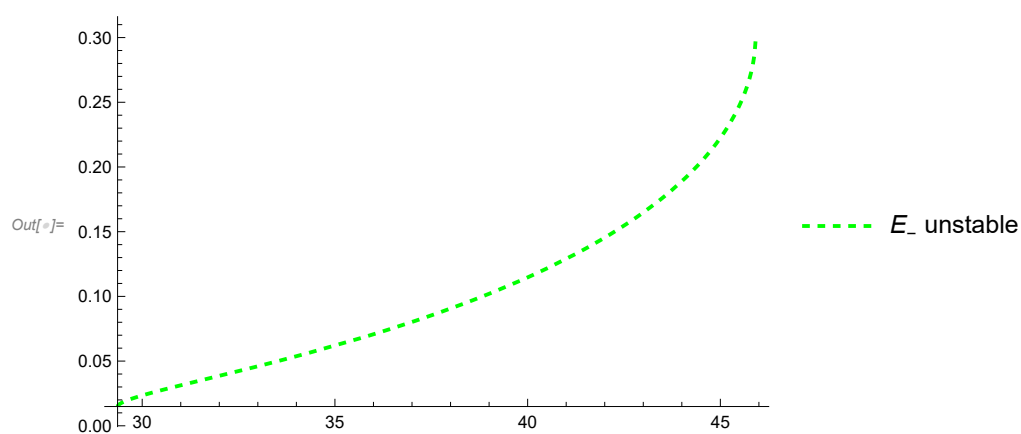
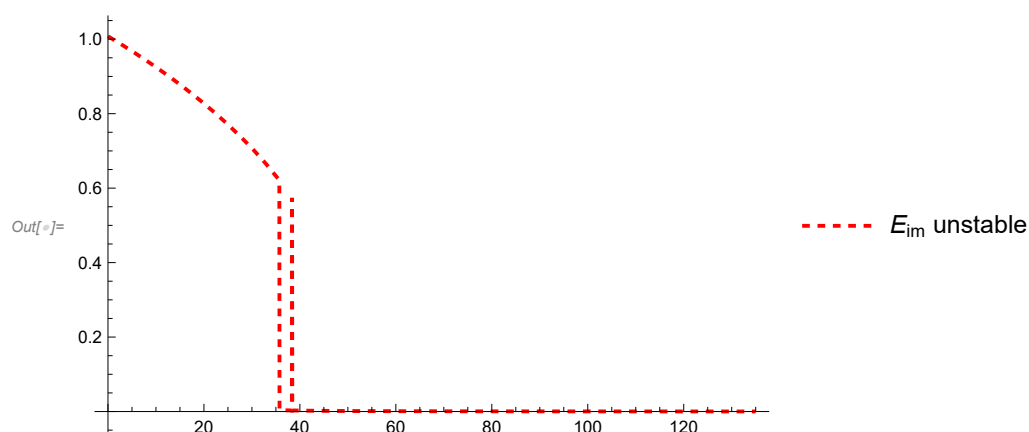
bifep1=Show[{pxs2,pxi,pxp,pxK1,pxK2,pxm2,pxb,li3,li1,li2,li4},
Epilog→{Text["b0",Offset[{10,10},{b0n//.cut,0}],{PointSize[Large],
Style[Point[{b0n//.cut,0}],Black]},
Text["b1",Offset[{-8,10},{bc1,0}],{PointSize[Large],
Style[Point[{bc1,0}],Purple]},
Text["b2",Offset[{10,10},{bc2,0}],{PointSize[Large],
Style[Point[{bc2,0}],Yellow]},Text["bb",Offset[{0,10},{b1,0}],{PointSize[Large],
Style[Point[{b1,0}],Magenta]}],AxesLabel→{"b","x"},PlotRange→{{-0.1,bL},{-0.1,max}}]
Export["Bif1x.pdf",bifep1]

```

Out[]:= 128.989

bH=1.152364263





Out[]= Bif1x.pdf