Deterministic model with Logistic growth (4 dim when $\epsilon = 1$)

In[•]:=
```
SetDirectory[NotebookDirectory[]];
AppendTo[$Path, Directory];
Clear["Global`*"];
(*Some aliases*)
Format[μv] := Subscript[μ, v]; Format[μy] := Subscript[μ, y];
pars = {β, λ, γ, δ, μy, μv, b, K, s, c};
parE = {β > 0, λ > 0, γ > 0, δ > 0, μy > 0, μv > 0, b > 1, K > 0, s > 0, c > 0};

R0 = b β K / (β K + δ);
Clear[K];
(****** Four dim Deterministic  epidemic model with Logistic growth ****)
x1 = λ x (1 - (x + y) / K) -  β x v ;
y1 = β x v - μy y z - γ y;
v1 = -β x v - μv v z + b γ y - δ v;
z1 = z (s y - c z);
ye = c / s; vM = λ (1 - ye) / β;
dyn = {x1, y1, v1, z1};
dyn3 = {x1, y1, v1} /. z → 0;(*3dim case*)
          x'
Print[" (y' )=", dyn // FullSimplify // MatrixForm]
          v'
          z'
Print["b0=", b0 = b /. Apart[Solve[R0 == 1, b][[1]] // FullSimplify]]
(***Fixed point when z→0**)
eq = Thread[dyn3 == {0, 0, 0}];
sol = Solve[eq, {x, y, v}] // FullSimplify;
Es = {x, y, v} /. sol[[3]];(*Endemic point with z=0*);
Print["E*=", Est = Es // FullSimplify]
(*Interior equilibria*)
x1s = λ (1 - (x + y) / K) -  β v ;
z1s = s y - c z;
dyns = {x1s, y1, v1, z1s};
el = Eliminate[Thread[dyns == {0, 0, 0, 0}], {x, v, z}];
Qybyelim = Factor[el[[1, 1]] - el[[1, 2]]] / y // FullSimplify;
Print["Coefficients of  Qy by elim polynomial are:"]
cof = CoefficientList[Qybyelim, y] // FullSimplify
so = y /. Solve[Qybyelim == 0, y];(*Third order roots*)
Print["Qybyelim/K,K->Infinity"]
QGuo = Limit[Qybyelim / K, K → Infinity] // Simplify
Print["second order Q of Guo "]
QGuocol = Collect[-QGuo / (c β ((-1 + b) c γ - s y μy)), y]
(*****Fixed points of Eric using P(y)***)
fy = (c γ (b - 1) - y μy s);
gy = ( μv s y + c δ); hy = (γ + y s μy / c);
xe = hy gy / (β fy); ve = y fy / gy; ze =  s y / c;
Qy = λ fy gy (1 -  y / K) - λ hy gy^2 / (β K) - y β fy^2 // FullSimplify;
QR = Solve[Qy == 0, y];
ym =  y /. QR[[1]];
yp =  y /. QR[[2]];
yi =  y /. QR[[3]];
```

$$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -v\,x\,\beta + x\left(1 - \frac{x+y}{K}\right)\lambda \\ v\,x\,\beta - y\left(\gamma + z\,\mu_y\right) \\ b\,y\,\gamma - v\,(x\,\beta + \delta + z\,\mu_v) \\ z\,(s\,y - c\,z) \end{pmatrix}$$

$$b0 = 1 + \frac{\delta}{K\,\beta}$$

$$E* = \left\{\frac{\delta}{(-1+b)\,\beta},\ \frac{((-1+b)\,K\,\beta - \delta)\,\delta\,\lambda}{(-1+b)\,\beta\,((-1+b)\,K\,\beta\,\gamma + \delta\,\lambda)},\ \frac{\gamma\,((-1+b)\,K\,\beta - \delta)\,\lambda}{\beta\,((-1+b)\,K\,\beta\,\gamma + \delta\,\lambda)}\right\}$$

Coefficients of Qy by elim polynomial are:

Out[•]=
$$\left\{c^3\,\gamma\,\delta\,(K\,(\beta - b\,\beta) + \delta)\,\lambda,\right.$$
$$c^2\left((-1+b)\,c\,\beta\,\gamma\,((-1+b)\,K\,\beta\,\gamma + \delta\,\lambda) + s\,\lambda\left(\gamma\,(K\,(\beta - b\,\beta) + 2\,\delta)\,\mu_v + \delta\,(K\,\beta + \delta)\,\mu_y\right)\right),$$
$$c\,s\left(s\,\lambda\,\mu_v\left(\gamma\,\mu_v + K\,\beta\,\mu_y + 2\,\delta\,\mu_y\right) + c\,\beta\left(-\delta\,\lambda\,\mu_y + (-1+b)\,\gamma\left(\lambda\,\mu_v - 2\,K\,\beta\,\mu_y\right)\right)\right),$$
$$\left.s^2\,\mu_y\left(s\,\lambda\,\mu_v^2 + c\,\beta\left(-\lambda\,\mu_v + K\,\beta\,\mu_y\right)\right)\right\}$$

Qybyelim/K,K->Infinity

Out[•]=
$$c\,\beta\left((-1+b)\,c\,\gamma - s\,y\,\mu_y\right)\left((-1+b)\,c\,y\,\beta\,\gamma - c\,\delta\,\lambda - s\,y\left(\lambda\,\mu_v + y\,\beta\,\mu_y\right)\right)$$

second order Q of Guo

Out[•]=
$$c\,\delta\,\lambda + y\,((1-b)\,c\,\beta\,\gamma + s\,\lambda\,\mu_v) + s\,y^2\,\beta\,\mu_y$$

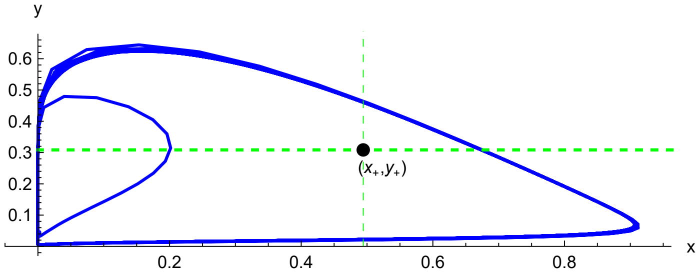
In[•]:=
```
(************************************ bH < b=
 42 < b2* *********************************)
(*cn={b→42,μv→1, μy→1,K→1,γ→1/128,β→87/2,λ→1,δ→1/2,s→1, c→1};*)
ClearParameters; tf = 2000;
b = 42;
μv = 1; μy = 1;
K = 1;
γ = 1 / 128;
β = 87 / 2;
λ = 1;
δ = 1 / 2;
s = 1;
c = 1;
Print["E*=", Es = Est // N];
Print["E+=", Ep = Chop[{xe, y, ve, ze} /. y → yp // N]]
Print["Ei=", Ei = Chop[{xe, y, ve, ze} /. y → yi // N]]
Print["E-=", Em = Chop[{xe, y, ve, ze} /. y → ym // N]]
X = {x, y, v, z};
Xt = Map[#[t] &, X];
Thread[X → Xt];
dynt = dyn /. Thread[X → Xt];
x1 = dynt[[1]] ;
y1 = dynt[[2]];
v1 = dynt[[3]];
z1 = dynt[[4]];
(*x0=0.8; y0=0.02;v0=0.01;z0=0.01;*)
x0 = 0.0022; y0 = 0.042; v0 = 0.02; z0 = 0.04;
ode1 = {x'[t] == x1, y'[t] == y1, v'[t] == v1,
   z'[t] == z1, x[0] == x0, y[0] == y0, v[0] == v0, z[0] == z0};
```

```
sol1 = NDSolve[ode1, {x, y, v, z}, {t, 0, tf}];
(*****..Parametric plot conditions***)
ppb3 = ParametricPlot[{ x[t], (y[t])} /. sol1, {t, 0, tf},
   AxesLabel → {"x", "y"}, PlotRange → Full, PlotStyle → {Blue}, AspectRatio → 1 / 3];
py3 = Plot[y /. y → Ep⟦2⟧, {t, 0, 400}, PlotStyle → {Dashed, Green}];
pb3 = Show[{ppb3, py3,
   Graphics[{Green, Dashed, Line[{{x /. x → Ep⟦1⟧, 0}, {x /. x → Ep⟦1⟧, 1}}]}] },
  Epilog → {{Thick, Text["(x+,y+)", Offset[{10, -10}, {x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}]],
     {PointSize[Large], Style[Point[{x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}], Black]}}(*,
    {PointSize[Large],Point[{x0,y0}]},Text["(x0,y0)",Offset[{-20,10},{x0,y0}]]*)}]
(************************************    Different initial
 values   ***********************************)
x01 = 0.1; y01 = 0.05; v01 = 0.0043; z01 = 0.1954;
ode1 = {x'[t] == x1, y'[t] == y1, v'[t] == v1,
   z'[t] == z1, x[0] == x01, y[0] == y01, v[0] == v01, z[0] == z01};
sol2 = NDSolve[ode1, {x, y, v, z}, {t, 0, tf}];

ppb3n = ParametricPlot[{ x[t], (y[t])} /. sol2, {t, 0, tf},
   AxesLabel → {"x", "y"}, PlotRange → Full, PlotStyle → {Red}, AspectRatio → 1 / 3];
py3n = Plot[y /. y → Ep⟦2⟧, {t, 0, 400}, PlotStyle → {Dashed, Green}];
pb3n = Show[{ppb3n, py3n,
   Graphics[{Green, Dashed, Line[{{x /. x → Ep⟦1⟧, 0}, {x /. x → Ep⟦1⟧, 1}}]}] },
  Epilog → {{Thick, Text["(x+,y+)", Offset[{10, -10}, {x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}]],
     {PointSize[Large], Style[Point[{x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}], Black]}},
    {PointSize[Large], Point[{x01, y01}]},
    Text["(x0',y0')", Offset[{15, 10}, {x01, y01}]]}}]
(************************************    Different initial
 values   ***********************************)
x0i = 0.14; y0i = 0.28; v0i = 0.013; z0i = 0.28;
ode1 = {x'[t] == x1, y'[t] == y1, v'[t] == v1,
   z'[t] == z1, x[0] == x0i, y[0] == y0i, v[0] == v0i, z[0] == z0i};
soli = NDSolve[ode1, {x, y, v, z}, {t, 0, tf}];

ppb3n = ParametricPlot[{ x[t], (y[t])} /. soli, {t, 0, tf},
   AxesLabel → {"x", "y"}, PlotRange → Full, PlotStyle → {Brown}, AspectRatio → 1 / 3];
py3n = Plot[y /. y → Ep⟦2⟧, {t, 0, 400}, PlotStyle → {Dashed, Green}];
pb4 = Show[{ppb3n, py3n,
   Graphics[{Green, Dashed, Line[{{x /. x → Ep⟦1⟧, 0}, {x /. x → Ep⟦1⟧, 1}}]}] },
  Epilog → {{Thick, Text["(x+,y+)", Offset[{10, -10}, {x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}]],
     {PointSize[Large], Style[Point[{x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}], Black]}},
    {PointSize[Large], Point[{x0i, y0i}]},
    Text["(x0'',y0'')", Offset[{15, 10}, {x0i, y0i}]]}}]

(************************************    Different
 initial values   ***********************************)
x0f = Chop[Evaluate[x[t] /. sol1 /. t → 50 // N]]⟦1⟧;
y0f = Chop[Evaluate[y[t] /. sol1 /. t → 50 // N]]⟦1⟧;
v0f = Chop[Evaluate[v[t] /. sol1 /. t → 50 // N]]⟦1⟧;
z0f = Chop[Evaluate[z[t] /. sol1 /. t → 50 // N]]⟦1⟧;
ode1 = {x'[t] == x1, y'[t] == y1, v'[t] == v1,
```

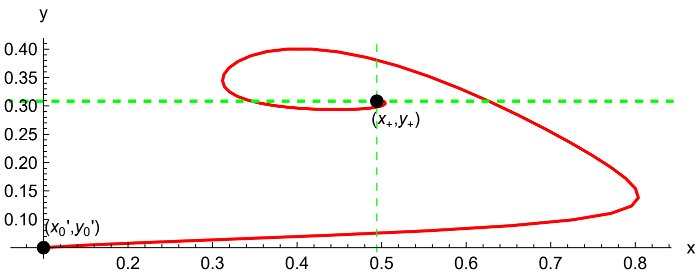
```
   z'[t] == z1, x[0] == x0f, y[0] == y0f, v[0] == v0f, z[0] == z0f};
solf = NDSolve[ode1, {x, y, v, z}, {t, 0, tf}];

ppb3n = ParametricPlot[{ x[t], (y[t])} /. solf, {t, 0, tf}, AxesLabel → {"x", "y"},
   PlotRange → Full, PlotStyle → {Orange}, AspectRatio → 1 / 3];
py3n = Plot[y /. y → Ep[[2]], {t, 0, 400}, PlotStyle → {Dashed, Green}];
pbf = Show[{ppb3n, py3n,
   Graphics[{Green, Dashed, Line[{{x /. x → Ep[[1]], 0}, {x /. x → Ep[[1]], 1}}]}] },
  Epilog → {{Thick, Text["(x+,y+)", Offset[{10, -10}, {x /. x → Ep[[1]], y /. y → Ep[[2]]}]],
      {PointSize[Large], Style[Point[{x /. x → Ep[[1]], y /. y → Ep[[2]]}], Black]}},
     {PointSize[Large], Point[{x0f, y0f}]},
     Text["(x0',y0')", Offset[{15, 10}, {x0f, y0f}]]}]

epi = {{Thick, Text["(x+,y+)", Offset[{10, -10}, {x /. x → Ep[[1]], y /. y → Ep[[2]]}]],
     {PointSize[Large], Style[Point[{x /. x → Ep[[1]], y /. y → Ep[[2]]}], Black]}},
   {PointSize[Large], Point[{x01, y01}]},
   Text["(x0,y0)", Offset[{15, 10}, {x01, y01}]],
   {PointSize[Large], Style[Point[{Em[[1]], Em[[2]]}], Magenta]},
   Text["(x-,y-)", Offset[{10, 4}, {Em[[1]], Em[[2]]}]],
   {PointSize[Large], Style[Point[{Ei[[1]], Ei[[2]]}], Magenta]},
   Text["(xim,yim)", Offset[{-20, 10}, {Ei[[1]], Ei[[2]]}]] (*,
   {PointSize[Large],Style[Point[{Es[[1]],Es[[2]]}],Magenta]},
   Text["(x*,y*)",Offset[{-20,10},{Es[[1]],Es[[2]]}]]*)};
y11 = Show[{pb3, pb3n, pb4}, Epilog → epi]
Export["ppG-2LC.pdf", y11]
Jac = Grad[dyn, {x, y, v, z}];
Jacs  = Jac //. {x → Es[[1]], y → Es[[2]], v → Es[[3]], z → 0} // N;
Print["The eigenvalues of E* when b=42 are: ", Eigenvalues[Jacs]]
Jacp  = Jac //. {x → Ep[[1]], y → Ep[[2]], v → Ep[[3]], z → Ep[[4]]} // N;
Print["The eigenvalues of E+ when b=42 are: ", Eigenvalues[Jacp]]
Jaci  = Jac //. {x → Ei[[1]], y → Ei[[2]], v → Ei[[3]], z → Ei[[4]]} // N;
Print["The eigenvalues of Ei when b=42 are: ", Eigenvalues[Jaci]]
Jacm  = Jac //. {x → Em[[1]], y → Em[[2]], v → Em[[3]], z → Em[[4]]} // N;
Print["The eigenvalues of E- when b=42 are: ", Eigenvalues[Jacm]]
```

```
E*={0.000280348, 0.0346317, 0.0221859}

E+={0.494238, 0.308421, 0.00453657, 0.308421}

Ei={0.145387, 0.284118, 0.0131148, 0.284118}

E-={0.00228546, 0.042969, 0.0219482, 0.042969}
```

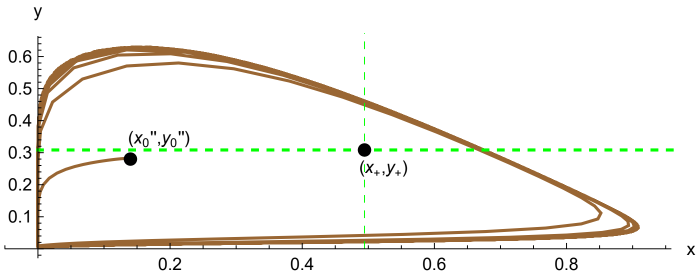
Out[∘]=



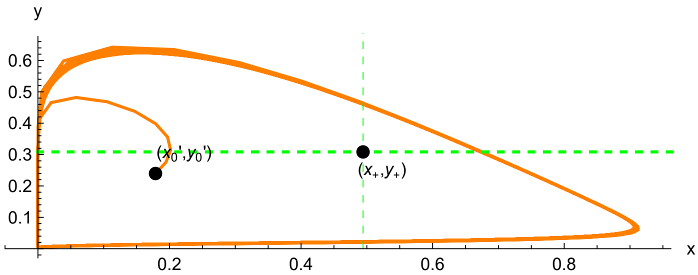
Out[∘]=



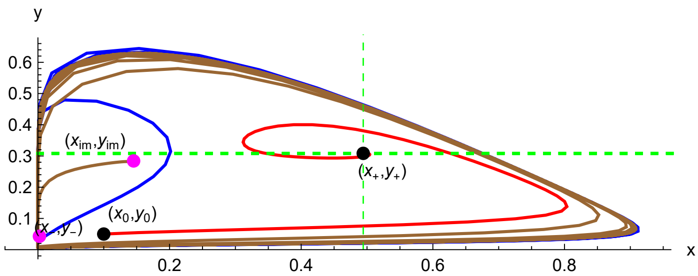
Out[∘]=



Out[∘]=



Out[∘]=



Out[∘]= ppG-2LC.pdf

The eigenvalues of E⋆ when b=42 are:
{-0.553543 + 0. i, 0.0166273 + 0.0823309 i, 0.0166273 - 0.0823309 i, 0.0346317 + 0. i}

The eigenvalues of E+ when b=42 are:
{-22.81 + 0. i, -0.157506 + 0.275785 i, -0.157506 - 0.275785 i, -0.301641 + 0. i}

The eigenvalues of Ei when b=42 are:
{-7.86054 + 0. i, -0.116738 + 0.254012 i, -0.116738 - 0.254012 i, 0.264137 + 0. i}

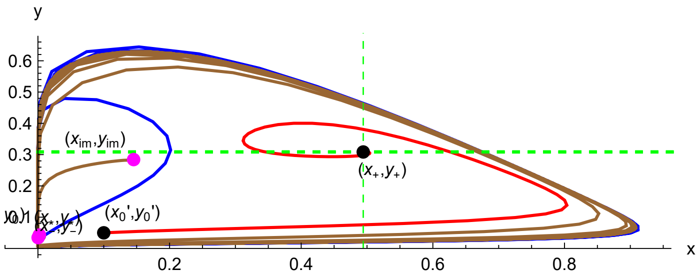
The eigenvalues of E- when b=42 are:
{-0.842468 + 0. i, 0.068671 + 0.167599 i, 0.068671 - 0.167599 i, -0.0332964 + 0. i}

In[ ]:=

```
Print["E*=",Es]
epi={{Thick,Text["(x+,y+)",Offset[{10,-10},{x/.x→Ep⟦1⟧,y/.y→Ep⟦2⟧}]],{PointSize[Large],
Style[Point[{x/.x→Ep⟦1⟧,y/.y→Ep⟦2⟧}],Black]}},{PointSize[Large],Point[{x01,y01}]},
Text["(x0',y0')",Offset[{15,10},{x01,y01}]],{PointSize[Large],Point[{x0,y0}]},
Text["(x0,y0)",Offset[{-20,10},{x0,y0}]],{PointSize[Large],Style[Point[{Em⟦1⟧,Em⟦2⟧}],Magenta
Text["(x-,y-)",Offset[{10,4},{Em⟦1⟧,Em⟦2⟧}]],{PointSize[Large],Style[Point[{Ei⟦1⟧,Ei⟦2⟧}],Ma
Text["(xim,yim)",Offset[{-20,10},{Ei⟦1⟧,Ei⟦2⟧}]],{PointSize[Large],Style[Point[{Es⟦1⟧,Es⟦2⟧}]
Text["(x*,y*)",Offset[{10,10},{Es⟦1⟧,Es⟦2⟧}]]}};
xy42=Show[{pb3,pb3n,pb4},Epilog→epi]
```

```
E*={0.000280348, 0.0346317, 0.0221859}
```

Out[ ]=

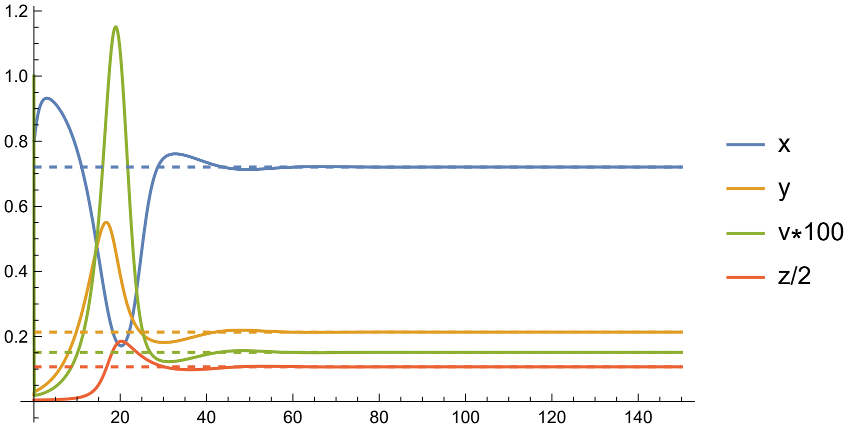


In[ ]:=

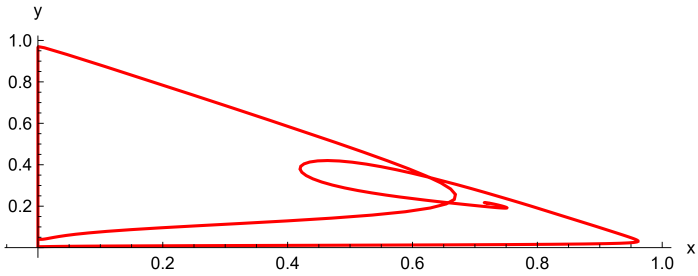
```
(************************************ b=
 29 < bH **********************************)
Clear["b"];
b = 29;
Print["E*=", Es = Est // N];
Print["E+=", Ep = Chop[{xe, y, ve, ze} /. y → yp // N]]
Print["Ei=", Ei = Chop[{xe, y, ve, ze} /. y → yi // N]]
Print["E-=", Em = Chop[{xe, y, ve, ze} /. y → ym // N]]
X = {x, y, v, z};
Xt = Map[#[t] &, X ];
Thread[X → Xt];
dynt = dyn /. Thread[X → Xt];
x1 = dynt⟦1⟧ ;
y1 = dynt⟦2⟧;
v1 = dynt⟦3⟧;
z1 = dynt⟦4⟧;
x0 = 0.8; y0 = 0.02; v0 = 0.01; z0 = 0.01;
ode1 = {x'[t] == x1, y'[t] == y1, v'[t] == v1,
   z'[t] == z1, x[0] == x0, y[0] == y0, v[0] == v0, z[0] == z0};
sol1 = NDSolve[ode1, {x, y, v, z}, {t, 0, tf}];
pdy1 = Plot[{x[t] /. sol1, y[t] /. sol1, v[t] * 100 /. sol1, z[t] / 2 /. sol1},
   {t, 0, 150}, PlotLegends → {"x", "y", "v*100", "z/2"}];
pEs1 = Plot[{x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧, v * 100 /. v → Ep⟦3⟧, z / 2 /. z → Ep⟦4⟧},
   {t, 0, 150}, PlotStyle → {Dashed}];
Dyn01 = Show[pdy1, pEs1]
Export["ptH.pdf", Dyn01]
(*****..Parametric plot conditions***)
ppb3 = ParametricPlot[{ x[t], (y[t])} /. sol1, {t, 0, tf},
   AxesLabel → {"x", "y"}, PlotRange → Full, PlotStyle → {Blue}, AspectRatio → 1 / 3];
py3 = Plot[y /. y → Ep⟦2⟧, {t, 0, 400}, PlotStyle → {Dashed, Green}];
pb3 = Show[{ppb3, py3, Graphics[
```

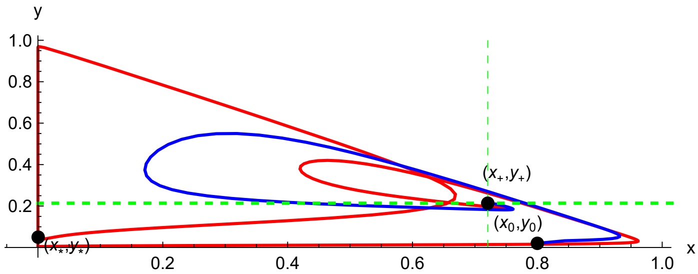
```
     {Green, Dashed, Line[{{x /. x → Ep⟦1⟧, 0}, {x /. x → Ep⟦1⟧, 1}}]}] }, Epilog →
    {{Thick, Text["(x₊,y₊)", Offset[{10, -10}, {x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}]],
      {PointSize[Large], Style[Point[{x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}], Black]}},
     {PointSize[Large], Point[{x0, y0}]},
     Text["(x₀,y₀)", Offset[{-20, 10}, {x0, y0}]]}];


(************************************    Different
 initial values   **********************************)
x0i = 0.004; y0i = 0.04; v0i = 0.00001; z0i = 0.00001;
ode1 = {x'[t] == x1, y'[t] == y1, v'[t] == v1,
   z'[t] == z1, x[0] == x0i, y[0] == y0i, v[0] == v0i, z[0] == z0i};
soli = NDSolve[ode1, {x, y, v, z}, {t, 0, tf}];

ppb3n = ParametricPlot[{ x[t], (y[t])} /. soli, {t, 0, tf},
  AxesLabel → {"x", "y"}, PlotRange → Full, PlotStyle → {Red}, AspectRatio → 1 / 3]
epi = {{Thick, Text["(x₊,y₊)", Offset[{10, 15}, {x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}]],
    {PointSize[Large], Style[Point[{x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}], Black]}},
   {PointSize[Large], Point[{x0, y0}]}, Text["(x₀,y₀)", Offset[{-10, 10}, {x0, y0}]],
   {PointSize[Large], Point[{Es⟦1⟧, Es⟦2⟧}]},
   Text["(x⋆,y⋆)", Offset[{15, -5}, {Es⟦1⟧, Es⟦2⟧}]]};

y11 = Show[{ppb3n, pb3}, Epilog → epi]
Export["pp29.pdf", y11]
Jac = Grad[dyn, {x, y, v, z}];
Jacs = Jac //. {x → Es⟦1⟧, y → Es⟦2⟧, v → Es⟦3⟧, z → 0} // N;
Print["The eigenvalues of E* when b=29 are: ", Eigenvalues[Jacs]]
Jacp = Jac //. {x → Ep⟦1⟧, y → Ep⟦2⟧, v → Ep⟦3⟧, z → Ep⟦4⟧} // N;
Print["The eigenvalues of E+ when b=29 are: ", Eigenvalues[Jacp]]
Jaci = Jac //. {x → Ei⟦1⟧, y → Ei⟦2⟧, v → Ei⟦3⟧, z → Ei⟦4⟧} // N;
Print["The eigenvalues of Ei when b=29 are: ", Eigenvalues[Jaci]]
Jacm = Jac //. {x → Em⟦1⟧, y → Em⟦2⟧, v → Em⟦3⟧, z → Em⟦4⟧} // N;
Print["The eigenvalues of E- when b=29 are: ", Eigenvalues[Jacm]]
```

```
E*={0.000410509, 0.0499015, 0.0218319}

E+={0.720598, 0.213706, 0.00151024, 0.213706}

Ei={0.0137981 - 0.00495629 i,
  0.108199 - 0.0179962 i, 0.020184 + 0.000527643 i, 0.108199 - 0.0179962 i}

E-={0.0137981 + 0.00495629 i,
  0.108199 + 0.0179962 i, 0.020184 - 0.000527643 i, 0.108199 + 0.0179962 i}
```

Out[∘]=


Out[∘]= ptH.pdf

Out[∘]=


Out[∘]=


Out[∘]= pp29.pdf

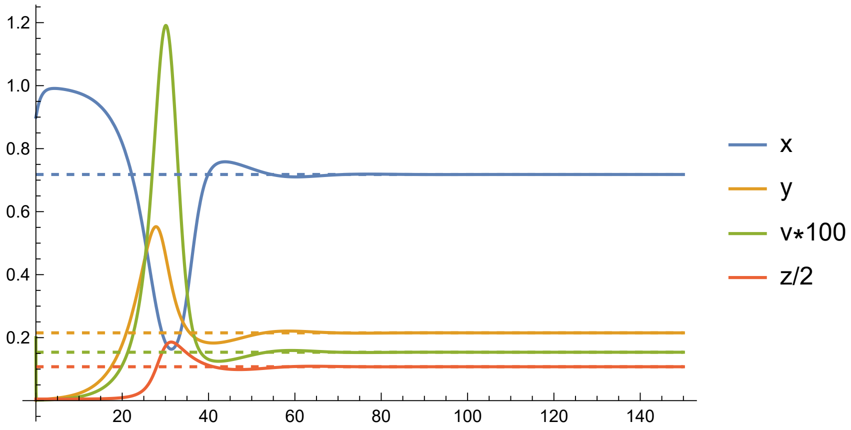
```
The eigenvalues of E* when b=29 are:
 {-0.567061 + 0. i, 0.0204906 + 0.0804108 i, 0.0204906 - 0.0804108 i, 0.0499015 + 0. i}
The eigenvalues of E+ when b=29 are:
 {-32.3449 + 0. i, -0.654873 + 0. i, -0.107884 + 0.185716 i, -0.107884 - 0.185716 i}
The eigenvalues of Ei when b=29 are:
 {-1.67085 + 0.294654 i, 0.183885 - 0.106515 i, 0.0707757 + 0.115845 i, -0.0302383 - 0.0294416 i}
The eigenvalues of E- when b=29 are:
 {-1.67085 - 0.294654 i, 0.183885 + 0.106515 i, 0.0707757 - 0.115845 i, -0.0302383 + 0.0294416 i}
```

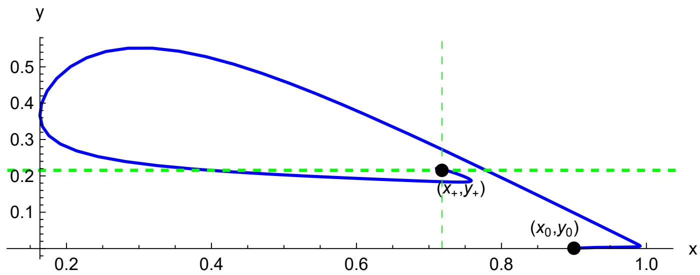
In[∘]:=
```
(************************************ bH < b=
 29.2 < b1* **********************************)

ClearParameters; tf = 1500;
b = 29.2;
μv = 1;  μy = 1;
K = 1;
γ = 1 / 128;
β = 87 / 2;
λ = 1;
```

```
δ = 1 / 2;
s = 1;
c = 1;
Print["E*=", Es = Est // N];
Print["E+=", Ep = Chop[{xe, y, ve, ze} /. y → yp // N]]
Print["Ei=", Ei = Chop[{xe, y, ve, ze} /. y → yi // N]]
Print["E-=", Em = Chop[{xe, y, ve, ze} /. y → ym // N]]
X = {x, y, v, z};
Xt = Map[#[t] &, X ];
Thread[X → Xt];
dynt = dyn /. Thread[X → Xt];
x1 = dynt⟦1⟧ ;
y1 = dynt⟦2⟧;
v1 = dynt⟦3⟧;
z1 = dynt⟦4⟧;
x0 = 0.9; y0 = 0.001; v0 = 0.002; z0 = 0.01;
ode1 = {x'[t] == x1, y'[t] == y1, v'[t] == v1,
   z'[t] == z1, x[0] == x0, y[0] == y0, v[0] == v0, z[0] == z0};
sol1 = NDSolve[ode1, {x, y, v, z}, {t, 0, tf}];
pdy1 = Plot[{x[t] /. sol1, y[t] /. sol1, v[t] * 100 /. sol1, z[t] / 2 /. sol1},
   {t, 0, 150}, PlotLegends → {"x", "y", "v*100", "z/2"}];
pEs1 = Plot[{x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧, v * 100 /. v → Ep⟦3⟧, z / 2 /. z → Ep⟦4⟧},
   {t, 0, 150}, PlotStyle → {Dashed}];
Dyn01 = Show[pdy1, pEs1]
(*****..Parametric plot conditions***)
ppb3 = ParametricPlot[{ x[t], (y[t])} /. sol1, {t, 0, tf},
   AxesLabel → {"x", "y"}, PlotRange → Full, PlotStyle → {Blue}, AspectRatio → 1 / 3];
py3 = Plot[y /. y → Ep⟦2⟧, {t, 0, 400}, PlotStyle → {Dashed, Green}];
pb3 = Show[{ppb3, py3,
   Graphics[{Green, Dashed, Line[{{x /. x → Ep⟦1⟧, 0}, {x /. x → Ep⟦1⟧, 1}}]}] },
  Epilog → {{Thick, Text["(x+,y+)", Offset[{10, -10}, {x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}]],
     {PointSize[Large], Style[Point[{x /. x → Ep⟦1⟧, y /. y → Ep⟦2⟧}], Black]}},
    {PointSize[Large], Point[{x0, y0}]},
    Text["(x0,y0)", Offset[{-10, 10}, {x0, y0}]]}]
Export["ppH1.pdf", pb3]
Export["ptH1.pdf", Dyn01]
Jac = Grad[dyn, {x, y, v, z}];
Jacs = Jac //. {x → Es⟦1⟧, y → Es⟦2⟧, v → Es⟦3⟧, z → 0} // N;
Print["The eigenvalues of E* when b=29.2 are: ", Eigenvalues[Jacs]]
Jacp = Jac //. {x → Ep⟦1⟧, y → Ep⟦2⟧, v → Ep⟦3⟧, z → Ep⟦4⟧} // N;
Print["The eigenvalues of E+ when b=29.2 are: ", Eigenvalues[Jacp]]
Jaci = Jac //. {x → Ei⟦1⟧, y → Ei⟦2⟧, v → Ei⟦3⟧, z → Ei⟦4⟧} // N;
Print["The eigenvalues of Ei when b=29.2 are: ", Eigenvalues[Jaci]]
Jacm = Jac //. {x → Em⟦1⟧, y → Em⟦2⟧, v → Em⟦3⟧, z → Em⟦4⟧} // N;
Print["The eigenvalues of E- when b=29.2 are: ", Eigenvalues[Jacm]]
```

```
E*={0.000407598, 0.0495653, 0.0218397}

E+={0.717945, 0.215205, 0.00153678, 0.215205}

Ei=
 {0.0143072 - 0.00335392 i, 0.10903 - 0.0120421 i, 0.0201532 + 0.000353933 i, 0.10903 - 0.0120421 i}
```

E−=
{0.0143072 + 0.00335392 i, 0.10903 + 0.0120421 i, 0.0201532 − 0.000353933 i, 0.10903 + 0.0120421 i}

Out[ ]=


Out[ ]=


Out[ ]= ppH1.pdf

Out[ ]= ptH1.pdf

The eigenvalues of E* when b=29.2 are:
{−0.566765 + 0. i, 0.0204074 + 0.0804544 i, 0.0204074 − 0.0804544 i, 0.0495653 + 0. i}

The eigenvalues of E+ when b=29.2 are:
{−32.2336 + 0. i, −0.651062 + 0. i, −0.108661 + 0.187036 i, −0.108661 − 0.187036 i}

The eigenvalues of Ei when b=29.2 are:
{−1.69485 + 0.198587 i, 0.168324 − 0.0905302 i, 0.0750338 + 0.103646 i, −0.0200824 − 0.0263267 i}

The eigenvalues of E− when b=29.2 are:
{−1.69485 − 0.198587 i, 0.168324 + 0.0905302 i, 0.0750338 − 0.103646 i, −0.0200824 + 0.0263267 i}

Now, Computations of the Jacobians and Eigenvalues using EcoEvo package:

In[ ]:=
```
<<EcoEvo`
(*EcoEvoDocs;*)
ClearParameters;
SetModel[{Pop[x]→{Equation:→dyn[[1]],Color→Red},Pop[y]→{Equation:→dyn[[2]],Color→Green},
Pop[v]→{Equation:→ dyn[[3]],Color→Blue},Pop[z]→{Equation:→ dyn[[4]],Color→Purple}}]

fpT=SolveEcoEq[]//N
Jp=EcoJacobian[fpT[[6]]]//FullSimplify;
Jim=EcoJacobian[fpT[[7]]]//FullSimplify;
Jm=EcoJacobian[fpT[[8]]]//FullSimplify;
Jst=EcoJacobian[fpT[[4]]]//FullSimplify;
Print["Jac(E+)=",Jp//MatrixForm]
Print["Jac(E-)=",Jm//MatrixForm]
Print["Jac(Eim)=",Jim//MatrixForm]
Print["Eigenvalues of Ep are:",eip=EcoEigenvalues[fpT[[6]]]//FullSimplify]
```

Out[ ]= EcoEvo Package Version 1.6.4 (November 5, 2021)
Christopher A. Klausmeier <christopher.klausmeier@gmail.com>

Out[ ]= {{x → 0., y → 0., v → 0., z → 0.}, {x → 1., y → 0., v → 0., z → 0.},
{x → 1., y → 0., v → 0., z → 0.}, {x → 0.000280348, y → 0.0346317, v → 0.0221859, z → 0.},
{x → 0., y → -0.0078125, v → -0.00520833, z → -0.0078125},
{x → 0.494238, y → 0.308421, v → 0.00453657, z → 0.308421},
{x → 0.145387, y → 0.284118, v → 0.0131148, z → 0.284118},
{x → 0.00228546, y → 0.042969, v → 0.0219482, z → 0.042969}}

$$\mathrm{Jac}(E_+) = \begin{pmatrix} -0.494238 & -0.494238 & -21.4993 & 0 \\ 0.197341 & -0.316234 & 21.4993 & -0.308421 \\ -0.197341 & \frac{21}{64} & -22.3078 & -0.00453657 \\ 0 & 0.308421 & 0 & -0.308421 \end{pmatrix}$$

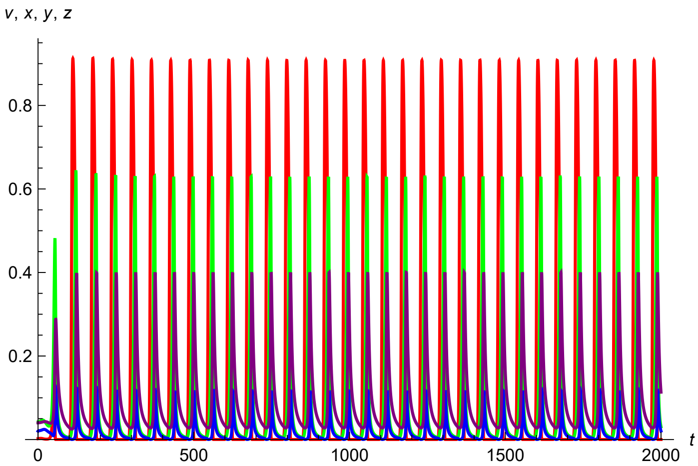
$$\mathrm{Jac}(E_-) = \begin{pmatrix} -0.00228546 & -0.00228546 & -0.0994175 & 0 \\ 0.954746 & -0.0507815 & 0.0994175 & -0.042969 \\ -0.954746 & \frac{21}{64} & -0.642387 & -0.0219482 \\ 0 & 0.042969 & 0 & -0.042969 \end{pmatrix}$$

$$\mathrm{Jac}(E_{im}) = \begin{pmatrix} -0.145387 & -0.145387 & -6.32433 & 0 \\ 0.570495 & -0.29193 & 6.32433 & -0.284118 \\ -0.570495 & \frac{21}{64} & -7.10845 & -0.0131148 \\ 0 & 0.284118 & 0 & -0.284118 \end{pmatrix}$$

Eigenvalues of E_p are:{-22.81, -0.157506 + 0.275785 i, -0.157506 - 0.275785 i, -0.301641}

In[ ]:=
```
(*Time plot corresponding to the blue curve**)
fpT⟦6⟧;
Epn=RuleListTweak[{x→x0,y→y0},{y},{0}]
sol=EcoSim[Epn,1500];
pt=PlotDynamics[sol1,PlotPoints→200]
lc=FindEcoCycle[FinalSlice[{sol1⟦1,1⟧,sol1⟦1,2⟧}]];
Print["The floquet exponents are"]
Chop[Evaluate[EcoEigenvalues[lc]]]
Export["soltF.pdf",pt]
```

Out[ ]= {x → 0.0022, y → 0.042}

Out[ ]=


The floquet exponents are

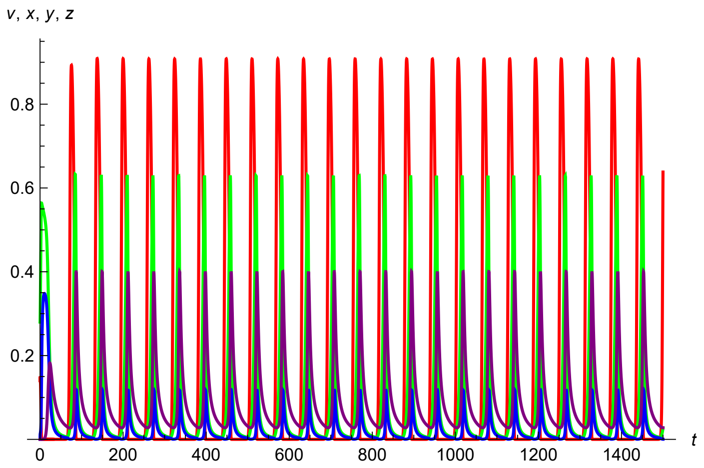
Out[ ]= {0.314283, 0.0000123842, -0.999987, -44.3216}

Out[ ]= soltF.pdf

In[ ]:=
```
(*Time plot corresponding to the brown curve**)
Epn=RuleListTweak[{x→x01,y→y01},{y},{0}]
sol=EcoSim[Epn,tf];
pt=PlotDynamics[soli,PlotPoints→200]
lc=FindEcoCycle[FinalSlice[{soli⟦1,1⟧,soli⟦1,2⟧}]];
Print["The floquet exponents are"]
Chop[Evaluate[EcoEigenvalues[lc]]]
Export["soltB.pdf",pt]
```

Out[ ]= {x → 0.1, y → 0.05}

Out[ ]=



The floquet exponents are

Out[ ]= {0.314283, 0.0000108179, -0.999989, -44.3216}

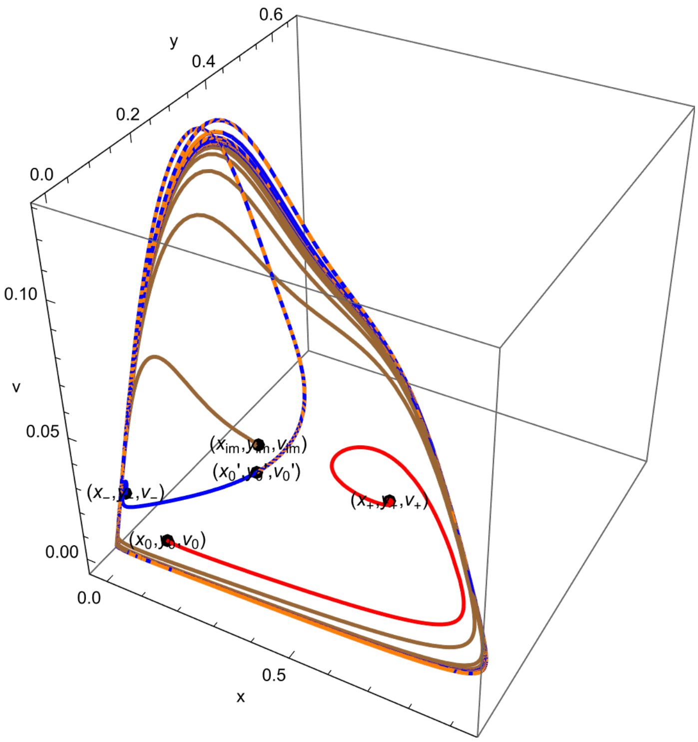
Out[ ]= soltB.pdf

In[ ]:=
```
X0={x0,y0,v0};
X01={x01,y01,v01};
X0i={x0i,y0i,v0i};
X0f={x0f,y0f,v0f};

Ep3={x,y,v}//.Drop[fpT[[6]],-1]
Epi={x,y,v}//.Drop[fpT[[7]],-1]
pp3D=ParametricPlot3D[{{  x[t],y[t],v[t]}/.sol1,{  x[t],y[t],v[t]}/.sol2,{  x[t],y[t],v[t]},
{  x[t],y[t],v[t]}/.solf},{t,0,tf},
BoxRatios→{1,1,1},AxesLabel→{"x","y","v"},PlotPoints→200, PlotRange→Full,PlotStyle→{Blue, Re
pt0={PointSize@.02, Style[Point[X0],Black]};pt0s=Text[Style["(x₋,y₋,v₋)", 10], X0];
pt01={PointSize@.02, Style[Point[X01],Black]};pt01s=Text[Style["(x₀,y₀,v₀)", 10], X01];
pt0i={PointSize@.02, Style[Point[Epi],Black]};pt0is=Text[Style["(xim,yim,vim)", 10], Epi];
pts={PointSize@.02, Style[Point[Ep3],Black]}; ptss=Text[Style["(x₊,y₊,v₊)", 10] ,Ep3
];
ptf={PointSize@.02, Style[Point[X0f],Black]}; ptfs=Text[Style["(x₀',y₀',v₀')", 10] ,X0f];
pts={ptss,pt0,pt0s,pts,pt01,pt01s,pt0i,pt0is,ptf,ptfs};
cy3=Show[{pp3D}, Graphics3D[pts], Axes → True,
 BoxRatios → 1]
 (*Export["cy3D.pdf",cy3]*)
```

Out[ ]= {0.494238, 0.308421, 0.00453657}

Out[ ]= {0.145387, 0.284118, 0.0131148}

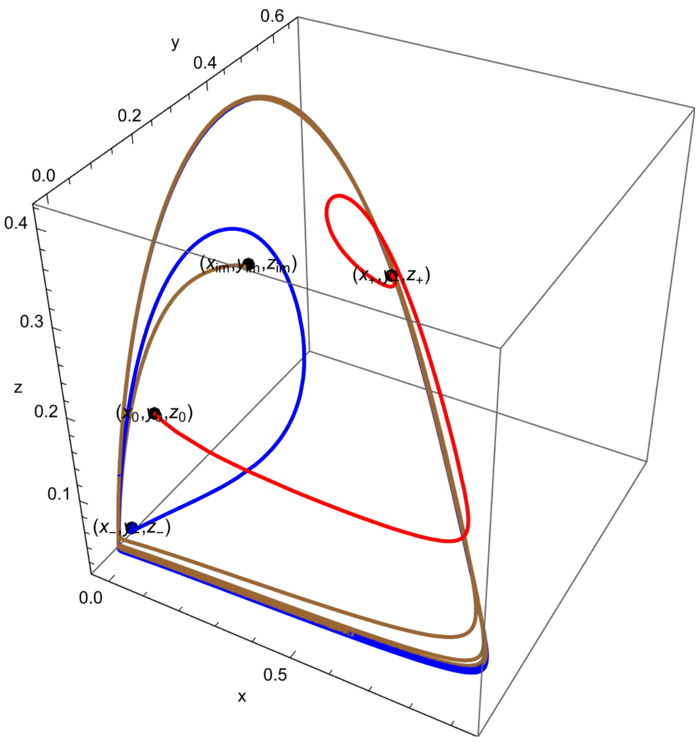
Out[ ]=



In[ ]:=
```
X0={x0,y0,z0};
X01={x01,y01,z01};
X0i={x0i,y0i,z0i};
Ep3={x,y,z}//.Drop[fpT[[6]],{3}]
Epi={x,y,z}//.Drop[fpT[[7]],{3}]
pp3D=ParametricPlot3D[{{  x[t],y[t],z[t]}/.sol1,{  x[t],y[t],z[t]}/.sol2,{  x[t],y[t],z[t]},
BoxRatios→{1,1,1},AxesLabel→{"x","y","z"},PlotPoints→200, PlotRange→Full,PlotStyle→{Blue, Re
pt0={PointSize@.02, Style[Point[X0],Black]};pt0s=Text[Style["(x₋,y₋,z₋)", 10], X0];
pt01={PointSize@.02, Style[Point[X01],Black]};pt01s=Text[Style["(x₀,y₀,z₀)", 10], X01];
pt0i={PointSize@.02, Style[Point[Epi],Black]};pt0is=Text[Style["(xim,yim,zim)", 10], Epi];
pts={PointSize@.02, Style[Point[Ep3],Black]}; ptss=Text[Style["(x₊,y₊,z₊)", 10] ,Ep3 ];
pts={ptss,pt0,pt0s,pts,pt01,pt01s,pt0i,pt0is};
cy3=Show[{pp3D}, Graphics3D[pts], Axes → True,
 BoxRatios → 1]
 Export["cy3Dz.pdf",cy3]
```

Out[ ]= {0.494238, 0.308421, 0.308421}

Out[ ]= {0.145387, 0.284118, 0.284118}

Out[ ]=


Out[ ]= cy3Dz.pdf