

On a three-dimensional tumor-virus compartmental model

This Mathematica Notebook is a supplementary material to the paper “On a three-dimensional and two four-dimensional oncolytic viro-therapy models”. It contains some of the calculations and illustrations appearing in the paper.

1) Section 2 (in paper): Deterministic model with Logistic growth [Tian2011]

0) Definition of the model [Tian11]:

In[]:=

```
SetDirectory[NotebookDirectory[]];
AppendTo[$Path,Directory];
Clear["Global`*"];
(*Some aliases*)
Format[βv]:=Subscript[β,v];Format[βy]:=Subscript[β,y];
(***** Four dim Deterministic epidemic model with Logistic growth *****)
x1=λ x(1-(x+y)/K)- β x v ;
y1=β x v -βy y z- γ y;
v1=-β x v - βv v z+ b γ y - δ v;

par={λ,δ,β,b};X={x,y,v};
val={0.36,0.44,0.11,28};
parN=Thread[par→val];
par2=Thread[{λ,δ}→{0.36,0.44}];
cp={λ>0,δ>0,β>0,b>1};
c3dim={βv→0,βy→0,γ→1, K→1};
cnTian={λ→0.36,β→0.11,δ→0.44,K→1,γ→1,βv→0, βy→0}(*Numerical values of Tian*);

dyn={x1,y1,v1}/.βy→0/.βv→0(*Tian 11 case with K>0*);
dynKeq1=dyn/.c3dim;
Print["
      x'
      (y')=",dyn//FullSimplify//MatrixForm,
      v'

"and the reparametrized dynamics [Tian 2011] are
      x'
      (y')=",
      v'

dynKeq1//FullSimplify//MatrixForm]

R0=b β K/(β K+δ);b0=1+δ/(β K);
Print["R0=", R0, " , b0=", b0]
```

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -v x \beta + x \left(1 - \frac{x+y}{K}\right) \lambda \\ v x \beta - y \gamma \\ b y \gamma - v (x \beta + \delta) \end{pmatrix}$$

and the reparametrized dynamics [Tian 2011] are $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -x (v \beta + (-1 + x + y) \lambda) \\ -y + v x \beta \\ b y - v (x \beta + \delta) \end{pmatrix}$

$$R_0 = \frac{b K \beta}{K \beta + \delta}, \quad b_0 = 1 + \frac{\delta}{K \beta}$$

Fixed points and analysis of the Stability via Routh Hurwitz:

In[]:=

```

cfp=Solve[Thread[dyn=={0,0,0}],{x,y,v}]/FullSimplify;
fp=X/.cfp;

Print["Number of fixed points is ",Length[fp]," The third FP is E*="]
fp[[3]]//FullSimplify
fpN=fp/.c3dim/.parN;
Print["in first fig. E*=",fpN[[3]]]
(*"Jacobian is"*)
Jac=Grad[dyn,{x,y,v}]/FullSimplify;
J0=Jac/.cfp[[1]];J0//MatrixForm;
Print["J(E_K) and its det are"]
J1=Jac/.cfp[[2]];J1//MatrixForm
Det[J1]//Factor
Print["J(E_*) "]
Jst=Jac/.cfp[[3]]//FullSimplify;Jst//MatrixForm
(*Jstcr=Jst/.b→b0//FullSimplify;*)

(*Routh Hurwitz conditions for the stability of E_**)
pc=Collect[Det[ψ IdentityMatrix[3]-Jst],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Print["a1=",a1=Apart[coT[[3]]]//.c3dim, ", a2=",a2=coT[[2]]//.c3dim, ", a3=",a3=coT[[1]]//.c3dim]
H2=a1*a2-a3;
Print["Hurw(b) when K=γ=1 is H2=", H2//.c3dim//FullSimplify]
Print["H2(b0) when K=γ=1 is",H2/.b→b0//.c3dim//FullSimplify]
Print["Denominator of H2 is posi",Denominator[Together[H2]]/.K→1//FullSimplify]
Together[H2]//FullSimplify];
φb=Collect[Numerator[Together[H2]]/(δ λ),b] /.K→1//FullSimplify;
(*Print["φ(b)=", Collect[φb,b]]*)
Print["value of φ(b) at crit b is posi "]
φb/.b→b0/.K→1//FullSimplify
cofi=CoefficientList[φb,b]//FullSimplify;
Print["Coefficients of numerator φ(b)are {B0,B1,B2,B3,B4}=",cofi]
(*φb/.b→1//FullSimplify*)
Print[" shifted coefs: "]
φbsh=Collect[φb/.b→(b0/.K→1)+b,b];
cofi=CoefficientList[φbsh,b]//FullSimplify
Print["There is no solution of B2<0 AND B3>0 for shifted pol: "]

eq=Join[{cofi[[4]]>0&&cofi[[3]]<0},cp];
Reduce[eq,λ]
FindInstance[Join[{cofi[[4]]>0&&cofi[[3]]<0},cp],par]
(*;
Reduce[Join[{cofi[[4]]>0&&cofi[[3]]<0},cp],par]//FullSimplify;*)
Print["There are solutions for BOTH B2<0 AND B3>0, separately: "]
eq=Join[{cofi[[4]]>0},cp];
Reduce[eq,λ]
eq=Join[{cofi[[3]]<0},cp];
Reduce[eq,λ]

```

Number of fixed points is 3 The third FP is E*=

$$Out[]:= \left\{ \frac{\delta}{(-1+b)\beta}, \frac{((-1+b)K\beta-\delta)\delta\lambda}{(-1+b)\beta((-1+b)K\beta\gamma+\delta\lambda)}, \frac{\gamma((-1+b)K\beta-\delta)\lambda}{\beta((-1+b)K\beta\gamma+\delta\lambda)} \right\}$$

in first fig. $E_* = \{0.148148, 0.0431317, 2.64672\}$

$J(E_K)$ and its det are

Out[*]=J/MatrixForm=

$$\begin{pmatrix} -\lambda & -\lambda & -K\beta \\ 0 & -\gamma & K\beta \\ 0 & b\gamma & -K\beta - \delta \end{pmatrix}$$

$$\text{Out[*]} = \gamma (-K\beta + bK\beta - \delta) \lambda$$

$J(E_*)$

Out[*]=J/MatrixForm=

$$\begin{pmatrix} \frac{\delta \lambda}{K\beta - bK\beta} & \frac{\delta \lambda}{K\beta - bK\beta} & -\frac{\delta}{-1+b} \\ \frac{\gamma((-1+b)K\beta - \delta)\lambda}{(-1+b)K\beta\gamma + \delta\lambda} & -\gamma & \frac{\delta}{-1+b} \\ \frac{\gamma(K(\beta - b\beta) + \delta)\lambda}{(-1+b)K\beta\gamma + \delta\lambda} & b\gamma & \frac{b\delta}{1-b} \end{pmatrix}$$

$$a_1 = \frac{-1+b+b\delta}{-1+b} + \frac{\delta\lambda}{(-1+b)\beta}, \quad a_2 =$$

$$\frac{\delta\lambda((-1+b)\beta(-1+b+\beta-b\beta+\delta+b\delta) + ((-1+b)^2\beta + b\delta^2)\lambda)}{(-1+b)^2\beta((-1+b)\beta + \delta\lambda)}, \quad a_3 = \delta \left(1 + \frac{\delta}{\beta - b\beta}\right) \lambda$$

$$\text{Hurw}(b) \text{ when } K=\gamma=1 \text{ is } H_2 = \delta\lambda \left(-1 + \frac{\delta}{(-1+b)\beta} - \frac{(\beta(-1+b+b\delta) + \delta\lambda)((-1+b)^2\beta^2 - b\delta^2\lambda - (-1+b)\beta(-1+\delta-\lambda+b(1+\delta+\lambda)))}{(-1+b)^3\beta^2((-1+b)\beta + \delta\lambda)} \right)$$

$H_2(b_0)$ when $K=\gamma=1$ is $(1+\beta+\delta)\lambda(1+\beta+\delta+\lambda)$

Denominator of H_2 is $\text{posi}((-1+b)^3\beta^2((-1+b)\beta + \delta\lambda))$

value of $\phi(b)$ at crit b is posi

$$\text{Out[*]} = \frac{\delta^3(1+\beta+\delta) \times (1+\lambda) \times (1+\beta+\delta+\lambda)}{\beta}$$

Coefficients of numerator $\phi(b)$ are $\{B_0, B_1, B_2, B_3, B_4\} =$

$$\left\{ \beta(1+\lambda)(-\beta+\delta\lambda), -\beta^3(-1+\delta) + \delta^3\lambda^2 - \beta\delta\lambda(2+3\delta+2\lambda) + \beta^2(3+3\delta-\delta^2+3\lambda), \right. \\ \left. \beta(\beta^2(-3+2\delta) - 3\beta(1+2\delta+\lambda) + \delta\lambda(1+\delta(3+\delta)+\lambda)), \beta^2(1-\beta(-3+\delta) + \delta(3+\delta)+\lambda), -\beta^3 \right\}$$

shifted coefs:

$$\text{Out[*]} = \left\{ \frac{\delta^3(1+\beta+\delta) \times (1+\lambda) \times (1+\beta+\delta+\lambda)}{\beta}, \right. \\ \left. \delta^2(3+\delta(5+3\delta) + 5\lambda + 2\delta(3+\delta)\lambda + (2+\delta)\lambda^2 + \beta(3+3\delta+3\lambda+2\delta\lambda)), \right. \\ \left. \beta\delta(3-\beta^2+3\delta(1+\delta) + 4\lambda + \delta(3+\delta)\lambda + \lambda^2), \beta^2(1+(-1+\delta)\delta - \beta(1+\delta)+\lambda), -\beta^3 \right\}$$

There is no solution of $B_2 < 0$ AND $B_3 > 0$ for shifted pol:

Out[*]= False

Out[*]= { }

There are solutions for BOTH $B_2 < 0$ AND $B_3 > 0$, separately:

$$\begin{aligned}
\text{Out}[*]= & \left(\delta > 0 \ \&\& \ 0 < \beta \leq \frac{1 - \delta + \delta^2}{1 + \delta} \ \&\& \ b > 1 \ \&\& \ \lambda > 0 \right) \ || \\
& \left(\delta > 0 \ \&\& \ \beta > \frac{1 - \delta + \delta^2}{1 + \delta} \ \&\& \ b > 1 \ \&\& \ \lambda > -1 + \beta + \delta + \beta \delta - \delta^2 \right) \\
\text{Out}[*]= & \delta > 0 \ \&\& \ \beta > \sqrt{3 + 3\delta + 3\delta^2} \ \&\& \ b > 1 \ \&\& \\
& 0 < \lambda < \frac{1}{2} \times (-4 - 3\delta - \delta^2) + \frac{1}{2} \sqrt{4 + 4\beta^2 + 12\delta + 5\delta^2 + 6\delta^3 + \delta^4}
\end{aligned}$$

Bifurcation diagram of codimension 2:

In[]:=

```

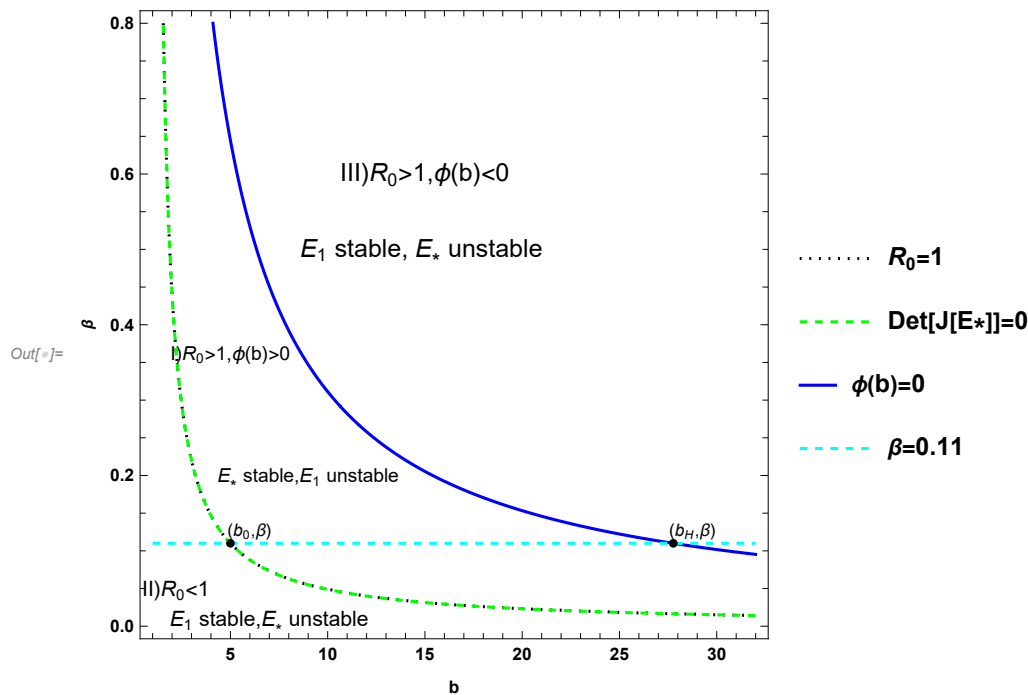
bm=1; bM=32; ym=0; yM=0.8;
R01=R0/.c3dim/.par2;
a3N=a3/.par2;
a1N=a1/.par2;
phiN=phi/.par2//FullSimplify;

pt1s=Text["(b0,β)",Offset[{10,6},{b0/.c3dim/.par2/.β→0.11,0.11}]];
pt1={PointSize[Medium],Style[Point[{b0/.c3dim/.par2/.β→0.11,0.11}],Black]};
ptHs=Text["(bH,β)",Offset[{10,6},{27.7664,0.11}]];
ptH1={PointSize[Medium],Style[Point[{27.7664,0.11}],Black]};
R1=Style[Text["I) R0>1, φ(b)>0",{5,0.36}],10];R1a=Style[Text["E* stable,E1 unstable",{9,0.2}],1
R2=Style[Text["II) R0<1",{2.2,0.05}],11];R2a=Style[Text["E1 stable,E* unstable",{7,0.01}],11];
R3=Style[Text["III) R0>1, φ(b)<0",{15,0.6}],13];R3a=Style[Text["E1 stable, E* unstable",{15,0.
epi={pt1,pt1s,ptHs,ptH1,R1,R1a,R2,R2a,R3,R3a};

pR0=ContourPlot[R01==1,{b,bm,bM},{β,ym,yM],ContourStyle→{Black,Dotted},
  FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"R0=1"}];
pphi1=ContourPlot[phiN==0,{b,bm,bM},{β,ym,yM],ContourStyle→{Blue,Bold},PlotPoints→200,
  FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"φ(b)=0"}];
pa1=ContourPlot[a1N==0,{b,bm,bM},{β,ym,yM],ContourStyle→{Red,Bold},
  FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Tr[J[E*]]=0"}];
pa3=ContourPlot[a3N==0,{b,bm,bM},{β,ym,yM],ContourStyle→{Green,Dashed},PlotPoints→180,
  FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"Det[J[E*]]=0"}];
pbeta=ContourPlot[β==0.11,{b,bm,bM},{β,ym,yM],ContourStyle→{Cyan,Dashed},PlotPoints→180,
  FrameLabel→{"b","β"},LabelStyle→{Black,Bold},Frame→True,PlotLegends→{"β=0.11"}];

peq={pR0,pa3,(*pa1,*)pphi1,pbeta};
pcut=Show[peq,Epilog→epi]
Export["map3.pdf",pcut]

```



Out[]:= map3.pdf

3D parametric plot:

```

In[ ]:=
Xt=Map[# [t]&,X ];
ct=Thread[X→Xt];
dynt=dynKeq1/.ct;
x1=dynt[[1]] ;
y1=dynt[[2]];
v1=dynt[[3]];
X0={0.4,0.05,0.05};
ode={x'[t]==x1,y'[t]==y1,v'[t]==v1,x[0]==X0[[1]],y[0]==X0[[2]],v[0]==X0[[3]]};
T=2800;
odeN=ode/.parN
solr=NDSolve[odeN,{x,y,v},{t,0,T}];

cyR=ParametricPlot3D[{ x[t],y[t],v[t] }/.solr,{t,0,T}, BoxRatios→{1,1,1},AxesLabel→{"x","y",
PlotRange→Full,PlotStyle→{Red,Dashed}}];
pt0={PointSize@.02, Style[Point[X0],Black]};pt0s=Text[Style["(x0,y0,v0)", 10], X0];
pts={PointSize@.02, Style[Point[fpN[[3]]],Black]}; ptss=Text[Style["(x*,y*,v*)", 10],fpN[[3]]
];
pts={ptss,pt0,pt0s,pts};

```

```

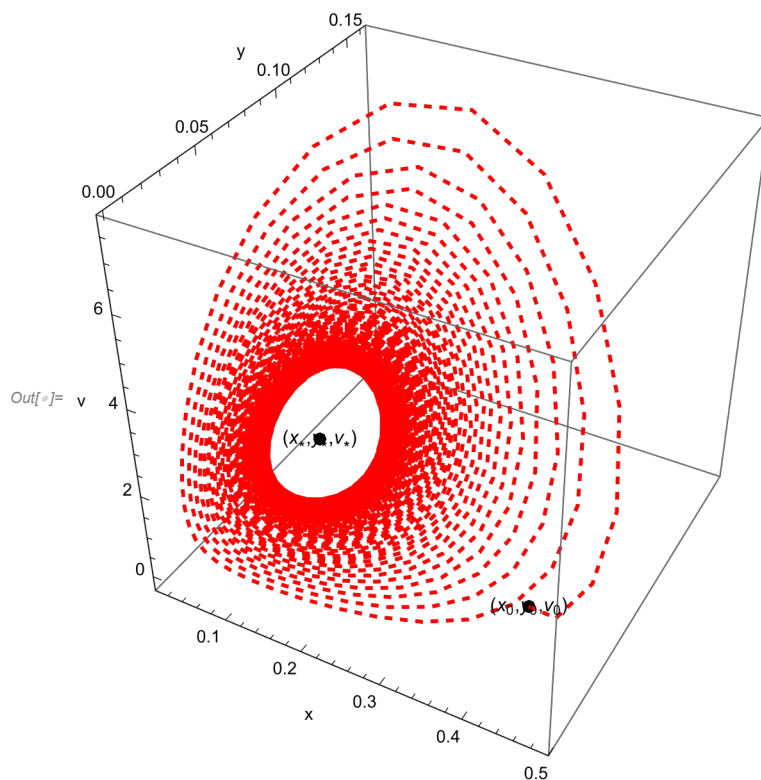
Out[ ]:= {x'[t] == -0.11 v[t] × x[t] + 0.36 x[t] (1 - x[t] - y[t]), y'[t] == 0.11 v[t] × x[t] - y[t],
v'[t] == -0.44 v[t] - 0.11 v[t] × x[t] + 28 y[t], x[0] == 0.4, y[0] == 0.05, v[0] == 0.05}

```

```

In[ ]:=
cyR3=Show[{cyR}, Graphics3D[pts], Axes → True,
BoxRatios → 1]

```



```

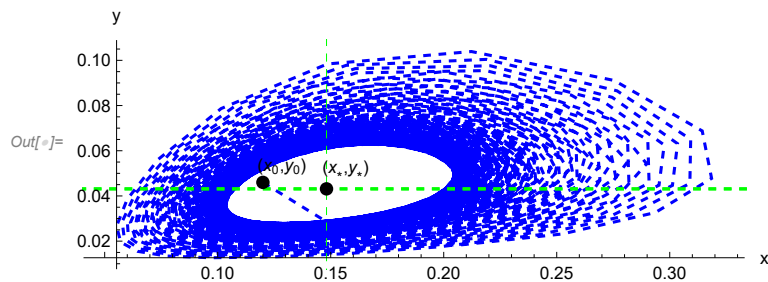
(*****b=28; different initial values *****)
b=28;T1=4000;
x0n=0.12; y0n=0.046;v0n=0.01;
ode={x'[t]==x1,y'[t]==y1,v'[t]==v1,x[0]==x0n,y[0]==y0n,v[0]==v0n};
sol=NDSolve[ode/.cnTian,{x,y,v},{t,0,T1}];
(*****Parametric plot conditions***)

cyB=ParametricPlot[{ x[t],(y[t])}/.sol,{t,0,T1}, AxesLabel->{"x","y"},
PlotRange->Full,PlotStyle->{Dashed,Blue}];
pyn=Plot[(fp[[3,2]]/.cnTian),{t,0,T1},PlotStyle->{Dashed,Green}];
cyBall=Show[{cyB,pyn, Graphics[{Green,Dashed,
Line[{ {fp[[3,1]]/.cnTian,0},{fp[[3,1]]/.cnTian,1} } ]},
Epilog->{{Text["(x*,y*)",Offset[{10,10},{fp[[3,1]]/.cnTian},{fp[[3,2]]/.cnTian}]}},
{PointSize[Large],Style[Point[{ {fp[[3,1]]/.cnTian},{fp[[3,2]]/.cnTian} },Black]}},
{PointSize[Large],Point[{x0n,y0n}],Text["(x0,y0)",Offset[{10,8},{x0n,y0n}]}]}]}];

cyr=ParametricPlot[{ x[t],(y[t])}/.solr,{t,0,T}, AxesLabel->{"x","y"},PlotStyle->{Red}];
cyRall=Show[{cyr},PlotRange->{{0.05,0.35},{0,0.12}},Epilog->
{{PointSize[Large],Point[{X0[[1]],X0[[2]]}],Text["(x0,y0)",Offset[{10,8},{X0[[1]],X0[[2]]}]}]}];

Export["T11R.pdf",cyr]
Export["T11B.pdf",cyB]

```



Out[]= T11R.pdf

Out[]= T11B.pdf

Computations of the Jacobians and Eigenvalues using EcoEvo package:

In[]:=

```

<<EcoEvo`
(*EcoEvoDocs;*)
(*****Analysis of the Model, K=γ=1; must redefine fp, cp***)
ClearParameters;Clear["b"];
SetModel[{Pop[x]→{Equation→dynKeq1[[1]],Color→Red},Pop[y]→
{Equation→(dynKeq1[[2]]),Color→Green},
Pop[v]→{Equation→(dynKeq1[[3]]),Color→Purple},
Parameters→(cp)}]

fpT=SolveEcoEq[]//FullSimplify

J0T=EcoJacobian[fpT[[1]]]//FullSimplify;
J1T=EcoJacobian[fpT[[2]]]//FullSimplify;
Jst=EcoJacobian[fpT[[3]]]//FullSimplify;
Print["Jac(E₀)=",J0T//MatrixForm]
Print["Jac(E₁)=",J1T//MatrixForm]
Print["Jac(Eₛ)=",Jst//MatrixForm]
Print["Eigenvalues of E₁ are:",eiT=EcoEigenvalues[fpT[[2]]]//FullSimplify]

Print["b₀=bₛ₁=",bs1=Apart[Last[Last[Reduce[Join[{eiT[[2]]>0},cp],b]]]]]

```

Out[]:= EcoEvo Package Version 1.6.4 (November 5, 2021)

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$$Out[]:= \left\{ \{x \rightarrow 0, y \rightarrow 0, v \rightarrow 0\}, \{x \rightarrow 1, y \rightarrow 0, v \rightarrow 0\}, \right. \\ \left. \left\{ x \rightarrow \frac{\delta}{(-1+b)\beta}, y \rightarrow \frac{((-1+b)\beta - \delta)\delta\lambda}{(-1+b)\beta((-1+b)\beta + \delta\lambda)}, v \rightarrow \frac{((-1+b)\beta - \delta)\lambda}{\beta((-1+b)\beta + \delta\lambda)} \right\} \right\}$$

$$Jac(E_0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -1 & 0 \\ 0 & b & -\delta \end{pmatrix}$$

$$Jac(E_1) = \begin{pmatrix} -\lambda & -\lambda & -\beta \\ 0 & -1 & \beta \\ 0 & b & -\beta - \delta \end{pmatrix}$$

$$Jac(E_s) = \begin{pmatrix} \frac{\delta\lambda}{\beta-b\beta} & \frac{\delta\lambda}{\beta-b\beta} & -\frac{\delta}{-1+b} \\ \frac{((-1+b)\beta-\delta)\lambda}{(-1+b)\beta+\delta\lambda} & -1 & \frac{\delta}{-1+b} \\ \frac{(\beta-b\beta+\delta)\lambda}{(-1+b)\beta+\delta\lambda} & b & \frac{b\delta}{1-b} \end{pmatrix}$$

Eigenvalues of E₁ are:

$$\left\{ \frac{1}{2} \times \left(-1 - \beta - \delta - \sqrt{(1 + \beta + \delta)^2 - 4(\beta - b\beta + \delta)} \right), \frac{1}{2} \times \left(-1 - \beta - \delta + \sqrt{(1 + \beta + \delta)^2 - 4(\beta - b\beta + \delta)} \right), -\lambda \right\}$$

$$b_0 = b_{s1} = 1 + \frac{\delta}{\beta}$$

```

In[ ]:= λ=0.36;β=0.11;δ=0.44;b=28;K=1;γ=1;T1=4000;T2=2000;T=800;
X0={x→0.5,y→0.5,v→1.5};
fpN=fpT(*Numerical values of FP*)
Print["Eigenvalues corresponding to E* are:"]
EcoEigenvalues[fpN[[3]]](*Eigenvalues corresponding to E* **)
solE3=EcoSim[RuleListAdd[(fpN[[3]]),X0],T2];

Fig5T=PlotDynamics[{solE3[[1]],solE3[[2]]},PlotRange→{{0,T2},{0,0.7}}]
Export["Fig5T.pdf",Fig5T]

```

```

Out[ ]:= {{x → 0, y → 0, v → 0}, {x → 1, y → 0, v → 0}, {x → 0.148148, y → 0.0431317, v → 2.64672}}

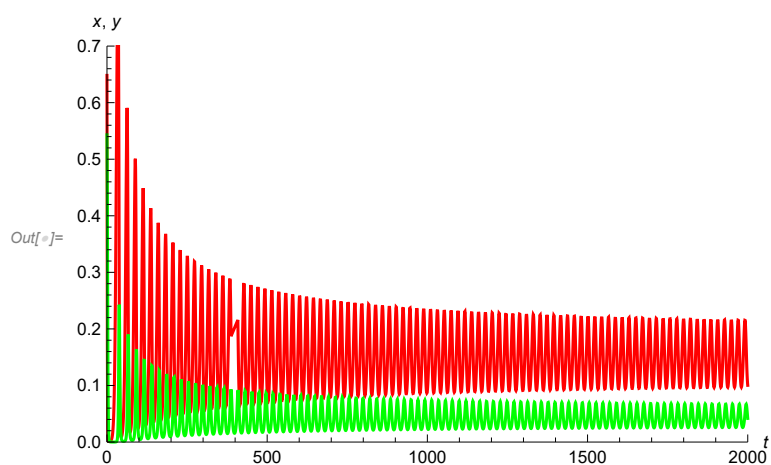
```

Eigenvalues corresponding to E* are:

```

Out[ ]:= {-1.51022, 0.000296187 + 0.298909 i, 0.000296187 - 0.298909 i}

```



```

Out[ ]:= Fig5T.pdf

```

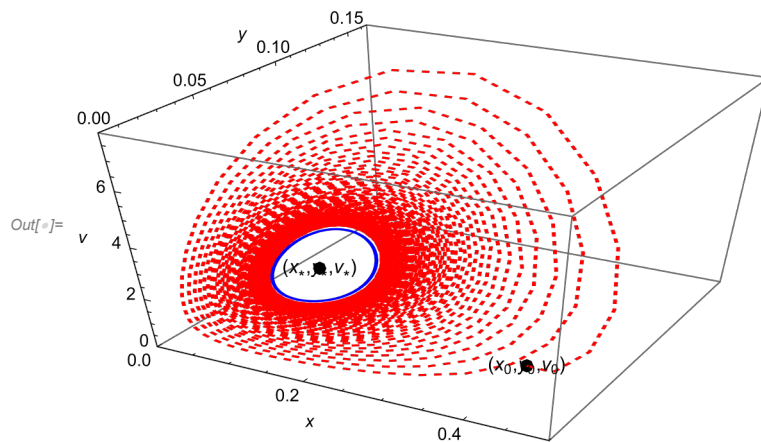
In[]:=

```
E2n = RuleListTweak[fpT[[3]], {x, y}, {0.1, 0.1}]
Print["Time plot around E2"]
sol = EcoSim[E2n, T2];
lc = FindEcoCycle[FinalSlice[sol]];
cy3D = RuleListPlot[lc, {x, y, v}, PlotStyle -> Blue];
cy113D = Show[{cy3D, cyR3}]
Export["cy113D.pdf", cy113D]
```

Out[]:= {v -> 2.64672, x -> 0.248148, y -> 0.143132}

Time plot around E2

FindEcoCycle: Failed to converge to the requested accuracy within 100 iterations.



Out[]:= cy113D.pdf

1) Numerical simulations:

Bifurcation Diagram:

In[]:=

```

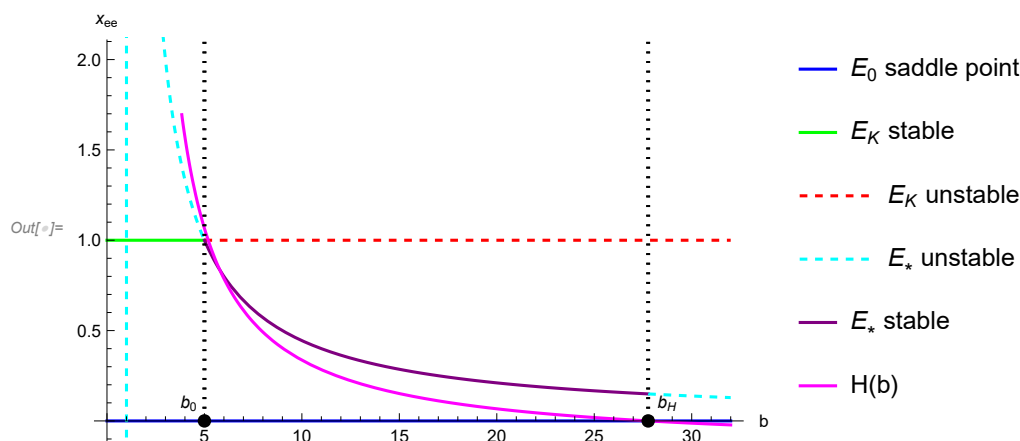
Clear["b"];
bL=32;
Print["b0=",bs1//N]
cb=NSolve[( $\phi$ b)==0,b,WorkingPrecision->10]
bM=Max[Table[Re[b/.cb[[i]]],{i,Length[cb]}]];
Print["bH=",bH=N[bM,30]]
fpT//N;
linb0=Line[{{bs1,0},{bs1,10}}];
lib0=Graphics[{Thick,Black,Dotted,linb0}];
linbH=Line[{{bH,0},{bH,10}}];
libH=Graphics[{Thick,Black,Dotted,linbH}];
px0=Plot[0,{b,0,bL},PlotStyle->{Blue},PlotRange->All,PlotLegends->{"E0 saddle point"}];
px1a=Plot[{1},{b,0,bs1},PlotStyle->{Green},PlotRange->All,PlotLegends->{"EK stable"}];
px1b=Plot[1,{b,bs1,bL},PlotStyle->{Red,Dashed},PlotRange->All,
PlotLegends->{"EK unstable"}];
pxe1=Plot[{x/.fpT[[3]]},{b,0,bs1},PlotStyle->{Cyan,Dashed},PlotRange->All,
PlotLegends->{"E* unstable"}];
pxe2=Plot[{x/.fpT[[3]]},{b,bs1,bH},PlotStyle->{Purple},PlotRange->All,
PlotLegends->{"E* stable"}];
pxe2a=Plot[{x/.fpT[[3]]},{b,bH,bL},PlotStyle->{Cyan,Dashed},PlotRange->All];
epi={Text["b0",Offset[{-8,10},{bs1,0}],
{PointSize[Large],Style[Point[{bs1,0}],Black]},Text["bH",Offset[{10,10},{bH,0}],
{PointSize[Large],Style[Point[{bH,0}],Black]}}];
pHb=Plot[{H2},{b,0,bL},PlotStyle->{Magenta},Epilog->epi,AxesLabel->{"b","H(b)"},PlotLegends->{"H(b)"}];
bif11T=Show[{px0,px1a,px1b,pxe1,pxe2,pxe2a,pHb,lib0,libH},Epilog->epi,
PlotRange->{{0,bL},{0,2}},AxesLabel->{"b","xee"}];
pHbS=Show[{pHb,lib0}];
Export["pHb.pdf",pHbS]
Export["bif11T.pdf",bif11T]

```

b0=5.

Out[]:= {{b → -0.347032}, {b → 0.63211 - 0.406197 i }, {b → 0.63211 + 0.406197 i }, {b → 27.7664}}

bH=27.7664



Out[]:= pHb.pdf

Out[]:= bif11T.pdf

Time plot when b=28:

```

In[ ]:= xm = 0; ym = 0; xM = 1; yM = 1; b = 28;
E2n = RuleListTweak[fpT[[3]], {x, y}, {0.2, 0.01}]
Print["Time plot around E2"]
sol = EcoSim[E2n, T]
psol = PlotDynamics[sol]
Print["Finding the cycle using time slice"]
lc = FindEcoCycle[FinalSlice[sol]]
Print["Final time slice is"]
FinalTime[lc]

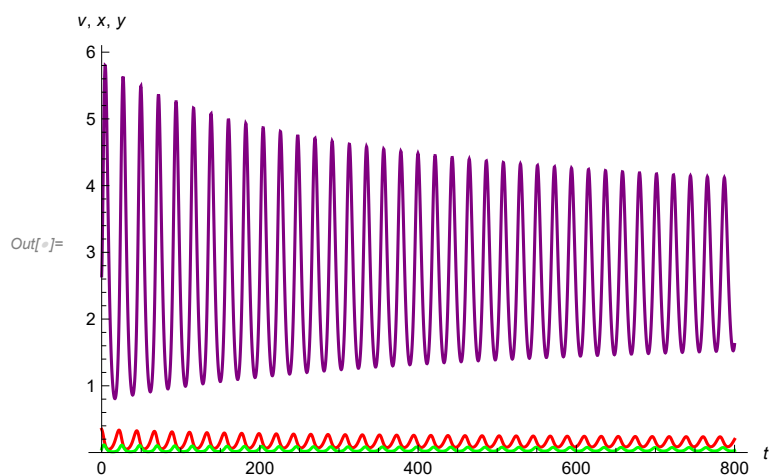
eq = {Drop[fpT[[1]], {3}], Drop[fpT[[2]], {3}], Drop[fpT[[3]], {3}]}
Print["FP="]
fpT
Print["Eigenvalues of FP:", EcoEigenvalues[fpT]]
pp = Show[PlotEcoPhasePlane[{x, xm, xM}, {y, ym, yM}], PlotPoints → 180],
RuleListPlot[eq, PlotMarkers → EcoStableQ[fpT]]];
(*sp11=Show[pp,RuleListPlot[Drop[sol,{3}],PlotStyle→Blue]]
Export["sp11.pdf",sp11]*)
(*RuleListPlot[Append[eq,lc]]*)
cy1B = RuleListPlot[Drop[sol,{3}], PlotStyle → Blue];

```

Out[]:= {v → 2.64672, x → 0.348148, y → 0.0531317}

Time plot around E2




Out[]:= {x → InterpolatingFunction[ Domain: {{0., 800.}}
Output: scalar],
y → InterpolatingFunction[ Domain: {{0., 800.}}
Output: scalar],
v → InterpolatingFunction[ Domain: {{0., 800.}}
Output: scalar]}



Finding the cycle using time slice

FindEcoCycle: Failed to converge to the requested accuracy within 100 iterations.

```

Out[ ]:= {v → InterpolatingFunction[ Domain: {{0., 21.2}}  
Output: scalar],  
x → InterpolatingFunction[ Domain: {{0., 21.2}}  
Output: scalar],  
y → InterpolatingFunction[ Domain: {{0., 21.2}}  
Output: scalar]}

```

Final time slice is

```
Out[ ]:= 21.1984
```

```
Out[ ]:= {{x → 0, y → 0}, {x → 1, y → 0}, {x → 0.148148, y → 0.0431317}}
```

FP=

```
Out[ ]:= {{x → 0, y → 0, v → 0}, {x → 1, y → 0, v → 0}, {x → 0.148148, y → 0.0431317, v → 2.64672}}
```

Eigenvalues of FP: $\{-1., -0.44, 0.36\}$, $\{-2.54436, 0.994357, -0.36\}$,
 $\{-1.51022, 0.000296187 + 0.298909 i, 0.000296187 - 0.298909 i\}$

```
In[ ]:= T = 3000;
```

```
E2nb = RuleListTweak[fpT[[3]], {x, y}, {0.1, 0.01}]
```

```
solb = EcoSim[E2nb, T];
```

```
lcb = FindEcoCycle[FinalSlice[solb]];
```

```
lcbp = RuleListPlot[lcb, {x, y}, PlotStyle → {Black}];
```

```
cy1B = RuleListPlot[Drop[solb, {3}], PlotStyle → {Dashed, Blue}];
```

```
cy11E = Show[
```

```
{cyBall, RuleListPlot[lc, {x, y}, PlotStyle → {Red}], pyn, Graphics[{Green, Dashed,  
Line[{x /. x → fp[[3, 1]], 0}, {x /. x → fp[[3, 1]], 1}]}],
```

```
PlotRange → {{0, 0.35}, {0, 0.1}}, Epilog → {Text["(x*,y*)",
```

```
Offset[{10, 10}, {x /. x → fp[[3, 1]], (y /. y → fp[[3, 2]])}],
```

```
{PointSize[Large], Style[Point[{x /. x → fp[[3, 1]], (y /. y → fp[[3, 2]])}, Black]},
```

```
{PointSize[Large], Point[{x0n, y0n}]},
```

```
Text["(x0,y0)", Offset[{10, 8}, {x0n, y0n}]]}]
```

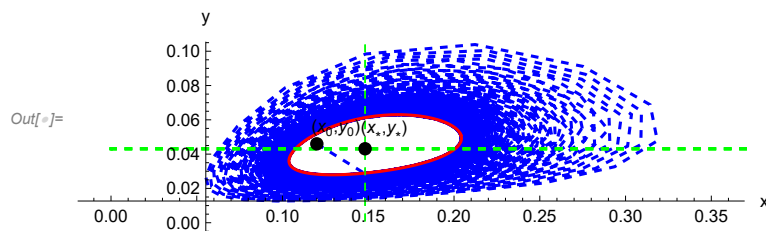
```
(*cy11F=Show[{cy11,RuleListPlot[lc,{x,y},PlotStyle→{Red}]}]*)
```

```
Export["cy11E.pdf", cy11E]
```

```
(*Export["cy11F.pdf", cy11F]*)
```

```
Out[ ]:= {v → 2.64672, x → 0.248148, y → 0.0531317}
```

FindEcoCycle: Failed to converge to the requested accuracy within 100 iterations.



```
Out[ ]:= cy11E.pdf
```

Linear birth model

```

In[ ]:= x1 = λ x - β x v ;
y1 = β x v - γ y ;
v1 = -β x v + b γ y - δ v ;
dyn = {x1, y1, v1} /. β y → 0 /. β v → 0 (*Tian 11 case with K>0*)
dynKeq1 = dyn /. c3dim ;

cfp = Solve[Thread[dyn == {0, 0, 0}], {x, y, v}] // FullSimplify;
fp = X /. cfp

Print["Number of fixed points is ", Length[fp], " The ee FP is E*="]
fp[[2]] // FullSimplify
fpN = fp /. c3dim /. parN;

(*"Jacobian is"*)
Jac = Grad[dyn, {x, y, v}] // FullSimplify;
Jst = Jac /. cfp[[2]];
Print["J(E_*) = ", Jst // MatrixForm, " det= ", Det[Jst] // Factor]

(*Routh Hurwitz conditions for the stability of E_**)
pc = Collect[Det[ψ IdentityMatrix[3] - Jst], ψ];
coT = CoefficientList[pc, ψ] // FullSimplify;
Print["a1=", a1 = Apart[coT[[3]]] // . c3dim,
      ", a2=", a2 = coT[[2]] // . c3dim, ", a3=", a3 = coT[[1]] // . c3dim]
H2 = a1 * a2 - a3;
Print["Hurw(b) when K=γ=1 is H2=", H2 // . c3dim // FullSimplify]
Print["H2(b0) when K=γ=1 is", H2 /. b → b0 // . c3dim // FullSimplify]
Print["Denominator of H2 is posi", Denominator[Together[H2]] /. K → 1 // FullSimplify]
Together[H2 // FullSimplify];
φb = Collect[Numerator[Together[H2]] / (δ λ), b] /. K → 1 // FullSimplify;
Print["φ(b)=", Collect[φb, b]]
cofi = CoefficientList[φb, b] // FullSimplify;
Print["Coefficients of numerator φ(b) are {B0,B1,B2}=", cofi]
bH = Solve[φb == 0, b] [[2]]

Out[ ]:= {-v x β + x λ, v x β - y γ, -v x β + b y γ - v δ}

Out[ ]:= {{0, 0, 0}, { $\frac{\delta}{(-1+b)\beta}$ ,  $\frac{\delta\lambda}{(-1+b)\beta\gamma}$ ,  $\frac{\lambda}{\beta}$ }}
```

Number of fixed points is 2 The ee FP is E*=

```

Out[ ]:= { $\frac{\delta}{(-1+b)\beta}$ ,  $\frac{\delta\lambda}{(-1+b)\beta\gamma}$ ,  $\frac{\lambda}{\beta}$ }
```

$$J(E_*) = \begin{pmatrix} 0 & 0 & -\frac{\delta}{-1+b} \\ \lambda & -\gamma & \frac{\delta}{-1+b} \\ -\lambda & b\gamma & -\delta - \frac{\delta}{-1+b} \end{pmatrix} \quad \det = -\gamma \delta \lambda$$

$$a_1 = 1 + \frac{b\delta}{-1+b}, \quad a_2 = \frac{\delta\lambda}{1-b}, \quad a_3 = \delta\lambda$$

$$\text{Hurw}(b) \text{ when } K=\gamma=1 \text{ is } H_2 = -\frac{b\delta(-1+b+\delta)\lambda}{(-1+b)^2}$$

$$H_2(b_0) \text{ when } K=\gamma=1 \text{ is } -((1+\beta)(\beta+\delta)\lambda)$$

$$\text{Denominator of } H_2 \text{ is } (-1+b)^2$$

$$\phi(b) = -b^2 - b(-1+\delta)$$

$$\text{Coefficients of numerator } \phi(b) \text{ are } \{B_0, B_1, B_2\} = \{0, 1-\delta, -1\}$$

$$\text{Out}[*] = \{b \rightarrow 1-\delta\}$$