

On a three-dimensional tumor-virus compartmental model, and two four-dimensional oncolytic Virotherapy models

This Mathematica Notebook is a supplementary material to the paper which has the same title as this document. It contains some of the calculations and illustrations appearing in the paper.

1) Section 2 (in paper): Deterministic model with Logistic growth [Tian2011]

0) Definition of the model [Tian11]:

```
In[ ]:=
SetDirectory[NotebookDirectory[]];
AppendTo[$Path,Directory];
Clear["Global`*"];
(*Some aliases*)
Format[μv]:=Subscript[μ,v];Format[μy]:=Subscript[μ,y];
parT={β>0,λ>0,δ>0,b>1};
cparT={ μv→0,μy→0,γ→1, K→1};
cnTian={ μv→0, μy→0,K→1,γ→1,λ→0.36,β→0.11,δ→0.44}(*Numerical values of Tian*);

(***** Four dim Deterministic epidemic model with Logistic growth *****)
x1=λ x(1-(x+y)/K)- β x v ;
y1=β x v -μy y z- γ y;
v1=-β x v - μv v z+ b γ y - δ v;

dyn={x1,y1,v1}/.μy→0/.μv→0(*Tian case with K>0*);

Print["
      x'
      (y')=",dyn//FullSimplify//MatrixForm,
      v'

", and the reparametrized dynamics [Tian 2011] are
      x'
      (y')=",
      v'

dyn/.cparT//FullSimplify//MatrixForm]
```

$$\begin{pmatrix} x' \\ y' \\ v' \end{pmatrix} = \begin{pmatrix} -v x \beta + x \left(1 - \frac{x+y}{K}\right) \lambda \\ v x \beta - y \gamma \\ b y \gamma - v (x \beta + \delta) \end{pmatrix}$$

$$, \text{ and the reparametrized dynamics [Tian 2011] are } \begin{pmatrix} x' \\ y' \\ v' \end{pmatrix} = \begin{pmatrix} -x (v \beta + (-1 + x + y) \lambda) \\ -y + v x \beta \\ b y - v (x \beta + \delta) \end{pmatrix}$$

Fixed points and analysis of the Stability via Routh Hurwitz:

In[]:=

```

cfp=Solve[Thread[dyn=={0,0,0}],{x,y,v}]/FullSimplify;
fp={x,y,v}/.cfp;
Print[Length[fp]," fixed points, the third is E*="]
fp[[3]]//FullSimplify
R0=b β K/(β K+δ);bcrit=1+δ/(β K);(*Reduce[Join[{R0>1},pars],δ]*)
Print["R0=", R0, " , b0=", bcrit]
(*cR0={K→Rz(β K+δ)/(b β)};
fp[[3]]/.cR0//FullSimplify*)
(*"Jacobian is"*)
Jac=Grad[dyn,{x,y,v}]/FullSimplify;
J0=Jac/.cfp[[1]];J0//MatrixForm;
Print["J(E_K) is"]
J1=Jac/.cfp[[2]];J1//MatrixForm
Eigenvalues[J1]
Print["J(E_*) is"]
Jst=Jac/.cfp[[3]]//FullSimplify;Jst//MatrixForm
Jstcr=Jst/.b→bcrit//FullSimplify;
Print["J(E_*)/.b→b0 is",Jstcr//MatrixForm," eigvals are ",Eigenvalues[Jstcr]]

(*Routh Hurwitz conditions for the stability of E_**)
pc=Collect[Det[ψ IdentityMatrix[3]-Jst],ψ];
coT=CoefficientList[pc,ψ]//FullSimplify;
Print["a1=",a1=Apart[coT[[3]]]//.cparT, " , a2=",a2=coT[[2]]//.cparT, " , a3=",a3=coT[[1]]//.cparT]
H2=a1*a2-a3;
Print["H(b) when K=γ=1 is", H2//.cparT//FullSimplify]
Print["H2(b0) when K=γ=1 is",H2/.b→bcrit//.cparT//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H2]]/.K→1//FullSimplify]
Together[H2//FullSimplify];
φb=Collect[Numerator[Together[H2]]/(δ λ),b]/.K→1//FullSimplify;
Print["φ(b)=", Collect[φb,b]]
Print["Coefficients of φ(b)are:",cofi=CoefficientList[φb,b]//FullSimplify]
(*φb/.b→1//FullSimplify*)
Print["value of φ(b) at crit b is "]
φb/.b→bcrit/.K→1//FullSimplify

```

3 fixed points, the third is E*=

$$Out[] = \left\{ \frac{\delta}{(-1+b)\beta}, \frac{((-1+b)K\beta - \delta)\delta\lambda}{(-1+b)\beta((-1+b)K\beta\gamma + \delta\lambda)}, \frac{\gamma((-1+b)K\beta - \delta)\lambda}{\beta((-1+b)K\beta\gamma + \delta\lambda)} \right\}$$

$$R0 = \frac{bK\beta}{K\beta + \delta}, \quad b0 = 1 + \frac{\delta}{K\beta}$$

J(E_K) is

Out[]//MatrixForm=

$$\begin{pmatrix} -\lambda & -\lambda & -K\beta \\ 0 & -\gamma & K\beta \\ 0 & b\gamma & -K\beta - \delta \end{pmatrix}$$

$$Out[] = \left\{ \frac{1}{2} \left(-K\beta - \gamma - \delta - \sqrt{(K\beta + \gamma + \delta)^2 - 4(K\beta\gamma - bK\beta\gamma + \gamma\delta)} \right), \right. \\ \left. \frac{1}{2} \left(-K\beta - \gamma - \delta + \sqrt{(K\beta + \gamma + \delta)^2 - 4(K\beta\gamma - bK\beta\gamma + \gamma\delta)} \right), -\lambda \right\}$$

J(E_*) is

Out[]//MatrixForm=

$$\begin{pmatrix} \frac{\delta \lambda}{K \beta - b K \beta} & \frac{\delta \lambda}{K \beta - b K \beta} & -\frac{\delta}{-1+b} \\ \frac{\gamma ((-1+b) K \beta - \delta) \lambda}{(-1+b) K \beta \gamma + \delta \lambda} & -\gamma & \frac{\delta}{-1+b} \\ \frac{\gamma (K (\beta - b \beta) + \delta) \lambda}{(-1+b) K \beta \gamma + \delta \lambda} & b \gamma & \frac{b \delta}{1-b} \end{pmatrix}$$

$$J(E^*) /. b \rightarrow b0 \text{ is } \begin{pmatrix} -\lambda & -\lambda & -K \beta \\ 0 & -\gamma & K \beta \\ 0 & \gamma + \frac{\gamma \delta}{K \beta} & -K \beta - \delta \end{pmatrix} \text{ eigvals are } \{0, -K \beta - \gamma - \delta, -\lambda\}$$

$$a_1 = \frac{-1+b+b \delta}{-1+b} + \frac{\delta \lambda}{(-1+b) \beta}, \quad a_2 =$$

$$\frac{\delta \lambda ((-1+b) \beta (-1+b+\beta-b \beta+\delta+b \delta) + ((-1+b)^2 \beta + b \delta^2) \lambda)}{(-1+b)^2 \beta ((-1+b) \beta + \delta \lambda)}, \quad a_3 = \delta \left(1 + \frac{\delta}{\beta - b \beta} \right) \lambda$$

$$H(b) \text{ when } K=\gamma=1 \text{ is } \delta \lambda \left(-1 + \frac{\delta}{(-1+b) \beta} - \frac{(\beta (-1+b+b \delta) + \delta \lambda) ((-1+b)^2 \beta^2 - b \delta^2 \lambda - (-1+b) \beta (-1+\delta-\lambda+b (1+\delta+\lambda)))}{(-1+b)^3 \beta^2 ((-1+b) \beta + \delta \lambda)} \right)$$

$$H2(b0) \text{ when } K=\gamma=1 \text{ is } (1+\beta+\delta) \lambda (1+\beta+\delta+\lambda)$$

$$\text{Denominator of } H2 \text{ is } (-1+b)^3 \beta^2 ((-1+b) \beta + \delta \lambda)$$

$$\begin{aligned} \phi(b) = & -\beta^2 - b^4 \beta^3 - \beta^2 \lambda + \beta \delta \lambda + \beta \delta \lambda^2 + \\ & b (\beta^2 + \beta^3 - \beta^3 \delta - \beta^2 (-3+\delta) \delta + \beta^2 \lambda - 2 \beta \delta \lambda - 3 \beta \delta^2 \lambda - 2 \beta \delta \lambda^2 + \delta^3 \lambda^2 + 2 \beta^2 (1+\lambda)) + \\ & b^3 (3 \beta^3 - \beta^3 \delta + \beta^2 (1+\delta (3+\delta) + \lambda)) + \\ & b^2 (-3 \beta^3 + 2 \beta^3 \delta + \beta^2 (-3+\delta) \delta - 2 \beta^2 (1+\lambda) - \beta^2 (1+\delta (3+\delta) + \lambda) + \beta \delta \lambda (1+\delta (3+\delta) + \lambda)) \end{aligned}$$

Coefficients of $\phi(b)$ are:

$$\left\{ \beta (1+\lambda) (-\beta + \delta \lambda), -\beta^3 (-1+\delta) + \delta^3 \lambda^2 - \beta \delta \lambda (2+3 \delta + 2 \lambda) + \beta^2 (3+3 \delta - \delta^2 + 3 \lambda), \right. \\ \left. \beta (\beta^2 (-3+2 \delta) - 3 \beta (1+2 \delta + \lambda) + \delta \lambda (1+\delta (3+\delta) + \lambda)), \beta^2 (1-\beta (-3+\delta) + \delta (3+\delta) + \lambda), -\beta^3 \right\}$$

value of $\phi(b)$ at crit b is

$$\text{Out[]}= \frac{\delta^3 (1+\beta+\delta) \times (1+\lambda) \times (1+\beta+\delta+\lambda)}{\beta}$$

Computations of the Jacobians and Eigenvalues using EcoEvo package:

In[]:=

```

<<EcoEvo`
(*EcoEvoDocs;*)
(*****Analysis of the Model, K=γ=1****)
dynKeq1=dyn/.cpaT;
ClearParameters;
UnsetModel;
SetModel[{Pop[x]→{Equation→dynKeq1[[1]],Color→Red},Pop[y]→
{Equation→(dynKeq1[[2]]),Color→Green},
Pop[v]→{Equation→(dynKeq1[[3]]),Color→Purple},
Parameters→(cp=parT)}]

fpT=SolveEcoEq[]//FullSimplify;

J0T=EcoJacobian[fpT[[1]]]//FullSimplify;
J1T=EcoJacobian[fpT[[2]]]//FullSimplify;
Jst=EcoJacobian[fpT[[3]]]//FullSimplify;
Print["Jac (E0)=",J0T//MatrixForm]
Print["Jac (E1)=",J1T//MatrixForm]
Print["Jac (E*)=",Jst//MatrixForm]
Print["Eigenvalues of E1 are:",eiT=EcoEigenvalues[fpT[[2]]]//FullSimplify]

Print["b0=bs1=",bs1=Apart[Last[Last[Reduce[Join[{eiT[[2]]>0},parT],b]]]]]

```

Out[]:= EcoEvo Package Version 1.6.4 (November 5, 2021)
 Christopher A. Klausmeier <christopher.klausmeier@gmail.com>

$$\text{Jac}(E_0) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -1 & 0 \\ 0 & b & -\delta \end{pmatrix}$$

$$\text{Jac}(E_1) = \begin{pmatrix} -\lambda & -\lambda & -\beta \\ 0 & -1 & \beta \\ 0 & b & -\beta - \delta \end{pmatrix}$$

$$\text{Jac}(E_*) = \begin{pmatrix} \frac{\delta \lambda}{\beta - b \beta} & \frac{\delta \lambda}{\beta - b \beta} & -\frac{\delta}{-1 + b} \\ \frac{((-1 + b) \beta - \delta) \lambda}{(-1 + b) \beta + \delta \lambda} & -1 & \frac{\delta}{-1 + b} \\ \frac{(\beta - b \beta + \delta) \lambda}{(-1 + b) \beta + \delta \lambda} & b & \frac{b \delta}{1 - b} \end{pmatrix}$$

Eigenvalues of E₁ are:

$$\left\{ \frac{1}{2} \times \left(-1 - \beta - \delta - \sqrt{(1 + \beta + \delta)^2 - 4(\beta - b \beta + \delta)} \right), \frac{1}{2} \times \left(-1 - \beta - \delta + \sqrt{(1 + \beta + \delta)^2 - 4(\beta - b \beta + \delta)} \right), -\lambda \right\}$$

$$b_0 = b_{s1} = 1 + \frac{\delta}{\beta}$$

1) Numerical simulations:

Bifurcation Diagram:

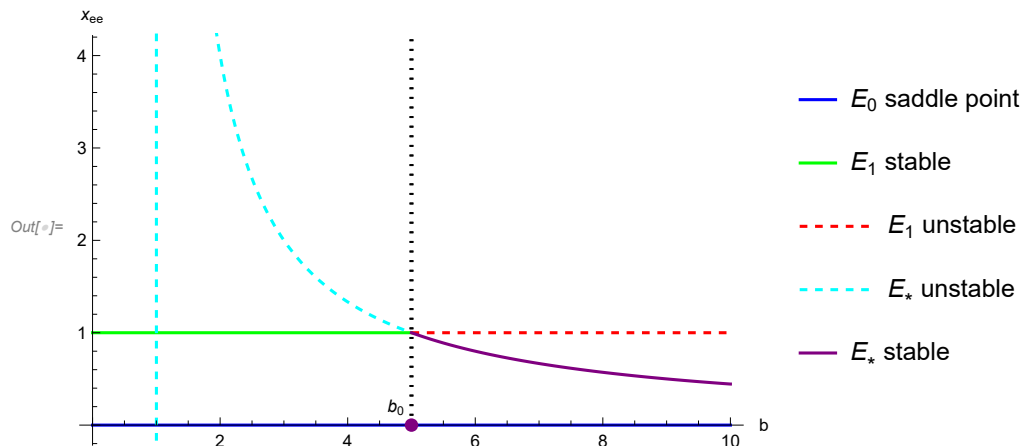
In[]:=

```

ClearParameters;
λ=0.36;β=0.11;δ=0.44;bL=10;
Print["b0=",bs1/N]
fpT/N;
linb0=Line[{{bs1,0},{bs1,10}}];
lib0=Graphics[{Thick,Black,Dotted,linb0}];
px0=Plot[0,{b,0,bL},PlotStyle→{Blue},PlotRange→All,PlotLegends→{"E0 saddle point"}];
px1a=Plot[1,{b,0,bs1},PlotStyle→{Green},PlotRange→All,PlotLegends→{"E1 stable"}];
px1b=Plot[1,{b,bs1,bL},PlotStyle→{Red,Dashed},PlotRange→All,
PlotLegends→{"E1 unstable"}];
pxe1=Plot[{x/.fpT[[3]},{b,0,bs1},PlotStyle→{Cyan,Dashed},PlotRange→All,
PlotLegends→{"E* unstable"}];
pxe2=Plot[{x/.fpT[[3]},{b,bs1,bL},PlotStyle→{Purple},PlotRange→All,
PlotLegends→{"E* stable"}];
bif11T=Show[{px0,px1a,px1b,pxe1,pxe2,lib0},Epilog→{Text["b0",Offset[{-8,10},{bs1,0}],
{PointSize[Large],Style[Point[{bs1,0}],Purple]}},
PlotRange→{{0,10},{0,4}},AxesLabel→{"b","xee"}]
Export["bif11T.pdf",bif11T]

```

b0=5.



Out[]:= bif11T.pdf

Periodic x,y values when b=28:

In[]:=

```

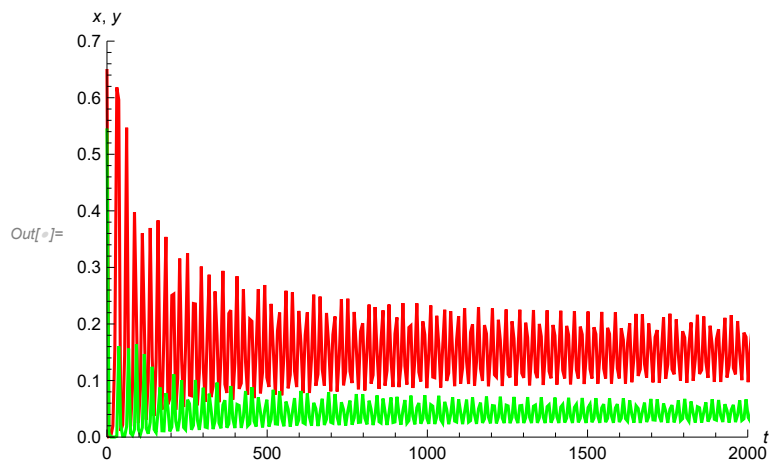
ClearParameters;
λ=0.36;β=0.11;δ=0.44;b=28;
in={x→0.5,y→0.5,v→1.5};
fpT//N
EcoEigenvalues[fpT[[3]]] (*Eigenvalues corresponding to E* **)
solE3=EcoSim[RuleListAdd[fpT[[3]],in],20000];

Fig5T=PlotDynamics[{solE3[[1]],solE3[[2]]},PlotRange→{{0,2000},{0,0.7}}]
Export["Fig5T.pdf",Fig5T]

```

Out[]:= $\{\{x \rightarrow 0., y \rightarrow 0., v \rightarrow 0.\}, \{x \rightarrow 1., y \rightarrow 0., v \rightarrow 0.\},$
 $\{x \rightarrow 0.148148, y \rightarrow 0.0431317, v \rightarrow 2.64672\}\}$

Out[]:= $\{-1.51022, 0.000296187 + 0.298909 i, 0.000296187 - 0.298909 i\}$



Out[]:= Fig5T.pdf

Numerical illustrations (Parametric plot) when $b=28$:

```

In[ ]:= ClearParameters;
λ=0.36;β=0.11;δ=0.44;b=28;K=1;γ=1;
Print["E*",fp[[3]]/N]
x1=λ x[t] (1-(x[t]+y[t]))- β x[t]×v[t] ;
y1=β x[t]×v[t] - y[t];
v1=-β x[t]×v[t] + b y[t] - δ v[t];

ode={x'[t]==x1,y'[t]==y1,v'[t]==v1,x[0]==0.5,y[0]==0.5,v[0]==1.5};
sol=NDSolve[ode,{x,y,v},{t,0,400}];

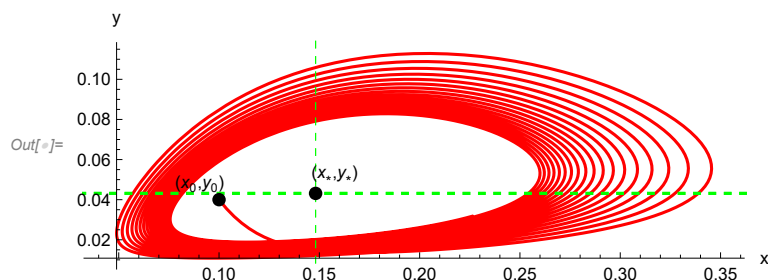
x0=0.5; y0=0.5;v0=1.5;
ppb28=ParametricPlot[{ x[t],(y[t]) }/.sol,{t,0,400}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py=Plot[(y/.y→fp[[3,2]]),{t,0,400},PlotStyle→{Dashed,Green}];
pb28=Show[{ppb28,py, Graphics[{Green,Dashed,
Line[{ {x/.x→fp[[3,1]],0},{x/.x→fp[[3,1]],1}]}]}],
Epilog→{{Text["(x*,y*)",Offset[{10,10},{x/.x→fp[[3,1]],(y/.y→fp[[3,2]])}],
{PointSize[Large],Style[Point[{ {x/.x→fp[[3,1]],(y/.y→fp[[3,2]])}],Black}],
{PointSize[Large],Point[{x0,y0}],Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}}];
(*****b=28; different initial values *****)
ClearParameters;
λ=0.36;β=0.11;δ=0.44;b=28;K=1;γ=1;

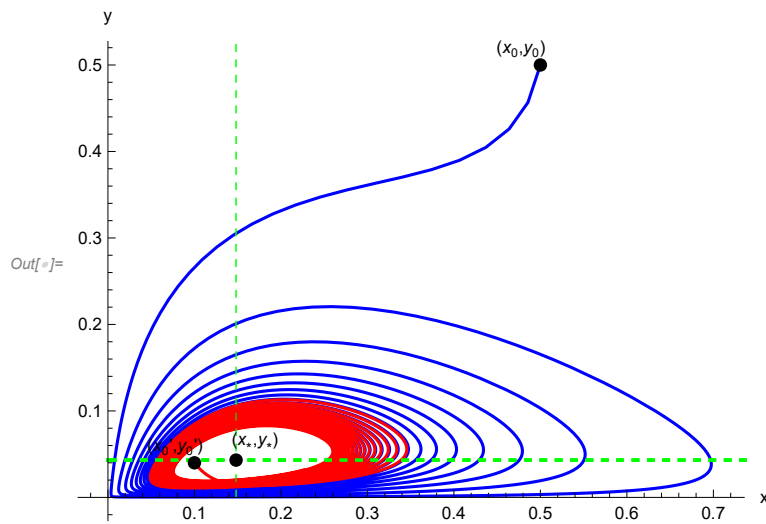
ode={x'[t]==x1,y'[t]==y1,v'[t]==v1,x[0]==0.1,y[0]==0.04,v[0]==0.01};
sol=NDSolve[ode,{x,y,v},{t,0,400}];

(*****Parametric plot conditions***)
x0=0.1; y0=0.04;v0=0.01;
ppb28n=ParametricPlot[{ x[t],(y[t]) }/.sol,{t,0,400}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Red}];
pyn=Plot[(y/.y→fp[[3,2]]),{t,0,200},PlotStyle→{Dashed,Green}];
pb28n=Show[{ppb28n,pyn, Graphics[{Green,Dashed,
Line[{ {x/.x→fp[[3,1]],0},{x/.x→fp[[3,1]],1}]}]}],
Epilog→{{Text["(x*,y*)",Offset[{10,10},{x/.x→fp[[3,1]],(y/.y→fp[[3,2]])}],
{PointSize[Large],Style[Point[{ {x/.x→fp[[3,1]],(y/.y→fp[[3,2]])}],Black}],
{PointSize[Large],Point[{x0,y0}],Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}}];
cy11=Show[{pb28,pb28n},Epilog→{Text["(x*,y*)",
Offset[{10,10},{x/.x→fp[[3,1]],(y/.y→fp[[3,2]])}],
{PointSize[Large],Style[Point[{ {x/.x→fp[[3,1]],(y/.y→fp[[3,2]])}],Black}],
{PointSize[Large],Point[{0.1,0.04}],Text["(x0',y0')",Offset[{-10,8},{0.1,0.04}]]},
{PointSize[Large],Point[{0.5,0.5}],Text["(x0,y0)",Offset[{-10,8},{0.5,0.5}]]}}];
Export["cy11.pdf",cy11]

```

E*{0.148148, 0.0431317, 2.64672}





Out[*]= cy11.pdf

2) Sections 3 and 4(in paper): Four-dim viro-therapy model with Logistic growth (when $\epsilon=0$)

```
In[1]:= SetDirectory[NotebookDirectory[]];
AppendTo[$Path, Directory];
Clear["Global`*"];
(*Some aliases*)
Format[μv] := Subscript[μ, v]; Format[μy] := Subscript[μ, y];
parE = {β > 0, λ > 0, γ > 0, δ > 0, μ > 0, μv > 0, b > 1, K > 0, s > 0, c > 0};
cKga1 = {ε → 0, K → 1, γ → 1};
cE = {ε → 1};
R0 = b β K / (β K + δ) (* Reproduction number*);
cnT17 = {μv → 0.16, μy → 0.48, K → 1, γ → 1, b → 9, β → 0.11,
  λ → 0.36, δ → 0.2, s → 0.6, c → 0.036} (*Numerical values of Tian17*);
```

2-1) Description of the model when $\epsilon=0$ and analysis of the stability of the fixed point when $z \rightarrow 0$

```
In[59]:= (***** Four dim Deterministic epidemic model with Logistic growth *****)
x1= $\lambda$  x(1-(x+y)/K)- $\beta$  x v ;
y1= $\beta$  x v - $\mu$  y y z- $\gamma$  y;
v1=- $\beta$  x v - $\mu$  v v z+ b  $\gamma$  y - $\delta$  v;
z1=z(s y - c );
ye=c/s; vM= $\lambda$ (1-ye)/ $\beta$ ; vMN=vM/.cnT17;
dyn={x1,y1,v1,z1};
dyn3={x1,y1,v1}/.z->0; (*3dim case*)

Print[" $\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \text{dyn} // \text{FullSimplify} // \text{MatrixForm}$ , " the reparametrized model is " $\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} =$ ",
dyn/.cKga1//FullSimplify//MatrixForm]
Print["b0=",b0=b/.Apart[Solve[R0==1,b][[1]]//FullSimplify]]
(*****Fixed points of Tian17 using the elimination when K=1,  $\gamma=1$ ****)

fv=(ye (b-1)-v  $\delta$ ); gv=(ye  $\mu$  y+v  $\mu$  v); hv=(1-ye-v  $\beta/\lambda$ );

Print["xe= ",xe=hv," ze =", ze=fv/gv]

Pv=Numerator[Together[v  $\beta$  xe-ye (1+ $\mu$  y fv/gv)]]/(-s2  $\beta^2$   $\mu$  v);
Print["P(v)=",pc=Collect[Together[Pv],v]," coefs are"]
coP=CoefficientList[pc,v]//Simplify

(***Fixed point when z->0***)
eq=Thread[dyn3=={0,0,0}];
sol=Solve[eq,{x,y,v}]/FullSimplify;
Es={x,y,v}/.sol[[3]]; (*Endemic point with z=0*);
Print[" E*=",Es//FullSimplify]
Est=Es/.cKga1//FullSimplify(* E* when K= $\gamma=1$ ****);
Print["Check: when K= $\gamma=1$ , E*=",Est]

bn=b/.Solve[Est[[2]]==ye,b]; bnn=bn/.cnT17;
bcn=b/.Solve[Est[[2]][[1]]==ye,b];
bcnn=bn/.cnT17;

Dis=Chop[Collect[Discriminant[Numerator[Pv],v],b]];
solb=Solve[Dis==0,b];
Jac=Grad[dyn/.cKga1,{x,y,v,z}]/FullSimplify;
```

$$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -v x \beta + x \left(1 - \frac{x+y}{K}\right) \lambda \\ v x \beta - y (\gamma + z \mu_y) \\ b y \gamma - v (x \beta + \delta + z \mu_v) \\ (-c + s y) z \end{pmatrix} \quad \text{the reparametrized model is} \quad \begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -x (v \beta + (-1 + x + y) \lambda) \\ v x \beta - y (1 + z \mu_y) \\ b y - v (x \beta + \delta + z \mu_v) \\ (-c + s y) z \end{pmatrix}$$

$$b\theta = 1 + \frac{\delta}{K \beta}$$

$$x_e = 1 - \frac{c}{s} - \frac{v \beta}{\lambda} \quad z_e = \frac{\frac{(-1+b)c}{s} - v \delta}{v \mu_v + \frac{c \mu_y}{s}}$$

$$P(v) = v^3 + \frac{b c^2 \lambda \mu_y}{s^2 \beta^2 \mu_v} + \frac{v^2 (c s \beta \lambda \mu_v - s^2 \beta \lambda \mu_v + c s \beta^2 \mu_y)}{s^2 \beta^2 \mu_v} + \frac{v (c s \lambda \mu_v + c^2 \beta \lambda \mu_y - c s \beta \lambda \mu_y - c s \delta \lambda \mu_y)}{s^2 \beta^2 \mu_v} \quad \text{coefs are}$$

$$\text{Out[72]=} \left\{ \frac{b c^2 \lambda \mu_y}{s^2 \beta^2 \mu_v}, \frac{c \lambda (c \beta \mu_y + s (\mu_v - (\beta + \delta) \mu_y))}{s^2 \beta^2 \mu_v}, \frac{c \lambda \mu_v - s \lambda \mu_v + c \beta \mu_y}{s \beta \mu_v}, 1 \right\}$$

$$E_* = \left\{ \frac{\delta}{(-1+b) \beta}, \frac{((-1+b) K \beta - \delta) \delta \lambda}{(-1+b) \beta ((-1+b) K \beta \gamma + \delta \lambda)}, \frac{\gamma ((-1+b) K \beta - \delta) \lambda}{\beta ((-1+b) K \beta \gamma + \delta \lambda)} \right\}$$

$$\text{Check: when } K=\gamma=1, E_* = \left\{ \frac{\delta}{(-1+b) \beta}, \frac{((-1+b) \beta - \delta) \delta \lambda}{(-1+b) \beta ((-1+b) \beta + \delta \lambda)}, \frac{((-1+b) \beta - \delta) \lambda}{\beta ((-1+b) \beta + \delta \lambda)} \right\}$$

2-2) Interior equilibrium

Analysis of the stability of the interior point E_* :

In[85]:=

```
Jac3=Grad[dyn3/.cKga1,{x,y,v}]/FullSimplify;
bcrit=1+δ/(β);(*Reduce[Join[{R0>1},pars],δ]*)
Print["J(E_*) is"]
Jst=(Jac3/.x→Est[[1]]/.y→Est[[2]]/.v→Est[[3]])/FullSimplify;Jst//MatrixForm
Trs=Tr[Jst];
pc=Collect[Det[ψ IdentityMatrix[3]-Jst],ψ];
coT=CoefficientList[pc,ψ]/FullSimplify;
Print["a1=",a1=coT[[3]],",",a2=",a2=coT[[2]],",",a3=",a3=coT[[1]]]
H2=a1*a2-a3;
Print["H2(bθ)=",H2/.b→bcrit/FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H2]]/.cKga1/FullSimplify]
φb=Collect[Numerator[Together[H2]]/(δ λ),b]/.cKga1;
cofi=CoefficientList[φb,b](*Coefficients of φ(b)*);
Print["value of φ(b) at crit b is "]
φb/.b→bcrit/.cKga1/FullSimplify
```

J(E_*) is

Out[88]//MatrixForm=

$$\begin{pmatrix} \frac{\delta \lambda}{\beta - b \beta} & \frac{\delta \lambda}{\beta - b \beta} & -\frac{\delta}{-1+b} \\ \frac{((-1+b) \beta - \delta) \lambda}{(-1+b) \beta + \delta \lambda} & -1 & \frac{\delta}{-1+b} \\ \frac{(\beta - b \beta + \delta) \lambda}{(-1+b) \beta + \delta \lambda} & b & \frac{b \delta}{1-b} \end{pmatrix}$$

$$a_1 = \frac{\beta(-1+b+b\delta) + \delta\lambda}{(-1+b)\beta}, \quad a_2 = \frac{\delta\lambda((-1+b)\beta(-1+\beta+\delta+b(1-\beta+\delta)) + ((-1+b)^2\beta + b\delta^2)\lambda)}{(-1+b)^2\beta((-1+b)\beta + \delta\lambda)}, \quad a_3 = \delta \left(1 + \frac{\delta}{\beta - b\beta}\right)\lambda$$

$$H_2(b_0) = (1 + \beta + \delta)\lambda(1 + \beta + \delta + \lambda)$$

$$\text{Denominator of } H_2 \text{ is } (-1+b)^3\beta^2((-1+b)\beta + \delta\lambda)$$

value of $\phi(b)$ at crit b is

$$\text{Out[99]} = \frac{\delta^3(1 + \beta + \delta) \times (1 + \lambda) \times (1 + \beta + \delta + \lambda)}{\beta}$$

Numerical values:

```
In[ ]:=
cn=cnT17;
cb=NSolve[(ϕb/.Drop[cnT17,{5}])&==0,b,WorkingPrecision->10]
bM=Max[Table[Re[b/.cb[[i]]],{i,Length[cb]}]];
Print["bH=",bH=N[bM,30]]
PR=Solve[Pv==0,v];
vn=v/.PR[[2]];vi=v/.PR[[3]];vp=v/.PR[[1]];
Chop[{vn,vi,vp}/.cnT17];
PR=Solve[Pv==0,v];
vn=v/.PR[[2]];vp=v/.PR[[1]];vi=v/.PR[[3]];
Print["b0=",b0=.cn//N," ",b1=", bnn[[1]], ", b2=", bnn[[2]], ", bH=",bH]
PRN=Chop[PR/.cn//N](*values of the roots v*);
Print["E*",Es=Es/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v->vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v->vi/.cn//N]]

Jiv=Jac/{x->xe,y->ye,z->ze};
JEi=Jiv/.v->vi/.cn//N//FullSimplify;
JEp=Jiv/.v->vp/.cn//N//FullSimplify;
JEif=Jac/.x->1/.y->0/.z->0/.v->0/.cn//N//FullSimplify;
Print["Eigv of E*:",Append[Eigenvalues[Jst]/.cn//N,Es[[2]]-ye/.cnT17]," Eigv of E+:",
Re[Eigenvalues[JEp]/N], " Eigv of Ei:",Re[Eigenvalues[JEi]/N],
" , Eigv of Eif:",Eigenvalues[JEif]/N]]

Out[ ]:= {{b -> 0.299052}, {b -> 0.835329 - 0.231156 i}, {b -> 0.835329 + 0.231156 i}, {b -> 19.0121}}

bH=19.0121

b0=2.81818 , b1=3.58676, b2=8.66779 ,bH=19.0121

E*={0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}

Eigv of E*:{-1.25056, -0.0281268 - 0.20904 i, -0.0281268 + 0.20904 i, -0.00155844}
Eigv of E+:{-1.29833, -0.0332218, -0.0332218, 0.000834552}
Eigv of Ei:{-1.69849, -0.076539, -0.076539, -0.00803072}
, Eigv of Eif:{-1.7081, 0.398103, -0.36, -0.036}
```

3)Sections 3.1---3.3(in paper):Figures used in the manuscript (*Run the previous cell*)

Numerical illustrations when $\epsilon=0$ (Bifurcations diagrams, parametric plots, and 3D plot)

Bifurcation diagram when b varies:

In[]:=

```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=9;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ε=0;
Print["{b2*,b1*}=",{b/.solb[[1]],b/.solb[[2]]} ," ,", and b0=",b0, " , bH=",bH,
" ,b1=", bnn[[1]], " , b2=", bnn[[2]]]
Clear["b"];
vs=
$$\frac{((-1+b) \beta - \delta) \lambda}{\beta ((-1+b) \beta + \delta \lambda)}$$
 (*v of E* * ****);
bL=28; max=3.5;
bc2=b/.solb[[1]];
bc1=b/.solb[[2]];
lin1=Line[{{ bc1,0},{ bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{ bc2,0},{ bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{ b0,0},{ b0,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
linH=Line[{{ bH,0},{ bH,max}}];
liH=Graphics[{Thick,Black,Dashed,linH}];
linb1=Line[{{ bnn[[1]],0},{ bnn[[1]],max}}];
lib1=Graphics[{Thick,Black,Dashed,linb1}];
linb2=Line[{{ bnn[[2]],0},{ bnn[[2]],max}}];
lib2=Graphics[{Thick,Black,Dashed,linb2}];
linb9=Line[{{ 9,0},{ 9,max}}];
lib9=Graphics[{Thick,Black,Dashed,linb2}];
pn=Plot[{vn},{b,0,bL},PlotStyle→{Green,Dotted},PlotRange→All,PlotLegends→{"E- negative"}];
p0=Plot[0,{b,0,bL},PlotStyle→{Brown,Thick},PlotRange→All,PlotLegends→{"E0 saddle point"}];
ppa=Plot[{vp},{b,0, bnn[[2]]},PlotStyle→{Purple,Dotted},PlotRange→All,
PlotLegends→{"E+ outside the domain"}];
ppb=Plot[{vp},{b,bnn[[2]], bL},PlotStyle→{Purple,Thick,Dashed},
PlotRange→All,PlotLegends→{"E+ unstable"}];
pi1=Plot[{vi},{b,0,bnn[[1]]},PlotStyle→{Blue,Dotted},
PlotRange→All,PlotLegends→{"Ei outside the domain"}];
pi2=Plot[{vi},{b,bnn[[1]],bL},PlotStyle→{Red,Thick},
PlotRange→All,PlotLegends→{"Ei stable"}];
pi3=Plot[{vi},{b,bnn[[2]],bL},PlotStyle→{Blue,Thick,Dashed},
PlotRange→All(*,PlotLegends→{"Ei unstable"}*)];
ps1=Plot[{vs},{b,b0,bnn[[1]]},PlotStyle→{Brown,Thick},
PlotRange→{{0,bL},{0,max}},PlotLegends→{"E+ stable"}];
ps2=Plot[{vs},{b,bnn[[1]],bnn[[2]]},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{0,bL},{0,max}},PlotLegends→{"E+ unstable"}];
ps3=Plot[{vs},{b,bnn[[2]],bH},PlotStyle→{Brown,Thick},
PlotRange→{{0,bL},{0,max}}(*,PlotLegends→{"E+ stable"}*)];
ps4=Plot[{vs},{b,bH,bL},PlotStyle→{Cyan,Thick,Dashed},
PlotRange→{{0,bL},{0,max}}(*,PlotLegends→{"E+ unstable"}*)];
pdf1=Plot[{0},{b,0,b0},PlotStyle→{Magenta, Thick},
PlotRange→{{0,200},{0,max}},PlotLegends→{"E1 stable"}];
pdf2=Plot[{0},{b,b0,bL},PlotStyle→{Orange, Thick,Dashed},
PlotRange→{{0,200},{0,max}},PlotLegends→{"E1 unstable"}];
pvmax=Plot[{c (b-1)/(s δ),(λ(1-c/s))/β},{b,0,bL},PlotStyle→{Yellow, Pink},
```

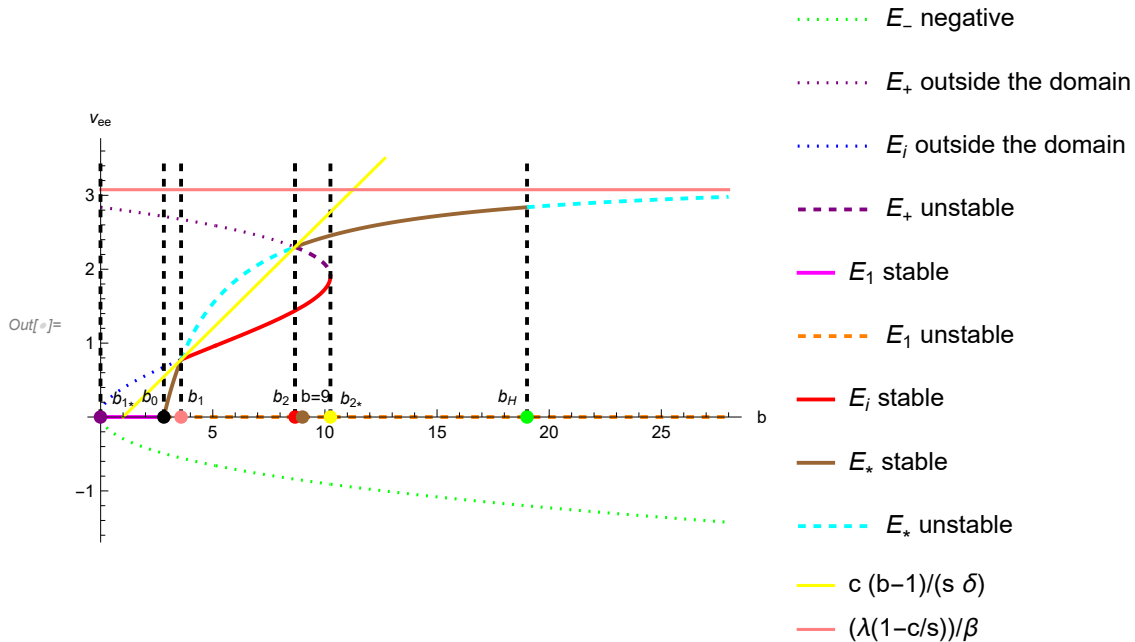
```

PlotRange→{{0,200},{0,max}},
PlotLegends→{"c (b-1)/(s δ)", "(λ(1-c/s))/β"};

bifT=Show[{pn,ppa,pi1,ppb,pdf1,pdf2,pi2,ps1,ps2,ps3,ps4,li1,li2,li3,
lib2,lib1,lib9,pvmax,liH},
Epilog→{Text["b1*",Offset[{12,10},{bc1,0}]],{PointSize[Large],
Style[Point[{bc1,0}],Purple]},
Text["b2*",Offset[{11,10},{bc2,0}]],{PointSize[Large],Style[Point[{bc2,0}],Yellow]},
Text["b0",Offset[{-7,11},{b0,0}]],{PointSize[Large],Style[Point[{b0,0}],Black]},
Text["bH",Offset[{-10,11},{bH,0}]],{PointSize[Large],Style[Point[{bH,0}],Green]},
Text["b1",Offset[{8,11},{bnn[1],0}]],{PointSize[Large],Style[Point[{bnn[1],0}],Pink]},
Text["b2",Offset[{-7,11},{bnn[2],0}]],{PointSize[Large],Style[Point[{bnn[2],0}],Red]},
Text["b=9",Offset[{7,11},{9,0}]],{PointSize[Large],Style[Point[{9,0}],Brown]}},
PlotRange→All,AxesLabel→{"b","vee"}]
Export["BiifT17.pdf",bifT]

```

$\{b_{2*}, b_{1*}\} = \{10.2462, -0.00697038\}$, and $b_0 =$
 2.81818 , $b_H = 19.012107471368075382$, $b_1 = 3.58676$, $b_2 = 8.66779$



Out[]= BiifT17.pdf

Parametric plots at the intervals of stability:

When $b_0 < b < b_1$:

In[]:=

```

ClearParameters;
 $\mu v = 0.16$ ;  $\mu y = 0.48$ ;  $K = 1$ ;  $b = 3$ ;  $\gamma = 1$ ;  $\lambda = 0.36$ ;  $\beta = 0.11$ ;  $\delta = 0.2$ ;  $s = 0.6$ ;  $c = 0.036$ ;  $\epsilon = 0$ ;

Print["E*", Es = Est /. cn /. N]
Print["E+=", Ep = Chop[{xe, ye, v, ze} /. v -> vp /. cn /. N]
Print["Ei=", Ei = Chop[{xe, ye, v, ze} /. v -> vi /. cn /. N]
 $x_1 = \lambda x[t] (1 - (x[t] + y[t]) / K) - \beta x[t] \times v[t]$ ;
 $y_1 = \beta x[t] \times v[t] - \mu y[t] \times z[t] - \gamma y[t]$ ;
 $v_1 = -\beta x[t] \times v[t] - \mu v[t] \times z[t] + b \gamma y[t] - \delta v[t]$ ;
 $z_1 = z[t] (s y[t] - c)$ ;
ode1 = {x'[t] == x1, y'[t] == y1, v'[t] == v1, z'[t] == z1, x[0] == 0.9, y[0] == 0.01, v[0] == 0.01,
z[0] == 0.01};
sol01 = NDSolve[ode1, {x, y, v, z}, {t, 0, 500}];
pdy1 = Plot[{x[t] / 100 /. sol01, y[t] /. sol01, v[t] / 100 /. sol01, z[t] /. sol01},
{t, 0, 600}, PlotLegends -> {"x/100", "y", "v/100", "z"}];
pEs1 = Plot[{x / 100 /. x -> Es[[1]], y /. y -> Es[[2]], v / 100 /. v -> Es[[3]], z /. z -> 0}, {t, 0, 1000},
PlotStyle -> {Dashed}];
Dyn01 = Show[pdy1, pEs1]

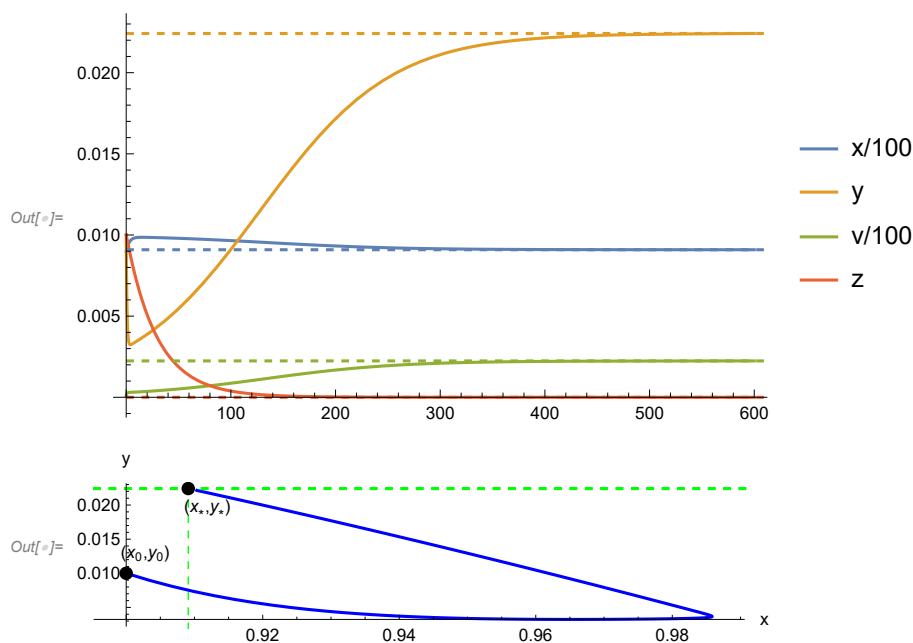
(*****Parametric plot conditions****)
x0 = 0.9; y0 = 0.01; v0 = 0.01; z0 = 0.01;
ppb3 = ParametricPlot[{x[t], (y[t]) /. sol01}, {t, 0, 400}, AxesLabel -> {"x", "y"},
PlotRange -> Full, PlotStyle -> {Blue}];
py3 = Plot[y /. y -> Es[[2]], {t, 0, 400}, PlotStyle -> {Dashed, Green}];
pb3 = Show[{ppb3, py3, Graphics[{Green, Dashed, Line[{x /. x -> Es[[1]], 0}, {x /. x -> Es[[1]], 1}]}]},
Epilog -> {{Thick, Text["(x*, y*)", Offset[{10, -10}, {x /. x -> Es[[1]], y /. y -> Es[[2]]]}],
{PointSize[Large], Style[Point[{x /. x -> Es[[1]], y /. y -> Es[[2]]}], Black]}},
{PointSize[Large], Point[{x0, y0}]}, Text["(x0, y0)", Offset[{10, 10}, {x0, y0}]}]}]
Export["pb3.pdf", pb3]
Export["Dyn01.pdf", Dyn01]

```

E* {0.909091, 0.0224159, 0.224159}

E+ = {0.113348, 0.06, 2.70541, -0.912093}

Ei = {0.726116, 0.06, 0.699984, -0.142026}



Out[]= pb3.pdf

Out[]= Dyn01.pdf

When $b_1 < b = 6 < b_2$:

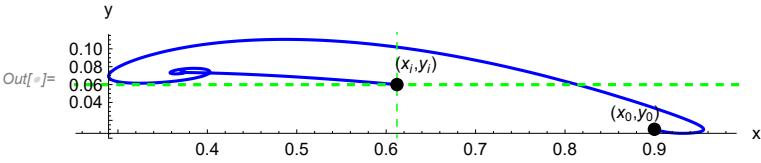
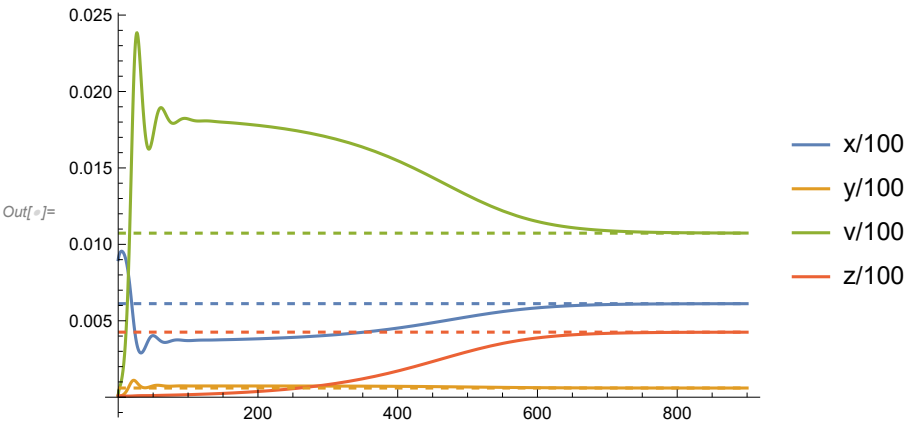
```
In[ ]:= ClearParameters;
μv=0.16; μy=0.48;K=1;b=6;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ε=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ x[t] (1-(x[t]+y[t])/K)- β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv v[t]×z[t]+ b γ y[t] - δ v[t];
z1=z[t] (s y[t] - c );
ode2={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol12=NDSolve[ode2,{x,y,v,z},{t,0,1000}];
pdy1=Plot[{x[t]/100/. sol12,y[t]/100/. sol12,v[t]/100/. sol12,z[t]/100/. sol12},
{t,0,900},PlotLegends→{"x/100","y/100","v/100","z/100"}];
pEs1=Plot[{x/100/.x→Ei[[1]],y/100/.y→Ei[[2]],v/100/.v→Ei[[3]],z/100/.z→Ei[[4]]},{t,0,900},
PlotStyle→{Dashed}];
Dyn12=Show[pdy1,pEs1]
(*****Parametric plot conditions****)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
startP=Epilog[{{PointSize[Large],Point[{0.9,0.01}]}},Text["(x0,y0)",
Offset[{0,10},{0.9,0.01}]}];
bnd =Thread[{x[0],y[0],v[0],z[0]}=={x0,y0,v0,z0}]( *Starting point of the Paramateric Plot**);
ppb6=ParametricPlot[{ x[t],(y[t]) }/.sol12,{t,0,900}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py6=Plot[{y/.y→Ei[[2]]},{t,0,1000},PlotRange→{{0,0.612},{0,0.08}},PlotStyle→{Dashed,Green}];
pb6=Show[{ppb6,py6, Graphics[{Green,Dashed,Line[{x/.x→Ei[[1]],0},{x/.x→Ei[[1]],1}]}]},
Epilog→{{Text["(xi,yi)",Offset[{10,10},{x/.x→Ei[[1]],(y/.y→Ei[[2]])}],
{PointSize[Large],Style[Point[{x/.x→Ei[[1]],(y/.y→Ei[[2]])}],Black]}},
{PointSize[Large],Point[{0.9,0.01}]}},Text["(x0,y0)",Offset[{-10,8},{0.9,0.01}]}]}]
Export["Dyn12.pdf",Dyn12]
Export["pb6.pdf",pb6]
```

E*{0.363636, 0.0736627, 1.84157}

E+={0.166196, 0.06, 2.53245, -0.475792}

Ei={0.612063, 0.06, 1.07325, 0.425644}



Out[*]= Dyn12.pdf

Out[*]= pb6.pdf

When $b_2 < b = 9 < b_2^*$:

```

ClearParameters;
μv=0.16; μy=0.48;K=1;b=9;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ε=0;
Print["E*=",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]

x1=λ x[t] (1-(x[t]+y[t])/K)- β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv v[t]×z[t]+ b γ y[t] - δ v[t];
z1=z[t] (s y[t] - c );
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol12=NDSolve[ode3,{x,y,v,z},{t,0,400}];
pdy2=Plot[{x[t]/100/. sol12,y[t]/. sol12,v[t]/100/. sol12,z[t]/. sol12},{t,0,400},
PlotLegends→{"x/100","y","v/100","z"}];
pEs2=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
pEi2=Plot[{x/100/.x→Ei[[1]],y/.y→Ei[[2]],v/100/.v→Ei[[3]],z/.z→Ei[[4]]},{t,0,1000},
PlotStyle→{Dashed}];
Dyni22=Show[pdy2,pEi2]
DyNs22=Show[pdy2,pEs2]

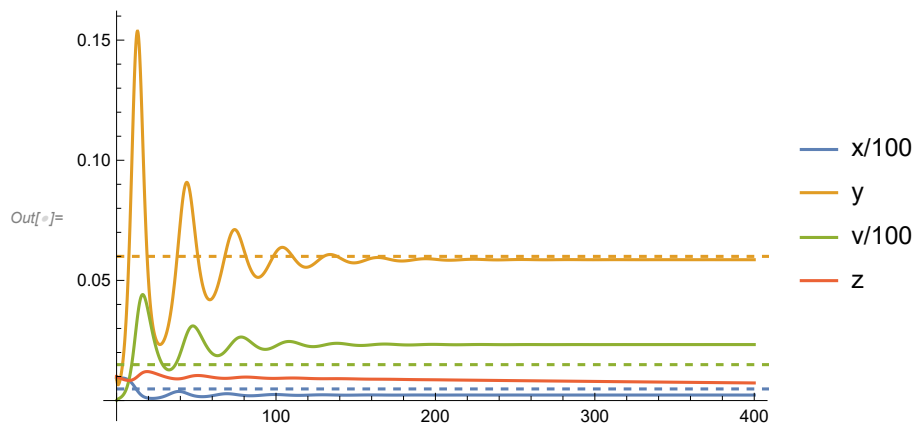
(*****Parametric plot conditions****)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb9=ParametricPlot[{ x[t],(y[t])}/.sol12,{t,0,200}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py9=Plot[(y/.y→Es[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb9=Show[{ppb9,py9, Graphics[{Green,Dashed,Line[{x/.x→Es[[1]],0},{x/.x→Es[[1]],1}]}]},
Epilog→{{Text["(x*,y*)",Offset[{10,10},{x/.x→Es[[1]]},{y/.y→Es[[2]]}]}},{PointSize[Large],
Style[Point[{x/.x→Es[[1]]},{y/.y→Es[[2]]}],Black]}},{PointSize[Large],Point[{0.9,0.01}]},
Text["(x0,y0)",Offset[{-10,8},{0.9,0.01}]}]}]
Export["pb9.pdf",pb9]
Export["Dyni22.pdf",Dyni22]
Export["DyNs22.pdf",DyNs22]

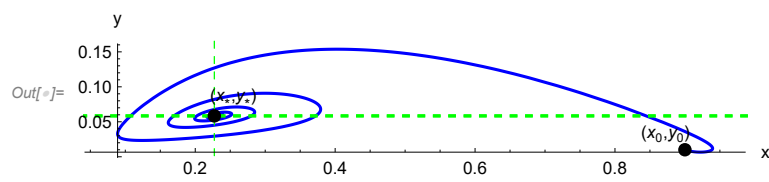
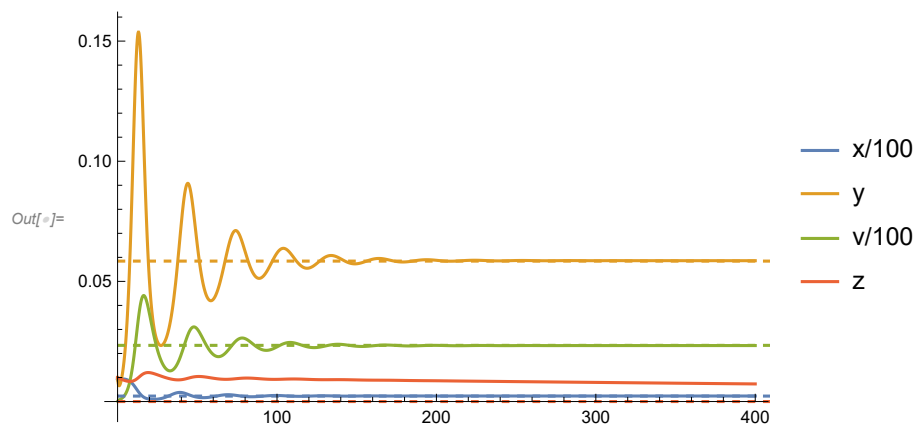
```

E*={0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}





$Out[t]=$ pb9.pdf

$Out[t]=$ Dyni22.pdf

$Out[t]=$ Dyns22.pdf

When $b_2 < b = 9 < b_2^*$ and different initial values :

In[]:=

```

ClearParameters;
μv=0.16; μy=0.48;K=1;b=9;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ε=0;
Print["E*=",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]

x1=λ x[t] (1-(x[t]+y[t])/K)- β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv v[t]×z[t]+ b γ y[t] - δ v[t];
z1=z[t] (s y[t] - c );
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.5,y[0]==0.01,v[0]==1.2,
z[0]==0.5};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,1000}];
pdy2=Plot[{x[t]/100/. sol22,y[t]/. sol22,z[t]/100/. sol22},{t,0,600},
PlotLegends→{"x/100","y","v/100","z/100"}];
pEs2=Plot[{x/100/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,1000},
PlotStyle→{Dashed}];
pEi2=Plot[{x/100/.x→Ei[[1]],y/.y→Ei[[2]],v/100/.v→Ei[[3]],z/.z→Ei[[4]]},{t,0,1000},
PlotStyle→{Dashed}];
Dyni22b=Show[pdy2,pEi2]
Dyns22=Show[pdy2,pEs2]

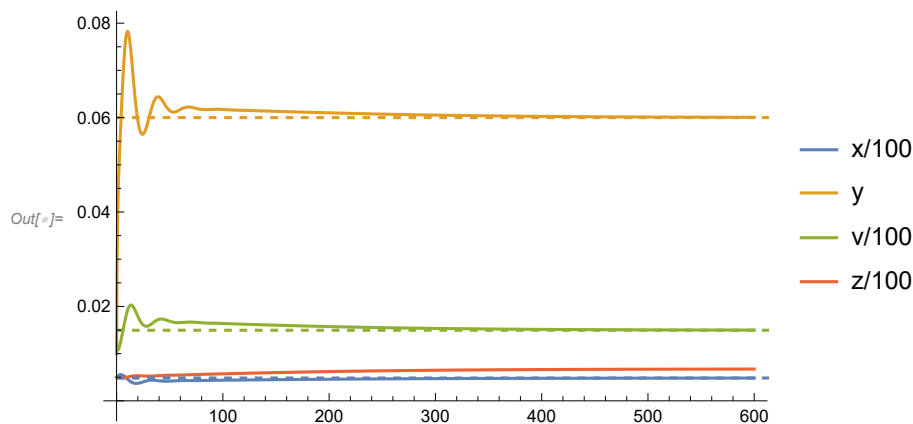
(*****Parametric plot conditions****)
x0=0.5; y0=0.01;v0=1.2;z0=0.5;
ppb9=ParametricPlot[{ x[t],(y[t])}/.sol22,{t,0,500}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py9=Plot[(y/.y→Ei[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb9i=Show[{ppb9,py9, Graphics[{Green,Dashed,Line[{x/.x→Ei[[1]],0},{x/.x→Ei[[1]],1}]}]},
Epilog→{{Text["(x*,y*)",Offset[{10,10},{x/.x→Ei[[1]],(y/.y→Ei[[2]])}],{PointSize[Large],
Style[Point[{x/.x→Ei[[1]],(y/.y→Ei[[2]])}],Black]}},{PointSize[Large],Point[{0.9,0.01}],
Text["(x0,y0)",Offset[{-10,8},{0.9,0.01}]]}}]
Export["pb9i.pdf",pb9i]
Export["Dyni22b.pdf",Dyni22b]

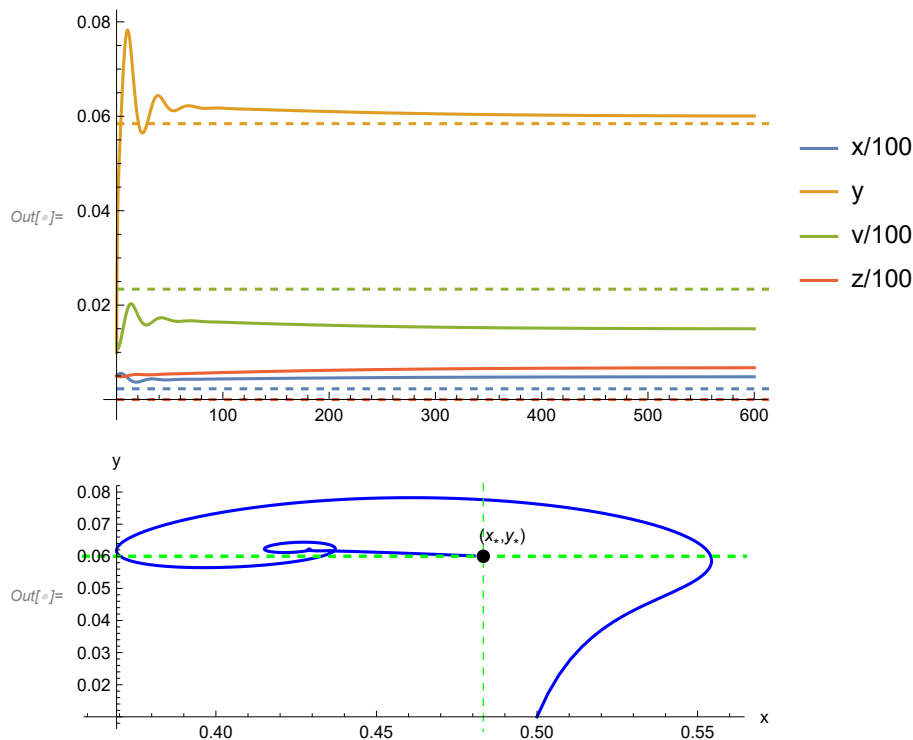
```

E*={0.227273, 0.0584416, 2.33766}

E+={0.249944, 0.06, 2.25837, 0.072607}

Ei={0.483284, 0.06, 1.49471, 0.675711}





Out[]= pb9i.pdf

Out[]= Dyni22b.pdf

When $b_2 < b = 10 < b_2^*$:

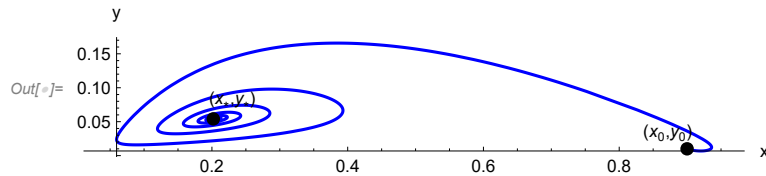
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=10;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ε=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ x[t] (1-(x[t]+y[t])/K)- β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv v[t]×z[t]+ b γ y[t] - δ v[t];
z1=z[t] (s y[t] - c );

ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,400}];

(*****Parametric plot conditions***)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb10=ParametricPlot[{ x[t],(y[t]) }/.sol22,{t,0,200}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py10=Plot[(y/.y→Es[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb10=Show[{ppb10,Epilog→{{Text["(x*,y*)",Offset[{10,10},{x/.x→Es[[1]],
(y/.y→Es[[2]])}]}},{PointSize[Large],Style[Point[{(x/.x→Es[[1]],(y/.y→Es[[2]])}],Black)}},
{PointSize[Large],Point[{0.9,0.01}],Text["(x0,y0)",Offset[{-10,8},{0.9,0.01}]}}]}]

Export["pb10.pdf",pb10]
```

$E_* = \{0.20202, 0.0541003, 2.43451\}$
 $E_+ = \{0.30876, 0.06, 2.06588, 0.352938\}$
 $E_i = \{0.411479, 0.06, 1.72971, 0.635108\}$


Out[]= pb10.pdf

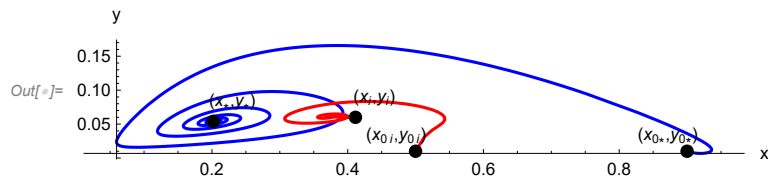
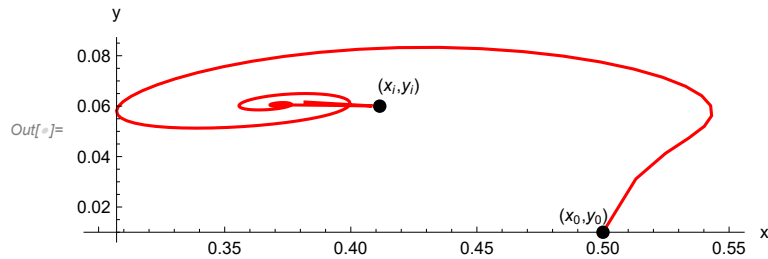
```
ClearParameters;
μv=0.16; μy=0.48;K=1;b=10;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ε=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ x[t] (1-(x[t]+y[t])/K)- β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv v[t]×z[t]+ b γ y[t] - δ v[t];
z1=z[t] (s y[t] - c );
ode3={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.5,y[0]==0.01,v[0]==1.2,
z[0]==0.5};
sol22=NDSolve[ode3,{x,y,v,z},{t,0,1000}];
x0=0.5; y0=0.01;v0=1.2;z0=0.5;
ppb10i=ParametricPlot[{ x[t],(y[t])}/.sol22,{t,0,1900}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Red}];
py10i=Plot[(y/.y→Ei[[2]]),{t,0,800},PlotStyle→{Dashed,Green}];
pb10i=Show[{ppb10i },Epilog→{{Text["(xi,yi)",Offset[{10,10},{(x/.x→Ei[[1])],
(y/.y→Ei[[2])}]}],
{PointSize[Large],Style[Point[{(x/.x→Ei[[1])],(y/.y→Ei[[2])}],Black]}},
{PointSize[Large],Point[{x0,y0}],Text["(x0,y0)",Offset[{-10,8},{x0,y0}]]}}]
pp10si=Show[{pb10,pb10i},Epilog→{{Text["(xi,yi)",Offset[{10,10},{(x/.x→Ei[[1])],
(y/.y→Ei[[2])}]}],{PointSize[Large],Style[Point[{(x/.x→Ei[[1])],(y/.y→Ei[[2])}],Black]}},
{PointSize[Large],Point[{x0,y0}],Text["(x0i,y0i)",Offset[{-10,8},{x0,y0}]]},
{Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1])],(y/.y→Es[[2])}]}],{PointSize[Large],
Style[Point[{(x/.x→Es[[1])],(y/.y→Es[[2])}],Black]}},{PointSize[Large],Point[{0.9,0.01}],
Text["(x0*,y0*)",Offset[{-10,8},{0.9,0.01}]]}}]
Export["pp10si.pdf",pp10si]
Export["pb10i.pdf",pb10i]
```

$E^* = \{0.20202, 0.0541003, 2.43451\}$

$E_+ = \{0.30876, 0.06, 2.06588, 0.352938\}$

$E_i = \{0.411479, 0.06, 1.72971, 0.635108\}$



Out[]= pp10si.pdf

Out[]= pb10i.pdf

When $b_2^* < b = 15 < b_H$:

```

In[ ]:= ClearParameters;
μv=0.16; μy=0.48;K=1;b=15;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ε=0;
Print["E*",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]

x1=λ x[t] (1-(x[t]+y[t])/K)- β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv v[t]×z[t]+ b γ y[t] - δ v[t];
z1=z[t] (s y[t] - c);
ode4={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
sol2H=NDSolve[ode4,{x,y,v,z},{t,0,800}];
pdy2H=Plot[{x[t]/.sol2H,y[t]/.sol2H,v[t]/100/.sol2H,z[t]/.sol2H},{t,0,800},
PlotLegends→{"x","y","v/100","z"}];
pEs2H=Plot[{x/.x→Es[[1]],y/.y→Es[[2]],v/100/.v→Es[[3]],z/.z→0},{t,0,800},
PlotStyle→{Dashed}];
Dyn2H=Show[pdy2H,pEs2H,PlotRange→All]

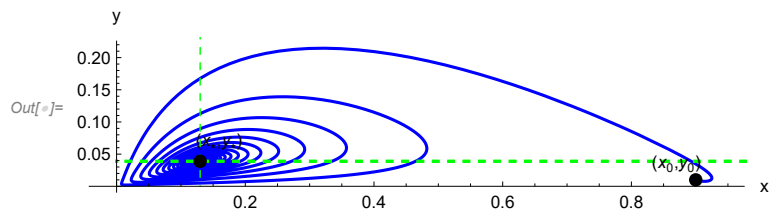
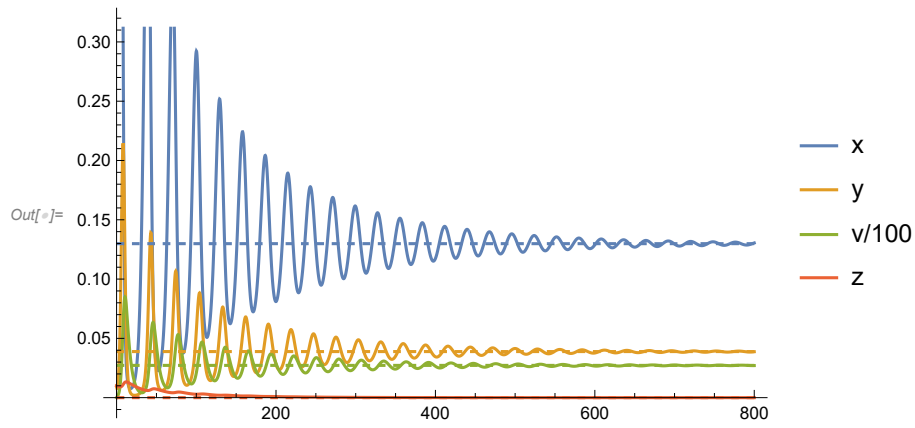
(*****Parametric plot conditions****)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb15=ParametricPlot[{ x[t],(y[t]) }/.sol2H,{t,0,500}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
py15=Plot[{y/.y→Es[[2]]},{t,0,400},PlotStyle→{Dashed,Green}];
pb15=Show[{ppb15,py15, Graphics[{Green,Dashed,Line[{x/.x→Es[[1]],0},
{x/.x→Es[[1]],1}]}]},Epilog→{{Thick,Text["(x_*,y_*)",Offset[{10,10},{x/.x→Es[[1]],
(y/.y→Es[[2]])}]}},{PointSize[Large],Style[Point[{x/.x→Es[[1]],(y/.y→Es[[2]])}],Black]}},
{PointSize[Large],Point[{x0,y0}],Text["(x_0,y_0)",Offset[{-10,8},{x0,y0}]]}]
Export["pb15.pdf",pb15]
Export["Dyn2H.pdf",Dyn2H]

```

$$E^* = \{0.12987, 0.0388644, 2.72051\}$$

$$E_+ = \{0.332366 + 0.217051 i, 0.06, 1.98862 - 0.710347 i, 1.03002 + 0.746836 i\}$$

$$E_- = \{0.332366 - 0.217051 i, 0.06, 1.98862 + 0.710347 i, 1.03002 - 0.746836 i\}$$



Out[]= pb15.pdf

Out[]= Dyn2H.pdf

When $b_H < b = 23 < b_\infty$:

In[]:=

```

ClearParameters;
μv=0.16; μy=0.48;K=1;b=23;γ=1;λ=0.36;β=0.11;δ=0.2;s=0.6; c=0.036;ε=0;
Print["E*=",Es=Est/.cn//N]
Print["E+=",Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]]
Print["Ei=",Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]]
x1=λ x[t] (1-(x[t]+y[t])/K)- β x[t]×v[t] ;
y1=β x[t]×v[t] -μy y[t]×z[t]- γ y[t];
v1=-β x[t]×v[t] - μv v[t]×z[t]+ b γ y[t] - δ v[t];
z1=z[t] (s y[t] - c );
ode5={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.9,y[0]==0.01,v[0]==0.01,
z[0]==0.01};
solHI=NDSolve[ode5,{x,y,v,z},{t,0,10000}];
pdyHI=Plot[{x[t]/. solHI,y[t]/. solHI,v[t]/. solHI,z[t]/. solHI},{t,0,1000},
PlotLegends→{"x","y","v","z"}];
DynHI=Show[pdyHI,PlotRange→All]

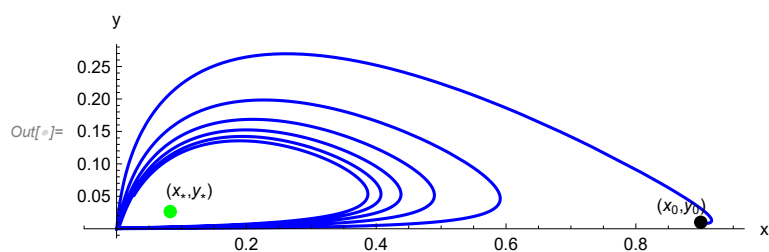
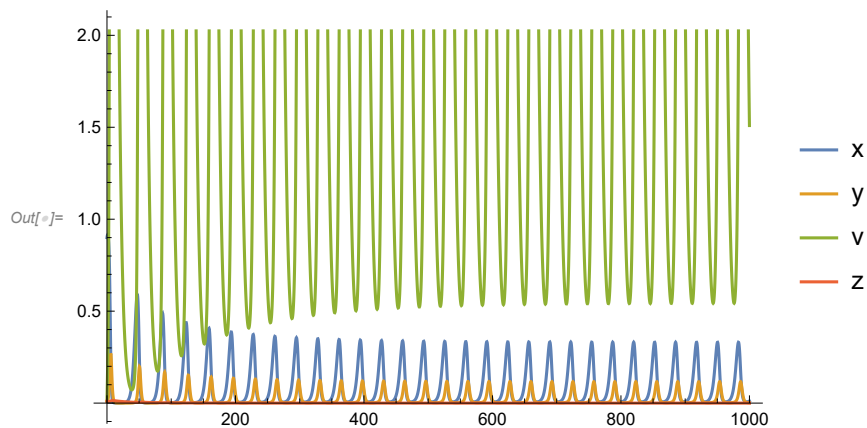
(*****Parametric plot conditions****)
x0=0.9; y0=0.01;v0=0.01;z0=0.01;
ppb23=ParametricPlot[{ x[t],(y[t])}/.solHI,{t,0,200}, AxesLabel→{"x","y"},
PlotRange→Full,PlotStyle→{Blue}];
NDSolve[{y'[t]==y[t]×(x[t]-1),x'[t]==x[t] (2-y[t]),x[0]==1,y[0]==2.7},{x,y},{t,0,10}];
py23=Plot[(y/.y→Es[[2]]},{t,0,400},PlotStyle→{Dashed,Green}];
pb23=Show[{ppb23,Epilog→{{Thick,Text["(x*,y*)",Offset[{10,10},{(x/.x→Es[[1])],
(y/.y→Es[[2])}]}],{PointSize[Large],Style[Point[{(x/.x→Es[[1])],(y/.y→Es[[2])}],Green}}},
{PointSize[Large],Point[{x0,y0}],Text["(x0,y0)",Offset[{-10,8},{x0,y0}]}]}]
Export["pb23.pdf",pb23]
Export["DynHI.pdf",DynHI]

```

E*={0.0826446, 0.0265046, 2.91551}

E+={0.297813 + 0.339863 i, 0.06, 2.1017 - 1.11228 i, 1.75115 + 1.463 i}

Ei={0.297813 - 0.339863 i, 0.06, 2.1017 + 1.11228 i, 1.75115 - 1.463 i}



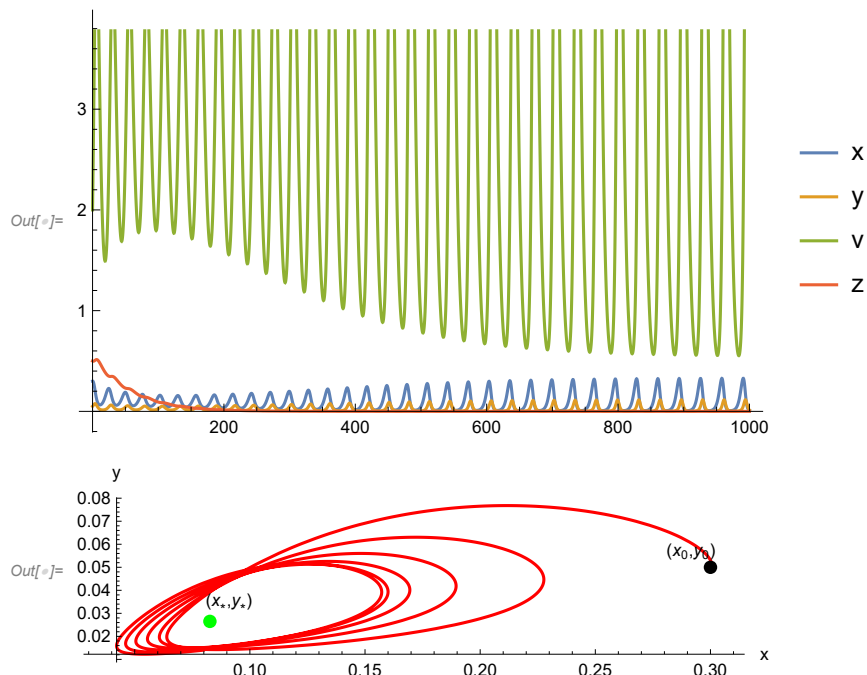
Out[]= pb23.pdf

Out[]= DynHI.pdf

```

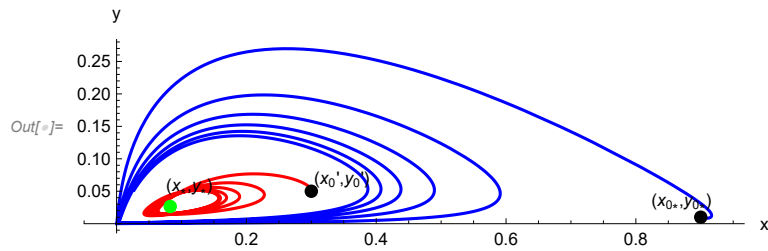
In[ ]:= ode5={x'[t]==x1,y'[t]==y1,v'[t]==v1,z'[t]==z1,x[0]==0.3,y[0]==0.05,v[0]==2,
z[0]==0.5};
solHI=NDSolve[ode5,{x,y,v,z},{t,0,10000}];
pdyHI=Plot[{x[t]/.solHI,y[t]/.solHI,v[t]/.solHI,z[t]/.solHI},{t,0,1000},
PlotLegends->{"x","y","v","z"}];
DynHIc=Show[pdyHI,PlotRange->All]
(*New initial conditions*)
x0=0.3; y0=0.05;v0=2;z0=0.5;
ppb23c=ParametricPlot[{x[t],(y[t])}/.solHI,{t,0,150}, AxesLabel->{"x","y"},
PlotRange->Full,PlotStyle->{Red}];
pb23c=Show[{ppb23c},Epilog->{{Thick,Text["(x*,y*)",Offset[{10,10},
{x/.x->Es[[1]],(y/.y->Es[[2]])}],{PointSize[Large],Style[Point[{x/.x->Es[[1]],
(y/.y->Es[[2]])}],Green}],{PointSize[Large],Point[{x0,y0}],Text["(x0,y0)",
Offset[{-10,8},{x0,y0}]]}}]
Export["pb23b.pdf",pb23c]
Export["DynHIb.pdf",DynHIc]
pp23cs=Show[{pb23,pb23c},Epilog->{{Thick,Text["(x*,y*)",Offset[{10,10},
{x/.x->Es[[1]],(y/.y->Es[[2]])}],{PointSize[Large],Style[Point[{x/.x->Es[[1]],(y/.y->Es[[2]])}],Green}],{
PointSize[Large],Point[{x0,y0}],Text["(x0',y0')",Offset[{16,6},{x0,y0}]],
{PointSize[Large],Point[{0.9,0.01}],Text["(x0*,y0*)",Offset[{-10,8},{0.9,0.01}]]}}]
Export["pp23s.pdf",pp23cs]

```



Out[]= pb23b.pdf

Out[]= DynHIb.pdf



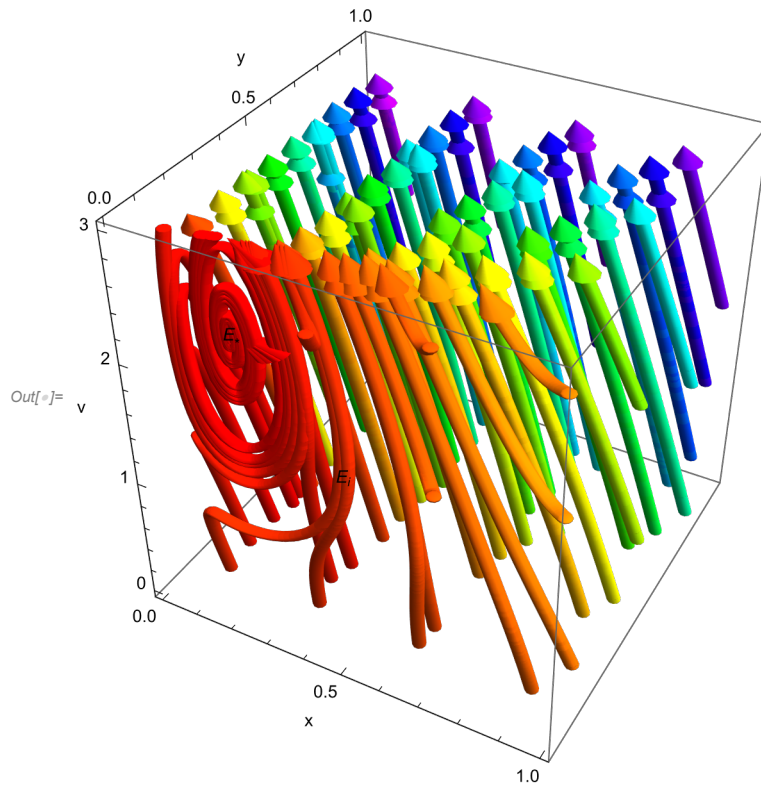
Out[]= pp23s.pdf

3D- Plot of the dynamic:

```
cn=cnT17;
Print["E*", Es=Es/.cn//N]
Print["E+=", Ep=Chop[{xe,ye,v,ze}/.v→vp/.cn//N]
Print["Ei=", Ei=Chop[{xe,ye,v,ze}/.v→vi/.cn//N]
epi={Text["E*", Offset[{-10,10},{Es[[1]],Es[[2]]}]]/.cn,
{PointSize[Large],Style[Point[{Es[[1]],Es[[2]]}],Blue]//.cn},Text["E_i",Offset[{0,10},
{Ei[[1]],Ei[[2]]}]],{PointSize[Large],Style[Point[{Ei[[1]],Ei[[2]]}],Purple]},
Text["E_+",Offset[{0,10},{Ep[[1]],Ep[[2]]}]],{PointSize[Large],
Style[Point[{Ep[[1]],Ep[[2]]}],Orange]}};
sp3=StreamPlot3D[dyn3//.cnT17,{x,0,1},{y,0,1},{v,0,3},AxesLabel→{"x","y","v"},
StreamColorFunction→Hue,PlotRange→All,StreamMarkers→"Arrow3D",StreamScale→Full];
sp3D=Show[{sp3},Graphics3D[Text[Style["E*",Black,Thick],Es//.cn],
{PointSize[0.06],Style[Point[Es],Black]}],Graphics3D[Text[Style["E_i",Black,Thick],
Drop[Ei,{4}]]/.cn,{PointSize[0.06],Style[Point[Drop[Ei,{4}]],Black]}]]
Export["sp3D.pdf",sp3D]
```

$$E^* = \{0.227273, 0.0584416, 2.33766\}$$

$$E_+ = \{0.249944, 0.06, 2.25837, 0.072607\}$$

$$E_i = \{0.483284, 0.06, 1.49471, 0.675711\}$$


Out[]:= sp3D.pdf

4)Section 3.4(in paper): 4-Dim.Viro-therapy model when $\epsilon=1$

4-1)Definition of the model and fixed points when $\epsilon=1$

```

In[181]:= SetDirectory[NotebookDirectory[]];
AppendTo[$Path,Directory];
Clear["Global`*"];
Clear["K"];
Format[μv]:=Subscript[μ,v];Format[μy]:=Subscript[μ,y];Format[La]:=Δ;(*La >0*)
pars={β,λ,γ,δ,μy, μv,b,K,s,c};
cpos={β>0,λ>0,γ>0,δ>0,μy>0, μv>0,b>1,K>0,s>0,c>0};
La=0; bL=100;cnb={b→50};
cEri={μy→1/48,K→2139.258, β→.0002,λ→.2062,γ→1/18,δ→.025, μv→2*10^(-8),c→10^(-3),s→.027};
(***** Four dim Deterministic epidemic model with Logistic growth *****)
x1=La+λ x(1-(x+y)/K)- β x v ;
y1=β x v -μy y z- γ y ;
v1=-β x v - μv v z+ b γ y - δ v;
z1=z(s y - c z);
x1s=λ (1-(x+y)/K)- β v ;
z1s=s y - c z;

```

```

dyns={x1s,y1,v1,z1s};
dyn={x1,y1,v1,z1};
dyn3={x1,y1,v1}/.z->0;
      x'
Print[" (y',)=",dyn//FullSimplify//MatrixForm]
      v'
      z'

(*Jacobian*)
Jac=Grad[(dyn),{x,y,v,z}]/FullSimplify;
Print["J(x,y,v,z)=",Jac//MatrixForm]
Print["J(E0)=",Jac//.Thread[{x,y,v,z}->{0,0,0,0}]/MatrixForm]
Jac3=Grad[(dyn3),{x,y,v}]/FullSimplify;
det=Det[Jac]/FullSimplify; tr=Tr[Jac]/FullSimplify;
R0=b β K/(β K+δ);bcrit=1+δ/(β K);(*Reduce[Join[{R0>1},pars],δ]*)
(****Endemic points in 3dims **)
eqE=Thread[dyn3=={0,0,0}];
Print["Three Fixed points in 3-dim case:",solE=Solve[eqE,{x,y,v}]/FullSimplify]

el=Eliminate[Thread[dyns=={0,0,0,0}],{x,v,z}];
Qybyelim=Factor[el[[1,1]]-el[[1,2]]]/y//FullSimplify;
Print["Coefficients of Qy by elim polynomial are:"]
cof=CoefficientList[Qybyelim,y]/FullSimplify
so=y/.Solve[Qybyelim==0,y];(*Third order roots*)

(*****Fixed points of 4-dim model using P(y)****)
fy=(c γ(b-1)-y μy s);
gy=( μv s y+c δ); hy=(γ +y s μy/c);
xe=hy gy/(β fy); ve=y fy/gy; ze= s y /c;
ys=y/.solE[[3]](* y of E* ****);

Py=λ(1-y/K)-β y fy/gy-λ hy gy/(β K fy); yb=c γ (b-1)/(μy s);
Qy=λ fy gy(1- y /K)- λ hy gy^2/(β K)-y β fy^2//FullSimplify;
Qycol=Collect[Qy,y];
Qycoef=CoefficientList[Qycol,y];
(*Print["Check Eric Qy -elim Qy=",Qybyelim+c K β Qy//FullSimplify]*)
Dis=Collect[Discriminant[Qy,y],b];
Discoef=CoefficientList[Dis,b];Length[Discoef];
DisE=Dis//.cEri//N;

```

$$\begin{pmatrix} x' \\ y' \\ v' \\ z' \end{pmatrix} = \begin{pmatrix} -v x \beta + x \left(1 - \frac{x+y}{K}\right) \lambda \\ v x \beta - y (\gamma + z \mu_y) \\ b y \gamma - v (x \beta + \delta + z \mu_v) \\ z (s y - c z) \end{pmatrix}$$

$$J(x,y,v,z) = \begin{pmatrix} -v \beta + \lambda - \frac{(2x+y) \lambda}{K} & -\frac{x \lambda}{K} & -x \beta & 0 \\ v \beta & -\gamma - z \mu_y & x \beta & -y \mu_y \\ -v \beta & b \gamma & -x \beta - \delta - z \mu_v & -v \mu_v \\ 0 & s z & 0 & s y - 2 c z \end{pmatrix}$$

$$J(E0) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \\ 0 & b \gamma & -\delta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Three Fixed points in 3-dim case: $\{x \rightarrow 0, y \rightarrow 0, v \rightarrow 0\}, \{x \rightarrow K, y \rightarrow 0, v \rightarrow 0\},$

$$\left\{ x \rightarrow \frac{\delta}{(-1+b)\beta}, y \rightarrow \frac{((-1+b)K\beta - \delta)\delta\lambda}{(-1+b)\beta((-1+b)K\beta\gamma + \delta\lambda)}, v \rightarrow \frac{\gamma((-1+b)K\beta - \delta)\lambda}{\beta((-1+b)K\beta\gamma + \delta\lambda)} \right\}$$

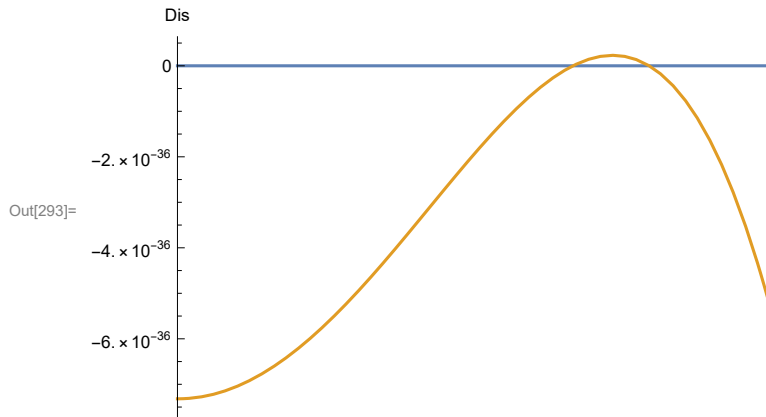
Coefficients of Qy by elim polynomial are:

$$\begin{aligned} \text{Out}[210]= & \left\{ c^3 \gamma \delta (K(\beta - b\beta) + \delta)\lambda, \right. \\ & c^2 \left((-1+b)c\beta\gamma((-1+b)K\beta\gamma + \delta\lambda) + s\lambda(\gamma(K(\beta - b\beta) + 2\delta)\mu_v + \delta(K\beta + \delta)\mu_y) \right), \\ & cs \left(s\lambda\mu_v(\gamma\mu_v + K\beta\mu_y + 2\delta\mu_y) + c\beta(-\delta\lambda\mu_y + (-1+b)\gamma(\lambda\mu_v - 2K\beta\mu_y)) \right), \\ & \left. s^2\mu_y(s\lambda\mu_v^2 + c\beta(-\lambda\mu_v + K\beta\mu_y)) \right\} \end{aligned}$$

```
In[293]:= Plot[{0,DisE},{b,0,2 bL},AxesLabel->{"b","Dis"}]
Print["Roots of Dis[b]=0 are: ",solbE=Solve[DisE==0,b]]
QR=Solve[Qy==0,y];
Print["critical b from R0=1 is:", bcrit//.cEri//N]
ym= y/.QR[[1]];
yp= y/.QR[[2]];yi= y/.QR[[3]];

{Chop[yp],Chop[ym]}//.cEri//Simplify;

jacEK=Jac/.x->K/.y->0/.v->0/.z->0;
Print["J(EK)=", jacEK//MatrixForm]
Print["Eig.val of J(EK) are:", Eigenvalues[jacEK]//FullSimplify]
```



Roots of Dis[b]=0 are:

$$\{b \rightarrow -468.749.\}, \{b \rightarrow -468.749.\}, \{b \rightarrow -159.797\}, \{b \rightarrow -132.005\}, \{b \rightarrow 133.421\}, \{b \rightarrow 159.122\}$$

critical b from R0=1 is:1.02299

$$J(EK) = \begin{pmatrix} -\lambda & -\lambda & -K\beta & 0 \\ 0 & -\gamma & K\beta & 0 \\ 0 & b\gamma & -K\beta - \delta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Eig.val of } J(EK) \text{ are: } \left\{ 0, \frac{1}{2} \left(-K\beta - \gamma - \delta - \sqrt{-4\gamma(K(\beta - b\beta) + \delta) + (K\beta + \gamma + \delta)^2} \right), \right.$$

$$\left. \frac{1}{2} \left(-K\beta - \gamma - \delta + \sqrt{-4\gamma(K(\beta - b\beta) + \delta) + (K\beta + \gamma + \delta)^2} \right), -\lambda \right\}$$

4-2) Stability of the interior points using Routh Hurwitz:

In[222]:=

```

jacE=Jac/.x->xe/.v->ve/.z->ze/.y->y;
jacE //FullSimplify//MatrixForm
Print["Det[Jac]=",Det[jacE]//FullSimplify, " , Trace[Jac]=",
Tr[jacE]//FullSimplify]
(*Reduce[Join[{Det[Jac]>0&&x>0&&y>0&&z>0&&v>0},cpos],β] (*take so long time**)*)
(*Reduce[Join[{Tr[jacE]<0&&x>0&&y>0&&z>0&&v>0},cpos],β];
poly=Collect[Together[Det[ψ IdentityMatrix[4]-jacE]],ψ];
coe=CoefficientList[poly,ψ];*)
(*Print["Coefficients of the Characteristic polynomial are:",coe]*)

```

Out[223]//MatrixForm=

$$\begin{pmatrix}
 \lambda + \frac{y\beta(c(\gamma-b\gamma)+s y\mu_y)}{c\delta+s y\mu_v} - \frac{\lambda\left(y + \frac{2(c\delta+s y\mu_v)\left(\gamma+\frac{s y\mu_y}{c}\right)}{\beta((-1+b)c\gamma-s y\mu_y)}\right)}{K} & -\frac{\lambda(c\delta+s y\mu_v)\left(\gamma+\frac{s y\mu_y}{c}\right)}{K\beta((-1+b)c\gamma-s y\mu_y)} & -\frac{(c\delta+s y\mu_v)\left(\gamma+\frac{s y\mu_y}{c}\right)}{(-1+b)c\gamma-s y\mu_y} & 0 \\
 \frac{y\beta((-1+b)c\gamma-s y\mu_y)}{c\delta+s y\mu_v} & -\frac{c\gamma+s y\mu_y}{c} & \frac{(c\delta+s y\mu_v)\left(\gamma+\frac{s y\mu_y}{c}\right)}{(-1+b)c\gamma-s y\mu_y} & -y\mu_y \\
 \frac{y\beta(c(\gamma-b\gamma)+s y\mu_y)}{c\delta+s y\mu_v} & b\gamma & \frac{b\gamma(c\delta+s y\mu_v)}{c(\gamma-b\gamma)+s y\mu_y} & \frac{y\mu_v(c(\gamma-b\gamma)+s y\mu_y)}{c\delta+s y\mu_v} \\
 0 & \frac{s^2 y}{c} & 0 & -s y
 \end{pmatrix}$$

Det[Jac]=

$$\begin{aligned}
 & -\frac{1}{c^3 K \beta((-1+b)c\gamma-s y\mu_y)^2} s y^2 \left(-(-1+b)^2 c^5 \beta \gamma^3((-1+b)K\beta\gamma+\delta\lambda) + 2 s^5 y^4 \lambda \mu_v^2 \mu_y^3 + c s^4 y^3 \mu_y^2 \right. \\
 & \quad \left(2 \times (3-2b)\gamma\lambda\mu_v^2 + 2(-y\beta+\delta)\lambda\mu_v\mu_y + K\beta\mu_y(\lambda\mu_v+2y\beta\mu_y) \right) - c^2 s^3 y^2 \mu_y \left(6 \times (-1+b)\gamma^2\lambda\mu_v^2 + \right. \\
 & \quad \left. y\beta\delta\lambda\mu_y^2 + \gamma\mu_y \left((3 \times (-1+b)K\beta + (6-5b)y\beta + 6 \times (-1+b)\delta)\lambda\mu_v + (-7+6b)Ky\beta^2\mu_y \right) \right) - \\
 & \quad c^3 s^2 y \gamma \left(2 \times (-1+b)\gamma^2\lambda\mu_v^2 + \delta(3y\beta+b(K\beta-3y\beta+2\delta))\lambda\mu_y^2 + \right. \\
 & \quad \left. \gamma\mu_y \left((3 \times (-2+b) \times (-1+b)y\beta - 6\delta+8b\delta)\lambda\mu_v - (-1+b) \times (-3+2b)K\beta(\lambda\mu_v+3y\beta\mu_y) \right) \right) + \\
 & \quad \left. c^4 s \gamma^2 \left(-2 \times (-1+b)\gamma((-1+b)y\beta+\delta)\lambda\mu_v - \delta((-1+b) \times (-3+2b)y\beta+2b\delta)\lambda\mu_y + \right. \right. \\
 & \quad \left. \left. (-1+b)K\beta(b\delta\lambda\mu_y + (-1+b)\gamma(\lambda\mu_v+(5-2b)y\beta\mu_y)) \right) \right) \\
 & , \text{Trace[Jac]} = -s y - \gamma - \delta + \lambda - \frac{s y\mu_v}{c} - \frac{s y\mu_y}{c} + \frac{y\beta(c(\gamma-b\gamma)+s y\mu_y)}{c\delta+s y\mu_v} - \\
 & \frac{(c\delta+s y\mu_v)\left(\gamma+\frac{s y\mu_y}{c}\right)}{(-1+b)c\gamma-s y\mu_y} - \\
 & \frac{\lambda\left(y + \frac{2(c\delta+s y\mu_v)\left(\gamma+\frac{s y\mu_y}{c}\right)}{\beta((-1+b)c\gamma-s y\mu_y)}\right)}{K}
 \end{aligned}$$

4-3) Stability of E* using Routh Hurwitz:

In[225]:=

```

jacEs=Jac/.x->xe/.v->ve/.z->0/.y->y;
Print["y*=",y/.solE[[3]]//FullSimplify]
jacEs //FullSimplify//MatrixForm
Print["Det [Jac]=",Det[jacEs]//FullSimplify, " ", Trace[Jac]=",
Tr[jacEs]//FullSimplify]
(*Reduce[Join[{Tr[jacEs]<0&&x>0&&y>0&&v>0},cpos],β];*)
poly=Collect[Together[Det[ψ IdentityMatrix[4]-jacEs]],ψ];
coe=CoefficientList[poly,ψ];

```

$$y^* = \frac{((-1+b) K \beta - \delta) \delta \lambda}{(-1+b) \beta ((-1+b) K \beta \gamma + \delta \lambda)}$$

Out[227]//MatrixForm=

$$\begin{pmatrix} \lambda + \frac{y \beta (c (\gamma - b \gamma) + s y \mu_y)}{c \delta + s y \mu_v} - \frac{\lambda \left(y + \frac{2 (c \delta + s y \mu_v) \left(\gamma + \frac{s y \mu_y}{c} \right)}{\beta ((-1+b) c \gamma - s y \mu_y)} \right)}{K} & -\frac{\lambda (c \delta + s y \mu_v) \left(\gamma + \frac{s y \mu_y}{c} \right)}{K \beta ((-1+b) c \gamma - s y \mu_y)} & -\frac{(c \delta + s y \mu_v) \left(\gamma + \frac{s y \mu_y}{c} \right)}{(-1+b) c \gamma - s y \mu_y} & 0 \\ \frac{y \beta ((-1+b) c \gamma - s y \mu_y)}{c \delta + s y \mu_v} & -\gamma & \frac{(c \delta + s y \mu_v) \left(\gamma + \frac{s y \mu_y}{c} \right)}{(-1+b) c \gamma - s y \mu_y} & -y \mu_y \\ \frac{y \beta (c (\gamma - b \gamma) + s y \mu_y)}{c \delta + s y \mu_v} & b \gamma & \frac{s y \mu_v}{c} + \frac{b \gamma (c \delta + s y \mu_v)}{c (\gamma - b \gamma) + s y \mu_y} & \frac{y \mu_v (c (\gamma - b \gamma) + s y \mu_y)}{c \delta + s y \mu_v} \\ 0 & 0 & 0 & s y \end{pmatrix}$$

$$\begin{aligned} \text{Det [Jac]} = & \left((s y (y \beta (c \delta + s y \mu_v) ((-1+b) c \gamma - s y \mu_y) (c \gamma + s y \mu_y) \right. \\ & (c K \beta \gamma ((-1+b) c \gamma - s y \mu_y) + \lambda (c \delta + s y \mu_v) (c \gamma + s y \mu_y)) - b \gamma \lambda (c \delta + s y \mu_v)^2 (c \gamma + s y \mu_y) \\ & (c K \beta ((-1+b) c \gamma - s y \mu_y) - 2 (c \delta + s y \mu_v) (c \gamma + s y \mu_y) + c y \beta (c (\gamma - b \gamma) + s y \mu_y)) - \\ & (b c^2 \gamma \delta + s y \mu_v (c \gamma + s y \mu_y)) (y \beta \lambda (c \delta + s y \mu_v) ((-1+b) c \gamma - s y \mu_y) (c \gamma + s y \mu_y) + \\ & \gamma (-c K \beta \lambda (c \delta + s y \mu_v) ((-1+b) c \gamma - s y \mu_y) + c K y \beta^2 ((-1+b) c \gamma - s y \mu_y)^2 + \\ & \left. \left. \left. \left. \lambda (c \delta + s y \mu_v) (c y \beta ((-1+b) c \gamma - s y \mu_y) + 2 (c \delta + s y \mu_v) (c \gamma + s y \mu_y)) \right) \right) \right) \right) / \\ & (c^2 K \beta (c \delta + s y \mu_v) ((-1+b) c \gamma - s y \mu_y)^2) \end{aligned}, \text{Trace [Jac]} = s$$

$$\begin{aligned} & y - \\ & \gamma - \\ & \delta + \\ & \lambda + \\ & \frac{y \beta (c (\gamma - b \gamma) + s y \mu_y)}{c \delta + s y \mu_v} - \\ & \frac{(c \delta + s y \mu_v) \left(\gamma + \frac{s y \mu_y}{c} \right)}{(-1+b) c \gamma - s y \mu_y} - \\ & \frac{\lambda \left(y + \frac{2 (c \delta + s y \mu_v) \left(\gamma + \frac{s y \mu_y}{c} \right)}{\beta ((-1+b) c \gamma - s y \mu_y)} \right)}{K} \end{aligned}$$

4-4) Stability of the interior points numerically :

Routh Hurwitz conditions for the stability of E₋ (4 dim)


```
In[231]:= Jac4=Grad[dyn//.cEri,{x,y,v,z}]/FullSimplify;
bcrit=1+δ/(β K);
Jst=(Jac4/.x→xe/.v→ve/.z→ze/.y→ym)//.cEri;Jst//N//MatrixForm;
Trs=Tr[Jst];
pc=Collect[Det[ψ IdentityMatrix[4]-Jst],ψ];
coT=CoefficientList[pc,ψ](*So long computations*);
```

Routh Hurwitz conditions for the stability of E_*

```
In[237]:= cF1={β→87/2,λ→1,γ→1/128,δ→1/2,μy→1,μv→1,K→1/2,s→1,c→1};
cEri=cF1;
Jac3=Grad[dyn3//.cEri,{x,y,v}]/FullSimplify;
bcrit=1+δ/(β K);(*Reduce[Join[{R0>1},pars],δ]*)
Print["J(E_*) is"]
Jst=(Jac3/.x→x/.solE[[3]]/.y→ys/.v→v/.solE[[3]])//.cEri//FullSimplify;Jst//MatrixForm

Trs=Tr[Jst];
pc=Collect[Det[ψ IdentityMatrix[3]-Jst],ψ];
coT=CoefficientList[pc,ψ]/FullSimplify;
Print["a1=",a1=coT[[3]]/.cEri," a2=",a2=coT[[2]]/.cEri," a3=",a3=coT[[1]]/.cEri]
H2=a1*a2-a3;
Print["H2(b0)=",H2/.b→bcrit//FullSimplify]
Print["Denominator of H2 is ",Denominator[Together[H2]]/.cEri//FullSimplify]
φb=Collect[Numerator[Together[H2]]/(δ λ),b]/.cEri;
cofi=CoefficientList[φb,b](*Coefficients of φ(b)*);
Print["value of φ(b) at crit b is "]
φb/.b→bcrit/.cEri//N
cb=NSolve[(H2//.cEri)==0,b,WorkingPrecision→20]
bM=Max[Table[Re[b/.cb[[i]]],{i,Length[cb]}]];
Print["bH=",bH=N[bM,30]]
```

$J(E_*)$ is

Out[242]//MatrixForm=

$$\begin{pmatrix} \frac{2}{87-87b} & \frac{2}{87-87b} & \frac{1}{2-2b} \\ 1-\frac{258}{169+87b} & -\frac{1}{128} & \frac{1}{2(-1+b)} \\ -1+\frac{258}{169+87b} & \frac{b}{128} & \frac{b}{2-2b} \end{pmatrix}$$

$$a_1 = \frac{65}{128} + \frac{91}{174(-1+b)}, \quad a_2 = \frac{-236553 + (478354 - 225417b)b}{5568(-1+b)^2(169+87b)}, \quad a_3 = \frac{87 - \frac{2}{-1+b}}{22272}$$

$$H2(b0) = -\frac{1}{256} + \frac{K\beta \left(-\frac{13695976827}{256K\beta+87\delta} + \frac{5963776K^2\beta^2+13781248K\beta\delta-76919571\delta^2}{\delta^3} \right)}{992083968}$$

Denominator of H2 is $62005248(-1+b)^3(169+87b)$

value of $\phi(b)$ at crit b is

Out[253]= 2.01216×10^8

Out[254]= $\{ \{b \rightarrow -61.53327023855142092\}, \{b \rightarrow 1.3349402618015729010\},$
 $\{b \rightarrow 0.7834975171549898868\}, \{b \rightarrow 0.0013985515488811205258\} \}$

bH=1.3349402618015729010

Numerical solution of the stability (Bifurcation diagram)

In[303]:=

```

cond=cF1;
Print["roots of Dis[b]=0:", bcE=NSolve[(Dis//.cond)==0,b]]
Print["b0=",bc=b/.Solve[R0==1,b][[1]]//.cond//N]
bL=100; max=2;
bc1=b/.bcE[[5]];
bc2=b/.bcE[[6]];
lin1=Line[{{bc1,0},{bc1,max}}];
li1=Graphics[{Thick,Black,Dashed,lin1}];
lin2=Line[{{bc2,0},{bc2,max}}];
li2=Graphics[{Thick,Black,Dashed,lin2}];
lin3=Line[{{bc,0},{bc,max}}];
li3=Graphics[{Thick,Black,Dashed,lin3}];
p1a=Plot[{ym} /. cond, {b,0,bc1},PlotStyle->{Dashed,Thick,Green},
PlotRange->All,PlotLegends->{"E_ unstable"}];
p1b=Plot[{ym} /. cond, {b,bc1,bL},PlotStyle->{Green},PlotRange->All,
PlotLegends->{"E_ stable"}];
p01=Plot[0, {b,0,bc},PlotStyle->{Magenta},PlotRange->All,PlotLegends->{"E_1 stable"}];
p02=Plot[0, {b,bc,bL},PlotStyle->{Blue},PlotRange->All,PlotLegends->{"E_1 unstable"}];
pp=Plot[{yi} /. cond, {b,0,bL},PlotStyle->{Purple},PlotRange->All,
PlotLegends->{"E_i unstable"}];
pm=Plot[{yp} /. cond, {b,0,bL},PlotStyle->{Red},PlotRange->All,
PlotLegends->{"E_+ unstable"}];
ps1=Plot[{ys} /. cond, {b,0,13.45},PlotStyle->{Orange,Dotted},
PlotRange->{{0,200},{0,max}},
PlotLegends->{"E_* outside the domain"}];
ps2=Plot[{ys} /. cond, {b,13.45,bL},PlotStyle->{Orange,Thick,Dashed},
PlotRange->{{0,200},{0,max}},
PlotLegends->{"E_* unstable"}];

pyb=Plot[{yb} /. cond, {b,0,bL},PlotStyle->{Dashed,Thick,Cyan},
PlotRange->{{0,200},{0,max}},PlotLegends->{"y_b "}]];
Print["y*'(b0)=",D[ys,b]/.b->bc//.cond//N//FullSimplify]
Chop[ys/.b->bc//.cond//N] (*Check*)];
bifE2=Show[{p01,p02,ps1,ps2,pyb,li3},PlotRange->{{0,3},{0,1}},
Epilog->{Text["b0",Offset[{10,11},{bc//.cond,0}]],{PointSize[Large],
Style[Point[{bc//.cond,0}],Black]}},
PlotRange->All,AxesLabel->{"b","y_ee"}]
Export["bifEE.pdf",bifE2]

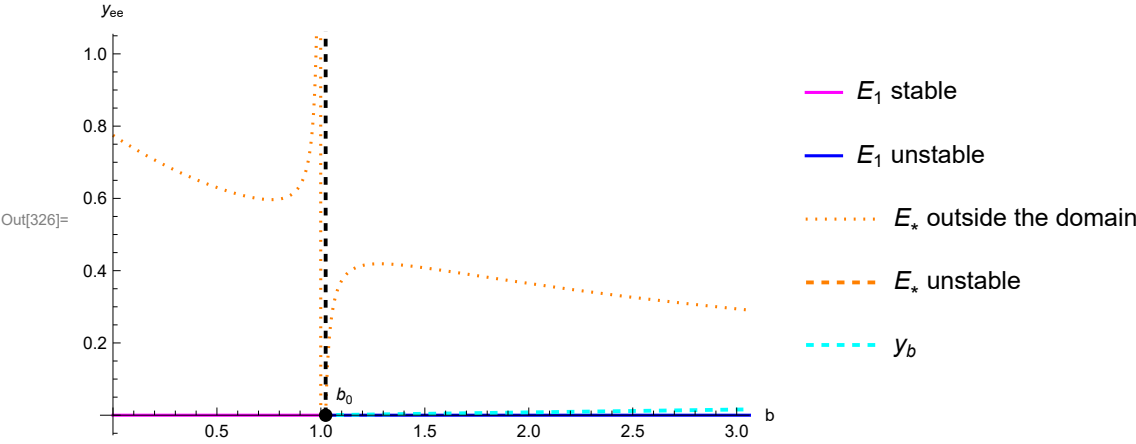
```

roots of Dis[b]=0:

{ {b → -152.24}, {b → -63.}, {b → -63.}, {b → -25.6124}, {b → 27.2273}, {b → 30.5559} }

b0=1.02299

y*'(b0)=21.5814



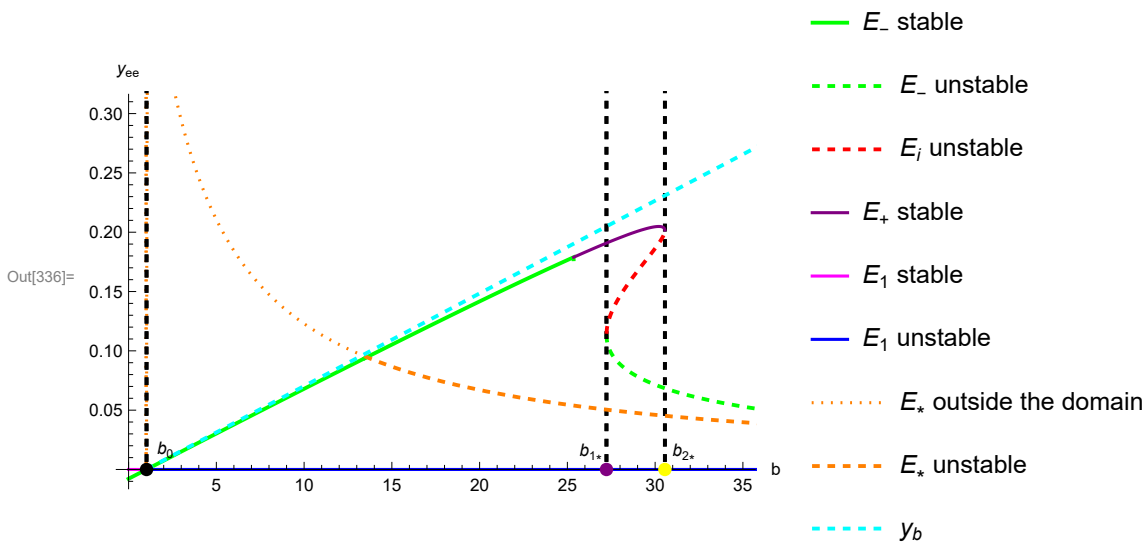
Out[327]= bifEE.pdf

In[328]=

```

p1a=Plot[{ym} /. cond, {b, 0, 13.45}, PlotStyle -> {Thick, Green}, PlotRange -> All];
p1c=Plot[{ym} /. cond, {b, 13.45, bc1}, PlotStyle -> {Thick, Green},
PlotRange -> All, PlotLegends -> {"E_ stable"}];
p1d=Plot[{ym} /. cond, {b, bc1, bL}, PlotStyle -> {Dashed, Thick, Green}, PlotRange -> All,
PlotLegends -> {"E_ unstable"}];
pi=Plot[{yi} /. cond, {b, 0, bL}, PlotStyle -> {Dashed, Thick, Red}, PlotRange -> All,
PlotLegends -> {"E_i unstable"}];
pp1=Plot[{yp} /. cond, {b, 0, bL}, PlotStyle -> {Purple}, PlotRange -> All,
PlotLegends -> {"E_+ stable"}];
shon=Show[{p1a, p1c, p1d, li1, li2}, PlotRange -> {{0, 60}, {0, 0.2}}, Epilog ->
{{Text["b1*", Offset[{-8, 10}, {bc1, 0}], {PointSize[Large],
Style[Point[{bc1, 0}], Purple]}, Text["b2*", Offset[{10, 10}, {bc2, 0}],
{PointSize[Large], Style[Point[{bc2, 0}], Yellow]}]
}, AxesLabel -> {"b", "yee"}];
shoip=Show[{pi, pp1}, PlotRange -> All, AxesLabel -> {"b", "yee"}];
Bnip=Show[shon, shoip];
EriB=Show[shon, shoip, bifE2, PlotRange -> {{0, 35}, {0, 0.3}}, Epilog -> {
Text["b0", Offset[{10, 11}, {bc /. cond, 0}], {PointSize[Large],
Style[Point[{bc /. cond, 0}], Black]},
Text["b1*", Offset[{-8, 10}, {bc1, 0}], {PointSize[Large],
Style[Point[{bc1, 0}], Purple]},
Text["b2*", Offset[{10, 10}, {bc2, 0}], {PointSize[Large],
Style[Point[{bc2, 0}], Yellow]}]}]
Export["Bnip.pdf", Bnip]
Export["EriB.pdf", EriB]

```



Out[337]= Bnip.pdf

Out[338]= EriB.pdf