Convex Portfolio Optimization

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Abstract

In this paper we explore using the Markowitz portfolio optimization method on S&P 500 data from recent years. Our prediction is that more volatile years such as 2020 will result in a worse performance using the optimal policy. However, from our experiments we found nothing of the sort. We found that the performance of our policy was more closely correlated with the minimum requested return as well as whether or not we allowed for negative percentages in our portfolio, also known as shorting.

1 Introduction

1.1 Motivation

Any experienced investor would be familiar with the risk-return trade-off of investing: higher return investment strategies are increasingly risky. Investors aim to find the optimal strategy that maximizes their expected return on investment while accounting for the risk-return trade off by looking at the range of strategies that do not exceed their risk tolerance. With this in mind, we explore using convex optimization to find the optimal asset portfolio with an expected return that meets or exceeds our expectations while minimizing risk. We are also interested in seeing the actual returns of these portfolios if they were used in action through back-testing on S&P 500 stocks. One could similarly seek to maximize return given a maximum risk, or seek to maximize a weighted combination of risk and return. However, for our experimentation we settled on minimizing risk constrained by a minimum return as we can easily compare the change in risk levels for increasingly high performing portfolios.

1.2 Previous Work

Given N assets, we seek to invest a percentage of our capital in each asset to create a weighted portfolio that meets our expected return and minimizes risk. Assets are very volatile so to monetize on these fluctuations in price we make purchases on assets that we expect to increase in price and short assets that we expect to drop. As established by Harry Markowitz, the change in price of each asset can be treated as a random variable so our portfolio is a weighted sum of random variables, P, where the expected return and risk of our portfolio are akin to the expected output and variance of P (Markowitz, 1952). Letting X represent a vector of random variables for each asset, then

$$P = \sum_{i=1}^{N} w_i X_i \Rightarrow E[P] = \sum_{i=1}^{N} w_i E[X_i] = w^T \bar{p}$$

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Where w_i and $\bar{p}_i$ are the percentage of our portfolio invested and expected change in price of asset i. Additionally,

$$var(P) = E[P^2] - E[P]^2 = E[\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j X_i X_j] - \sum_{k=1}^{N}\sum_{l=1}^{N} E[w_k X_k]E[w_l X_l]$$

$$= \sum_{i=1}^{N} w_i^2(E[X_i^2] - E[X_i]^2) + 2\sum_{j=1}^{N}\sum_{k>j} w_j w_k(E[X_j X_k] - E[X_j]E[X_k])$$

$$= \sum_{i=1}^{N} w_i^2 var(X_i) + 2\sum_{j=1}^{N}\sum_{k>j} w_j w_k cov(X_j, X_k) = w^T \Sigma w$$

Whereas $\Sigma \in S_n$ is a co-variance matrix where $\Sigma_{i,j}$ is the co-variance between asset i and j. These are the two necessary equations for us to construct our primal problem and begin our optimization.

1.3 Contribution

In this paper we wish to analyze the accuracy of this method on recent market data. With the onset of coronavirus, the market has been particularly volatile in recent years. We wish to compute the efficacy of these methods over different time frames and compare the results. We also seek to compare the results of allowing our policy to place negative weights on our portfolio. A negative weight would mean that the overall predicted percentage change will be negative and taking such a position is often referred to as a short. Thus the short position is quite risky because it can result in other positions that sum to over 100% of our portfolio. We wish to compare policies over different years with allowing and disallowing for short positions.

1.4 Organization

In this paper we will first detail the primal/dual formulations and state the Karush–Kuhn–Tucker conditions. We will then describe our approach to the experiments we ran on S&P 500 data. Finally we will discuss our results, conclusions, and ideas for possible future work.

2 Problem Formulation

For our problem formulation we will define w to be a vector of the percentages of our portfolio we should devote to each asset. We will define $\bar{p}$ to be the expected price change of each asset and Σ to be the co-variance of the price changes for each asset. Finally we define r to be the minimum return on our investment.

2.1 Primal Problem

$$\min\ w^T \Sigma w$$
$$s.t.\ w^T \bar{p} \geq r, w^T 1_n = 1, (w \geq 0)$$

By taking the quadratic product of the co-variance and our asset distribution, we are finding the variance of our entire portfolio. In order to calculate the return, we must weight the expected price change of each asset by its distribution in the portfolio. We then want to minimize variance, or risk, subject to a minimum weighted price change, also known as the return of our portfolio. We also include the constraint that the sum of the weights must sum to one as we want to ensure our portfolio is fully invested. If we do not want to allow shorting then we can simply add the constraint that no weight can be negative.

2.2 Dual Problem

$$\min\ \frac{1}{4}(\lambda\bar{p} + v1_n)^T \Sigma^{-1}(\lambda\bar{p} + v1_n) + \lambda r + v$$
$$s.t.\ \lambda \geq 0$$

The solution to the dual problem will always provide a lower bound for the solution to the primal problem. This is especially useful when the primal problem is non-convex and therefore cannot be solved using a convex optimization algorithm. If the primal problem is convex, then the solution to the dual problem will be exactly equal to the solution to the primal problem. In our case, our primal problem is in fact convex and therefore we did not need to consider the dual problem when finding our solutions.

2.3 KKT Conditions

$$w^{*T}\bar{p} - r \leq 0 \tag{1}$$

$$w^{*T}1_n - 1 = 0 \tag{2}$$

$$\lambda^* \geq 0 \tag{3}$$

$$\lambda^*(w^T\bar{p} - r) = 0 \tag{4}$$

$$2\Sigma w^* + \lambda^*\bar{p} + v^* = 0 \tag{5}$$

The Karush–Kuhn–Tucker (KKT) conditions are the conditions that must be met for any solution to the primal problem or the dual problem. Once you find a solution to the primal problem or the dual problem, you can verify that your solution is valid by plugging it in to these conditions. Equations 1, 2, 3, and 4 are simply a reflection of constraints found in the original problem. Equation 5 states that the gradient of the Lagrangian must be 0 at the optimal solution. Elementary calculus states that critical points occur when the gradient of a function is 0. The optimal value of a given problem will occur when the Lagrangian function for that problem reaches a critical point. Thus in order to have an optimal solution, the gradient of the Lagrangian must be 0.

3 Approach

The SP 500 is an index that tracks the 500 largest assets on the New York Stock Exchange. We decided to use this set of assets for our data set because we believe that they provide a good representation of the market as a whole during a given time frame. We recorded the price c for each asset from yahoo finance for a time range of Δt starting at time t. First we split the data into the two time periods $[t, t + \Delta t/2]$ and $[t + \Delta t/2, t + \Delta t]$. We use the first time period to compute the expected returns for each stock and the co-variance matrix that are needed to train the model, and then use the second time period to evaluate our results. To calculate the predicted returns, we first compute the percent change $p_{i,k}$ for each asset i and week k in the first time period by computing the change in price of the current week to the next week.

$$p_{i,k} = (c_{i,k+1} - c_{i,k})/c_{i,k}$$

The expected return of asset i is the average weekly change in price, $\bar{p}_i = \sum_{k=1}^{\Delta t} p_{i,k}/\Delta t$. We can then form our percentage matrix P by stacking each p_i vector, then we simply derive Σ and $\bar{p}$ that are needed for our primal equation:

$$\bar{p} = avg(P)$$
$$\Sigma = cov(P)$$

We now have all the necessary components to train our model, but in order to measure the effectiveness of our optimization method we will compare the predicted percentage return of our portfolio with the actual return of our portfolio over the second half of the given time period. We can calculate the actual return a_i of asset i as follows:

$$a_i = (c_{i,t+\Delta t/2} - c_{i,t+\Delta t})/c_{i,t+\Delta t/2}$$

Finally, when we compute the optimal portfolio weights w we can compare the expected $\bar{p}^T w^*$ and the actual returns $a^T w^*$ of our policy.

4 Results

4.1 General Trends

As shown in figure 1 and figure 2, we ran tests for years 2016 to 2020 without allowing for shorting and with allowing for shorting. The minimum requested return is on the x axis and the expected and

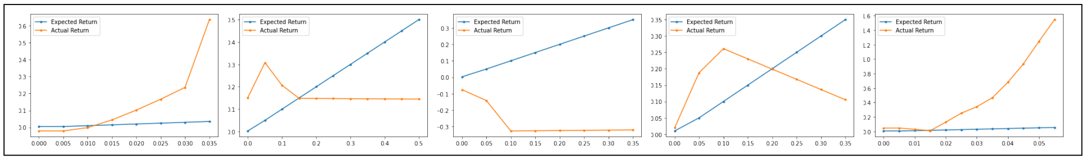

Figure 1: S&P 500 Without Shorting: 2016, 2017, 2018, 2019, 2020

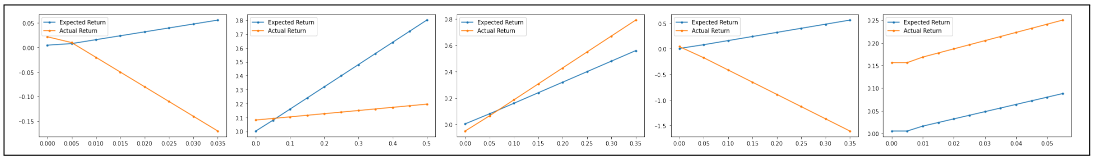

Figure 2: S&P 500 With Shorting: 2016, 2017, 2018, 2019, 2020

actual returns are shown in blue and orange respectively. It appears that our model does particularly well with relatively low minimum returns. We believe this is because the model can produce a safer policy that corresponds to a lower variance in outcomes. It also appears that the model is more accurate when we do not allow for shorting. Because the model cannot short assets to instantly gain more capital, it is forced to work with what it has and therefore make safer bets.

4.2 Inflection Point

In the experiments we ran, there often appears to be an inflection point where the actual return crosses the expected return. For every experiment we ran that contains an inflection point, the expected return is much closer to the actual return before this inflection point. We believe that this is when the minimum requested return becomes too high, the positions become too risky, and the model is no longer able to ensure an accurate result. As we can see the inflection point appears to occur a lot earlier when we allow for short positions than when we disallow for short positions.

4.3 Future Work

In the future we might want to explore other methods for computing our co-variance matrix. The main issue with the feasibility of this problem is the fact that it is extremely difficult to predict the co-variance of stocks over a given time frame. To expand on this project we might want to explore other methods of finding the co-variance such as using a Markov regime or a regression whitener.

5 References

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