Option pricing and Hedging using Monte Carlo Simulation

A Project report submitted to
Indian Institute of Technology Hyderabad
Department of Mathematics
Probability Theory in Finance

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Abstract

In modern finance, accurately pricing options and managing risks is essential. This project models stock prices using Geometric Brownian Motion (GBM) and prices options with the Black-Scholes model, which offers a closed-form solution under ideal conditions. When exact solutions are difficult, we apply Monte Carlo Simulation, generating multiple stock price paths to estimate option values. To improve accuracy and reduce computational effort, variance reduction techniques like Stratified Sampling is used. These methods help achieve more reliable pricing with fewer simulations.

We further enhance the simulation using Quasi-Monte Carlo methods, replacing random sampling with low-discrepancy sequences like Sobol, leading to faster and smoother convergence. Additional variance reduction within Quasi-Monte Carlo, such as Owen Scrambling optimizes results. Using simulation outputs, we compute Greeks (Delta, Vega, Theta), helping traders hedge risks effectively. This project compares Black-Scholes, Monte Carlo, and Quasi-Monte Carlo methods, demonstrating that Quasi-Monte Carlo provides faster, more stable pricing and hedging, offering practical advantages for traders managing real-world portfolios.

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1 Problem Statement

In the options market, traders constantly seek to identify whether an option is fairly priced. Mispricing can lead to profitable trading opportunities or significant financial risk. A trader must estimate the fair value of an option based on theoretical models before making decisions to buy or sell.

However, real-world factors like stock price randomness (Brownian motion), market volatility, and time decay complicate fair valuation. Traditional models like the Black-Scholes formula offer closed-form solutions but rely on assumptions. Monte Carlo methods offer flexibility but can suffer from high variance. Therefore, combining different simulation techniques and variance reduction methods is essential for accurate, efficient option pricing.

Objective: Develop a robust model to fairly price European options using:

- Black-Scholes Model
- Monte Carlo Simulation
- Quasi-Monte Carlo Simulation (with improved efficiency)

  and use this fair value to help traders make buy/sell decisions.

2 Data Collection of Stocks

2.1 HDFC Bank

One of India's largest and most trusted private sector banks, offering services in retail banking, corporate banking, and wealth management. It is known for its strong financial performance, consistent growth, and focus on digital banking innovations.

Source : yahoofinance

Date	Close	High	Low	Open	Volume
2024-03-28	1428.3947	1440.8249	1421.2916	1421.2916	27796071
2024-04-01	1450.6902	1453.9458	1435.9909	1438.3586	12599785
2024-04-02	1460.2102	1474.7121	1443.5378	1445.4123	20612723
2024-04-03	1462.3313	1475.5014	1451.5781	1452.2687	2272193
2024-04-04	1507.0210	1509.3887	1483.7389	1483.7389	44467533

Table 1: Stock Data for Selected Dates

3 From Market Data to Risk Control: The Full Journey of Option Pricing

3.1 What is an Option? (Call/Put)

An **option** is a type of financial contract that gives the holder the right (but not the obligation) to **buy** or **sell** an asset at a predetermined price (*strike price*) before or at a specific time (*expiry date*).

3.2 Call Option

Right to **buy** an asset at the strike price.
⇒ Traders buy call options if they expect the price of the asset to rise.

3.3 Put Option

Right to **sell** an asset at the strike price.
⇒ Traders buy put options if they expect the price of the asset to fall.

Options are widely used for **speculation** (betting on price movements) or **hedging** (protecting portfolios against losses).

4 Why Option Pricing is Important

Accurately pricing an option is crucial because:

- It helps traders and investors determine whether the option is fairly priced, overvalued, or undervalued in the market.
- Incorrect pricing can lead to wrong trading decisions and financial losses.
- Traders use the theoretical price to compare with the market price and decide whether to buy, sell, or hold an option.

For example:

If the market price of a call option is Rs.50 but the model shows it should be Rs.55, the option is undervalued — a good opportunity to buy.

5 Basic Idea of Randomness in Stock Price Movement → Brownian Motion

Stock prices do not move in a straight line — they are random and unpredictable. This randomness can be modeled mathematically using **Brownian Motion**.

Brownian Motion is a continuous random movement where future movements are independent of the past.

In finance, **Geometric Brownian Motion (GBM)** is used to model stock prices because prices can't be negative, and they grow exponentially over time.

Mathematically, under GBM, the stock price $S(t)$ follows:

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t)$$

Where:

- μ = expected return (drift)
- σ = volatility (randomness)
- $W(t)$ = standard Brownian motion (Wiener process)

In simple words: Stock prices move up or down every instant randomly, but on average, they may trend upward (drift) with some fluctuations (volatility).

This randomness is the foundation for building models like **Monte Carlo Simulation**, **Black-Scholes Model**, and **Quasi-Monte Carlo Simulation** to predict option prices and manage risk.

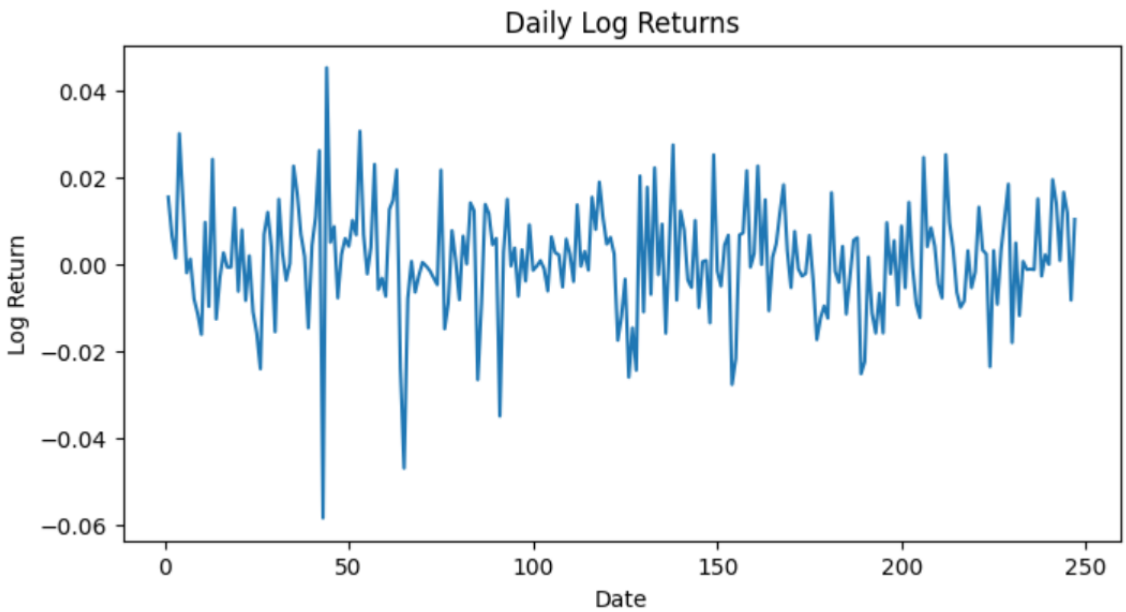
Calculate daily log returns

To model stock price movements using Geometric Brownian Motion (GBM), we first need to understand how returns behave over time. A common way to analyze stock returns is by calculating daily log returns.

Log returns are preferred in finance because they make calculations easier and have nice mathematical properties — for example, they can be easily added across time periods. We calculate log returns by taking the natural logarithm of the ratio of today's closing price to yesterday's closing price.

After calculating these returns, we plot them over time. This plot of daily log returns helps visualize how much and how randomly the stock price fluctuates each day. If the log returns appear to fluctuate randomly around zero, it supports the assumption that stock prices can be modeled as a random process like Brownian Motion.

This is an important step because in Geometric Brownian Motion, we assume that the stock's log returns are normally distributed and independent across time.



Estimation of Parameters for Geometric Brownian Motion (GBM)

After calculating the daily log returns, the next step is to estimate two key parameters needed for **Geometric Brownian Motion (GBM)**:

- **Drift** (μ): This represents the expected average return of the stock over time. It captures the general upward (or downward) trend in the stock price.
- **Volatility** (σ): This measures how much the stock price fluctuates around the drift. It reflects the uncertainty or risk associated with the stock.

To estimate these parameters, we calculate:

- The **mean** of the daily log returns (multiplied by 252 to annualize it, since there are about 252 trading days in a year) for the drift.
- The **standard deviation** of the daily log returns (multiplied by the square root of 252) for the annualized volatility.

These two values — drift and volatility — are then used to simulate future stock price paths under the GBM model. Estimating them accurately is crucial because they define the behavior of the simulated stock prices.

Simulating Stock Prices using Geometric Brownian Motion (GBM)

Once we have estimated the drift (μ) and volatility (σ), we can simulate the future path of a stock price using the Geometric Brownian Motion (GBM) model.

First, we set up the simulation environment by defining:

- The total time period (1 year),
- The number of trading days (approximately 248–252),

- The initial stock price (the most recent closing price).

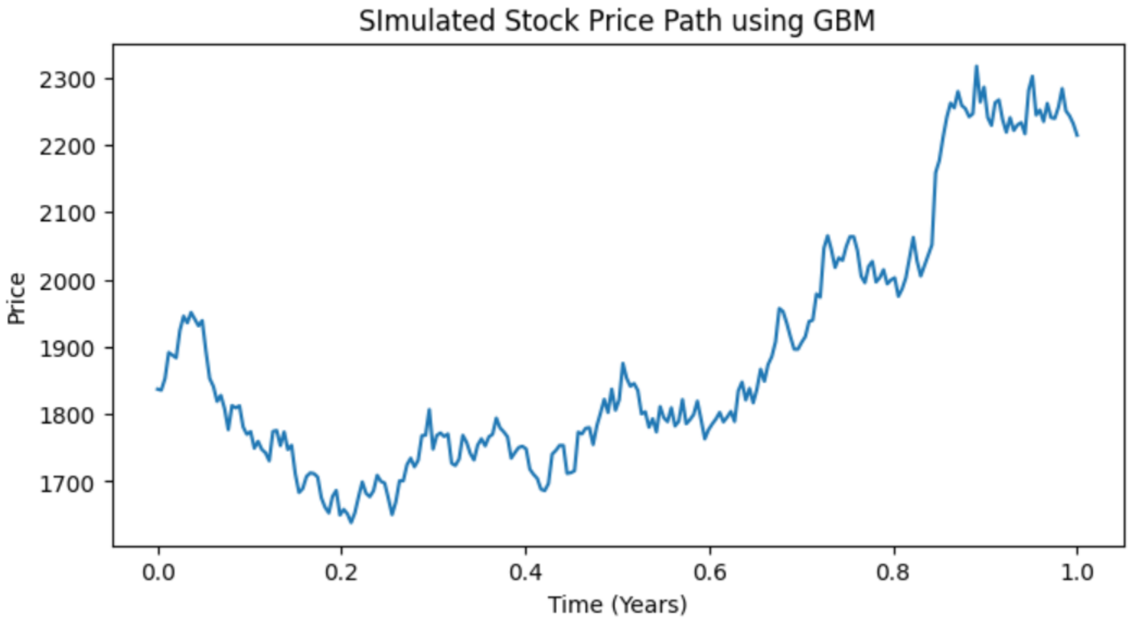
Then, we generate random movements (called Brownian increments) to model the randomness in stock prices. By taking the cumulative sum of these random increments, we create a random path known as **Brownian Motion** $W(t)$.

Using the GBM formula:

$$S(t) = S_0 \times \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right],$$

we simulate the stock price over time.

Finally, the simulated stock price path is plotted, which shows how the stock price could evolve randomly under the GBM model. This helps in understanding potential future stock movements and forms the basis for pricing options through simulation methods.



Black-Scholes Model for Option Pricing

The Black-Scholes model is a mathematical model used to estimate the fair price of European call and put options. It assumes that stock prices follow a random process known as **Geometric Brownian Motion (GBM)**.

Under GBM, the stock price $S(t)$ evolves according to the stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

where:

- μ is the drift (expected return),
- σ is the volatility (randomness in returns),
- $W(t)$ is the standard Brownian motion (Wiener process).

The Black-Scholes formula derives the option price by solving this stochastic process under a risk-neutral measure, where the drift μ is replaced by the risk-free rate r. The idea is that, in a risk-neutral world, all assets grow at the risk-free rate.

The Black-Scholes formulas for the price of a European Call and Put option are:

$$C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$
$$P = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

where:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

and:

- C = Call option price,
- P = Put option price,
- S_0 = Current stock price,
- K = Strike price,
- r = Risk-free interest rate,
- T = Time to maturity (in years),
- $\Phi(\cdot)$ = Cumulative distribution function (CDF) of the standard normal distribution.

In summary: The Black-Scholes model uses the randomness captured by Brownian Motion to predict the probability distribution of future stock prices, and then computes the present value of expected payoffs to determine the option's fair price. This approach helps traders and investors make informed decisions when buying or selling options.

Theoretical Option Pricing using Black-Scholes Model

Using the Black-Scholes model under ideal market conditions (i.e., no transaction costs, continuous trading, constant volatility, and a risk-free interest rate), the theoretical price of a European call option was calculated.

Based on the given input parameters:

- Initial Stock Price (S_0): \$1428.39
- Strike Price (K): \$1825.35
- Risk-Free Rate (r): 5% per annum
- Volatility (σ): 20.46% per annum
- Time to Maturity (T): 1 year

the **Black-Scholes analytical price** for the European call option was found to be:

$$\text{Call Price} = 29.1757$$

This value represents the fair price of the option under the assumptions of the Black-Scholes model. It assumes that markets are perfectly efficient, the underlying asset follows a Geometric Brownian Motion with constant drift and volatility, and there are no dividends paid during the life of the option.

The result serves as a theoretical benchmark for comparison with real-world market prices and highlights how idealized models can be used for pricing and risk management in finance.

Option Pricing using Monte Carlo Simulation

Monte Carlo simulation is a numerical method used to estimate the price of options, especially when analytical solutions (like the Black-Scholes formula) are difficult or impossible to apply. It relies on simulating a large number of possible future stock price paths and calculating the corresponding option payoffs.

Steps Involved

1. **Model the Stock Price Dynamics:** The stock price is assumed to follow a **Geometric Brownian Motion (GBM)**, described by:

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$$

   where μ is the drift (expected return), σ is the volatility, and $W(t)$ is a standard Brownian motion.

2. **Simulate Random Paths:** For each simulation, the stock price at maturity T can be approximated as:

$$S(T) = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right)$$

   where:

   - S_0 is the initial stock price,
   - r is the risk-free interest rate,
   - Z is a standard normal random variable ($Z \sim \mathcal{N}(0, 1)$).

   This formula incorporates the randomness via Z and models how the stock price evolves according to Brownian motion.

3. **Calculate Payoffs:** For a European call option, the payoff at maturity for each simulated path is:

$$\text{Payoff} = \max(S(T) - K, 0)$$

   where K is the strike price.

4. **Estimate the Option Price:** The option price is the discounted average of all simulated payoffs:

$$\text{Call Price} = e^{-rT} \times \text{Average Payoff}$$

   The discount factor e^{-rT} accounts for the time value of money.

Advantages of Monte Carlo Simulation

- It can handle more complex features like path-dependent options (e.g., Asian or Barrier options).
- It is flexible and can model different types of stochastic processes beyond the simple GBM.

Limitations

- It is computationally intensive and requires a large number of simulations for accurate results.
- Accuracy depends on the number of simulated paths and quality of random number generation.

Monte Carlo simulation provides a powerful and flexible approach to option pricing by explicitly modeling the randomness in stock price movements using Brownian motion. While it may not be as fast as analytical solutions like the Black-Scholes formula, it is particularly useful when dealing with more complex financial instruments where no closed-form solutions exist.

Monte Carlo Simulation in Option Pricing

The purpose of this simulation is to estimate the price of both a European call and put option using Monte Carlo methods, enhanced with stratified sampling. This approach aims to reduce the variance of the estimation, making the results more accurate compared to standard Monte Carlo simulations. Below is a detailed breakdown of each step involved in this process.

1. Parameter Setup

The parameters required for this simulation are based on the Geometric Brownian Motion (GBM) model used for modeling asset prices and pricing options. The specific parameters are as follows:

- **Initial Stock Price (S_0)**: The starting price of the underlying asset (stock), which is 1428.39.
- **Strike Price (K)**: The price at which the option holder can exercise the option, which is 1825.35.
- **Risk-free Interest Rate (r)**: The rate at which money can be borrowed or lent without risk, assumed to be 5% (0.05) per annum.
- **Volatility ()**: The annualized standard deviation of the stock's returns, given as 20.46% (0.2046).
- **Time to Maturity (T)**: The time remaining until the option expires, given as 1 year.
- **Number of Paths (M)**: The number of simulations to run for the option pricing, set to 10,000. A higher number of paths improves the accuracy of the estimate.

These parameters are essential for simulating the stock price dynamics under the GBM model and for calculating the payoff of the European options.

2. Stratified Sampling

Traditional Monte Carlo simulations rely on random sampling to generate the inputs for the simulation. Stratified sampling is an enhancement to this process, where the range of the random variable is divided into several intervals, and samples are drawn from each interval. This approach ensures a more even coverage of the entire sample space, reducing the variance of the estimation.

- **Uniform Sampling Over Intervals**: In this approach, the range [0,1] is divided into M intervals. A random value within each interval is selected to generate stratified samples.

- **Inverse CDF Transformation**: The uniform samples generated from stratified sampling are then transformed into standard normal samples using the inverse cumulative distribution function (CDF) of the normal distribution. This is done to match the normal distribution required for simulating the stock price dynamics in the GBM model.

The stratified sampling ensures that the sampling process is more uniform and reduces the variance in the simulated stock price paths compared to purely random sampling.

3. Simulate Terminal Stock Prices Using GBM

Once the stratified samples have been transformed into standard normal variables, the Geometric Brownian Motion (GBM) model is used to simulate the terminal stock prices. The GBM model is widely used for modeling the dynamics of asset prices, and is given by the following formula:

$$S_T = S_0 \cdot \exp\left(\left(r - \frac{1}{2}\sigma^2\right) \cdot T + \sigma \cdot \sqrt{T} \cdot Z\right)$$

Where:

- S_T: The terminal stock price at maturity T.
- S_0: The initial stock price.
- r: The risk-free interest rate.
- σ: The volatility of the stock.
- T: The time to maturity (1 year in this case).
- Z: The standard normal random variables generated using the stratified sampling method.

This formula models the asset price as a continuous process, where the price evolves over time with a deterministic drift term and a stochastic volatility term. The random component Z is drawn from the standard normal distribution and represents the random fluctuations in the stock price.

4. Payoff Calculation for European Call and Put Options

For each simulated terminal stock price S_T, the payoff of the European option is calculated. The payoff for a European call option is given by:

$$\text{Payoff} = \max(S_T - K, 0)$$

Where:

- S_T: The simulated terminal stock price.
- K: The strike price of the option.

The payoff is zero if the stock price at maturity is less than the strike price (out-of-the-money), and it is the difference between the stock price and strike price if the stock price exceeds the strike price (in-the-money).

For a European put option, the payoff is calculated as:

$$\text{Payoff} = \max(K - S_T, 0)$$

The put option's payoff is zero if the stock price is above the strike price, and it is the difference between the strike price and the stock price if the stock price is below the strike price.

5. Discounted Expected Payoff

The expected payoff is then discounted to present value using the risk-free interest rate r. The formula for the discounted expected payoff is given by:

$$\text{Option Price} = \exp(-r \cdot T) \cdot \text{mean(payoffs)}$$

Where:

- $\exp(-r \cdot T)$: This factor discounts the payoff to its present value, accounting for the time value of money.
- mean(payoffs): The average of all simulated payoffs, which represents the expected payoff of the option.

This results in the estimated price of the European call or put option, which is the present value of the expected payoff.

6. Result Display

Finally, the results give us the estimated prices for the European call option based on the Monte Carlo simulation : 29.1803

Quasi-Monte Carlo Simulation: Concept and Applications

Introduction

Quasi-Monte Carlo (QMC) methods are an enhancement over traditional Monte Carlo (MC) simulations. While standard Monte Carlo simulations use purely random numbers to simulate stochastic processes, Quasi-Monte Carlo methods use specially constructed deterministic sequences called **low-discrepancy sequences**, such as **Sobol** or **Halton** sequences.

The key idea is to *cover the sample space more uniformly* than random numbers, leading to faster convergence and more accurate results.

Working of Quasi-Monte Carlo Simulation

In Quasi-Monte Carlo simulation:

- Instead of using random numbers, a low-discrepancy sequence is generated.
- The sequence is optionally scrambled to introduce randomness and prevent pattern formation.
- These sequences are transformed from a uniform distribution $[0, 1]$ to a standard normal distribution, which is required for simulating Brownian motion.
- The normal samples are then used to simulate paths of the underlying stochastic process, such as Geometric Brownian Motion for stock prices.
- Finally, payoffs (for example, option payoffs) are calculated and discounted back to obtain present values.

Advantages of Quasi-Monte Carlo Simulation

- **Faster Convergence**: QMC methods converge at a rate of approximately $\mathcal{O}(1/N)$, compared to the slower $\mathcal{O}(1/\sqrt{N})$ rate of standard Monte Carlo, where N is the number of samples.
- **More Accurate Results**: By covering the sample space more evenly, QMC reduces gaps and clustering effects seen in random sampling, leading to more reliable estimates.
- **Reduced Variance**: Combining QMC with techniques like scrambling and variance reduction (e.g., antithetic variates) further improves the stability of the simulation results.
- **Efficiency in High Dimensions**: Although traditional QMC methods perform best in moderate dimensions, modern versions like scrambled Sobol sequences work well even for complex, high-dimensional problems.

Applications of Quasi-Monte Carlo Simulation

Quasi-Monte Carlo methods are widely used in areas requiring the simulation of stochastic processes, especially when accuracy and speed are critical:

- **Financial Engineering**: Pricing complex derivatives like European and American options, mortgage-backed securities, credit derivatives, and risk management (Value-at-Risk calculations).
- **Insurance**: Simulation of policyholder behaviors, claim distributions, and solvency analysis.
- **Physics and Engineering**: Solving high-dimensional integration problems in fields such as particle physics and structural reliability analysis.
- **Computer Graphics**: Rendering techniques such as ray tracing rely on QMC to sample light paths more efficiently.

Quasi-Monte Carlo Simulation for Option Pricing

The goal of this simulation is to estimate the price of a European call option using Quasi-Monte Carlo (QMC) methods, specifically employing a 1D Sobol sequence with Owen scrambling. Below is a detailed breakdown of each step of the process, which allows for a more accurate and efficient estimate of the option price compared to standard Monte Carlo methods.

1. Parameter Setup

The parameters used in the Geometric Brownian Motion (GBM) model are as follows:

- **Initial Stock Price (S_0)**: The starting price of the underlying asset (stock), which is given as 1428.39.
- **Strike Price (K)**: The price at which the option holder can exercise the option, which is 1825.35.
- **Risk-free Interest Rate (r)**: The rate at which money can be borrowed or lent without risk, assumed to be 5% (0.05) per annum.
- **Volatility ()**: The annualized standard deviation of the stock's returns, given as 20.46% (0.2046).
- **Time to Maturity (T)**: The time remaining until the option expires, given as 1 year.
- **Number of Paths (M)**: The number of simulations to run for the option pricing, set to 10,000. More paths improve accuracy.

These parameters are necessary for the Geometric Brownian Motion (GBM) model, which is commonly used to model the price of financial assets, and for pricing derivatives like options.

2. 1D Sobol Sequence with Owen Scrambling

In traditional Monte Carlo simulations, random samples are drawn from a uniform distribution. However, for improved accuracy and faster convergence, Quasi-Monte Carlo (QMC) methods are used, which rely on low-discrepancy sequences like Sobol sequences. These sequences are designed to cover the space more evenly than purely random samples, reducing the variance in the estimate.

- **Sobol Sequence**: A low-discrepancy sequence that is used to generate quasi-random numbers in a more uniform manner.
- **1D Sobol Sequence**: In this case, the Sobol sequence is used for just the terminal time step (i.e., the final price of the asset at maturity) in the simulation. This means that instead of using a multidimensional sequence for all time steps, the Sobol sequence is applied to only the final time step for simplicity and efficiency.
- **Owen Scrambling**: This technique is applied to the Sobol sequence to further randomize the sample points. Scrambling reduces any inherent regularity or patterns in the sequence, ensuring a more randomized distribution of sample points, which helps in improving the uniformity and independence of the samples. Owen scrambling ensures that the Sobol sequence maintains its low-discrepancy property while also removing any unwanted correlations between samples.

3. Transformation to Normal Distribution (Inverse CDF of Normal)

The Sobol sequence generates samples in the range [0, 1], which are uniformly distributed. Since the GBM model requires normal (Gaussian) random variables for the Brownian motion component, the samples are transformed using the inverse CDF of the normal distribution (also known as the quantile function or norm.ppf).

This transformation converts the uniformly distributed Sobol samples into samples from a standard normal distribution (mean = 0, standard deviation = 1), which can be used for modeling the stochastic component of the GBM process.

4. Simulate Terminal GBM Values

Once the normal random variables are generated, the Geometric Brownian Motion (GBM) model is used to simulate the terminal stock price at maturity:

$$S_T = S_0 \cdot \exp\left(\left(r - \frac{1}{2}\sigma^2\right) \cdot T + \sigma \cdot \sqrt{T} \cdot Z\right)$$

Where:

- S_T: The terminal stock price at time T (maturity).
- S_0: The initial stock price.
- r: The risk-free interest rate.
- σ: The volatility of the stock.

- T: The time to maturity (1 year in this case).
- Z: The normal random variables generated using the Sobol sequence with Owen scrambling.

The formula reflects the standard approach for modeling asset prices under the GBM assumption, where the stock price evolves as a continuous process influenced by both the drift (expected return) and the random component (volatility).

5. Payoff Calculation for European Call Option

The payoff of a European call option is calculated as:

$$\text{Payoff} = \max(S_T - K, 0)$$

Where:

- S_T: The simulated terminal stock price.
- K: The strike price of the option.

The payoff is zero if the stock price at maturity is less than the strike price (out-of-the-money), and it is the difference between the stock price and strike price if the stock price exceeds the strike price (in-the-money).

6. Discounted Expected Payoff

The expected payoff is then discounted to account for the time value of money, using the risk-free interest rate r. The discounted payoff is given by:

$$\text{Option Price} = \exp(-r \cdot T) \cdot \text{mean(payoffs)}$$

Where:

- $\exp(-r \cdot T)$: This factor discounts the payoff to present value.
- mean(payoffs): The average of all simulated payoffs, representing the expected payoff.

This results in the estimated price of the European call option, which is the present value of its expected payoff.

7. Result Display

Finally, the result of the calculation is displayed, which is the estimated price of the European call option based on the 1D Sobol sequence with Owen scrambling. This price is the final estimate derived from the Quasi-Monte Carlo simulation and represents a more efficient and accurate estimate compared to traditional Monte Carlo methods.

Quasi Monte Carlo Call Price: 29.17

Interpretation of Option Pricing Results

1. Black-Scholes Price

The **Black-Scholes price** represents the theoretical value of a European call option derived from the Black-Scholes model. This model provides a closed-form solution under the following assumptions:

- Constant volatility of the underlying asset.
- Lognormal distribution of stock prices.
- No arbitrage opportunities.
- Continuous trading with no transaction costs.

The Black-Scholes price is regarded as the *true value* in an idealized financial environment without real-world imperfections.

2. Monte Carlo Simulation with Stratified Sampling

This method estimates the option price through Monte Carlo (MC) simulation enhanced by **stratified sampling**, which aims to reduce the variance of the estimate by:

- Dividing the sampling space into equal-probability intervals.
- Sampling once from each interval to ensure uniform coverage.

The result obtained using this method (29.1803) is very close to the Black-Scholes price, reflecting the improved accuracy due to stratification. It achieves better performance compared to standard MC, especially for lower sample sizes.

3. Quasi-Monte Carlo (Sobol + Owen Scrambling)

This approach applies a **Quasi-Monte Carlo** method, using **Sobol sequences**—a type of low-discrepancy sequence—and **Owen scrambling** to enhance randomness:

- Sobol sequences provide evenly spread sample points across the integration space.
- Owen scrambling adds randomness while maintaining the low-discrepancy properties.

The result from this method (29.15) is again very close to the Black-Scholes price, and slightly better in terms of convergence behavior. QMC techniques are known for faster convergence and higher accuracy in multi-dimensional problems.

4. Comparative Analysis

Accuracy:

- The Black-Scholes price serves as the benchmark.
- Both MC with Stratified Sampling and QMC provide very close approximations (29.1803 and 29.17 respectively vs. 29.1757 from Black-Scholes).

Speed of Convergence:

- **QMC (Sobol + Owen Scrambling)** converges faster due to low-discrepancy sequences, requiring fewer samples for high accuracy.
- **MC with Stratified Sampling** reduces variance but generally needs more paths to match QMC's efficiency.

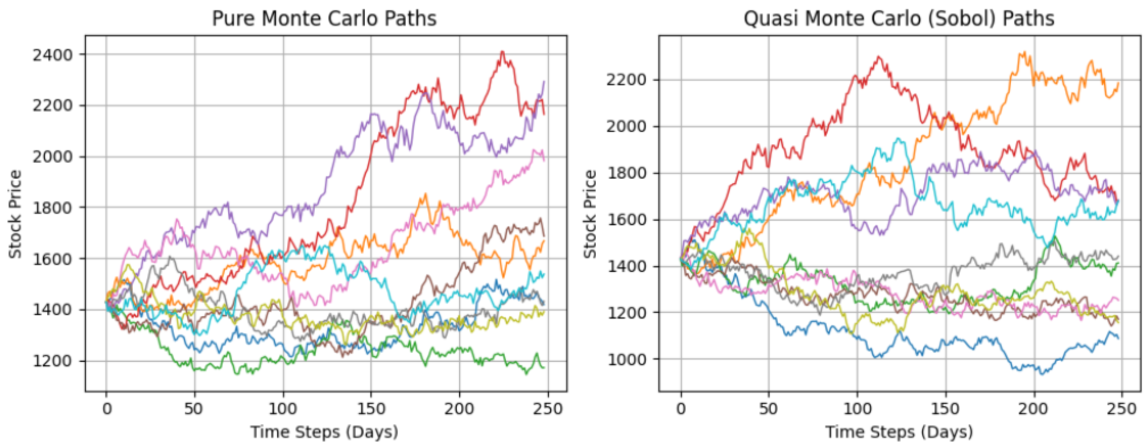
Best for Real-World Use:

- QMC is ideal for high-dimensional, complex problems, such as pricing exotic options or multiple correlated assets.
- MC with Stratified Sampling is suitable for simpler, lower-dimensional problems and still achieves good accuracy.

5. Conclusion

- **QMC (Sobol + Owen Scrambling)** is typically the preferred method for achieving high accuracy and faster convergence in computational finance.
- **MC with Stratified Sampling** remains a robust and efficient choice for simpler models.
- In this case, both methods provide results that are very close to the Black-Scholes price, validating their effectiveness. QMC offers a slightly better result with fewer samples.

Comparison of Monte Carlo vs Quasi Monte Carlo (Sobol) Simulations



Left Plot: Pure Monte Carlo Paths

What it shows:

- Each colored line is a single simulation path of a stock price over time.
- These paths are generated using **standard pseudo-random normal numbers** for the Brownian motion in GBM.

Characteristics:

- **Random behavior:** The paths appear more "noisy" and are spread unevenly.
- **Clustering:** Some paths are bunched together, while others are widely scattered.
- **Inefficient sampling:** Randomness isn't evenly distributed; some regions are over- or under-sampled.
- **Slower convergence:** Leads to higher variance and makes Monte Carlo less efficient for option pricing with fewer samples.

Right Plot: Quasi Monte Carlo (Sobol) Paths

What it shows:

- Each colored line again represents a stock path.
- These paths are generated using **Sobol sequences** — quasi-random low-discrepancy numbers.

Characteristics:

- **More evenly spread paths:** The paths are smoother and more uniformly distributed.

- **Better coverage of scenarios:** Less overlapping or clumping of paths.
- **Less randomness, more structure:** Sobol sequences reduce randomness while preserving variability.
- **Faster convergence:** Fewer simulations are needed for accurate option pricing compared to regular Monte Carlo.

Hedging and the Role of Quasi-Monte Carlo Simulation

Hedging: Concept and Importance

Hedging is a risk management strategy used to offset potential losses in investments by taking an opposite position in a related asset. The goal of hedging is not to generate profit but to **reduce or eliminate risk**.

In financial markets, hedging typically involves the use of derivatives like options, futures, or swaps. For example, an investor holding a stock might buy a put option to protect against a decline in the stock's price. If the stock price falls, the loss in the stock is offset by the gain in the put option.

Mathematically, hedging is often based on calculating the **Greeks** such as **Delta**, **Gamma**, **Vega**, and **Theta**, which measure the sensitivity of an option's price to various factors. **Delta hedging**, for instance, involves adjusting the position in the underlying asset to maintain a portfolio's delta at zero, thus making it insensitive to small movements in the underlying asset's price.

Challenges in Hedging

Effective hedging requires accurate estimation of option prices and sensitivities (Greeks). However, when dealing with complex derivatives or large portfolios, the computational cost and errors associated with standard simulation methods become significant. Traditional Monte Carlo simulations, while useful, often converge slowly and can produce noisy estimates, making real-time hedging adjustments difficult.

Using Quasi-Monte Carlo Simulation for Hedging

Quasi-Monte Carlo (QMC) simulation provides a powerful tool for improving the accuracy and efficiency of hedging calculations.

Why Quasi-Monte Carlo Simulation?

- **Faster Convergence**: QMC simulations converge faster than standard Monte Carlo simulations, reducing the number of simulations needed to achieve a given accuracy.
- **Lower Variance**: Using low-discrepancy sequences (such as Sobol sequences) ensures that the sample space is covered more uniformly, leading to more stable and less noisy estimates of option prices and Greeks.
- **Efficient Calculation of Greeks**: Greeks like Delta and Vega, essential for dynamic hedging, can be computed more accurately with QMC methods, improving hedging effectiveness.

Process of Hedging Using Quasi-Monte Carlo Simulation

The general steps to apply QMC simulation for hedging purposes are:

1. **Generate Sobol Sequence**: Instead of random numbers, generate a Sobol sequence to ensure even coverage of the probability space.

2. **Scramble the Sequence**: Apply scrambling techniques to introduce randomness and avoid structured patterns, which improves the robustness of results.

3. **Transform to Normal Distribution**: Convert the uniformly distributed samples into standard normal variables, necessary for simulating Brownian motion in asset price modeling.

4. **Simulate Asset Paths**: Use the normal variables to simulate asset price paths using Geometric Brownian Motion or other stochastic models.

5. **Compute Option Payoffs**: For each simulated path, compute the option payoff at maturity.

6. **Estimate Greeks**: Slightly perturb input parameters (like underlying asset price or volatility) and use finite difference methods to calculate sensitivities like Delta, Vega, and Theta.

7. **Design the Hedge**: Use the computed Greeks to determine the optimal hedge ratios and adjust the portfolio to neutralize risks.

8. **Monitor and Rebalance**: As market conditions change, repeat the simulation and adjust the hedge accordingly to maintain risk neutrality.

Benefits of Using QMC for Hedging

- Provides more accurate estimates of option prices and Greeks with fewer simulations.

- Enhances the effectiveness of dynamic hedging strategies by reducing errors.

- Saves computational resources and time, making real-time risk management more feasible.

Hedging is a fundamental strategy in risk management, aiming to protect investments against unfavorable market movements. Accurate estimation of option prices and sensitivities is crucial for effective hedging. Quasi-Monte Carlo simulation, by providing faster convergence and more accurate results than standard Monte Carlo methods, greatly enhances hedging strategies, especially in complex or high-dimensional financial environments.

5.1 Greeks Calculation

In financial mathematics, *Greeks* measure the sensitivity of an option's price to various factors such as the underlying asset price, volatility, and time. In this project, we use the **finite difference method** to estimate Delta, Vega, and Theta of a European option.

The general approach is:

- Slightly perturb the input parameter (e.g., spot price, volatility, or time),

- Recompute the option price,
- Estimate the Greek by taking the difference between perturbed prices and dividing by the size of the perturbation.

The following Greeks were calculated:

5.2 Delta (Δ)

Delta measures the sensitivity of the option price with respect to changes in the underlying asset price S.

Mathematically, Delta is approximated as:

$$\Delta \approx \frac{V(S+\epsilon_S) - V(S-\epsilon_S)}{2\epsilon_S} \tag{1}$$

where:

- $V(S+\epsilon_S)$ is the option value with a small increase ϵ_S in the asset price,
- $V(S-\epsilon_S)$ is the option value with a small decrease ϵ_S in the asset price,
- ϵ_S is the perturbation size in the spot price (typically small, e.g., 1).

5.3 Vega (ν)

Vega measures the sensitivity of the option price to changes in volatility σ.

It is approximated using:

$$\nu \approx \frac{V(\sigma+\epsilon_\sigma) - V(\sigma-\epsilon_\sigma)}{2\epsilon_\sigma} \tag{2}$$

where:

- $V(\sigma+\epsilon_\sigma)$ is the option value with a small increase ϵ_σ in volatility,
- $V(\sigma-\epsilon_\sigma)$ is the option value with a small decrease ϵ_σ in volatility,
- ϵ_σ is the perturbation size in volatility (e.g., 0.01).

5.4 Theta (Θ)

Theta measures the sensitivity of the option price to the passage of time, often interpreted as the option's *time decay*.

Theta is approximated by reducing the time to maturity T by a small amount ϵ_T:

$$\Theta \approx \frac{V(T-\epsilon_T) - V(T)}{\epsilon_T} \tag{3}$$

where:

- $V(T-\epsilon_T)$ is the option value with a small reduction ϵ_T in time to maturity,
- $V(T)$ is the original option value,
- ϵ_T is a small perturbation in time, typically one day (i.e., $\epsilon_T = \frac{1}{365}$).

These approximations are based on the finite difference method and provide estimates of the Greeks when closed-form analytical formulas are unavailable.

Inputs for Quasi-Monte Carlo Simulation

- Initial stock price (S_0): 1428.39 (HDFC stock)
- Strike price (K): 1825.35
- Risk-free interest rate (r): 5%
- Annual volatility (σ): 20.46%
- Time to maturity (T): 1 year
- Number of time steps (N): 248 (trading days)
- Number of simulation paths (M): 2^{14} (16,384 paths)
- Option type: Call

Results

- **Delta**: 0.1999
- **Vega**: 396.6274
- **Theta**: -53.4302

Delta, Theta, and Vega Hedging from a Trader's Perspective

When trading options, traders use the concept of **hedging** to manage risks associated with changes in the price of the underlying asset and the passage of time. The Greeks, including *Delta*, *Theta*, and *Vega*, provide essential information about the sensitivity of an option's price to various factors. These sensitivities are used to create strategies that can hedge the risk inherent in option positions.

Delta Hedging

A trader uses **delta hedging** to neutralize the risk arising from changes in the underlying asset's price. This is achieved by taking an offsetting position in the underlying asset.

Example:

If a trader holds a call option with a delta of 0.60, they would need to sell 60 shares of the underlying asset for every option contract they own. This offsetting position helps the trader remain neutral to small movements in the underlying asset's price.

In other words, delta hedging involves maintaining a position in the underlying asset such that any price movement in the asset is offset by an opposite movement in the option's position. This way, the overall portfolio is "neutralized" in terms of price movement risk.

Theta Hedging

Options lose value over time, especially as expiration nears, and **theta hedging** involves strategies to offset this time decay. Traders who are long options will face a negative theta (since options lose value as time passes), while traders who are short options benefit from this time decay (since they are collecting premium).

Example:

Suppose a trader is long on a call option with a negative theta. To hedge this risk, they might take a short position in the underlying asset to offset potential losses due to time decay. This helps the trader reduce the adverse effects of time decay on their option position.

Vega Hedging

Vega (ν) measures the sensitivity of the option's price to changes in volatility. It tells the trader how much the price of the option will change for a 1% change in implied volatility.

Example:

If a trader holds an option with a high vega, they are exposed to fluctuations in volatility. To hedge this risk, the trader might engage in strategies such as volatility arbitrage or adjust their portfolio's exposure to implied volatility by taking positions in other assets or options.

Vega hedging helps traders manage the risk of volatility changes that could affect the pricing of the options. This is especially important in periods of high market uncertainty when volatility can fluctuate significantly.

Practical Use of Delta, Theta, and Vega in Hedging

Delta hedging is used to manage the risk from price movements of the underlying asset, **Theta hedging** is used to manage the risk from time decay, and **Vega hedging** is used to manage the risk from changes in volatility.

Traders may use a combination of these Greeks in a dynamic and flexible strategy, adjusting positions as market conditions evolve. For instance, in an options portfolio, a trader might adjust the hedge as the price of the underlying asset changes, the time to maturity shortens, or the market volatility increases or decreases.

In conclusion, the Greeks (Delta, Theta, and Vega) are powerful tools for traders to manage and hedge the risks inherent in options trading. By understanding the sensitivity of options to the underlying asset price, time decay, and volatility, traders can make informed decisions about hedging their portfolios effectively. Each of these hedging strategies helps mitigate different types of risk, and when combined, they provide a comprehensive approach to managing an options portfolio.

Conclusion

In this project, we explored various methods for calculating the fair price of options and managing the associated risks from a trader's perspective. The models used included the **Black-Scholes Model**, **Monte Carlo Simulation**, and **Quasi-Monte Carlo Simulation**. Additionally, we applied hedging strategies based on the **Greeks** (Delta, Theta, Vega) to mitigate trading risks and to develop a systematic approach for dynamic risk management.

Black-Scholes Model

The **Black-Scholes Model** provided an analytical solution for European-style options under idealized assumptions such as constant volatility, continuous trading, and no arbitrage. From a trader's perspective, this model serves as the theoretical benchmark for option pricing. It offers a closed-form formula that is computationally efficient and widely accepted in financial markets.

However, the Black-Scholes Model has limitations, particularly its assumptions of constant volatility and lognormal asset price distribution. Real-world markets often exhibit volatility smiles and stochastic behavior, making it essential for traders to consider more robust and flexible pricing approaches.

Monte Carlo Simulation

To address the limitations of the Black-Scholes Model, we implemented a **Monte Carlo Simulation** approach. Monte Carlo methods simulate a large number of possible price paths for the underlying asset, capturing the stochastic nature of financial markets more realistically.

From a trader's perspective, Monte Carlo simulations are valuable because they allow for the modeling of complex derivative structures and non-standard payoffs. However, the method is computationally intensive and may require a significant number of paths to achieve high accuracy, especially when estimating Greeks for hedging purposes.

Quasi-Monte Carlo Simulation

The **Quasi-Monte Carlo Simulation** was employed to enhance the efficiency and accuracy of traditional Monte Carlo simulations. By using low-discrepancy sequences (such as Sobol sequences), Quasi-Monte Carlo reduces the variance of the estimator, providing faster convergence to the true option price.

For traders, Quasi-Monte Carlo methods offer a more accurate and computationally efficient means of pricing options, especially when dealing with complex portfolios. This leads to better pricing strategies and reduces the computational costs associated with frequent recalibrations in dynamic trading environments.

Hedging Using Greeks

Effective risk management is critical for traders. We applied **Delta**, **Theta**, and **Vega** hedging strategies to dynamically manage the risk exposure associated with changes in the underlying asset's price, the passage of time, and changes in market volatility.

- **Delta Hedging** was used to neutralize the risk arising from small movements in the underlying asset price by maintaining a delta-neutral portfolio.
- **Theta Hedging** helped manage the risk of time decay in options, particularly for traders who hold long option positions.
- **Vega Hedging** was critical for managing exposure to changes in implied volatility, a significant factor affecting option prices during periods of market uncertainty.

Through dynamic hedging, traders could maintain a market-neutral position, thus protecting themselves from adverse market movements and ensuring the stability of their portfolios.

Final Remarks

Overall, the project demonstrated that a combination of theoretical models and numerical simulations, supplemented with robust hedging strategies, provides a comprehensive framework for option pricing and risk management.

From a trader's perspective:

- The Black-Scholes model serves as a foundational tool for quick, theoretical pricing under idealized conditions.
- Monte Carlo simulations offer flexibility for modeling real-world complexities but require substantial computational resources.
- Quasi-Monte Carlo simulations enhance the efficiency and accuracy of simulations, making them suitable for real-time trading decisions.
- Hedging based on the Greeks allows traders to dynamically manage and minimize portfolio risks, ensuring sustainable and profitable trading strategies.

By using these techniques collectively, traders can not only determine the fair price of options but also protect their portfolios against various market risks, leading to better decision-making and improved financial outcomes in the highly dynamic and volatile trading environment.

References

- Risk Management and Financial Institutions by John C. Hull
- Options, Futures, and Other Derivatives by John C. Hull