Part-A

Soft Thresholding Operator:
$S_\lambda(y)$=arg min$\{\frac{1}{2}(y-\theta)^2+\lambda|\theta|\}$

f(x)=|x| is a continuous function and piecewise differentiable.

$$f(x)=\begin{cases}-1 & \text{if } x<0\\ 1 & \text{if } x>0\end{cases}$$

Similarly, $\frac{1}{2}(y-\theta)^2+\lambda|\theta|$ is a piecewise differentiable function.

Case-1: $f(\theta)=\frac{1}{2}(y-\theta)^2-\lambda\theta$.

$\frac{d}{d\theta}f(\theta^*)$=0 $\rightarrow \theta^*=y+\lambda\in(-\infty,0)$.

$\theta^*=y+\lambda$ is a candidate $\iff$ y $<-\lambda$.

Case-2: $f(\theta)=\frac{1}{2}(y-\theta)^2+\lambda\theta$.

$\frac{d}{d\theta}f(\theta^*)$=0 $\rightarrow \theta^*=y-\lambda\in(0,\infty)$.

$\theta^*=y-\lambda$ is a candidate $\iff$ y $>\lambda$

$$\theta^*=\begin{cases}0 & \text{if } -\lambda\le y\le\lambda\\ y+\lambda & \text{if } y<-\lambda\\ y-\lambda & \text{if } y>\lambda\end{cases}$$

Any real number can be written as y=sign(y) |y|

where

$$\text{sign}(y)=\begin{cases}-1 & \text{if } y<0\\ 0 & \text{if } y=0\\ 1 & \text{if } y>0\end{cases}$$

$$S_\lambda(y)=\begin{cases}\text{sign}(y).(|y|-\lambda) & \text{if } |y|>\lambda\\ 0 & \text{otherwise}\end{cases}$$

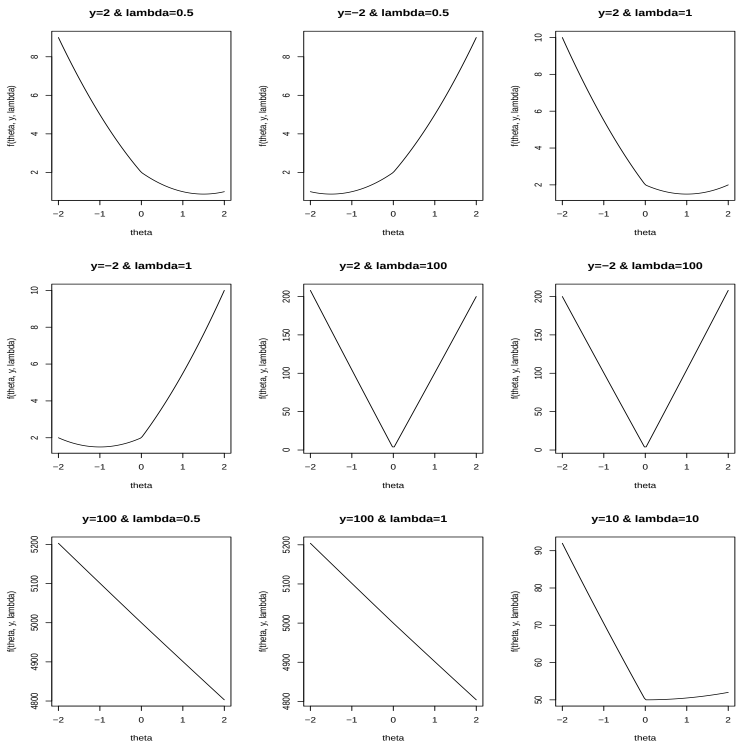

Figure 1: Plot of objective function for various y and lambda

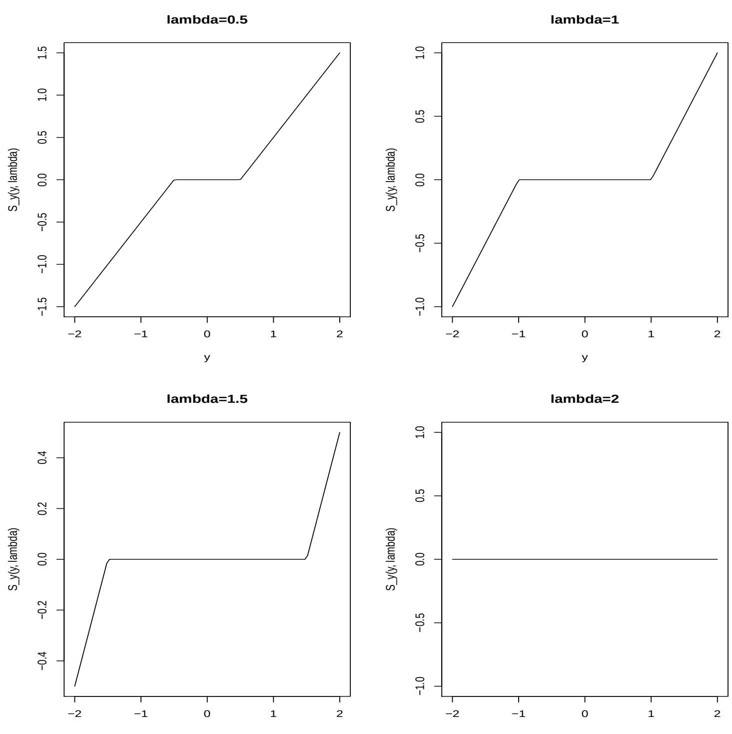

Figure 2: Plot of soft thresholding function for different lambda