Exercise 1.2

$X_1, ..., X_M$: Finite sequence.

For $N \leq M$, $\mathrm{P}(\sum_{i=1}^{N} X_i = s | \sum_{i=1}^{M} X_i = t) = \frac{\binom{t}{s}\binom{M-t}{N-s}}{\binom{M}{N}}$.

Proof:
$\mathrm{P}(\sum_{i=1}^{N} X_i = s | \sum_{i=1}^{M} X_i = t)$
$\rightarrow \frac{\mathrm{P}(\sum_{i=1}^{N} X_i = s, \sum_{i=1}^{M} X_i = t)}{\mathrm{P}(\sum_{i=1}^{M} X_i = t)}$

$\rightarrow \frac{\mathrm{P}(\sum_{i=1}^{N} X_i = s, \sum_{i=1}^{M} X_i = t-s)}{\mathrm{P}(\sum_{i=1}^{M} X_i = t)}$

$\rightarrow \frac{\mathrm{P}(\sum_{i=1}^{N} X_i = s)\mathrm{P}(\sum_{i=1}^{M} X_i = t-s)}{\mathrm{P}(\sum_{i=1}^{M} X_i = t)}$

Now it is given that X_i is binary and independent of each other, i.e., $\sum_{i=1}^{M} X_i \sim$ binomial distribution.
$\rightarrow \frac{\binom{N}{s}\binom{M-N}{t-s}}{\binom{M}{t}}$

If we manipulate the expression little bit, we will get $\frac{\binom{t}{s}\binom{M-t}{N-s}}{\binom{M}{N}}$.

Thus we can think of an urn containing M items, of which t are 1's and $M-t$ are 0's. We pick N items without replacement. Then we get the expression: $\frac{\binom{t}{s}\binom{M-t}{N-s}}{\binom{M}{N}}$ which is the probability of obtaining s 1's and $N-s$ 0's.

Exercise 1.4

$p(x|\lambda) = \exp(\log\left(\frac{\lambda^x e^{-\lambda}}{x!}\right))$

$\rightarrow \exp(x \log\lambda - \lambda - \log x!)$

$\rightarrow \frac{\exp(x\log\lambda - \lambda)}{x!}$

Therefore, $\eta = \log\lambda$, $T(x) = x$, $A(\eta) = \exp(\log\lambda) = \exp(\eta)$, $h(x) = \frac{1}{x!}$

$p(\underset{\sim}{x}|\lambda) = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}$
$\rightarrow (\prod_{i=1}^{n} \frac{1}{x_i!})\exp\{\sum_{i=1}^{n} x_i \log\lambda - n\lambda\}$

Therefore, $\eta = \log\lambda$, $T(x) = \sum_{i=1}^{n} x_i$, $A(\eta) = n\exp(\eta)$, $h(x) = \prod_{i=1}^{n} \frac{1}{x_i!}$

Exercise 1.5

$X_i \sim \text{Gamma}(\alpha, \beta)$

$p(\underset{\sim}{x}|\alpha, \beta) = \prod_{i=1}^n \frac{\beta^\alpha x_i^{\alpha-1} e^{-\beta x_i}}{\Gamma(\alpha)}$

$\rightarrow \frac{\beta^{n\alpha} \prod (x_i^{\alpha-1}) e^{-\beta \sum x_i}}{(\Gamma(\alpha))^n}$

$\rightarrow \exp\{n\alpha \log \beta - n \log \Gamma(\alpha) + (\alpha - 1) \sum \log x_i - \beta \sum x_i\}$

$$\text{Let } \underset{\sim}{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \alpha - 1 \\ -\beta \end{pmatrix},$$

$$T(x) = \begin{pmatrix} \sum \log x_i \\ \sum x_i \end{pmatrix},$$

$$h(x) = 1,\ A(\underset{\sim}{\eta}) = n \log \Gamma(\alpha) - n\alpha \log \beta = n\{\log \Gamma(\eta_1 + 1) - (\eta_1 + 1) \log(-\eta_2)\}.$$

Exercise 1.6

$X \sim \exp(\lambda)$

$f(x|\lambda) = \lambda e^{-\lambda x}, x \geq 0$

$f(x|\lambda) = \exp\{\log \lambda e^{-\lambda x}\} = \exp\{-\lambda x + \log \lambda\}.$

Therefore, $\eta = -\lambda,\ T(x) = x,\ A(\eta) = -\log(-\eta),\ h(x) = 1.$

$f(x|\eta) = -\eta e^{\eta x}, x < 0$

$\text{E}(e^{sT(x)}|\eta) = -\int_{-\infty}^0 e^{sx} \eta e^{\eta x} dx$

$\rightarrow -\int_{-\infty}^0 \eta e^{(\eta+s)x} dx = \frac{\eta}{\eta+s} \int_{-\infty}^0 -(\eta + s) e^{(\eta+s)x} dx = \frac{\eta}{\eta+s}.$

Now $A(\eta + s) - A(\eta) = -\log(-\eta - s) + \log(-\eta) = -\log\left(\frac{(-\eta-s)}{(-\eta)}\right) = \log \frac{\eta}{(\eta+s)}.$

Exercise 1.7 MGF

$\text{E}(e^{sX}) = \sum_{x=0}^\infty \frac{e^{sx} \lambda^x e^{-\lambda}}{x!}$

$\rightarrow e^{-\lambda} \sum_{x=0}^\infty \frac{(\lambda e^s)^x}{x!}$

$\rightarrow e^{-\lambda} e^{\lambda e^s}$

$\rightarrow e^{-\lambda(1-e^s)}$

$m_1 = \frac{d}{ds}\mathrm{E}(e^{sX})|_{s=0} = \frac{d}{ds}e^{-\lambda(1-e^s)}|_{s=0} = e^{-\lambda(1-e^s)}\lambda e^s|_{s=0} = \lambda = \mathrm{E}(X)$

$m_2 = \frac{d^2}{ds^2}\mathrm{E}(e^{sX})|_{s=0} = \frac{d}{ds}e^{-\lambda(1-e^s)}\lambda e^s|_{s=0} = e^{-\lambda(1-e^s)}\lambda e^s + e^{-\lambda(1-e^s)}\lambda^2 e^{2s}|_{s=0} = \lambda + \lambda^2 = \mathrm{E}(X^2)$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}^2(X) = \lambda + \lambda^2 - \lambda^2 = \lambda$

Thus by MGF, $\mathrm{E}(X) = \lambda$ and $\mathrm{Var}(X) = \lambda$.

Exercise 1.7 CGF

$\mathrm{C}_X(s) = \log \mathrm{E}(e^{sX})$

$\log(\mathrm{M}_X(s)) = -\lambda(1 - e^s)$

$\frac{d}{ds}\log(\mathrm{M}_X(s))|_{s=0} = \lambda e^s|_{s=0} = \lambda = \mathrm{E}(X)$

$\frac{d^2}{ds^2}\log(\mathrm{M}_X(s))|_{s=0} = \lambda e^s|_{s=0} = \lambda = \mathrm{Var}(X)$