Exercise 5.1

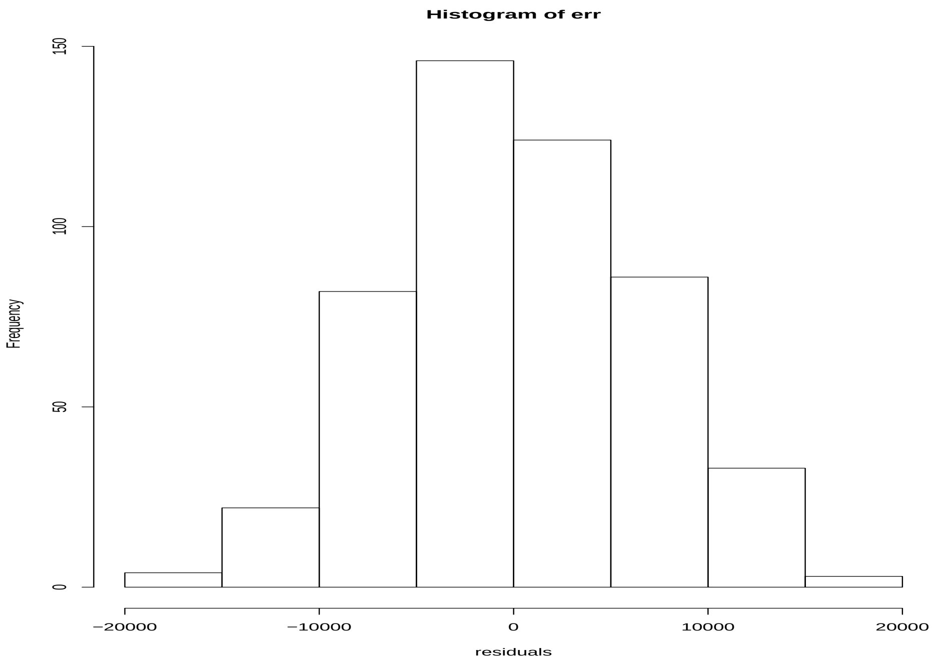

Figure 1: plot of residuals

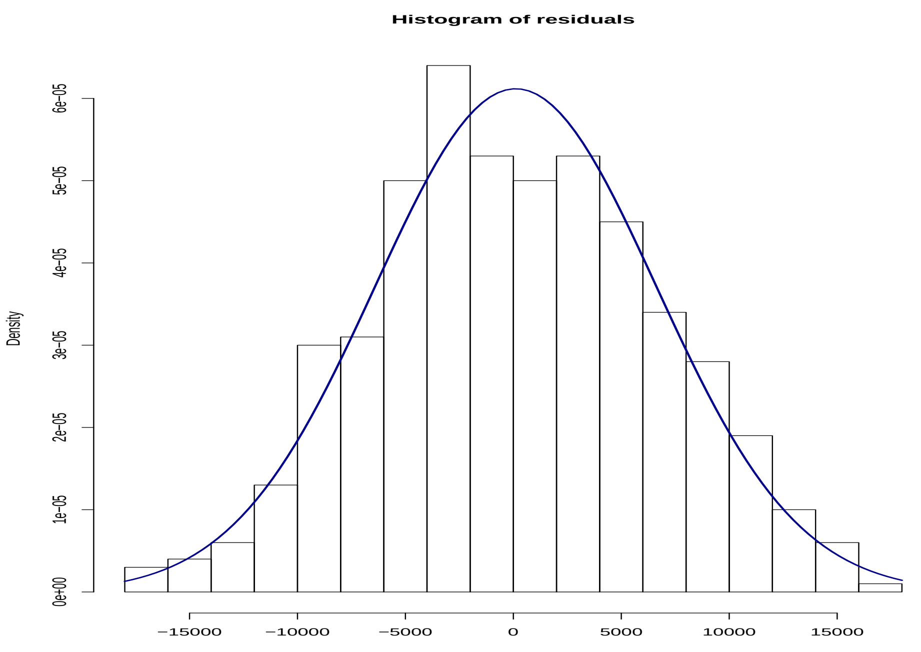

Figure 2: plot of residuals with Normal density curve

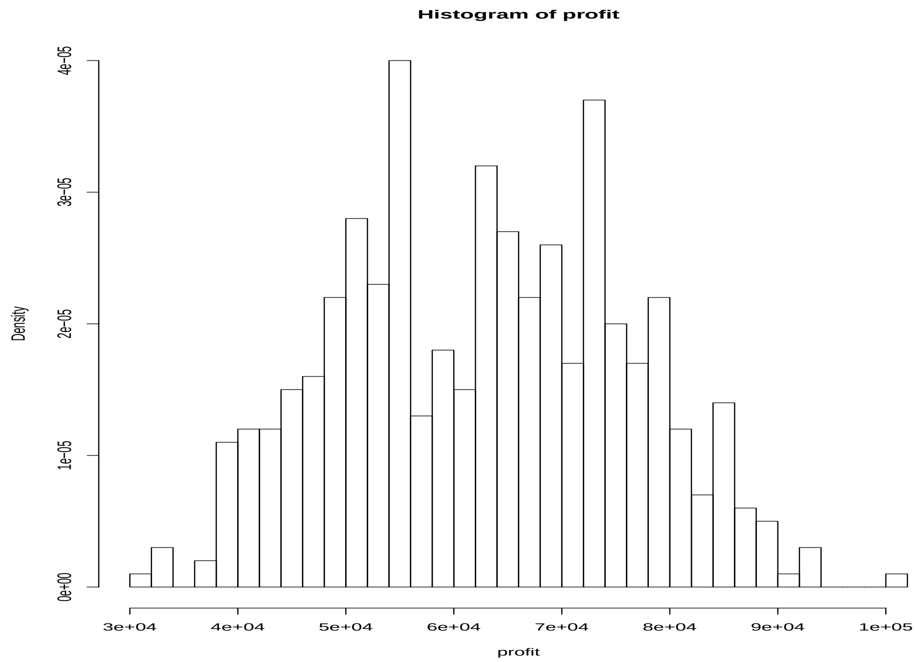

Figure 3: plot of raw data

It seems from Figure 1 and Figure 2 plots that the residuals resemble the nature of a Normal distribution. Figure 3 plot represents the distribution of raw profit data. It seems that the plot does not exactly resemble the nature of a Normal distribution but it is close enough.

Exercise 5.2

Model: $g(x|\theta) = \pi N(x|\mu_1, \sigma^2) + (1-\pi)N(x|\mu_2, \sigma^2)$ with priors on $\lambda = \frac{1}{\sigma^2} \sim \text{Gamma}(1, 1)$, $f(\mu_1) = N(0, 100)$ and $f(\mu_2) = N(0, 100)$ and $\pi = 0.5$. The latent variable here is the d which determines which group each observation came from. So $d = 0$ indicates sample from $N(x|\mu_1, \sigma^2)$ and $d = 1$ indicates sample from $N(x|\mu_2, \sigma^2)$.
$g(x, 0|\theta) = \pi N(x|\mu_1, \sigma^2), g(x, 1|\theta) = (1-\pi)N(x|\mu_2, \sigma^2)$

We derive all the posterior distributions for d, μ_1, μ_2, λ.

Computation of posteriors:

$\boldsymbol{d_i|\mu_1, \mu_2, \lambda, x_1, ..., x_n \sim}$ **Bernoulli**$(\boldsymbol{p})$ where $p = \frac{\pi N(x_i|\mu_1,\sigma^2)}{\pi N(x_i|\mu_1,\sigma^2)+(1-\pi)N(x_i|\mu_2,\sigma^2)}$.

$f(\mu_1|...) \propto \exp\{-\frac{1}{2\sigma^2}\sum_{d_i=0}(x_i - \mu_1)^2\}f(\mu_1)$
$\boldsymbol{\mu_1|... \quad \sim N(\frac{\lambda \sum x_i}{n_1\lambda + 0.01}, (n_1\lambda + 0.01)^{-1})}$, $\sum x_i$ over $d_i = 0$.
Similarly, in this case, $\boldsymbol{\mu_2|... \quad \sim N(\frac{\lambda \sum x_i}{n_2\lambda + 0.01}, (n_2\lambda + 0.01)^{-1})}$, $\sum x_i$ over $d_i = 1$.

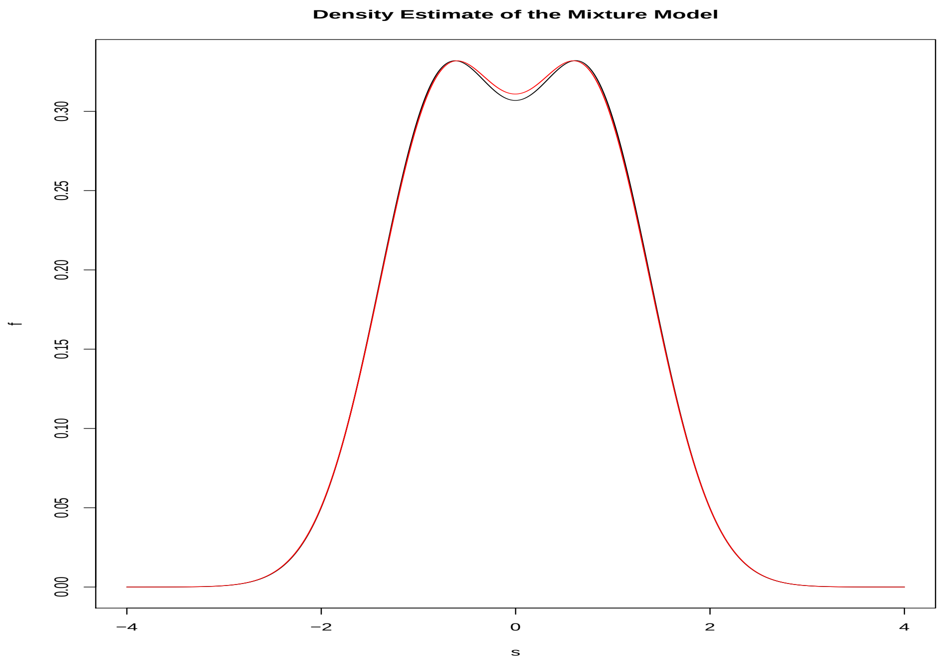

Figure 4: plot of estimated density assuming λ is same for μ_1 and μ_2

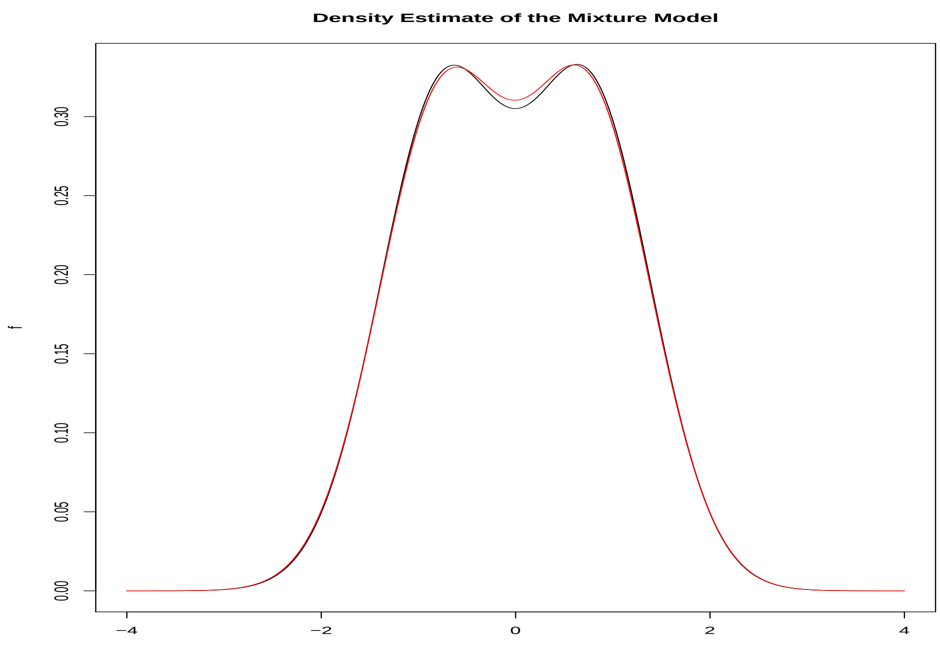

Figure 5: plot of estimated density assuming λ is not same for μ_1 and μ_2

$f(\lambda|...) \propto f(\lambda) \prod N(x_i|\mu_{d_i}, 1/\lambda)$
$\boldsymbol{\lambda|... \sim \mathrm{Gamma}(\frac{n}{2}+1, 1+\frac{1}{2}\sum(x_i-\mu_{d_i})^2)}$

where $\sum(x_i - \mu_{d_i})^2 = \sum x_i^2 - 2(\mu_1 \sum x_i + \mu_2 \sum x_i) + n_1\mu_1^2 + n_2\mu_2^2$

After getting all the full conditionals, I run Gibbs sampling with 5000 simulations. Next, I run the sampler using the given dataset. The two plots show the estimated densities.

Exercise 5.3

$p(Z_i|Z_{-i}, X_i, \theta, \alpha, \beta) = \int p(Z_i|X_i, \pi, \theta)p(\pi|Z_{-i}, \alpha, \beta)d\pi$
$= \int \pi^{Z_i}(1-\pi)^{1-Z_i}N(X_i|\mu(Z_i), \sigma^2(Z_i))\text{Beta}(\alpha + \sum_{j\neq i} Z_j, \beta + n - \sum_{j\neq i} Z_j)d\pi$
Let $\alpha + \sum_{j\neq i} Z_j = \alpha^*$ and $\beta + n - \sum_{j\neq i} Z_j = \beta^*$.
$= N(X_i|\mu(Z_i), \sigma^2(Z_i)) \int \frac{\pi^{\alpha^*+Z_i-1}(1-\pi)^{\beta^*+1-Z_i-1}}{\text{Beta}(\alpha^*, \beta^*)}d\pi$
$= N(X_i|\mu(Z_i), \sigma^2(Z_i))\frac{\Gamma(\alpha^*+Z_i)\Gamma(\beta^*+1-Z_i)}{(\alpha^*+\beta^*)\Gamma(\alpha^*)\Gamma(\beta^*)}$
Let $p = \frac{\alpha^*}{\alpha^*+\beta^*}$
$= N(X_i|\mu(Z_i), \sigma^2(Z_i))p^{Z_i}(1-p)^{1-Z_i}$

Exercise 5.6

Let $p_1, ..., p_k \sim \text{Dir}(\alpha_1, .., \alpha_k)$
$y_1, .., y_k \sim \text{Mult}(p_1, ..., p_k)$

$f(p_1, ..., p_k|...) \propto f(y_1, .., y_k|p_1, ..., p_k)f(p_1, ..., p_k|\alpha_1, .., \alpha_k)$
$\propto \prod p_j^{\alpha_j - 1 + \sum_i I(y_i = j)}$
Thus, we see that the above density is exactly a Dirichlet distribution with $\alpha_j^* = \alpha_j + \sum_i I(y_i = j)$