Bernoulli Distribution:

Consider an experiment with two possible outcomes call them Success (S) and failure (f) [Alive or dead, Sweet or not sweet, Defective or non-defective, Solved or un solved] with Probabilities

$$P(S) = p \quad \text{and} \quad P(f) = 1-p = q$$

Let X be a discrete random variable takes value 0 if failure occur and 1 if success occur with probabilities

$$P(X=0) = 1-p = q$$
$$P(X=1) = p.$$

The probability distribution of the values which the random variable X takes is given by

$$P(X=x) = p^x (1-p)^{1-x} \quad ; \quad x = 0,1$$
$$= p^x q^{1-x} \qquad p+q=1$$

or point binomial dist

is called a Bernoulli distribution and the random variable is called a Bernoulli variate

Moments of Bernoulli Distribution:

By definition

$$\mu'_r = E[X^r] = \sum_x x^r P(X=x)$$

$$\mu'_1 = E[X] = \sum_{x=0,1} x\, P(X=x)$$

$$= \sum_{x=0,1} x \cdot p^x q^{1-x}$$

$$= 0 \cdot p^0 q^{1-0} + 1 \cdot p^1 q^{1-1}$$

$$\boxed{\mu'_1 = p}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_4'^4$$

$$= p - 4p^2 + 6p^3 - 3p^4$$

$$= p\{1 - 4p + 6p^2 - 3p^3]$$

$$\boxed{\mu_4 = p(1-p)(3p^2 - 3p + 1)}$$

Binomial Experiment:

A binomial experiment is one that possess the following properties.

a) The experiment consists of n repeated trials
b) Each trial results in an outcomes that may be classified as success or a failure.
c) The probability of success, denoted by p remains constant from trial to trial.
d) The repeated trials are independent.

Binomial Distribution:

consider an experiment with two possible outcomes call them success (S) and failure (f) with $P(S) = p$ and $P(f) = f$ such that $p+q=1$. Let 'x' be a random variable denotes the number of successes in n independent repeated trials, e.g.

consider

$$\overbrace{\underbrace{S.S....S}_{\substack{x \\ \text{success}}} \quad \underbrace{f.f....f}_{\substack{n-x \\ \text{failure}}}}^{n \text{ trials}}$$

The probability distribution of the particular sequence (by multiplicative law of independent events) is

$$p^{x} q^{(n-x)} \quad \text{or} \quad p^{x}(1-p)^{n-x}$$

The number of sequence in which 'x' successes and $n-x$ failures are observed in some order is $\binom{n}{x}$ ways. which is binomial coefficient.

Thus the probability Distribution that exactly x successes and $n-x$ failures occur in n independent trials is

$$b(x; n, p) = \binom{n}{x} p^{x} q^{n-x} \quad ; x = 0, 1, 2, \dots, n$$

which is known as binomial distribution with index 'n' and parameter p.

$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n} b^n$

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The distribution is known as binomial distribution with index n and parameter p.

Prove that $\sum_{x=0}^{n} b(x; n, p) = 1$

$$= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= q^n + \binom{n}{1} p q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots + \binom{n}{n} p^n$$

$$= [q+p]^n$$

$$= 1^n = 1 \qquad \because p+q = 1$$

Proved.

Moments:

Let 'X' be a random variable with the binomial distribution $b(x; n, p)$. The moments about origion are given by the relation

$$\mu_r' = E[X^r]$$

at $r=1$

$$\mu_1' = E[x] = \sum_{x=0}^{n} x\, b(x; n, p)$$

$$= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x q^{n-x}$$

$$= 0 \cdot q^n + 1 \cdot \binom{n}{1} q^{n-1} p + 2 \cdot \binom{n}{2} q^{n-2} p^2 + \dots + \binom{n}{n} p^n$$

$$= npq^{n-1} + 2\frac{n \cdot (n-1)(n-2)!}{(n-2)!\, 2!} p^2 q^{n-2} + \dots + p^n$$

$$= np\left[q^{n-1} + \binom{n-1}{1} p q^{n-2} + \dots + p^{n-1}\right]$$

$\binom{n-1}{2} p^2 q^{n-3}$

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$$\mu_1' = np\left[q^{n-1} + \binom{n-1}{1} p q^{n-2} + \cdots + p^{n-1}\right]$$

$$= np\,[q+p]^{n-1}$$

$$\boxed{\mu_1' = np} \qquad \because q+p=1$$

at r=2

$$\mu_2' = E[x^2] = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x + x(x-1)] \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} + \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x}$$

$$= np + 0 + 0 + 2\cdot1\binom{n}{2} p^2 q^{n-2} + 3\cdot2\binom{n}{3} p^3 q^{n-3} + \cdots + n(n-1)\binom{n}{n} p^n q^{n-n}$$

$$= np + n(n-1)p^2\left[q^{n-2} + \binom{n-2}{1} p q^{n-3} + \cdots + p^{n-2}\right]$$

$$= np + n(n-1)p^2\,[q+p]^{n-2}$$

$$= np + n(n-1)p^2$$

$$= np[1+(n-1)p] = np[1+np-p]$$

$$= np[q+np]$$

$$\boxed{\mu_2' = npq + n^2p^2}$$

Similarly Proceeding we get

$$\mu_3' = E[x^3] = \sum_{x=0}^{n} x^3 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x(x-1)(x-2) + 3x(x-1) + x] \binom{n}{x} p^x q^{n-x}$$

$$= n(n-1)(n-2)p^3(q+p)^{n-3} + 3n(n-1)p^2(p+q)^{n-2} + np(q+p)^{n-1}$$

$$\boxed{\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np}$$

(6)

$$\mu_4' = E[X^4] = \sum_{x=0}^{n} x^4 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x] \times \binom{n}{x} p^x q^{n-x}$$

$$\boxed{\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.}$$

<u>Moment about the mean:</u>

$$\mu_r = \frac{1}{n}\sum (x_i - \bar{x})^r$$

$$\mu_1 = \mu_1' - \mu_1' = 0.$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$\sigma^2 = \boxed{\mu_2 = npq} \Rightarrow \sigma = \sqrt{\mu_2} = \sqrt{npq}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= [n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np]$$

$$-3[n(n-1)p^2 + np]np + 2n^3p^3$$

$$= n[n^2 - 3n + 2]p^3 + 3[n^2 - n]p^2 + np - 3n^2p^2 - 3n^2p^3(n-1) + 2n^3p^3$$

$$= \cancel{n^3p^3} - \cancel{3n^2p^3} + 2np^3 + \cancel{3n^2p^2} - 3np^2 + np - \cancel{3n^2p^2} - \cancel{3n^3p^3} + \cancel{3n^2p^3} + \cancel{2n^3p^3}$$

$$2np^3 - 3np^2 + np = np[1 - 3p + 2p^2]$$

$= np[1-p)(1-2p)$

$= np(1-p)(1-p-p)$

$$\boxed{\mu_3 = npq\,(q-p)}$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4$$

$$= [n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np]$$
$$-4np[n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np]$$
$$+6n^2p^2[np + n(n-1)p^2] - 3n^4p^4$$

$$= n^4[p^4 - q^4] + n^3[-6p^4 + 6p^3 + 6p^3 - 6p^4] + n^4[11p^4 - 18p^3$$
$$+ 7p^2 - 4p^2 + 12p^3 - 8p^4] + n[-6p^4 + 12p^3 - 7p^2 + p]$$
$$= 3n^2p^2(1-p)^2 + np(1-p)(1-6p+6p^2)$$

$$\boxed{\mu_4 = 3n^2p^2q^2 + npq(1-6pq)}$$

$$\boxed{\mu_4 = npq[1 + 3(n-2)pq]}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{[npq(q-p)]^2}{[npq]^3} = \frac{(q-p)^2}{npq} = \boxed{\frac{(1-2p)^2}{npq} = \beta_1}$$

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{(1-2p)^2}{npq}} = \boxed{\frac{1-2p}{\sqrt{npq}} = \gamma_1}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3n^2p^2q^2 + npq(1-6pq)}{(npq)^2} = \boxed{3 + \frac{1-6pq}{npq} = \beta_2}$$

$$\gamma_2 = \beta_2 - 3 = \boxed{\frac{1-6pq}{npq} = \gamma_2}$$

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M.G.F

The m.g.f of the bionomial distribution $b(x; n, p)$ is derived as

$$M_x(t) = E[e^{tx}] \quad \text{(By defination)}$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} [pe^t]^x q^{n-x}$$

$$= \binom{n}{0}(pe^t)^0 q^{n-0} + \binom{n}{1}(pe^t)^1 q^{n-1} + \binom{n}{2}(pe^t)^2 q^{n-2} + \cdots$$
$$\cdots + \binom{n}{n}(pe^t)^n q^{n-n}$$

$$= q^n + \binom{n}{1}(pe^t) q^{n-1} + \binom{n}{2}(pe^t)^2 q^{n-2} + \cdots + \binom{n}{n}(pe^t)^n 0$$

$$\boxed{M_x(t) = [q + pe^t]^n}$$

Which is the simplified form of M.g.f of bionomial distribution.

Moments of Bionomial (by mgf):

The moments of bionomial distribution is obtained by differentiating $M_x(t)$ rth times with respect to t and putting $t=0$. thus:

$$\mu_r' = E(X^r) = \left[\frac{d^r}{dt^r}(q+pe^t)^n\right]_{t=0}$$

at $r=1$ (9)

$$\mu_1' = E[x] = \frac{d}{dt}\left[(q+pe^t)^n\right]_{t=0}$$

$$= \left[n(q+pe^t)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= np\left[e^t(q+pe^t)^{n-1}\right]_{t=0}$$

$$= np\left[e^0(q+pe^0)^{n-1}\right]$$

$$= np\left[1\cdot(q+p\cdot 1)^{n-1}\right]$$

$$= np(q+p)^{n-1} \qquad \because q+p=1$$

$$\boxed{\mu_1' = np}$$

$$\mu_2' = E[x^2] = \frac{d^2}{dt^2}\left[(q+pe^t)^n\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{d}{dt}\left\{(q+pe^t)^n\right\}\right]_{t=0}$$

$$= \frac{d}{dt}\left[npe^t(q+pe^t)^{n-1}\right]$$

$$= \left[npe^t(q+pe^t)^{n-1}\right]_{t=0} + \left[n(n-1)p^2e^{2t}(q+pe^t)^{n-2}\right]_{t=0}$$

$$= np(q+p)^{n-1} + n(n-1)p^2(q+p)^{n-2}$$

$$= np + n(n-1)p^2$$

$$= np + n^2p^2 - np^2$$

$$= n^2p^2 + np(1-p)$$

$$\boxed{\mu_2' = n^2p^2 + npq}$$

Similarly, differentiating 3 times or four times to get.

$$\mu_3' = E(x^3) = \left[\frac{d^3}{dt^3}\left\{(q+pe^t)^n\right\}\right]_{t=0}$$

$$= \left[npe^t(q+pe^t)^{n-1}\right]_{t=0} + 3\left[n(n-1)p^2e^{2t}(q+pe^t)^{n-2}\right]_{t=0}$$

$$+ \left[n(n-1)(n-2)p^3e^{3t}(q+pe^t)^{n-3}\right]_{t=0}$$

$$\boxed{\mu_3' = np + 3n(n-1)p^2 + n(n-1)(n-2)p^3}$$

$$\mu_4' = E\left[x^4\right] = \left[\frac{d^4}{dt^4}\left\{(q+pe^t)^n\right\}\right]_{t=0}$$

$$\mu_4' = np + 7n(n-1)p^2 + 6n(n-1)(n-2)p^3 + n(n-1)(n-2)(n-3)p^4.$$

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Problem: Binomial Distribution

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to contracted this rare blood disease, what is the probability that

a) at least 10 survive?

b) from 3 to 8 survive?

c) Exactly 5 survive

d) Fewer than 5 survive

Sol.: Let 'X' be the random variable denote the number of patient that survive.

a) $P(X \geq 10) = 1 - P(X < 10) = 1 - \sum_{x=0}^{9} b(x; n, p)$

$= 1 - \sum_{x=0}^{9} b(x; n=15, p=0.4)$ from cumulative binomial table

$= 1 - 0.9662 = 0.0338.$

b) $P(3 \leq X \leq 8) = \sum_{x=0}^{8} b(x; n=15, p=0.4) - \sum_{x=0}^{2} b(x; n=15, p=0.4)$

$= 0.9050 - 0.0271 = 0.8779$

c) $P(X=5) = b(X=5; n=15, p=0.4) = 0.1859$

$= \sum_{x=0}^{5} b(x; n=15, p=0.4) - \sum_{x=0}^{4} b(x; n=15, p=0.4)$

$= 0.4032 - 0.2173 = 0.1859$

d) $P(X < 5) = \sum_{x=0}^{4} b(x; n=15, p=0.4)$

$= 0.2173$ (from Table)

Problem. (Binomial Distⁿ)

computer

The probability that a patient recovers from a rare disease is 0.4. If 15 ~~such~~ people are known to have contracted this disease, what is the probability

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a) at least 10 survive?

b) from 3 to 8 survive?

c) Exactly 5 survive.?

d) Fewer than 5 survive?

$P(S) = 0.4$.

Let 'X': the number of people that survive

Sol: a) $P(X \geq 10) = 1 - P(X < 10)$

$= 1 - \sum_{x=0}^{9} b(x; n=15, p=0$

$= 1 - 0.9662 = 0.0338$

b) $P(3 \leq X \leq 8) = \sum_{x=3}^{8} b(x; n=15, p=0.4)$

$= \sum_{x=0}^{8} b(x; n=15, p=0.4) - \sum_{x=0}^{2} b(x, n, p$

$= 0.9050 - 0.0271$

$= 0.8779$

c) $P(X=5) = b(x; n=15, p=0.4)$

$\sum_{x=0}^{5} b(x; n=15, p=0.4) - \sum_{x=0}^{4} b(x; n, p)$

$= 0.4032 - 0.2173$

$= 0.1859$

d) $P(X<5) = \sum_{x=0}^{4} b(x; n=15, p=0.4)$

$= 0.2173$

Q. 7 A survey is conducted to determine eye and hair color in a population. The results sre given below

HAIRCOLOR	BLUE EYES	BROWNEYES	TOTAL
BLOND	10	30	40
BLACK	40	100	140
RED	5	25	30
Totals	55	155	210

(a) what's the probability that a person selected at random from this survey group is a blue-eyed blond
(b) what' s the probability that a person has red hair? P(R)
(c) What's s the probability that a person has blue eyes ? P(A)

Q. 8 If a penny is flipped seven time's what's the probability of getting: b(x, 7, ½)
(a) Exactly four heads?
(b) Exactly five heads?
(c) Four or five head's?

Q. 9 A restaurant prepares a tossed salad containing on the average 5 vegetables Find the probability that the salad contains more than 5 vegetables
(a) on a given day;
(b) on 3 of the next 4 days;
~~(c) for the first time in April on April 5~~

Q.10 The probability that a person dies from a certain respiratory infection is 0.002. Find the probability that fewer than 5 of the next 2000 so infected will ~~bie~~ die.

Q.11 On the average a certain intersection results in 3 traffic accidents per month. What is the probability that in any given month at this intersection
(a) exactly 5 accidents will occur?
(b) less that 3 accidents will occur?
(c) at least 2 accidents will occur?

Q. 12 Acertain area of the eastern United States is, on the average, hit by 6 hurricanes a year. Find the probability that in a given year this area will be hit by
(a) fewer than 4 hurricanes; λ = 6
(b) anywhere from 6 to 8 hurricanes

Q. 13 A secretary makes 2 errors per page on the average. What is the probability that on the next page she makes λ = 2
(a) 4 or more errors?
(b) no errors?

Q.14 A die is rolled five times and a 5 or 6 is considered a success. Find the probability of
(a) no success, (b) atleast 2 successes,
(c) atleast one but not more than 3 successes

Q.15 The probability that a patient recovers from a delicate heart operation is 0.9. What is the probability that exactly five of the next 7 patients having this operation survive? b(x, 7, 0.9)

Q.16 A doctor receives an average of 3 telephone calls from 9 p.m. until 9 a.m. the next morning . What is the probability that the doctor will not be disturbed by a call if she goes to bed at midnight and rises at 6 a.m.?

Q.17 The incidence of occupational disease in an industry is such that the workmen have a 20% chance of suffering from it. What is the probability that out of 6 workmen
(1) not more than 2,
(2) 4 or more will catch the disease.
P = 20% = 0.2
b(x; 6, 0.2)

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(1) not more than 2,

(2) 4 or more will catch the disease.

$P = 20\% = 0.2$

$b(x, 6, 0.2)$

(11)

<u>M.G.F</u>:

The moment generating function (m.g.f) of a random variable X (about origin) having a probability function $P(x)$ is given by:

$$M_x(t) = E[e^{tx}] = \sum_x e^{tx} P(x)$$

$$= E\left[1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^r x^r}{r!}\right]$$

$$= 1 + t\,E(x) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

The coefficient of $\frac{t^r}{r!}$ in $M_x(t)$ gives μ_r' (moments ~~ago~~ about origin). Since $M_x(t)$ generates moments, it is known as mgf.

$$\left[\frac{d^r}{dt^r}\{M_x(t)\}\right]_{t=0} = \left[\frac{\mu_r'}{r!}\cdot r! + \mu_{r+1}' t + \mu_{r+2}'\frac{t^2}{2!} + \cdots\right]_{t=0}$$

In general, the moment generating function of X about the point $X=a$ is defined as:

$$M_x(t) \text{ (about } X=a) = E\left[e^{t(x-a)}\right]$$

$$= E\left[1 + t(x-a) + \frac{t^2}{2!}(x-a)^2 + \cdots + \frac{t^r}{r!}(x-a)^r + \cdots\right]$$

$$= 1 + t\mu_1^{a} + \frac{t^2}{2!}\mu_2^{a} + \cdots + \frac{t^r}{r!}\mu_r^{a} + \cdots$$

where $\mu_r^{a} = E[(x-a)^r]$ is the rth moment about $X=a$.

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Recurrence Relation for the Probabilities of Binomial Dist

$$P(x) = \binom{n}{x} p^x q^{n-x} \; ; \quad x = 0, 1, 2, \dots, n \qquad \text{—— (1)}$$

$$P(x+1) = \binom{n}{x+1} p^{x+1} q^{n-x-1} \; ; \quad x = 0, 1, 2, \dots n \qquad \text{—— (2)}$$

$$\frac{P(x+1)}{P(x)} = \frac{\binom{n}{x+1} p^{x+1} q^{n-x-1}}{\binom{n}{x} p^{x} q^{n-x}} = \frac{n!}{(x+1)!\,(n-x-1)!} \times \frac{x!\,(n-x)!}{n!} \frac{p}{q}$$

$$= \frac{n-x}{x+1} \cdot \frac{p}{q}$$

$$\boxed{P(x+1) = \left[\frac{n-x}{x+1} \cdot \frac{p}{q}\right] P(x)} \qquad \text{—— (3)}$$

Which is the recurrence formula of Binomial Dist.

When $x = 0$ eq (1) $\Rightarrow$. $P(0) = q^n$

$\because$ $\bar{x} = np$

$p = \frac{\bar{x}}{n}$ and $q = 1 - p$

The remaining probabilities, can easily be obtained using eq (3).

$x = 0$.

$$P(1) = \left[\frac{n-1}{x+1} \cdot \frac{p}{q}\right] P(0)$$

and so on.