Mathematical Expectation:

Let 'X' be a random variable associated with probabilities $f(x)$ or $P(X=x)$. The p.d. is

X	x_1	x_2	$\cdots$	x_k	Total
$f(x)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_k)$	1

The mean or expected value of X is

$$\mu = E(X) = \sum_{\text{for } \forall x} x\, f(x) = \sum_{\text{all } x} x\, P(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

if X is continuous.

Properties:

1. $E(X) = \mu = \bar{X}$

Proof:

As $\bar{x} = \mu = \dfrac{\sum f_i \cdot x_i}{\sum f_i} = \sum \dfrac{f_i \, x_i}{N} = \sum x_i \cdot \dfrac{f_i}{N}$

$= \sum x_i \cdot P(x_i)$

$= E(X)$

2. $E(\text{constant}) = \text{Constant}$

$E(c) = \sum c \; P(x_i)$

$= c \sum P(x) = c \times 1$

$= c$

3. The variance of a random variable X is

$Var(X) = \sigma^2 = E(X^2) - \{E(X)\}^2$

$= E(X^2) - \mu^2$

Proof

$\sigma^2 = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$ //

4. The expected value of the sum or difference of two or more function of the random variables X and Y is the sum or the difference of the expected values of the functions, i.e

$E(X \pm Y) = E(X) \pm E(Y)$

Proof:

$E[X \pm Y] = \sum_{\text{all } x} \sum_{\text{all } y} (x \pm y) \, P(x, y)$

$= \sum_x \sum_y x \, P(x,y) \pm \sum_x \sum_y y \, P(x,y)$

$= \sum_x x \, P(x) \pm \sum y \, P(y) = E(X) \pm E(Y)$

5. If X and Y are discrete random variables and a and b are constants, then

$$E[aX \pm bY] = a\,E(X) \pm b\,E(Y)$$

Proof:

Let $U = aX$ and $V = bY$

$$E[U \pm V] = E[U] \pm E[V]$$
$$= E[aX] \pm E[bY]$$
$$= a\,E(X) \pm b\,E(Y)$$

6. Let X and Y be the two independent random variables then

$$E(XY) = E(X)\,E(Y)$$

Proof:

$$E[XY] = \sum_{\text{all } x}\sum_{\text{all } y} xy\, P(x,y)$$

$\because X$ and Y independent $P(x,y) = P(x)\,P(y)$

$$= \sum_{x}\sum_{y} xy\, P(x)\,P(y)$$
$$= \sum x\,P(x) \sum y\,P(y)$$
$$= E(X)\,E(Y)$$

7. Let X be a discrete r.v. with p.d. $f(x)$. The mean or expected value of the random variable $g(x)$ is

$$E[g(x)] = \sum_{\text{all } x} g(x)\,f(x)$$

Problem:

A race car driver wishes to insure his car for the racing season for \$ 50,000. The insurance company estimates a total loss may occur with probability 0.002, a 50% loss with probability 0.01, and a 25% loss with probability 0.1. Ignoring all other partial losses, what premium should the insurance company charge each season to realize an average profit of \$ 500?

Sol:

Let 'X' be the profit to the company

'A' be the amount of premium paid for every season.

Given: $E(X) = \$\,500$

X	$P(X=x)$
A-50,000	0.002
A-25,000	0.01
A-12,500	0.1
A	0.888
Total	1.000

$$E(X) = (A-50{,}000)(0.002) + (A-25{,}000)(0.01) + (A-12500)(0.1) + A(0.888) = 500$$

$$\Rightarrow A(1) - 1600 = 500$$

$$A = 1600 + 500$$

$$\boxed{A = \$\,21{,}00}$$

Decision:

The company charge \$ 21,00 every season in order to get an average profit of \$ 500.

Problem: In a gambling game a man is paid \$5 if he gets all heads or all tails when three coins are tossed and he pays out \$3 if either one or two heads show. What is his expected gain?

Sol:

The random variable of interest X, the amount the gambler can win; and the possible values of X are \$5 if events

$E_1 = \{HHH, TTT\}$ occurs and $-\$3$ if events

$E_2 = \{HHT, HTH, HTT, THH, THT, TTH\}$ occurs

with probabilities

$$P(E_1) = \frac{1}{4} = \frac{2}{8} \qquad P(E_2) = \frac{3}{4} = \frac{6}{8}$$

Thus

X	5	-3	Total
f(X)	$\frac{1}{4}$	$\frac{3}{4}$	1

$$\text{Mean} = \mu = E(X) = \sum_{\text{all } x} x\, f(x) = 5 \times \frac{1}{4} + (-3)\left(\frac{3}{4}\right)$$
$$= \frac{5}{4} - \frac{9}{4} = -1$$

Decision:

In this game gambler will on the average lose \$1 per toss of the three coins. It the gambler is wise not to play this game

Problem:

Suppose that the number of cars, X, that pass through a car wash between 4:00 pm and 5:00 pm on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Sol:

By property

$$E[g(X)] = \sum_{\text{all } x} g(x) f(x)$$

$$= \sum_{x=4}^{9} (2x-1) f(x)$$

$$= (7)\left(\frac{1}{12}\right) + (9)\left(\frac{1}{12}\right) + (11)\left(\frac{1}{4}\right)$$

$$+ (13)\left(\frac{1}{4}\right) + (15)\left(\frac{1}{6}\right) + (17)\left(\frac{1}{6}\right)$$

$$= \$12.67$$