PROBABILITY.

Set Theory:

SETS: A set is a well-defined collection of objects or items.

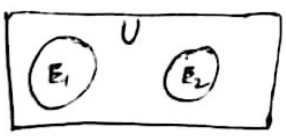
Universal set: It is defined as the set of all objects under consideration. Universal set is the superset of every set.

Subsets: Part of the Universal set or largest set U.

Superset: Let E be a subset of U, then U is a superset of E and we express it in symbols as $U \supset E$.

Thus $E \subset U \Leftrightarrow U \supset E$

Disjoint set: Two subsets E_1 and E_2 are said to be disjoint if they do not have any common element.



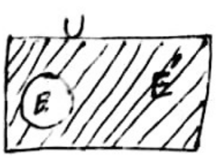
Example: $E_1 = \{2,4,6,8\}$ $\quad E_1 \subset U$

$E_2 = \{1,3,5,7\}$ $\quad E_2 \subset U$

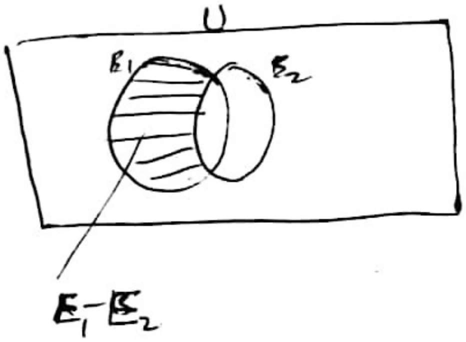
$U = \{1,2,3,4,5,6,7,8\}$

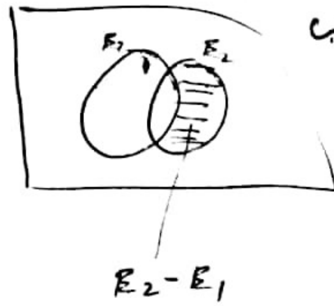
$\because$ $E_1 \cap E_2 = \phi$, E_1 and E_2 are disjoint.

Complement sets: If E is a subset of the Universal set U, then the complement of E with respect to U is the set of all elements of U that are not in E we denote the complement of E by E'.



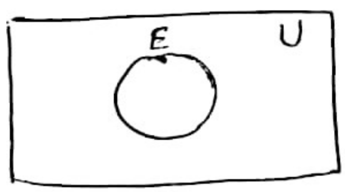
Overlapping sets:





Venn Diagram:

A Venn diagram is a diagram related to set theory in mathematics by which the events that can occur in a particular experiment can be portrayed.



Statistical Concepts:

Sample Space: It is defined as the set of all possible outcomes (results) of a certain statistical or random experiment.

Experiment: An experiment is defined as an action which we conceive, do or intend to do.

Deterministic Experiments:

Any Experiment which results into a single result and that result may be predicted with certainity, prior to the performance of the experiment called predictable or Deterministic or Non-probabilistic experiment. e.g

i) For a perfect gas $PV = \text{constant}$

ii) Tomorrow Sun will rise

These types of Experiments we donot consider.

Statistical Experiments: (Non-deterministic)

Any experiment which results into more then one results and if these results may not be predicted With Certainity prior to the performance of the experiment is called probabilistic or Statistical Experiment. e.g

In tossing a single coin, the sample space 'S' consists of only two possible results i.e $S = \{\text{Head}, \text{Tail}\}$ or $\{H, T\}$, but none of these can be predicted with certainity

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Random Experiment : An experiment generated more than one results when conducted repeatedly under independent and identical conditions, the results is not unique but may be any one defined in the sample space of the various possible outcomes, is called as random experiment

Examples :

1. Tossing a fair coin
2. Rolling an unbiased die
3. Drawing a card from a pack of cards
4. Drawing balls from an Urn.
5. Distribution of boys and girls in a family having three children.

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Sample Point: Any Element of the sample space is called as sample point

Event: Any outcome or a set of outcomes of some particular interest called as an event

Equally Likely Events: Two or more events or outcomes are said to be equally likely if they occur with same chance. e.g

Rolling a die once

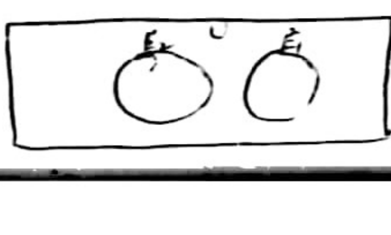
$S = \{1, 2, 3, 4, 5, 6\}$

outcomes	1	2	3	4	5	6
Chance	1/6	1/6	1/6	1/6	1/6	1/6

Mutually Exclusive Events: Two events E_1, and E_2 are said to be 'mutually exclusive' events, if the occurance of one precludes the occurance of the other or if simultaneous occurance of both is not possible. e.g

In tossing a single coin once, either Head or a tail would occur but simultaneous occurance of both is impossible. Hece H and T are mutually exclusive (Disjoint) outcomes

$E_1 = \{H\}, \ E_2 = \{T\} \ \therefore \ E_1 \cap E_2 = \phi.$



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Non-mutually Exclusive Events: Two events E_1 and E_2 are said to be 'non-mutually exclusive' if their simultaneous occurance is possible i-e $E_1 \cap E_2 \neq \phi$

e-g. Suppose in a single draw of cards, from an ordinary deck of 52 cards, we are required to obtain 'a black card' or 'an ace'

i) There are 26 black cards from which a black card can be drawn

ii) There are 4 aces from which an ace can be drawn

iii) There are 2 cards which are aces as well as black i-e $E_1 \cap E_2 = 2 \neq \phi$

Hence an ace and black card are two non-mutually exclusive events.

PROBABILITY: is the study of random or statistical experiments. It is a measure of uncertainty.
e.g. If a coin is tossed in the air, then it is certain that the coin will come down, but it is not certain that head occur. Thus, It is that branch of science which deals with chances.

Approaches of Probability

a) Mathematical or Classical or a Priori probability:
If a random experiment results in N exhaustive, mutually exclusive and equally likely outcomes out of which m are favourable to the happening of an event E, then the probability of occurance of E usually denoted by $P(E)$ is given by

$$P(E) = \frac{n(E)}{n(S)} = \frac{m}{N}$$

e.g. In rolling of two dice once, find out the Probability that the sum is 7

Die I \ Die II	1	2	3	4	5	6
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)

Let E: sum is 7 $\quad n(E) = 6$
S: Sample space $= 36 = n(S)$.

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

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b. Statistical or Empirical or Relative or Posteriori

If an experiment is repeated a large number of times under uniform conditions, the limiting value of the ratio of the number of times the event E happens to the total number of trials of the experiments as the number of trials increases indefinitely, is called the probability of the occurance of E.

suppose that the event E occur m times in N repetitions of a random experiment. Then

$$P(E) = \lim_{N \to \infty} \frac{m}{N}$$

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Subjective Approach

The most simple and natural interpretation is that the proabilities referred to are measures of the individual's belief in the statement for the occurance of an event. This interpretation of probability as being a measure of one's belief is often referred to as the personal or subjective view of probability.

Axioms on Probability

i) $P(\phi) = 0$; That is If the event certain not to occur, its probability is zero

ii) $P(S) = 1$; If the event certain to occur, its probability is one.

iii) $0 \leq P(E) \leq 1$, for every event E

iv) $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, If both events are mutually exclusive

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LAW OF ADDITION OF PROBABILITY FOR ANY TWO EVENTS E_1 & E_2 :

Statement: If E_1 and E_2 are two non-mutually exclusive events defined in sample space S, then the probability that E_1 and E_2 occurs or probability that atleast one of them occurs is given by

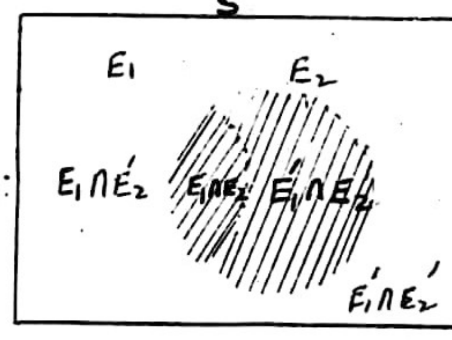
$$\frac{n(E_1 \cup E_2)}{n(S)} = \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)} \quad — (1)$$

$$\text{OR} \quad P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \quad — (2)$$

$$= P(E_1) + P(E_2) \quad \text{If } E_1 \,\&\, E_2 \text{ are mutually exclusive events}$$

Proof .1: Consider E_1 and E_2 are two non-mutually exclusive events defined in sample space S also shown in fig.1.

Let the number of favourable cases to occur E_1 is denoted by $n(E_1)$, number of favourable case to occur E_2 is by $n(E_2)$ and number of favourable cases to occur E_1 and E_2 or atleast one of them is given by $n(E_1 \cap E_2)$. Then by definition of non-mutually exclusive events we have



$$n(E_1 \cup E_2) = n(E_1) + n(E_2) - n(E_1 \cap E_2) \quad — (3).$$

Dividing by $n(S)$ the sample points in sample space we have

$$\frac{n(E_1 \cup E_2)}{n(S)} = \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)}$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= P(E_1) + P(E_2) \quad \text{If } E_1 \,\&\, E_2 \text{ are}$$

Proof 2: From Fig 1.

$$E_1 \cup E_2 = E_1 \cup (E_1' \cap E_2)$$

$$P(E_1 \cup E_2) = P[E_1 \cup (E_1' \cap E_2)]$$

$$= P(E_1) + P(E_1' \cap E_2)$$

$$= P(E_1) + P(E_1' \cap E_2) + P(E_1 \cap E_2) - P(E_1 \cap E_2)$$

$$= P(E_1) + P[(E_1' \cap E_2) \cup (E_1 \cap E_2)] - P(E_1 \cap E_2)$$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

which completes the theorem. Proved.

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Law of addition of probability for any three events E_1, E_2 & E_3:

Statement: If E_1, E_2 & E_3 are three non-mutually exclusive events defined in the sample space S, then the probability that 'E_1 or E_2 or E_3' occures or the probability that atleast one of them occures is given by

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3).$$

Proof 1:

Consider the case of three non-mutually exclusive events E_1; E_2 and E_3 with $n(E_1)$, favorable cases for event E_1, $n(E_2)$, $n(E_3)$, $n(E_1 \cap E_2)$, favorable cases for events E_1 and E_2 or atleast one of the two, $n(E_1 \cap E_2)$, $n(E_2 \cap E_3)$ and $n(E_1 \cap E_2 \cap E_3)$ favorable cases for event E_1, E_2 and E_3 or atleast one of the three, gives.

$$n(E_1 \cup E_2 \cup E_3) = n(E_1) + n(E_2) + n(E_3) - n(E_1 \cap E_2) - n(E_1 \cap E_3) - n(E_2 \cap E_3) + n(E_1 \cap E_2 \cap E_3)$$

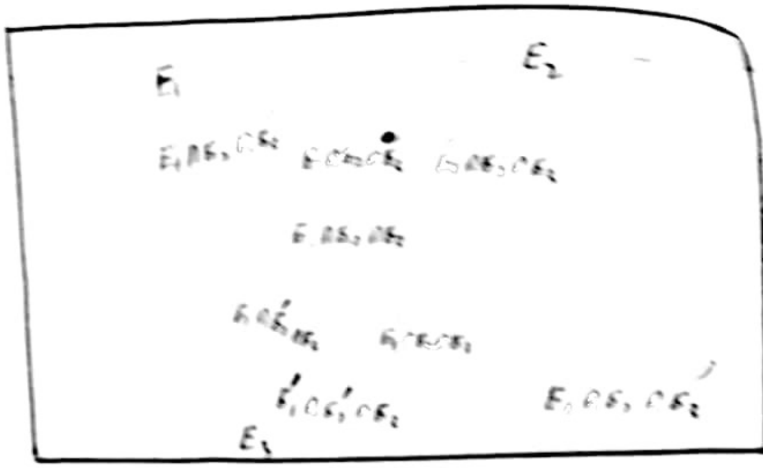
Dividing by $n(S)$ we get

$$\frac{n(E_1 \cup E_2 \cup E_3)}{n(S)} = \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} + \frac{n(E_3)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)} - \frac{n(E_1 \cap E_3)}{n(S)} - \frac{n(E_2 \cap E_3)}{n(S)} + \frac{n(E_1 \cap E_2 \cap E_3)}{n(S)}$$

$$\Rightarrow P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

Proved

Proof 2:



Let $E = E_1 \cup E_2$

$\Rightarrow P(E) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

By definition of non mutually exclusive events

Let E and E_3 are two events, then

$P(E \cup E_3) = P(E) + P(E_3) - P(E \cap E_3)$ By definition

Substituting the value of E in above equation, we have

$P(E \cup E_3) = P(E_1 \cup E_2 \cup E_3)$

$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E \cap E_3)$ — (5)

Now Consider

$E \cap E_3 = (E_1 \cup E_2) \cap E_3$

$= (E_1 \cap E_3) \cup (E_2 \cap E_3)$

$\Rightarrow P(E \cap E_3) = P(E_1 \cap E_3) + P(E_2 \cap E_3) - P(E_1 \cap E_2 \cap E_3)$ — (6)

Using eq (5) and eq (6) $\Rightarrow$

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2)$

$- P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

which completes the theorem

E_1: Teeth cleaned

E_2: Cavity filled

E_3: Tooth Extracted

$P(E_1) = 0.47$

$P(E_2) = 0.29$

$P(E_3) = 0.22$

$P(E_1 \cap E_2) = 0.08$

$P(E_1 \cap E_3) = 0.06$

$P(E_2 \cap E_3) =$ [illegible]

$P(E_1 \cap E_2 \cap E_3) = 0.03$

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= 0.47 + 0.29 + 0.22 - 0.08 - 0.06 - 0.07 + 0.03$$

$$= 0.98 - 0.21 + 0.03$$

$$= 1.01 - 0.21$$

$$= 0.8$$

Problem:

The probability that a person visiting his dentist will have his teeth cleaned, a cavity filled, a tooth extracted, his teeth cleaned and a cavity filled, his teeth cleaned and a tooth extracted, a cavity filled and a tooth extracted, or his teeth cleaned, a cavity filled and a tooth extracted are 0.47, 0.29, 0.22, 0.08, 0.06, 0.07, and 0.03. What is the probability that the person will have at least one of these things done ?.

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Problem:

A firm has 2 operating systems operating independently. The probability that a specific system is available when needed is 0.96.

a) What is the probability that neither is available when needed?

b) What is the probability that an operating system is available when needed?

E_1 : First operating system is available

E_2 : 2nd " " " "

a) $P(E_1) = P(E_2) = 0.96$ (Given)

$P(E_1') = P(E_2') = 1 - 0.96 = 0.04.$

$P(E_1' \cap E_2') = P(E_1')\,P(E_2')$

$= 0.04 \times 0.04 = 0.0016$ //

b) $P(E_1 \cup E_2) = 1 - P(E_1' \cap E_2')$

$= 1 - 0.0016 = 0.9984.$

or. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= 0.96 + 0.96 - 0.0016$

$= 0.9984$

b) $P(E_1 \cup E_2) = 1 - P(E_1 \cup E_2)$

$= 1 - P(E_1' \cap E_2')$

$= 1 - 0.04 \times 0.04$

$= 1 - 0.0016$

$= 0.9984.$

Independent Trials

Independent Events:

Two non-mutually Exclusive events E_1 and E_2 are said to be independent if chance of occurence of one event does not effect the chance of occurence of other. Then

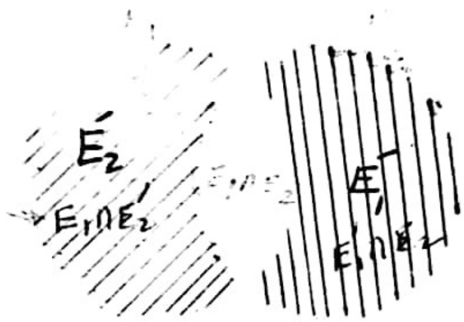
$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

If there are more than two events. If we study three or more events. If we say that there are n events then, indpendence mean.

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1) \cdot P(E_2) \cdots\cdots P(E_n).$$

Theorem:

Theorem: If E_1 and E_2 are independent then E_1
are also independent



Proof: Given that $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

To prove that $P(E_1 \cap E_2') = P(E_1) \cdot P(E_2')$

By definition.

$$P(E_1 \cap E_2') = P(E_1) - P(E_1 \cap E_2)$$

$$= P(E_1) - P(E_1) \cdot P(E_2) \quad \text{E_1 and E_2 are independent}$$

$$= P(E_1) [1 - P(E_2)]$$

$$= P(E_1) \cdot P(E_2').$$

$$\boxed{P(E_1 \cap E_2') = P(E_1) \cdot P(E_2')}$$ Proved.

Theorem: If two events E_1 and E_2 are independent then E_1' and E_2' are also independent.

Proof: Given that $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

To prove that $P(E_1' \cap E_2') = P(E_1') \cdot P(E_2')$

By Demorgan's Law we have

$$
\begin{aligned}
P(E_1' \cap E_2') &= P(E_1 \cup E_2)' \\
&= 1 - P(E_1 \cup E_2) \\
&= 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2) \\
&= 1 - P(E_1) - P(E_2) + P(E_1) \cdot P(E_2) \\
&= (1 - P(E_1)) - P(E_2)[1 - P(E_1)] \\
&= (1 - P(E_1))[1 - P(E_2)] \\
&= P(E_1') \cdot P(E_2')
\end{aligned}
$$

$$\boxed{P(E_1' \cap E_2') = P(E_1') \cdot P(E_2')}$$ Proved.

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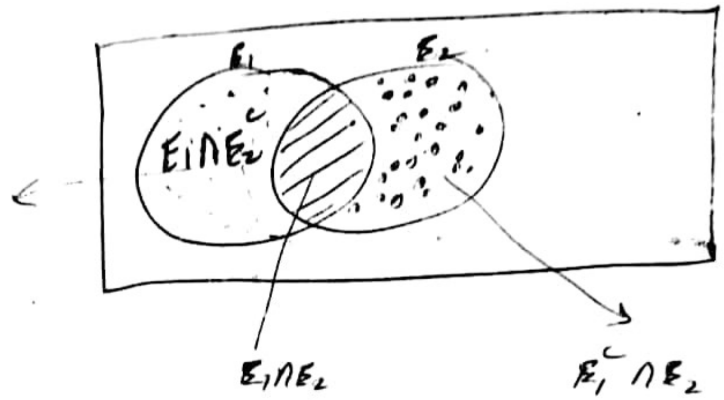
Prove

i) $P(E_1 \cap E_2^c) = P(E_1) - P(E_1 \cap E_2)$

ii) $P(E_1^c \cap E_2) = P(E_2) - P(E_1 \cap E_2)$

$E_1 \cap E_2^c$



i) Let $E_1 = (E_1 \cap E_2^c) \cup (E_1 \cap E_2)$

$\Rightarrow P(E_1) = P[(E_1 \cap E_2^c) \cup (E_1 \cap E_2)]$

$= P(E_1 \cap E_2^c) + P(E_1 \cap E_2)$

$\Rightarrow \boxed{P(E_1 \cap E_2^c) = P(E_1) - P(E_1 \cap E_2)}$

ii) Let $E_2 = (E_1^c \cap E_2) \cup (E_1 \cap E_2)$

$P(E_2) = P[(E_1^c \cap E_2) \cup (E_1 \cap E_2)]$

$= P(E_1^c \cap E_2) + P(E_1 \cap E_2)$

$\Rightarrow \boxed{P(E_1^c \cap E_2) = P(E_2) - P(E_1 \cap E_2)}$

A vendor's experience has shown that, in units of a particular product 4 out of 100 electronic components have fabrication errors and 3 out of 100 have both fabrication errors and the presence of impurities. If a particular bin contains components of which 8% have impurities, what is the probability of finding, in a unit from that bin

a) fabrication errors or impurities?

b) neither fabrication errors nor impurities?

c) only impurities?

Sol

I : presence of impurities $\qquad P(I) = 0.08$

$\qquad\qquad\qquad\qquad\qquad P(F) = 0.04$

F : Presence of fabrication $\qquad P(F \cap I) = 0.03$

a) $P(F \cup I) = P(F) + P(I) - P(F \cap I)$

$= 0.04 + 0.08 - 0.03 = 0.09$ //

b) $P(F \cup I)' = 1 - P(F \cup I) = 1 - 0.09 = 0.91$

c) $P(I \cap F') = P(I) - P(I \cap F) = 0.08 - 0.03 = 0.05$

Independent Events:

Two non-mutually exclusive events E_1 and E_2 are said to be independent if chance of occurance of one event does not effect the chance of occurance of other. Then

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) \quad \text{--- (1)}$$

If we study three or more events we may represent them by $E_1, E_2, \ldots, E_m$. If we say that these m events are independent provided the probability of their intersection is equal to the product of their respective probabilities:

$$P(E_1 \cap E_2 \cap \cdots \cap E_m) = P(E_1) \cdot P(E_2) \cdot \cdots \cdot P(E_m) \quad \text{--- (2)}$$

Mutual Independence or Complete Independence:

The m events are said to be completely independent iff every combination of these events, taken any number of time, is independent. For $m=3$, complete independence of E_1, E_2, E_3 means that the following equations are satisfied:

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3) &= P(E_1) \cdot P(E_2) \cdot P(E_3) \\ P(E_1 \cap E_2) &= P(E_1) \cdot P(E_2) \\ P(E_1 \cap E_3) &= P(E_1) \cdot P(E_3) \\ P(E_2 \cap E_3) &= P(E_2) \cdot P(E_3) \end{aligned} \quad \text{--- (3)}$$

If eqs (3) are satisfied then

$$P(E_1' \cap E_2 \cap E_3) = P(E_1') \cdot P(E_2) \cdot P(E_3)$$

or $P(E_1' \cap E_2' \cap E_3) = P(E_1') \cdot P(E_2') \cdot P(E_3)$

or $P(E_1' \cap E_2' \cap E_3') = P(E_1') \, P(E_2') \cdot P(E_3')$ must holds.

Pairwise Independence:

The ~~three~~ events are said to be Pairwise independent iff the following equations are satisfied:

$$P(E_1 \cap E_2 \cap E_3) \neq P(E_1) \cdot P(E_2) \cdot P(E_3)$$

$$\left.\begin{aligned} P(E_1 \cap E_2) &= P(E_1) \cdot P(E_2) \\ P(E_1 \cap E_3) &= P(E_1) \cdot P(E_3) \\ P(E_2 \cap E_3) &= P(E_2) \cdot P(E_3) \end{aligned}\right\} \quad (4)$$

Example: (Pairwise Independence):

Two coins are tossed. If E_1 is the event "head on first coin", E_2 the event "head on second coin", and E_3 the event "the coins match; both are heads or tails", we show that they are not completely independent there is a pairwise independence.

Sol:

E_1 : head on first coin $\qquad P(E_1) = \frac{1}{2}$

E_2 : head on second coin $\qquad P(E_2) = \frac{1}{2}$

E_3 : both have head or tail $\qquad P(E_3) = \frac{1}{2}$

$$\begin{bmatrix} HH \\ HT \\ TH \\ TT \end{bmatrix}$$

$P(E_1 \cap E_2) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$

$P(E_2 \cap E_3) = \frac{1}{4}$

$P(E_1 \cap E_3) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

$$P(E_1 \cap E_2 \cap E_3) = \frac{1}{4} \neq P(E_1) \cdot P(E_2) \cdot P(E_3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

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Example: (Complete or mutual Independence):

A coin is tossed, a card is drawn independently, from a deck, and a die is thrown. What is the probability that we observe a head on the coin, an ace from the deck, and a five on the die?

Sol:

$$P(\text{Head}) = \frac{1}{2}, \quad P(\text{ace}) = \frac{1}{13}, \quad P(5 \text{ on die}) = \frac{1}{6}.$$

$$P(\text{head and ace and } 5) = \frac{1}{2} \times \frac{1}{13} \times \frac{1}{6} = \frac{1}{156}$$

defect → crack

Example: Flawless Shoes:

In a shoe factory, uppers, soles, and heels are manufactured separately and randomly assembled into single shoes. Five percent of the uppers, four percent of the soles, and one percent of the heels have flaws. What percent of the pairs of shoes are flawless in these three parts?

Sol:

U : (flawed) upper $\quad U'$: (unflawed) upper

S : " sole $\quad S'$: (") sole

H : " heel $\quad H'$: (") heel

$P(U) = 0.05 \Rightarrow P(U') = 1 - 0.05 = 0.95$

$P(S) = 0.04 \Rightarrow P(S') = 1 - 0.04 = 0.96$

$P(H) = 0.01 \Rightarrow P(H') = 1 - 0.01 = 0.99$

$$P(U' \cap S' \cap H') = P(U') \cdot P(S') \cdot P(H')$$

$$= 0.95 \times 0.96 \times 0.99 \cong 0.903$$

$$\boxed{P(U' \cap S' \cap H') \simeq 0.903}$$

This is the probability that one shoe is unflawed.

Assuming that pairs are also randomly assembled, we would have.

P(both shoes unflawed) = P(Left and Right unflawed)

$= P(\text{Left unflawed}) \cdot P(\text{Right unflawed})$

$= 0.903 \times 0.903$

$\simeq 0.815$

Problem

A firm has 2 system engineers operating indep[endently]. The probability that a specific engineer is available when needed is 0.96

a) What is the probability that neither is available when needed?

b) What is the probability that an ~~extra engine~~ engineer is available when needed?

Sol:

E_1: 1st engineer is available

E_2: 2nd Engineer is available

a) $P(E_1) = P(E_2) = 0.96$

$P(E_1') = P(E_2') = 1 - P(E_2) = 1 - 0.96 = 0.04$

$P(E_1' \cap E_2') = P(E_1') P(E_2')$

$= 0.04 \times 0.04 = 0.0016$

b) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= 0.96 + 0.96 - 0.96 \times 0.96$

$= 0.9984$

or $\to 1 - P(E_1 \cup E_2)'$

$P(E_1 \cup E_2) = 1 - P(E_1' \cap E_2')$

$= 1 - 0.04 \times 0.04$

$= 0.9984$

Conditional Probability :

The probability of an event depending on the outcomes of another event. The sample space of the experiment must often be reduced when some additional information partaining to the outcomes of the experiment is received. The probabilities associated with such a reduced sample space are called conditional probabilities

If S be the sample space and E_1 and E_2 are the two events defined in S. The probability that E_2 occur given that E_1 has already occured is denoted by

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} \quad \text{or} \quad P(E_1 \cap E_2) = P(E_1) \cdot P(E_2/E_1)$$

and

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \quad \text{or} \quad P(E_1 \cap E_2) = P(E_2) \cdot P(E_1/E_2)$$

If E_1 and E_2 are independent events then

$P(E_2/E_1) = P(E_2)$ and $P(E_1/E_2) = P(E_1)$

$\Rightarrow P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

Example :

The probability that a die shows 3 given that die shows odd numbers ?

Let E_1 : Die shows odd numbers (Three cases)

E_2 : Die shows face 3 (one case)

$\Rightarrow n(E_1 \cap E_2) = 1$ and $n(E_1) = 3$

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{n(E_1 \cap E_2)}{n(E_1)} = 1/3$$

$P(E_1 \cap E_2 \cap \cdots \cap E_m) = P(E_1) \cdot P(E_2/E_1) \cdot P(E_3/E_1 \cap E_2) \cdots P(E_m/E_1 \cap E_2 \cap \cdots \cap E_{m-1})$

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The probability that a married man watches a certain television show is 0.4 and the probability that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7, Find the probability that

a) a married couple watches the show

M: man watches the show $\quad P(M) = 0.4$

WF: Woman watches the show $\quad P(W) = 0.5$

$P(M/W) = 0.7$

$$P(M/W) = \frac{P(M \cap W)}{P(W)}$$

$$P(M \cap W) = P(W) \cdot P(M/W) = 0.5 \times 0.7 = 0.35$$

b) a wife watches the show given that her husband does

$$P(W/M) = \frac{P(M \cap W)}{P(M)} = \frac{0.35}{0.40} = 7/8 \;//$$

c) Atleast 1 person of a married couple will watch the show

$$P(M \cup W) = P(M) + P(W) - P(M \cap W)$$
$$= 0.40 + 0.5 - 0.35$$
$$= 0.9 - 0.35$$
$$= 0.55 \;//$$

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Let 10% of males and 2% of females be Colour-blind, and that males and females each form 50% of the population. A research worker studying Colour blindness selects a Colour-blind person at random what is the probability that the person so selected is

a) male   b) female

Sol:

Sex \ Type	(C) Colour blind	(N) Normal	Total.
Male (M)	5	45	50
female (F)	1	49	50
Total	6	94	100

a) $P(M/C) = \frac{P(M \cap C)}{P(C)} = \frac{5/100}{6/100} = \frac{5}{6}$ //

b) $P(F/C) = \frac{P(F \cap C)}{P(C)} = \frac{1/100}{6/100} = 1/6$ //

m :

Let 10% of males and 2% of females be colour-blind, and that males and females each form 50% of the population. A research worker studying colour blindness selects a colour-blind person at random what is the probability that the person so selected is

a) male b) female

Sol :

Sex \ Type	(C) colour blind	(N) Normal.	Total.
Male (M)	5	45	50
female (F)	1	49	50
Total	6	94	100

a) $$P(M/C) = \frac{P(M \cap C)}{P(C)} = \frac{5/100}{6/100} = \frac{5}{6}$$

b) $$P(F/C) = \frac{P(F \cap C)}{P(C)} = \frac{1/100}{6/100} = 1/6$$

Problem :

Suppose that of a group of people surveyed 30% own have both house and a car, 40% own have a car but not a house, 10% a house but not a car and 20% own neither of them. Suppose it is known that a randomly selected person owns a car, find the probability that he owns a house. Are the two events independent ?

Sol :

E_1 : Person has own car E_1' : not has own car

E_2 : Person has own house E_2' : not has own house

Car / house	E_1 (C)	E_1' (C')	P(H) P(E_2)
E_2 H	0.30	0.10	0.40
E_2' H'	0.40	0.20	0.60
P(E_1) P(C)	0.70	0.30	1.00

$$P(H/C) = \frac{P(H \cap C)}{P(C)} = \frac{0.30}{0.70} = \frac{3}{7}$$

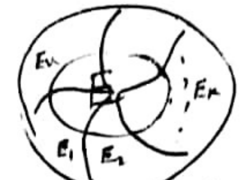
checking. $P(E_1 \cap E_2) = 0.30$, $P(C \cap H) = P(C) \times P(H)$

By multiplication rule of independent event

$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) = 0.70 \times 0.40 = 0.0028$

$\Rightarrow \; P(E_1 \cap E_2) \neq 0.0028$

$0.30 \neq 0.0028$. They are dependent events



1-5 slides

Baye's Theorem:

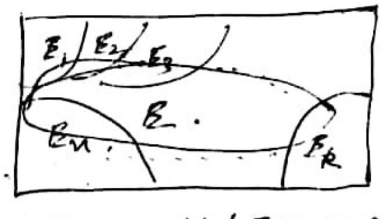
If the events $E_1, E_2, \ldots, E_n$ constitute n partition of the sample space S, where $P(E_i) \neq 0$ for $i = 1, 2, \ldots, n$, then for any event E define in the sample space S such that $P(E) \neq 0$. The probability that an event E_k occur given that event E has occurred is given by.

$$P(E_k/E) = \frac{P(E_k)\cdot P(E/E_k)}{P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2) + \cdots + P(E_n)\cdot P(E/E_n)}$$

$$\boxed{P(E_k/E) = \frac{P(E_k)\cdot P(E/E_k)}{\sum_{i=1}^{n} P(E_i)\cdot P(E/E_i)}} \quad \text{—— (1)}$$

Proof:

Suppose 'S' be the sample space and let events $E_1, E_2, \ldots, E_n$ be the n partitions (mutually exclusive events) of the sample space S. Let E be any other event (subset of S) such that $P(E) > 0$, Then



$$E = E \cap S$$

$$= E \cap (E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n)$$

By distributive law, we have

$$E = (E \cap E_1) \cup (E \cap E_2) \cup (E \cap E_3) \cup \cdots \cup (E \cap E_n)$$

By additional law of mutually Exclusive events

$$P(E) = P(E \cap E_1) + P(E \cap E_2) + P(E \cap E_3) + \cdots + P(E \cap E_n)$$

By the definition of Conditional probability

$$P(E \cap E_1) = P(E_1)\cdot P(E/E_1),\ P(E/E_2) = P(E_2)\cdot P(E/E_2),$$

$$\cdots\ P(E \cap E_n) = P(E_n)\cdot P(E/E_n)$$

$$\Rightarrow P(E) = P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2) + P(E_3)\cdot P(E/E_3) + \cdots$$

$$\cdots + P(E_n)\cdot P(E/E_n) \quad \text{—— (2)}$$

For any kth event, the conditional probability of E_k given E is defined as

$$P(E_k/E) = \frac{P(E_k \cap E)}{P(E)} \qquad \text{③}$$

Substituting the value of $P(E)$ from eq ② in eq ③ we get

$$P(E_k/E) = \frac{P(E_k) \cdot P(E/E_k)}{P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) + \cdots + P(E_n) \cdot P(E/E_n)}$$

$$\boxed{P(E_k/E) = \frac{P(E_k) \cdot P(E/E_k)}{\sum_{i=1}^{n} P(E_i) \cdot P(E/E_i)}}$$

//.

where $P(E_k \cap E) = P(E_k) \cdot P(E/E_k)$.

Pairwise exclusive and exhaustive events

Two or more events occur simultaneously and they are exclusive to each other and the union of all events comprises together to form a whole population are called pairwise exclusive and exhaustive events.

Problem. vet

Problem:

Let 10% of males and 2% of females be colour-blind, and that males and females each form 50% of the population. A research worker studying colour blindness. Selects a colour-blind person at random. What is the probability that the person so selected is a) Male b) Female.

Sol:

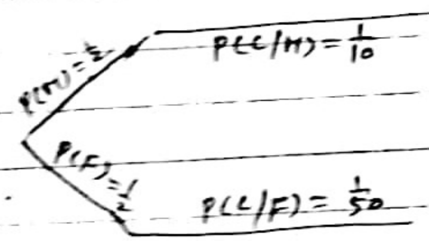
M : Male is selected

F : Female is selected

C : Colour-blind person is selected.

$P(M) = \frac{50}{100} = \frac{1}{2}$; $P(F) = \frac{50}{100} = \frac{1}{2}$

$P(C/M) = \frac{10}{100} = \frac{1}{10}$; $P(C/F) = \frac{2}{100} = \frac{1}{50}$



$P(M)\,P(C/M) = \frac{1}{2} \times \frac{1}{10} = \frac{1}{20}$

$P(F) \cdot P(C/F) = \frac{1}{2} \times \frac{1}{50} = \frac{1}{100}$

Using Baye's Theorem

a) $$P(M/C) = \frac{P(M) \cdot P(C/M)}{P(C)} = \frac{\frac{1}{20}}{\frac{1}{20} + \frac{1}{100}} = \frac{1}{20} \times \frac{\overset{5}{100}}{6}$$

$$\boxed{P(M/C) = \frac{5}{6}}$$

b) $$P(F/C) = \frac{P(F) \cdot P(C/F)}{P(C)} = \frac{\frac{1}{100}}{\frac{1}{20} + \frac{1}{100}} = \frac{1}{100} \times \frac{100}{6}$$

$$\boxed{P(F/C) = \frac{1}{6}}$$

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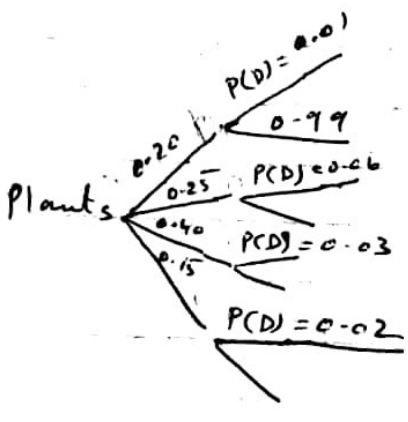
Problem

A company producing electric motors has 4 ass[illegible] plants $P_i (i=1,2,3,4)$ producing 20%, 25%, 40% and 15[illegible] respectively, of its output. The probabilities that a mo[illegible] produced by these plants is defective are 0.01, 0.06, 0.03, and 0.02, respectively. If a motor is chosen at random, what is the probability that it is defective? If a motor, chosen at random, is found to be defective, what is the probability that it came from plant P_3?

solution



Let D: be the event that the motor chosen is defective

E_i: be the event that motor came from plant P_i, $i = 1,2,3,4$.

$$P(D) = P(E_1) \cdot P(D/E_1) + P(E_2) \cdot P(D/E_2) + P(E_3) \cdot P(D/E_3) + P(E_4) \cdot P(D/E_4)$$

$$= (0.20)(0.01) + (0.25)(0.06) + (0.40)(0.03) + (0.15)(0.02)$$

$$P(D) = 0.032$$

$$P(E_3/D) = \frac{P(E_3) \cdot P(D/E_3)}{P(D)} = \frac{P(E_3) \cdot P(D/E_3)}{\sum_{i=1}^{4} P(E_i) \cdot P(D/E_i)}$$

$$= \frac{(0.40)(0.03)}{0.032} = \frac{0.012}{0.032} = 0.375$$

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Q5 Suppose that coloured balls are distributed in three indistinguishable boxes as follows.

Colour	Box 1	Box 2	Box 3	Total
Red	2	4	3	9
White	3	1	4	8
Blue	5	3	3	11
Total	10	8	10	28

A box is selected at random from which a ball is drawn at random.

a) Find the probability that the ball is red.

b) Given that the ball is red, what is the probability that box 3 was selected?

Sol:

E : A red ball is selected

B_1 : Box 1 is selected

B_2 : Box 2 is selected

B_3 : Box 3 is selected.

$P(B_1) = 1/3$ $\quad P(E/B_1) = \frac{2}{10} = \frac{1}{5}$ $\quad P(E_1) \cdot P(E/E_1) = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$

$P(B_2) = 1/3$ $\quad P(E/B_2) = \frac{4}{8} = 1/2$ $\quad P(E_2) \cdot P(E/E_2) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$P(B_3) = \frac{1}{3}$ $\quad P(E/B_3) = \frac{3}{10}$ $\quad P(E_3) \cdot P(E/E_3) = \frac{1}{3} \times \frac{3}{10} = \frac{1}{10}$

$$P(E) = P(E_1) \cdot P(E/B_1) + P(B_2) \cdot P(E/B_2) + P(B_3) \cdot P(E/B_3) =$$

$$= \frac{1}{15} + \frac{1}{6} + \frac{1}{10} = \frac{2+5+3}{30} = \frac{10}{30} = \frac{1}{3}$$

$$\therefore \frac{1}{3}$$

b) What is the Probability that the box 3 was selected?

$$P(B_3/E) = \frac{P(B_3)\cdot P(E/B_3)}{P(B_1)\cdot P(E/B_1) + P(B_2)\cdot P(E/B_2) + P(B_3)\cdot P(E/B_3)}$$

$$= \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{3}{10}$$

$$\boxed{P(B_3/E) = \frac{3}{10}}$$