Poisson Distribution :

Suppose that an experiment with two possible outcomes S and f and $P(S) = P$ and $P(f) = 1-P$. is repeated independently and indefinitely. Let P be small ($p \to 0$), such that $np \to \lambda$ as $n \to \infty$. Then, the probability distribution of the number of successes is

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \quad ; \quad x = 0, 1, 2, \dots, \infty.$$

Which is called as poisson distribution with parameter λ.

Derivation :-

The probability of x sucesses in n trials in the bionomial distribution $\cdot$ $b(x; n, p)$ is given by

$$P(X=x) = \binom{n}{x} p^{x} q^{n-x} ;$$

$$= \frac{n(n-1)\cdots(n-x+1)(n-x)!}{(n-x)!\; x!} p^{x} q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} p^{x} q^{n-x}$$

Let $np = \lambda$, $\Rightarrow P = \frac{\lambda}{n}$ and $q = (1 - \frac{\lambda}{n})$

replacing P and q in terms of λ. $\Rightarrow$

$$P(X=x) = \frac{n(n-1)\cdots(n-x+1)}{x!} \left(\frac{\lambda}{n}\right)^{x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$P(X=x) = \frac{\lambda^{x}}{x!} \cdot \frac{n(n-1)\cdots(n-x+1)}{n^{x}} \cdot \left(1-\frac{\lambda}{n}\right)^{n} \cdot \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$x) = \frac{\lambda^x}{x!}\frac{n^x}{n^x}\cdot\left[1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x}{n}+\frac{1}{n}\right)\right]\cdot\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\cdot\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

Since λ remains fixed and n becomes large we observe that each of the term.

$\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right] \longrightarrow 1$ and.

$\left(1-\frac{\lambda}{n}\right)^{-x} \longrightarrow 1$ for large n.

$$P(X=x) = \frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n$$

The term $\left(1-\frac{\lambda}{n}\right)^n = \left[\left(1-\frac{\lambda}{n}\right)^{\frac{n}{\lambda}}\right]^\lambda$ $\quad \because k=\frac{n}{\lambda}$, $\frac{\lambda}{n}=\frac{1}{k}$

$$\left(1-\frac{\lambda}{n}\right)^n = \left[\left(1-\frac{\lambda}{n}\right)^k\right]^\lambda$$

If n increases indefinitely, so for each k.

$\left[\left(1-\frac{1}{k}\right)^k\right]$ tends to e^{-1} where $e = 2.71828$

Thus.

$$\left[\left(1-\frac{1}{k}\right)^k\right]^\lambda \rightarrow e^{-\lambda}$$

$$\Rightarrow \lim_{n\to\infty} P(X=x) = \lim_{n\to\infty}\frac{\lambda^x}{x!}\left[\left(1-\frac{\lambda}{k}\right)^k\right]^\lambda$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left[\left(1-\frac{1}{k}\right)^k\right]^\lambda$$

$$\boxed{P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda}}$$

$\therefore \lim_{n\to\infty}\left(1-\frac{1}{k}\right)^k = e^{-1}$

$k=\frac{n}{\lambda}$.

Moments :

Let X be a random variable with the Poisson distribution $P(X;\lambda)$ then.

$$\mu_r' = E(X^r) = \sum_{x=0}^{\infty} X^r P(X;\lambda)$$

$\underline{r=1}$ $\mu_1' = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, \ldots \infty.$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{x\lambda^x}{x!}$$

$$= e^{-\lambda}\left[0 + 1 \cdot \frac{\lambda}{1!} + 2 \cdot \frac{\lambda^2}{2!} + 3 \cdot \frac{\lambda^3}{3!} + \cdots\right]$$

$$= e^{-\lambda}\left[\lambda + \lambda^2 + \frac{\lambda^3}{2!} + \frac{\lambda^4}{4!} + \cdots\right]$$

$$= \lambda e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} \qquad \therefore \left[e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots\right]$$

$$\boxed{\mu_1' = \lambda} = \text{Mean} = E[x]$$

$$\mu_2' = E[X^2] = E[X(X-1) + X] = E[X(X-1)] + E[X]$$

$$= E[X(X-1)] + \lambda.$$

$$E[X(X-1)] = \sum_{x=0}^{\infty} X(X-1)\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[0 + 0 + 2 \cdot 1\frac{\lambda^2}{2!} + 3 \cdot 2\frac{\lambda^3}{3!} + 4 \cdot 3\frac{\lambda^4}{4!} + 5 \cdot 4\frac{\lambda^5}{5!} + \cdots\right]$$

$$= \lambda^2 e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= \lambda^2 e^{-\lambda} \cdot e^{\lambda} = \boxed{\lambda^2 = E[X(X-1)]}$$

$$\boxed{E(X^2) = \lambda + \lambda^2}$$

$$\mu_3' = E[X^3] = E[X(X-1)(X-2) + 3X(X-1) + X]$$

$$= E[X(X-1)(X-2)] + 3E[X(X-1)] + E[X]$$

$$= E[X(X-1)(X-2)] + 3[\lambda^2] + \lambda$$

$$E[X(X-1)(X-2)] = \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[\frac{3\cdot 2\cdot 1\,\lambda^3}{3!} + \frac{4\cdot 3\cdot 2\,\lambda^4}{4!} + \frac{5\cdot 4\cdot 3\,\lambda^5}{5!} + \frac{6\cdot 5\cdot 4\,\lambda^6}{6!} + \cdots\right]$$

$$= e^{-\lambda}\left[\lambda^3 + \frac{\lambda^4}{1!} + \frac{\lambda^5}{2!} + \frac{\lambda^6}{3!} + \cdots\right]$$

$$= \lambda^3 e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= \lambda^3 e^{-\lambda} e^{\lambda}$$

$$\boxed{E[X(X-1)(X-2)] = \lambda^3}$$

$$\boxed{\mu_3' = \lambda^3 + 3\lambda^2 + \lambda}$$

$$\mu_4' = E[X^4] = E[X(X-1)(X-2)(X-3) + 6X(X-1)(X-2) + 7X(X-1) + X]$$

$$\mu_4' = E[X(X-1)(X-2)(X-3)] + 6\lambda^3 + 7\lambda^2 + \lambda.$$

$E[x(x-1)(x-2)(x-3)]$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)(x-3) \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda}\left[4\cdot3\cdot2\cdot1 \frac{\lambda^4}{4!} + 5\cdot4\cdot3\cdot2 \frac{\lambda^5}{5!} + 6\cdot5\cdot4\cdot3 \frac{\lambda^6}{6!} + 7\cdot6\cdot5\cdot4 \frac{\lambda^7}{7!} + \cdots\right]$$

$$= \lambda^4 e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= \lambda^4 e^{-\lambda} e^{\lambda}$$

$$= \boxed{\lambda^4 = E[x(x-1)(x-2)(x-3)]}$$

$$\Rightarrow \boxed{\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda.}$$

Moments about Mean :

$$\mu_1 = \mu_1' - \mu_1' = 0 \quad \text{(always)}$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\mu_2 = \lambda} = Var(x) \Rightarrow \boxed{S\cdot D = \sqrt{\lambda}}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= (\lambda^3 + 3\lambda^2 + \lambda) - 3(\lambda^2 + \lambda)(\lambda) + 2\lambda^3$$

$$= \cancel{\lambda^3} + \cancel{3\lambda^2} + \lambda - \cancel{3\lambda^3} - \cancel{3\lambda^2} + \cancel{2\lambda^3}$$

$$\boxed{\mu_3 = \lambda}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) - 4\lambda(\lambda^3 + 3\lambda^2 + \lambda) + 6\lambda^2(\lambda^2 + \lambda) - 3\lambda^4$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda - 4\lambda^4 - 12\lambda^3 - 4\lambda^2 + 6\lambda^4 + 6\lambda^3 - 3\lambda^4$$

$$\boxed{\mu_4 = 3\lambda^2 + \lambda}$$

Measure of Skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \Rightarrow \boxed{\beta_1 = \frac{1}{\lambda}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \frac{1}{\lambda}$$

$$\boxed{\beta_2 = 3 + \frac{1}{\lambda}}$$

Measure of Kurtosis

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{1}{\lambda}} = \boxed{\frac{1}{\sqrt{\lambda}} = \gamma_1}$$

$$\gamma_2 = \beta_2 - 3 = 3 + \frac{1}{\lambda} - 3 \qquad \boxed{\gamma_2 = \frac{1}{\lambda}}$$

POISSON

em: Q8

A Secretary makes 2 typing errors per page on the average. What is the probability that on the next page she makes

a) 4 or more errors   b) no errors

c) Atleast 2 errors.

Sol:

a) $P(X \geq 4)$ Mean $= \lambda = 2$

$$= \sum_{x=4}^{\infty} P(x; 2) = 1 - P(X < 4)$$

$$= 1 - \sum_{x=0}^{3} P(x; 2)$$

$$= 1 - 0.8571$$

$$= 0.1429$$

b) $P(X=0) = P(0; 2) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-2} 2^{0}}{0!} = e^{-2} = 0.1355$

c) $P(X \geq 2) = 1 - P(X<2) = \sum_{x=2}^{\infty} P(x; \lambda)$

$$= 1 - \sum_{x=0}^{1} P(x; 2)$$

$$= 1 - \frac{e^{-2} 2^{0}}{0!} - \frac{e^{-2} 2^{1}}{1!}$$

$$= 1 - 0.1353 - 0.2707$$

$$= 1 - 0.4060$$

$$= 0.5940$$

OF POISSON DISTRIBUTION:

blem:

The Distribution of typing mistakes committed by a typist is given below. a) Assuming poisson model, find out the expected frequencies

Mistakes per page	0	1	2	3	4	5
No of pages	142	156	69	27	5	1

Sol:

$$\text{Mean} = \frac{\Sigma f x}{\Sigma f} = \frac{400}{400} = \boxed{1 = \lambda} = np.$$

$$f(x) = N\,P(x) = 400 \times \frac{e^{-1}}{x!} \quad ; \; x = 0, 1, 2, \ldots, 5$$

$$P(0) = e^{-\lambda} = e^{-\lambda} = 0.3679$$

X	$P(X) = \frac{e^{-\lambda}\lambda^{x}}{x!}$		Expected frequency $f(x) = N\,P(x)$
0	0.3679	0.3679	$400 \times 0.3679 = 147$
1	0.3679	0.3679	$147.16 \simeq 147$
2	$\frac{0.3679 \times 1^2}{2!}$	$= 0.18395$	$73.58 \simeq 74$
3	$0.3679/3!$	$= 0.061317$	$24.52 \simeq 25$
4	$0.3679/4!$	$= 0.01533$	$6.13 \simeq 6$
5	$0.3679/5!$	$= 0.003066$	$1.22 \simeq 1$

~~b) Is the Skewness between observed and Expected data set remains same~~

~~c) Is the data set Skewed~~

b) Does observed and Expected data suggests equal skewness and Kurtosis?

Q7. On the average a certain intersection results in 3 traffic accidents per month. What is the probability that in any given month at this intersection.

a) exactly 5 accidents will occur?

b) Less than 3 accidents will occur?

c) at least 2 accidents will occur?

Sol:

$\bar{x} = 3 \Rightarrow np = \boxed{3 = \lambda}$

we have to find

a) $P(x=5, \lambda=3) = \sum_{x=0}^{5} p(x;\lambda) - \sum_{x=0}^{4} p(x;\lambda)$

$=$

$= 0.1008$

b) $P(X<3) = \sum_{x=0}^{2} P(x;\lambda) = 0.4232$

c) $P(X \geq 2) = 1 - \sum_{x=0}^{1} P(x;\lambda) = 1 - 0.1991 = 0.8009$