TEST OF HYPOTHESES

by

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Test of Hypotheses

The most important area of decision theory

Hypothesis: Simply a statement about one or more population (about some universe characteristics)

Statistical hypothesis: A statistical hypothesis is an assumption or statement, which may or may not be true, concerning one or more population.

Null hypothesis: A statistical hypothesis stated for the purpose of possible acceptance is called null hypothesis and it is denoted by H_0, e.g.,

$$H_0 : \mu = 65 \text{ mm}$$

(Is the average rain fall at Karachi City equal to 65mm?)

Alternative hypothesis: An alternative hypothesis is any other hypothesis which we may accept when null hypothesis is rejected. It is usually denoted by H_A or H_1. e.g.,

$$H_A : \mu > 65 \text{ mm}$$

Test Statistic:
Any statistic that provides the basis for testing a statistical hypothesis is called a test statistic. Some commonly used test statistics are z, t, χ^2 and F.

Z - Statistic: [When σ^2 is known]
When sampling is from a normally distributed population and population variance is known. The test statistic used for testing $H_0: \mu = \mu_0$ is

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Provided n is sufficiently large. When H_0 is true $\bar{x}$ is distributed as standard normal distribution $N(0,1)$.

t - Statistic: [when σ^2 is unknown]
when sampling is from a normally distributed population with an unknown variance, the test statistic used for testing the $H_0: \mu = \mu_0$ is

$$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Provided n is small. When H_0 is true, T is distributed as student's t-distribution with $n-1$ degrees of freedom.

Simple hypothesis: A hypothesis, whether null or alternative, might specify just a single value say μ_0, for the population parameter μ. In that case the hypothesis is said to be simple say:

$$H_0 : \mu = \mu_0$$

Composite hypothesis: A hypothesis may also be specify for a range of values for the unknown population parameter such a hypothesis is said to be composite

$$\underbrace{\begin{matrix} H_0 : \mu = \mu_0 \\ H_A : \mu < \mu_0 \end{matrix} \quad \text{or} \quad \begin{matrix} H_0 : \mu = \mu_0 \\ H_A : \mu > \mu_0 \end{matrix}}_{\text{one-sided}} \qquad \underbrace{\begin{matrix} H_0 : \mu = \mu_0 \\ H_A : \mu \neq \mu_0 \end{matrix}}_{\text{Two-sided}}$$

Normal Distribution:

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \;; \quad \begin{matrix} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{matrix}$$

Let $z = \frac{x-\mu}{\sigma}$

$$f(z;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \;; \quad \begin{matrix} -\infty < z < \infty \\ \mu = 0 \\ \sigma^2 = 1 \end{matrix}$$

which is known as Standard Normal Distribution

Student t - Distribution

$$f(t,\nu) = \frac{1}{B\left(\frac{\nu}{2},\frac{1}{2}\right)\sqrt{\nu}} \left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} \;; \; -\infty < t < \infty$$

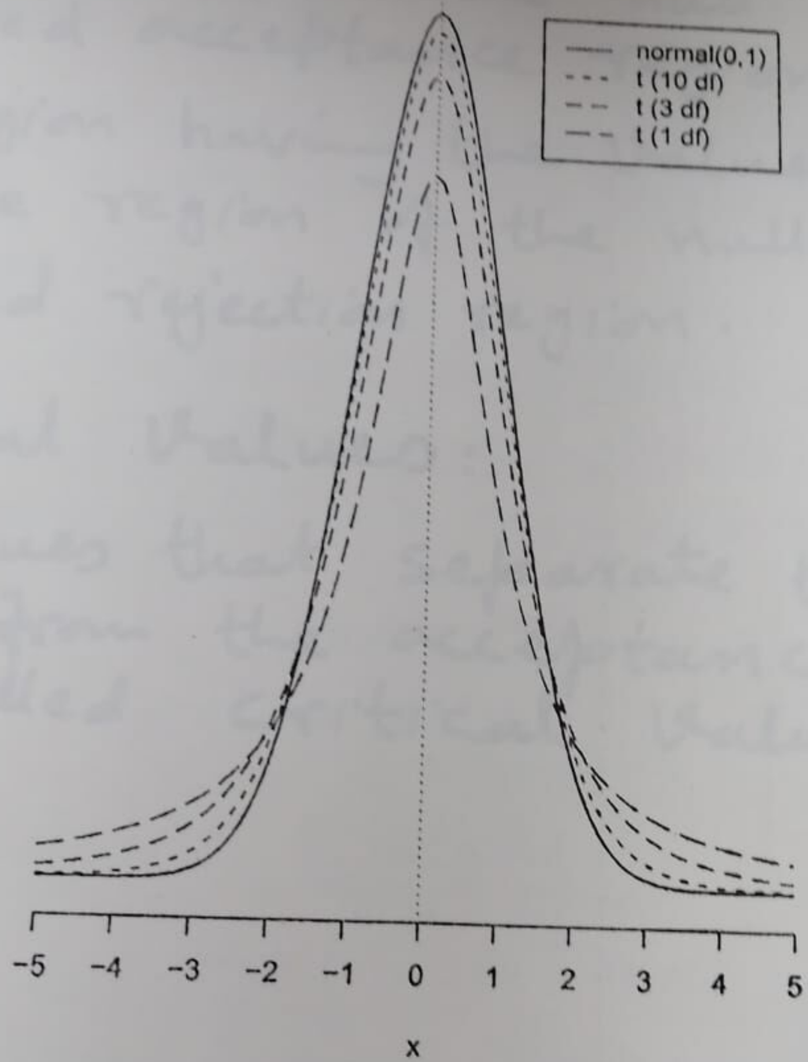
Chi - Square Distribution:

$$f(\chi^2,\nu) = \frac{1}{2^{\nu/2}\,\Gamma(\nu/2)} (\chi^2)^{\frac{\nu}{2}-1} e^{-\chi^2/2} \;, \; 0 \le \chi^2 \le \infty$$

F - Distribution:

$$f(x;\nu_1,\nu_2) = \frac{(\nu_1/\nu_2)^{\frac{\nu_1}{2}}\, x^{\frac{\nu_2}{2}-1}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)\left(1+\frac{\nu_1}{\nu_2}x\right)^{\frac{\nu_1+\nu_2}{2}}} \;; \; 0 \le x \le \infty$$

Distribution	Mean	Variance
Normal	μ	σ^2
t - dist	0	$\frac{\nu}{\nu-2}$, $\nu > 2$
χ^2 - dist	ν	2ν.
F - dist	$\frac{\nu_2}{\nu_2-2}$, $(\nu_2 > 2)$	$\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$ $(\nu_2 > 4)$


normal(0,1)
t (10 df)
t (3 df)
t (1 df)
-5 -4 -3 -2 -1 0 1 2 3 4 5
x

Acceptance and Rejection Region:

The region consisting of the values of a test statistic that appear to be consistent with the null hypothesis is called acceptance region. Whereas, the region having the values that lead to the region of the null hypothesis is called rejection region.

Critical Values:

The values that separate the critical region from the acceptance region are called critical values.

Decision Rules:

Every decision which we make in our life is based on a set of assumptions usually, we try to make the "best" decision given that what we know.

Decision regarding the environment can also be put into the hypothesis testing framework.

Table 1. displays the framework in the context of deciding whether contamination is present in the environment

Reality	Your Decision: Contamination	Your Decision: No Contamination
No Contamination	Mistake Type I Error $\alpha = Pr[\text{Type I Error}]$	Correct Decision $1-\alpha = Pr[\text{Correct Decision}]$
Contamination	Correct Decision $1-\beta = Pr[\text{Correct Decision}]$	Mistake Type II Error $\beta = Pr[\text{Type II Error}]$

Table 1: Hypothesis testing framework for deciding on the presence of contamination in the Environment when the null hypothesis is "no contamination".

Thus the decision to accept or reject the null hypothesis is made on the basis of the information supplied by the observed sample observations. The conclusion drawn on the basis of a particular sample may not always be true in respect of population.

Actual Population ↓	Decision from Sample → Reject H_0	Accept H_0
H_0 is true	Wrong decision $\alpha = P_r[\text{Type I Error}]$	Correct decision $1-\alpha = P_r[\text{Correct decision}]$
H_0 is false	Correct decision $1-\beta = P_r[\text{Correct decision}]$	Wrong decision $\beta = P_r[\text{Type II Error}]$

In any testing problem we are liable to commit two types of Errors:

Type I Error: The error of rejecting H_0 (accepting H_A) when H_0 is true is called Type I error.

$$\alpha = P_r[\text{Type I Error}]$$

This probability is called the false positive rate.

Type II Error: The error of accepting H_0 when H_0 is false (H_A is true) is called Type II error

$\beta = P_r[\text{Type II Error}]$ → is called the false negative rate.

How to Calculate Probabilities of Type I and Type II Errors ?

Problem 1:

An environmental engineer working on water contamination has taken 100 random samples of an area over last few months. Assuming that the pH of the samples follow a normal distribution with unknown mean μ and known variance 36. He wishes to test the hypothesis that

$$H_0 : \mu = 6.5 \text{ pH}$$

$$H_A : \mu > 6.5 \text{ pH}$$

He decides on the following criteria:

accept H_0 iff sample mean $\bar{x} \leq 7.5$ pH

reject H_0 iff sample mean $\bar{x} > 7.5$ pH

a) Find the probability of Type I Error

b) Find the probability of Type II Error if he uses $H_A : \mu = 7.6$.

c) Is this a good test procedure?

Solution:

a) $\alpha = Pr[\text{Type I Error}]$

$= Pr[\text{Rejecting } H_0 \mid H_0: \mu = 6.5]$

$= Pr[\bar{x} > 7.5 \mid H_0: \mu = 6.5]$

$= 1 - Pr[\bar{x} \le 7.5 \mid H_0: \mu = 6.5]$

$= 1 - Pr\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le \frac{7.5-6.5}{6/10}\right]$

$= 1 - Pr[Z \le \frac{10}{6}] = 1 - Pr[Z \le \frac{5}{3}]$

$= 1 - Pr[Z \le 1.67]$

$= 1 - 0.9525$ [from Normal Table]

$\boxed{\alpha = 0.0475}$

b) $\beta = Pr[\text{Type II Error}]$

$= Pr[\text{Accepting } H_0 \mid H_A: \mu = 7.6]$

$= Pr[\bar{x} \le 7.5 \mid H_A: \mu = 7.6]$

$= Pr\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le \frac{7.5-7.5}{6/10}\right] = Pr[Z \le -\frac{5}{3}]$

$= Pr[Z \le -1.67]$

$\boxed{\beta = 0.0475}$

c) Small values of α and β indicate that the test procedure is good.

roblem 2:

A shirt issued for military has an average useful life of 85 washings, when used in a moderate climate but will a tropical climate reduces its useful life?. A sample of 60 such shirts worn by soldiers in a tropical climate indicates an average life of 76.4 washings, with a standard deviation of 12.8. At the 0.01 level of significance can we conclude that the use of these shirts in a tropical climate reduces their average useful life?

Sol:

1. Assumption:
2. Hypotheses: $H_0: \mu = 85$ washings
   $H_A: \mu < 85$ washings
2. Level of significance $\alpha = 0.01$
3. Test statistic used:

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

4. Critical Region:

Reject iff $Z_{cal} < Z_\alpha$ (table value) $= -2.31$

5. Computation:

$$Z = \frac{76.4 - 85}{12.8/\sqrt{60}} = -5.2 \ (0.0099)$$

6. Conclusion:

Reject Ho and conclude that these shirts in a tropical climate ~~[illegible]~~ reduces their average useful life.

Problem 3:

The guidance document USEPA (1989b, p. 6-4) contains observations of aldicarb at three compliance wells. These data are displayed in Table 2:

Month	Well 1	Well 2	Well 3
Jan.	19.9	23.7	5.6
Feb.	29.6	21.9	3.3
Mar.	18.7	26.9	2.3
Apr.	24.2	26.1	6.9

Table 2: Aldicarb Concentration (ppb) at the three compliance well, from USEPA (1989b, p. 6-4).

The maximum Contaminant Level (MCL) has been set at 7 ppb. For each well test the null hypothesis that the average aldicarb concentration is less than or equal to the MCL of 7 ppb. Use $\alpha = 0.05$.

Solution:

1) Assumptions
2) Formulate the hypothesis
   1. $H_0 : \mu \leq 7$ ppb
      $H_A : \mu > 7$ ppb
   2. Level of significance $\alpha = 0.05$

Test statistic used:

$$T = \frac{(\bar{x} - \mu_0)}{s/\sqrt{n}}$$; with $n-1$ degrees of freedom.

4. Reject H_0 iff

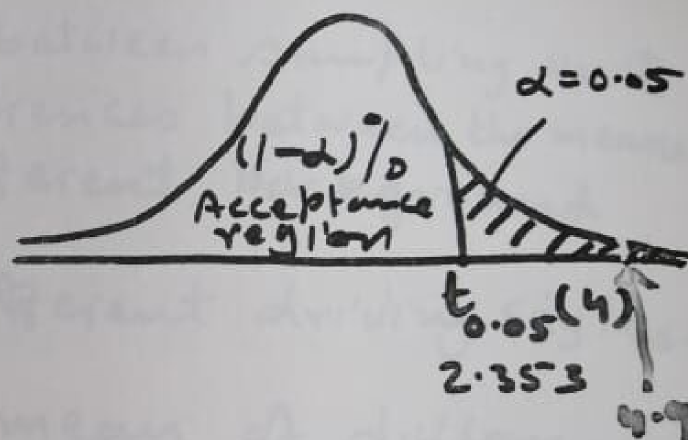
$$T_{cal} > t_\alpha(n-1) = t_{0.05}(3) = 2.353$$

5. Computation:

Well 1) $T_{cal} = \dfrac{23.1 - 7}{4.9/\sqrt{4}} = 4.9$

Well 2) $T_{cal} =$

Well 3)



6. Conclusion:

The average concentration of well 1 is significantly greater than 7 ppb ($p < 0.01$)

Comparing Two Groups

Paired t-test:

- Each observation is a pair of measurements
  e.g.
  1. Water quality upstream and downstream of a road crossing.
  2. Fuel mileage by a taxi driver using two brands of gasoline.
- Natural variability between sampling units might swamp differences between the means
  1. Streams have different background water quality.
  2. Drivers have different driving styles.
- Apply t-test for mean of differences.

Problem 4:

Fuel mileage by taxi drivers using two fuel types:

Driver	1	2	3	4	5	6
Gas A	26.95	20.44	25.05	26.32	29.56	26.6
Gas B	27.01	20	23.41	25.22	30.11	25.44

Driver	7	8	9	10
Gas A	22.93	20.23	33.95	26.01
Gas B	22.23	19.78	33.45	25.22

P-Value :

A P-value (or probability value) is the probability of getting a value of the sample statistic that is at least as extreme as the one found from the sample data, assuming the null hypothesis is true.

Decision

- Reject the null hypothesis if the P-value is less than or equal to the value of level of significance α.
- Accept the null hypothesis if the P-value is greater than α.

Conclusion (Guide Line)

P-value	Interpretation
• Less than 0.01	Highly statistically significant Very strong evidence against Ho.
• 0.01 to 0.05	statistically significant Adequate evidence against Ho.
• Greater than 0.05	In sufficient evidence against Ho.

- Always report P-value, so that other can draw own conclusions.

1. Assumption
2. Hypotheses :

$$H_0 : \mu_d = 0$$
$$H_A : \mu_d > 0$$

3. Level of significance

$$\alpha = 0.05$$

4. Test statistic used

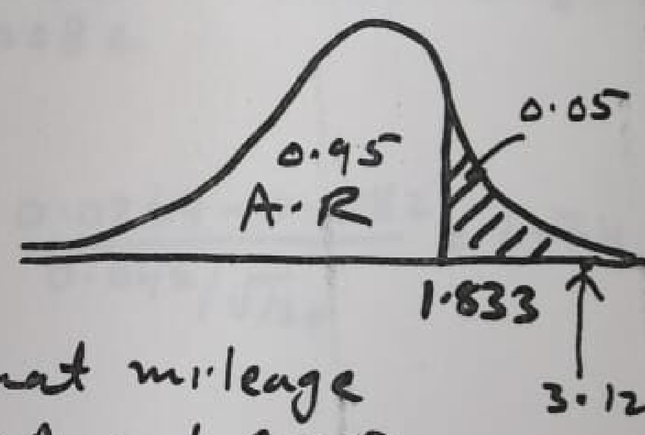
$$T = \frac{\bar{D} - d_0}{S_d/\sqrt{n}} \; ; \quad \nu = n-1 \text{ d.f.}$$

5. Critical Region :

Reject if $T_{cal} > t_{0.05}(n-1) = t_{0.05}(9)$
$= 1.833$

6. Computation :

$$T = \frac{0.606 - 0}{0.614/\sqrt{10}} = 3.12$$



7. Conclusion :

strong evidence that mileage differs between gas A and gas B.

Thus the p-value is 0.003,

Conclusion:

The p-value is less than the significance level $\alpha = 0.05$. so we reject $H_0 : \mu = 0.0082$.

Comparing Two Groups

- Samples are independent
- Each sample observation is roughly the shape of Normal distribution $N(\mu, \sigma^2)$

Case I: Population Variances are known

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}}$$

When H_0 is true then Z is $N(0,1)$ under the assumption $H_0 : \mu_1 = \mu_2$

Case II: Population Variances are unknown

a) $\sigma_1^2 = \sigma_2^2$ (assumed to be equal)

$$T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$

When $H_0 : \mu_1 = \mu_2$ is true

$T \approx$ student's t-distribution with $\nu = n_1 + n_2 - 2$ degrees of freedom

b) $\sigma_1^2 \neq \sigma_2^2$ (assumed to be unequal)

$$T' = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}}$$

with

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left[\frac{(s_1^2/n_1)^2}{(n_1-1)} + \frac{(s_2^2/n_2)^2}{(n_2-1)}\right]}$$

$T' \approx$ when $H_0: \mu_1 = \mu_2$ is true Student's t-distribution with ν degrees of freedom