X_1 depends on X_2

Multiple Regression Equation:

When the values of one variable, are associated with or influenced by other variable, method of Least Square can be used to measure the relationship in terms of an equation, which is of the form

$$X_1 = a + b_{12} X_2$$

Sometimes there is interrelation between many variables and the value of one variable may be influenced by many other, e.g, the yield of crop per acre say (X_1) depends upon quality of seed (X_2), fertility of soil (X_3), fertilizer used (X_4), weather conditions (X_5) and so on. Whenever we are interested in studying the joint effect of a group of variables upon a variable not included in that group, our study is that of multiple correlation and multiple regression

Let us consider a distribution involving three random variables X_1, X_2 and X_3 in which the variable X_1 depends on the values of X_2 and X_3. The equation of multiple regression of X_1 on X_2 and X_3 is of the form.

$$X_1 = a + b_{12.3} X_2 + b_{13.2} X_3 \quad —(1)$$

Multiple Correlation:

The simple correlation between dependent variable and the joint effect of all independent variables on dependent variable is called as multiple correlation.

In a trivariate case, having dependent variable X_1 and independent variable X_2 and X_3 and containing n observations The multiple correlations are

$$R^2_{1.23} = \frac{r_{12}^2 + r_{13}^2 - 2 r_{12} r_{13} r_{23}}{1 - r_{23}^2}$$

similarly

$$R^2_{2.31} = \frac{r_{23}^2 + r_{21}^2 - 2 r_{12} r_{23} r_{31}}{1 - r_{31}^2}$$

and

$$R^2_{3.12} = \frac{r_{31}^2 + r_{32}^2 - 2 r_{12} r_{23} r_{31}}{1 - r_{12}^2}$$

The coefficients of regressions, b's in eq (1) are determined by the principle of LS's i.e., by minimising the sum of the squared of the residuals,

$$S = \sum X_{1.23}^2 = \sum (X_1 - \hat{X}_1)^2$$

error of the estimates or residual.

$$= \sum (X_1 - a - b_{12.3}X_2 - b_{13.2}X_3)^2 \quad \text{--- (2)}$$

The normal equations are obtained by differentiating equation (2) with respect to a, $b_{12.3}$ and $b_{13.2}$, we have

$$\frac{\partial S}{\partial a} = 2\sum (X_1 - a - b_{12.3}X_2 - b_{13.2}X_3)(-1) = 0$$

$$\frac{\partial S}{\partial b_{12.3}} = 2\sum (X_1 - a - b_{12.3}X_2 - b_{13.2}X_3)(-X_2) = 0$$

$$\frac{\partial S}{\partial b_{13.2}} = 2\sum (X_1 - a - b_{12.3}X_2 - b_{13.2}X_3)(-X_3) = 0$$

$$\Rightarrow \quad \sum (X_1 - a - b_{12.3}X_2 - b_{13.2}X_3) = 0$$

$$\sum X_2(X_1 - a - b_{12.3}X_2 - b_{13.2}X_3) = 0$$

$$\sum X_3(X_1 - a - b_{12.3}X_2 - b_{13.2}X_3) = 0$$

$$\Rightarrow \quad \sum X_{1.23} = 0 \quad \text{--- (i)}$$

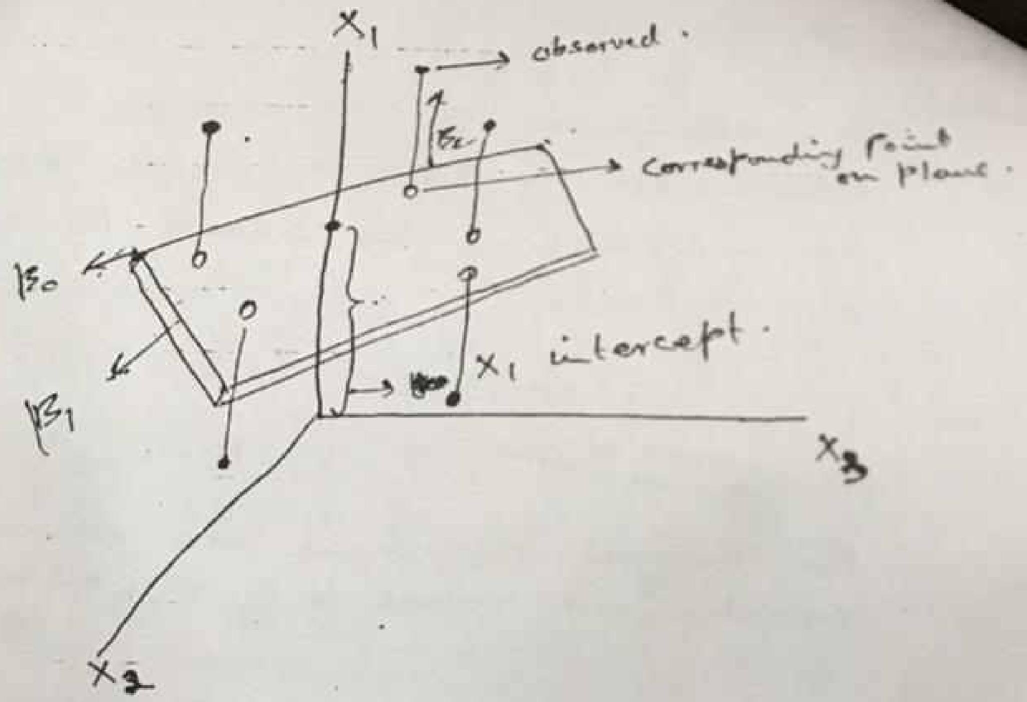
$$\sum X_2 X_{1.23} = 0 \quad \text{--- (ii)}$$

$$\sum X_3 X_{1.23} = 0 \quad \text{--- (iii)}$$

We assume that the variables X_1, X_2, and X_3 have been measured from their respective means, so that.

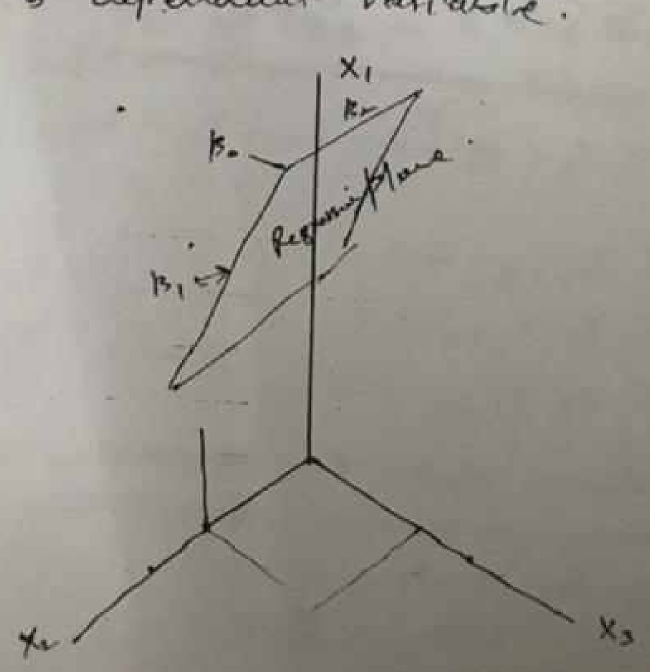
$$E(X_1) = E(X_2) = E(X_3) = 0.$$

Eq (i) gives $a = 0$. $\Rightarrow$



$$\hat{X}_1 = a + b_{12\cdot3}X_2 + b_{13\cdot2}X_3 .$$

$\hat{X}_1$ is the estimated values
a is The intercept
$b_{12\cdot3}$ and $b_{13\cdot2}$ slopes associated with X_2 and X_3.
X_2 and X_3 are independent variables
X_1 is dependent variable.



and Eq (ii) and (iii) gives.

$$\left.\begin{aligned}\sum X_1 X_2 - b_{12\cdot3}\sum X_2^2 - b_{13\cdot2}\sum X_2X_3 = 0\\ \sum X_1X_3 - b_{12\cdot3}\sum X_2X_3 - b_{13\cdot2}\sum X_3^2 = 0\end{aligned}\right] \quad (3)$$

Since X_i's are measured from their respective means, we have.

$$\sigma_i^2 = \frac{1}{N}\sum X_i^2$$

and

$$Cov(X_i, X_j) = \frac{1}{N}\sum X_i X_j$$

and.

$$r_{ij} = \frac{Cov(X_i, X_j)}{\sigma_i \sigma_j}$$

Hence from eq (3), we get.

$$N\,Cov(X_1,X_2) - N b_{12\cdot3}\,\sigma_2^2 - N b_{13\cdot2}\,Cov(X_2,X_3) = 0$$
$$N\,Cov(X_1,X_3) - N b_{12\cdot3}\,Cov(X_2,X_3) - N b_{13\cdot2}\,\sigma_3^2 = 0$$

$$\Rightarrow Cov(X_1,X_2) - b_{12\cdot3}\,\sigma_2^2 - b_{13\cdot2}\,Cov(X_2,X_3) = 0$$
$$Cov(X_1,X_3) - b_{12\cdot3}\,Cov(X_2,X_3) - b_{13\cdot2}\,\sigma_3^2 = 0$$

$$\left.\begin{aligned}r_{12}\sigma_1\sigma_2 - b_{12\cdot3}\sigma_2^2 - b_{13\cdot2}\, r_{23}\sigma_2\sigma_3 = 0\\ r_{13}\sigma_1\sigma_3 - b_{12\cdot3}\, r_{23}\sigma_2\sigma_3 - b_{13\cdot2}\,\sigma_3^2 = 0\end{aligned}\right] - (4)$$

Since there are two equations and two unknowns using cramer's rule we have.

$$\left.\begin{aligned}r_{12}\sigma_1 - b_{12\cdot3}\sigma_2 - b_{13\cdot2}\, r_{23}\sigma_3 = 0\\ r_{13}\sigma_1 - b_{12\cdot3}\, r_{23}\sigma_2 - b_{13\cdot2}\,\sigma_3 = 0\end{aligned}\right] - (5)$$

$$\omega_{12} = -\begin{vmatrix} r_{21} & r_{23} \\ r_{31} & r_{33} \end{vmatrix} \quad \omega_{13} = -\begin{vmatrix} r_{22} & r_{21} \\ r_{32} & r_{31} \end{vmatrix} \quad \omega_{11} = \begin{vmatrix} 1 & r_{23} \\ r_{32} & 1 \end{vmatrix}$$

If we use

$$W = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} \quad \text{—— (8)}$$

and ω_{ij} is the cofactor of the element in the ith and jth column of W, we have from equation (6) and (7).

$$b_{12.3} = -\frac{\sigma_1}{\sigma_2} \cdot \frac{\omega_{12}}{\omega_{11}} \quad \text{and} \quad b_{13.2} = -\frac{\sigma_1}{\sigma_3} \frac{\omega_{13}}{\omega_{11}} \quad \text{—— (9)}$$

Substituting this value in eq (1) when $a = 0$, gives

=> The requation of the plane of regression of X_1 on X_2 and X_3

$$X_1 = -\frac{\sigma_1}{\sigma_2} \cdot \frac{\omega_{12}}{\omega_{11}} X_2 - \frac{\sigma_1}{\sigma_3} \cdot \frac{\omega_{13}}{\omega_{11}} X_3$$

$$\frac{X_1}{\sigma_1} \omega_{11} + \frac{X_2}{\sigma_2} \omega_{12} + \frac{X_3}{\sigma_3} \omega_{13} = 0 \quad \text{—— (10)}$$

Similarly the other two regression equations are regression equation X_2 on X_1 and X_3 is

$$\frac{X_2}{\sigma_2} W_{22} + \frac{X_3}{\sigma_3} \omega_{23} + \frac{X_1}{\sigma_1} \omega_{21} = 0$$

and the regression equation X_3 on X_1 and X_2 is

$$\frac{X_3}{\sigma_3} W_{33} + \frac{X_1}{\sigma_1} W_{31} + \frac{X_2}{\sigma_2} \omega_{32} = 0$$

and interms of original values the regression equation X_1 on X_2 and X_3 is

$$\frac{x_1 - \bar{x}_1}{\sigma_1} w_{11} + \frac{x_2 - \bar{x}_2}{\sigma_2} w_{12} + \frac{x_3 - \bar{x}_3}{\sigma_3} w_{13} = 0$$

and

regression equation X_2 on X_1 and X_3 is

$$\frac{x_2 - \bar{x}_2}{\sigma_2} w_{22} + \frac{x_3 - \bar{x}_3}{\sigma_3} w_{23} + \frac{x_1 - \bar{x}_1}{\sigma_1} w_{21} = 0$$

regression equation X_3 on X_1 and X_2 is

$$\frac{x_3 - \bar{x}_3}{\sigma_3} w_{33} + \frac{x_1 - \bar{x}_1}{\sigma_1} w_{31} + \frac{x_2 - \bar{x}_2}{\sigma_2} w_{32} = 0.$$

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$$\begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}$$

$$\left.\begin{aligned} r_{12}\sigma_1 &= b_{12\cdot3}\,\sigma_2 + b_{13\cdot2}\,r_{23}\sigma_3 \\ r_{13}\sigma_1 &= b_{12\cdot3}\,r_{23}\sigma_2 + b_{13\cdot2}\,\sigma_3 \end{aligned}\right] \quad (5)$$

$$b_{12\cdot3} = \frac{\begin{vmatrix} r_{12}\sigma_1 & r_{23}\sigma_3 \\ r_{13}\sigma_1 & \sigma_3 \end{vmatrix}}{\begin{vmatrix} \sigma_2 & r_{23}\sigma_3 \\ r_{23}\sigma_2 & \sigma_3 \end{vmatrix}} = \frac{\sigma_1}{\sigma_2}\,\frac{\begin{vmatrix} r_{12} & r_{23} \\ r_{13} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & r_{23} \\ r_{23} & 1 \end{vmatrix}}$$

$$\boxed{b_{12\cdot3} = \frac{\sigma_1}{\sigma_2}\left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2}\right)} \quad \text{——— } (6)$$

similarly, we get.

$$b_{13\cdot2} = \frac{\begin{vmatrix} \sigma_2 & r_{12}\sigma_1 \\ r_{23}\sigma_2 & r_{13}\sigma_1 \end{vmatrix}}{\begin{vmatrix} \sigma_2 & r_{23}\sigma_3 \\ r_{23}\sigma_2 & \sigma_3 \end{vmatrix}} = \frac{\sigma_1}{\sigma_3}\,\frac{\begin{vmatrix} 1 & r_{12} \\ r_{23} & r_{13} \end{vmatrix}}{\begin{vmatrix} 1 & r_{23} \\ r_{23} & 1 \end{vmatrix}}$$

$$\boxed{b_{13\cdot2} = \frac{\sigma_1}{\sigma_3}\left(\frac{r_{13} - r_{12}r_{23}}{1 - r_{23}^2}\right)} \quad \text{——— } (7)$$

Partial Correlation Coefficient:

Sometimes the correlation between two variable x_1 and x_2 may be partly due to the correlation of a third variable x_3 with both x_1 and x_2. In such a situation one may want to know what the correlation between x_1 and x_2 would be if the linear effect of x_3 on each of x_1 and x_2 were eliminated. This correlation is called as partial correlation.

The linear regression equations

x_1 on x_3 is $x_1 = b_{13} x_3$

x_2 on x_3 is $x_2 = b_{23} x_3$

Removing the linear effect of x_3 from x_1 and x_2 we get the residuals

$$x_{1\cdot3} = x_1 - b_{13} x_3$$

$$x_{2\cdot3} = x_2 - b_{23} x_3$$

The correlation between x_1 and x_2 after eliminating the linear effect of x_3 is

$$r_{12\cdot3} = \frac{Cov(x_{1\cdot3}, x_{2\cdot3})}{\sqrt{Var(x_{1\cdot3})\, Var(x_{2\cdot3})}}$$

We have

$$Cov(x_{1\cdot3}, x_{2\cdot3}) = \frac{1}{n}\sum x_{1\cdot3}\, x_{2\cdot3}$$

Since x's are measured from their respective means

$$= \frac{1}{n}\sum\left[(x_1 - b_{13}x_3)(x_2 - b_{23}x_3)\right]$$

$$= \frac{1}{n}\sum\left[x_1x_2 - b_{13}x_2x_3 - b_{23}x_1x_3 + b_{13}b_{23}x_3^2\right]$$

$$= \frac{1}{n}\sum x_1x_2 - b_{13}\cdot\frac{1}{n}\sum x_2x_3 - b_{23}\cdot\frac{1}{n}\sum x_1x_3 + b_{13}b_{23}\frac{1}{n}\sum x_3^2$$

$$Cov(x_{1\cdot3}, x_{2\cdot3}) = r_{12}\sigma_1\sigma_2 - b_{13} r_{23}\sigma_2\sigma_3 - b_{23} r_{13}\sigma_1\sigma_3 + b_{13}b_{23}\sigma_3^2$$

We know that

$$b_{13} = r_{13}\frac{\sigma_1}{\sigma_3} \quad \text{and} \quad b_{23} = r_{23}\frac{\sigma_2}{\sigma_3}$$

$$\therefore Cov(x_{1\cdot3}, x_{2\cdot3}) = r_{12}\sigma_1\sigma_2 - r_{13}\frac{\sigma_1}{\cancel{\sigma_3}}\cdot r_{23}\sigma_2\cancel{\sigma_3} - r_{23}\frac{\sigma_2}{\cancel{\sigma_3}} r_{13}\sigma_1\cancel{\sigma_3} + r_{13}\frac{\sigma_1}{\cancel{\sigma_3}}\cdot r_{23}\frac{\sigma_2}{\cancel{\sigma_3}}\cdot\cancel{\sigma_3^2}$$

$$= r_{12}\sigma_1\sigma_2 - r_{13}r_{23}\sigma_1\sigma_2 \cancel{- r_{23}r_{13}\sigma_1\sigma_2} \cancel{+ r_{13}\sigma_1 r_{23}\sigma_2}$$

$$\boxed{Cov(x_{1\cdot3}, x_{2\cdot3}) = (r_{12} - r_{13}r_{23})\,\sigma_1\sigma_2}$$

$$Var(X_{1.3}) = \frac{1}{n}\sum X_{1.3}^2 = \frac{1}{n}\sum X_{1.3}\cdot X_{1.3} = \frac{1}{n}\sum X_1 X_{1.3}$$

$$= \frac{1}{n}\sum X_1(X_1 - b_{13}X_3)$$

$$= \frac{1}{n}\sum X_1^2 - b_{13}\cdot \underbrace{\frac{1}{n}\sum X_1X_3}_{Cov(X_1, X_3)}$$

$$= \sigma_1^2 - b_{13}\, r_{13}\,\sigma_1\sigma_3$$

$$= \sigma_1^2 - r_{13}\frac{\sigma_1}{\cancel{\sigma_3}}\cdot r_{13}\,\sigma_1\cancel{\sigma_3}$$

$$\boxed{Var(X_{1.3}) = \sigma_1^2(1-r_{13}^2)}$$

Similarly.

$$\boxed{Var(X_{2.3}) = \sigma_2^2(1-r_{23}^2)}$$

Hence $r_{12.3} = \dfrac{Cov(X_{1.3}, X_{2.3})}{\sqrt{Var(X_{1.3})\,Var(X_{2.3})}} = \dfrac{(r_{12} - r_{13}r_{23})\,\sigma_1\sigma_2}{\sqrt{\sigma_1^2(1-r_{13}^2)\cdot\sigma_2^2(1-r_{23}^2)}}$

$$\boxed{r_{12.3} = \frac{(r_{12} - r_{13}r_{23})}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}}$$

Similarly

$$\boxed{r_{13.2} = \frac{(r_{13} - r_{12}r_{23})}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}}} \text{ and } \boxed{r_{23.1} = \frac{(r_{23} - r_{12}r_{13})}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}}}$$

Test of Significance.

To test the null hypothesis that $H_0: \rho_{12.3} = 0$, we may calculate

$$t = \frac{r_{12.3}}{\sqrt{1-r_{12.3}^2}}\cdot\sqrt{n-3}$$

with $n-3$ d.f.

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Multiple Correlation Coefficient:

The simple correlation between dependent variable and the joint effect of all independent variables on dependent variable is called as multiple correlation coefficient

In a trivariate distribution, having x_1 dependent variable x_1 and independent variable x_2 and x_3 and containing n observations. The simple correlation is

$$R_{1.23} = \frac{Cov(x_1, e_{1.23})}{\sqrt{Var(x_1)\ Var(e_{1.23})}} = \frac{\Sigma x_1 e_{1.23}}{\sqrt{\Sigma x_1^2\ \Sigma e_{1.23}^2}}$$

where

$$e_{1.23} = b_{12.3}x_2 + b_{13.2}x_3$$

and

$$x_{1.23} = x_1 - b_{12.3}x_2 - b_{13.2}x_3$$

$$= x_1 - e_{1.23}$$

We assume that all x's are measured from their respective mean

$\Rightarrow E(x) = 0 \Rightarrow E(e_{1.23}) = 0$ and $E(x_{1.23}) = 0$

$$\Sigma x_1 e_{1.23}$$
$$= \Sigma x_1 (x_1 - x_{1.23})$$
$$= \Sigma x_1^2 - \Sigma x_1 x_{1.23}$$
$$= \Sigma x_1^2 - \Sigma x_{1.23}^2$$
$$= n\sigma_1^2 - n\sigma_{1.23}^2$$
$$\boxed{\Sigma x_1 e_{1.23} = n(\sigma_1^2 - \sigma_{1.23}^2)}$$

$$\because \Sigma x_{1.23}^2 = \Sigma x_{1.23}(x_1 - b_{12.3}x_2 - b_{13.2}x_3)$$
$$= \Sigma x_1 x_{1.23} - b_{12.3}\Sigma x_2 x_{1.23} - b_{13.2}\Sigma x_3 x_{1.23}$$

In multiple regression analysis, normal equations are

$$\left.\begin{array}{l}\Sigma x_{1.23} = 0\\ \Sigma x_2 x_{1.23} = 0\\ \Sigma x_3 x_{1.23} = 0\end{array}\right\} \Rightarrow \boxed{\Sigma x_{1.23}^2 = \Sigma x_1 x_{1.23}}$$

$$\boxed{\Sigma x_1^2 = n\sigma_1^2}$$

$$\Sigma e_{1.23}^2 = \Sigma (x_1 - x_{1.23})^2$$
$$= \Sigma (x_1^2 - 2x_1 x_{1.23} + x_{1.23}^2)$$
$$= \Sigma x_1^2 - 2\Sigma x_1 x_{1.23} + \Sigma x_{1.23}^2$$
$$= \Sigma x_1^2 - 2\Sigma x_{1.23}^2 + \Sigma x_{1.23}^2$$
$$= \Sigma x_1^2 - \Sigma x_{1.23}^2$$
$$= n\sigma_1^2 - n\sigma_{1.23}^2$$
$$\boxed{\Sigma e_{1.23}^2 = n(\sigma_1^2 - \sigma_{1.23}^2)}$$

$$R_{1.23} = \frac{n(\sigma_1^2 - \sigma_{1.23}^2)}{\sqrt{[n\sigma_1^2][n(\sigma_1^2 - \sigma_{1.23}^2)]}}$$
$$= \frac{n(\sigma_1^2 - \sigma_{1.23}^2)}{n\sigma_1\sqrt{(\sigma_1^2 - \sigma_{1.23}^2)}}$$
$$= \frac{\sqrt{\sigma_1^2 - \sigma_{1.23}^2}}{\sigma_1}$$
$$= \sqrt{\frac{\sigma_1^2 - \sigma_{1.23}^2}{\sigma_1^2}} = \sqrt{1 - \frac{\sigma_{1.23}^2}{\sigma_1^2}}$$
$$\boxed{R_{1.23}^2 = 1 - \frac{\sigma_{1.23}^2}{\sigma_1^2}} = \boxed{1 - \frac{W}{W_{11}} = R_{1.23}^2}$$

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$$W = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix} = 1(1-r_{23}^2) - r_{12}(r_{21} - r_{13}r_{23}) + r_{13}(r_{12}r_{32} - r_{31})$$

$$= 1 - r_{23}^2 - r_{12}^2 + r_{12}r_{13}r_{23} + r_{12}r_{13}r_{23} - r_{13}^2$$

$$\boxed{W = 1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23}}$$

$$W_{11} = \begin{vmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{vmatrix} = \begin{vmatrix} 1 & r_{23} \\ r_{32} & 1 \end{vmatrix}$$

$$\boxed{W_{11} = 1 - r_{23}^2}$$

$$R_{1.23}^2 = 1 - \frac{W}{W_{11}} = 1 - \frac{1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}$$

$$= 1 - \frac{(1 - r_{23}^2) - r_{12}^2 - r_{13}^2 + 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}$$

$$= \cancel{1} - \cancel{1} + \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}$$

$$\boxed{R_{1.23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}}$$

Similarly

$$\boxed{R_{2.31}^2 = 1 - \frac{W}{W_{22}} = \frac{r_{23}^2 + r_{21}^2 - 2r_{12}r_{23}r_{31}}{1 - r_{31}^2}}$$

$$\boxed{R_{3.12}^2 = 1 - \frac{W}{W_{33}} = \frac{r_{31}^2 + r_{32}^2 - 2r_{12}r_{23}r_{31}}{1 - r_{12}^2}}$$

[redacted]

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PRACTICAL 9

PROBLEM: A collector of antique clocks believes that the price received for the clock at an antique auction increases with the age of the clocks and with the number of bidders. The following model is suggested:

$$X_1 = \beta_0 + \beta_1 X_2 + \beta_2 X_3 + \epsilon$$

where X_1 = Auction Price X_3 = Age of clock (years) X_2 = No. of bidders

A sample of 32 auction prices of grandfather clocks, along with their ages and the number of bidders, is given below:

Age X_3	Number of bidders X_2	Auction Price X_1	Age X_3	Number of bidders X_2	Auction Price X_1
127	13	1235	170	14	2131
115	12	1080	182	8	1550
127	7	845	162	11	1884
150	9	1522	184	10	2041
156	6	1047	143	6	854
192	11	1979	159	9	1483
156	12	1822	108	14	1055
132	10	1253	175	8	1545
137	9	1297	108	6	729
113	9	946	179	9	1792
137	15	1713	111	15	1175
117	11	1024	187	8	1593
137	8	1147	111	7	755
153	6	1092	115	7	744
117	13	1152	194	5	1356
126	10	1336	168	7	1262

a) Fit the regression model x_1 on x_2 and x_3.

b) Is there evidence to indicate that the overall model is useful? Test at 5% Level of significance.

c) Construct the regression model interms of correlations. The correlation matrix is

$$\begin{array}{ccc} & x_1 & x_2 \\ x_2 & \multicolumn{2}{l}{\left(\begin{array}{ll} 0.39464 & \\ 0.73023 & -0.25375 \end{array}\right)} \\ x_3 & & \end{array}$$

d) Calculate the multiple correlation coefficient $R_{1.23}$ of x_1 on x_2 and x_3.

e) Estimate the Auction Price (x_1) if an antique clock of Age (x_3) 150 years passes through 16 bidders (x_2).

f) Plot the observed and fitted values on the same graph paper and comment on the results.