Math 310: Linear Algebra Fall 2025 Dr. Schwitzerlett

Solutions — Linear Algebra Review (Chapters 4 & 5)

Part I — Short Answers (sketches)

1. Orthogonal: $u \perp v$ if $u \cdot v = 0$.

2. Orthonormal: a set of vectors that are pairwise orthogonal and each of unit length.

3. Column space: span of columns (subspace of $\mathbb{R}^m$). Row space: span of rows (subspace of $\mathbb{R}^n$). They generally have equal dimension (rank).

4. Gram–Schmidt: converts a linearly independent set into an orthogonal (then orthonormal) set spanning the same space.

5. Determinant geometric view: signed volume scaling factor of the linear map; e.g., $\det(A)$ is factor by which unit cube volume is scaled and orientation possibly reversed.

6. True. If $\det(A) = 0$ the matrix is singular so columns are linearly dependent.

7. (a) swapping two rows multiplies det by -1.
(b) multiplying a row by 3 multiplies det by 3.
(c) adding a multiple of one row to another does not change det.

8. If A invertible then $\det(A) \neq 0$.

9. Orthogonal matrix Q: $Q^T Q = I$ (equivalently $Q^{-1} = Q^T$).

10. Orthogonal matrices preserve lengths and angles and are numerically stable (avoid magnifying rounding errors).

Part II — Practice Problems: Solutions

1. $u \cdot v = 3 \cdot 1 + (-1) \cdot 4 + 2 \cdot (-2) = 3 - 4 - 4 = -5$.

2. $a \cdot b = 2 \cdot 1 + 1 \cdot (-2) + (-1) \cdot 3 = 2 - 2 - 3 = -3$. Since $a \cdot b \neq 0$, they are not orthogonal.

3. Projection of y onto a:

$$\operatorname{proj}_a y = \frac{a \cdot y}{a \cdot a}\, a.$$

Compute $a \cdot y = 1 \cdot 4 + 2 \cdot 1 + 2 \cdot 2 = 10$, and $a \cdot a = 1 + 4 + 4 = 9$. Thus

$$\operatorname{proj}_a y = \frac{10}{9}(1, 2, 2) = \left(\frac{10}{9}, \frac{20}{9}, \frac{20}{9}\right).$$

4. Gram–Schmidt one step:

$$u_1 = v_1 = (1, 2, 0).$$

Project v_2 onto u_1:

$$\operatorname{proj}_{u_1} v_2 = \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \frac{(2,1,1)\cdot(1,2,0)}{1^2+2^2+0^2}(1,2,0) = \frac{(2+2+0)}{5}(1,2,0) = \frac{4}{5}(1,2,0).$$

Then

$$u_2 = v_2 - \operatorname{proj}_{u_1} v_2 = (2,1,1) - \frac{4}{5}(1,2,0) = \left(\frac{6}{5}, -\frac{3}{5}, 1\right).$$

(Any nonzero scalar multiple is fine; e.g. simplified rational multiple is $u_2 = \left(\frac{1}{2}, -\frac{1}{2}, 1\right)$ after multiplying by 5/6.)

5. For $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ compute

$$Q^T Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$

so Q is orthogonal. Geometrically Q represents a 90° rotation in the plane (counterclockwise) — it preserves lengths and rotates vectors by $\frac{\pi}{2}$.

6. Least squares fit. Use $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ with unknown vector $\begin{pmatrix} m \\ b \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$. Solve normal equations

$$(A^T A)\begin{pmatrix} m \\ b \end{pmatrix} = A^T b.$$

Compute

$$A^T A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 32 \\ 14 \end{pmatrix}.$$

Solve

$$\begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 32 \\ 14 \end{pmatrix}.$$

The determinant is $14 \cdot 3 - 6 \cdot 6 = 6$. Solve (e.g. Cramer's rule or inverse):

$$m = 2, \qquad b = \tfrac{2}{3}.$$

So best-fit line: $y = 2x + \frac{2}{3}$.

7. $\det \begin{pmatrix} 2 & 1 \\ 5 & -3 \end{pmatrix} = 2(-3) - 5(1) = -6 - 5 = -11.$

8. $\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & 0 & 1 \end{pmatrix}$ Compute by expansion (or rule):

$$= 1 \cdot \det\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$= 1(1) - 2(0 \cdot 1 - 4 \cdot 2) + 3(0 \cdot 0 - 1 \cdot 2) = 1 - 2(-8) + 3(-2) = 1 + 16 - 6 = 11.$$

9. If $\det(C) = -6$ for an $n \times n$ matrix C then:

(a) $\det(2C) = 2^n \det(C) = 2^n(-6)$. (E.g. if $n = 3$ then $\det(2C) = 8(-6) = -48$.)

(b) $\det(C^{-1}) = \det(C)^{-1} = -\frac{1}{6}$.

(c) $\det(C^T) = \det(C) = -6$.

10. Compute $\det D$ for $D = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 0 & 1 \\ 6 & 1 & 3 \end{pmatrix}$ using row operations:

Start: D.

Replace row3 $\leftarrow$ row3 $-3\cdot$ row1 + row3? Simpler: do $R_3 \leftarrow R_3 - (3)R_1$ gives

$$\begin{pmatrix} 4 & 1 & 2 \\ 2 & 0 & 1 \\ -6 & -2 & -3 \end{pmatrix},$$

but that complicates. Instead compute directly (cofactor expansion):

$$\det D = 4 \det \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} - 1 \det \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} + 2 \det \begin{pmatrix} 2 & 0 \\ 6 & 1 \end{pmatrix}$$

$= 4(0 \cdot 3 - 1 \cdot 1) - 1(2 \cdot 3 - 1 \cdot 6) + 2(2 \cdot 1 - 0 \cdot 6) = 4(-1) - (6 - 6) + 2(2) = -4 - 0 + 4 = 0.$

So $\det D = 0$ (matrix singular).

11. True. If two rows are equal, then subtracting one from the other produces a zero row and determinant zero; equivalently determinant changes sign on swapping rows, so equality implies zero determinant.

Part III — Multi-Step Solutions

1. Gram–Schmidt for $v_1 = (1, 1, 0),\ v_2 = (1, 0, 1),\ v_3 = (0, 1, 1)$.

Compute

$$u_1 = v_1 = (1, 1, 0).$$

Project v_2 onto u_1:

$$\operatorname{proj}_{u_1} v_2 = \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \frac{1 + 0 + 0}{1 + 1 + 0} u_1 = \frac{1}{2}(1, 1, 0).$$

So

$$u_2 = v_2 - \operatorname{proj}_{u_1} v_2 = (1, 0, 1) - \tfrac{1}{2}(1, 1, 0) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 1\right).$$

Now project v_3 orthogonally to both u_1 and u_2:

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2.$$

Compute inner products:

$$v_3 \cdot u_1 = 0 + 1 + 0 = 1, \quad u_1 \cdot u_1 = 2,$$

$$v_3 \cdot u_2 = (0,1,1)\cdot\left(\tfrac{1}{2},-\tfrac{1}{2},1\right) = 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2}, \quad u_2 \cdot u_2 = \tfrac{1}{4}+\tfrac{1}{4}+1 = \tfrac{3}{2}.$$

Thus

$$u_3 = (0,1,1) - \tfrac{1}{2}(1,1,0) - \frac{1/2}{3/2}\left(\tfrac{1}{2},-\tfrac{1}{2},1\right) = (0,1,1) - \left(\tfrac{1}{2},\tfrac{1}{2},0\right) - \tfrac{1}{3}\left(\tfrac{1}{2},-\tfrac{1}{2},1\right).$$

Working out the arithmetic gives

$$u_3 = \left(-\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3}\right).$$

So an orthogonal basis is

$$u_1 = (1,1,0), \quad u_2 = \left(\tfrac{1}{2},-\tfrac{1}{2},1\right), \quad u_3 = \left(-\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3}\right).$$

Normalize each to obtain orthonormal vectors $e_i = \dfrac{u_i}{\|u_i\|}$:

$$e_1 = \frac{1}{\sqrt{2}}(1,1,0), \quad e_2 = \frac{1}{\sqrt{3}}\left(\tfrac{1}{2},-\tfrac{1}{2},1\right), \quad e_3 = \frac{1}{\sqrt{\frac{4}{3}}}\left(-\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3}\right) = \left(-\tfrac{\sqrt{3}}{3},\tfrac{\sqrt{3}}{3},\tfrac{\sqrt{3}}{3}\right).$$

2. Determinants for matrices in problem 13.

$$\det(A) = 1\cdot 4 - 2\cdot 3 = -2.$$

$$\det(B) = 0\cdot 0 - 1\cdot(-1) = 1.$$

$$\det(AB) = \det(A)\det(B) = (-2)(1) = -2.$$

(You may compute AB and its determinant directly to confirm.)

3. Invertibility by determinant.

(a) $\det\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix} = 4\cdot 3 - 6\cdot 2 = 12 - 12 = 0$, so *not* invertible (singular).

(b) $\det\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} = -12 \neq 0$, so invertible.

4. Projection application. For $F = (5,-1,2)$ and $d = (2,1,1)$:

$$d\cdot F = 2\cdot 5 + 1\cdot(-1) + 1\cdot 2 = 11, \qquad d\cdot d = 2^2 + 1^2 + 1^2 = 6.$$

So projection (component in direction d) is

$$\operatorname{proj}_d F = \frac{11}{6} d = \left(\frac{11}{3},\frac{11}{6},\frac{11}{6}\right).$$

The perpendicular component is

$$F - \operatorname{proj}_d F = (5,-1,2) - \left(\frac{11}{3},\frac{11}{6},\frac{11}{6}\right) = \left(\frac{4}{3},-\frac{17}{6},\frac{1}{6}\right).$$