Chapter 4 Practice Key

Show each pair of sets have equal cardinalities.

1. $\mathbb{Z}$ and S = $\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$

   …, -2, -1, 0, 1, 2, 3, 4, 5, …

   $\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots$

2. $\{0, 1\} \times \mathbb{N}$ and $\mathbb{N}$

   (0, 1), (1, 1), (0, 2), (1, 2), (0, 3), (1, 3), …

   1, 2, 3, 4, 5, 6, …

3. $\{0, 1\} \times \mathbb{N}$ and $\mathbb{Z}$

   …, (0, 4), (0, 3), (0, 2), (0, 1), (1, 1), (1, 2), (1, 3), (1, 4), …

   …, -3, -2, -1, 0, 1, 2, 3, 4, …

4. Odd integers and even integers

   …, -3, -1, 1, 3, 5, 7, 9, …

   …, -4, -2, 0, 2, 4, 6, 8, …

5. $A = \{3k : k \in \mathbb{Z}\}$ and $B = \{7k : k \in \mathbb{Z}\}$.

   …, -6, -3, 0, 3, 6, 9, …

   …, -14, -7, 0, 7, 14, 21, …

Prove each of the following.

6. The union of any two countably infinite sets is countably infinite.

   $\mathbb{N} \cup \mathbb{N} =$ 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, …

7. The union of any three countably infinite sets is countably infinite.

   $\mathbb{N} \cup \mathbb{N} \cup \mathbb{N} =$ 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, …

8. The union of any four countably infinite sets is countably infinite.

   $\mathbb{N} \cup \mathbb{N} \cup \mathbb{N} \cup \mathbb{N} =$ 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, …

9. The union of countably many infinite sets is countable. (Hint: Think about the proof that the set of all rationals is countable.)

1	2	3	4	5	…
1	2	3	4	5	…
1	2	3	4	5	…
1	2	3	4	5	…
1	2	3	4	5	…
…	…	…	…	…	…

10. The set of all irrationals is uncountable.

    Cantor's Diagonalization Argument

    0.$\underline{0}$0110854…

    0.1$\underline{1}$987654…

    0.22$\underline{8}$89615…

    0.339$\underline{8}$5724...

    …

If I pick 0.0303030…, I have found a number that is not in the set and not the same as the diagonalization of 0.0188… I can always find a number like this that is not the same as what is in the set above and not on the diagonal.