Theory of Computation Chapter 5

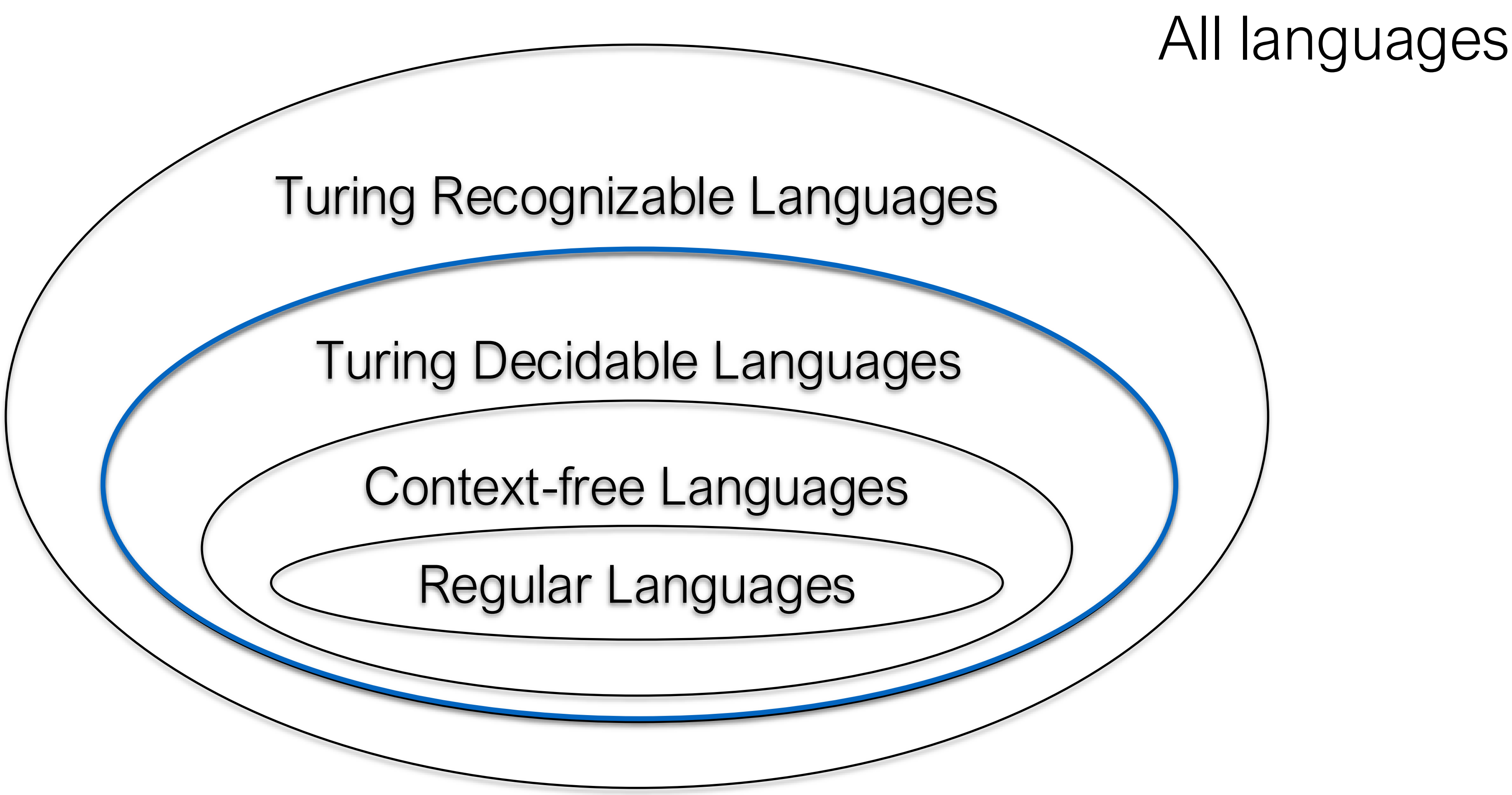
Using Rice's Theorem

School of Engineering | Computer Science

Kurt Godel
1906-1978

- His “incompleteness theorems” revolutionized mathematics and philosophy
- First to prove that some true mathematics statements must be unproveable
  - No formal system can be both complete and consistent
  - This statement is not provable
- Founded metamathematics
- Contributed to fields of of recursive theory, computability, general relativity, etc.
- Fled Nazis in 1939; settled at Princeton’s Insti for Advanced Study

Decidability



Review of Rice's Theorem

- Rice's Theorem: Questions about Turing Machines are often, but not always, undecidable

- Every non-trivial property of the language of TM's is undecidable

Review of Rice's Theorem

- Rice's Theorem:
  - Let L be a language of the form: L = {<M> | L(M) has some property P} where:
    1. P is non-trivial (there exists at least one machine M such that <M> ∈ L and at least one machine M such that <M> ∉ L)
    2. P is indeed a property of the language of TM's (whenever $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in L$ if and only if $\langle M_2 \rangle \in L$)
  - Then L is undecidable (Note: L can still be Turing-Recognizable)

School of Engineering | Computer Science

Using Rice's Theorem

- Ex 1: $ALL_{TM} = \{<M> \mid M \text{ is a TM and } L(M) = \Sigma^*\}$ is undecidable
    1. By Rice's Theorem:
        a. The property of P in this case is "$L(M) = \Sigma^*$"
        b. Therefore, P is non-trivial since there is at least one TM that belongs to ALL_{TM} (a TM that will accept everything) and at least one machine that doesn't belong to ALL_{TM} (a TM that rejects everything)

Using Rice's Theorem

- Ex 1 cont.: $ALL_{TM} = \{<M> \mid M \text{ is a TM and } L(M) = \Sigma^*\}$ is undecidable

  2. We must show that P is a property of the language of TMs. Let M_1 and M_2 be Turing machines such that $L(M_1) = L(M_2)$

$$
\begin{aligned}
M_1 \in ALL_{TM} &\Leftrightarrow L(M_1) = \Sigma^* \\
&\Leftrightarrow L(M_2) = \Sigma^* \\
&\Leftrightarrow M_2 \in ALL_{TM}
\end{aligned}
$$

Using Rice's Theorem

- Ex 2: CFL_{TM} = {<M> | M is a TM and L(M) is context-free} is undecidable.
  1. By Rice's Theorem:
     a. The property P in this case is "L(M) is context-free".
     b. P is non-trivial because there is at least one TM that belongs to CFL_{TM} (a machine that accepts $a^i b^j c^k \mid i, j, k \geq 0$ and $i = j$ or $j = k$) and at least one machine that doesn't belong to CFL_{TM} (a machine that accepts $a^n b^n c^n \mid n \geq 0$)

VCU School of Engineering | Computer Science

Using Rice's Theorem

- Ex 2 cont.: $CFL_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M)$ is context-free$\}$ is undecidable

  2. P is a property of the language of TM's since for any two machines M_1 and M_2 such that $L(M_1) = L(M_2)$

$$\begin{aligned} M_1 \in CFL_{TM} &\Leftrightarrow L(M_1) = \text{is context-free} \\ &\Leftrightarrow L(M_2) = \text{is context-free} \\ &\Leftrightarrow M_2 \in CFL_{TM} \end{aligned}$$

Using Rice's Theorem

- Ex 3: $M3_{TM}$ = {<M> | M is a TM and L(M) consists of at least 37 strings} is undecidable.
  1. By Rice's Theorem:
     a. The property P in this case is "L(M) consists of at least 37 strings".
     b. P is non-trivial because there is at least one TM that belongs to $M3_{TM}$ (a machine that accepts all strings) and at least one machine that doesn't belong to $M3_{TM}$ (a machine that accepts no strings)

Using Rice's Theorem

- <u>Ex 3 cont.</u>: $M3_{TM}$ = {<M> | M is a TM and L(M) consists of at least 37 strings} is undecidable

  2. P is a property of the language of TM's since for any two machines M_1 and M_2 such that $L(M_1) = L(M_2)$

$M_1 \in M3_{TM}$ $\Leftrightarrow$ $L(M_1)$ = consists of at least 37 strings

$\Leftrightarrow$ $L(M_2)$ = consists of at least 37 strings

$\Leftrightarrow$ $M_2 \in M3_{TM}$

VCU School of Engineering | Computer Science

Using Rice's Theorem

- Ex 4: $M4_{TM}$ = {<M> | M is a TM that has 37 states} is undecidable.
  - What is the property P of this language?
    - We don't know, so Rice's Theorem does not apply

Using Rice's Theorem

- Ex 5: $M5_{TM}$ = {<M> | M is a TM that accepts the string 01 in exactly a perfect square number of steps} is undecidable.
  - What is the property P of this language?
    - We don't know, so Rice's Theorem does not apply

Using Rice's Theorem

- <u>Ex 6</u>: $M6_{TM}$ = {<M> | M is a TM and L(M) is recognized by some TM having an even number of states} is undecidable.
  - What is the property P of this language?
    - The property of P is: "the language is recognized by a TM"
    - What is wrong here?
      - This is trivial. I can always add a state to my TM if it has an even number of states.

Using Rice's Theorem

- Ex 7: $M7_{TM}$ = {<M> | M is a TM and L(M) is an infinite language} is undecidable.
- What is the property P?
  - P is "L(M) is an infinite language".

Using Rice's Theorem

- Ex 7: $M7_{TM}$ = {<M> | M is a TM and L(M) is an infinite language} is undecidable.
  1. By Rice's Theorem:
     a. The property P in this case is "L(M) is an infinite language".
     b. P is non-trivial because there is at least one TM that belongs to $M7_{TM}$ (a machine that accepts all strings) and at least one machine that doesn't belong to $M7_{TM}$ (a machine that accepts no strings)

Using Rice's Theorem

- Ex 7 cont.: $M7_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is an infinite language}\}$ is undecidable
  2. P is a property of the language of TM's since for any two machines M_1 and M_2 such that $L(M_1) = L(M_2)$

$$
\begin{aligned}
M_1 \in M7_{TM} &\Leftrightarrow L(M_1) = \text{infinite language} \\
&\Leftrightarrow L(M_2) = \text{infinite language} \\
&\Leftrightarrow M_2 \in M7_{TM}
\end{aligned}
$$

Using Rice's Theorem

- Ex 8: $M8_{TM}$ = {<M> | M is a TM and L(M) is recognized by some TM having at least 10 states and at least 10 tape symbols} is undecidable.
- What is the property P?
  - P is "L(M) is recognized by some TM ".
  - What is the problem with this?
    - This is trivial. I can always add states and tape symbols.

Post's Correspondence Problem (PCP)

- The Post's Correspondence Problem is a famous undecidable problem concerning the manipulation of strings
- It is a puzzle with a collection of dominos, each containing two strings like: $\left\{\left[\frac{a}{ab}\right],\left[\frac{abc}{c}\right],\left[\frac{ca}{a}\right],\left[\frac{b}{ca}\right]\right\}$.
- The goal is to make a list of these dominos (allowing repetitions) so that the string we get by reading the top symbols is the same string we get by reading the bottom symbols. This list would then be a match.
    - Ex: $\left[\frac{a}{ab}\right]\left[\frac{b}{ca}\right]\left[\frac{ca}{a}\right]\left[\frac{a}{ab}\right]\left[\frac{abc}{c}\right] \Rightarrow \frac{abcaaabc}{abcaaabc}$

Post's Correspondence Problem (PCP)

- Not all collections make a match, such as: $\left\{\left[\frac{abc}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{acc}{ba}\right]\right\}$. There are no possible matches since the top values are larger than the bottom values.
- The Post Correspondence Problem is to determine whether a collection of dominos has a match.
- This problem is unsolvable or undecidable
- Proof (next slide)

Post's Correspondence Problem (PCP)

- Proof Idea:
  1. Make a modified PCP problem (MPCP) where the first domino is the start domino for the match.
  - Now we can show that A_{TM} reduces to MPCP, thus MPCP is undecidable. ($A_{TM} \leq$ MPCP)
    - Consider <M, w> in A_{TM}, the idea is to design a set of dominos such that a solution or match to the tiles is a computation string for M on w. (As you read across the top or bottom of the dominos, you should see a sequence of configurations of the TM leading to acceptance of the string w.

Post's Correspondence Problem (PCP)

- Proof Idea cont.:
  2. Show that MPCP reduces to PCP
  - To do this, we add a special symbol to the collection of dominos in a way that the first domino must start the match based on this special symbol
  - Thus, $A_{TM} \leq MPCP \leq PCP$, so PCP is unsolvable or undecidable.
  - The full proof is on pages 228 – 233 of your textbook.

Try It

1. Use Rice's Theorem if applicable to show that M_9 = {<M> | M is a TM and 1011 ∈ L(M)} is undecidable.

2. Find a match in the following instance of the Post Correspondence Problem: $\left\{\left[\frac{ab}{abab}\right], \left[\frac{b}{a}\right], \left[\frac{ab}{b}\right], \left[\frac{aa}{a}\right]\right\}$

Try It

1. Use Rice's Theorem if applicable to show that $M9_{TM}$ = {<M> | M is a TM and 1011 $\in$ L(M)} is undecidable.
   1. By Rice's Theorem:
      a. The property P = "L(M) contains 1011"
      b. P is non-trivial since there is at least on TM that accepts a string containing 1011, the TM that accepts all strings, and at least one that does not accept 1011, the TM that accepts nothing
   2. P is a property of the language of TM's since for any two machines M_1 and M_2 such that $L(M_1) = L(M_2)$

$$
\begin{aligned}
M_1 \in M9_{TM} &\Leftrightarrow L(M_1) = \text{contains } 1011 \\
&\Leftrightarrow L(M_2) = \text{contains } 1011 \\
&\Leftrightarrow M_2 \in M9_{TM}
\end{aligned}
$$

Try It

2. Find a match in the following instance of the Post Correspondence Problem: $\left\{\left[\frac{ab}{abab}\right], \left[\frac{b}{a}\right], \left[\frac{ab}{b}\right], \left[\frac{aa}{a}\right]\right\}$

- $\left[\frac{aa}{a}\right] \left[\frac{b}{a}\right] \left[\frac{ab}{b}\right] \left[\frac{ab}{abab}\right] \Rightarrow \frac{aababab}{aababab}$