Chapter 7.2 Practice Key

Show that the following problems are in NP by describing a non-deterministic TM and a verifier for each.

1. VERTEX COVER: A graph $G = (V, E)$ and a positive integer $k \leq |V|$ where there is a subset $V' \subseteq V$ such that $|V'| \leq k$ and, for each edge $uv \in E$, at least one of u and v belongs to V'.

   Non-deterministic TM:

   $N =$ "On input $< G, k >$ (G is the graph, k is a positive integer, where $k \leq |V|$, V' is the string passed in):

   1. Non-deterministically select a subset V' such that $|V'| \leq k$ of nodes in G.
   2. Test whether for each edge $uv \in E$, at least one of u and v belongs to V'.
   3. If yes, accept; otherwise, reject."

   Verifier:

   $V =$ "On input $\ll G, k >, V' >$ (G is the graph, k is a positive integer, where $k \leq |V|$, V' is the input string):

   1. Test if V' is a subgraph with k of nodes in G.
   2. Test if V' contains at least one of u and v for each edge $uv \in E$.
   3. If both pass, accept; otherwise, reject."

2. HAMILTON CIRCUIT: A graph $G = (V, E)$. G contain a Hamiltonian circuit, that is, an ordering $< v_1, v_2, \ldots, v_n >$ of the vertices of G, where $n = |V|$, such that $v_n v_1 \in E$ and $v_i v_{i+1} \in E$ whenever $1 \leq i \leq n - 1$.

   Non-deterministic TM:

   $N =$ "On input $< G, n >$ (G is the graph, n is a positive integer, where $n = |V|$, $< v_1, v_2, \ldots, v_n >$ is the string passed in):

   1. Non-deterministically select an ordering $< v_1, v_2, \ldots, v_n >$ such that that $v_n v_1 \in E$ and $v_i v_{i+1} \in E$ whenever $1 \leq i \leq n - 1$ of nodes in G.
   2. Test whether $< v_1, v_2, \ldots, v_n >$ is a Hamilton circuit of G, each vertex is listed once and there is a path from the starting vertex back to the starting vertex, covering all vertices of G.
   3. If yes, accept; otherwise, reject."

   Verifier:

   $V =$ "On input $\ll G, n >, < v_1, v_2, \ldots, v_n \gg$ (G is the graph, n is a positive integer, where $n = |V|$, $< v_1, v_2, \ldots, v_n >$ is the input string):

   1. Test if $< v_1, v_2, \ldots, v_n >$ is an ordering such that that $v_n v_1 \in E$ and $v_i v_{i+1} \in E$ whenever $1 \leq i \leq n - 1$ of nodes in G.
   2. Test if $< v_1, v_2, \ldots, v_n >$ is a Hamilton circuit of G, each vertex is listed once and there is a path from the starting vertex back to the starting vertex, covering all vertices of G.
   3. If both pass, accept; otherwise, reject."

3. PARTITION: A multiset A of positive integer, and there is a subset $A' \subseteq$ such that the sum of the elements of $A' =$ the sum of the elements of A that are not in A'.

   Non-deterministic TM:

   $N =$ "On input $< A >$ (A is a set of positive integers, A' is the string passed in):

   1. Non-deterministically select a subset A' such that $A' \subseteq A$.
   2. Test whether the sum of the elements of $A' =$ the sum of the elements of A that are not in A'.
   3. If yes, accept; otherwise, reject."

   Verifier:

   $V =$ "On input $\ll A >, A' >$ (A is a set of positive integers, A' is the input string):

   1. Test if A' is a subset such that $A' \subseteq A$.
   2. Test if the sum of the elements of $A' =$ the sum of the elements of A that are not in A'.
   3. If both pass, accept; otherwise, reject."

4. 3-DIMENSIONAL MATCHING: A set $M \subseteq W \times X \times Y$, where W, X, and Y are disjoint sets with the same number q of elements, where M contain a subset $M' \subseteq M$ such that $|M'| = q$ and no two elements of M' agree in any coordinate.

   Non-deterministic TM:

   $N =$ "On input $< M >$ (M is a set on 3 dimensions, M' is the string passed in):

   1. Non-deterministically select a subset M' such that $M' \subseteq M$ and $|M'| = q$.
   2. Test whether no two elements of M' agree in any coordinate.
   3. If yes, accept; otherwise, reject."

   Verifier:

   $V =$ "On input $\ll M >, M' >$ (M is a set on 3 dimensions, M' is the input string):

   1. Test if M' is a subset such that $M' \subseteq M$ and $|M'| = q$.
   2. Test whether no two elements of M' agree in any coordinate.
   3. If both pass, accept; otherwise, reject."

5. FEEDBACK VERTEX SET: A directed graph $G = (V, E)$, integer $k \leq |V|$ where there is a subset $V' \subseteq V$ with $|V'| \leq k$ such that every directed cycle in G includes at least one vertex from V'.

   Non-deterministic TM:

   $N =$ "On input $< G, k >$ (G is the graph, k is a positive integer, where $k \leq |V|$, V' is the string passed in):

   1. Non-deterministically select a subset V' such that $|V'| \leq k$ of nodes in G.
   2. Test whether every directed cycle in G includes at least one vertex from V'.
   3. If yes, accept; otherwise, reject."

Verifier:

V = "On input $\ll G, k >, V' >$ (G is the graph, k is a positive integer, where $k \leq |V|$, V' is the input string):

1. Test if V' is a subgraph such that $|V'| \leq k$ of nodes in G.
2. Test if every directed cycle in G includes at least one vertex from V'.
3. If both pass, accept; otherwise, reject."