CMSC 303 Introduction to Theory of Computation, VCU
Assignment 6

Turned in electronically in PDF, PNG or Word format

Total marks: 52 marks + 3 marks bonus for all the answers typed out.

Unless otherwise noted, the alphabet for all questions below is assumed to be $\Sigma = \{0, 1\}$. This assignment will get you primarily to practice reductions in the context of decidability.

1. [10 marks] We begin with some mathematics regarding uncountability. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denote the set of natural numbers.
   (a) [5 marks] Prove that the set of binary numbers has the same size as $\mathbb{N}$ by giving a bijection between the binary numbers and $\mathbb{N}$.
   (b) [5 marks] Let B denote the set of all infinite sequences over the English alphabet. Show that B is uncountable using a proof by diagonalization.
2. [15 marks] We next move to a warmup question regarding reductions.
   (a) [3 marks] Intuitively, what does the notation $A \leq B$ mean for problems A and B?
   (b) [3 marks] What is a mapping reduction $A \leq_m B$ from language A to language B? Give both a formal definition, and a brief intuitive explanation in your own words.
   (c) [3 marks] What is a computable function? Give both a formal definition, and a brief intuitive explanation in your own words.
   (d) [6 marks] Suppose $A \leq_m B$ for languages A and B. Please answer each of the following with a brief explanation.
      i. If B is decidable, is A decidable?
      ii. If A is undecidable, is B undecidable?
      iii. If B is undecidable, is A undecidable?
3. [5 marks] Show that if L = $\{0^n 1^n \mid n \geq 0\}$ is Turing-recogizable and L $\leq \overline{L}$, then L is decidable.
4. [12 marks] Prove and discuss the following reductions.
   (a) [5 marks] Walk through the proof to show that the problem of proving the language of a Turning Machine is a context-free language is undecidable. (Do not use Rice's theorem as a black box and note that this is not the same problem as Theorem 5.13 in the textbook.)
   (b) [5 marks] Use mapping reductions to prove that L = $\{$<M> $\mid$ M is a Turing Machine and M accepts $\epsilon\}$ is undecidable.
   (c) [2 marks] How are these two proofs different?

5. [10 marks] Use Rice's Theorem if possible to show the following problems are undecidable. If it is not possible to use Rice's Theorem, explain why not.
   (a) [5 marks] $M_{1TM} = \{< M >|$ M is a TM and L(M) is finite$\}$.
   (b) [5 marks] $M_{2TM} = \{< M >|$ M is a TM and L(M) is a subset of $\Sigma^*\}$.
6. [Bonus +3 marks] Find a match to the following Post Correspondence Problem set:
   $\{ \frac{ab}{abab}, \frac{b}{a}, \frac{aba}{b}, \frac{aa}{a} \}$