Key

CMSC 303 Introduction to Theory of Computing

Chapters 4&5 Review

Note: Unless otherwise noted, the alphabet for all questions below is assumed to be $\Sigma = \{0,1\}$.

1. Explain the concept of countable sets and give of some countable sets and uncountable sets.

Sets are countable if:
① set is finite
② set is infinite, but same size as set $\mathbb{N}$

2. Let A denote the set of Σ^*, where $\Sigma = \{0\}$. Show that A is countable.

0 = ε
1 - 0
2 - 00
3 - 000
4 - 0000
⋮

3. Let B denote the set of irrational numbers. Show that B is uncountable using a proof by diagonalization?

① Create set L of irrational numbers

$$L = \begin{cases} 0.\underline{3}333\ldots \\ 3.1\underline{4}159\ldots \\ 0.66\underline{6}6\ldots \\ 1.414\underline{2}1\ldots \\ \vdots \end{cases}$$

② Set diagonal to $d = 0.3462\ldots$

③ Find $x \in L$ + $x \neq d$, $x = 0.121212\ldots$

The ith digit of x is created to be different from ever ith digit of L_i. So $x \neq L_i$ for all i. L is not a complete list + uncountable.

4. Explain what mapping reduction provides for us that reductions do not. Understand the proof for a mapping reduction and how it differs from a proof showing reduction.

It allows the reduction to provide the answer for the other problem. $A \leq_m B$. B gives the answer for A. There is a direct relationship.

5. What does it mean to reduce a Turing Machine A to a Turing Machine B?

We convert A into B so that the Solution to B helps us to solve A.

6. Given $A \leq_M B$. If A is decidable, what do we know about A and B?

Nothing about B. Know all the implications

7. Use Rice's theorem to prove

$$L_{TM} = \{< M, w > \mid M \text{ is a TM and } L(M) \text{ is infinite}\}$$

is undecidable.

① P is L(M) is infinite, thus P is nontrivial. There is a TM in P, ALL$_{TM}$, and a TM not in P, TM that accepts nothing

② P is a property of the language of TM since for any two machines, M_1 + M_2 such that $L(M_1) = L(M_2)$

$\langle M_1 \rangle \in L_{TM} \iff L(M_1)$ is infinite

$\iff L(M_2)$ is infinite

$\iff \langle M_2 \rangle \in L_{TM}$

8. Define the languages:

$$A_{TM} = \{< M, w > \mid M \text{ accepts input } w\}$$
$$L = \{< M, w > \mid M \text{ is a TM and } 010 \in L(M)\}.$$

In class, we showed that ATM is undecidable. We now show L is undecidable by sketching a reduction from A_{TM} to L.

a. Given any $< M, w >$ (i.e. an input for A_{TM}), we define the Turing machine N as follows, for which you are to fill in the missing blank:

   i. "On input $x \in \{0,1\}^*$

   ii. If x $\neq 010$, reject x.

   iii. Else, run M on w.

   iv. If M accepts w, then accept x.

   v. Else, reject x."

b. If$< M, w > \in A_{TM}$, what do you know about the language recognized by N?

$L(N) = \{010\}$

All strings $x \neq 010$ are rejected in step ii

Since M accepts w, Step iv, accepts $x = 010$

c. If $< M, w > \notin A_{TM}$, what do you know about the language recognized by N?

$L(N) = \emptyset$

Line ii rejects all $x \neq 010$

Either line iii loops forever on input $x = 010$ or Line v rejects $x = 010$

d. Why does the ability to decide L imply the ability to decide A_{TM}?

Since $<M, w> \in A_{TM}$ if + only if $<N> \in L$

Thus the ability to decide the latter implies the ability to decide the former.

9. **What is a problem in the context of Turing Machines?**

The membership of strings in a language

10. **What do we know about problems in terms of decidability?**

Most (almost all) problems are undecidable