CMSC303 Introduction to Theory of Computation, VCU
Assignment 1 Solutions

Total marks: 26 marks + 3 bonus marks for all the answers typed out.

NOTE: As this is a warmup assignment intended to refresh your memory on background material, it will be marked only for completeness, *not* correctness. It is your responsibility to compare your answers with the solutions (to be posted after the due date) to gauge how well you understand the concepts on this assignment.

1 Exercises

1. (6 marks) Sipser, Ex. 0.3: Let A be the set $\{x, y\}$ and B be the set $\{x, y, z\}$.

   (a) (1 mark) Is A a subset of B?

   (b) (1 mark) Is B a subset of A?

   (c) (1 mark) What is $A \cup B$?

   (d) (1 mark) What is $A \cap B$?

   (e) (1 mark) What is $A \times B$?

   (f) (1 mark) What is the power set of B?

   Solution: (a) Yes. (b) No. (c) $A \cup B = \{x, y, z\}$. (d) $A \cap B = \{x, y\}$. (e) $A \times B = \{(x, x), (x, y), (x, z), (y, x), (y, y), (y, z)\}$. (f) $P(B) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$.

2. (2 marks) Sipser, Ex. 0.4: If A has a elements and B has b elements, how many elements are in $A \times B$? Explain your answer.
   Solution: $|A \times B| = ab$. Proof: The elements of $A \times B$ are obtained by considering all possible pairings between some $x \in A$ and some $y \in B$. Since A and B have sizes a and b, respectively, the total number of such pairings is ab.

3. (2 marks) Sipser, Ex. 0.5: If C is a set with c elements, how many elements are in the power set of C? Explain your answer.
   Solution: $P(C) = 2^c$. Proof: The elements of $P(C)$ are obtained by considering all possible subsets of C. To generate all possible subsets, for each element $x \in C$, we must consider two options: Either we include x in the subset, or we do not. Thus, the total number of ways of generating a subset is 2^c, since there are c elements, and each offers us two distinct ways to create a subset. (Alternatively, one can view this as assigning a bit to each $x \in C$. If and only if the bit i is set to 1, we think of x_i as being in our subset. The number of distinct settings for c bits is then 2^c.)

4. (7 marks) Sipser, Ex. 0.6: Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. The unary function $f : X \mapsto Y$ and the binary function $g : X \times Y \mapsto Y$ are described in the following tables.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

   (a) (1 mark) What is the value of $f(3)$?

(b) (2 marks) What are the domain and co-domain of f?

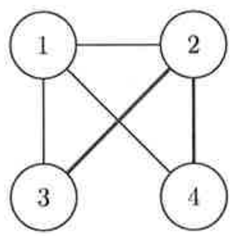
(c) (1 mark) What is the value of $g(3, 9)$?

(d) (2 marks) What are the domain and co-domain of g?

(e) (1 mark) What is the value of $g(4, f(4))$?

Solution: (a) $f(3) = 6$. (b) Co-Domain(f) $= \{6, 7, 8, 9, 10\} = Y$. Domain(f) $= X$. (c) $g(3, 9) = 8$. (d) Co-Domain(g) $= Y$. Domain(g) $= X \times Y$. (e) $g(5, f(5)) = 6$.

5. (3 marks) Sipser, Ex. 0.8. Consider the undirected graph $G = \{V, E\}$, where V, the set of nodes, is $\{1, 2, 3, 4\}$ and E, the set of edges, is $\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{1, 4\}\}$. Draw the graph G. What are the degrees of each node? Indicate a path from node 3 to node 4 on your drawing of G.
Solution:



(a) Degree of Node 1 is 3
(b) Degree of Node 2 is 3
(c) Degree of Node 3 is 2
(d) Degree of Node 4 is 2
(e) A path from Node 3 to Node 4 is indicated above. It can consist of any path through the graph that starts on Node 3 and ends on Node 4 since the graph is not a directed graph.

2 Problems

1. (2 marks) Sipser, Prob. 0.12 (0.11 in 2nd edition): Find the error in the following proof that all horses are the same color.
CLAIM: In any set of h horses, all horses are the same color.
PROOF: By induction on h.

Base Case: For $h = 1$. In any set containing just one horse, all horses clearly are the same color.

Induction Step: For $k \geq 1$ assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$. Take any set H of $k + 1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all the horses in H_2 are the same color. Therefore all horses in H must be the same color, and the proof is complete.

Solution: The problem is that the inductive step does *not* work for all $h \geq 2$; in particular, it fails for $h = 2$. This is because when $h = 2$, we have $H_1 \cap H_2 = \emptyset$, and the argument here works only if we can guarantee $H_1 \cap H_2 \neq \emptyset$ (can you see why?).

2. (4 marks) Prove using induction that

$$\sum_{m=0}^{n} m = \frac{n(n+1)}{2}.$$

Solution: Observation: Since when $m = 0$, no value is contributed to the sum, we can instead evaluate the sum $\sum_{m=1}^{n} m = \frac{n(n+1)}{2}$.

Base Case ($n = 1$): Here, $\sum_{m=1}^{1} m = 1$, which also equals $1 \cdot (1+1)/2$.

Inductive Hypothesis: Assume the claim holds for $n = k$ for any $k \geq 1$.

Inductive Step: We prove the claim holds for $n = k+1$. Specifically,

$$\sum_{m=1}^{k+1} m = (1 + 2 + \cdots + k) + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{k^2 + k + 2k + 2}{2} = \frac{(k+1)(k+2)}{2},$$

as required, where the second equality holds by the Induction Hypothesis. Hence, by the principle of Mathematical Induction, the claim holds for all $n \geq 1$.